The Power of Commuting with Finite Sets of Words

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Outline:

0. Language inequalities.
1. Proving regularity of maximal solutions
   - by constructing finite recognizers.
   - using well quasi-orders.
2. Systems having non-regular largest solutions.
3. Commutation equations.
4. Non-regularity of solutions—sketch of the proof.

finite alphabet $A = \{a, b, \dots\}$

A^* …the monoid of finite words over A

finite set of variables $\mathcal{V} = \{X_1, \dots, X_n\}$

Explicit polynomial equations:

$X_i = P_i, \quad i = 1, \dots, n$

$P_i \subseteq (A \cup \mathcal{V})^*$ finite

smallest solutions …context-free languages

Example: $X = aXb \cup \{\varepsilon\} \implies X = \{a^k b^k \mid k \in \mathbb{N}_0\}$

Constructing finite recognizers

$P_j \subseteq L_j$, $P_j = L_j$

$L_j \subseteq A^*$ regular, $P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary

Conway 1971:

- finitely many maximal solutions, regular
- for regular P_j algorithmically regular

Example: $X \cdot Y \subseteq L$

$\sim \ldots$ congruence of A^* of finite index recognizing L,

i.e. $u \sim v \implies (u \in L \iff v \in L)$

$M \cdot N \subseteq L \implies (M\sim) \cdot (N\sim) \subseteq L$

$K_0 \cup K_1X_1 \cup \cdots \cup K_nX_n \subseteq L_0 \cup L_1X_1 \cup \cdots \cup L_nX_n$

K_i arbitrary, L_i regular

Baader & Küsters 2001, MK 2004:

- largest solution regular
- for regular K_i algorithmically regular

Well quasi-orders

Wqo $\leq$ on A^* contains neither $\begin{array}{c}\bullet\\ |\\ \bullet\\ |\\ \bullet\\ \vdots\end{array}$ nor $\bullet\ \bullet\ \bullet \cdots$

Monotone: $u \leq v \;\&\; \tilde{u} \leq \tilde{v} \implies u\tilde{u} \leq v\tilde{v}$

Example: (scattered) subword partial order

Theorem: Ehrenfeucht, Haussler, Rozenberg 1983

$L \subseteq A^*$ is regular $\iff$ L is upward closed with respect to a monotone well quasi-order on A^*

Example:

Congruence of finite index is a monotone wqo.

upward closed = recognized by the congruence

Applying well quasi-orders to inequalities

Method:

Construct a wqo on A^* such that every solution is contained in an upward closed solution.

$XK \subseteq LX$

K arbitrary, L regular

has largest solution:

$$M_iK \subseteq LM_i \implies (\bigcup M_i)K \subseteq L(\bigcup M_i)$$

MK 2004:

- largest solution regular
- for regular K algorithmically recursive

Systems with non-regular largest solutions

Wqo's for two inequalities cannot be combined
(not even with congruences).

$$\begin{array}{ccc} XK \subseteq LX & XK \subseteq LX & XK \subseteq LX \\ X \subseteq M & XM \subseteq NX & MX \subseteq XN \end{array}$$

Largest solutions can be non-r.e. (MK 2004)

Problem: $XL = LX$ (Conway 1971)

$\mathcal{C}(L)$ ... largest solution
$\mathcal{C}_+(L)$ ... largest solution without ε

Main Theorem:

There exists a finite set of words L such that
$\mathcal{C}(L)$ and $\mathcal{C}_+(L)$ are not recursively enumerable.

Difference between $\mathcal{C}(L)$ and $\mathcal{C}_+(L)$ can be non-r.e.

Let's play

$L = \{a, ab, ba, bb\}$

$\mathcal{C}(L) = \{a, b\}^*$

$\mathcal{C}_+(L) = \{a, b\}^+ \setminus \{b\}$

$b \in \mathcal{C}(L)$: attacker defender

$$
\begin{tikzcd}
 & \underline{a \cdot b} \arrow[r, blue] & \varepsilon \in \mathcal{C}(L) \\
b\underline{b \cdot b} \arrow[r, blue, shift right] & b \arrow[l, red, shift right] \arrow[u, red] \arrow[r, red] \arrow[d, red, shift right] & a\underline{b \cdot b} \arrow[d, blue] \\
 & b\underline{a \cdot b} \arrow[u, blue, shift right] & a \in L \subseteq \mathcal{C}(L)
\end{tikzcd}
$$

$\mathcal{C}(L)$ . . . all winning positions for the defender

$b \notin \mathcal{C}_+(L) : \quad b \longrightarrow a \cdot b \notin \{a, b\}^+ \cdot L$

Let us do mathematics

$A = \{a, b, c, e, \hat{e}, f, \hat{f}, g, \hat{g}\}$

$$
\begin{aligned}
L = \{c, ef, ga, e, fg, \hat{f}\hat{e}, a\hat{g}, \hat{e}, \hat{g}\hat{f}, fgba\hat{g}\} \cup cM \cup Mc \\
\cup A^*bA^*bA^* \cup (A \setminus \{c\})^*b(A \setminus \{c\})^* \setminus N \\
M = efga^+ba^* \cup ga^*ba^*\hat{g}\hat{f} \cup a^*ba^*\hat{g}\hat{f}\hat{e} \cup fga^*ba^*\hat{g} \\
N = \{efg, fg, g, \varepsilon\} \cdot a^*ba^* \cdot \{\varepsilon, \hat{g}, \hat{g}\hat{f}, \hat{g}\hat{f}\hat{e}\}
\end{aligned}
$$

$\mathcal{C}(L)$ is not regular

Configurations: $[[[e]f]g]a^m ba^n[\hat{g}[\hat{f}[\hat{e}]]]$

$L \setminus A^*bA^* = \{c, ef, ga, e, fg, \hat{f}\hat{e}, a\hat{g}, \hat{e}, \hat{g}\hat{f}\}$

Let us attack

$$L = \{c, ef, ga, e, fg, \hat{f}\hat{e}, a\hat{g}, \hat{e}, \hat{g}\hat{f}, fgba\hat{g}\} \cup cM \cup Mc$$
$$\cup\, A^*bA^*bA^* \cup (A \setminus \{c\})^* b (A \setminus \{c\})^* \setminus N$$
$$M = efga^+ba^* \cup ga^*ba^*\hat{g}\hat{f} \cup a^*ba^*\hat{g}\hat{f}\hat{e} \cup fga^*ba^*\hat{g}$$

By induction on m: $\quad m < n \implies efga^mba^n \notin \mathcal{C}(L)$

Basis: $\quad n > 0 \implies efgba^n \longrightarrow c \cdot efgba^n \notin A^* \cdot L$

Induction step:

$$\begin{tikzcd}
efga^mba^n \arrow[d] & \\
\underline{ef}ga^mba^n \cdot \hat{g}\hat{f} \arrow[r] \arrow[d] & fga^mba^n\hat{g}\hat{f} \arrow[d, Rightarrow] \\
ga^mba^n\hat{g}\hat{f} \arrow[d] & fga^mba^n\hat{g}\hat{f} \cdot c \cdot c \notin L^2 \cdot A^* \\
\underline{ga}a^{m-1}ba^n\hat{g}\hat{f} \cdot \hat{e} \arrow[d] & \\
a^{m-1}ba^n\hat{g}\hat{f}\hat{e} \arrow[d] & \\
\vdots \arrow[d] & \\
efga^{m-1}ba^{n-1} &
\end{tikzcd}$$

Let us defend

$$
\begin{aligned}
L = &\{c, ef, ga, e, fg, \hat{f}\hat{e}, a\hat{g}, \hat{e}, \hat{g}\hat{f}, fgba\hat{g}\} \cup cM \cup Mc \\
&\cup A^*bA^*bA^* \cup (A \setminus \{c\})^* b (A \setminus \{c\})^* \setminus N
\end{aligned}
$$

$$M = efga^+ba^* \cup ga^*ba^*\hat{g}\hat{f} \cup a^*ba^*\hat{g}\hat{f}\hat{e} \cup fga^*ba^*\hat{g}$$

$$N = \{efg, fg, g, \varepsilon\} \cdot a^*ba^* \cdot \{\varepsilon, \hat{g}, \hat{g}\hat{f}, \hat{g}\hat{f}\hat{e}\}$$

Claim: $efga^nba^n \in \mathcal{C}(L)$

The following language commutes with L:

$L^* \cup \{efga^nba^n, ga^nba^n\hat{g}\hat{f}, a^{n-1}ba^n\hat{g}\hat{f}\hat{e}, fga^{n-1}ba^n\hat{g} \mid n \in \mathbb{N}\}$

For $n \geq 2$:

$$
\begin{tikzcd}
c \cdot fga^{n-1}ba^n\hat{g} \in cM & \\
fga^{n-1}ba^n\hat{g} \arrow[u] \arrow[r] \arrow[d] & ef \cdot fga^{n-1}ba^n\hat{g} \notin N \\
e \cdot fga^{n-1}ba^{n-1}\underline{a\hat{g}} \arrow[r] & efga^{n-1}ba^{n-1}
\end{tikzcd}
$$

Attacker chooses direction of computation:

$$
\begin{tikzcd}
& a^{n-1}ba^n\hat{g}\hat{f}\hat{e} \arrow[dl] \arrow[dr] & \\
ga \cdot a^{n-1}ba^n\hat{g}\hat{f}\underline{\hat{e}} \arrow[d] & & fg \cdot a^{n-1}ba^n\hat{g}\underline{\hat{f}\hat{e}} \arrow[d] \\
ga^nba^n\hat{g}\hat{f} & & fga^{n-1}ba^n\hat{g}
\end{tikzcd}
$$

1. $m < n \implies efga^mba^n \notin \mathcal{C}(L)$
2. $efga^nba^n \in \mathcal{C}(L)$

Pumping lemma $\implies$ $\mathcal{C}(L)$ is not regular.

Possible future research:

1. Regularity of solutions of similar systems of inequalities, for instance:

   $KXL \subseteq MX$

   $KX \subseteq LX, \ XM \subseteq XN$

2. Existence of algorithms for computing regular solutions.