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2-kategorie

Diplomová práce

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Vedoucí práce: John Denis Bourke, PhD. **Brno 2020**

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Abstrakt

V práci se zabýváme výsledky o koherenci v teorii 2-kategorií. V obecném pojetí jsou to podmínky, za kterých jsou pseudoalgebry pro danou 2-monádu ekvivalentní striktním algebrám. Například každá monoidální kategorie je monoidálně ekvivalentní striktní monoidální kategorii.

Kromě těchto výsledků o koherenci pseudoalgeber se v práci zaměřujeme i na větu o koherenci pro laxní algebry, která byla prvně dokázána Stevem Lackem, a uvádíme její alternativní důkaz. Tato věta zobecňuje výsledky o adjunkci mezi kategorií a její Kleisliho kategorií pro danou monádu a také tvrzení, že existuje kanonická adjunkce mezi laxním funktorem a striktním 2-funktorem.

Abstract

The thesis is devoted to coherence results in 2-category theory. In the general setting, these assert conditions under which are pseudoalgebras for a 2-monad equivalent to strict ones. For example every monoidal category is monoidally equivalent to a strict monoidal category.

In this thesis we give exposition on such results. We moreover focus on coherence theorem for lax algebras that has first been proven by Steve Lack and we give a more conceptual proof of this theorem. This result generalizes results about free-forgetful adjunction of a category and its Kleisli category for a monad, as well as the fact that there is a canonical adjunction between a lax functor and a strict one.

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Oficiální zadání:
Many of the key structures in category theory - adjoints, Kan extensions, monads, limits and so on - are naturally understood 2-categorically and the subject of 2-category theory has been in development since the late 1960's/early 1970's. The first aim of this project will be to understand aspects of classical 2-category theory. A possible further aim will be to investigate the recent application of classical 2-category theory (including that of Riehl-Verity) to studying weak higher dimensional structures, such the (infinity,1)-categories of Joyal, Lurie et al.

Literatura: (1) Emily Riehl. Category theory in Context. Dover Publications. 2016. (2) Steve Lack. A 2-categories companion. In "Towards higher categories". Springer 2010. (3) Riehl, E. and Verity, D. The 2-category theory of quasi-categories. Advances in Mathematics. 2015.

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Poděkování

Děkuji svým rodičům za podporu během celého mého studia. Děkuji svému vedoucímu za návrh tématu a za všechny rady, připomínky a návrhy.

Prohlášení

Prohlašuji, že jsem svoji diplomovou práci vypracoval samostatně pod vedením vedoucího práce s využitím informačních zdrojů, které jsou v práci citovány.

Brno 17. srpna 2020

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Miloslav Štěpán

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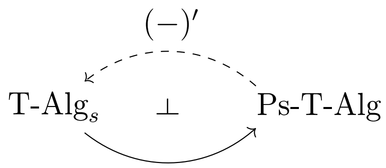
Introduction

Coherence theorems appear in several places in category theory, the most famous example being the coherence for monoidal categories that says that every diagram for a monoidal category built using the defining associativity/unit isomorphisms commutes.

The modern approach to coherence is to move from category theory to 2-category theory (where in addition to objects and morphisms we also have "morphisms between morphisms" or the *2-cells*) and exhibit categorical structures as algebras for a 2-monad. Although the definition of a 2-monad T is almost the same as that for ordinary monads, the higher dimension allows us to not just consider ordinary (strict) T-algebras but now we also have lax T-algebras (pseudo-T-algebras) where the algebra laws do not hold strictly but only "up to a 2-cell" ("up to an invertible 2-cell").

With this weaker notion, we can exhibit non-strict categorical structures (such monoidal categories, symmetric monoidal categories, pseudofunctors, bicategories) as pseudo-algebras for a 2-monad, with the strict structures (strict monoidal category, strict symmetric monoidal category, 2-functor, 2-category) being strict algebras for the 2-monad. Coherence theorem then says that under certain conditions, every pseudo-algebra for a 2-monad is equivalent to a strict one.

A canonical way to turn a pseudo-algebra to a strict one is to provide a left 2-adjoint to the inclusion 2-functor of strict algebras into pseudo-algebras:



It turns out that this 2-adjoint exists if the category of strict algebras admits a certain kind of 2-categorical colimit called a *codescent object*. Next, a canonical way to obtain an equivalence $A \simeq A'$ between every pseudoalgebra A and its strictification A' is to show that the unit of this 2-adjunction is an equivalence. This happens if the forgetful 2-functor from the 2-category of strict algebras to the base 2-category preserves said codescent objects.

The thesis is structured as follows. The first chapter serves as an elementary introduction to 2-category theory. In the second chapter we provide the definitions of a 2-monad and its various notions of algebras and we also prove some of their properties. In the third chapter, in sections 3.1 and 3.2, we give an exposition on codescent objects and their uses in the theory of coherence in 2-category theory. In the section 3.3 we prove a lax analogue of the coherence result and develop the underlying theory a bit further. In the final section 3.4 we apply the results we obtained in the previous section to the cases of identity 2-monad and lax functor 2-monad.

Proposition 2.1, Theorem 2.2 and Theorem 3.2 are folklore. Theorem 3.14, Theorem 3.2, and the proof of Theorem 3.16 appear to be new.

Chapter 1

Introduction to 2-category theory

In this chapter we provide the very basics necessary to understand the next two chapters, as well as provide examples of 2-categorical structures one may come across. Main references used for this chapter are [10], [14],[18]. The third listed provides an overview of almost all 2-categorical concepts.

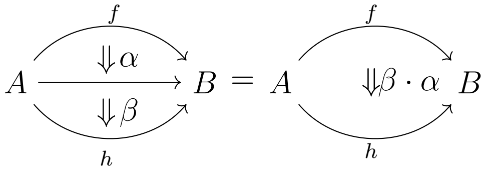
1.1 Language of 2-categories

Definition 1. A (strict) 2-category $\mathcal{K}$ consists of the following data:

- a class of objects (0-cells) ob $\mathcal{K}$, denoted $A, B, \ldots$
- a class of morphisms (1-cells) $\mathcal{K}(A, B)$ for each two objects $A, B \in \mathcal{K}$, denoted $f : A \to B, g, h \ldots$
- a class of 2-cells, i.e. morphisms between 1-cells, denoted $\alpha : f \Rightarrow g, \beta, \gamma \ldots$

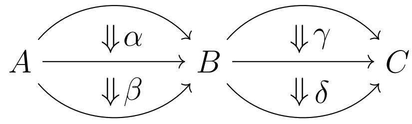
such that:

- objects and morphisms form a category $\mathcal{K}_0$ with identities $1_A : A \to A$
- for each $A, B \in$ ob$\mathcal{K}$ the class $\mathcal{K}(A, B)$ is a category, objects being 1-cells $f : A \to B$, morphisms 2-cells $\alpha : f \Rightarrow f'$. The composition of 2-cells in this category will be referred to as *vertical composition* and denoted $\alpha \cdot \beta$ (some authors may use $\alpha\beta$). Identity 2-cells will be denoted as 1_f.



- There is also a category whose objects are objects of $\mathcal{K}$ and a morphism $C \to D$ is a 2-cell $\alpha : f \Rightarrow g$ (f, g being 1-cells $C \to D$). Composition of morphisms in this category will be referred to as *horizontal composition* and denoted $\beta\alpha$ (some authors use $\beta * \alpha$). Identity $C \to C$ in this category then must be equal to $1_{1:C\to C}$ defined in the previous step.

- Given a diagram:



The order in which we compose doesn't matter, i.e. the *middle-four interchange law* holds:

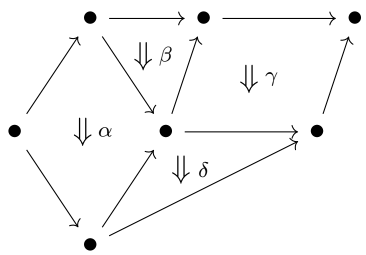
$$(\delta \cdot \gamma)(\beta \cdot \alpha) = (\delta\beta) \cdot (\gamma\alpha)$$

Definition 2. Given a 2-category $\mathcal{K}$, there are 2-categories $\mathcal{K}^{op}$ - where we reverse the direction of all 1-cells, and $\mathcal{K}^{co}$ where all 2-cells are reversed. Thus $\mathcal{K}(A, B) = \mathcal{K}^{op}(B, A)$ and $\mathcal{K}^{co}(A, B) = \mathcal{K}(A, B)^{op}$. Reversing 1-cells and 2-cells, we get $\mathcal{K}^{coop} = \mathcal{K}^{opco}$.

The notation $\mathcal{K}_0$ is used to denote the *underlying 1-category*, which has the same objects and 1-cells as $\mathcal{K}$, but we ignore the 2-cells.

Remark 1 (Pasting diagrams). By *whiskering* of a 2-cell $\alpha : h_1 \Rightarrow h_2 : B \to C$ with 1-cells $f : A \to B$, $g : C \to D$, denoted $g\alpha f$, is meant a horizontal composition $(1_g)\alpha(1_f)$. A horizontal composition of general 2-cells can be then defined using whiskering.

Given any directed planar graph with a *source* and a *sink* ("all paths start in the source and end in the sink"), we may label its vertices, edges and interior faces with objects, 1-cells and 2-cells of a 2-category $\mathcal{K}$ (all 2-cells have to go in the same direction), this is roughly what a *pasting diagram* is.
For example consider a diagram like this:



Using whiskering, it is simple to find out what the composition of these 2-cells should be. Like in this picture, there are usually more choices for the order in which to compose the 2-cells. What is important though is the fact that no matter the choice of the order, the final composition always ends up being the same. This is referred to as a *pasting theorem*. Its formulation and proof has not been made precise until the 90's in the short paper [28], which uses induction and basic graph theory.

The category of all small categories, denoted Cat, is symmetric monoidal cartesian closed, the hom object $[\mathcal{C}, \mathcal{D}]$ is the functor category. A strict 2-category (with small hom categories) is then precisely a Cat-category, and many 2-categorical concepts such as 2-functors, 2-natural transformations, 2-adjoints, 2-categorical Yoneda lemma, 2-monadicity theorem can be obtained from enriched category theory by setting $\mathcal{V} = \text{Cat}$.

Example 1. A fundamental example is the 2-category Cat with small categories, functors and natural transformations being objects, 1-cells and 2-cells respectively. As a symmetric monoidal closed category, there is a canonical structure of a Cat-category (or 2-category) on Cat. The vertical composition of two natural transformations $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ is just composition of their components, while the component of horizontal transformation of

$$C \underset{G_1}{\overset{F_1}{\Downarrow \gamma}} D \underset{G_2}{\overset{F_2}{\Downarrow \delta}} E$$

is a transformation $\delta\gamma : F_2F_1 \Rightarrow G_2G_1$ whose component at $c \in C$ is defined as either leg of this commutative square:

$$\begin{array}{ccc} F_2F_1c & \xrightarrow{\delta_{F_1c}} & G_2F_1c \\ {\scriptstyle F_2\gamma_c}\downarrow & & \downarrow{\scriptstyle G_2\gamma_c} \\ F_2G_1c & \xrightarrow[\delta_{G_1c}]{} & G_2G_1c \end{array}$$

It comes from the internal composition morphism $M : [\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}, \mathcal{C}]$.

Generally speaking, Cat has the same relationship to 2-category theory as Set has to ordinary category theory. Representable 2-functors will be Cat valued, the 2-categorical Yoneda lemma provides an isomorphism of categories (and not just sets), Cat is 2-categorically complete just as Set is complete, and so on ...

Example 2. There is a 2-category of small categories, functors and **natural isomorphisms**, denoted Cat_g.

Example 3. Any category is trivially a 2-category whose only 2-cells are identities on each morphism (we say that it's *locally discrete*).

Example 4. Given a general monoidal category $\mathcal{V}$, there is a 2-category $\mathcal{V}$-Cat of small $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. Special case is of course $\mathcal{V} = \text{Cat}$, in which we obtain the 2-category 2-Cat of locally small 2-categories, 2-functors, 2-natural transformations.

Example 5. A class of categories with finite products, product preserving functors and natural transformations form a 2-category. Same goes for any other (classes) of limits or colimits.

Example 6. Given a category $\mathcal{E}$ with pullbacks, we have a notion of a *category internal* to $\mathcal{E}$ as well as the notions of *internal functors* and *internal natural transformations*. These form a 2-category $\text{Cat}(\mathcal{E})$. For the definitions and properties of internal categories see [3][Section 2.5] for example.

An internal category to Cat is precisely a *double category* (see [10] for the notion), an internal category to Set is a small category. An internal category to Vect is what's called a Baez–Crans *2-vector space*.

Example 7. A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ consists of a category $\mathcal{V}_0$ together with an object $I \in \mathcal{V}_0$ and natural isomorphisms

$$a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$
$$l : (I \otimes -) \Rightarrow 1_{\mathcal{V}_0}$$
$$r : (- \otimes I) \Rightarrow 1_{\mathcal{V}_0}$$

subject to two *coherence axioms*, see [15]. If all of the a, l, r are identities, we call the structure a *strict monoidal category*. A *lax monoidal functor* $(\mathcal{V}, \otimes) \to (\mathcal{V}', \otimes')$ between monoidal categories is a functor $F : \mathcal{V}_0 \to \mathcal{V}_0'$ together with a morphism $\overline{F}_0 : I \to FI$ and a natural transformation

$$\overline{F}_{A,B} : FA \otimes FB \to F(A \otimes B)$$

that is subject to two further *coherence axioms*. If $\overline{F}_0$ is an isomorphism and $\overline{F}$ is a natural isomorphism, we call such functor just a *monoidal functor*. If they're identities (so that we have $F(A \otimes B) = FA \otimes FB$ and $FI = I$), it's called a *strict monoidal functor*. The same story goes with transformation of lax monoidal functors (a *monoidal natural transformation*). It is a natural transformation α between underlying functors that further satisfies certain compatibility conditions with $\overline{F}$ and $\overline{F}_0$[1].

All of these form objects, morphisms and 2-cells for various 2-categories. In particular we have 2-categories MonCat (of monoidal categories, monoidal functors, monoidal natural transformations) and StrMonCat (of strict monoidal categories, strict monoidal functors and monoidal natural transformations). *Coherence results* then tell us that every monoidal category is monoidally equivalent to a strict monoidal category. We will see this in the third chapter.

Example 8. A *multicategory* $\boldsymbol{M}$ consists of a class of objects (denoted $X, Y, Z \dots$), class of morphisms (denoted $f : (X_1, \dots, X_n) \to Y$), each of which has a domain consisting of a finite sequence of objects in $\boldsymbol{M}$. There is identity morphisms for each object and a **composition operation** that is subject to associativity and unit axioms. A *morphism* of multicategories maps objects to objects and arrows to arrows, preserving units and compositions. There is also a notion of a *transformation* of morphisms, which is like a natural transformation but it is "multilinear" in a sense. These form a 2-category Multicat. See [6] for the notions.

Example 9. The collection Top of topological spaces, continuous functions and *homotopies* of continuous maps does not form a 2-category because composition of homotopies as 2-cells is not associative. It is however associative up to homotopy equivalence of homotopies, so if we instead consider 2-cells as equivalence classes of this equivalence, we get a 2-category. Note that all 2-cells are then invertible.

[1] We will see what all these conditions are in greater generality in the next chapter.

A 2-category $\mathcal{K}$ is *locally preordered* if between each two morphisms $f, g : A \to B$ there is at most one 2-cell, which is denoted as $f \sqsubseteq g$ (this means that each hom category is preordered). Horizontal composition of 2-cells gives us that if $f \sqsubseteq g$ and $f' \sqsubseteq g'$, then $ff' \sqsubseteq gg'$ (if those 1-cells are composable). If Ord denotes the category of preorders, $\mathcal{K}$ is equally an Ord enriched category.

Example 10. There is a locally preordered[2] 2-category Rel whose objects are sets, a morphism $f : X \to Y$ is a relation from X to Y (i.e. a subset $R \subseteq X \times Y$). A 2-cell $R \sqsubseteq R'$ between relations from X to Y exists if and only if $R \subseteq R'$.

Definition 3. Let $\mathcal{K}$, $\mathcal{L}$ be 2-categories. A *2-functor* $F : \mathcal{K} \to \mathcal{L}$ is given by:

- a functor $F_0 : \mathcal{K}_0 \to \mathcal{L}_0$ on the underlying categories
- a family of functors for each pair of objects A, B :

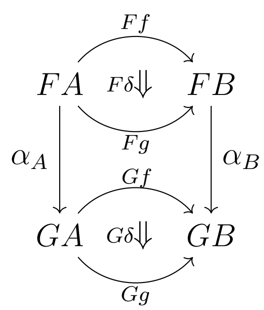
$$F_{A,B} : \mathcal{K}(A, B) \to \mathcal{K}(FA, FB) \tag{1.1}$$
$$(f : A \to B) \mapsto (Ff : F_0A \to F_0B) \tag{1.2}$$

such that the conditions $F(1_f) = 1_{Ff}$ and $F(\alpha\beta) = F\alpha F\beta$ are satisfied for all morphisms and suitable 2-cells.

Example 11. A 2-functor $F : \mathcal{K} \to \mathcal{L}$ between locally preordered 2-categories is an ordinary functor that also respects the partial order on each hom category, i.e. $f \sqsubseteq g$ implies $Ff \sqsubseteq Fg$.

Example 12. Given a locally small 2-category $\mathcal{K}$, there is a pair of *representable 2-functors* $\mathcal{K}(A, -) : \mathcal{K} \to \mathrm{Cat}$, $\mathcal{K}(-, A) : \mathcal{K}^{op} \to \mathrm{Cat}$. A good (and easy) exercise is to think about how they're defined.

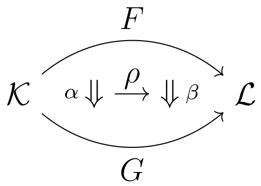
Definition 4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be 2-functors and let $\alpha : F \to G$ be a natural transformation of the underlying 1-functors. We say that it's *2-natural* if for any 2-cell $\delta : f \Rightarrow g$ the following meta-diagram commutes; i.e. $\alpha_B F\delta = G\delta\alpha_A$:



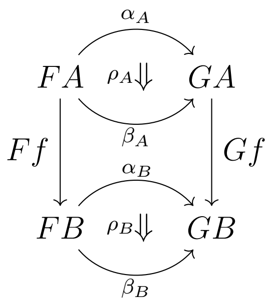
Just as the class of functors $[\mathcal{C}, \mathcal{D}]$ (for ordinary categories $\mathcal{C}, \mathcal{D}$) has the structure of a category, the class of 2-functors between 2-categories $\mathcal{K}, \mathcal{L}$ will have a structure of a 2-category. We need a notion of a morphism between 2-natural transformations, and this is precisely what a modification is.

[2]Its hom-categories are actually posets so we may call it being *locally posetal*.

Definition 5. Given 2-natural transformations α, β between 2-functors F, G between 2-categories $\mathcal{K}, \mathcal{L}$, a *modification* between them, denoted $\rho : \alpha \to \beta$,



is a collection of 2-cells $\rho_A : \alpha_A \Rightarrow \beta_A$ (in $\mathcal{L}$) for each object A in $\mathcal{K}$, such that the following meta-diagram commutes for all $f : A \to B$, i.e. $Gf\rho_A = \rho_B Ff$:



Example 13. If $\mathcal{K}, \mathcal{L}$ are locally preordered, every two pasting diagrams with the same domain and codomain are the same, so the 2-natural transformation α is just an ordinary natural transformation. An existence of a modification

$$\rho : \alpha \to \beta : F \Rightarrow G : \mathcal{K} \to \mathcal{L}$$

reduces down to the assertion that for all objects A in $\mathcal{K}$, we have

$$\alpha_A \sqsubseteq \beta_A.$$

We can do horizontal and vertical compositions of modifications, we can also whisker them with natural transformations from both sides (as well as whiskering them with 2-functors) in an obvious way. The next definition thus makes sense.

Definition 6. Given two 2-categories $\mathcal{K}, \mathcal{L}$, denote by $[\mathcal{K}, \mathcal{L}]$ the 2-category of 2-functors, 2-natural transformations and modifications.

Remark 2. 2-Cat is again cartesian closed symmetric monoidal category and thus has a canonical structure of a 2-Cat-category, or *a (strict) 3-category.*

Example 14. The 2-category $[1, \mathcal{K}]$ for a 2-category $\mathcal{K}$ is isomorphic to $\mathcal{K}$.

Definition 7. Given a pair of 2-functors between 2-categories $F : \mathcal{K} \to \mathcal{L}$, $G : \mathcal{L} \to \mathcal{K}$, we say that F is left 2-adjoint to G, denoted $F \dashv G$, if there is an isomorphism

$$\mathcal{L}(FA, B) \cong \mathcal{K}(A, GB)$$

that is 2-natural in A, B.

Remark 3. The 2-adjunction can be expressed in many equivalent ways, analogous to the 1-dimensional case. See the beginning of [5] for the general, $\mathcal{V}$-enriched setting. For example the adjunction $F \dashv U$ between 2-functors may be specified by two 2-natural transformations $\epsilon : FU \Rightarrow 1$, $\eta : 1 \Rightarrow UF$ satisfying the triangle identities.

A 2-adjunction can be weakened to the notion of a *pseudoadjunction* if we only require that ϵ, η are pseudonatural and moreover satisfy the triangle laws only up to certain invertible modifications. See [22, Definition 1.1]. By Yoneda lemma for bicategories, this can be equally expressed as an equivalence of hom categories:

$$\mathcal{L}(FA, B) \simeq \mathcal{K}(A, GB),$$

that is pseudonatural in each A, B.

Example 15. There is a trivial 2-adjunction:

$$\text{2-Cat} \underset{F_*}{\overset{(-)_0}{\rightleftarrows}} \text{Cat} \qquad (\top) \tag{1.3}$$

in which $\mathcal{A}_0$ for a 2-category $\mathcal{A}$ is its underlying category, and $F_*\mathcal{C}$ regards a category $\mathcal{C}$ as a 2-category that only has identity 2-cells[3].

Definition 8. We say that 2-categories $\mathcal{K}, \mathcal{L}$ are *2-equivalent* if we have 2-functors $F : \mathcal{K} \to \mathcal{L}$ and $G : \mathcal{L} \to \mathcal{K}$ and 2-natural isomorphisms $\eta : 1 \cong GF$, $\epsilon : FG \cong 1$[4].

Utilizing the existence of 2-cells in a 2-category, many notions can be weakened. Take the definition of a 2-functor for example. We could assume that $F(fg)$ is not equal to $Ff \cdot Fg$, but rather there is an isomorphism or just a 2-cell $FfFg \Rightarrow F(fg)$ as well as a 2-cell $1_{FA} \Rightarrow F(1_A)$ that satisfy certain coherence axioms. We would obtain the notions of a *pseudofunctor* or a *lax functor* (see [1][Chapter 4]). One natural example of a pseudofunctor is:

Example 16. Fix a class Φ of small diagrams. Given a small category $\mathcal{A}$, its *free completion under* Φ*-shaped colimits*, denoted $\Phi\mathcal{A}$, can be realized as the closure under Φ-colimits of its image in the Yoneda embedding $\mathcal{A} \hookrightarrow [\mathcal{A}^{op}, \text{Set}]$. This provides us with a pseudofunctor:

$$\Phi : \text{Cat} \to \Phi\text{-Cat}$$

given by left Kan extensions. The 2-category on the right hand side consists of small Φ-cocomplete categories, Φ-colimit preserving functors and natural transformations. It can be shown that this **pseudofunctor** is pseudoadjoint to the inclusion 2-functor $\Phi\text{-Cat} \hookrightarrow \text{Cat}$.

---

[3] If we have a monoidal category $\mathcal{V}$ with good properties (in particular if it has coproducts), this example is easily generalized to a 2-adjunction between $\mathcal{V}$-Cat and Cat, see [15, 2.5].

[4] Any 2-equivalence can be promoted to 2-adjoint 2-equivalence in an obvious way.

Weakening the notion of a 2-natural transformation, we get:

Definition 9. Given two 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, a *lax natural transformation* $\alpha : F \Rightarrow G$ consists of:

- a 1-cell $\alpha_A : FA \to GA$ for each object A,
- a 2-cell α_f for each $f : A \to B$:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd} \qquad \Downarrow \alpha_f$$

satisfying:

- (*Unity*): $\alpha_{1_A} = 1_{\alpha_A}$
- (*Composition*): for any pair of composable morphisms f, g, we have:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] \arrow[d, "Fg"'] & GB \arrow[d, "Gg"] \\ FC \arrow[r, "\alpha_C"'] & GC \end{tikzcd} \ (\Downarrow \alpha_f, \Downarrow \alpha_g) \quad = \quad \begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "F(gf)"'] & GA \arrow[d, "G(gf)"] \\ FC \arrow[r, "\alpha_C"'] & GC \end{tikzcd} \ (\Downarrow \alpha_{gf})$$

- (*Naturality*): Collection $\alpha := (\alpha_f)_f$ forms a 1-natural transformation between 1-functors $\mathcal{D}(\alpha_A, 1) \circ G_{AB} \Rightarrow \mathcal{D}(1, \alpha_B) \circ F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, GB)$, meaning that the following commutes[5]:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"] \arrow[d, bend right=60, "Fg"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd} \ (F\delta \Downarrow,\ \alpha_f \Downarrow) \quad = \quad \begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Fg"'] & GA \arrow[d, "Gg"'] \arrow[d, bend left=60, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd} \ (\alpha_g \Downarrow,\ G\delta \Downarrow) \tag{1.4}$$

[5] Note that if α is a 2-natural transformation, i.e. α_f's are identities, then the equation 1.4 is exactly the 2-naturality condition.

We say that the lax transformation is *pseudonatural* if α_f is invertible for all morphisms f. Reversing the direction of all 2-cells, we get an *oplax natural transformation.* There is an obvious way of (vertical) composition of those weak natural transformations. We thus have:

Definition 10. Denote $\mathrm{Lax}[\mathcal{K}, \mathcal{L}]$ the 2-category of 2-functors, lax natural transformations and modifications. Denote $\mathrm{Psd}[\mathcal{K}, \mathcal{L}]$ the 2-category of 2-functors, pseudonatural transformations and modifications.

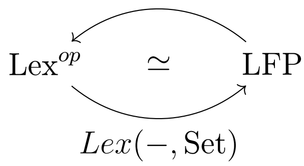
An appropriate weak version of 2-adjunction and 2-equivalence are following:

Definition 11. We say that a pseudofunctor $F : \mathcal{K} \to \mathcal{L}$ is a *biequivalence* if each hom functor $F_{A,B} : \mathcal{K}(A, B) \to \mathcal{L}(FA, FB)$ is an equivalence of categories and moreover F is "biessentially surjective" on objects, meaning that for any $C \in \mathcal{L}$ there exists $K \in \mathcal{K}$ such that $FK \simeq C$.

Example 17. (Gabriel-Ulmer duality) Denote by

- **Lex** the 2-category of small finitely complete categories, finite limit preserving functors and natural transformations, and
- **LFP** the 2-category of locally finitely presentable categories, finitary right adjoint functors and natural transformations

Gabriel-Ulmer duality then states that there is a biequivalence:

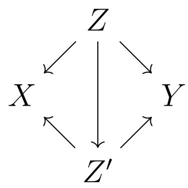


We can weaken the notion of a 2-category itself as well. A *bicategory* has again objects, 1-cells and 2-cells. The 2-cells again have vertical and horizontal composition, but the composition of 1-cells is associative and unital only up to an isomorphism, i.e. there are isomorphism 2-cells:

$$a_{hgf} : (hg)f \cong h(gf)$$
$$r_f : f \cdot 1_A \cong f$$
$$l_f : 1_B \cdot f \cong f$$

that are natural in $f : A \to B$ and are further required to satisfy certain *coherence axioms* (see [1]). Where 2-category theory has isomorphisms, bicategory theory will have equivalences. For example bicategorical Yoneda lemma gives a pseudonatural equivalence of categories and bicategorical limit is defined only up to a pseudonatural equivalence. A morphism of bicategories (also referred to as a *homomorphism*) is a variant of either a pseudofunctor or a lax functor, similarly with lax and oplax natural transformations. We can define a biadjunction as well as a biequivalence between pseudofunctors in a similar way as above.

Example 18. Given a category $\mathcal{E}$ with pullbacks, there is a bicategory $\mathrm{Spn}(\mathcal{E})$ whose objects are objects from $\mathcal{E}$. A morphism $X \to Y$ is a *span* $X \leftarrow Z \to Y$. A 2-cell between spans $X \leftarrow Z \to Y$ and $X \leftarrow Z' \to Y$ is a morphism $Z \to Z'$ commuting with the legs of the span:



Vertical and horizontal composition can be easily defined using pullbacks.

Example 19. Given a monoidal category $(\mathcal{V}, \otimes, a, l, r, I)$, define its *suspension* $\sum \mathcal{V}$ as a bicategory with a single object. A 1-cell x is an object of $\mathcal{V}$, a composition of 1-cells x, y being $x \otimes y$. A 2-cell $x \Rightarrow y$ is a morphism $x \to y$ in $\mathcal{V}$. A vertical composition of 2-cells is ordinary composition of morphisms in $\mathcal{V}$. A horizontal composition of 2-cells $x \to y$ and $x' \to y'$ is the tensor product of those morphisms $x \otimes x' \to y \otimes y'$ in $\mathcal{V}$.

Remark 4. Small bicategories, their homomorphisms and (op)lax natural transformations do **not** form a 2-category due to several obstructions. They do however form a 2-category once we use *icons* as 2-cells. The notion was introduced in the paper [19] and has since served as a useful tool to study bicategories using 2-category theory.

This 2-cell between homomorphisms $F, G : \mathcal{A} \to \mathcal{B}$ of bicategories is defined only when $FA = GA$ for all objects $A \in \mathcal{A}$, and is defined to be collection of natural transformations $F_{A,B} \Rightarrow G_{A,B} : \mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$ for each tuple of objects in $\mathcal{A}$ that satisfies two further axioms. The resulting 2-category, denoted Bicat_2 in the paper, has a number of interesting properties. For example:

- There is a full embedding $\mathrm{MonCat} \hookrightarrow \mathrm{Bicat}_2$,
- Bicat_2 is biequivalent to Cat_2 - the 2-category of (small) 2-categories, pseudofunctors and icons,
- It is the 2-category of algebras for some 2-monad T on the 2-category of Cat-graphs.

See [19]: Theorem 4.1, Theorem 4.4, Section 6.2.

1.2 Yoneda, limits and colimits

Given a 2-category $\mathcal{K}$ and an object A, we have a pair of representable 2-functors $\mathcal{K}(A, -), \mathcal{K}(-, A)$ that are now valued in the 2-category Cat.

Theorem 1.1 (Yoneda lemma for 2-categories)**.** *Let $\mathcal{K}$ be a 2-category. Let $A \in \mathcal{K}$ be an object and $F : \mathcal{K} \to Cat$ a 2-functor. Then there is an isomorphism of categories:*

$$[\mathcal{K}, Cat\,](\mathcal{K}(A, -), F) \cong F(A) \tag{1.5}$$

Sketch of a proof. 2-natural transformation $\alpha : \mathcal{K}(A, -) \Rightarrow F$ will be again sent to an object $\alpha_A(1_A) \in F(A)$. Considering a modification $\rho : \alpha \to \beta$, its component at $A \in \mathcal{K}$ is a natural transformation ρ_A between functors α_A and $\beta_A : \mathcal{K}(A, A) \to F(A)$. This modification will ρ will then be sent to its component at 1_A;

$$(\rho_A)_{1_A} : \alpha_A(1_A) \to \beta_A(1_A)$$

On the other hand, any object $x \in FA$ induces a 2-natural transformation β, whose component at B is defined as:

$$\beta_B(f) = Ff(x)$$
$$\beta_B(\delta : f \Rightarrow g) = (F\delta)_x$$

□

Remark 5. The isomorphism in the Yoneda lemma can further be proven to be 2-natural in objects A and 2-functors F, similar to the 1-categorical situation described in [29][Theorem 2.2.4, Remark 2.2.7.]. This fact (as well as the Yoneda lemma) is really true in enriched category theory (see [15, 2.4]).

Remark 6. We have two 2-functors $y : A \mapsto \mathcal{K}(A, -)$, $y : A \mapsto \mathcal{K}(-, A)$ that are called *Yoneda embeddings*. They can be proven to be *fully faithful*, i.e. inducing isomorphism on hom categories.

Definition 12. Let $\mathcal{K}$ be a 2-category and $F : \mathcal{P} \to \mathcal{K}, W : \mathcal{P} \to \mathrm{Cat}$ be 2-functors. The limit of F weighted by W is an object $\{F, W\} \in \mathcal{K}$ together with an isomorphism of categories for each B:

$$\Phi_B : \mathcal{K}(B, \{F, W\}) \cong [\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}(B, F-))$$

that is furthermore 2-natural in $B \in \mathcal{K}$. In other words, it induces a 2-natural isomorphism in $[\mathcal{K}^{op}, \mathrm{Cat}]$:

$$\Phi : \mathcal{K}(-, \{F, W\}) \cong [\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}(-, F?)) \tag{1.6}$$

Remark 7. By the Yoneda lemma, this 2-natural transformation Φ is fully determined by a certain 2-natural transformation $\eta : W \Rightarrow \mathcal{K}(\{F, W\}, F-)$ (which we call *a limit cone*). The fact that each Φ_B is **bijection on objects** and **fully faithful** means that η satisfies one-dimensional and two-dimensional universal properties respectively:

- (*One-dimensional universal property*): Given any 2-natural transformation $\gamma : W \Rightarrow \mathcal{K}(B, F-)$, there is a unique 1-cell $\theta : B \to \{F, W\}$ such that $\mathcal{K}(\theta, F-) \cdot \eta = \gamma$.

- (*Two-dimensional universal property*): Given any modification

  $$\rho : \mathcal{K}(\theta, F-) \cdot \eta \to \mathcal{K}(\theta', F-) \cdot \eta,$$

  there is a unique 2-cell $\overline{\theta}$ such that[6]

  $$\mathcal{K}(\overline{\theta}, F-)\eta = \rho.$$

[6]Note that $\mathcal{K}(\overline{\theta}, F-)$ is a modification that is being whiskered by η.

Remark 8. Given a product (or any other limit) $A \times B$ of A, B in a 1-category $\mathcal{C}$, in order to show that two maps $f, g : C \to A \times B$ are equal, according to the universal property of products it is enough to show that their post composition with product projections $p_i f, p_i g$ are equal.

The same thing works for 2-categorical limits but in addition we can do this for 2-cells as well. For example in order to show that 2-cells

$$\gamma, \delta : \theta \Rightarrow \theta' : A \to \{W, F\}$$

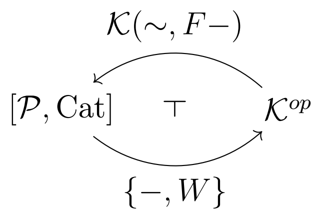
are equal, it is enough to show that their whiskers with η are equal:

$$\mathcal{K}(\gamma, F-)\eta = \mathcal{K}(\delta, F-)\eta$$

Remark 9. One-dimensional universal property guarantees that the object $\{F, W\}$ is unique up to an isomorphism, again by the Yoneda lemma.

Remark 10. When $W : \mathcal{P} \to \mathrm{Cat}$ is constant functor at 1, we call the limit $\{W, F\}$ *conical*. It is denoted as $\lim F$. In ordinary category theory, the notion of weighted limit can be reduced to that of ordinary (conical) limit (see [15, 3.4]). This is not possible in 2-category theory (or in general in $\mathcal{V}$-category theory), see [15][Section 3.9]. In the 2-category theory at least, conical limits are not expressive enough for most purposes.

Remark 11. If the 2-category $\mathcal{K}$ has all limits for a particular weight W, this may equally be seen as a 2-adjunction:



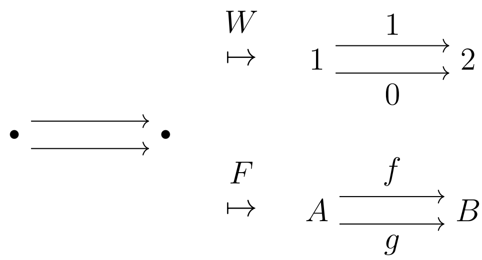
w ith unit of the 2-adjunction being again η and $\{-, W\}$ being a 2-functor defined in an obvious way.

Definition 13. A *colimit* of $F : \mathcal{P}^{op} \to \mathcal{K}$ weighted by $W : \mathcal{P} \to \mathrm{Cat}$, denoted $F * W$, is a limit of F^{op} in $\mathcal{K}^{op}$ weighted by W. Namely, there is a 2-natural isomorphism:

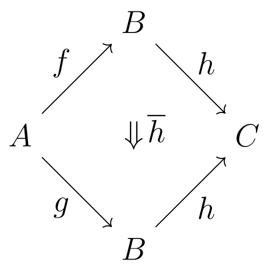
$$\mathcal{K}(F * W, -) \cong [\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}(F?, -)) \tag{1.7}$$

Remark 12. If $\mathcal{P}$, F, W are simple enough, one can usually simplify the definition of a given (co)limit and state it in more elementary terms. We now introduce several examples of 2-categorical (co)limits and will mention their simplified form only.

Definition 14. A *coinserter* of a pair of parallel morphisms is a colimit of a 2-functor F weighted by W, both given as:

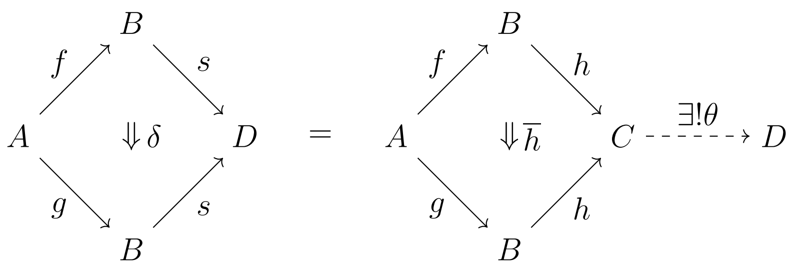


A colimit cone is a universal 2-cell:

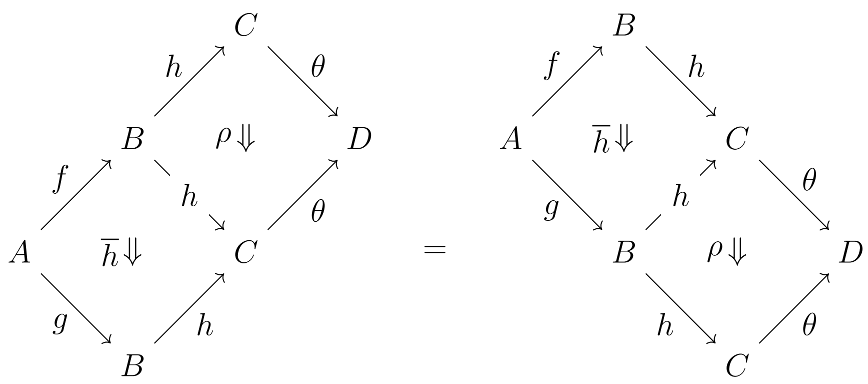


Meaning that:

- *(One-dim. UP)* Given any other cocone $\delta : sf \Rightarrow sg : B \to D$, there is a unique 1-cell $\theta : C \to D$ such that $\theta h = s$, $\theta \overline{h} = \delta$.

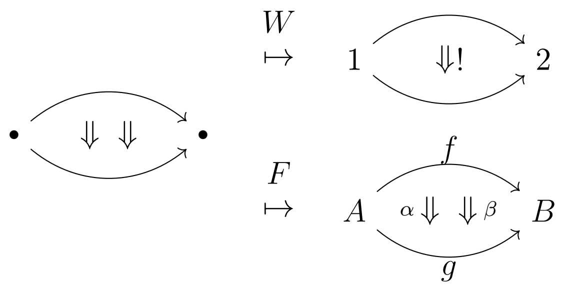


- *(Two-dim. UP)* Given a 2-cell $\rho : \theta h \Rightarrow \theta' h$ satisfying:



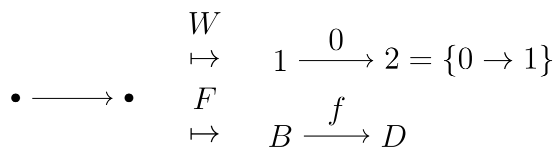
Then there is a unique 2-cell $\overline{\theta} : \theta \to \theta'$ such that $\overline{\theta} h = \rho$.

Definition 15. A *coequifier* of two 2-cells α, β is a colimit of functor F weighted by W, both given by:

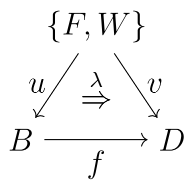


It is a universal arrow c such that $c\alpha = c\beta$. See [3, After Example 2.6]

Definition 16. A *lax limit of an arrow* $f : B \to D$ is a limit of a functor F weighted by W, both defined as:

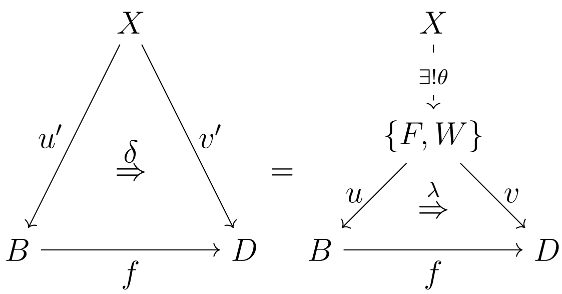


It is a universal a 2-cell:



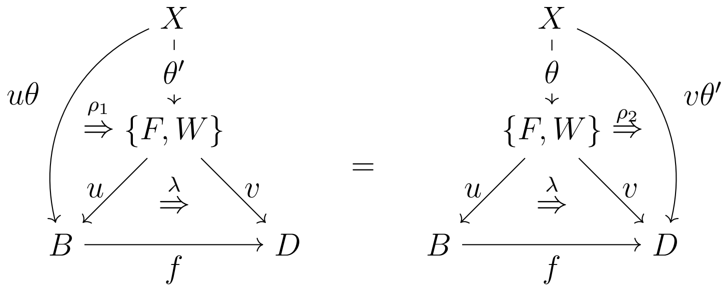
Meaning that:

- (*One-dim. UP):* Given any 2-cell $\delta u' f \Rightarrow v'$, there is a unique 1-cell $\theta : X \to \{F, W\}$ such that $u\theta = u'$, $v\theta = v'$ and $\lambda\theta = \delta$.



- (*Two-dim. UP*): Given a tuple of 2-cells $(\rho_1 : u\theta \Rightarrow u\theta', \rho_2 : v\theta \Rightarrow v\theta')$

satisfying[7]:



(1.8)

There is a unique 2-cell $\overline{\theta} : \theta \Rightarrow \theta'$ such that:

$$u\overline{\theta} = \rho_1$$
$$v\overline{\theta} = \rho_2$$

Remark 13. *Pseudo limits* and *lax limits* (denoted $\{F, W\}_p, \{F, W\}_l$) in [14]) are defined as in 1.7, except $[\mathcal{P}, \mathrm{Cat}]$ is replaced by $\mathrm{Psd}[\mathcal{P}, \mathrm{Cat}]$ and $\mathrm{Lax}[\mathcal{P}, \mathrm{Cat}]$ respectively. Note that in these cases η will be pseudo-natural transformation or just lax natural. It is true that if a 2-category admits products, inserters, equifiers, then it admits all lax and pseudo limits ([14][Proposition 5.2]).

Also note that these definitions can be further generalized. We can define lax natural transformations between lax functors as well and we can thus obtain a **lax limit of a lax functor**.

Remark 14. Any lax or pseudo limit of a (lax) functor F can be reduced to an ordinary 2-limit of a (strict) 2-functor F' for a certain F'. See Theorem 3.6.

Remark 15. We can also generalize the notion in a different way. A *bilimit* of F weighted by W, denoted $\{F, W\}_b$, is defined the same as limit, except the isomorphism 1.7 in $[\mathcal{K}^{op}, \mathrm{Cat}]$ is replaced by an equivalence in $\mathrm{Psd}[\mathcal{K}^{op}, \mathrm{Cat}]$:

$$\Phi : \mathcal{K}(-, \{F, W\}_b) \simeq \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}(-, F?)).$$

This means that Φ is a pseudonatural transformation for which there is pseudonatural transformation $\Psi : \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}(-, F?)) \to \mathcal{K}(-, \{F, W\}_b)$ and invertible modifications $\rho_1 : \Phi\Psi \to 1$, $\rho_2 : \Psi\Phi \to 1$. This is a special case of a limit in bicategories.

Definition 17. A 2-category is *complete* when it admits all limits for any weight $W : \mathcal{P} \to \mathrm{Cat}$ where $\mathcal{P}$ is small. 2-category $\mathcal{K}$ is *cocomplete* if $\mathcal{K}^{op}$ is complete.

Considering the 2-category Cat, plugging in $A = 1$ into the limit equation suggests that the objects of a limit category $\{F, W\}$ might be natural transformations $W \Rightarrow F$. And really:

Theorem 1.2. *Cat is complete. Any limit weighted by $W : \mathcal{P} \to Cat$, where $\mathcal{P}$ is small can be calculated as $\{F, W\} = [\mathcal{P}, Cat](W, F)$.*

[7]This diagram says that this tuple is modification of cones $\lambda\theta \to \lambda\theta'$.

Sketch of a proof. We need to find a 2-natural transformation with components (for $P \in \mathcal{P}$) being functors:

$$WP \to \mathrm{Cat}([\mathcal{P}, \mathrm{Cat}](W, F), FP).$$

It can be shown that

$$(x \in Wa) \mapsto (\alpha \mapsto \alpha_a(x))$$

is 2-natural and satisfies both universal properties. □

Thus, analogous to the one-dimensional case ([29, Theorem 3.4.2]), we obtain for a 2-category $\mathcal{K}$ an isomorphism that expresses the **representable nature of 2-categorical limits**:

$$\mathcal{K}(A, \{W, F\}) \cong \{W, \mathcal{K}(A, F-)\} \tag{1.9}$$

Definition 18. We say that a 2-functor $H : \mathcal{K} \to \mathcal{L}$ *preserves* limits weighted by $W : \mathcal{P} \to \mathrm{Cat}$ if, given a limit $\{F, W\}$ of $F : \mathcal{P} \to \mathcal{K}$ with a limit cone η, the induced cone:

$$H\eta := H_{\{F,W\},F-} \cdot \eta : W \Rightarrow \mathcal{L}(H\{F, W\}, HF-)$$

exhibits $H\{F, W\}$ as the limit for the 2-functor $HF : \mathcal{P} \to \mathcal{L}$ weighted by W.

Example 20. From the isomorphism 1.9 it follows that representable 2-functors $\mathcal{K}(A, -)$ preserve limits.

Remark 16. It is easy to think of what the notion of a (strict) creation of a limit should be. It's also true that 2-categorical limits and colimits in the 2-category $[\mathcal{K}, \mathcal{L}]$ are calculated pointwise (i.e. each evaluation 2-functor $\mathrm{ev}_a : [\mathcal{K}, \mathcal{L}] \to \mathcal{L}$ creates limits), giving us in particular that $\mathcal{L}$ complete implies that $[\mathcal{K}, \mathcal{L}]$ is complete. We also have that **left 2-adjoints preserve colimits** and its dual as well.

1.3 Internal structures in a 2-category

2-categories have rich enough structure to allow us to define a lot of 1-categorical things in them. For example we can develop the theory of internal monads as well as internal categories. Let us work in an ambient 2-category $\mathcal{K}$ in this section.

Definition 19. We say that a 1-cell $f : B \to A$ is a *left adjoint* to a 1-cell $g : A \to B$ in $\mathcal{K}$, written $f \dashv g$, if there are two 2-cells $\epsilon : fg \Rightarrow 1_A, \eta : 1 \Rightarrow gf$ (*counit* and *unit* of the adjunction) satisfying the two *triangle identities*:

$$g\epsilon \cdot \eta g = 1_f \tag{1.10}$$

$$\epsilon f \cdot f\eta = 1_g \tag{1.11}$$

The objects A, B are said to be *equivalent*, denoted $A \simeq B$, if ϵ, η are invertible.

Remark 17. Any internal adjunction $(\epsilon, \eta) : f \dashv u : A \to B$ in a 2-category $\mathcal{K}$ gives rise to an adjunction in Cat once we apply a 2-functor $\mathcal{K}(C, -)$ to it (as 2-functors preserve 1-cell and 2-cell compositions). This results in an adjunction

$$\mathcal{K}(C, f) \dashv \mathcal{K}(C, u) : \mathcal{K}(C, A) \to \mathcal{K}(C, B)$$

which provides us with a *representable definition* of an internal adjunction via the isomorphism of hom sets of hom categories

$$\mathcal{K}(C, A)(fy, y') \cong \mathcal{K}(C, B)(y, uy')$$

given by $\alpha \mapsto u\alpha \cdot \eta y$.

Remark 18. Analogous to ordinary category theory, adjunctions may be composed and an adjoint is unique up to an invertible 2-cell.

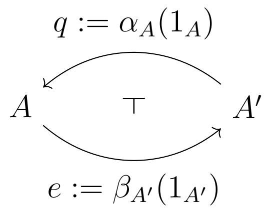
We say that a morphism $f : A \to B$ in a 2-category is *fully faithful* if $\mathcal{K}(X, f)$ is a fully faithful functor for all X. In ordinary category theory it is a well known fact that if the unit of the adjunction is an isomorphism, the left adjoint is fully faithful. From the above remark we thus deduce:

Lemma 1.3. *Let $\mathcal{K}$ be a 2-category and let $(\epsilon, \eta) : f \dashv g$ be an internal adjunction. If η is an invertible 2-cell, f is fully faithful.*

The following fact will also be useful:

Lemma 1.4. *Let $\mathcal{K}$ be a locally small 2-category and let $\alpha : \mathcal{K}(A, -) \Rightarrow \mathcal{K}(A', -)$ be a left 2-adjoint to $\beta : \mathcal{K}(A', -) \Rightarrow \mathcal{K}(A, -)$ in $[\mathcal{K}, Cat]$ (i.e. there are modifications $\rho : 1_{\mathcal{K}(A,-)} \to \beta\alpha$, $\Psi : \alpha\beta \to 1_{\mathcal{K}(A',-)}$ satisfying triangle identities). Then this 2-adjunction is determined by a unique internal adjunction of A, A' in $\mathcal{K}$.*

Proof. This follows directly from Yoneda lemma, the adjoint 1-cells are given by



and unit and counit 2-cells are given by $(\rho_A)_{1_A} : 1 \Rightarrow qe$ and $(\Psi_{A'})_{1_{A'}} : eq \Rightarrow 1$. □

Remark 19. The Lemma demonstrates that there really is an another dimension in the 2-categorical Yoneda lemma. By the same arguments, an equivalence of 2-functors $\mathcal{K}(A', -) \simeq \mathcal{K}(A, -)$ is given by an equivalence $A' \simeq A$ in $\mathcal{K}$.

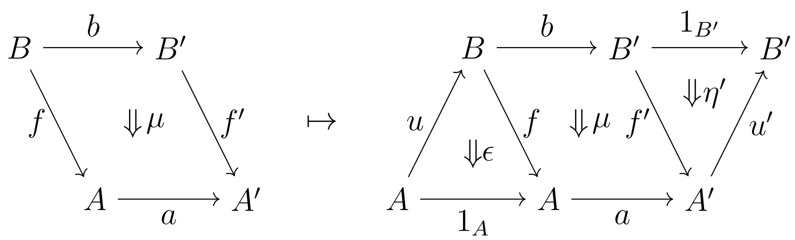
Lemma 1.5. *Assume we have two adjunctions $u \vdash f : B \to A$, $u' \vdash f' : B' \to A'$ with units η, η' and counits ϵ, ϵ'. Given 1-cells $a : A \to A', b : B \to B'$, there's a bijection between 2-cells*

$$\mu : f'b \Rightarrow af$$

and 2-cells

$$\lambda : bu \Rightarrow u'a$$

given by:



With the inverse given by whiskering with η from the left side and with ϵ' from the right side.

Definition 20. The 2-cells above λ, μ are said to be *mates* under the adjunction.

Definition 21. An (internal) *monad* in a 2-category $\mathcal{K}$, denoted (B, t) or (B, t, μ, η), consists of a 1-cell $t : B \to B$ together with two 2-cells $\mu : t{\cdot}t \Rightarrow t$, $\eta : 1 \Rightarrow t$ satisfying:

$$\mu \cdot \mu t = \mu \cdot t\mu \tag{1.12}$$

$$\mu \cdot \eta t = \mu \cdot t\eta = 1_t \tag{1.13}$$

Remark 20. As in ordinary category theory, any adjunction $(\epsilon, \eta) : f \dashv u : A \to B$ induces a monad on B, namely $(uf : B \to B, u\epsilon f, \eta)$.

Example 21. Monad in Cat is just an ordinary monad. Monad in 2-Cat is a (strict) 2-monad; we will study them in the next chapter in greater detail.

Example 22. A monad in the bicategory $\mathrm{Spn}(\mathcal{E})$ of spans (see Example 18) is an internal category in $\mathcal{E}$. In particular for $\mathcal{E} = \mathrm{Set}$ we obtain small ategories as internal monads.

Example 23. Consider a free monoid (1-)monad T on Set. There is a bicategory $\mathrm{Spn}_T(\mathrm{Set})$ of spans in Set of the form $TX \leftarrow Z \to Y$. A monad in this bicategory is precisely a multicategory.

Example 24. Given a monoidal category $\mathcal{V}$, a monad in the bicategory $\sum \mathcal{V}$ (see Example 19) is precisely a monoid in $\mathcal{V}$.

Example 25. A monad in the 2-category OpMon of monoidal categories, opmonoidal functors and opmonoidal transformation is what is called an *opmonoidal monad* or a *Hopf monad* in [25]. They have a number of pretty properties, one that is easily seen is that if S is a Hopf monad on a monoidal category $(\mathcal{V}, \otimes)$, the tensor product $\otimes$ lifts up to the category S-Alg of S-algebras for this monad.

Remark 21. Given any 2-category $\mathcal{K}$, the hom category $\mathcal{K}(B, B)$ has the structure of a strict monoidal category where the functor $\otimes : \mathcal{K}(B, B) \times \mathcal{K}(B, B) \to \mathcal{K}(B, B)$ is a horizontal composition of 2-cells and the unit object I is the identity on B. Monoid in $\mathcal{K}$ is then just a monad in this monoidal category, generalizing the well known joke about monads in Cat.

Definition 22. Let $\mathcal{K}$ be a 2-category and (B, t, μ, η) a monad. Given $A \in \mathcal{K}$, define the *category of t-algebras* Alg(t, A) as follows: The objects are t-algebras, i.e. tuples $(s : A \to B, \nu : ts \Rightarrow s)$ satisfying:

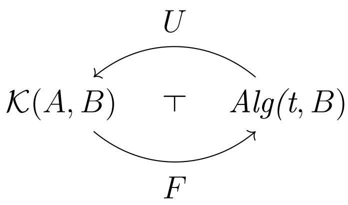
$$\nu \cdot \eta s = 1_s, \tag{1.14}$$

$$\nu \cdot \mu s = \nu \cdot t\nu, \tag{1.15}$$

and morphisms $(s, \nu) \to (s', \nu')$ is a 2-cell $\sigma : s \Rightarrow s'$ satisfying:

$$\nu' \cdot t\sigma = \sigma \cdot \nu$$

Theorem 1.6. *Given a monad $t : B \to B$, there is an adjunction*



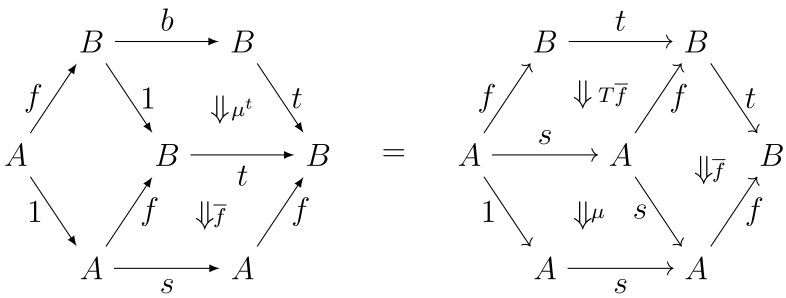
(1.16)

where U is the forgetful functor $(s : B \to A, \nu) \mapsto s$ and F sends a 1-cell r to the ***free t-algebra*** *on r, i.e. $Fr = (tr, \mu r)$.*

Remark 22. All of this is a generalization of the classical notion of a category of T-algebras for a monad $T : \mathcal{C} \to \mathcal{C}$. We obtain the classical notion if $\mathcal{K} =$ Cat and $B = 1$.

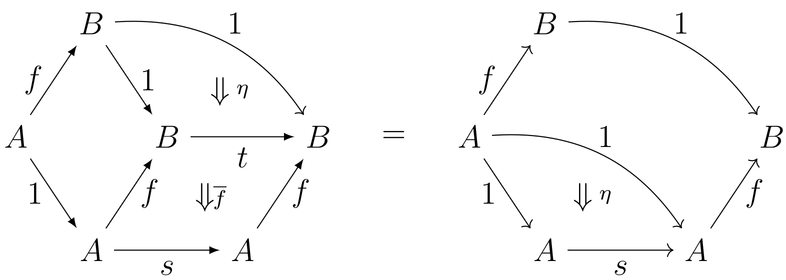
Another approach is to study the collection of **all** monads in a 2-category $\mathcal{K}$, this was done in the paper [31]. Let's sketch some of the ideas from that paper that will be relevant later.

Definition 23. Given monads $(s : A \to A, \mu, \eta)$, $(t : B \to B, \mu^t, \eta^t)$, a *monad functor* $(f, \overline{f}) : s \to t$ consists of 1-cell $f : A \to B$ and a 2-cell $\overline{f} : tf \Rightarrow fs$ satisfying:



(1.17)

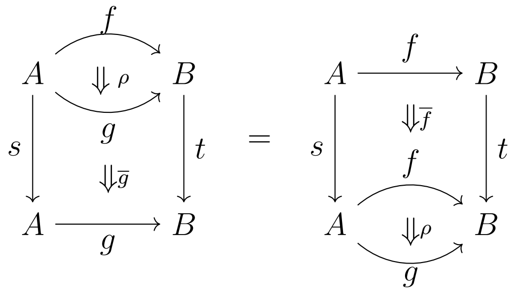
and[8]:



(1.18)

[8]You will see why those diagrams are shaped in this particular way in the next chapter.

Definition 24. A *monad functor transformation* $\rho : (f, \overline{f}) \Rightarrow (g, \overline{g})$ between monad functors is a 2-cell in $\mathcal{K}$ satisfying:



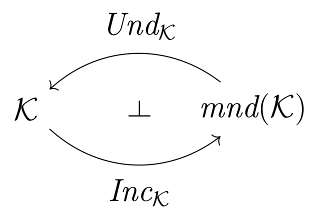
(1.19)

It is not hard to verify that monads, monad functors and monad functor transformations form a 2-category. Morphism composition is given by composing the 1-cell components and placing 2-cell components next to each other. Horizontal and vertical composition of 2-cells is inherited from $\mathcal{K}$.

Definition 25. Denote $\mathrm{mnd}(\mathcal{K})$ the 2-category of monads, monad functors and monad functor transformations in $\mathcal{K}$.

The next theorem follows straight from the definitions:

Theorem 1.7. *There is a 2-adjunction:*



(1.20)

where $Und_{\mathcal{K}}$ is the forgetful 2-functor:

$$Und_{\mathcal{K}} : (\sigma : (u, \psi) \Rightarrow (u', \psi')) \mapsto (\sigma : u \Rightarrow u' : A \to B)$$

and $Inc_{\mathcal{K}}$ sending an object to identity monad on that object:

$$Inc_{\mathcal{K}} : (\delta : f \Rightarrow g : A \to B) \mapsto (\delta : (f, 1) \Rightarrow (g, 1) : (A, 1) \to (B, 1))$$

Definition 26. Let $\mathcal{K}$ be a 2-category. We say that it *admits the construction of algebras* when the inclusion 2-functor $\mathrm{Inc}_{\mathcal{K}}$ has a **right** 2-adjoint. In other words, for any objects $B \in \mathcal{K}$ there is an object B^t and an isomorphism (2-natural in A):

$$\mathcal{K}(A, B^t) \cong \mathrm{mnd}(\mathcal{K})((A, 1), (B, t)).$$

We call B^t the *Eilenberg-Moore (EM) object*. We also say $\mathcal{K}$ *admits Kleisli objects* if $\mathcal{K}^{op}$ admits the construction of algebras and we refer to the EM-object in $\mathcal{K}^{op}$ as a *Kleisli object* in $\mathcal{K}$ and denote it B_t.

Remark 23. A lax functor $1 \to \mathcal{K}$ is precisely a monad in $\mathcal{K}$. It can be verified that the lax limit of this lax functor is then exactly the EM-object of this monad. As a lax limit, it can equally be seen as a limit of (strict) 2-functor, and this 2-functor is explicitly described in [18, 8.4].

EM-objects and Kleisli objects are particularly pretty in the case of Cat and Cat^{op} (see [31][Theorem 7 and 13]):

Theorem 1.8. *Cat admits the construction of algebras. Given a monad $T : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$, $\mathcal{C}^T$ equals to the Eilenberg-moore category of T-algebras.*

Theorem 1.9. *Cat^{op} admits the construction of algebras. Given a monad T in Cat^{op} (which is the same as monad on Cat), the category $\mathcal{C}^T$ is equal to the Kleisli category.*

Remark 24. Because $\mathcal{K}(A, -)$ is a 2-functor, any monad t in $\mathcal{K}$ induces a monad $\mathcal{K}(A, t)$ in Cat for any object $A \in \mathcal{K}$. If 2-category $\mathcal{K}$ admits the construction of algebras, we moreover have a following 2-natural isomorphism:

$$\mathcal{K}(Y, X^t) \cong \mathcal{K}(Y, X)^{\mathcal{K}(Y,t)}$$

It's easy to see directly, but it can also be derived from the fact that EM-object is a certain limit and representable 2-functors preserve limits.

Remark 25. It is known that if a 2-category admits inserters and equifiers, it admits construction of algebras. We will obtain this result in the last section of the third chapter.

Remark 26. It can further be proven that an internal adjunction $f \dashv u : D \to B$ is *(internally) monadic* (meaning in particular there is an internal equivalence of D and B^{uf}) if and only if $\mathcal{K}(C, f) \dashv \mathcal{K}(C, u)$ is monadic for all C. See [31, Corollary 8.1].

Chapter 2

2-monads

Our notion of a 2-monad will be *strict*, so the axioms are analogous to those of 1-category theory. In the higher dimension however, we do not have just one kind of algebra for a 2-monad, but multiple ones. There are the strict algebras with the usual algebra axioms, but we now also have a weaker notions of a pseudoalgebra and a lax algebra, where the axioms only hold up to an invertible 2-cell or just a 2-cell.

Throughout the section, basic familiarity with monads in ordinary category theory (at least in the scope of [29][Chapter 5]) is welcome.

2.1 Monads and their algebras

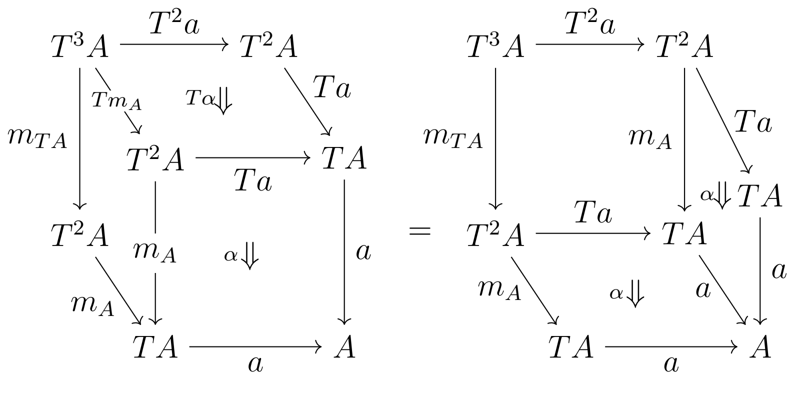
Definition 27. Let T be an endo-2-functor on a 2-category $\mathcal{K}$ and let $m : T^2 \Rightarrow T$, $i : 1 \Rightarrow T$ be 2-natural transformations. We say that the triple (T, μ, η) is a *2-monad* (also called a *doctrine* in older papers) if the following diagrams commute[1]:

$$\begin{array}{ccccc} TA & \xrightarrow{i_{TA}} & T^2A & \xleftarrow{Ti_A} & TA \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle m_A} & \swarrow{\scriptstyle 1} & \\ & & TA & & \end{array} \qquad \begin{array}{ccc} T^2A & \xrightarrow{m_{TA}} & TA \\ {\scriptstyle Tm_A}\downarrow & & \downarrow{\scriptstyle m_A} \\ TA & \xrightarrow[m_A]{} & A \end{array} \tag{2.1}$$

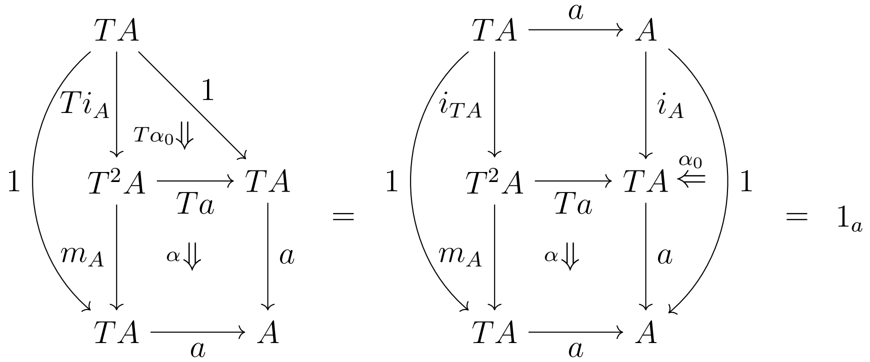
Just as for ordinary monads, m is called *the multiplication* and i *the unit* of the 2-monad T.

Definition 28. A lax $T-$algebra is a tuple (A, a, α, α_0), where A is an object of $\mathcal{K}$, $a : TA \to A$ is a morphism, $\alpha : aTa \Rightarrow a\mu_A$, $\alpha_0 : 1 \Rightarrow a \cdot i_A$ are 2-cells such that:

[1] If we ignore size issues, this is precisely the definition of an internal monad in the 2-category 2-Cat



(2.2)

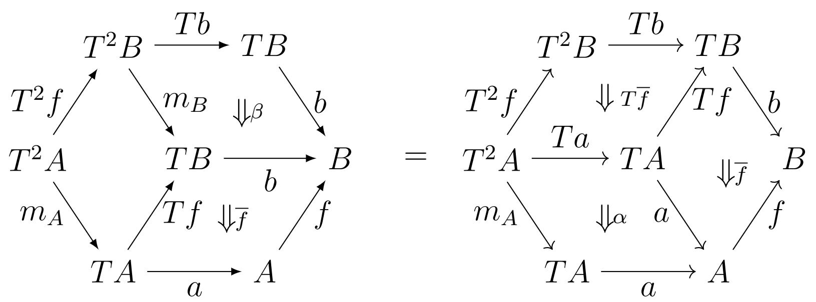


(2.3)

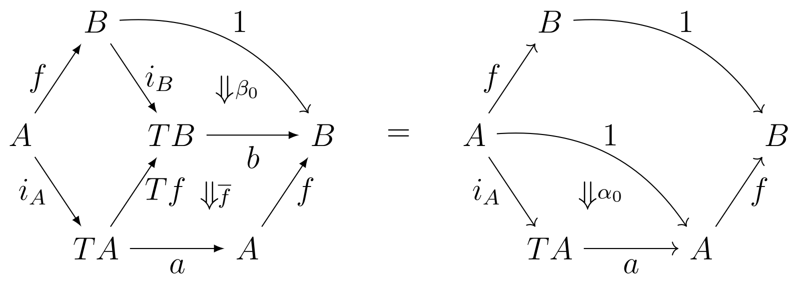
Definition 29. We say that a lax algebra is *normal* when α_0 equals the identity 2-cell. When α, α_0 are isomorphisms, we say the T-algebra is a *pseudo* T-algebra. When α, α_0 are identity 2-cells, we have the notion of a *strict* T-algebra and we denote it just (A, a). Reversing the direction of α, α_0, we get a notion of a *colax* T-algebra, but we won't use them in this thesis.

Remark 27. Note that when (A, a) is a strict T-algebra, equations 2.2 and 2.3 boil down to the assertion that $am_A = aTa$ and $ai_A = 1_A$. These are the same as associativity and unit laws for algebras for an ordinary monad ([29, Definition 5.2.4]). Thus 2.2 is some sort of higher-dimensional associativity law and 2.3 is some sort of higher-dimensional unit law.

Definition 30. A *lax morphism* $(f, \overline{f}) : (A, a, \alpha, \alpha_0) \to (B, b, \beta, \beta_0)$ of lax T-algebras consists of morphism $f : A \to B$ and a 2-cell $\overline{f} : b \cdot Tf \Rightarrow f \cdot a$ satisfying:

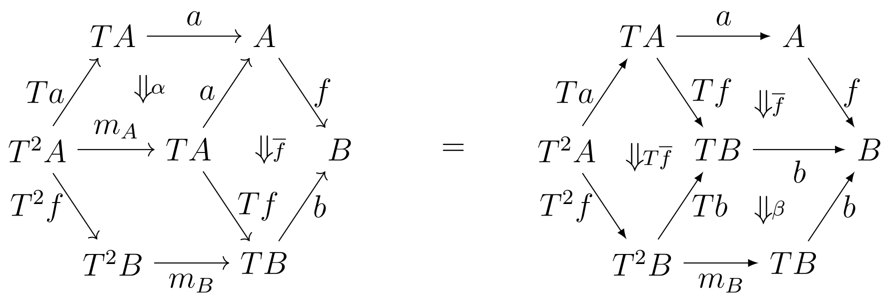


(2.4)

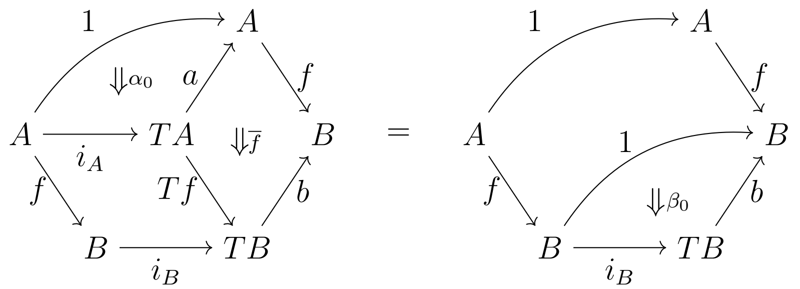


(2.5)

Definition 31. A *colax morphism* $(f, \overline{f}) : (A, a, \alpha, \alpha_0) \to (B, b, \beta, \beta_0)$ of lax T-algebras consists of morphism $f : A \to B$ and a 2-cell $\overline{f} : f \cdot a \Rightarrow b \cdot Tf$



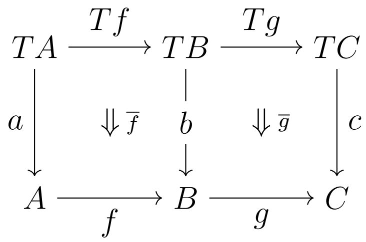
(2.6)



(2.7)

If $\overline{f}$ is invertible, we call such a morphism a *pseudo-morphism*. If $\overline{f}$ is the identity, we call the morphism *strict*.

Composition of lax morphisms $(f, \overline{f}) : A \to B$, $(g, \overline{g}) : B \to C$ is a morphism $(gf, g\overline{f} \cdot \overline{g}Tf)$; we just paste the 2-cells together:

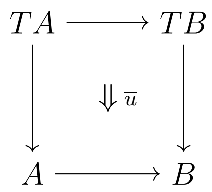


The same goes for colax morphisms. We also have an obvious (co)lax identity morphism $(1_A, 1_a)$ for each algebra.

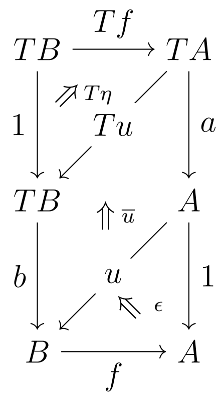
Remark 28 (Justification of the colax definition). If $\overline{f}$ is invertible, $(f, \overline{f})$ is a lax morphism if and only if $(f, \overline{f}^{-1})$ is colax.

Another relationship between lax and colax morphisms follows from the property of mates under adjunction and was first observed in [8]. It is referred to as a *doctrinal adjunction*:

Lemma 2.1. *Let $(A, a), (B, b)$ be strict T-algebras in a 2-category $\mathcal{K}$ and let $(\epsilon, \eta) : u \vdash f : B \to A$ be an adjunction in $\mathcal{K}$. Then:*



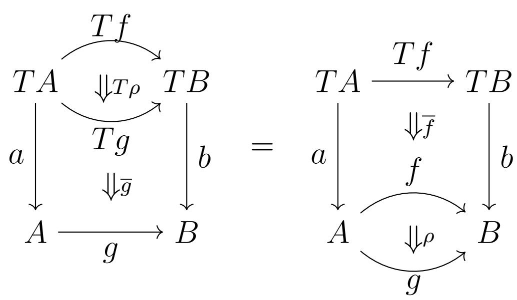
is a lax T-algebra morphism if and only if



is a colax T-algebra morphism.

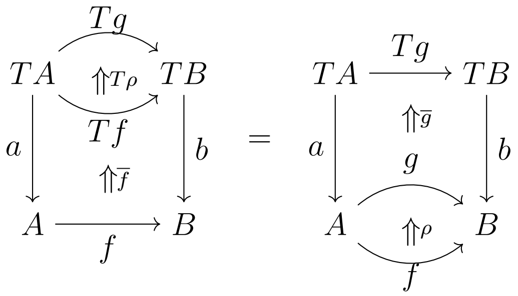
Proof. [8][Lemma 1.1] □

Definition 32. A *transformation* $\rho : (f, \overline{f}) \Rightarrow (g, \overline{g})$ between lax morphisms of lax algebras $(A, a, \alpha, \alpha_0) \to (B, b, \beta, \beta_0)$ is a 2-cell $\rho : f \Rightarrow g$ in $\mathcal{K}$ satisfying:



(2.8)

Definition 33. A *transformation* $\rho : (f, \overline{f}) \Rightarrow (g, \overline{g})$ between **colax** morphisms of lax algebras $(A, a, \alpha, \alpha_0) \to (B, b, \beta, \beta_0)$ is a 2-cell $\rho : f \Rightarrow g$ in $\mathcal{K}$ satisfying:



(2.9)

We can again compose those 2-cells horizontally, vertically and whisker them from both sides with algebra morphisms. All is done the same way as in $\mathcal{K}$ (although we can not compose lax and colax 2-cells, nor can we whisker lax 2-cell with a colax morphism and so on).

We have just mentioned several kinds of algebras, algebra morphisms and algebra morphism transformations. There will serve as 0,1,2-cells for a bunch of 2-categories we wil now define as well:

Definition 34. Let $\mathcal{K}$ be a 2-category. Define the following 2-categories:

2-category	**objects**	**morphisms**	**2-cells**
T-Alg$_s$	strict algebras	strict morphisms	morphism transformations
T-Alg		pseudo morphisms	
T-Alg$_l$		lax morphisms	
Ps-T-Alg	pseudo algebras	pseudo morphisms	
Lax-T-Alg	lax algebras	lax morphisms	
Lax-T-Alg$_\mathrm{c}$		colax morphisms	colax morphism transformations
T-Alg$_c$	strict algebras		

Remark 29. Lax-T-Alg$_\mathrm{c}$ is not to be confused with Lax-T-Alg^{co}, the dual category to Lax-T-Alg. What is however true is that given a 2-monad T on $\mathcal{K}$, there is a 2-monad T^{co} on $\mathcal{K}^{co}$ and (Lax-T-Alg)^{co} = CoLax-T^{co}-Alg[2]. T-Alg$_c$ = T^{co}-Alg however.

Remark 30. If we have a 2-monad T on a 2-category $\mathcal{K}$, a natural question to ask is whether there exists a 2-monad T' with the property that T'-Alg$_s$ = Ps-T-Alg. Under nice conditions (T being finitary and $\mathcal{K}$ locally finitely presentable) the answer is yes. See [18, 7.4].

2.2 Examples of 2-monads

There's plenty of examples of 2-monads that arise in practice. Just as ordinary monads are a useful tool to describe algebraic structures, 2-monads are good at

[2]The 2-category of colax algebras, colax morphisms and colax morphism transformation. But we shall not burden ourselves with them in this thesis.

describing categorical structures. Main reference for this section is [2, Section 6] which contains a large number of examples.

Example 26 (*Identity 2-monad*)**.** Let $\mathcal{K}$ be a 2-category. The simplest example is the identity 2-functor $1_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ that becomes a 2-monad if we take multiplication and unit 2-transformations to be identities.

The category T-Alg$_s$ is then just $\mathcal{K}$. What's more interesting is the 2-category Lax-T-Alg, which is precisely the 2-category mnd($\mathcal{K}$) of monads, monad functor, monad functor transformations that we introduced earlier.

Also, as is easily verified, Lax-T-Alg$_c$ is equal to 2-category mnd$(\mathcal{K}^{op})^{op}$, which is the 2-category of monads, *monad opfunctors* and *monad opfunctor transformations* (the notion introduced in [31]).

Example 27 (Terminal object 2-monad)**.** Consider a 2-functor $(-)_+ : \mathrm{Cat} \to \mathrm{Cat}$ that **freely adds** a terminal object to a category $\mathcal{C}$; i.e. ob $\mathcal{C}_+ = \mathrm{ob}\, \mathcal{C} \coprod \{*\}$ and $\mathcal{C}_+(a, *)$ consists of a unique morphism for each object a. The multiplication functor $m_{\mathcal{C}} : \mathcal{C}_{++} \to \mathcal{C}_+$ squeezes both added terminal objects into one and is identity on $\mathcal{C}$. The unit $i_{\mathcal{C}}$ is an inclusion of a category $\mathcal{C}$ into $\mathcal{C}_+$.

Then a strict T-algebra $(\mathcal{A}, a)$ is a category with chosen terminal object t and a strict T-algebra morphism is a functor that strictly preserves those terminal objects. Pseudo-morphisms in this case are functors that preserve the limit (terminal object) in the usual sense. It is also easily seen that any functor $F : (\mathcal{A}, t_1) \to (\mathcal{B}, t_2)$ between such categories is a **lax morphism**, for there is always a canonical morphism $F(t_1) \to t_2$.

Example 28 (*Monoidal 2-monad*)**.** Consider the 2-functor $T : \mathrm{Cat} \to \mathrm{Cat}$ given by:

$$T\mathcal{A} = \coprod_{n \geq 0} \mathcal{A}^n$$

This is a 2-monad with the multiplication being concatenation of lists of objects and unit being the inclusion of $x \mapsto (x)$ as a singleton list.

Consider strict T-algebra $(\mathcal{A}, a)$. Thanks to Remark 27, the situation is essentially the same as for **free monoid monad** on Set. Associativity and unit laws tell us that the algebra multiplication $a : \coprod_{n \geq 0} \mathcal{A}^n \to \mathcal{A}$ is induced by a binary associative operation (functor) $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a nullary operation, an object $I \in \mathcal{A}$ that acts as an identity with respect to $\otimes$. We thus obtain that T-Alg$_s$ = StrMonCat, the 2-category of strict monoidal categories, strict monoidal functors and strict monoidal transformations.

For a lax-T-algebra (A, a, α, α_0) the situation is more complicated. We are given a small category $\mathcal{A}$ and an n-ary operation $\otimes^n : \mathcal{A}^n \to \mathcal{A}$ for each $n \in \mathbb{N}$. Moreover, for every list of lists $((a_1^1, \dots, a_1^{k_1}), \dots, (a_n^1, \dots, a_n^{k_n}))$ and an element $x \in \mathcal{A}$ we have natural morphisms:

$$\otimes^n(\otimes^{k_1}(a_1^1, \dots, a_1^{k_1}), \dots, \otimes^{k_n}(a_n^1, \dots, a_n^{k_n})) \to \otimes^{k_1 + \dots + k_n}(a_1^1, \dots, a_1^{k_1}, \dots, a_n^1, \dots, a_n^{k_n})$$
$$x \to \otimes^1(x) \tag{2.10}$$

that satisfy the lax algebra identities. This is what's called a *lax monoidal category*. If A is a pseudoalgebra, the above morphisms are isomorphisms and we call the structure an *unbiased monoidal category*.

Note that these "new" notions of a monoidal category contain ordinary monoidal categories. Given a monoidal category $(\mathcal{V}, \otimes, I, a, l, r)$ we can simply define the n-ary tensor product $\otimes^n$ by iterating the binary tensor product $\otimes$ and build maps 2.10 using monoidal category isomorphisms a, l, r. Coherence for monoidal categories then guarantees that they will satisfy the axioms for an unbiased monoidal category. See Corollary 3.9 and Remark 40.

Lax and colax morphisms of T-algebras are what's referred to as a *lax* and *oplax monoidal functors*, they are an "unbiased" version of what we've seen in Example 7. Similarly with algebra 2-cells.

It is worth it to mention a natural example of an oplax monoidal functor. Consider the forgetful functor $U : (\mathrm{Ab}, \otimes) \to (\mathrm{Set}, \times)$ between the monoidal categories of abelian groups and sets. It is in a no way is it true that for two abelian groups A, B, the underlying set of their tensor product $A \otimes B$ is isomorphic to the product $A \times B$. There is however a canonical map $A \times B \to A \otimes B$ as well as a canonical map $1 \to \mathbb{Z}$. And these turn U into an oplax morphism.

Remark 31. There is a significant difference between the last two examples in that there can be many different monoidal structures on a category $\mathcal{A}$, but there is *essentially unique* structure of a "category with a terminal object" on a category $\mathcal{A}$[3]. 2-monads whose algebras have essentially unique structure are called *property-like* (as for example having terminal object is a property, rather than structure) and have been studied in detail in [9].

Example 29 (Lax functor 2-monad)**.** Let $\mathcal{J}$ be a small 2-category. Then the restriction res_J of a 2-functor $X : \mathcal{J} \to \mathrm{Cat}$ to objects has a left 2-adjoint given by the left Kan extension.

$$[\mathcal{J}, \mathrm{Cat}] \underset{\mathrm{res}_J}{\overset{\mathrm{lan}_J}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} [\mathrm{ob}\mathcal{J}, \mathrm{Cat}] \tag{2.11}$$

In this case, the 2-functor $\mathrm{lan}_J X$ is given by[4]

$$\mathrm{lan}_J X(b) = \sum_{j \in \mathcal{J}} (\mathcal{J}(j, b) \times Xj)$$

The composition $T = \mathrm{res}_J \cdot \mathrm{lan}_J$ is then a 2-monad on $[\mathrm{ob}\mathcal{J}, \mathcal{K}]$. To see this, we must find the multiplication $m : T^2 \Rightarrow T$ and the unit $i : 1 \Rightarrow T$, whose components

[3] Meaning that given two "categories with terminal object" $(\mathcal{A}, t), (\mathcal{A}, t')$ with the same underlying category, there is a unique pseudomorphism $(1_A, \overline{f}) : (\mathcal{A}, t) \to (\mathcal{A}, t')$ that is obviously an isomorphism.

[4] Try to think where 1-cells and 2-cells of $\mathcal{J}$ are being sent.

at each functor X are 2-natural transformations. Define the component of m_X at $b \in \mathcal{J}$:

$$(m_X)_b : T^2X(b) = \sum_{j'} \mathcal{J}(j', b) \times (\sum_j J(j, j') \times Xj) \to \sum_j J(j, b) \times Xj$$

As a composition $(f : j' \to b, (g : j \to j', o)) \mapsto (fg, o)$. For the unit, define the component of i_X at $b \in \mathcal{J}$:

$$(i_X)_b : Xb \to \sum_j J(j, b) \times Xj$$

As an inclusion $o \mapsto (1_b, o) \in J(b, b) \times Xj$ (that further sends a morphism $o \to o'$ to $(1_{1_b}, o \to o')$). Monad axioms (2.1) then boil down to left composition with identity, right composition with identity, and associativity - and are satisfied.

Assume now we're given a strict T-algebra (X, x), that is, a small category Xb for each $b \in \mathcal{J}$ and a collection of functors $x_b : TX(b) \to X(b)$ for each $b \in \mathcal{J}$. The unit algebra axiom 2.3 means that the functor:

$$x_b : \sum_j \mathcal{J}(j, b) \times Xj \to Xb$$

Satisfies $x_b(1_b, -) = 1_{Xb}$. The algebra multiplication axiom 2.2 reveals that $x_b(f'f, --) = x_b(f', x_{j'}(f, -))$ for all $f' : j' \to b, f : j \to j'$. But this means exactly that X extends to a 2-functor $\mathcal{J} \to \mathrm{Cat}$ given by

$$X(f : k \to l) = x_l(f, -) : Xk \to Xl.$$

Given a strict morphism $h : (X, x) \to (Y, y)$, the condition 2.4 amounts to the commutativity of the square (for each b):

$$\begin{array}{ccc} TXb & \xrightarrow{(Th)_b} & TYb \\ \downarrow{\scriptstyle x_b} & & \downarrow{\scriptstyle y_b} \\ Xb & \xrightarrow[h_b]{} & Yb \end{array}$$

This means that for any object of the top left category $(f : j \to b, o \in Xj)$ we have:

$$h_b(x_b(f, o)) = y_b(f, h_j(o))$$

This is precisely the requirement that the collection of functors $h_b : Xb \to Yb$ is natural[5]. Continuing the approach, we find that the 2-cells in T-Alg_s are precisely modifications. All in all, $\text{T-Alg}_s = [\mathcal{J}, \mathrm{Cat}]$.

Let's find out what a lax morphism $(h, \overline{h}) : (X, x) \to (Y, y)$ is. It consists of a set of functors $h_b : Xb \to Yb$ for each object b as well as a modification $\overline{h} : yTh \to hx$,

[5] Similarly with 2-naturality, you don't send 1-cell f around but a 2-cell Δ.

which just boils down to a set of natural transformations $\overline{h}_b : y_b T h_b \Rightarrow h_b x_b$ for each b. Let's define for each $f : j \to b$ the natural transformation $\overline{g}_f := \overline{h}_{b,(f,-)} : Yf \cdot \cdot h_j \Rightarrow h_b \cdot Xf$. At each component b, the axiom 2.4 becomes this equality of natural transformations:

$$\overline{h}_b m_{X,b} = \overline{h}_b (Tx)_b \cdot y_b (T\overline{h}_b)$$

Evaluating this at a component $(f_1, f_2, o) \in T^2X$, we obtain and denoting $\overline{g}_f := = \overline{h}_{b,(f,-)} : Yf \cdot h_j \Rightarrow h_b \cdot Xf$ for each $f : j \to b$, we arrive at:

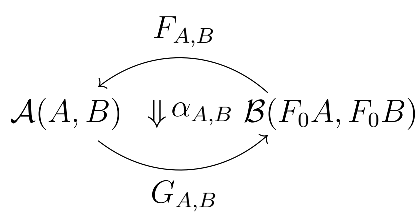
$$\overline{h}_{b,(f_1 f_2,o)} = \overline{h}_{b,(f_1,x_j(f_2,o))} \cdot y_b(f_1, \overline{h}_{b,(f_2,o)})$$
$$\overline{g}_{f_1 f_2} = \overline{g}_{f_1} X f_2 \cdot Y f_1 \overline{g}_{f_2}$$

Similarly, axiom 2.5 gives us that $\overline{g}_{1_k} = 1_{h_k}$. But this is precisely a lax natural transformation! Thus $\text{T-Alg}_l = \text{Lax}[\mathcal{J}, \text{Cat}]$ as well as $\text{T-Alg} = \text{Psd}[\mathcal{J}, \text{Cat}]$. It is also possible[6] to further verify that $\text{Hom}[\mathcal{J}, \text{Cat}] = \text{Ps-T-Alg}$ i.e. that pseudoalgebras are pseudofunctors, and that lax algebras are lax functors.

We will return to this example at the end of the thesis. Also, this example can be further generalized by replacing Cat with any cocomplete 2-category $\mathcal{K}$, see [2, 6.6].

Remark 32. There is also non-elementary demonstration of the fact that strict algebras for this 2-monad are precisely 2-functors. Because 2-categorical colim its are computed pointwise, the 2-functor res_J strictly creates all colimits. res_J is thus (strictly) 2-monadic and we have a 2-isomorphism $\text{T-Alg}_s \cong [\mathcal{J}, \text{Cat}]$ so these 2-categories could be identified for most purposes.

Example 30 (2-category 2-monad)**.** A *Cat-graph* $\mathcal{A}$ consists of a set of objects $\mathcal{A}_0$ together with a category $\mathcal{A}(A, B)$ for each $A, B \in \mathcal{A}_0$. A *Cat-graph morphism* $\mathcal{A} \to \mathcal{B}$ between Cat-graphs consists of a function $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ as well as a functor $F_{A,B} : \mathcal{A}(A, B) \to \mathcal{B}(F_0 A, F_0 B)$ for each $A, B \in \mathcal{A}_0$. We define a 2-cell $\alpha : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$ between morphisms for which $F_0 = G_0$ as a collection of natural transformations $\alpha_{A,B}$ for each $A, B \in \mathcal{A}_0$:



These form a 2-category $\mathcal{G}(\text{Cat})$ of Cat-graphs. Every 2-category $\mathcal{K}$ has an underlying Cat-graph $U\mathcal{K}$, and to every Cat-graph $\mathcal{A}$ we can assign a **free 2-category** $F\mathcal{A}$ generated by this graph. We obtain 2-functors U, F that form a 2-adjunction[7]

[6] Although it is more headache inducing.

[7] 2-Cat_2 is the 2-category of small 2-categories, 2-functors and icons.

$$\mathcal{G}(\mathrm{Cat}) \underset{F}{\overset{U}{\rightleftarrows}} \text{2-Cat}_2 \quad (\top) \tag{2.12}$$

It can be shown that this adjunction is 2-monadic, meaning that for the induced 2-monad $T = UF$ we have a (strict) 2-equivalence T-Alg$_s \simeq$ 2-Cat$_2$ and can thus identify 2-categories as strict algebras for this 2-monad. Pseudoalgebras for this 2-monad are what's referred in [30, Page 2030] as an *unbiased bicategory*. Ordinary bicategories again give rise to unbiased ones.

Example 31. Consider the category Δ of finite ordinals and order preserving maps. Defining $[n] \oplus [m] := [n+m]$, it is easily verified that $(\Delta, \oplus, [0])$ has the structure of a strict monoidal category. Moreover, **it's the free strict monoidal category containing a monoid**. Thus, to give a monad $H : \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ is to give a monoidal functor:

$$\Delta \to [\mathcal{C}, \mathcal{C}]$$

which is equivalent to giving an action:

$$\oplus : \Delta \times \mathcal{C} \to \mathcal{C}$$

That is, a functor such that for all objects C we have:

$$[n] \oplus ([m] \oplus C) = [n+m] \oplus C,$$
$$[0] \oplus C = C.$$

We obtain a finitary 2-functor $\Delta \times - :$ Cat $\to$ Cat which has the structure of a 2-monad (multiplication and identity given by those in M). Its **strict** algebras are precisely monads on small categories, and a lax morphism between them is a monad functor.

The above 2-monad is an example of a construction called a *club*. With this notion introduced in [11] more structures can be exhibited as the algebras for a 2-monad such as categories that admit coproducts or monoidal functors between monoidal categories. To recognize yet wider array of structures as algebras, a more general type of a 2-monad has been studied:

Definition 35. A 2-monad (T, m, i) is said to be *finitary* if the 2-functor T is finitary, i.e. preserves filtered colimits[8].

[8] Here by filtered colimit is meant a conical colimit of a 2-functor $F : \mathcal{P} \to \mathcal{K}$ whose domain is a filtered category regarded as a 2-category.

For finitary 2-monads T on Cat it was shown in [12] that a T-algebra $\mathcal{A}$ is precisely a category $\mathcal{A}$ with c-ary operations, i.e. functors $\mathcal{A}^c \to \mathcal{A}$ with c being a locally finitely presentable category), together with:

1. equations between them
2. natural transformations between their various iterations
3. equations between iterations of those natural transformations

Example 32. There is a finitary 2-monad on Cat whose algebras are categories that admit a class of colimits of given shape. This is briefly described in the introduction to [13]. The idea is, if we want our algebras for a 2-monad to have limits of shape M, we choose $c := M$. The functor computing the limit, $L : [M, \mathcal{A}] \to \mathcal{A}$ is the right adjoint to the diagonal $\Delta : \mathcal{A} \to [M, \mathcal{A}]$, and as such can be described as an M-operation, two natural transformations (unit and the counit of the adjunction) and an equation between iterations of those (the triangle identities).

Example 33 (Bicategory 2-monad)**.** Because a bicategory can be specified as a Cat-graph together with several operations of various arity (and equations between them. . .), we may construct a finitary 2-monad on Cat-graphs such that its algebras are bicategories, see [21, Section 4] for the explicit construction.

The last two examples come from different areas alltogether - order theory and topology. These have been studied in the paper [4].

Example 34. Consider the locally posetal 2-category Rel from Example 10. For the identity 2-monad $T = 1_{\mathrm{Rel}}$ it can be shown that Lax-T-Alg is the 2-category of preordered sets and monotone maps.

Example 35. Recall the ultrafilter monad on Set. It admits an extension to a 2-monad T on Rel, for which lax algebras are **topological spaces**.

Some categorical structures have been proven to **not** be the algebras for any 2-monad T on Cat. For example symmetric monoidal closed categories and cartesian closed categories. Both of these can be described as algebras for an ordinary monad on Cat_0 (the underlying category of Cat). See [2, 6.4] and [18, 5.8].

2.3 2-categories of algebras

The 2-category T-Alg$_s$ of strict T-algebras for a 2-monad T behaves very much like the Eilenberg-Moore category of algebras $\mathcal{C}^{T'}$ for an ordinary monad T'. Thus we have the following theorems that are straightforward generalizations of results from 1-category theory (in fact they hold in any $\mathcal{V}$-category, see [5]).

Theorem 2.2. *There is a 2-adjunction*

$$\mathcal{K} \underset{F^T}{\overset{U^T}{\leftrightarrows}} T\text{-}Alg_s \quad (\top) \tag{2.13}$$

where U^T is the forgetful 2-functor and F^T sends an object A to the strict T-algebra (TA, m_A).

Proof. This is analogous to the 1-dimensional case [29, Lemma 5.2.8]. The isomorphism of categories

$$\mathcal{K}(A, B) \cong \text{T-Alg}_s((TA, m_A), (B, b)) \tag{2.14}$$

is given by:

$$(\theta : f \Rightarrow f' : A \to B) \mapsto (bT\theta : bTf \Rightarrow bTf' : (TA, m_A) \to (B, b)) \tag{2.15}$$

with the inverse being given by pre-composition with a 1-cell i_A. □

Remark 33. This can be extended to a *biadjunction*[9] between the forgetful functor $U' : \text{T-Alg} \to \mathcal{K}$ and F composed with the inclusion $\text{T-Alg}_s \hookrightarrow \text{T-Alg}$. The same goes for Ps-T-Alg, see [3, Remark 6.6]. For Lax-T-Alg, we only obtain a "lax adjunction".

Definition 36. We call $F^T(A) = (TA, m_A)$ the *free* T-algebra on A.

Theorem 2.3. *Let $\mathcal{K}$ be a 2-category and T a 2-monad on it. Then the forgetful 2-functor $U : \text{T-Alg}_s \to \mathcal{K}$ strictly creates*

- *limits that $\mathcal{K}$ has*
- *colimits that T and its square preserve.*

Sketch of a proof. This is again analogous to the 1-dimensional case [29, Theorem 5.6.5.]. Let's sketch the first part of the proof. Assume we're given a 2-functor $G : \mathcal{P} \to \text{T-Alg}_s$ and a weight $W : \mathcal{P} \to \text{Cat}$ such that the limit $A := \{UG, W\}$ exists in $\mathcal{K}$. Denote the limit cone as η.

For short, we denote the T-algebra GP (for $P \in \mathcal{P}$) as (GP, Gp). The fact that G is a 2-functor means that the collection of algebra multiplications $Gp_i : TGP_i \to GP_i$ form components of a 2-natural transformation $Gp_{(-)} : TUG \Rightarrow UG : \mathcal{P} \to \mathcal{K}$. Consider the cone

$$W \Rightarrow \mathcal{K}(A, UG-) \Rightarrow \mathcal{K}(TA, TUG-) \Rightarrow \mathcal{K}(TA, UG-)$$

Given by $\mathcal{K}(TA, Gp_{(-)}) \cdot T_{A,UG-} \cdot \eta$. There is a unique map $a : TA \to A$ that commutes with those cones. Using universal properties, it can be shown that this a makes (A, a) into a strict T-algebra. Fixing $P \in \mathcal{P}, x \in WP$, the following diagram

$$\begin{array}{ccc} TA & \xrightarrow{T\eta_P(x)} & TGP \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle Gp} \\ A & \xrightarrow[\eta_P(x)]{} & GP \end{array}$$

[9]Unit and counit of a biadjunction are pseudonatural rather than natural.

commutes, i.e. $\eta_P(x) : (A, a) \to (GP, Gp)$ is an algebra morphism. This means that η lifts to a cone $\hat{\eta} : W \Rightarrow \text{T-Alg}_s((A, a), G-)$. It can be shown that it has the required universal property in T-Alg_s. □

Corollary 2.1. *Let $\mathcal{K}$ be complete 2-category. Then T-Alg$_s$ is complete.*

Cocompleteness is again more complicated than completeness (now) in the 2-category of algebras. We have the following analogue of [29, Theorem 5.6.12]:

Theorem 2.4. *Let $\mathcal{K}$ be complete and cocomplete 2-category and let assume T has a rank (preserves α-filtered colimits for some regular cardinal α). Then T-Alg$_s$ is cocomplete.*

Proof. [2, Theorem 3.8] □

It can be further proven that the free-forgetful adjunction between $\mathcal{K}$ and T-Alg_s is terminal amongst those 2-adjunctions $F \dashv U : \mathcal{D} \to \mathcal{C}$ that generate the 2-monad T. Unique such arrow that commutes with both left and right adjoints is called a *canonical comparison arrow*. We say that the 2-functor U is *monadic* if the canonical comparison arrow is an equivalence of 2-categories. A version of Beck's monadicity theorem, a criterion which says when exactly is U monadic, can be proven for 2-categories. See [5, Theorem II.2.1]. Also, the 2-category T-Alg_s is precisely an EM-object $\mathcal{C}^T$ that was mentioned in section 3 of the first chapter.

A big part of the paper [2] was showing that the 2-category T-Alg admits all kinds of 2-categorical limits. To summarize:

Theorem 2.5. *Let $\mathcal{K}$ be a 2-category and $T : \mathcal{K} \to \mathcal{K}$ a 2-monad. Then the forgetful 2-functor U : T-Alg $\to \mathcal{K}$ creates products, inserters, equifiers, therefore also inverters, cotensor products, lax and pseudo limits.*

Our interest is the lax case. Moving onto the 2-category T-Alg_l, we find that it admits a lot less limits than its pseudo-cousin (see [2, Remark 2.9] for an example). Limits in T-Alg_l and T-Alg_c have been studied in the paper [17][10]. The results in this paper can be easily generalized to results about Lax-T-Alg and Lax-T-Alg_c , as we demonstrate with the following propositions. Their proofs are not important for the rest of the thesis so the reader may find them in the Appendix.

Proposition 2.1. *The inclusion J : T-Alg$_s \hookrightarrow$ Lax-T-Alg$_c$ preserves all existing limits.*

Proof. Link to the Appendix: (3.4). □

Note that in the good cases that we will encounter in the next chapter, this inclusion J has a left adjoint and thus preservation of limits is automatic.

Proposition 2.2. *Given a 2-monad T on a 2-category $\mathcal{K}$, the forgetful 2-functor Lax-T-Alg$_c \to \mathcal{K}$ creates lax limits of arrows.*

[10] You may also see [18, 8.5] for the summary.

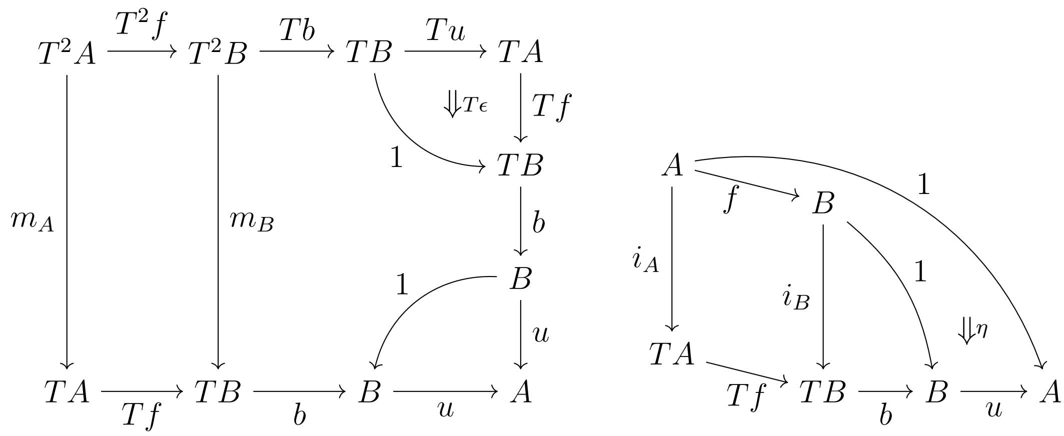
Proof. Link to the Appendix: (3.4). □

Theorem 2.6. *The forgetful 2-functor U : Lax-T-Alg$_c$ $\to \mathcal{K}$ creates lax limits. In particular it creates EM-objects.*

Proof. [17, Proposition 4.5, Theorem 4.8]. □

With the doctrinal adjunction (Lemma 2.1), we've seen a relationship between adjoints and lax and colax morphisms. As is known (or see Remark 20), an adjunction generates a monad. What's more, an adjunction gives rise to a lax algebra provided that domain of the right adjoint is a lax algebra in the first place:

Theorem 2.7. *Let (B, b) be a strict T-algebra and assume we have an adjunction $(\eta, \epsilon) : f \dashv u : B \to A$. Then we can equip A with the structure of a lax T-algebra (A, a, α, α_0), where $a = ubTf$ and α, α_0 are given by:*



Proof. See [20, Theorem 8.3], which is the dual version of this theorem. □

Definition 37. We may refer to the above as a *transport of structure along an adjunction*[11] or say that the adjunction $f \dashv u$ *generates* the T-algebra (A, a, α, α_0).

[11] This is a categorification of transporting structure of an algebra (in the sense of universal algebra) along a bijection.

Chapter 3

Codescent objects and coherence

3.1 Codescent objects

Descent objects (and more importantly, codescent objects) appear in several places in two-dimensional (i.e. categorified) universal algebra. They make an appearance in descent theory as well, see also recent works of Fernando Lucatelli Nunes. They behave similarly to how coequalizers do in ordinary category theory. We leave their most significant use (coherence results) for the next chapter.

Definition 38. Consider the following truncated cosimplicial graph:

$$\begin{array}{ccccc} & \xrightarrow{\quad p \quad} & & \xrightarrow{\quad d \quad} & \\ X_3 & \xrightarrow{\quad q \quad} & X_2 & \xleftarrow{\quad e \quad} & X_1 \\ & \xrightarrow{\quad r \quad} & & \xrightarrow{\quad c \quad} & \end{array}$$

and let:

- Δ_s be the 2-category generated by the above 1-cells that are subject to the cosimplicial identities $de = 1$, $1 = ce$, $dp = dq$, $cr = cq$, $cp = dr$, and whose only 2-cells are the identities.
- Δ_l be the 2-category whose morphisms are freely generated by morphisms p, q, r, d, e, c and whose 2-cells are freely generated by the names $\delta : de \Rightarrow 1$, $\gamma : 1. \Rightarrow ce$, $\kappa : dp \Rightarrow dq$, $\lambda : cr \Rightarrow cq$, $\rho : cp \Rightarrow dr$,
- Δ_p be the 2-category whose morphisms are freely generated by morphisms p, q, r, d, e, c and whose 2-cells are generated by the above 2-cells with the ad-dditional property that between any pair of parallel morphisms there is a unique 2-cell.

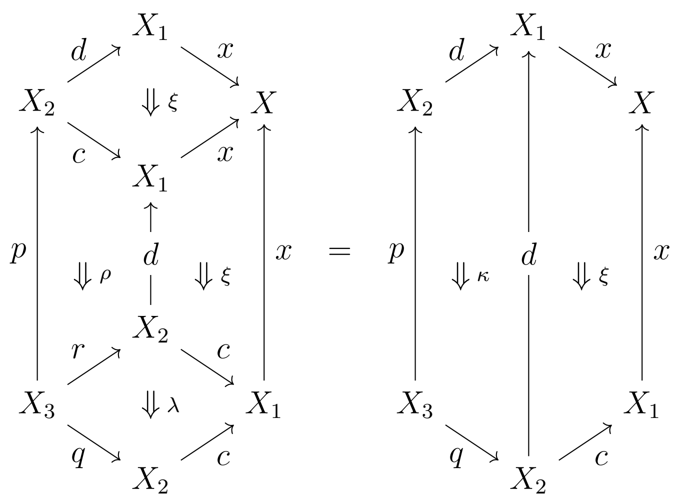
Let $\mathcal{K}$ be a 2-category.

- A *strict coherence data* is a 2-functor $\Delta_s \to \mathcal{K}$,
- A *lax coherence data* is a 2-functor $\Delta_l \to \mathcal{K}$,
- A *coherence data* is a 2-functor $\Delta_p \to \mathcal{K}$.

By (strict, lax) coherence data *in* $\mathcal{K}$ we mean the image of the corresponding 2-functor. We also abuse the notation and denote $p, q, \ldots$ and $\delta, \gamma, \ldots$ the images of these 2-functors in $\mathcal{K}$.

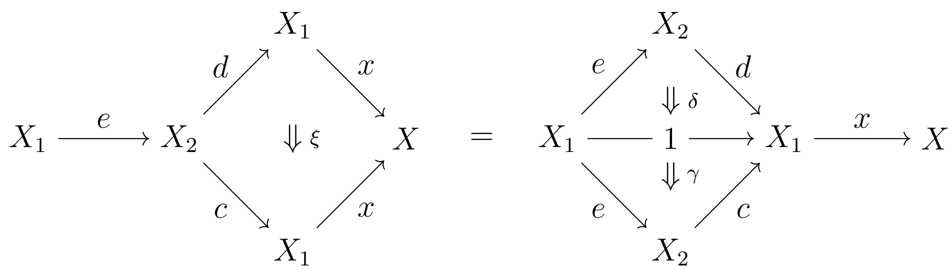
Remark 34. The 2-category Δ_s can be realized as a full subcategory of Δ^{op}, where Δ is a simplicial category of finite ordinals $[n] = \{0, \ldots, n-1\}$ with the usual face and degeneracy maps. For the free 2-categories, the basic reference is [33]. For the construction of Δ_p, see [3, Remark 6.24].

Definition 39. Given a coherence data in $\mathcal{K}$, a *coherence cocone* is a pair $(x : X_1 \to X, \xi : xd \Rightarrow xc)$ of a morphism and an invertible 2-cell that satisfies:



(3.1)

and



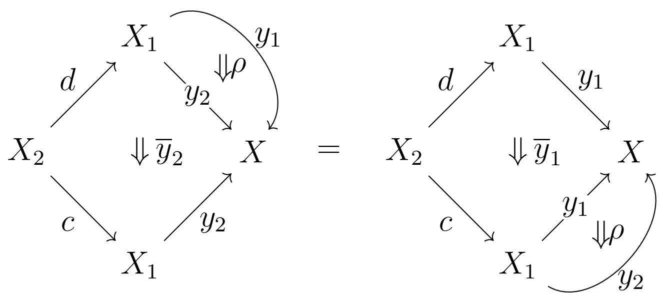
(3.2)

Definition 40. Given lax coherence data as in Definition 38, a *lax codescent object* $(x : X_1 \to X, \xi : xd \Rightarrow xc)$ is the *universal* coherence cocone. Namely, one-dimensional and two-dimensional properties are satisfied.

1. (*One-dimensional universal property*) Given any coherence cocone $(y : X_1 \to Y, \overline{y} : yd \Rightarrow yc)$, there is a unique morphism $\theta : x \to y$ satisfying:

$$\theta x = y$$
$$\theta \xi = \overline{y}$$

2. (*Two-dimensional universal property*): Given any *morphism of coherence cocones* $\rho : (\theta_1 x, \theta_1 \xi) \to (\theta_2 x, \theta_2 \xi)$, that is, a 2-cell $\rho : \theta_1 x \Rightarrow \theta_2 x$ satisfying:



then there is a unique 2-cell $\rho' : \theta_1 \Rightarrow \theta_2$ such that $\rho' x = \rho$.

We may abuse the notation and call $x : X_1 \to X$ or just X a *lax codescent object.*

Definition 41. Given a 2-category $\mathcal{K}$, a codescent object in $\mathcal{K}^{op}$ is called a *descent object* in $\mathcal{K}$.

Definition 42. Let (A, a, α, α_0) be a lax T-algebra. By its *resolution*, denoted $\mathrm{Res}(A, a, \alpha, \alpha_0)$, is meant the following diagram in $\mathcal{K}$:

$$\begin{array}{ccccc} & \xrightarrow{\quad m_{TA} \quad} & & \xrightarrow{\quad m_A \quad} & \\ T^3A & \xrightarrow{\quad Tm_A \quad} & T^2A & \xleftarrow{\quad Ti_A \quad} & TA \\ & \xrightarrow{\quad T^2A \quad} & & \xrightarrow{\quad Ta \quad} & \end{array}$$

together with 2-cells $T\alpha : TaT^2a \Rightarrow TaTm_A$ and $T\alpha_0 : 1 \Rightarrow TaTi_A$ (other 2-cells being identities).

Remark 35. This diagram is clearly a lax coherence data in $\mathcal{K}$. It lifts in an obvious way to T-Alg$_s$, so we will use its location interchangeably. Note that if α is invertible, (a, α^{-1}) is a coherence cocone.

Let's now discuss three neat instances of (co)descent objects in Cat.
Given a small category $\mathcal{C} \in$ Cat, we may regard $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ (the set of objects, morphisms and composable pairs of morphisms) as discrete categories. We have then two projection functors p_1, p_2, composition functor comp : $(f, g) \mapsto gf$, domain and codomain functors d, c and identity morphism functor $i : A \mapsto 1_A$. These satisfy the usual category axioms. It is readily verified that they then form into a **strict** coherence data:

$$\begin{array}{ccccc} & \xrightarrow{\quad p_1 \quad} & & \xrightarrow{\quad d \quad} & \\ \mathcal{C}_2 & \xrightarrow{\quad \mathrm{comp} \quad} & \mathcal{C}_1 & \xleftarrow{\quad i \quad} & \mathcal{C}_0 \\ & \xrightarrow{\quad p_2 \quad} & & \xrightarrow{\quad c \quad} & \end{array}$$

And the following holds:

Theorem 3.1. *A small category $\mathcal{C}$ is a codescent object of its associated coherence data in Cat.*

Proof. The codescent object $(\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}, \xi : \iota d \Rightarrow \iota c)$ consists of an inclusion of objects and a natural transformation, whose component at a morphism $f : A \to B$ is the morphism f, i.e. $\xi_f = f$. Given any coherence cocone $(y : \mathcal{C}_0 \to Y, \delta : yd \Rightarrow yc)$, the equation 3.1 forces for all $f : A \to B$, $g : B \to C$:

$$\delta_g \cdot \delta_f = \delta_{gf} : yA \to yC$$

While 3.2 forces $\delta_{1_A} = 1_{yA}$. This tells us that the cocone is basically a functor $\mathcal{C} \to Y$, and universal property forces the unique functor $\mathcal{C} \to Y$ to be equal to (y, δ). □

Note that in this easy example, the 1-cell functor of the codescent object, $\iota : \mathcal{C}_0 \to \mathcal{C}$, is bijective on object. This is true for any codescent object morphism in Cat (as well as for coinserters, coequifiers, coinverters, see [3, Corollary 2.44]). What's more, codescent objects provide us a way to factorize any functor $F : \mathcal{A} \to \mathcal{C}$ between small categories as bijective on objects followed by a fully faithful functor (BO-FF factorisation), see [3, 2.6].

A version of the following Theorem for the 2-category T-Alg$_l$ in place of Lax-T-Alg is the first documented appearance of (then unnamed) descent object. It has been observed by Ross Street in 1975 at the end of the paper [33].

Theorem 3.2. *Assume we're given a lax T-algebra (A, a, α, α_0) and a strict T-algebra (B, b) for a 2-monad (T, m, i) in a 2-category $\mathcal{K}$. Then the category Lax-T-Alg$((A, a, \alpha, \alpha_0), (B, b))$ is the* ***descent*** *object of the following op-coherence data*[1]*:*

$$\mathcal{K}(T^2A, B) \;\begin{array}{c} \xleftarrow{\;\mathcal{K}(1,b)\cdot T_{TA,TB}\;} \\ \xleftarrow{\;\mathcal{K}(m_A,1)\;} \\ \xleftarrow{\;\mathcal{K}(Ta,1)\;} \end{array}\; \mathcal{K}(TA, B) \;\begin{array}{c} \xleftarrow{\;\mathcal{K}(1,b)\cdot T_{A,B}\;} \\ \xrightarrow{\;\mathcal{K}(i_A,1)\;} \\ \xleftarrow{\;\mathcal{K}(a,1)\;} \end{array}\; \mathcal{K}(A, B)$$

with $\delta = 1$, $\gamma = \mathcal{K}(\alpha_0, B)$, $\kappa = 1$, $\lambda = \mathcal{K}(\alpha, B)$, $\rho = 1$.

Proof. Given a descent cone $(Y : \mathcal{Y} \to \mathcal{K}(A, B), \delta : \mathcal{K}(1, b)T_{A,B}Y \Rightarrow \mathcal{K}(a, 1)Y)$ the conditions 3.1, 3.2 amount exactly to (YA, δ_D) being a lax T-algebra morphism $(A, a, \alpha, \alpha_0) \to (B, b)$ for each $D \in \text{ob}\mathcal{Y}$! The descent object consists of a forgetful functor U : Lax-T-Alg$((A, a, \alpha, \alpha_0), (B, b)) \to \mathcal{K}(A, B)$ and a natural transformation

$$\xi : \mathcal{K}(1, b)T_{AB}U \Rightarrow \mathcal{K}(a, 1)U : \text{Lax-T-Alg}((A, a, \alpha, \alpha_0), (B, b)) \to \mathcal{K}(TA, B)$$

Whose component at a lax morphism $(f, \overline{f})$ is the 2-cell $\overline{f} : bTf \Rightarrow fa$.

If there was a functor $K : \mathcal{Y} \to$ Lax-T-Alg$((A, a, \alpha, \alpha_0), (B, b))$, $K : D \mapsto (g_D, \overline{g}_D)$ such that:

$$Y = UK$$
$$\delta = \xi K$$

We clearly must have $YD = g_D$, $\delta_D = \overline{g}_D$. But we know that this is a lax algebra morphism from before. □

[1] Coherence data in $\mathcal{K}^{op}$.

Given a 2-category $\mathcal{K}$, an object B and coherence data S in $\mathcal{K}$, there is a category $\mathrm{Coh}(\mathcal{K}, S, B)$ whose objects are coherence cocones $(x : TA \to B, \overline{x} : xd \Rightarrow xc)$. Decription of this category is analogous to that of "colax codescent objects" that will be introduced in Section 3.3, so we omit it at this point.

Theorem 3.3. *Let (A, a, α, α_0) be a pseudo T-algebra in a 2-category $\mathcal{K}$. The canonical mapping $\kappa : (\theta : A \to B) \mapsto (\theta a, \theta\alpha^{-1})$ induces an equivalence of categories:*

$$\mathcal{K}(A, B) \simeq \mathit{Coh}(\mathcal{K}, \mathit{Res}(A, a, \alpha, \alpha_0), B)$$

Sketch of a proof. Using simple pasting diagrams it can be shown that κ is fully faithful and essentially surjective. The proof is in [22, Lemma 2.3]. □

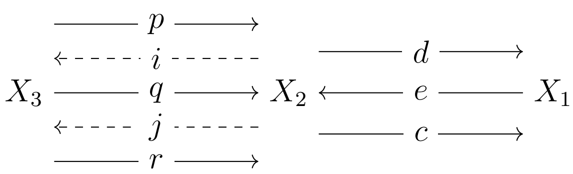
Remark 36. Codescent object can be defined using coinserters and coequifiers [16, Proposition 2.1], and so are themselves a colimit. We can thus find a weight $W : J \to \mathrm{Cat}$ and a 2-functor $F : J \to \mathcal{K}$ so that the category $\mathrm{Coh}(\mathcal{K}, S, B)$ is isomorphic to $[J, \mathrm{Cat}](W, \mathcal{K}(B, F-))$.

It can also be shown that the category $[J, \mathrm{Cat}](W, \mathcal{K}(B, F-))$ is equivalent to $\mathrm{Psd}[W, \mathcal{K}(B, F-)]$[2] and moreover that this equivalence is 2-natural in B. This implies that A is a **bicolimit** of F weighted by W. We obtain:

Corollary 3.1. *Given a pseudo T-algebra (A, a, α, α_0) in a 2-category $\mathcal{K}$, the underlying object A of the algebra is the bicolimit of its resolution.*

Let's introduce an important certain important subclass of codescent objects - those that are reflexive.

Definition 43. We say the lax coherence data in a 2-category $\mathcal{K}$ is *reflexive* if there are morphisms $i, j : X_2 \to X_3$:



together with 2-cells $1 \Rightarrow pi$, $1 \Rightarrow qi$, $1 \Rightarrow qj$, $1 \Rightarrow rj$ and $ri \Rightarrow ec$, $pj \Rightarrow ed$ and $ie \Rightarrow je$. A *lax codescent object* of lax reflexive coherence data is just a codescent object of the underlying lax coherence data.

Example 36. Given a lax T-algebra (A, a, α, α_0), its resolution is reflexive coherence data both in $\mathcal{K}$ and T-Alg_s. We have:

[2] This is assumed implicitly in [22]

$$\begin{array}{ccccc} & \xrightarrow{m_{TA}} & & & \\ & \xleftarrow{Ti_{TA}} & & \xrightarrow{m_A} & \\ T^3A & \xrightarrow{Tm_A} & T^2A & \xleftarrow{Ti_A} & TA \\ & \xleftarrow{T^2 i_A} & & \xrightarrow{Ta} & \\ & \xrightarrow{T^2a} & & & \end{array} \tag{3.3}$$

with a 2-cell $1 \Rightarrow rj = T^2(ai_A)$ being $T^2\alpha_0$ and other 2-cells being identities.

Remark 37. If (A, a, α, α_0) is a pseudoalgebra, its resolution is not just a lax coherence data but **a** coherence data, meaning in particular that all pasting diagrams built using 1-cells $m_{TA}, Tm_A, T^2a, m_A, Ti_A, Ta$ and 2-cells $T\alpha, T\alpha_0$ with the same domain and codomain morphisms are equal. This isn't immediately clear but becomes obvious if one understand an explanation of where the resolution for a pseudo algebra really comes from, see [3, Remark 6.29].

For the lax algebras, nothing of sorts is true. Consider the identity 2-monad on Cat and its lax algebra (t, μ, η) (a monad). In a no way is it true that $t\eta = \eta t$ unless the monad is idempotent.

Theorem 3.4. *Codescent objects of reflexive coherence data commute with finite products in Cat.*

Proof. Two possible proofs were first sketched in [16, Proposition 4.3], for a full proof see [3, Proposition 8.41]. □

There is a note-worthy interaction between codescent objects and a certain stronger version of finitary 2-monads on Cat. Algebras for these 2-monads can be described as small categories together with n-ary operations, equations and natural transformations between the iterations of these operations, and equations between iterations of these transformations. Compare with the talk after Definition 35.

Definition 44. A 2-functor T : Cat $\to$ Cat is said to be *strongly finitary* if it is the left Kan extension of itself restricted to Set_f - the skeletal category of finite sets regarded as a locally discrete 2-category. A 2-monad is said to be *strongly finitary* if its endo-2-functor part is so.

In the coend notation, T satisfies (n being the unique set with n elements):

$$T = \int^{n \in \mathrm{Set}_f} (-)^n \times Tn$$

Corollary 3.2. *Any strongly-finitary 2-monad T : Cat $\to$ Cat preserves codescent objects of reflexive coherence data.*

Proof. From 3.4 it follows that the product 2-functor $\prod^n$: $\mathrm{Cat}^n \to \mathrm{Cat}$ preserves codescent objects of reflexive coherence data for $n \geq 2$. The diagonal 2-functor $\Delta : \mathrm{Cat} \to \mathrm{Cat}^n$, $\mathcal{A} \mapsto \mathcal{A}^n$, preserves colimits as it is a left adjoint so it preserves all colimits. Thus $(-)^n = \prod^n \Delta$ preserves codescent objects of reflexive coherence data. Since $(-) \times Tn$ preserves colimits in Cat and T is defined using colimits, we obtain that it preserves codescent objects of relexive coherence data. □

Theorem 3.5. *Cat is the free completion of Set_f under filtered colimits and codescent objects of strict reflexive coherence data.*

Proof. [3, Theorem 8.31]. □

The following characterization of strongly-finitary 2-monads is possible:

Corollary 3.3. *Let T be a 2-functor on Cat. The following are equivalent:*

- *T is strongly finitary,*
- *T preserves filtered colimits and codescent objects of strict reflexive coherence data.*

Codescent objects are like a 2-dimensional analogue of coequalizers in ordinary category theory, consider that:

- Corollary 3.1 is an analogue of the fact that every algebra for a 1-monad can be expressed as canonical coequalizer of free algebras on its underlying object (see [29][Example 5.4.7]),
- Theorem 3.4 is an analogue of the fact that **reflexive coequalizers commute with finite products** in Set,
- Moreover, in the paper [22], codescent objects have been successfully used to prove a version of Beck's monadicity theorem for pseudomonads. It can be shown that a pseudoadjunction $F \dashv U : \mathcal{D} \to \mathcal{C}$ generates a pseudomonad $T = = UF$. We can also define a canonical comparison 2-functor $\mathcal{D} \to$ Ps-T-Alg. It can be then shown that this 2-functor is a biequivalence if and only if U reflects adjoint equivalences, and moreover $\mathcal{D}$ admits certain (U-absolute) codescent objects and U preserves them.

3.2 Coherence

In our setting, a *coherence result* or a *coherence theorem* is a theorem that describes under which conditions on a 2-monad T on a 2-category is every pseudo-T-algebra equivalent to a strict T-algebra. Proving coherence results is useful because it allows us to study non-strict structures in terms of the strict ones. We also don't lose any generality if we choose to work with strict algebras instead of pseudoalgebras once we have coherence results in our hands.

What people usually mean by *coherence* (and what came first) is that in any monoidal category (or closed/symmetric monoidal category, or a bicategory), any diagram built using the defining associativity/unit isomorphisms commutes. These notions are related, see Remark 40.

A universal way to obtain a strict algebra (A', a') from a pseudoalgebra (or a lax algebra) (A, a, α, α_0) is by finding a left adjoint to the inclusion T-Alg$_s \hookrightarrow$ Ps-T-Alg (or T-Alg$_s \hookrightarrow$ Lax-T-Alg). This A' then may be called a *pseudo* (or *lax*) *morphism classifier* because every pseudo (or lax) morphism from of A to a strict algebra factors uniquely through a strict morphism:

$$\begin{array}{ccc} (A,a,\alpha,\alpha_0) & \longrightarrow & (A',a') \\ {\scriptstyle (f,\overline{f})}\downarrow & \swarrow {\scriptstyle \exists!(g,1)} & \\ (B,b) & & \end{array}$$

Given a general 2-monad T on a 2-category $\mathcal{K}$, the existence of left adjoints to inclusions T-Alg$_s \hookrightarrow$ T-Alg and T-Alg$_s \hookrightarrow$ T-Alg$_l$ has first been proven in [2, Theorem 3.13] under the assumption that $\mathcal{K}$ is complete and cocomplete and T has a rank (i.e. preserves κ-filtered colimits). In fact, much less is necessary, we only need T-Alg$_s$ to admit codescent objects to obtain either of these adjoints, as well as adjoints to T-Alg$_s \hookrightarrow$ Ps-T-Alg and T-Alg$_s \hookrightarrow$ Lax-T-Alg.

Let's demonstrate how can such result be useful. Recall the functor 2-monad from Example 29. Clearly, T-Alg$_s = [\mathcal{J}, \mathrm{Cat}]$ admits all colimits and in particular codescent objects. We obtain a left adjoint:

$$[\mathcal{J}, \mathrm{Cat}] \underset{\hookrightarrow}{\overset{(-)'}{\leftarrow}} \perp \ \mathrm{Lax}[\mathcal{J}, \mathrm{Cat}]$$

And now:

Theorem 3.6. *Given 2-category $\mathcal{K}$, any weighted lax limit of a 2-functor F can be expressed as (an ordinary) limit for some 2-functor F'. The same is true for pseudo limits.*

Proof. Consider the weight $W : \mathcal{J} \to \mathrm{Cat}$ and a 2-functor $F : \mathcal{J} \to \mathcal{K}$. Because of the 2-adjunction have 2-natural isomorphisms:

$$\mathcal{K}(-, \{W, F\}) \cong \mathrm{Lax}[\mathcal{J}, \mathrm{Cat}](F, \mathcal{K}(-, G?)) \cong [\mathcal{J}, \mathrm{Cat}](F', \mathcal{K}(-, G?)),$$

expressing a lax limit of F as a strict limit of F'. □

Let's now be more specific. By a *coherence result* we mean a theorem of the following form:

Theorem-Schema 3.1. *Let $\mathcal{K}$ be a 2-category and T a 2-monad. The inclusion*

$$T\text{-}Alg_s \hookrightarrow Ps\text{-}T\text{-}Alg$$

admits a left adjoint (denoted $(-)'$), and the component of the unit of the adjunction at each object $(A, a, \alpha, \alpha_0) \in$ Ps-T-Alg is an equivalence in Ps-T-Alg.

The following coherence result has been proven by Lack in 2002 in his significant paper [16]. In Section 3.3 we will prove its analogue for the lax case with all the details, so its proof is omited here.

Theorem 3.7. *Let $\mathcal{K}$ be a 2-category and a 2-monad T on $\mathcal{K}$ such that T-Alg$_s$ admits codescent objects of resolutions of pseudo-algebras and U preserves them. Then Theorem schema holds.*

Proof. The proof can be found in [16, Theorem 3.2]. Note that what you obtain in the proof is an equivalence in $\mathcal{K}$, but according to (well known and easy to prove) [22, Proposition 3.4] the equivalence lifts up to Ps-T-Alg. □

The following has been observed by John Bourke. It in particular shows that it is enough to only assume that U : T-Alg$_s \to \mathcal{K}$ only preserves codescent objects as a bicolimit, i.e. "up to an equivalence". It assumes familiarity with Lack's proof so we only sketch this.

Proposition 3.1. *Let $\mathcal{K}$ be a 2-category and T a 2-monad. The Theorem Schema holds if and only if T-Alg$_s$ admits codescent objects of resolutions of algebras and the forgetful 2-functor U : T-Alg$_s \to \mathcal{K}$ preserves them as a bicolimit.*

Sketch of a proof. By Lack's construction, the inclusion T-Alg$_s \hookrightarrow$ Ps-T-Alg admits a left adjoint $(-)'$ if and only if T-Alg$_s$ admits codescent objects of resolutions of algebras, and this left adjoint applied to pseudo algebra (A, a, α, α_0) calculates codescent object of its resolution Res(A, a, α, α_0).

" $\Rightarrow$ ": As the left adjoint $(-)'$: Ps-T-Alg $\to \mathcal{K}$ applied to a pseudo algebra (A, a, α, α_0) calculates the codescent object $(A', a') \in$ T-Alg$_s$ of the resolution Res(A, a, α, α_0), T-Alg$_s$ clearly admits these codescent objects. Denote by $\eta_{(A,a,\alpha,\alpha_0)} : (A, a, \alpha, \alpha_0) \to (A', a')$ the unit of the adjunction in Ps-T-Alg. By the assumption, it is an equivalence. By Corollary 3.1, A is a bicolimit of its resolution in $\mathcal{K}$. Since it is equivalent to A' and bicolimits are defined up to an equivalence, the result follows.

" $\Leftarrow$ ": Let $(x : (TA, m_A) \to (A', a'), \xi)$ be a codescent object of resolution of pseudoalgebra (A, a, α, α_0) in T-Alg$_s$ (this is equally a colimit cocone by Remark 36). Because U preserves codescent objects as a bicolimit, the cocone $(x : TA \to A', \xi)$ exhibits A' as a bicolimit in $\mathcal{K}$. Because A is also a bicolimit of the same diagram (and bicolimits are unique up to an equivalence), we obtain that A is equivalent to A'. Furthermore, using the unit $i_{A'}$ of the algebra (A', a') (which is a section of a'), it can be shown that the equivalence morphism is precisely the unit of the adjunction of U and $(-)'$ (the adjoint $(-)'$ exists because T-Alg$_s$ admits the codescent objects of resolutions of algebras, again by Lack's construction). □

Remark 38. We can obtain Theorem 3.8 as a corollary of a more general result on *biadjoint triangles* (the generalisation of adjoint triangles in ordinary category theory), see [26, 8.2. Corollary]. The proof of the general biadjoint triangle theorem also makes use of codescent objects.

Another general coherence theorem says the following: If T is a 2-functor on Cat that preserves bijections on objects, then every pseudoalgebra is equivalent to a strict one. In [16] Lack has shown that it is an instance of the **Theorem schema**[3].

Theorem 3.8. *Let X be a set and let T be a 2-monad on Cat^X that preserves bijections on objects (meaning that each component of the image of X-indexed family of bijection on objects functors is bijection on objects). Then every pseudoalgebra is equivalent to a strict one.*

Sketch of a proof. Note that g is essentially surjective on objects as well: from the definition of invertible 2-cells $\alpha_0, \alpha_0 g$ we have isomorphisms $x \cong ghi_A(x)$ and $ghi_A(g(x)) \cong g(x)$. Thus g is an equivalence in Cat.

Assume that (A, a, α, α_0) is a pseudoalgebra. Factor the functor $a : TA \to A$ as $a = gh$, where g is fully faithful and h is bijection on objects. It can be shown that α admits a unique factorization

$$\begin{array}{ccccc} T^2A & \xrightarrow{Th} & TB & \xrightarrow{Tg} & TA \\ {\scriptstyle m_A}\big\downarrow & & \Downarrow\alpha & & \big\downarrow{\scriptstyle a} \\ TA & \xrightarrow[h]{} & B & \xrightarrow[g]{} & A \end{array} \quad = \quad \begin{array}{ccccc} T^2A & \xrightarrow{Th} & TB & \xrightarrow{Tg} & TA \\ \big\downarrow{\scriptstyle m_A} & & {\scriptstyle b}\big\downarrow & \Downarrow\overline{g} & \big\downarrow{\scriptstyle a} \\ TA & \xrightarrow[h]{} & B & \xrightarrow[g]{} & A \end{array}$$

such that (B, b) is a strict algebra and $(g, \overline{g}) : (A, a, \alpha, \alpha_0) \to (B, b)$ is a pseudomorphism of algebras. As equivalences in $\mathcal{K}$ lift to Ps-T-Alg, the result follows. □

Proof. [27, 3.4 Theorem] □

Corollary 3.4. *Let X be a set and T be a 2-monad on Cat^X that preserves bijections on objects. Then Theorem schema holds.*

Proof. [16, Theorem 4.10]. □

Examples of coherence results

Let's begin with (unbiased) monoidal categories.

Theorem 3.9. *The inclusion of the 2-category of strict monoidal categories into the 2-category of (unbiased) monoidal categories*

$$StrictMonCat \hookrightarrow MonCat$$

has a left adjoint Q, and unit of this adjunction is an equivalence in MonCat. This means that for any monoidal category $\mathcal{A}$ there exists a strict monoidal category $Q\mathcal{A}$ and a monoidal equivalence of categories $\mathcal{A} \simeq Q\mathcal{A}$.

[3] He stated this result in more general form, Cat is replaced by general 2-category that admits an *enhanced factorisation system* and T preserves the first class of morphisms for this system.

First proof. Cat is complete and cocomplete and the monoidal 2-monad $T = \coprod(-)^n$ clearly preserves filtered colimits and thus has a rank. By Theorem 2.4, StrictMonCat is cocomplete, in particular admits codescent objects. Next, T is defined using products and colimits and codescent objects of reflexive coherence data commute with products (Theorem 3.4), so T preserves codescent objects. But because $T = UF$ and F is cocontinous, U preserves codescent objects too. The rest follows from Theorem 3.7. □

Second proof. T preserves bijections on objects so we may apply Theorem 3.4. □

Remark 39. Yet another way to prove this would be to explicitly describe the strictification. Given a monoidal category $(\mathcal{V}, \otimes, I, a, l, r)$, define the objects of the strict monoidal category $(\mathcal{V}', \otimes', I')$ as sequences of objects $[X_1, \ldots, X_n]$ of $\mathcal{V}$, the morphisms $[X_1, \ldots, X_n] \to [Y_1, \ldots, Y_m]$ as the morphisms

$$((X_1 \otimes X_2) \otimes \cdots \otimes X_{n-1}) \otimes X_n \to ((Y_1 \otimes Y_2) \otimes \cdots \otimes Y_{m-1}) \otimes Y_m$$

in $\mathcal{V}$. On object, define $\otimes'$ as concatenation of lists, the empty list $I' = [\,]$ being the unit. On morphisms, $\otimes'$ is defined using an induction. The unit monoidal functor $\mathcal{V} \to \mathcal{V}'$ then sends an object X to the singleton list $[X]$. This example is described in detail in [7, XI.5]. The version for unbiased monoidal categories is in [24, Theorem 3.1.6].

Remark 40. By coherence in a monoidal category $(\mathcal{V}, \otimes, a, l, r, I)$ we may also mean the fact that any diagram built of isomorphisms $l, r,$, their inverses, compositions and tensor products, commutes. It can be shown that this follows from the fact that $\mathcal{V}$ is monoidally equivalent to a strict monoidal category $\mathcal{V}'$. In $\mathcal{V}'$, clearly any diagram built with a', l', r' commutes (because they're identities) and monoidal equivalence transports this identity. See for example [23, 2.4] (it is stated for bicategories but the idea is the same).

Consider now the 2-monad $T = \mathrm{res}_J \cdot \mathrm{lan}_J$ on a 2-category $[\mathrm{ob}\,\mathcal{J}, \mathrm{Cat}]$ introduced in Example 29. The 2-functor res_J also has a **right** 2-adjoint given by right Kan extensions. It thus preserves all colimits and so does T. We obtain[4]:

Theorem 3.10. *Let $\mathcal{J}$ be small and $\mathcal{K}$ complete. Denote Hom$(\mathcal{J}, \mathcal{K})$ for the 2-category of pseudofunctors, pseudonatural transformations and modifications. Then the inclusion of 2-functors into pseudofunctors:*

$$[\mathcal{J}, \mathcal{K}] \hookrightarrow \mathit{Hom}(\mathcal{J}, \mathcal{K})$$

admits a left adjoint whose unit is an equivalence of pseudofunctors. This means that every pseudofunctor is biequivalent to a strict 2-functor.

Remark 41. For a general 2-monad T on a 2-category $\mathcal{K}$, not every pseudoalgebra is equivalent to a strict one, see [16, Example 3.1] for a simple counter-example.

[4]Again, we can also prove this by showing that T preserves bijections on objects.

Even under "nice" conditions ($\mathcal{K}$ being locally finitely presentable[5] and T finitary), a counter-example to the Theorem Schema has been found in the year 2011 in the paper [30]. Recall the Example 30. In [30] this example is generalized to obtain a finitary 2-monad T_{Cat} on a locally finitely presentable 2-category of Cat-enriched Cat-graphs, whose strict algebras are strict 3-categories and pseudo algebras are certain weak 3-categories. The result then ultimately follows from the fact that a *Gray-category* (certain weak version of a 3-category) is not in general triequivalent to a strict 3-category.

Going one dimension lower, every bicategory **is** biequivalent to a (strict) 2-category. Let's sketch two ways one can prove this. The first proof is elementary, the second one uses Theorem 3.4. They can be found in [23, 2.3 Theorem], [27, 4.3].

Theorem 3.11. *Every small bicategory is biequivalent to a 2-category.*

Sketch of a first proof. It can be shown that the Yoneda embedding $y : \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{op}, \mathrm{Cat})$ is a homomorphism that is locally an equivalence and that $y(\mathcal{C})$ is a 2-category. Restricting to $y' : \mathcal{C} \to y(\mathcal{C})$ gives us a homomorphism that is surjective on objects and locally an equivalence, so it is a biequivalence. □

Sketch of a second proof. Let X be a set. Consider the following modification of Example 30: Denote 2-Cat_X to be the 2-category of small 2-categories with object set X, morphisms 2-functors that are identity on objects and 2-cells being icons. 2-category of Cat-graphs with object set X is then isomorphic to $\mathrm{Cat}^{X\times X}$ and there is a 2-adjunction

$$\mathrm{Cat}^{X\times X} \underset{F}{\overset{U}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} 2\text{-}\mathrm{Cat}_X \tag{3.4}$$

between the forgetful 2-functor and the "free 2-category on a Cat-graph" 2-functor. This adjunction is 2-monadic and $T = UF$ clearly preserves bijections on objects. The result follows from Theorem 3.4. □

Note that in the second proof we've actually proven that every (small) bicategory is biequivalent to a 2-category **with the same object-set**.

Remark 42. By Corollary 3.2, any strongly finitary 2-monad on Cat preserves codescent objects or reflexive coherence data. As coherence data of pseudoalgebras are always reflexive and Cat is cocomplete, Theorem Schema holds for this class of 2-monads. It also holds for clubs, see [27, 4.1].

Let's briefly mention the coherence result for multicategories. In the paper [6], it has been proven that there's a 2-adjunction:

[5] See [18, 5.3] for the definition.

$$\mathrm{Multicat} \underset{F}{\overset{U}{\leftrightarrows}} \mathrm{MonCat} \qquad (F \dashv U)$$

that induces a 2-monad $T = UF$ on 2-category Multicat of multicategories. A multicategory M is said to be *representable* if for every tuple $\overline{x} = (x_1, \dots, x_n)$ there's an object $\otimes\overline{x}$ and an arrow $\pi_{\overline{x}} : (x_1, \dots, x_n) \to \otimes\overline{x}$ that induces a natural isomorphism:

$$M((x_1, \dots, x_n), y) \cong M((\otimes\overline{x}), y)$$

It was shown that a multicategory is representable if and only if the unit $\eta_M : M \to UFM$ of the adjunction admits a left adjoint. Moreover, Ps-T-Alg $\cong$ RepMulticat. Strict T-algebras are what's called a *strict representable multicategory*. Coherence result for multicategories, although not an instance of theorem-schema, reads as:

Theorem 3.12. *The inclusion StrictReprMulticat $\hookrightarrow$ RepMulticat admits a left biadjoint whose unit is a pseudo-natural equivalence.*

Proof. [6, 10.8 Theorem]. □

3.3 Lax coherence result

Given a pseudoalgebra (A, a, α, α_0), the pair $(a : TA \to A, \alpha^{-1} : am_A \Rightarrow aTa)$ is a coherence cocone for the resolution $\mathrm{Res}(A, a, \alpha, \alpha_0)$. This convenient fact is necessary to formulate and prove Theorem 3.3 and was also used in Lack's proof of Theorem 3.7. If we wish to prove analogues of these theorems for lax algebras, α is not invertible and we have to introduce a new notion (as well as use colax morphisms instead of lax ones). This is a motivation for what we call a *colax codescent object*. The main goal of this section is to prove the analogues of these two theorems.

We call the coherence theorem in the lax setting a *lax coherence result*. It is a statement of the form:

Theorem-Schema 3.2. *Let $\mathcal{K}$ be a 2-category and T a 2-monad. The inclusion*

$$T\text{-}Alg_s \hookrightarrow Lax\text{-}T\text{-}Alg_c$$

admits a left adjoint (denoted $(-)'$), and the component of the unit of the adjunction at each object $(A, a, \alpha, \alpha_0) \in Lax\text{-}T\text{-}Alg_c$ has an internal right adjoint in $\mathcal{K}$.

Definition 45. Consider the following graph:

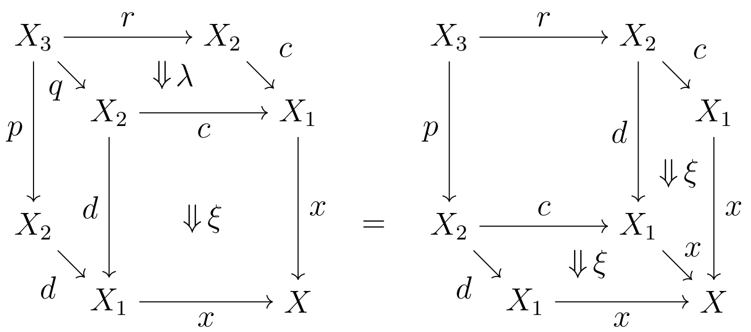
$$\begin{array}{ccccc} & \xrightarrow{\quad p \quad} & & \xrightarrow{\quad d \quad} & \\ X_3 & \xrightarrow{\quad q \quad} & X_2 & \xleftarrow{\quad e \quad} & X_1 \\ & \xrightarrow{\quad r \quad} & & \xrightarrow{\quad c \quad} & \end{array}$$

together with identities $cp = dr$,$de = 1$, $dp = dq$ and 2-cells $\gamma : 1 \Rightarrow ce$, $\lambda : cr \Rightarrow cq$. Let Δ_c be the free 2-category whose morphisms are generated by this graph subject to these relations, and whose 2-cells are freely generated by γ, λ.

A *colax coherence data* for a 2-category $\mathcal{K}$ is a 2-functor $\Delta_c \to \mathcal{K}$. Colax coherence data *in* $\mathcal{K}$ is the image of Δ_c in $\mathcal{K}$.

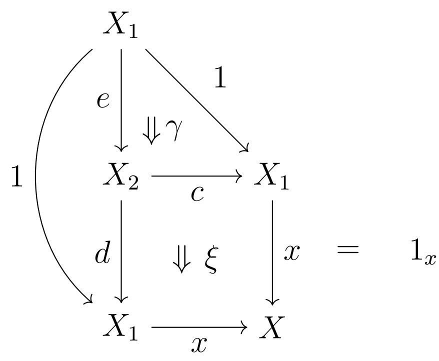
We again use the same letters $p, q, \ldots, \gamma, \lambda$ for their image under this 2-functor in $\mathcal{K}$.

Definition 46. Let X be an object of $\mathcal{K}$ and S colax coherence data. The *(colax) coherence category* $\mathrm{Coh}(\mathcal{K}, S, X)$ is a category with objects being tuples $(x : X_1 \to X, \xi : xc \Rightarrow xd)$ satisfying:



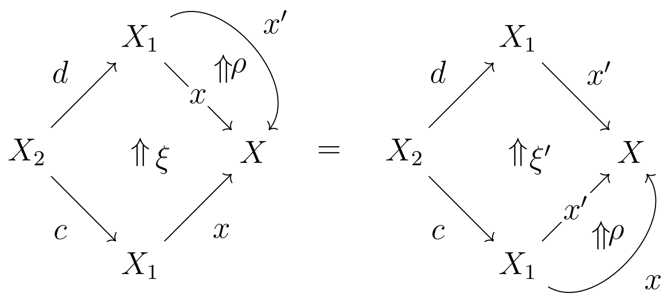
(3.5)

and also:



(3.6)

A morphism $\rho : (x, \xi) \to (x', \xi')$ in this category is a 2-cell $\rho : x \Rightarrow x'$ in $\mathcal{K}$ such that the following holds:



(3.7)

Remark 43. The construction above is 2-functorial. You can verify that given a locally small 2-category $\mathcal{K}$, there is a 2-functor

$$\mathrm{Coh}(\mathcal{K}, S, -) : \mathcal{K} \to \mathrm{Cat}$$

defined in an obvious way.

Example 37. Given a lax T-algebra (A, a, α, α_0), its resolution $\mathrm{Res}(A, a, \alpha, \alpha_0)$ (Definition 42) defines colax coherence data (the directions of 2-cells γ, λ is the same as for lax coherence data). If $(x : TA \to X, \xi : xTa \Rightarrow xm_A)$ is an element of $\mathrm{Coh}(\mathcal{K}, \mathrm{Res}(A, a, \alpha, \alpha_0), B)$, the equations 3.5, 3.6 become:

$$\xi T m_A \cdot xT\alpha = \xi m_{TA} \cdot \xi T^2 a, \tag{3.8}$$
$$\xi T i_A \cdot xT\alpha_0 = 1_x, \tag{3.9}$$

and a morphism $\rho : (x, \xi) \to (x', \xi')$ satisfies:

$$\rho m_A \cdot \xi = \xi' \cdot \rho T a. \tag{3.10}$$

Definition 47. We call the elements of $\mathrm{Coh}(\mathcal{K}, S, X)$ the *(colax) coherence cocones.* Such an object $(x : X_1 \to X, \xi : xc \Rightarrow xd)$ is called *colax codescent object* if the following universal properties are satisfied:

1. (*One-dimensional universal property*): For any $(y : X_1 \to Y, \overline{y} : yc \Rightarrow yd)$ there is a unique morphism $z : X \to Y$ such that:
$$zx = y,$$
$$z\xi = \sigma.$$
2. (*Two-dimensional universal property*): Given a morphism of cocones $\rho : (zx, z\xi) \to (z'x, z'\xi)$ in $\mathrm{Coh}(S, Y)$, there is a unique 2-cell $\overline{z} : z \Rightarrow z'$ in $\mathcal{K}$ such that
$$\overline{z}x = \rho.$$

Remark 44. Given some coherence data S in a 2-category $\mathcal{K}$, having a codescent object $(e : X_1 \to E, \overline{e} : ec \Rightarrow ed)$ amounts exactly to there being a 2-natural isomorphism:

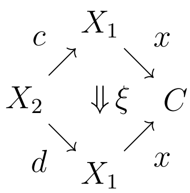
$$\mathcal{K}(E, -) \cong \mathrm{Coh}(\mathcal{K}, S, -) \tag{3.11}$$

whose component at B is given by $(\theta : E \to B) \mapsto (\theta e, \theta \overline{e})$.

Analogous to lax codescent objects, colax codescent objects can be calculated using colimits and thus it itself must be a colimit. $\mathrm{Coh}(\mathcal{K}, S, B)$ can be then viewed as a "category of cocones" for some 2-functor F and weight W.

Theorem 3.13. *Let $\mathcal{K}$ be a 2-category admitting coinserters and coequifiers. Then it admits colax codescent objects.*

Sketch of a proof. Assume we're given coherence data as in the beginning of this chapter. First take the coinserter of d, c:

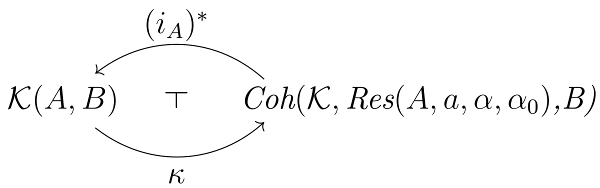


Then take coequifier $w_1 : C \to W_1$ of the LHS and RHS 2-cells of 3.5 (we want these to be equal), and then take coequifier $w_2 : W_1 \to W_2$ of the LHS and RHS of 3.6 post-composed with w_1. Then $(w_2 w_1 x, w_2 w_1 \xi)$ has the universal properties of a colax codescent object. □

Remark 45. Again analogous to [16][Proposition 2.2], we may find an explicit description for the weight and a 2-functor whose colimit the colax codescent object is. This is an instance of a general procedure of finding a 2-functor F and a weight W, see [18][6.9].

In the lax case, it would be too much to hope for the equivalence (as in Theorem 3.3) between $\mathcal{K}(A, B)$ and the Coh category especially because the unit of the equivalence is α_0, which doesn't have to be invertible in the lax case. We do however have:

Theorem 3.14. *Let (A, a, α, α_0) be a lax T-algebra. There is an adjunction:*



Where the functors are defined on objects as $\kappa : (\theta : A \to B) \mapsto (\theta a : TA \to B, \theta\alpha)$, $i_A^* : (x : TA \to X, \xi) \mapsto x \cdot i_A$ and are defined on morphisms in an obvious way.

Proof. We need to find two natural transformations

$$\eta : 1 \Rightarrow (i_A)^*\kappa$$
$$\epsilon : \kappa(i_A)^* \Rightarrow 1$$

satisfying the triangle identities:

$$\epsilon\kappa \cdot \kappa\eta = 1_\kappa \tag{3.12}$$
$$(i_A)^*\epsilon \cdot \eta(i_A)^* = 1_{(i_A)^*} \tag{3.13}$$

For a morphism $\theta : A \to B$, put $\eta_\theta := \theta\alpha_0$. Then η is clearly a natural transformation thanks to the middle-four interchange law. Next, note that for a cocone $(y, \overline{y}) = (y : TA \to Y, \overline{y} : yTa \Rightarrow ym_A)$, we have $\kappa(i_A)^*(y, \overline{y}) = (yi_A a, yi_A\alpha)$. Define the component of ϵ at $(y, \overline{y})$ as:

$$\epsilon_{(y,\overline{y})} := \overline{y} i_{TA} : (yi_A a, yi_A\alpha) \to (y, \overline{y}).$$

In order for this to be a morphism in $\mathrm{Coh}(\mathrm{Res}(A, a, \alpha, \alpha_0)), B)$, we'd need the following to hold (3.10):

$$\overline{y}i_{TA}m_A \cdot yi_A\alpha = \overline{y} \cdot \overline{y}i_{TA}Ta$$

It holds because:

$$\begin{aligned}
\overline{y}i_{TA}m_A \cdot yi_A\alpha &= \overline{y}Tm_Ai_{T^2A} \cdot yi_A\alpha \\
&= \overline{y}Tm_Ai_{T^2A} \cdot yT\alpha i_{T^2A} \\
&= (\overline{y}Tm_A \cdot yT\alpha)i_{T^2A} \\
&\overset{3.8}{=} (\overline{y}m_{TA} \cdot \overline{y}T^2a)i_{T^2A} \\
&= \overline{y}m_{TA}i_{T^2A} \cdot \overline{y}T^2ai_{T^2A} \\
&= \overline{y} \cdot \overline{y}i_{TA}Ta
\end{aligned}$$

Next, ϵ is natural - the naturality in morphisms $\rho : (y, \overline{y}) \to (z, \overline{z})$:

$$\begin{array}{ccc}
(yi_Aa, yi_A\alpha) & \xrightarrow{\overline{y}i_{TA}} & (y, \overline{y}) \\
{\scriptstyle \rho i_A a}\downarrow & & \downarrow{\scriptstyle \rho} \\
(zi_Aa, zi_A\alpha) & \xrightarrow[\overline{z}i_{TA}]{} & (z, \overline{z})
\end{array}$$

holds thanks to the morphism axiom of ρ:

$$\begin{aligned}
\rho \cdot \overline{y}i_{TA} &= \rho m_Ai_{TA} \cdot \overline{y}i_{TA} \\
&= (\rho m_A \cdot \overline{y})i_{TA} \\
&\overset{3.10}{=} (\overline{z} \cdot \rho Ta)i_{TA} \\
&= \overline{z}i_{TA} \cdot \rho Tai_{TA} \\
&= \overline{z}i_{TA} \cdot \rho i_Aa
\end{aligned}$$

Next, the LHS of 3.12 is:

$$\epsilon_{(\theta a, \theta\alpha)} \cdot \theta\alpha_0 a = \theta\alpha i_{TA} \cdot \theta\alpha_0 a \overset{2.3}{=} 1$$

The LHS of 3.13 is:

$$\epsilon_{(y,\overline{y})}i_A \cdot yi_A\alpha_0 = \overline{y}i_{TA}i_A \cdot yi_A\alpha_0 = \overline{y}Ti_Ai_A \cdot yT\alpha_0i_A = (\overline{y}Ti_A \cdot yT\alpha_0)i_A \overset{3.9}{=} 1$$

□

It is not difficult to verify that the functors

$$(i_A)^* : \mathrm{Coh}(\mathcal{K}, \mathrm{Res}(A, a, \alpha, \alpha_0), B) \to \mathcal{K}(A, B) \text{ for each } B$$

form the components of a 2-natural transformation (similarly with κ):

$$(i_A)^* : \mathrm{Coh}(\mathcal{K}, \mathrm{Res}(A, a, \alpha, \alpha_0), -) \Rightarrow \mathcal{K}(A, -),$$

and that the unit and counit η, ϵ of the adjunction lift to form **modifications**. We obtain:

Corollary 3.5. *There is an adjunction in* $[\mathcal{K}, Cat]$*:*

$$\mathcal{K}(A,-) \;\; \underset{\kappa}{\overset{(i_A)^*}{\leftrightarrows}} \;\; Coh(\mathcal{K}, Res(A,a,\alpha,\alpha_0),-) \qquad \mathcal{K}(A,-) \top Coh(\mathcal{K}, Res(A,a,\alpha,\alpha_0),-)$$

Let's now focus on the lax coherence result. First we show that the inclusion 2-functor $\text{T-Alg}_s \hookrightarrow \text{Lax-T-Alg}_c$ has a left adjoint provided T-Alg_s admits colax codescent objects of resolutions of lax algebras. This approach is analogous to that of [16] for pseudoalgebras and the proof essentially boils down to writing down all the definitons.

Let $\mathbb{A} = (A, a, \alpha, \alpha_0)$ be a lax T-algebra and (B, b) a strict T-algebra. Consider a pair $(f : A \to B, \overline{f} : fa \Rightarrow bTf) : \mathbb{A} \to (B, b)$ of a 1-cell and 2-cell in $\mathcal{K}$.
Denote by:

$$g := bTf : (TA, m_A) \to (B, b),$$
$$\overline{g} := bT\overline{f} : bTfTa = gTa \Rightarrow bTbT^2f = bm_BT^2f = bTfm_A = gm_A$$

the images of $f, \overline{f}$ under the isomorphism $bT(-)$ from 2.14. With this isomorphism, it is easy to see that $(f, \overline{f})$ is a colax algebra morphism (i.e. 2.6 and 2.7 hold in $\mathcal{K}$):

$$bT\overline{f} \cdot \overline{f}Ta = \overline{f}m_A \cdot f\alpha \tag{3.14}$$
$$\overline{f}i_A \cdot f\alpha_0 = 1_{f:A\to B} \tag{3.15}$$

if and only if the following identities hold in T-Alg_s[6]:

$$gm_{TA} \cdot \overline{g}T^2a = \overline{g}Tm_A \cdot gT\alpha, \tag{3.16}$$
$$\overline{g}Ti_A \cdot gT\alpha_0 = 1_g. \tag{3.17}$$

Notice that these are precisely the identities we require in the definition of the coherence cocone, the 3.8 and 3.9.

Similarly, given two colax morphisms $(f_1, \overline{f_1})$, $(f_2, \overline{f_2})$ between (A, a, α, α_0) and (B, b) together with a 2-cell $\phi : f_1 \Rightarrow f_2$, we can see that ϕ it is a colax morphism transformation (2.9 is satisfied) if and only if the 2-cell $\phi' := bT\phi : g_1 \Rightarrow g_2$ satisfies cocone morphism condition 3.10.

It can be further proven that this correspondence between coherence cocones and colax morphisms is functorial and 2-natural in (B, b), thus estabilishing the following isomorphism of categories:

$$\text{Lax-T-Alg}_c((A, a, \alpha, \alpha_0), (B, b)) \cong \text{Coh}(\text{T-Alg}_s, Res(A, a, \alpha, \alpha_0), (B, b)) \tag{3.18}$$

[6] Note that this equation make sense in T-Alg_s, take the term $\overline{g}Ti_A$ for example. $\overline{g}$ is a 2-cell in T-Alg_s and Ti_A is a morphism in T-Alg_s and whiskering operation in T-Alg_s is the same as in 2-category $\mathcal{K}$.

Assume now that T-Alg_s admits a colax codescent object $(e : (TA, m_A) \to (A', a'), \bar{e})$. By remark 3.11, this means that we have 2-natural isomorphism:

$$\text{T-Alg}_s((A', a'), (B, b)) \cong \text{Coh}(\text{T-Alg}_s, Res(A, a, \alpha, \alpha_0), (B, b)). \tag{3.19}$$

Composing these two isomorphisms we arrive at:

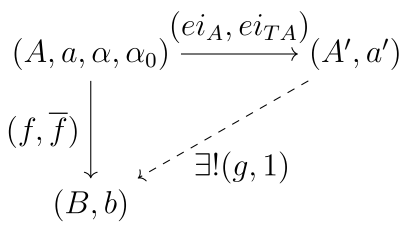
Theorem 3.15. *If T is a 2-monad, then $T\text{-}Alg_s$ admits colax codescent objects of resolutions of lax algebras if and only if the inclusion $T\text{-}Alg_s \hookrightarrow Lax\text{-}T\text{-}Alg_c$ admits a left adjoint.*

Remark 46. The component of the unit of this adjunction at a lax algebra (A, a, α, α_0) is the colax morphism:

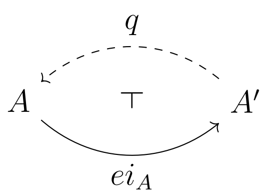
$$(ei_A, \bar{e}i_{TA}) : (A, a, \alpha, \alpha_0) \to (A', a'),$$

where $(e : (TA, m_A) \to (A', a'), \bar{e})$ is the colax codescent object of the resolution $\text{Res}(A, a, \alpha, \alpha_0)$ in T-Alg_s.

Definition 48. Given a lax T-algebra (A, a, α, α_0), we call the image of the left adjoint (A', a') a *strictification* of the lax T-algebra A. It is the codescent object of the resolution $\text{Res}(A, a, \alpha, \alpha_0)$ in T-Alg_s. We may also call it a *colax morphism classifier* because every colax morphism admits a factorisation through a strict morphism:



We now aim to prove the lax coherence result: we wish to show that each component of the unit of the adjunction $\text{T-Alg}_s \hookrightarrow \text{Lax-T-Alg}_c$ has a **right adjoint** in $\mathcal{K}$:



(3.20)

In other words, we have to find a 1-cell q and two 2-cells $\eta_A : 1 \Rightarrow qei_A$, $\epsilon_A : ei_Aq \Rightarrow 1$ satisfying triangle identities. Turns out this is rather easy when we have Theorem 1.4 at our disposal:

Theorem 3.16. *Assume what we did in Theorem 3.15. If moreover $U : T\text{-}Alg_s \to \mathcal{K}$ preserves colax codescent objects, the components of the unit of the adjunction have (internal) right adjoints in $\mathcal{K}$.*

Proof. Assume (A', a') is a colax codescent object in T-Alg_s. If U preserves it, A' is colax codescent object in $\mathcal{K}$ and there is 2-natural isomorphism:

$$\text{Coh}(\mathcal{K}, \text{Res}(A, a, \alpha, \alpha_0), -) \cong \mathcal{K}(A', -) \tag{3.21}$$

Composing with the adjunction from Remark 3.5, we obtain an adjunction:

$$\mathcal{K}(A, -) \quad \top \quad \mathcal{K}(A', -) \tag{3.22}$$

By Lemma 1.4, this adjunction is given by an internal adjunction $q \vdash ei_A$ in $\mathcal{K}$. □

Explicitly, q is the unique morphism such that:

$$qe = a$$
$$q\bar{e} = \alpha$$

The unit of the adjunction is $\alpha_0 : 1 \Rightarrow ai_A = qei_A$. Next, note that the counit of the adjunction from Remark 3.5 evaluated at the cocone $(e, \bar{e})$ is:

$$\epsilon_{(e,\bar{e})} = \bar{e}i_{TA} : (ei_A a, ei_A \alpha) \to (e, \bar{e})$$

Consider the isomorphic image (under 3.21) of $\epsilon_{(e,\bar{e})}$. We obtain a component of the counit of the adjunction 3.22 evaluated at $1_{A'}$. It is the unique 2-cell ϵ such that:

$$\epsilon e = \bar{e}i_{TA}.$$

and this is the counit of the adjunction $(q \vdash ei_A)$.

Remark 47. Note that Theorem 3.16 can be proven without the knowledge of Theorem 3.5 if we use the same approach to the one in [16, Theorem 3.2]. With this approach, Lack sketched a proof of (the dual of) Theorem 3.16 in [20, Theorem 8.6].

Remark 48. Using Theorem 3.13, we may modify the assumption of U preserving colax codescent objects to U preserving coequifiers and coinserters.

Remark 49. Note that the adjunction $q \vdash ei_A$ lives in $\mathcal{K}$ and can't be lifted up to Lax-T-Alg_c (as opposed to the case of pseudo-T-algebras). We can however equip q with the structure of a **lax** morphism. See Lemma 2.1.

Let's mention two variations of Theorem 3.15 (these can be done for the classical coherence results as well). Denote NLax-T-Alg_{co} the 2-category of **normal** lax algebras, colax algebra morphisms and colax morphism transformations. Both of the inclusions:

$$\text{T-Alg}_c \hookrightarrow \text{Lax-T-Alg}_c \qquad \text{and} \qquad \text{NLax-T-Alg}_{co} \hookrightarrow \text{Lax-T-Alg}_c$$

are fully faithful. Assume what we did in Theorem 3.15. It is now immediate to verify that the adjunction from this Theorem restricts to an adjunction between T-Alg_s and T-Alg_c (or NLax-T-Alg_{co}) and we also get an internal adjunction $(\epsilon, \alpha_0) : ei_A \dashv q$ between algebra A and its strictification. Moreover, in both of these cases α_0 is invertible, so by Lemma 1.3 we obtain that ei_A is a fully faithful arrow. To summarize:

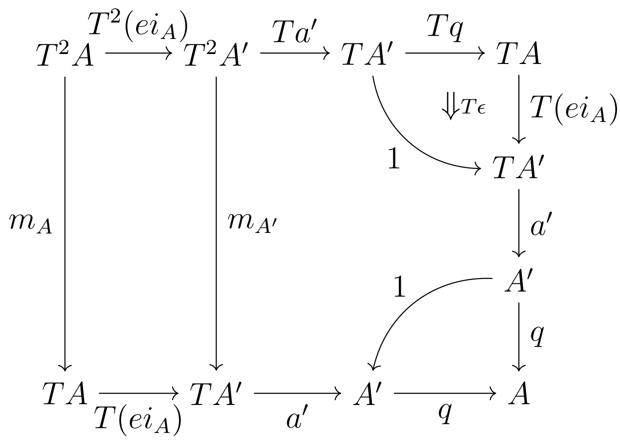
Corollary 3.6. *If T is a 2-monad for which $T\text{-}Alg_s$ admits colax codescent objects, then the inclusions $T\text{-}Alg_s \hookrightarrow T\text{-}Alg_c$, $T\text{-}Alg_s \hookrightarrow NLax\text{-}T\text{-}Alg_{co}$ admit a left adjoint. If moreover $U : T\text{-}Alg_s \to \mathcal{K}$ preserves colax codescent objects, each component of the unit of the adjunctions admits (an internal) right adjoint and is (internally) fully faihtful.*

It can be verified that the adjunction $(\epsilon, \alpha_0) : q \vdash ei_A : A \to A'$ generates an internal monad $(ai_A, \alpha i_{TA} i_A, \alpha_0)$ in $\mathcal{K}$. Furthermore, the adjunction generates **the whole algebra** (A, a, α, α_0) in the sense of Definition 37. To see this, note that we'd need to have:

$$a \;=\; A \xleftarrow{q} A' \xleftarrow{a'} TA' \xleftarrow{Te} T^2A \xleftarrow{Ti_A} TA \tag{3.23}$$

as well as:

$\alpha \quad =$



Note that since e is a strict morphism of algebras $(TA, m_A) \to (A', a')$, we have $a'Te = em_A$, from which (along with the fact that $qe = a$) 3.23 follows. Also:

$$\begin{aligned}
qa'T\epsilon Ta'T^2eT^2i_A &= qa'T\epsilon T(em_A)T^2i_A \\
&= qa'T\epsilon Te \\
&= qa'T(\epsilon e) \\
&= qa'T(\bar{e}i_{TA}) \\
&= qa'T\bar{e}Ti_{TA} \\
&\overset{(*)}{=} q\bar{e}m_{TA}Ti_{TA} \\
&= \alpha
\end{aligned}$$

Where $(*)$ is due to the fact that $\bar{e}$ is a 2-cell in T-Alg$_s$.

More is true. The 2-cell of the morphism $(ei_A, \bar{e}i_{TA})$ is equal to the 2-cell $\epsilon a'T(ei_A)$. This is what we refer to as a *bounding condition*. The adjunction $q \vdash ei_A$ is then initial amongst those adjunctions with bounding condition that generate the algebra A. This is an analogue to the fact that the adjunction generating the Kleisli category $\mathcal{C}_T$ for a monad T on a category $\mathcal{C}$ is initial amongst the adjunctions generating the monad T. We will see this in the next section.

Proposition 3.2. *Let $(\epsilon, \alpha_0) : q \vdash ei_A : A \to A'$ be the adjunction from Theorem 3.16 (hence assume everything we did then). Let (D, d) be a strict T-algebra and $(l, \bar{l}) : (A, a, \alpha, \alpha_0) \to (D, d)$ a colax morphism that is also a left adjoint $(\epsilon', \alpha_0) : r \vdash l : A \to D$ in $\mathcal{K}$. Assume that it generates the algebra A in the sense of Definition 37:*

$$a = rdTl \tag{3.24}$$
$$\alpha = rdT\epsilon' TdT^2 l \tag{3.25}$$

Then there is a unique strict T-algebra map $\theta : (A', a') \to (D, d)$ satisfying:

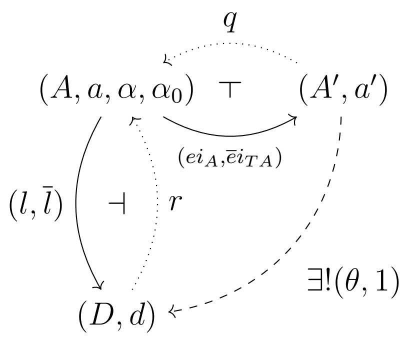
$$(l, \bar{l}) = \theta(ei_A, \bar{e}i_{TA}) \tag{3.26}$$

If moreover the counit ϵ' of the adjunction satisfies the bounding condition

$$\bar{l} = \epsilon' dTl, \tag{B}$$

the map θ commutes with right adjoints as well.

Proof. From the existence of the left adjoint to T-Alg$_s$ $\hookrightarrow$ Lax-T-Alg$_c$ we immediately see that there is a unique strict morphism $\theta : (A', a') \to (D, d)$ commuting with colax algebra maps (3.26).
Look at this picture.



Assume now that the adjunction satisfies the bounding condition. We want to show that it commutes with dotted arrows as well. According to the universal property of $(e, \bar{e})$, it is enough to show that:

$$r\theta e = a$$
$$r\theta\bar{e} = \alpha$$

Thanks to 3.24, 3.25, these equations will follow if we in particular show that:

$$\theta e = dTl \tag{3.27}$$
$$\theta\bar{e} = dT\epsilon' TdT^2 l \tag{3.28}$$

Since both θe and dTl are strict morphisms $(TA, m_A) \to (D, d)$, knowing the isomorphism 2.14, it is enough to show that these morphisms are equal after pre-composing with i_A. For 3.27 we get:

$$\theta e i_A = l = di_D l = dTli_A$$

Similarly for 3.28: Both $\theta\bar{e}$ and $dT\epsilon' TdT^2 l$ are 2-cells in T-Alg$_s$. Again, by isomorphism 2.14 (with (T^2A, m_{TA}) in place of (TA, m_A)), it is enough to prove that they are equal after precomposing them with i_{TA}. We get:

$$\begin{aligned}
\theta\bar{e}i_{TA} &\overset{3.26}{=} \bar{l} \\
&\overset{B}{=} \epsilon' dTl \\
&= di_D\epsilon' dTl \\
&= dT\epsilon' i_D dTl \\
&= dT\epsilon' Tdi_{TD}Tl \\
&= dT\epsilon' TdT^2 li_{TA}
\end{aligned}$$

and the proof is complete. □

3.4 Examples of lax coherence

In this section we give the exposition on two instances of a lax coherence that have been proven in the past, namely identity 2-monad and lax functor 2-monad. We show how Theorem 3.16 is a generalization of these results. We also compare other known results from Section 1.3 and from [32] with what we've proven in the previous section.

Identity 2-monad

Consider the identity 2-monad $1_{\mathcal{K}^{op}}$ (Example 26) on a base 2-category $\mathcal{K}^{op}$ that admits colax codescent objects. $1_{\mathcal{K}}$ then clearly preserves them. It's true that T-Alg$_s$ = $\mathcal{K}^{op}$ and Lax-T-Alg$_c$ = $\mathrm{mnd}(\mathcal{K})^{op}$. The existence of a left adjoint to the inclusion $\mathcal{K}^{op} \hookrightarrow \mathrm{mnd}(\mathcal{K})^{op}$:

$$\mathcal{K}^{op} \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathrm{mnd}(\mathcal{K})^{op} \tag{3.29}$$

is equivalent to the existence of a right adjoint to the inclusion $\mathcal{K} \hookrightarrow \mathrm{mnd}(\mathcal{K})$:

$$\mathcal{K} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathrm{mnd}(\mathcal{K}) \tag{3.30}$$

By Definition 26, this adjunction, if it exists, computes Kleisli objects in $\mathcal{K}^{op}$ (EM-objects in $\mathcal{K}$). We thus obtain that the category $(A', 1_A)$ (the strictification of the lax 1-algebra (A, t, μ, η) and the Kleisli object A_t for a monad (A, t, μ, η) (regarded as a monad in $(\mathcal{K}^{op})^{op} = \mathcal{K}$) are the same thing. We also have:

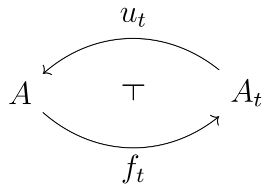
- A 2-category $\mathcal{K}$ admits the construction of algebras if it admits colax descent objects.

- A 2-category $\mathcal{K}$ admits Kleisli objects if it admits colax codescent objects.

Next, given a 2-category $\mathcal{K}^{op}$ that admits codescent objects and an internal monad $(t : A \to A, \mu, \eta)$, we find that the category $\mathrm{Coh}(\mathcal{K}^{op}, \mathrm{Res}(t, \mu, \eta), B)$ is precisely the category $\mathrm{Alg}(\mathrm{t}, B)$ (Definition 22) for $\mathcal{K}^{op}$. Theorem 3.14 can be thus seen as a generalization of Theorem 1.6. We also gain some intuition on what the coherence category $\mathrm{Coh}(\mathcal{K}, \mathrm{Res}(A, a, \alpha, \alpha_0), B)$ for a general 2-monad T is. It's some kind of a "category of algebras for a lax T-algebra"!

Assume now that $\mathcal{K} = \mathrm{Cat}$. We've mentioned that in the paper [31], A_t was observed to be the Kleisli category of a monad. Recall this fact about the Kleisli category:

Lemma 3.17. *For a monad (t, μ, η) acting on a category A, there is a canonical adjunction:*



(3.31)

that furthermore generates this monad.

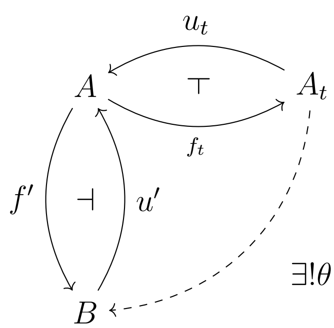
Proof. See [29, Lemma 5.2.11] for the proof and definitions of f_t, u_t. □

In [31] (just above Theorem 13) the author also identifies the unit of the adjunction 3.29 to be this functor $f_t : A \to A_t$. As adjoints are unique up to an isomorphism, we lose nothing if we identify our q with u_t and denote $f_t := ei_A = e$. Our lax coherence Theorem 3.16 for the case of identity 2-monad is then essentially just the assertion that:

Proposition 3.3. *For a monad (t, μ, η) acting on a category A, the free functor $f_t : A \to A_t$ (defined in [29, Lemma 5.2.11]) admits a right adjoint.*

It is a well known fact that the Kleisli category A_t is initial amongst the adjunctions that generate the monad t:

Proposition 3.4. *The adjunction $f_t \dashv u_t$ (defined in [29, Lemma 5.2.11]) generates the monad t. Moreover, if an adjunction $(\epsilon', \eta') : f' \dashv u' : B \to A$ generates the monad t, there is a unique functor $\theta : A_t \to B$ that commutes with left and right adjoints:*



i.e. $\theta f_t = f'$, $u_t = u\theta$.

Proof. [29, Proposition 5.2.12]. □

Let's study the relationship between this fact and Theorem 3.2. Assume again that we're given a category A and a monad (t, μ, η) on A and that $(\epsilon, \eta) : f_t \dashv u_t : A_t \to A$ is the adjunction from Lemma 3.17. Assume that there is another adjunction $(\epsilon', \eta') : f' \dashv u' : B \to A$ that generates the monad t. Recall that this means that:

$$t = u'f'$$
$$\mu = u'\epsilon' f'$$
$$\eta = \eta'$$

From the point of view of the identity 2-monad 1_{Cat}, the category A is a lax 1_{Cat}-algebra (A, t, μ, η) and B is a strict 1_{Cat}-algebra $(B, 1_B)$. We would like to equip f' with a 2-cell $\overline{f} : f' \Rightarrow f't$ such that $(f', \overline{f})$ is a colax morphism of algebras:

$$(f', \overline{f}) : (A, a, \mu, \eta) \to (B, 1_B)$$

The colax morphism equations 2.6, 2.7 dictate us that such $\overline{f}$ would have to satisfy:

$$\overline{f} \cdot f'\mu = \overline{f} \cdot \overline{f}t$$
$$\overline{f} \cdot f'\eta = 1$$

As $\eta = \eta'$, this suggests that we might put $\overline{f} := \epsilon' f'$ so that the second equation follows from the triangle identity. The first equation becomes:

$$\epsilon' f' \cdot f'u'\epsilon' f' = \epsilon' f' \cdot \epsilon' f'u'f'$$

which is true because of the middle four interchange law (you can also draw this).

Moreover, the defining equation $\overline{f} := \epsilon' f$ is precisely the bounding condition of Proposition 3.2. We conclude that the Proposition 3.2 is a **stronger** version of Proposition 3.4: the adjunction $f_t \dashv u_t$ is not just the initial object in the category of adjunctions generating the monad t, but in a larger category.

Lax functor 2-monad

Considering the lax functor 2-monad (Example 29), we've previously noted that $\text{T-Alg}_s = [\mathcal{J}, \mathrm{Cat}]$ is cocomplete and that this 2-monad preserves all colimits (in particular codescent objects). The lax coherence result for lax functors reads as follows:

Theorem 3.18. *Let $\mathcal{J}$ be a small 2-category. Denote $LaxHom[\mathcal{J}, Cat]_c$ the 2-category of lax functors, oplax natural transformations and modifications. The inclusion $[\mathcal{J}, Cat] \hookrightarrow LaxHom[\mathcal{J}, Cat]_c$ has a left 2-adjoint. Each component of the unit of this adjunction has a right adjoint in $[ob\mathcal{J}, Cat]$.*

Oplax natural transformations have been called *right lax natural transformations* in the paper [32]. In this paper, the author gives an explicit description of the strictification F' of a lax functor $F : \mathcal{J} \to \mathrm{Cat}$ (without the use of codescent objects). Moreover:

- [32, Theorem 3] describes the existence of the left adjoint to the inclusion $[\mathcal{J}, \mathrm{Cat}] \hookrightarrow \mathrm{LaxHom}[\mathcal{J}, \mathrm{Cat}]_c$,
- [32, Theorem 2] describes a canonical adjunction between a lax functor F and its strictification. Note that their described adjunction is already "lifted" to form a lax/colax adjunction, as described by Lemma 2.1,
- [32, Corrolary of Theorem 3] shows that the canonical adjunction generating a lax functor is initial amongst the adjunctions generating it. It can be verified that Theorem 3.2 is a stronger result.

This example also reduces to the previous one if we put $\mathcal{J} = 1$.

Concluding remarks

There are several things that could be done in the future. As Theorem 3.16 is a generalization of the theorems about the Kleisli object C_t for a monad t, it could be examined which of the results from the formal theory of monads [31] have their analogues for non-identity 2-monad (such an analogue is Proposition 3.2).

The assumption in the coherence theorem 3.7 that U preserves codescent objects as colimits can be (by 3.1) weakened to the assumption that it preserves them only as a bicolimit. It's quite possible that the assumptions in lax coherence theorem 3.16 can be further weakened.

The lax coherence result could be applied to many more examples once it is proven that colax codescent objects commute with products, result analogous to [16, Proposition 4.3]. In particular we would obtain a canonical monoidal adjunction between a lax monoidal category and its strictification.

Appendix

Sketch of the proof of Proposition 2.1. Let $F : \mathcal{P} \to \text{T-Alg}_s$ be a 2-functor and $W : \mathcal{P} \to \text{Cat}$ a weight. Assume (L, l) is the limit of F weighted by W in T-Alg_s, that is exhibited by the limit cone $\eta : W \Rightarrow \text{T-Alg}_s((L, l), F-)$. Denote as $U : \text{T-Alg}_s \to \mathcal{K}$ forgetful 2-functors and by $U\eta$ the induced cones of UF and $\mathcal{K}$. We want to show that $J\eta$ is a limit cone, i.e. prove one-dimensional and two-dimensional universal properties.

Assume $\gamma : W \Rightarrow \text{Lax-T-Alg}_c((B, b), JF-)$ is a cone. Because U is a right adjoints, it preserves limits. Thus L is a limit of UF weighted by W and $U\eta$ is a limit cone. Applying U to γ we obtain a cone in $\mathcal{K}$. By the one-dimensional universal property of $U\eta$ there is a unique morphism $\theta : B \to L$ in $\mathcal{K}$ such that:

$$\mathcal{K}(\eta, UF-) \cdot U\eta = U\gamma.$$

For the components at $P \in \mathcal{P}$ and $x \in WP$ this reads as

$$U\eta_P(x) \cdot \theta = U\gamma_P(x). \tag{3.32}$$

We would like to find a 2-cell $\overline{\theta} : \theta b \Rightarrow lT\theta$ such that $(\theta, \overline{\theta}) : (B, b) \to (L, l)$ is a colax morphism of algebras. By the two-dimensional universal property of $U\eta$, it is enough to find a modification

$$\rho : \mathcal{K}(\theta b, UF-) \cdot U\eta \to \mathcal{K}(l\theta, UF-) \cdot U\eta : W \Rightarrow \mathcal{K}(B, UF-) : \mathcal{P} \to \text{Cat}.$$

A component at $P \in \mathcal{P}$ is a natural transformation:

$$\rho_P : U\eta_P \theta b \Rightarrow U\eta_P lT\theta : WP \to \mathcal{K}(B, UFP) \in \text{Cat},$$

component of which at $x \in WP$ is a 2-cell in $\mathcal{K}$:

$$\rho_{P,x} : U\gamma_P(x)\theta b \Rightarrow U\eta_P(x) lT\theta$$

or equivalently a 2-cell (because $\gamma_P(x)$ is a strict algebra morphism and 3.32):

$$\rho_{P,x} : \gamma_P(x) b \Rightarrow pT\eta_P(x)T\theta = pT(\gamma_P(x))$$

Denoting the strict algebra FP as (FP, p), note that $\gamma_P(x) : (B, b) \to (FP, p)$ is a colax morphism of algebras; denote its 2-cell component as $\overline{\gamma_P(x)}$. For $x \in WP$, put:

$$\rho_{P,x} := \overline{\gamma_P(x)}$$

It is easy to verify that ρ_P is a natural transformation - the naturality condition is precisely morphism transformation condition of $\gamma_P(r)$ for $r : x \to y \in \mathcal{P}$.

The modification condition for ρ says that for any $Q : P \to P' \in \mathcal{P}$ we must have:

$$\mathcal{K}(B, UF(Q))\rho_P = \rho_{P'} WQ$$

it can be verified that this is satisfied because γ is natural. Thus ρ is a modification. By the two-dimensional universal property, there is a unique 2-cell $\overline{\theta} : \theta b \Rightarrow lT\theta$ such that:

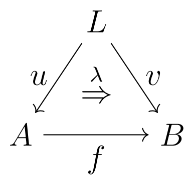
$$\mathcal{K}(\overline{\theta}, UF-)\eta = \rho.$$

We ommit the verification that $(\theta, \overline{\theta})$ is a unique colax morphism of lax algebras that commutes "with the legs" of the cone. We also ommit the verification of the two-dimensional universal property. □

Proof of Theorem 2.2. Assume we're given a colax morphism between lax algebras in Lax-T-Alg$_c$:

$$(A, a, \alpha, \alpha_0) \xrightarrow{(f, \overline{f})} (B, b, \beta, \beta_0)$$

Apply the forgetful 2-functor and form lax limit of an arrow in $\mathcal{K}$:



We need to do the following:

1. Show that there is a unique lax algebra structure $(L, l, \bar{l}, \bar{l}_0)$ on L,

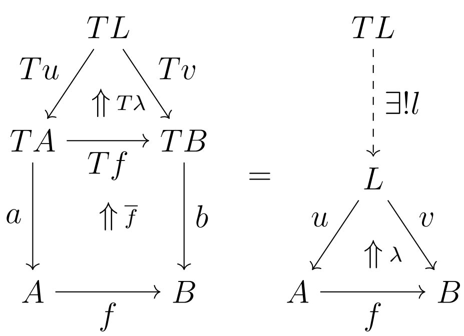
2. prove that u, v have a unique lift to algebra morphisms and that λ lifts to an algebra 2-cell,

3. prove one dimensional universal property,

4. prove two-dimensional universal property.

Let's start:

1. Consider the 2-cell on the left hand side.



By a universal property there is a unique $l : TL \to L$ such that:

$$ul = aTu \tag{3.33}$$
$$vl = bTv \tag{3.34}$$
$$\lambda l = bT\lambda \cdot \overline{f}Tu \tag{3.35}$$

Next, it can be shown that the tuple of 2-cells $(\alpha T^2 u, \beta T^2 v)$ satisfies the condition 1.8, and so by 2-dimensional universal property there exists a unique 2-cell $\bar{l} : lTl \Rightarrow lm_L$ such that:

$$u\bar{l} = \alpha T^2 u \tag{3.36}$$
$$v\bar{l} = \beta T^2 v \tag{3.37}$$

It can also be shown that the tuple of 2-cells $(\alpha_0 u, \beta_0 v)$ satisfies 1.8, and thus there is a unique $\bar{l}_0 : 1 \Rightarrow li_L$ such that:

$$u\bar{l}_0 = \alpha_0 u$$
$$v\bar{l}_0 = \beta_0 v$$

In order for $(L, l, \bar{l}, \bar{l}_0)$ to be a lax algebra, we need to show (2.2, 2.3):

$$\bar{l}Tm_L \cdot lT\bar{l} = \bar{l}m_{TL} \cdot \bar{l}T^2 l \tag{3.38}$$
$$\bar{l}Ti_L \cdot lT\bar{l}_0 = 1 \tag{3.39}$$
$$\bar{l}i_{TL} \cdot \bar{l}_0 l = 1 \tag{3.40}$$

Proof of this claim. Let's do 3.38, others are done in the same way. By two-dimensional universal property, it is enough to show that LHS and RHS are equal after composing them with u and also with v. Let's show it with u:

$$\begin{aligned}
u(\bar{l}Tm_L \cdot lT\bar{l}) &= \alpha T^2 uTm_L \cdot aTuT\bar{l} \\
&= \alpha Tm_A T^3 u \cdot aT\alpha T^3 u \\
&= \alpha Tm_A T^3 u \cdot aT\alpha T^3 u \\
&\overset{2.2}{=} (\alpha m_{TA} \cdot \alpha T^2 a)T^3 u \\
&= \alpha m_{TA} T^3 u \cdot \alpha T^2 a T^3 u \\
&= \alpha T^2 u m_{TL} \cdot \alpha T^2 u T^2 l \\
&= u(\bar{l}m_{TL} \cdot \bar{l}T^2 l)
\end{aligned}$$

□

2. The defining equations for $\bar{l}$ and $\bar{l}_0$ ensure that u, v are algebra morphisms (and are strict morphisms). The defining equation for l is precisely the requirement that λ is an algebra 2-cell $(fu, \overline{f}Tu) \Rightarrow (v, 1) : L \to B$.

3. Assume we're now given this 2-cell in Lax-T-Alg$_c$:

$$\begin{array}{ccc} & (K, k, \overline{k}, \overline{k}_0) & \\ {\scriptstyle (w,\overline{w})} \swarrow & \overset{\delta}{\Rightarrow} & \searrow {\scriptstyle (w',\overline{w}')} \\ (A, a, \alpha, \alpha_0) & \xrightarrow[(f,\overline{f})]{} & (B, b, \beta, \beta_0) \end{array}$$

Applying forgetful 2-functor again, we get that there's a unique 1-cell $\theta : K \to L$ such that:

$$\lambda\theta = \delta \tag{3.41}$$
$$u\theta = w \tag{3.42}$$
$$v\theta = w' \tag{3.43}$$

We want to find $\overline{\theta}$ so that $(\theta, \overline{\theta})$ is a colax morphism $(K, k, \overline{k}, \overline{k}_0) \to (L, l, \overline{l}, \overline{l}_0)$ satisfying:

$$(u, 1) \cdot (\theta, \overline{\theta}) = (w, \overline{w}) \tag{3.44}$$
$$(v, 1) \cdot (\theta, \overline{\theta}) = (w', \overline{w}') \tag{3.45}$$
$$\lambda(\theta, \overline{\theta}) = \delta \tag{3.46}$$

Note that the tuple $(\overline{w}, \overline{w}')$ is a modification $\lambda\theta k \to \lambda l T\theta$ and satisfies the condition 1.8.

Proof of this claim.

$$\begin{aligned} \lambda l T\theta \cdot f\overline{w} &\overset{3.35}{=} (bT\lambda \cdot \overline{f}Tu)T\theta \cdot f\overline{w} \\ &= bT\lambda T\theta \cdot \overline{f}TuT\theta \cdot f\overline{w} \\ &\overset{3.41}{=} bT\delta \cdot \overline{f}TuT\theta \cdot f\overline{w} \\ &\overset{3.42}{=} bT\delta \cdot \overline{f}Tw \cdot f\overline{w} \\ &\overset{(*)}{=} \overline{w}' \cdot \delta k \\ &= \overline{w}' \cdot \lambda\theta k \end{aligned}$$

where $(*)$ is due to the fact that δ is a 2-cell in $\text{Lax-T-Alg}_\text{c}$. □

There thus exists a unique 2-cell $\overline{\theta} : \theta k \Rightarrow l T\theta : TK \to L$ such that:

$$u\overline{\theta} = \overline{w} \tag{3.47}$$
$$v\overline{\theta} = \overline{w}' \tag{3.48}$$

We wish to show that $(\theta, \overline{\theta})$ is a colax morphism, i.e.:

$$\overline{\theta} m_L \cdot \theta\overline{k} = \overline{l}T^2\theta \cdot lT\overline{\theta} \cdot \overline{\theta}Tk$$
$$\overline{\theta} i_K \cdot \theta\overline{k}_0 = \overline{l}_0$$

According to two-dimensional universal property, it is enough to show that these are equal after whiskering with u and v. Moreover, from the construction it follows that $(\theta, \overline{\theta})$ satisfies 3.44, 3.45. Because in Lax-T-Alg$_c$ the result of whiskering λ and $(\theta, \overline{\theta})$ is $\lambda\theta$, from 3.41 we know that it equals δ. So 3.46 is satisfied as well.

4. Assume that for two colax morphisms

$$(\theta, \overline{\theta}), (\theta', \overline{\theta}') : (X, x, \overline{x}, \overline{x}_0) \to (L, l, \overline{l}, \overline{l}_0)$$

We're given a modification

$$\rho : \lambda(\theta, \overline{\theta}) \to \lambda(\theta', \overline{\theta}')$$

That is, algebra 2-cells

$$\rho_1 : (u\theta, u\overline{\theta}) \Rightarrow (u\theta', u\overline{\theta}'),$$
$$\rho_2 : (v\theta, v\overline{\theta}) \Rightarrow (v\theta', v\overline{\theta}'),$$

satisfying this pasting diagram equation in Lax-T-Alg$_c$:

$$\lambda(\theta', \overline{\theta}') \cdot (f, \overline{f})\rho_1 = \rho_2 \cdot \lambda(\theta, \overline{\theta})$$

As we said earlier, whiskering and composing algebra 2-cells is done the same as in $\mathcal{K}$, so ρ_1, ρ_2 are 2-cells in $\mathcal{K}$ satisfying:

$$\lambda\theta' \cdot f\rho_1 = \rho_2 \cdot \lambda\theta$$

And so ρ is also a modification in $\mathcal{K}$. By the two-dimensional universal property in $\mathcal{K}$, there is a unique 2-cell $\rho' : \theta \to \theta'$. Such that:

$$u\rho' = \rho_1$$
$$v\rho' = \rho_2$$

We wish to show that this is a 2-cell in Lax-T-Alg$_c$:

$$\rho' : (\theta, \overline{\theta}) \Rightarrow (\theta', \overline{\theta}') : (X, x, \overline{x}, \overline{x}_0) \to (L, l, \overline{l}, \overline{l}_0)$$

In other words, it satisfies:

$$lT\rho' \cdot \overline{\theta} = \overline{\theta}' \cdot \rho' x$$

Again, it is enough to show that this equality holds after whiskering with u and v. Let's show it for u:

$$\begin{aligned} u(lT\rho' \cdot \overline{\theta}) &= ulT\rho' \cdot u\overline{\theta} \\ &= aTuT\rho' \cdot u\overline{\theta} \\ &= aT\rho_1 \cdot u\overline{\theta} \\ &= u\overline{\theta}' \cdot \rho_1 x \\ &= u(\overline{\theta}' \cdot \rho' x) \end{aligned}$$

And the proof is complete.

□

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