M9302 Mathematical Models in Economics

Game Theory – Brief Introduction

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What is Game Theory?

- We do not live in vacuum.
- Whether we like it or not, all of us are strategists.
- ST is art but its foundations consist of some simple basic principles.
- The science of strategic thinking is called Game Theory.

Where is Game Theory coming from?

- Game Theory was created by

  Von Neumann and Morgenstern (1944)

in their classic book

The Theory of Games and Economic Behavior

- Two distinct approaches to the theory of games:

1. Strategic/Non-cooperative Approach
2. Coalition/Cooperative Approach

Where is Game Theory coming from?

- The key contributions of John Nash:

1. The notion of Nash equilibrium
2. Arguments for determining the two-person bargaining problems

- Other significant names:

N-Nash, A-Aumann, S-Shapley&Selten, H-Harsanyi

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1.1.Static Games of Complete Information

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The static (simultaneous-move) games

- Informally, the games of this class could be described as follows:

- First, players simultaneously choose a move (action).

- Then, based on the resulting combination of actions chosen in total, each player receives a given payoff.

Example: Students' Dilemma

- Strategic behaviour of students taking a course:

- First, each of you is forced to choose between studying HARD or taking it EASY.

- Then, you do your exam and get a GRADE.

Static Games of Complete Information

Standard assumptions:

- Players move (take an action or make a choice) simultaneously at a moment

– it is STATIC

- Each player knows what her payoff and the payoff of the other players will be at any combination of chosen actions

– it is COMPLETE INFORMATION

Example: Students' Dilemma

Standard assumptions:

- Students choose between HARD and EASY SIMULTANEOUSLY.
- Grading policy is announced in advance, so it is known by all the students.

Simplification assumptions:

- Performance depends on CHOICE.
- EQUAL EFFICIENCY of studies.

The static (simultaneous-move) games

- Game theory answers two standard questions:

1. How to describe a type of a game?

2. How to solve the resulting game-theoretic problem?

How to describe a game?

- The normal form representation of a game contains the following elements:

1. PLAYERS – generally of number *n*
2. STRATEGIES – $s_i \in S_i$, for *i = 1,…,n*
3. PAYOFFS – $u_i = u_i\left(s_1,\ldots,s_n\right)$, for *i = 1,…,n*

- We denote the game of n-players by

G = $\left\{S_1,\ldots,S_n;u_1,\ldots,u_n\right\}$

Example: Students' Dilemma

Normal Form Representation:

1. Reduce the players to 2 – YOU vs. OTHERS

2. Single choice symmetric strategies

$$S_i = \{Easy, Hard\}$$, for *i* = *1,…,n*

3. Payoff function:

$$u_i = u_i(s_i, s_{-i}) = LEISURE_i(s_i) - GRADE_i(s_i, s_{-i})$$

Example: Students’ Dilemma

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a WEAK PASS (Grade C, or 3) and

who is under the average

FAIL (Grade F, or 5).

Example: Students' Dilemma

Leisure Rule: HARD study schedule devotes twice more time (leisure = 1) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	3	**-1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	1	1	**0**
	All Hard	1	3	**-2**

Example: Students' Dilemma

Bi-matrix of payoffs:

OTHERS

		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	-2,-2

How to solve the GT problem?

Solution Concepts:

- Strategic Dominance
- Nash Equilibrium (NE)

in static games of complete information

- Subgame-Perfect Nash Equilibrium (SPNE)

in dynamic games of complete information

- Bayesian Nash Equilibrium (BNE)

in static games of incomplete information

- Perfect Bayesian Equilibrium (PBNE)

in dynamic games of incomplete information

Strategic Dominance

Definition of a *strictly dominated strategy:*

- *Consider the normal-form game G =* $\{S_1,\ldots,S_n;u_1,\ldots,u_n\}$
- *Feasible strategy* $s_i^{'}$ *is strictly dominated by strategy* $s_i^{''}$

if i's payoff from playing $s_i^{'}$ *is strictly less than i's payoff from playing* $s_i^{''}$*:*

$$u_i\left(s_1,\ldots,s_{i-1},s_i^{'},s_{i+1},\ldots,s_n\right)<u_i\left(s_1,\ldots,s_{i-1},s_i^{''},s_{i+1},\ldots,s_n\right)$$

for each feasible combination $\left(s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n\right)$ *that can be constructed from the other players' strategy spaces* $S_1,\ldots,S_{i-1},S_{i+1},\ldots,S_n$.

Strategic Dominance

<u>Solution Principle:</u> Rational players do not play strictly dominated strategies.

The solution process is called “*iterated elimination of strictly dominated strategies*”.

Example: Students' Dilemma

- Solution by iterated elimination of strictly dominated strategies:

OTHERS

YOU	~~Easy~~	Hard
~~Easy~~	-1,-1	-3,0
Hard	0,-3	-2,-2

Easy is strictly dominated by Hard for YOU.

Easy is strictly dominated by Hard for OTHERS.

After elimination a single strategy combination remains:

{HARD; HARD}

Weaknesses of IESDS

- Each step of elimination requires a further assumption about what the players know about each other's rationality
- The process often produces a very imprecise predictions about the play of the game

Example: Students' Dilemma -2

- <u>Leisure Rule:</u> HARD study schedule devotes all their time (leisure = 0) to studying.

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	3	**-1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	0 / 1	1	**-1 / 0**
	All Hard	0 / 1	3	**-3 / -2**

Example: Students' Dilemma -2

- Solution by iterated elimination of strictly dominated strategies:

OTHERS

YOU		Easy	Hard
	Easy	-1 , -1	-3 , -1
	Hard	-1 , -3	-3 , -3

No single strategy could be eliminated:

{EASY / HARD; EASY / HARD}

Nash Equilibrium

- Definition (NE): In the n-player normal form game

G = $\{S_1,\ldots,S_n;u_1,\ldots,u_n\}$

the strategies $\left(s_1^*,\ldots,s_n^*\right)$ are a Nash equilibrium if,

for each player *i*,

s_i^* is (at least tied for) player *i*'s best response to the strategies specified for the n-1 other players:

$$u_i\left(s_1^*,\ldots,s_{i-1}^*,s_i^*,s_{i+1}^*\ldots,s_n^*\right)\geq u_i\left(s_1^*,\ldots,s_{i-1}^*,s_i,s_{i+1}^*\ldots,s_n^*\right)$$

for every feasible strategy s_i in S_i; that is, s_i^* solves

$$\max_{s_i\in S_i} u_i\left(s_1^*,\ldots,s_{i-1}^*,s_i,s_{i+1}^*\ldots,s_n^*\right)$$

Relation between Strategic Dominance and Nash Equilibrium

- If a single solution is derived through iterated elimination of strictly dominated strategies it is also a unique NE.
- The players' strategies in a Nash equilibrium always survive iterated elimination of strictly dominated strategies.

Example: Students' Dilemma - 2

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a **PASS (Grade B, or 2)** and

who is under the average

FAIL (Grade F, or 5).

Example: Students'Dilemma - 2

- Leisure Rule: HARD study schedule devotes all their time (leisure = 0) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	2 / 3	**0 / -1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	0	1	**-1**
	All Hard	0	2 / 3	**-2 / -3**

Example: Students' Dilemma -2

- Solution by iterated elimination of strictly dominated strategies:

OTHERS

YOU		Easy	Hard
	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

No single strategy could be eliminated:

{EASY / HARD; EASY / HARD}

Example: Students' Dilemma -2

- Nash Equilibrium Solution:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

Two Nash Equilibria:

{EASY / EASY; HARD / HARD}

Example: Students' Dilemma - 2

Some useful policy implications:

- Harsh grading of the mediocre behavior would motivate the rational students to study hard.
- Extremely time-consuming studies discourage rational students and make them hesitant between taking it easy and studying hard.

Summary

- The simplest class of games is the class of Static Games of Complete Information.
- By ‘static’ it is meant that players choose their strategies simultaneously without observing each other’s choices.
- ‘Complete information’ implies that the payoffs of each combination of strategies available are known to all the players.
- Static games of complete information are usually represented in normal form consisting of bi-matrix of player’s payoffs.

Summary

- A strategy is strictly dominated if it yields lower payoff than another strategy available to a player irrespective of the strategic choice of the rest of the players.
- The weakest solution concept in game theory is the iterated elimination of strictly dominated strategies. It requires too strong assumptions for player's rationality and often gives imprecise predictions.
- Nash Equilibrium is a stronger solution concept that produces much tighter predictions in a very broad class of games.

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Dynamic Games of Complete and Perfect Information

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Fast Revision on Lecture 1

- Strategic Games of Complete Information:

- Description
- Normal Form Representation
- Solution Concepts – IESDS vs. NE

How to solve the GT problem?

Solution Concepts:

- Strategic Dominance
- Nash Equilibrium (NE)

in static games of complete information

- Backwards Induction
- Subgame-Perfect Nash Equilibrium (SPNE)

in dynamic games of complete information

- Bayesian Nash Equilibrium (BNE)

in static games of incomplete information

- Perfect Bayesian Equilibrium (PBNE)

in dynamic games of incomplete information

Revision: Students' Dilemma -2 (simultaneous-move solution)

- Nash Equilibrium Solution:

OTHERS

YOU

	Easy	Hard
Easy	<u>0</u>,<u>0</u>	-3,-1
Hard	-1,-3	<u>-2</u>,<u>-2</u>

Two Nash Equilibria:

{EASY/EASY; HARD/HARD}

Dynamic (sequential-move) games

- Informally, the games of this class could be described as follows:
- First, only one of the players chooses a move (action).
- Then, the other player(s) moves.
- Finally, based on the resulting combination of actions chosen in total, each player receives a given payoff.

Example 1: Students' Dilemma -2 (sequential version)

- Strategic behaviour of students taking a course:
- First, only YOU are forced to choose between studying HARD or taking it EASY.
- Then, the OTHERS observe what YOU have chosen and make their choice.
- Finally, both You and OTHERS do exam and get a GRADE.

Will the simultaneous-move prediction be defined?

The dynamic (sequential-move) games

- The aim of the first lecture is to show:

1. How to describe a dynamic game?

2. How to solve the simplest class of dynamic games with complete and perfect information?

How to describe a dynamic game?

- The extensive form representation of a game specifies:

1. Who are the PLAYERS.

2.1. When each player has the MOVE.

2.2. What each player KNOWS when she is on a move.

2.3. What ACTIONS each player can take.

3. What is the PAYOFF received by each player.

Example 1: Students' Dilemma (Sequential Version)

Extensive Form Representation:

1. Reduce the players to 2 – YOU vs. OTHERS

2.1. First YOU move, then – OTHERS.

2.2. OTHERS know what YOU have chosen when they are on a move but YOU don't.

2.3. Both YOU and OTHERS choose an ACTION from the set $A_i = \{Easy, Hard\}$, for $i = 1,\ldots,n$

3. *Payoffs:*

$$u_i = u_i(a_i, a_{-i}) = LEISURE(a_i) - GRADE_i(a_i, a_{-i})$$

Example 1: Students' Dilemma -2 (Sequential Version)

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a PASS (Grade B, or 2) and

who is under the average

FAIL (Grade F, or 5).

Example 1: Students' Dilemma – 2 (Sequential Version)

<u>Leisure Rule:</u> HARD study schedule devotes all the time (leisure = 0) to studying distinct from the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	-2	0
	At least one Hard	2	-5	-3
Hard	At least one Easy	0	-1	-1
	All Hard	0	-2	-2

Dynamic Games of Complete and Perfect Information

- The simple class of dynamic games of complete and prefect information has the following general description:

1. Player 1 chooses and action a_1 from the feasible set A_1.
2. Player 2 OBSERVES a_1 and then chooses an action a_2 from the feasible set A_2.
3. Payoffs are $u_1(a_1,a_2)$ and $u_2(a_1,a_2)$.

Dynamic Games of Complete and Perfect Information

Standard assumptions:

- Players move at different, sequential moments

– it is DYNAMIC

- The players' payoff functions are common knowledge

– it is COMPLETE INFORMATION

- At each move of the game the player with the move knows the full history how the game was played thus far

– it is PERFECT INFORMATION

Example 1: Students' Dilemma -2 (Sequential Version)

Standard assumptions:

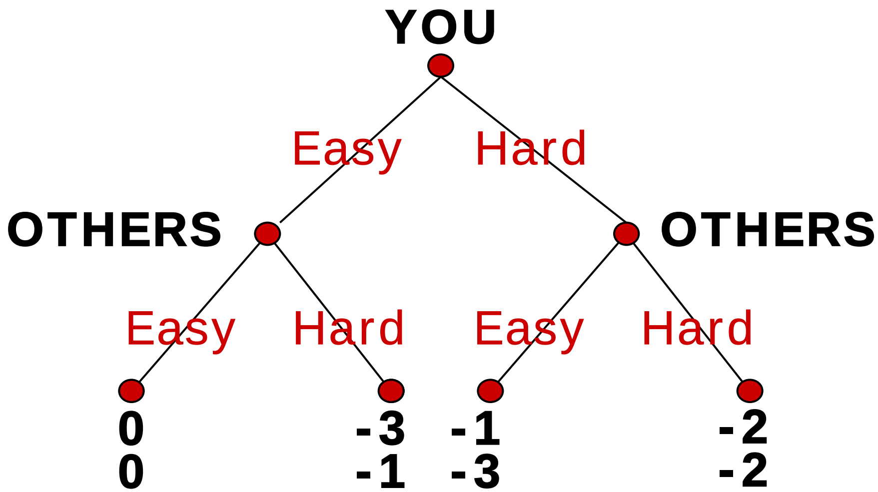
- Students choose between HARD and EASY SEQUENTIALLY.
- Grading is announced in advance, so it is COMMON KNOWLEDGE to all the students.
- Before making a choice in the second stage, OTHERS observe the choice of YOU in the first stage.

Simplification assumptions:

- Performance depends on CHOICE.
- EQUAL EFFICIENCY of studies.

Example 1: Students' Dilemma – 2 (Sequential Version)

Game Tree VS. Normal-Form



	(HARD, HARD)	**(HARD, EASY)**	**(EASY, HARD)**	**(EASY, EASY)**
HARD	-2,-2 (NE)	-2,-2	-1,-3	-1,-3
EASY	-3,-1	0,0 (NE)	-3,-1	0,0 (NE)

Backwards Induction

Solve the game from the last to the first stage:

- Suppose a unique solution to the second stage payoff-maximization:

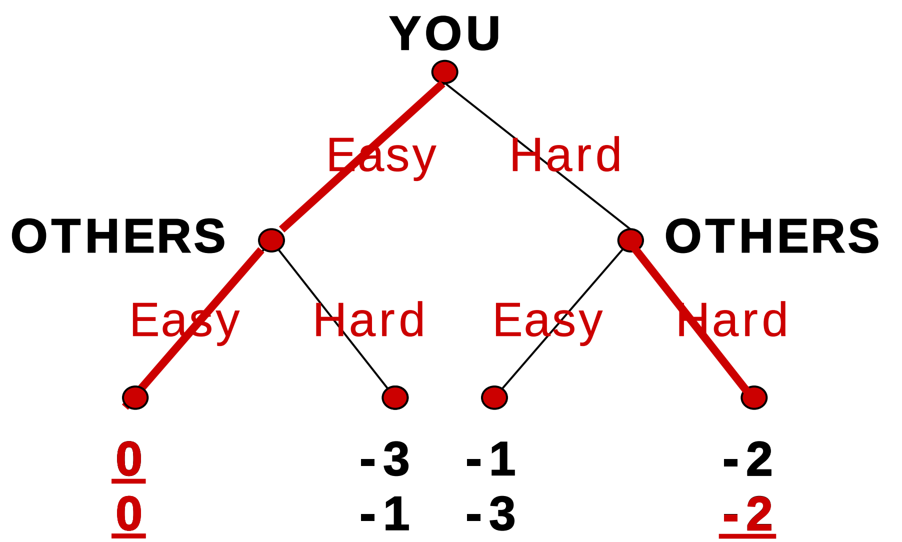
$$R_2(a_1) = \arg\max_{a_2 \in A_2} u_2(a_1, a_2)$$

- Then assume a unique solution to the first stage payoff-maximization:

$$a_1^* = \arg\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$$

- Call $(a_1^*, R_2(a_1^*))$ a *backwards-induction outcome.*

Example 1: Students' Dilemma – 2 (Sequential Version)



	(HARD,HARD)	(HARD,EASY)	(EASY,HARD)	(EASY,EASY)
HARD	-2,-2 (NE)	-2,-2	-1,-3	-1,-3
EASY	-3,-1	0,0 **(SPNE)**	-3,-1	**0,0 (NE)**

Example 2: Students' Dilemma -2 (with non-credible threat)

- Strategic behaviour of students taking a course:
- First, only YOU are forced to choose between studying HARD or taking it EASY.
- Then, the course instructor warns you:
- if YOU choose to study HARD in the first stage, all students get a WEAK PASS (C or 3)
- But if YOU choose to take it EASY, OTHERS still have a choice and YOU are on a threat to FAIL (F or 5)

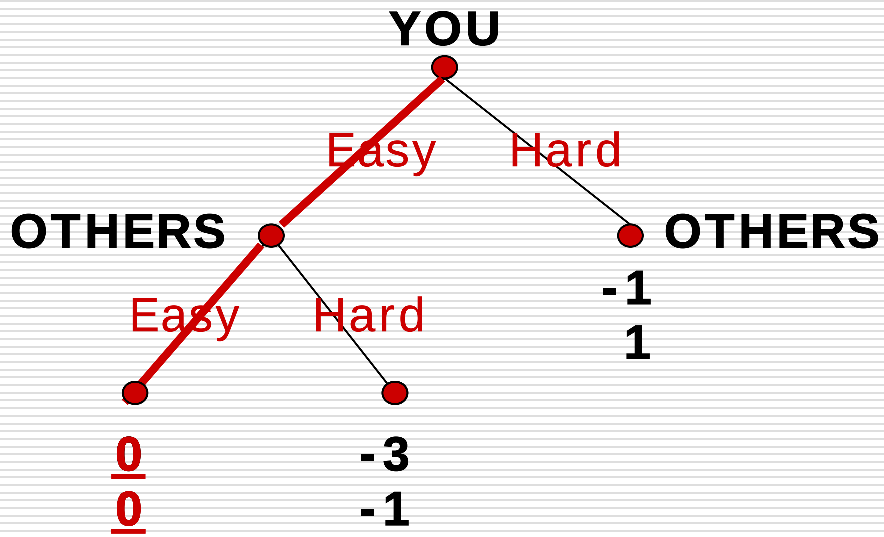
Is instructor's threat credible? Should YOU take it seriously?

Example 2: Students' Dilemma – 2 (with non-credible threat)

Leisure Rule: HARD study schedule devotes all the time (leisure = 0) to studying distinct from the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2,2	-2,-2	0,0
	At least one Hard	2,0	-5,-1	-3,-1
Hard	No Choice	0,2	-1,-1	-1,1

Example 2: Students' Dilemma – 2 (with non-credible threat)



Subgame Perfect Nash Equilibrium

Informal Definition:

- The only subgame-perfect Nash equilibrium is the backwards-induction outcome.
- The backwards-induction outcome does not involve non-credible threats.

Summary

- Dynamic (sequential-move) games represent strategic situations where one of the players moves before the other(s) allowing them to observe her move before making a decision how to move themselves.
- To represent a dynamic game it is more suitable to use extensive form in which in addition to players, their strategy spaces and payoffs, it is also shown **when each player moves** and **what she knows before moving**.

Summary

- Graphically a dynamic game could be represented by the so called “game tree”.
- the number of the subgames is equal to the number of decision nodes in the tree minus 1.
- Distinct from the static games of complete information, here the strategy set of the second player does not coincide with its set of feasible actions.
- Strategy in a dynamic game is a complete plan of action – it specifies a feasible action for each contingency (other player’s preceding move) in which given player might be called to act.

Summary

- Dynamic games of complete information are solved by backwards induction i.e. first the optimal outcome in the last stage of the game is defined to reduce the possible moves in the previous stages.
- Backwards induction outcome does not involve non-credible threats – it corresponds to the subgame-perfect Nash equilibrium as a refinement of the pure-strategy NE concept.

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Dynamic Games of Complete but Imperfect Information

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How to solve the GT problem?

Solution Concepts:

- Strategic Dominance
- Nash Equilibrium (NE)

in static games of complete information

- Backwards Induction
- Subgame-Perfect Nash Equilibrium (SPNE)

in dynamic games of complete information

- Bayesian Nash Equilibrium (BNE)

in static games of incomplete information

- Perfect Bayesian Equilibrium (PBNE)

in dynamic games of incomplete information

Revision

What is information set?

An information set for a player is a collection of decision nodes satisfying:

- the player has the move at every node in the information set, and
- when the play of the game reaches a node in the information set, the player with the move does not know which node in the information set is reached

Revision

- What does the extensive form representation of a game specifies?

1. Who are the PLAYERS.

2.1. When each player has the MOVE.

2.2. What each player KNOWS when she is on a move.

2.3. What ACTIONS each player can take.

3. What is the PAYOFF received by each player.

Dynamic games of complete but imperfect information

- Informally, the games of this class could be described as follows:
- First, Players 1 and 2 **simultaneously** choose actions a_1 and a_2 from feasible sets A_1 and A_2, respectively
- Second, players 3 and 4 observe the outcome of the first stage, (a_1, a_2), *and then simultaneously choose actions* a_3 and a_4 *from feasible sets* A_3 and A_4, respectively.
- Finally, based on the resulting combination of actions chosen in total, each player receives a given payoff $u_i(a_1,a_2,a_3,a_4)$ for $i=1,2,3,4$

Dynamic Games of Complete and Imperfect Information

Standard assumptions:

- Players move at different, sequential moments

– it is DYNAMIC

- The players' payoff functions are common knowledge

– it is COMPLETE INFORMATION

- At each stage of the game players move simultaneously

– it is IMPERFECT INFORMATION

Dynamic games of complete but imperfect information

- The aim of the third lecture is to show:

1. What is the difference between perfect and imperfect information?

2. How to solve games of complete but imperfect information?

Perfect vs. Imperfect Information

What is perfect information?

- when at each stage the player with the move knows the full history of the game thus far
- When each information set is a singleton

Then, what is imperfect information?

- When there is at least one non-singleton information set

How to solve dynamic games of imperfect information?

In a game of complete and perfect information BI eliminates noncredible threads. Why?

- Because each decision node represents a contingency in which a player might be called on to act.
- The process of working backwards thus amounts to forcing each player to consider carrying out each threat

How to solve dynamic games of imperfect information?

In a game of imperfect information BI does not work so simply. Why?

- Because working backwards would eventually lead us to a decision node in a non-singleton information set
- Then the player does not know whether or not that decision node is reached
- The player is forced to consider what it would eventually do if a node is really reached not in a contingency in which she is called on to act

How to solve dynamic games of imperfect information?

How to deal with the problem of nonsingleton information sets in BI?

Work backwards until a nonsingleton information set is encountered, then:

- Skip over it and proceed the tree until a singleton information set is found and solve for the subgame emanating from it - SGPNE
- Force the player with the move at the information set to consider what she would do if that information set was reached – Bayesian NE

Dynamic games of Complete but Imperfect Information – key terms

- Subgame – a piece of a game that remains to be played beginning at any point at which the complete history of the game thus far is common knowledge among the players, i.e.:

- begins at a singleton information set

- includes all the decision and terminal nodes following but not preceding the starting singleton decision node

- does not cut any (non-singleton) information sets.

Dynamic games of Complete but Imperfect Information – key terms

- Strategy – a complete plan of action – it specifies a feasible action which the player will take in each stage, for every possible history of play through the previous stage.

Dynamic games of Complete but Imperfect Information – SGPNE

- (Selten 1965) Subgame-perfect Nash Equilibrium (SGPNE)– a Nash equilibrium is subgame perfect if the players' strategies constitute a Nash equilibrium in every subgame.

Dynamic games of Complete but Imperfect Information – Summary

- BI fails to eliminate noncredible threads in the games of imperfect information because of the non-singleton information sets.
- Therefore a stronger solution concept called subgame-perfect N.E. is applied.
- SGPNE includes not only the best response to the unique action played in the first stage but full plan of action (strategy) how it would be best to respond to any possible action in the unobserved part of the game (subgame).

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Repeated Games

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Repeated Games

- The aim of the forth lecture is to describe a special subclass of dynamic games of complete and perfect information called repeated games
- Key question: Can threats and promises about future behavior influence current behavior in repeated relationships?

Repeated Games

- Let $G = \{A_1,\ldots,A_n; u_1,\ldots,u_n\}$ denote a static game of complete information in which player 1 through player n simultaneously choose actions a_1 through a_n

  from the action spaces A_1 through A_n.

Respectively, the payoffs are u(a,…,a) through u(a,…,a)

Allow for any finite number of repetitions.

- Then, G is called the stage game of the repeated game

Finitely Repeated Game

- Finitely repeated game: Given a stage game G, let G(T) denote the finitely repeated game in which G is played T times, with the outcomes of all preceding plays observed before the next play begins.
- The payoffs for G(T) are simply the sum of the payoffs from the T stage games.

Finitely Repeated Game

- In the finitely repeated game G(T), a subgame beginning at stage t+1 is the repeated game in which G is played $T\text{-}t$ times, denoted G($T\text{-}t$).
- There are many subgames that begin in stage $t+1$, one for each of the possible histories of play through stage t.
- The t^{th} stage of a repeated game ($t<T$) is not a subgame of the repeated game.

Example: 2-stage Students' Dilemma

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a WEAK PASS (Grade C, or 3) and

who is under the average

FAIL (Grade F, or 5).

Example: 2-stage Students' Dilemma

<u>Leisure Rule:</u> HARD study schedule devotes twice more time (leisure = 1) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	3	**-1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	1	1	**0**
	All Hard	1	3	**-2**

Example: 2-stage Students' Dilemma

Bi-matrix of payoffs:

		OTHERS	
		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	-2,-2

Repeat the stage game twice!

Example: 2-stage Students' Dilemma

Stage 2:

		OTHERS	
		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	-2,-2

Stage 1:

		OTHERS	
		Easy	Hard
YOU	Easy	-3,-3	-5,-2
	Hard	-2,-5	-4,-4

Finitely Repeated Game

- Proposition: If the stage game G has a unique Nash equilibrium then, for any finite T, the repeated game G(T) has a unique subgame-perfect outcome:
  - The Nash equilibrium of G is played in every stage.

Finitely Repeated Game

- What if the stage game has no unique solution?
  - If $G = \{A_1,\ldots,A_n; u_1,\ldots,u_n\}$ is a static game of complete information with multiple Nash equilibria, there may be subgame-perfect outcomes of the repeated game G(T) in which the outcome in stage $t<T$ is not a Nash equilibrium in G.

Example: 2-stage Students' Dilemma - 2

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a **PASS (Grade B, or 2)** and

who is under the average

FAIL (Grade F, or 5).

Example: 2-stage Students' Dilemma - 2

- <u>Leisure Rule:</u> HARD study schedule devotes all their time (leisure = 0) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	2	**0**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	0	1	**-1**
	All Hard	0	2	**-2**

Example: 2-stage Students' Dilemma - 2

OTHERS

		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

Suppose each player's strategy is:

- Play Easy in the 2nd stage if the 1st stage outcome is (Easy, Easy)
- Play Hard in the 2nd stage for any other 1st stage outcome

Example: 2-stage Students' Dilemma - 2

Stage 2:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

Stage 1:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-5,-3
	Hard	-3,-5	-4,-4

Example: 2-stage Students' Dilemma - 2

Stage 1:

OTHERS

		Easy	Hard
YOU	Easy	0,0	-5,-3
	Hard	-3,-5	-4,-4

The threat of player *i* to punish in the 2nd stage player j's cheating in the 1st stage is not credible.

Example: 2-stage Students' Dilemma - 2

Stage 1:

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

{Easy, Easy} Pareto-dominates {Hard, Hard} in the second stage. There is space for re-negotiation because punishment hurts punisher as well.

Example: 2-stage Students' Dilemma - 2

		OTHERS			
		Easy	Hard	C	O
YOU	Easy	0,0	-3,-1	-3,-3	-3,-3
	Hard	-1,-3	-2,-2	-3,-3	-3,3
	C	-3,-3	-3,-3	0,-2.5	-3,-3
	O	-3,-3	-3,-3	-3,-3	-2.5,0

Add 2 more actions and suppose each player's strategy is:

- Play Easy in the 2nd stage if the 1st stage outcome is (E, E)
- Play C in the 2nd stage if the 1st stage outcome is (E, w≠E)
- Play O in the 2nd stage if the 1st stage is (y≠E,z=E/H/C/O)
- Outcomes{C,C} and {O,O} are on the **Pareto frontier**.

Finitely Repeated Game

- Conclusion: Credible threats or promises about future behavior which leave no space for negotiation (Pareto improvement) in the final stage can influence current behavior in a finite repeated game.

Infinitely Repeated Game

- Given a stage-game G, let G(∞,δ) denote the infinitely repeated game in which G is repeated forever and the players share the discount factor δ.
- For each *t*, the outcomes of the t-1 preceding plays of G are observed.
- Each player's payoff in G(∞,δ) is the present value of the player's payoffs from the infinite sequence of stage games

Infinitely Repeated Game

- The history of play through stage *t* – in the finitely repeated game G(T) or the infinitely repeated game $G(\infty,\delta)$ – is the record of the player's choices in stages 1 through *t*.

Infinitely Repeated Game

- Strategy /in a repeated game/ - the sequence of actions the player will take in each stage, for each possible history of play through the previous stage.
- Subgame /in a repeated game/ - the piece of the game that remains to be played beginning at any point at which the complete history of the game thus far is common knowledge among the players.

Infinitely Repeated Game

- As in the finite-horizon case, there are as many subgames beginning at stage $t+1$ of $G(\infty,\delta)$ as there are possible histories through stage t.
- In the infinitely repeated game $G(\infty,\delta)$, each subgame beginning at stage $t+1$ is identical to the original game.

Infinitely Repeated Game

- How to compute the player's payoff of an infinitely repeated game?
  - Simply summing the payoffs of all stage-games does not provide a useful measure
  - Present value of the infinite sequence of payoffs:

$$\Pi_1 + \delta\Pi_2 + \delta^2\Pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1}\Pi_t$$

Infinitely Repeated Game

- Key result: Even when the stage game has a unique Nash equilibrium it does not need to be present in every stage of a SGP outcome of the infinitely repeated game.
- The result follows the argument of the analysis of the 2-stage repeated game with credible punishment.

Infinitely Repeated Game

		OTHERS		
	Easy	Hard	C	O
YOU Easy	0,0	-3,-1	-3,-3	-3,-3
Hard	-1,-3	-2,-2	-3,-3	-3,3
C	-3,-3	-3,-3	0,-2.5	-3,-3
O	-3,-3	-3,-3	-3,-3	-2.5,0

Instead of adding artificial equilibria that brings higher payoff tomorrow, the Pareto dominant action is played.

Example: Infinite Students' Dilemma

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a WEAK PASS (Grade C, or 3) and

who is under the average

FAIL (Grade F, or 5).

Example: Infinite Students' Dilemma

<u>Leisure Rule:</u> HARD study schedule devotes twice more time (leisure = 1) to studying than the EASY one (leisure = 2).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	2	3	**-1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	1	1	**0**
	All Hard	1	3	**-2**

Example: Infinite Students' Dilemma

Bi-matrix of payoffs:

OTHERS

		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	0,-3	<u>-2</u>,<u>-2</u>

Repeat the stage game infinitely!

Example: Infinite Students' Dilemma

- Consider the following trigger strategy:
  - Play Easy in the 1^{st} stage.
  - In the t^{th} stage if the outcome of all t-1 preceding stages has been (E, E) then play Easy,
  - Otherwise, play Hard in the t^{th} stage.
- Need to define δ for which the trigger strategy is SGPNE.

Example: Infinite Students' Dilemma

- Subgames could be grouped into 2 classes:
  - Subgames in which the outcome of at least one earlier stage differs from (E,E) – trigger strategy fails to induce cooperation
  - Subgames in which all the outcomes of the earlier stages have been (E,E) – trigger strategy induces cooperation

Example: Infinite Students' Dilemma

- If HARD is played in the 1st stage total payoff is:

$$0+(-2\delta)+(-2\delta^2)+\ldots=0-\frac{2\delta}{1-\delta}$$

- If EASY is played in the 1st stage, let the present discounted value be V:

$$V=-1+\delta V \Rightarrow V=-\frac{1}{1-\delta}$$

Example: Infinite Students' Dilemma

- In order to have a SGPE where (E, E) is played in all the stages till infinity the following inequality must hold:

$$-\frac{2\delta}{1-\delta} \leq V$$

- After substituting for V we get the following condition on δ:

$$\delta \geq \frac{1}{2}$$

Folk's Theorem

- In order to generalize the result of the SD to hold for all infinitely repeated games, several key terms need to be introduced:
- The payoffs $(x_1,\ldots,x_n)$ are called ***feasible*** in the stage game G if they are a convex (i.e. weighted average, with weights from 0 to1) combination of the pure-strategy payoffs of G.

Folk's Theorem

- The average payoff from an infinite sequence of stage-game payoffs is the payoff that would have to be received in every stage so as to yield the same present value as the player's infinite sequence of stage-game payoffs.
- Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\Pi_1, \Pi_2, \Pi_3 \ldots$ is: $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pi_t$

Folk's Theorem

- Folk's Theorem (Friedman 1971): Let G be a finite, static game of complete information. Let $(e_1,\ldots,e_n)$ denote the payoffs from a Nash Equilibrium of G, and let $(x_1,\ldots,x_n)$ denote any other feasible payoffs from G.
- If $x_i > e_i$ for every player *i* and if δ is sufficiently close to 1, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty,\delta)$ that achieves $(x_1,\ldots,x_n)$ as the average payoff.

Folk's Theorem

- Reservation payoff r_i – the largest payoff player i can guarantee receiving, no matter what the other players do.
- It must be that $r_i \leq e_i$, since if r_i were greater than e_i, it would not be a best response for player i to play her Nash equilibrium strategy.
- In SD, $r_i = e_i$ but in the Cournot Duopoly Game (and typically) $r_i < e_i$

Folk's Theorem

- Folk's Theorem (Fudenber & Maskin 1986): If (x_1, x_2) is a feasible payoff from G, with $x_i > r_i$ for each *i*, then for δ sufficiently close to 1, there exists a SGPNE of $G(\infty,\delta)$ that achieves (x_1, x_2) as the average payoff even if $x_i < e_i$ for one or both of the players.

Folk's Theorem

- What if δ is close to 0?
  - 1st Approach: After deviation follow the trigger strategy and play the stage-game equilibrium.
  - 2nd Approach (Abreu 1988): After deviation play the N.E. that yields the lowest payoff of all N.E. Average strategy can be lower than the one of the 1st approach if switching to stage game is not the strongest credible punishment.

Summary

- Key question that stays behind repeated games is whether threats or promises about future behavior can affect current behavior in repeated relationships.
- In finite games, if the stage game has a unique Nash Equilibrium, repetition makes the threat of deviation credible.
- If stage game has multiple equilibria however there could be a space for negotiating the punishment in the next stage after deviation.

Summary

- In infinitely repeated games, even when the stage game has a unique Nash equilibrium it does not need to be present in every stage of a SGP outcome of the infinitely repeated game.
- Folk's theorem implies that if there is a set of feasible payoffs that are larger than the payoffs from the stage game Nash equilibrium, and the discount factor is close to one, there is a SGPNE at which the set of higher feasible payoffs is achieved as an average payoff.

Summary

- Extension of the Folk's theorem implies that for 2-player infinitely repeated game if there is a set of feasible payoffs that exceed the reservation ones, the outcome that yields these feasible payoffs as an average payoff could constitute a SGPNE even if they are smaller than the payoffs from the stage game N.E., provided that the discount factor is close to 1.
- If the discount factor is close to 0, an alternative strategy to the trigger one (where the stage game equilibrium is played after deviation) is to play instead the N.E. that yields the lowest payoff of all N.E. This might be stronger credible punishment.

M9302 Mathematical Models in Economics

Static Games of Incomplete Information

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INVESTMENTS IN EDUCATION DEVELOPMENT

Revision

- When a combination of strategies $\left(s_1^*, \ldots, s_n^*\right)$ is a Nash equilibrium?
  - If for any player *i*, is player *i*'s best response to the strategies of the n-1 other players
- Following this definition we could easily find game that have no Nash equilibrium:
  - Example: Penny Game

Example: Penny Game

P2

P1		Heads	Tails
	Heads	-1,$\underline{1}$	$\underline{1}$,-1
	Tails	$\underline{1}$,-1	-1,$\underline{1}$

No pair of strategies can satisfy N.E.:
If match (H,H), (T,T) – P1 prefers to switch
If no match (H,T), (T,H) – P2 prefers to switch

Extended definition of Nash Equilibrium

- In the 2-player normal-form game $G=\{S_1,S_2;u_1,u_2\}$, the **MIXED** strategies $\left(p_1^*, p_2^*\right)$ are a Nash equilibrium if each player's mixed strategy is a best response to the other player's **MIXED** strategy
- Hereafter, let's refer to the strategies in S_i as player *i*'s **pure strategies**
- Then, a **mixed strategy** for player *i* is a probability distribution over the strategies in S_i

Example: Penny Game

- In Penny Game, S_i consists of the two **pure strategies** H and T
- A **mixed strategy** for player *i* is the probability distribution (*q,1-q*), where q is the probability of playing H, and *1-q* is the probability of playing T, $0 \le q \le 1$
- Note that the mixed strategy (0,1) is simply the pure strategy T, likewise, the mixed strategy (1,0) is the pure strategy H

Example: Penny Game

- Computing P1's best response to a mixed strategy by P2 represents P1's uncertainty about what P2 will do.
- Let (q,1-q) denote the mixed strategy in which P2 plays H with probability q.
- Let (r, 1-r) denote the mixed strategy in which P1 plays H with probability r.

Example: Penny Game

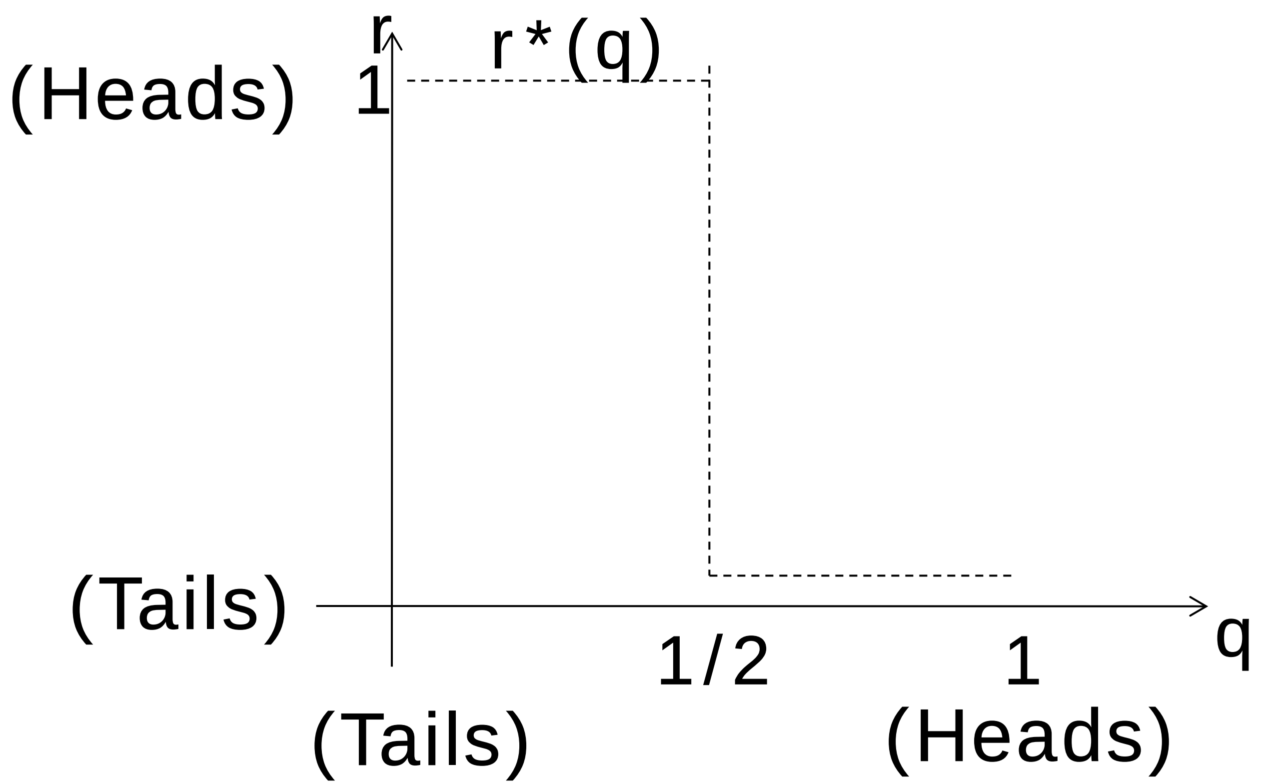
- P1's expected payoff from playing (r,1-r) when P2 plays (q,1-q) is:

$$rq\cdot(-1)+r(1-q)\cdot 1+(1-r)(1-q)\cdot(-1)+(1-r)\cdot q=$$

$$=(2q-1)+r(2-4q)$$

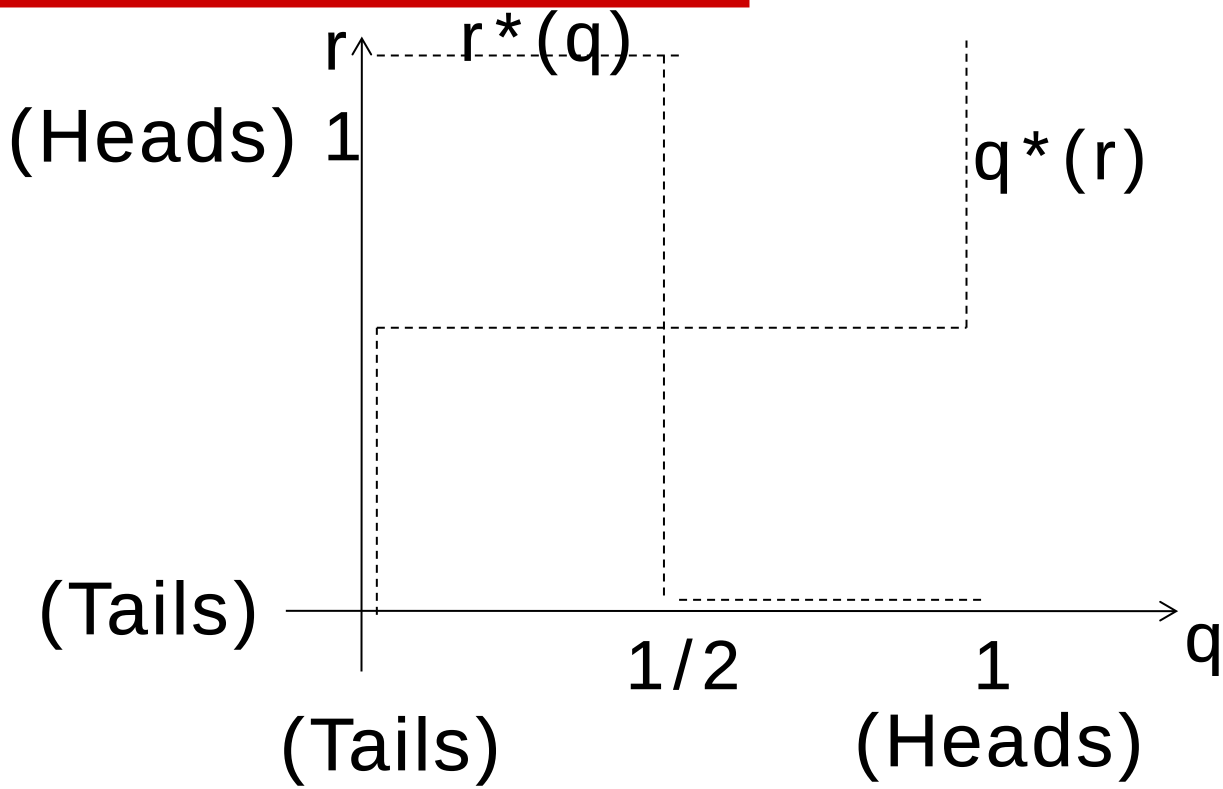
- which is increasing in r for q<1/2 (i.e. P1's best response is r=1) and decreasing in r for q>1/2 (i.e. P1's best response is r=0).
- P1 is indifferent among all mixed strategies (r,1-r) when q=1/2.

Example: Penny Game



Because there is a value of q such that r*(q) has more than one value, r*(q) is called P1's **best-response correspondence**.

Example: Penny Game



The intersection of the best-response correspondences r*(q) and q*(r)yields the mixed-strategy N.E. in Penny Game.

General Definition of Mixed Strategy

- Suppose that player i has K pure strategies, $S_i = \{s_{i1}, \ldots, s_{iK}\}$
- Then, a **mixed strategy** for player *i* is a

probability distribution $(p_{i1}, \ldots, p_{iK})$, where p_{ik} is the probability that player *i* will play strategy s_{ik}, k=1,...,K

- Respectively, $0 \leq p_{ik} \leq 1$ for k=1,...,K

and $p_{i1} + \ldots + p_{iK} = 1$

- Denote an arbitrary mixed strategy by p_i

General Definition of Nash Equilibrium

- Consider 2-player case where strategy sets of the two players are $S_1 = \{s_{11}, \ldots, s_{1J}\}$ and $S_1 = \{s_{11}, \ldots, s_{1K}\}$, respectively
- P1's expected payoff from playing the mixed strategies $p_1 = (p_{11}, \ldots, p_{1J})$ is:

$$v_1\left(p_1^*, p_2^*\right) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} \cdot p_{2k} u_1\left(s_{1j}, s_{2k}\right)$$

- P2's expected payoff from playing the mixed strategies $p_2 = (p_{21}, \ldots, p_{2K})$ is:

$$v_2\left(p_1^*, p_2^*\right) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} \cdot p_{2k} u_2\left(s_{1j}, s_{2k}\right)$$

General Definition of Nash Equilibrium

- For the pair of mixed strategies $\left(p_1^*, p_2^*\right)$ to be a Nash equilibrium, p_1^* must satisfy:

$$v_1\left(p_1^*, p_2^*\right) \geq v_1\left(p_1, p_2^*\right)$$

- for every probability distribution p_1 over S_1, and p_2^* must satisfy:

$$v_2\left(p_1^*, p_2^*\right) \geq v_2\left(p_1^*, p_2\right)$$

- for every probability distribution p_2 over S_2.

Existence of Nash Equilibrium

- Theorem (Nash 1950): In the n-player normal-form game $G=\{S_1,\ldots,S_n;u_1,\ldots,u_n)$, if n is finite and S_i is finite for every i then there exists at least one Nash equilibrium, possibly involving mixed strategies.
- Proof consists of 2 steps:
  - Step1: Show that any fixed point of a certain correspondence is a N.E.
  - Step 2: Use an appropriate fixed-point theorem to show that the correspondence must have a fixed point.

Revision

- What is a strictly dominated strategy?
  - If a strategy s_i is strictly dominated then there is no belief that player *i* could hold such that it would be optimal to play s_i.
- The converse is also true when mixed strategies are introduced
  - If there is no belief that player i could hold such that it would be optimal to play s_i, then there exists another strategy that strictly dominates s_i.

Example /mixed strategy dominance/:

		P2	
		B1	B2
P1	A1	3,—	0,—
	A2	0,—	3,—
	A3	1,—	1,—

For any belief of P1, A3 is not a best response even though it is not strictly dominated by any pure strategy. A3 is strictly dominated by a mixed strategy (½ , ½, 0)

Example /mixed strategy best response/:

		P2	
		B1	B2
P1	A1	3,—	0,—
	A2	0,—	3,—
	A3	2,—	2,—

For any belief of P1, A3 is not a best response to any pure strategy but it is the best response to mixed strategy (q,1-q) for $1/3<q<2/3$.

Introduction to Incomplete Information

- What is complete information?
- What must be incomplete information then?

Introduction to Incomplete Information

A game in which one of the players does not know for sure the payoff function of the other player is a game of INCOMPLETE INFORMATION

<u>Example:</u>

Cournot Duopoly with Asymmetric Information about Production

Static Games of Incomplete Information

- The aim of this lecture is to show:
  - How to represent a static game of incomplete information in normal form?
  - What solution concept is used to solve a static game of incomplete information?

Normal-form Representation

- ADD a TYPE parameter t_i to the payoff function -> $u_i(a_1,\ldots,a_n;\ t_i)$

A player is uncertain about

{other player's payoff function} = {other player's type t_{-i}}

where $t_{-i} = (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$

Normal-form Representation

- ADD probability measure of types to account for uncertainty:
- $p_i(t_{-i}|t_i)$ - player *i*'s belief about the other players' types (t_{-i}) given player *i's* knowledge of her own type, t_i.
- Bayesian Theorem

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)}$$

Normal-form Representation

- PLAYERS
- ACTIONS – $A_1, \ldots, A_n$; $A_i = \{a_{i1}, \ldots, a_{in}\}$
- TYPES – $T_i = \{t_{i1}, \ldots, t_{in}\}$
- System of BELIEFS - $p_i(t_{-i} / t_i)$
- PAYOFFS - $u_i(a_1, \ldots, a_n; t_i)$

which is briefly denoted as

$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$$

Timing of the Bayesian Games (Harsanyi, 1967)

- Stage 1: Nature draws a type vector

$t = (t_1,\ldots,t_n)$, where t_i is drawn from the set of possible types T_i.

- Stage 2: Nature reveals t_i to player i but not necessarily to the other players.
- Stage 3: Players simultaneously choose actions i.e. player i chooses a_i from the feasible set A_i.
- Stage 4: Payoffs $u_i(a_1,\ldots,a_n;\ t_i)$ are received.

Strategy in a Bayesian Game

- In a static Bayesian game, a strategy for player *i* is a function , where for each type t_i in Ti, $s_i(t_i)$ specifies the action from the feasible set A_i that type t_i would choose if drawn by nature.
- In a ***separating strategy***, each type t_i in T_i chooses a different action a_i from A_i.
- In a **pooling strategy**, in contrast, all types choose the same action.

How to solve a Bayesian game?

- Bayesian Nash Equilibrium:

In the static Bayesian game

$$G = \{A_1,\ldots,A_n;T_1,\ldots,T_n;p_1,\ldots,p_n;u_1,\ldots,u_n\}$$

the strategies $s^* = (s_1^*,\ldots,s_n^*)$ *are a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each of i's types ti in Ti,* $s_i^*(t_i)$ *solves:*

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(t_1),\ldots,s_{i-1}^*(t_{i-1}),a_i,s_{i+1}^*(t_{i+1}),\ldots,s_n^*(t_n);t\right)p_i\left(t_{-i}\middle|t_i\right)$$

That is, no player wants to change his or her strategy, even if the change involves only one action by one type.

Existence of a Bayesian Nash Equilibrium

- In a finite static Bayesian game (i.e., where n is finite and $(A_1,\ldots,A_n)$ and $(T_1,\ldots,T_n)$ are all finite sets), there exists a Bayesian Nash equilibrium, perhaps in mixed strategies.

<u>Mixed-strategy in a Bayesian game:</u>

Player *i* is uncertain about player *j's* choice not because it is random but rather because of **incomplete information about j's payoffs**.

Examples: Battle of Sexes; Cournot Competition with Asymmetric Information

Summary

- Game Theory distinguishes between pure and mixed strategy
- Mixed strategy is a probability distribution over the strategy set
- To be efficient in solving games including uncertainty, N.E. concept needs to be extended and defined for mixed strategies
- Games with uncertainty are called Bayesian games and their solution concept – Bayesian N.E.

M9302 Mathematical Models in Economics

Dynamic Games of Incomplete Information

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How to solve the GT problem?

Solution Concepts:

- Strategic Dominance
- Nash Equilibrium (NE)

in static games of complete information

- Backwards Induction
- Subgame-Perfect Nash Equilibrium (SPNE)

in dynamic games of complete information

- Bayesian Nash Equilibrium (BNE)

in static games of incomplete information

- Perfect Bayesian Equilibrium (PBNE)

in dynamic games of incomplete information

How to describe a dynamic game?

- The extensive form representation of a game specifies:

1. Who are the PLAYERS.

2.1. When each player has the MOVE.

2.2. What each player KNOWS when she is on a move.

2.3. What ACTIONS each player can take.

3. What is the PAYOFF received by each player.

How to describe a game of incomplete information?

- A game in which one of the players does not know for sure the payoff function of the other player is a game of INCOMPLETE INFORMATION
- Thanks to Harsanyi (1967) games of incomplete information could be represented as dynamic games of complete but imperfect information
- For the purpose, in the first stage a neutral player (Nature) is introduced to decide what will be the type (payoffs) of the players which is private information for at least one of them.

Normal-form Representation

- PLAYERS
- ACTIONS – $A_1, \ldots, A_n; A_i = \{a_{i1}, \ldots, a_{in}\}$
- TYPES – $T_i = \{t_{i1}, \ldots, t_{in}\}$
- System of BELIEFS - $p_i(t_{-i} / t_i)$
- PAYOFFS - $u_i(a_1, \ldots, a_n; t_i)$

which is briefly denoted as

$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$$

The static (simultaneous-move) games

- The aim of the sixth lecture is to show:

1. How to strengthen the Bayesian equilibrium to hold in dynamic games?

2. How to define the resulting solution concept?

Example: Students’ Dilemma - 4

- Strategic behaviour of students taking a course:

- First, YOU and OTHERS might be called on to choose between studying HARD or taking it EASY but YOU could reject if YOU feel UNSURE. Then, the game ends.

- If YOU do not reject, students do the exam and get a grade.

Example: Students' Dilemma

Standard assumptions:

- Students choose between HARD and EASY SIMULTANEOUSLY.
- Grading is announced in advance, so it is COMMON KNOWLEDGE to all the students.

Simplification assumptions:

- Performance depends on CHOICE.
- EQUAL EFFICIENCY of studies.

Example: Students' Dilemma

Normal Form Representation:

1. 2 players – YOU vs. OTHERS
2. Single choice strategies

$$S_{YOU} = \{Easy, Hard, Unsure\}$$

$$S_{OTHERS} = \{Easy, Hard\}$$

3. Payoff function:

$$u_i = u_i(s_i, s_{-i}) = \begin{cases} LEISURE_i(s_i) - GRADE_i(s_i, s_{-i}) \\ \text{Rejection Payoff}_i = \{\text{-1.5}\} \end{cases}$$

Example: Students' Dilemma

Grading Policy:

the students over the average have a STRONG PASS (Grade A, or 1),

the ones with average performance get a WEAK PASS (Grade C, or 3) and

who is under the average

FAIL (Grade F, or 5).

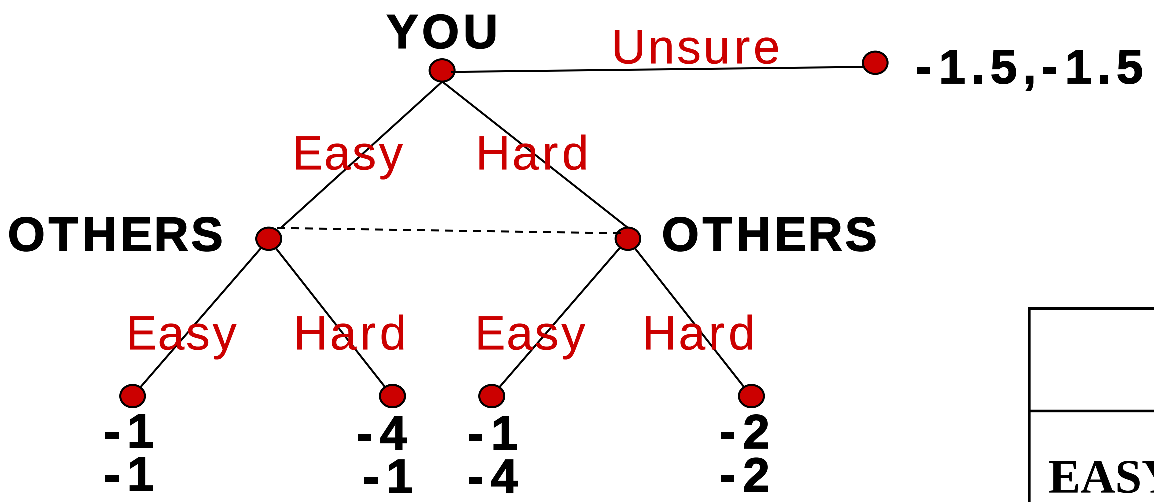
Example 1: Students' Dilemma – 4

<u>Leisure Rule:</u> HARD study schedule devotes all the time (leisure = 0) to studying distinct from the EASY one (leisure = 1).

Player i's choice	Others' choice	LEISURE	GRADE	Player i' payoff
Easy	All Easy	1	-2	-1
	At least one Hard	1	-5	-4
Hard	At least one Easy	0	-1	-1
	All Hard	0	-2	-2
Rejection				-1.5

Example 1: Students' Dilemma – 3

Game Tree VS. Normal-Form



	EASY	HARD
EASY	-1,-1 (NE)	-4,-1
HARD	-1,-4	-2,-2
UNSURE	-1.5,-1.5	-1.5,-1.5 (NE)

Example 1: Students' Dilemma – 4

- There is a single subgame with 2 NE: {E, E} and {U, H}
- However, if YOU do not reject,

{E,E} is not the only equilibrium

- How to strengthen the solution concept to allow for strict prediction?

Dynamic Games of Incomplete Information

One way to strengthen the equilibrium concept is to impose the following two requirements:

- Requirement 1: At each information set the player with the move must have a belief about which node in the information set has been reached by the play of the game.
- Requirement 2: Given their beliefs, the players' strategies must be sequentially rational. That is, at each information set the action taken by the player with the move must be optimal given the player's belief and the other player's subsequent strategies.

Example 1: Students' Dilemma – 4

- Given OTHER's belief that YOU would move EASY with probability *p*, their expected payoff of playing:
  - EASY is:

$$E\Pi_{OTHERS}(EASY) = (-1)\cdot p + (-4)\cdot(1-p) = 3p-4$$

  - HARD is:

$$E\Pi_{OTHERS}(HARD) = (-1)\cdot p + (-2)\cdot(1-p) = p-2$$

- HARD weakly dominates EASY which rules out {E,E} for $p<1$.

Dynamic Games of Incomplete Information

Requirements 1 and 2 insist that the players have beliefs and act optimally given these beliefs, but not that these beliefs be reasonable. For the solution to be strict, the following requirement must also hold:

- Requirement 3: At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

- Definition: For a given equilibrium in a given extensive-form game, an information set is **on (off) the equilibrium path** if it will (not) be reached with positive probability when the game is played according to the equilibrium strategies.

Example 1: Students' Dilemma – 4

- If the Nash equilibrium {U, H} holds, the belief of OTHERS should be $p<1$.
- It is not reasonable for OTHERS to choose H if they believe that YOU will play E for sure ($p=1$).
- Suppose there is a mixed-strategy equilibrium in which YOU plays E with probability q_1, H with probability q_2 and U with probability $1- q_1- q_2$.
- In conformity with the Bayes' rule,

  OTHERS' belief is forced by requirement 3 to be $p = q_1/(q_1+q_2)$.

Perfect Bayesian Equilibrium

- In simple dynamic games of incomplete information – including the signaling game – requirements 1 through 3 constitute the definition of a **perfect Bayesian equilibrium**.
- In richer games, however, more requirements need to be imposed:
  - Requirement 4: At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.
- Definition: A perfect Bayesian equilibrium consists of strategies and beliefs satisfying Requirements 1 through 4.

Summary

- In order to rule out non-credible threats or promises, the Bayesian equilibrium concept needs to be strengthened.
- In the definition of the perfect Bayesian Nash equilibrium (PBNE) beliefs are elevated to the level of importance of strategies.
- For the simple games of incomplete information, PBNE consists of 3 basic requirements on players' beliefs.
- Richer dynamic games have more specific requirements concerning the beliefs off the equilibrium path that also need to be satisfied.