On invariant calculi for parabolic geometries

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joint work with Andreas Čap

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Structure

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2. Parabolic Geometries and Weyl connections
   - Cartan connections as analogies to affine geometry on manifolds
   - Bundle of Weyl structures, Weyl connections, and Rho–tensors
   - The normalization
3. Nearly invariant calculus

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Conformal Riemannian and projective structures

Conformal structures

Conformal Riemannian manifolds $(M, [g])$ can be viewed as finite type G-structures with the structure group $CSO(p, q)$ $p + q = n = \dim M$, and the Cartan approach yields the prolongation $\mathbb{R}^n \oplus (\mathfrak{cso}(p, q) \oplus \mathbb{R}) \oplus \mathbb{R}^{n*} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The Levi-Civita connections from $[g]$ extend to the Weyl connections parametrized by one-forms on M.

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Projective structures

A projective structure on M is a class of all connections $[\gamma]$ sharing their geodesics. The projective structures are second order G-structure since $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$ in this case. Again, as well known, given one linear connection γ, the class $[\gamma]$ is parametrized by one-forms on M.

Cartan connection

Definition

Cartan geometries of type G/P are deformations of the homogeneous space $G \to G/P$ with the Maurer–Cartan form $\omega \in \Omega^1(G; \mathfrak{g})$:
absolute parallelism $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on a principal fiber bundle $\mathcal{G} \to M$ with structure group P, enjoying suitable invariance properties with respect to the principal action of P:

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- $\omega(\zeta_X)(u) = X$ for all $X \in \mathfrak{p}$, $u \in \mathcal{G}$ (the connection reproduces the fundamental vertical fields)

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- $(r^b)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ (the connection form is equivariant with respect to the principal action)
- $\omega_{|T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$ (the absolute parallelism condition).

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- *Curvature function*

$$\begin{aligned} \kappa(u)(X, Y) &= K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) \\ &= [X, Y] - \omega(u)([\omega^{-1}(X), \omega^{-1}(Y)]). \end{aligned}$$

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- Curvature is a horizontal 2–form.

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- Fixed grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ on Lie algebra $\mathfrak{g}$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$, $\mathfrak{g}_0$ is the reductive part in $\mathfrak{p}$,

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- Unique decomposition $g = g_0 \cdot \exp \Upsilon_1 \cdot \ldots \cdot \exp \Upsilon_k$, $g \in P$, $g_0 \in G_0$, and $\Upsilon_i \in \mathfrak{g}_i$.
- *Grading element* is the unique $E \in \mathfrak{g}_0$ with $\operatorname{ad} E_{|\mathfrak{g}_i} = i \cdot \operatorname{id}_{\mathfrak{g}_i}$ for all $i = -k, \dots, k$.

Weyl structures

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Weyl structure on a parabolic geometry $(\mathcal{G}, \omega)$ is a G_0 equivariant section $\bar{s} : \mathcal{G}_0 = \mathcal{G}/P_+ \to \mathcal{G}$, i.e., reduction of the parabolic structure group P to its reductive part G_0.

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- θ is a soldering form for M (identifying also $TM \simeq \operatorname{gr} TM$)
- γ is a linear connection form on M, $\theta + \gamma$ is the affine Weyl connection
- P is a one–form on M valued in T^*M (measuring the difference between ω and $\theta + \gamma$ on the image $\bar{s}(\mathcal{G}_0)$).

Natural bundles and tractor bundles

Natural bundles

P–representation λ on a vector space $\mathbb{V}$ provides the homogeneous vector bundle $G \times_P \mathbb{V}$ and, more generally, the associated vector bundles

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with standard fiber $\mathbb{V}$ over all manifolds with a parabolic geometry of the type G/P. These are the *natural bundles* $\mathcal{V}$.

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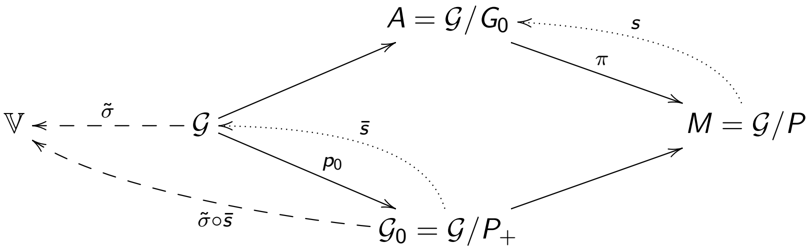
The adjoint representation $\mathcal{G}$ provides the *adjoint tractor bundles* $\mathcal{A}$, the standard representation of a matrix group on $\mathbb{R}^n$ provides the *standard tractors* $\mathcal{T}$. The curvature function is identified with $\mathcal{A}M$-valued 2-form on M.

The bundle of Weyl structures

Reductions of $\mathcal{G}$ to the structure group G_0 are equivalent to sections s of the bundle $A = \mathcal{G}/G_0 = \mathcal{G} \times_P P/G_0$, and this is an affine bundle modelled over 1-forms. All P-modules can be viewed as G_0-modules, thus $\Gamma(\mathcal{V}M) \subset \Gamma(\mathcal{V}A)$.



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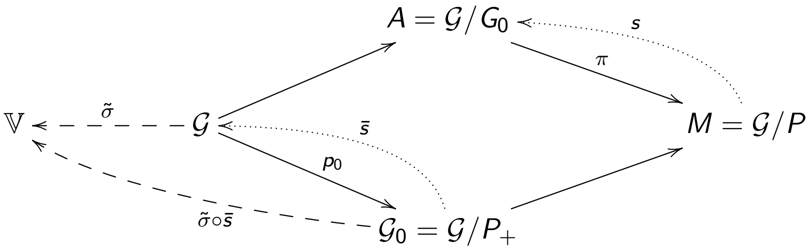

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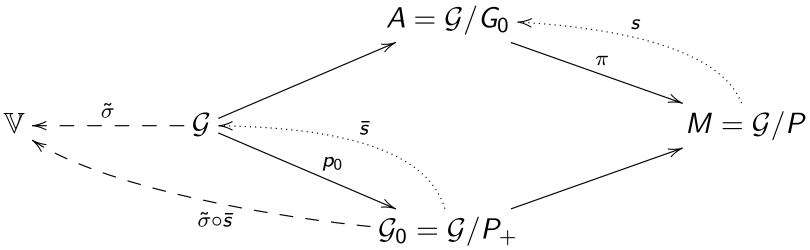

Now, $\omega = (\omega_- + \omega_+) + \omega_0$ is an affine connection on A! (noticed in the recent paper by Čap and Mettler). Moreover, $TA = L_- + L_+$ and there is P, the projection to $L^+ = \ker \pi_*$ along L_-, the torsion $T + Y$ and curvature W. We write $s^*\mathrm{P} = \mathrm{P}^s$.

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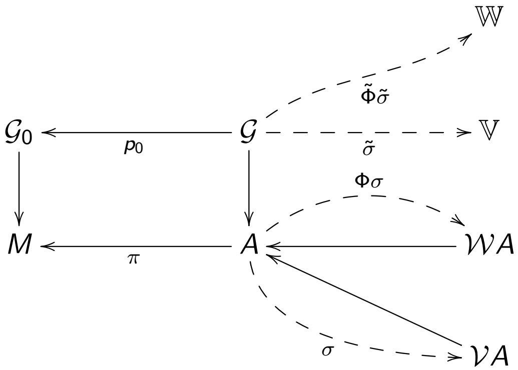

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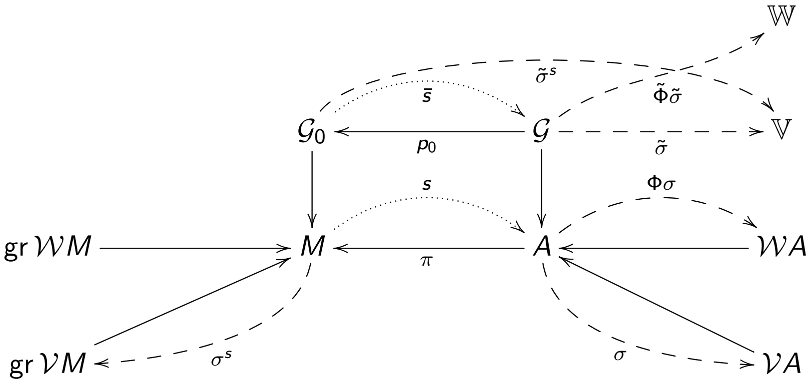

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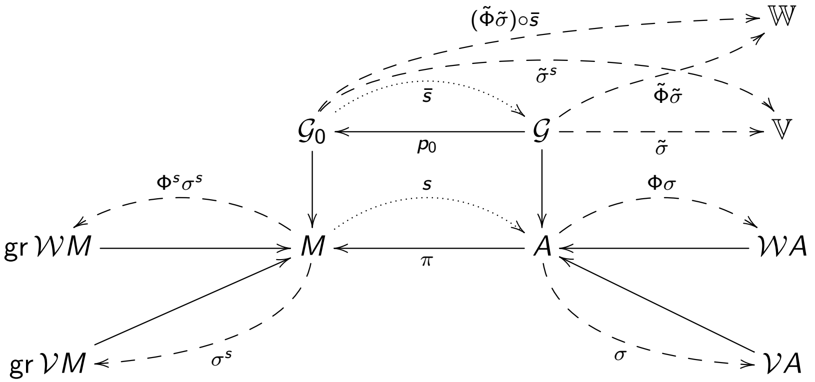

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Restricting to section σ in $\Gamma(\mathcal{V}M) \subset \Gamma(\mathcal{V}A)$, clearly the derivative in the vertical direction will be the algebraic action, while the difference between $\nabla^s\sigma$ and $\nabla^\omega\sigma$ comes from the algebraic action of P^s via the P_+-action on $\mathbb{V}$. We introduce the *Rho-corrected derivative* $\nabla^{\mathrm{P}^s} = \nabla^s + \mathrm{P}^s$. Actually, ∇^{P^s} is the pullback of ∇^ω restricted to $TM \simeq L_- \subset TA$.

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Theorem (1)

For each operator $\Phi : \Gamma(\mathcal{V}M) \to \Gamma(\mathcal{W}A)$ given in term of the affine connection ω on A, there is a universal formula for the operator Φ^s expressed in terms of the Weyl connection $\theta^s + \gamma^s$, the curvature of ω and P^s.

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Change of Weyl connections and gradings

If $\hat{s} = s \cdot \exp \Upsilon$, and $\sigma = \sigma_1 + \cdots + \sigma_\ell$ is a section of a natural bundle (corresponding the representation λ), then:

- $\hat{\sigma}_\ell = \sum_{\|\underline{i}\|+j=\ell} \frac{(-1)^{\underline{i}}}{\underline{i}!} \lambda(\Upsilon_k)^{i_k} \circ \ldots \circ \lambda(\Upsilon_1)^{i_1} \circ \sigma_j$
- $\hat{\nabla}^{\mathrm{P}^{\hat{s}}}_\xi \sigma = \nabla^{\mathrm{P}^s}_\xi \sigma + \sum_{\|\underline{i}\|+j \geq 0} \frac{(-1)^{\underline{i}}}{\underline{i}!} \big(\mathrm{ad}(\Upsilon_k)^{i_k} \circ \ldots \circ \mathrm{ad}(\Upsilon_1)^{i_1}(\xi_j)\big) \bullet \sigma$

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Here $\underline{i}$ is a multiindex $(i_1, \ldots, i_k)$ with $i_j \geq 0$. We put $\underline{i}! = i_1! \cdots i_k!$ and $\|\underline{i}\| = i_1 + 2i_2 + \cdots + ki_k$, while $(-1)^{\underline{i}} = (-1)^{i_1+\cdots+i_k}$

Tranformations – continued

Change of Rho–tensors

$$\hat{\mathsf{P}}_i(\xi) = \sum_{\|\underline{j}\|+\ell=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} \operatorname{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \operatorname{ad}(\Upsilon_1)^{j_1}(\xi_\ell) +$$

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Example: $|1|$-graded $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, irreducible $\mathbb{V}$:

$$\hat{\nabla}_\xi \sigma = \nabla_\xi \sigma - [\Upsilon, \xi] \bullet \sigma$$

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Iterating the differentiation, we face derivatives of Υ. Thus, adding "correction terms" based on P looks promising. This was the original motivation for introducing the Schouten's tensor (trace adjusted Ricci) in the conformal geometry nearly 100 years back.

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For $|1|$-graded cases, the Weyl connections are all connections with the given holonomy and the prescribed torsion T, shared by all of them (e.g., zero in conformal Riemannian or projective), and choosing one of them, i.e. the Weyl structure s, ω is given by

$$\mathsf{P}^s = -\square^{-1} \partial^* R^s,$$

where R^s is the curvature of the Weyl connection.

Structure

The derivatives (works for all Cartan connections!)

Invariant derivative

Similarly to covariant derivatives, we define the invariant derivative $\nabla^\omega : C^\infty(\mathcal{G}, \mathbb{V}) \to C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{V})$, $\nabla s(u)(X) = \omega^{-1}(X)(u) \cdot s$. This operation **does not** transform sections of $\mathcal{V}M$ into sections of $\mathcal{V}M$, in general!

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Fundamental derivative

The extension of the invariant derivative to arguments $X \in \mathfrak{g}$,
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- $D^\omega_{\xi+\zeta} s = \nabla^s_\xi s + \mathsf{P}^s(\xi) \bullet s - \zeta \bullet s$, where ξ a vector in TM, ζ a vertical vector on $\mathcal{G}$. (here again – parabolic geometries)

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- on all tractor bundles, the fundamental derivative is related to the invariant linear connection $\nabla^{\mathcal{V}}_\xi s = D^\omega_\xi s + \xi \bullet s$.

Invariant jet operator – still for all Cartan connections

Both invariant and fundamental derivatives allow iteration!

Write $J^1\mathbb{V}$ for the standard fiber $J^1(G \times_P \mathbb{V})_o$. Then $J^1\mathcal{V}M = \mathcal{G} \times_P J^1\mathbb{V}$ and $J^1\mathbb{V} = \mathbb{V} \oplus (g_-^* \otimes \mathbb{V})$.

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1st order invariant jets

$C^\infty(\mathcal{G}, \mathbb{V}) \ni s \mapsto (s, \nabla^\omega s) \in C^\infty(\mathcal{G}, \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V}))$ defines the natural 1st order prolongation operator $\mathcal{V}M \to J^1\mathcal{V}M$.

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higher orders

$C^\infty(\mathcal{G},\mathbb{V})^P\ni s\mapsto\bar{j}^k_\omega s=(s,\nabla^\omega s,\dots,(\nabla^\omega)^k s)\in C^\infty(\mathcal{G},\mathbb{V}\oplus\dots\oplus(\otimes^k\mathfrak{g}_-^*\otimes\mathbb{V}))^P$ is the invariant rth order prolongation operator valued in semi–holonomic jets $\bar{J}^k\mathcal{V}M$.

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Note: Symmetrization provides similar formulae for holonomic jets $j^k_\omega s$, but these are **not equivariant**!

Bianchi and Ricci identities

In terms of the invariant differnetial, the Bianchi identity reads

$$\sum_{\text{cycl}} \nabla^{\omega}_{Z}\kappa(X,Y) = \sum_{\text{cycl}} \big([\kappa(X,Y),Z] + \kappa([X,Y],Z) - \kappa(\kappa(X,Y),Z)\big)$$

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Note: also available in terms of the fundamental derivative.

Back to parabolic geometries

The invariant derivative ∇^ω is just the covariant derivative of ω on A, restricted to L^-. Thus, $\nabla^\omega : \Gamma(\mathcal{V}M) \to \Gamma(\mathcal{V}A)$.

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$$T^s + R^s + Y^s + \partial \mathrm{P}^s = s^*\kappa$$

where $Y = d^{\nabla^s}\mathrm{P}^s + \mathrm{P}^s([\cdot,\cdot]) + [\mathrm{P}^s, \mathrm{P}^s]$ is the Cotton York tensor, and $\partial \mathrm{P}^s$ the Lie algebra cohomology differential.

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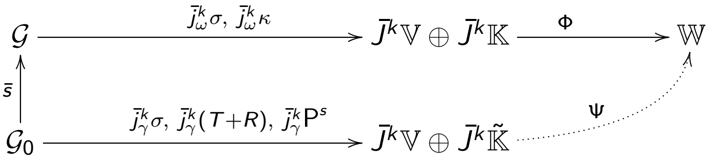
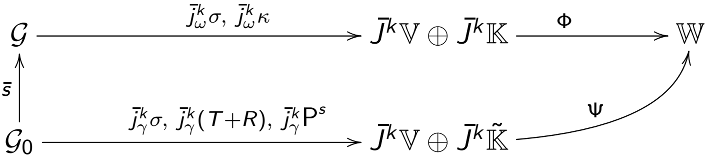
Looking at differential operators – affine invariants of the connecitons, i.e., Φ is G_0–equivariant:



Main observations and theorems



Our Theorem (1) on the expansion can be reformulated as:

Theorem (2)

Each affine differential invariant $\tilde{\Phi} : \Gamma(\mathcal{V}A) \to \Gamma(\mathcal{W}A)$ on A, restricted to L^-, i.e., given by a G_0-equivariant map Φ, can be expressed in a universal way by means of a G_0-equivariant map Ψ, i.e. in terms of affine invariants of ∇^s and P^s.

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And finally, the main theorem:

Theorem

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All affine invariants of Weyl connections and Rho tensors transforming algebraically in Υ are obtained this way.

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All affine invariants of Weyl connections and Rho tensors transforming algebraically in Υ are obtained this way.

The first implication might be surprising in view of the complicated transformation rules of the basic objects.
The proof of the second implication relies on (locally existing) special Weyl structures called normal - they mimic the concept of exponential coordinates in affine geometry. They enjoy the property that all symmetrized covariant derivatives of P^s vanish at the center of the coordinates.

Thanks for attention and patience!