Research statement
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1 Introduction

My focus in my Ph.D. has been on 2-category theory. My primary interests are **lax structures** and **double categories**. This includes turning lax algebras for a 2-monad into strict ones and computing lax morphism classifiers, the correspondence between factorization systems and double categories, lax aspects of KZ-monads with an eye towards applications in two-dimensional monad theory, the replacement of structure by property à la Claudio Hermida. I am also very open-minded about doing research on new topics in the future and broadening my horizons.

This section will serve as a brief introduction to these concepts, while in Section 2 I will outline my proven results and the work that's in progress.

1.1 Strictification of structures

A famous coherence theorem for monoidal categories is the statement that "every monoidal category is monoidally equivalent to a strict one". A modern approach to problems of coherence uses two-dimensional monad theory [1] - the statement becomes that under certain conditions, "every pseudo-T-algebra is equivalent to a strict T-algebra."

One precise formulation that has been proven in [2] uses the machinery of *codescent objects* - a certain weighted 2-categorical colimit that's a higher analogue of coequalizers.

Definition 1.1. Given a 2-category $\mathcal{K}$ and a simplicial object $X : \Delta^{op} \to \mathcal{K}$ in $\mathcal{K}$, the *codescent object* of X is a colimit of X weighted by a 2-functor $\iota : \Delta \to \mathrm{Cat}$ that regards every ordinal as a category.

By the *iso-codescent object* we mean the same colimit but the weight is given by composing ι with the reflection and then inclusion of Cat into the 2-category of grupoids.

Definition 1.2. Given a 2-monad T on a 2-category $\mathcal{K}$ and a strict T-algebra $\mathbb{A} = (A, a)$, by its *resolution* $\mathrm{Res}(\mathbb{A})$ we mean the simplicial object in $\mathrm{T\text{-}Alg}_s$ associated to the algebra, i.e. $\mathrm{Res}(\mathbb{A})_i := TA^{i+1}$.

Similarly one can define the resolution for a colax T-algebra, except this time we only get a "lax" simplicial object in $\mathrm{T\text{-}Alg}_s$.

Theorem 1.3 ([2])**.** Let T be a 2-monad on a 2-category $\mathcal{K}$ and assume that the 2-category of strict algebras and strict morphisms $\mathrm{T\text{-}Alg}_s$ admits codescent objects. Then the inclusion 2-functors of strict algebras into pseudo-algebras and strict algebras into strict algebras and pseudo morphisms have left 2-adjoints that send a T-algebra to the iso-codescent object of its resolution:

$$\text{T-Alg}_s \underset{\longrightarrow}{\overset{(-)'}{\underset{\bot}{\longleftarrow}}} \text{Ps-T-Alg} \qquad\qquad \text{T-Alg}_s \underset{\longrightarrow}{\overset{(-)'}{\underset{\bot}{\longleftarrow}}} \text{T-Alg}$$

If T preserves reflexive codescent objects, the unit of the above adjunction is an equivalence in Ps-T-Alg, making every pseudo-T-algebra $\mathbb{A} = (A, a, \gamma, \iota)$ equivalent to a strict T-algebra $\mathbb{A}' = (A', a')$.

There is also a less known lax version of the theorem that has been proven in [3]:

Theorem 1.4. Let T be a 2-monad on a 2-category $\mathcal{K}$ and assume that the 2-category of strict algebras and strict morphisms T-Alg_s admits codescent objects. Then the inclusion 2-functors of strict algebras into colax algebras and strict algebras into strict algebras and lax morphisms have left 2-adjoints that send a T-algebra to the codescent object of its resolution:

$$\text{T-Alg}_s \underset{\longrightarrow}{\overset{(-)'}{\underset{\bot}{\longleftarrow}}} \text{CoLax-T-Alg}_l \qquad\qquad \text{T-Alg}_s \underset{\longrightarrow}{\overset{(-)'}{\underset{\bot}{\longleftarrow}}} \text{T-Alg}_l$$

If T preserves reflexive codescent objects, the unit of the above adjunction is (an internal) right adjoint in CoLax-T-Alg_l, making every colax-T-algebra $\mathbb{A} = (A, a, \gamma, \iota)$ related to the T-algebra $\mathbb{A}' = (A', a')$ by an adjunction.

1.2 Internal algebra classifiers

Definition 1.5. Let $\mathbb{A} := (A, a)$ be a strict T-algebra. By an *internal T-algebra* in $\mathbb{A}$ we mean a lax T-algebra morphism $* \rightsquigarrow \mathbb{A}$ from the terminal T-algebra $*$.

The value of the left adjoint in Theorem 1.4 at the terminal algebra $*$ is the T-algebra $(*)'$ that is called an *internal algebra classifier*, meaning that internal algebras in a T-algebra $\mathbb{A}$ correspond to strict T-algebra morphisms out of $(*)' \to \mathbb{A}$. Thus $(*)'$ can be seen as a universal T-algebra containing an internal T-algebra.

For instance, given the free strict monoidal category 2-monad T on Cat, the internal algebra classifier is given by the category Δ of (possibly empty) ordinals and order-preserving maps.

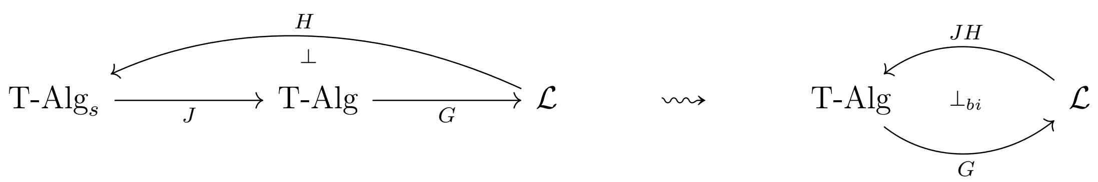
The questions of giving explicit description to various internal algebra classifiers has been studied by Mark Weber in [4]. By Theorem 1.4, these are given by certain codescent objects so the question becomes how to compute codescent objects of "nice" simplicial objects. Mark Weber defines a *crossed double category* X as a double category for which the domain functor $d_1 : X_1 \to X_0$ is a split opfibration (with additional axioms), and gave an explicit description of the codescent object of X (regarded as a simplicial object) in terms of pullbacks in Set. This is what he calls a *category of corners* $\mathrm{Cnr}(X)$ associated to X.

The 2-monads that Weber has been studying (that include the free monoidal category and free symmetric strict monoidal category 2-monads) have the property that the resolution $\mathrm{Res}(*)$ of a terminal T-algebra has the structure of a crossed double category, and thus his formula for the category of corners applies here.

1.3 Two-dimensional monad theory

In the milestone paper [1] the authors gave a systematic study of 2-monads. This paper gave a proof of the coherence theorem for strict algebras and pseudo morphisms (under different assumptions than those in Theorem 1.3), introduced the notions of *flexible* and *semiflexible* algebras and gave their characterizations, studied limits and colimits in T-Alg as well as various biadjunctions related to this 2-category. I highlight the following result and one of its corollaries:

Theorem 1.6. Any 2-adjunction pictured below-left gives rise to a biadjunction pictured below right:



Corollary 1.7. Assume that T-Alg$_s$ is cocomplete. Then T-Alg is bicocomplete.

1.4 Coherence via universality

In the 70's, Grothendieck estabilished what Claudio Hermida calls "*coherence via universality*" - the process of replacing pseudo-structure by a *property-like* structure. In Grothendieck's case, it was replacing pseudofunctors by functors with the **property** of being an opfibration. Another instance of this was given by Hermida in [5] where monoidal categories were described as multicategories with the **property** of being representable.

In a follow up paper [6] Hermida gave a general framework for this process, namely: given a 2-monad T on a 2-category $\mathcal{K}$ with good properties, there is a lax-idempotent 2-monad $T_{\dashv}$ and a morphism of monad inducing a biequivalence:

$$\text{Ps-T-Alg} \simeq \text{Ps-}T_{\dashv}\text{-Alg}.$$

One disadvantage of the approach in the paper is that it is fairly technical and involves bicategories and two-sided internal fibrations. A more conceptual approach to this problem has been promised to be done by the authors in the paper [7] but this has not been fullfilled since.

2 Current research

2.1 Lax coherence theorem

My focus has been on better understanding Theorem 1.4 - this began in my Master's thesis where I gave an alternative proof to this theorem. In my Ph.D. I had two goals regarding it:

- Find an explicit description for the strictification 2-functors $(-)' : \text{Lax-T-Alg}_c \to \text{T-Alg}_s$ and $(-)' : \text{T-Alg}_c \to \text{T-Alg}_s$,
- find a class of 2-monads that satisfy the assumptions of the theorem,

For various concrete examples, the answer to the first point has been given - for instance in [8] for lax functors (this specializes to the computation of the Kleisli category of a monad), in [9] for double categories and colax double functors, for 2-categories and lax functors in folklore, as mentioned in [10].

In my work I have extended the work of Mark Weber (described in Section 1.2) to compute not just strictifications of the terminal T-algebra $*$, but a general colax T-algebra $\mathbb{A}$. This is done as follows.

Starting with a cartesian 2-monad T on Cat, I defined the concept of a *codomain-lax category* - a double category-like structure for which the horizontal composition $f \circ g$ of horizontal morphisms f, g won't necessarily have the same codomain as f. I show that for this class of diagrams, the codescent object can be given using a generalized category of corners construction $\text{Cnr}(X)$ that uses generators and relations.

In the special case where T is a 2-monad of form $\text{Cat}(T')$ for a cartesian monad T' on Set, $\text{Cnr}(X)$ can be described only using pullbacks, which implies that it's automatically preserved by T. We obtain:

Theorem 2.1. Let T be a 2-monad on Cat that is of form $\text{Cat}(T')$ for a cartesian monad T on Set. Then Theorem 1.4 holds for T.

In fact, I prove this more generally where instead of Cat I have the 2-category $\text{Cat}(\mathcal{E})$ of categories internal to category $\mathcal{E}$ that is sufficiently cocomplete. All of the examples mentioned above can be seen to follow from this Theorem.

2.2 Factorization systems and double categories

In 2021 I started studying the relationship between orthogonal factorization systems and double categories, both fundamental concepts in category theory.

I showed that for every orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$ there is an associated double category $D_{\mathcal{E},\mathcal{M}}$ whose objects are the objects of $\mathcal{C}$, vertical morphisms are those of $\mathcal{E}$, horizontal morphisms are those of $\mathcal{M}$, and squares are commutative squares.

This double category has certain desirable properties, the most important of which is the fact that every top-right corner can be filled to a *bicartesian* square - a bidirectional analogue of cartesian squares used in the definition of crossed double categories of Mark Weber. I call these *factorization double categories.*

On the other hand, given a factorization double category, there is an associated category $\mathrm{Cnr}(X)$ (that's again given by an analogue to Weber's category of corners) and two classes of morphisms $(\mathcal{E}_X, \mathcal{M}_X)$ that form an orthogonal factorization system on $\mathrm{Cnr}(X)$.

These two processes are mutually equivalence-inverse, providing us with the equivalence of the category $\mathcal{OFS}$ of categories equipped with orthogonal factorization system and functors preserving both classes, and a full subcategory FactDbl of the category of small double categories spanned by factorization double categories.

Moreover, a similar equivalence holds between strict factorization systems and what I call *codomain-discrete double categories* - those for which the codomain functor $d_0 : X_1 \to X_0$ is a discrete opfibration. There's two applications of these concepts I had in mind:

- The equivalence $\mathcal{OFS} \simeq \mathrm{FactDbl}$ gives conceptual reasons for why categories like $\mathrm{Par}(\mathcal{C})$ (of objects and partial maps in $\mathcal{C}$) or $\mathrm{Cof}(\mathcal{E})$ (of categories and *cofunctors* internal to $\mathcal{E}$) admit orthogonal factorization systems - both can be described as the category of corners for a naturally occuring factorization double category).

- Because the resolution $\mathrm{Res}(\mathbb{A})$ of a T-algebra $\mathbb{A}$ for a 2-monad on Cat of form $\mathrm{Cat}(T')$ is codomain-discrete, together with the results from Subsection 2.1 this explains why lax morphism classifiers for these 2-monads always come equipped with a strict factorization system.

These results have been put into a preprint [11] that has been sent to a specialized journal.

2.3 Lax aspects of KZ pseudomonads

The next area of my study concerns *lax-idempotent pseudomonads* and certain aspects that have not been studied before - characterizations of pseudoalgebras for the pseudomonad in terms of *coreflectors* (morphisms in a 2-category that have a left adjoint with invertible unit), weak completeness of the Kleisli 2-category and the rise of various lax adjunctions between Kleisli 2-categories for such pseudomonads.

To properly motivate the following, let's start with examples from two-dimensional monad theory. Note that as with any 2-adjunction, the one in Theorem 1.4 between T-Alg$_s$ and T-Alg$_l$ generates a 2-comonad Q_l that's called a *lax morphism classifier 2-comonad*. Similarly we get a *pseudo-morphism classifier 2-comonad* Q_p generated by the 2-adjunction in Theorem 1.3. The first step for me was recognizing that the results mentioned in Subsection 1.2 hold more generally if we replace $Q_p : \text{T-Alg}_s \to \text{T-Alg}_s$ by a general pseudo-idempotent 2-comonad Q on a 2-category $\mathcal{K}$. The study of T-Alg then becomes the study of the Kleisli 2-category for the 2-comonad Q (in case $Q = Q_p$ this gives T-Alg). To study lax analogue of these results, I move from pseudo-idempotent 2-comonads to lax-idempotent 2-comonads, and to make exposition clearer and cover yet more examples this becomes the study of lax-idempotent pseudomonads.

Throughout this section, let (D, m, i) be a lax-idempotent pseudomonad on a 2-category $\mathcal{K}$ and denote by $J : \mathcal{K} \to \mathcal{K}_D$ the inclusion pseudofunctor to its Kleisli 2-category. Call a morphism $f : A \to B$ a *J-coreflector* if Jf is a coreflector in the Kleisli 2-category.

Theorem 2.2. The following are equivalent for an object $A \in \mathcal{K}$:

- A admits the structure of a pseudo-D-algebra,
- $\mathcal{K}(-, A) : \mathcal{K}^{op} \to \text{Cat}$ sends J-coreflectors in $\mathcal{K}^{op}$ to coreflectors in Cat,

For instance, when applied to a small presheaf pseudomonad P on the 2-category CAT of locally small categories, the statement becomes a folklore characterization of cocompleteness:

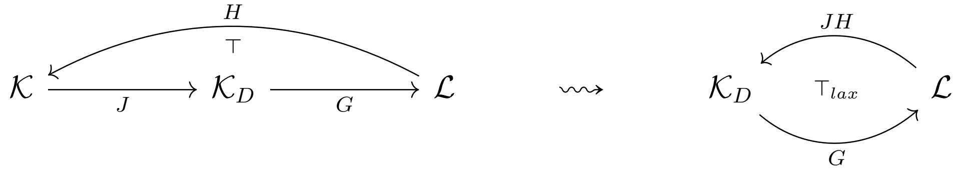
Corollary 2.3. The following are equivalent for a locally small category $\mathcal{C}$:

- $\mathcal{C}$ is cocomplete,
- left Kan extensions along small (also called *admissible*) fully faithful functors exist in $\mathcal{C}$.

When the dual of this theorem is applied to the pseudomorphism classifier 2-comonad Q_p, we obtain one characterization theorem for *semiflexible* T-algebras that has been given in [12, Theorem 20].

The generalization of Theorem 1.6 from 2-comonad Q_p to a general lax-idempotent pseudomonad is given as:

Theorem 2.4. Any biadjunction pictured below-left gives rise to a lax adjunction pictured below right:



One of the corollaries to this theorem is the fact that there is a canonical lax adjunction between the Kleisli 2-category and the 2-category of pseudo-D-algebras. The following corollary concerns my notion of *coreflector-limits*. These are a certain kind of *enriched weak limits* in the sense of [13, Section 4] (with $\mathcal{V} = \text{Cat}$, $\mathcal{E}$ being the class of coreflectors). Note that the word *weak* is understood here in the sense that the canonical comparison 1-cell to the limit from any other cone must exist, but is not required to be unique. In addition to the weak 1-dimensional universal property, these also satisfy a certain 2-categorical property.

Corollary 2.5. Assume that $\mathcal{K}$ is complete. Then $\mathcal{K}_D$ is coreflector-complete.

In the case of a small presheaf pseudomonad, this result says that the bicategory Prof of locally small categories and *small profunctors*[1] is complete in this sense.

In the case of a pseudo morphism classifier 2-comonad Q_p the dual of this corollary recovers Corollary 1.7.

In case of a lax morphism classifier 2-comonad Q_l this gives a result on weak cocompleteness of the 2-category T-Alg_l of strict algebras and lax morphisms. A particular example includes the 2-category of monoidal categories and lax monoidal functors and variations thereof. A result of this kind is interesting because colimits in these 2-categories have not been studied in the literature before (as opposed to limits, see [14], [15]).

2.4 Coherence via universality

My approach to what's been outlined in Section 1.4 is given as follows. Assume we're given a 2-monad T that is either finitary or preserves reflexive codescent objects, on a 2-category $\mathcal{K}$ that is sufficiently cocomplete. Consider the lax morphism classifier 2-comonad Q_l on T-Alg_s. It has an associated 2-monad $T_\dashv$ on the 2-category Q_l-Coalg_s of Q_l-coalgebras which is the sought replacement of T by a lax-idempotent 2-monad.

It can be shown that in case the 2-monad T is of form $\text{Cat}(T')$, the corresponding Q_l-coalgebras are equivalent to T'-multicategories. The equivalence of pseudo-T-algebras and pseudo-$T_\dashv$-algebras is then the equivalence of monoidal categories and representable multicategories as studied by Hermida [5].

Reference

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