Regular Solutions of Language Inequalities and Well Quasi-orders

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Outline:

1. Summary of known results.
2. Well quasi-orders of free monoids.
3. Inequalities $XK \subseteq LX$.
4. Inequalities with constants recognizable by simple semigroups.

finite alphabet $A = \{a, b, \dots\}$
finite set of variables $\mathcal{V} = \{X_1, \dots, X_n\}$
A^*, A^+ ... free monoid and free semigroup over A

Explicit polynomial equations:

$$X_i = P_i \qquad P_i \subseteq (A \cup \mathcal{V})^*,\ i = 1, \dots, n$$

smallest solutions ... context-free languages
Example: $X = aXb + 1 \implies X = \{a^k b^k \mid k \in \mathbb{N}_0\}$

$\sigma\colon A^* \to M$ monoid homomorphism
σ recognizes language $L \subseteq A^* \iff \sigma^{-1}\sigma(L) = L$
L regular $\iff$ exists a homomorphism to a finite monoid which recognizes L

$K_j, L_j \subseteq A^*$ regular, $P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary,

Q_j ... regular expression over variables and languages recognizable by finite simple semigroups

System of equations	Maximal solutions
$P_j \subseteq L_j$, $P_j = L_j$	finitely many, algorithmically regular (Conway 1971)
$K_0 + \sum K_i X_i =$ $L_0 + \sum L_i X_i$	one, algorithmically regular (Baader & Küsters 2001)
$XK \subseteq LX$ L regular	one, regular, for regular K algorithmically recursive (MK 2004)
$XL = LX$ L regular	one, non-r.e. (MK 2004)
$P_j \subseteq Q_j$	regular (MK 2004)

Example: $X_1 \cdot X_2 \subseteq L$, σ recognizes L

$$M_1 \cdot M_2 \subseteq L \implies \sigma^{-1}\sigma(M_1) \cdot \sigma^{-1}\sigma(M_2) \subseteq L$$

Well quasi-order (wqo) $\leq$ on A^*:

contains neither  nor $\bullet \bullet \bullet \cdots$

Equivalent definitions:

- Every upward closed language over A is finitely generated.
- There is no infinite ascending sequence of upward closed languages.

Ehrenfeucht, Haussler, Rozenberg 1983:
$L \subseteq A^*$ is regular $\iff$ L is upward closed with respect to a monotone well quasi-order

Our method:
Construct a wqo on A^* such that every solution is contained in an upward closed solution.

$XK \subseteq LX$, L regular

Game: position: $w \in A^*$

attacker: $u \in K$, $w \rightsquigarrow wu$

defender: $v \in L$, $wu = v\bar{w}$, $wu \rightsquigarrow \bar{w}$

Largest solution of $XK \subseteq LX$ . . . all winning positions for the defender

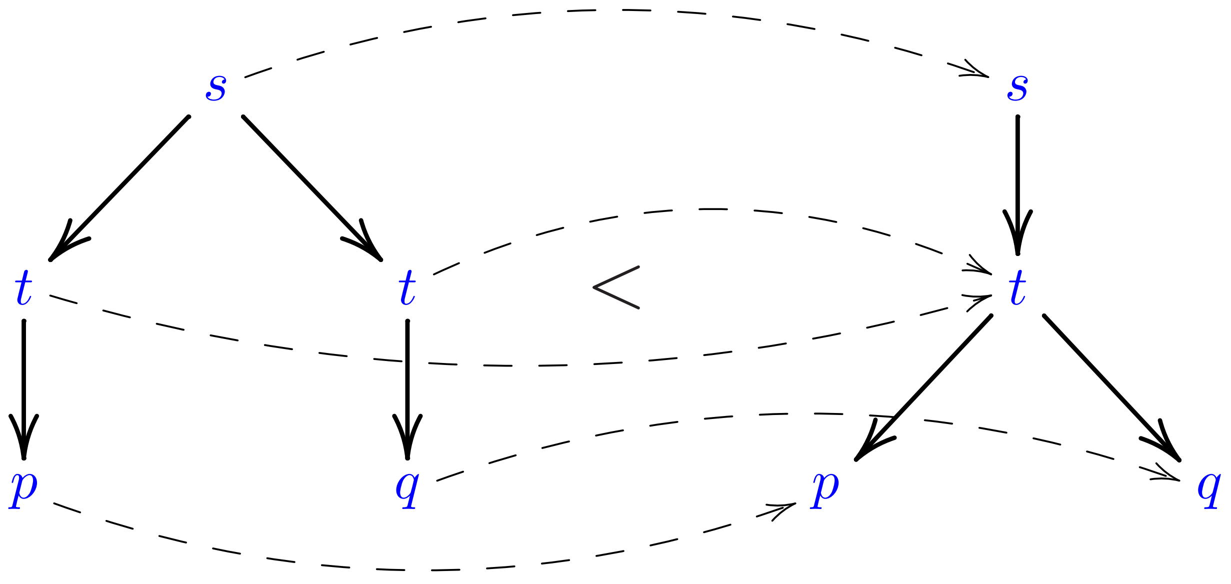
Example: $w = abcd$, $L = \{a, ab, abcde, bc, c, cd, da\}$,
$\sigma\colon A^* \to M$ recognizes L

$$\begin{tikzcd}
 & & \sigma(abcd) & (\,) \arrow[l, dotted] \arrow[dl] \arrow[dr] & & & \\
 & \sigma(bcd) & (a) \arrow[l, dotted] \arrow[dl] & & (ab) \arrow[r, dotted] \arrow[dl] \arrow[dr] & \sigma(cd) & \\
\sigma(d) & (a, bc) \arrow[l, dotted] & \sigma(d) & (ab, c) \arrow[l, dotted] & & (ab, cd) \arrow[r, dotted] & 1
\end{tikzcd}$$

Quasi-ordering of labelled trees:

$w \leq v$ ... defender's winning strategies for w can be used also for v

Example:



Largest solution is upward closed with respect to $\leq$

Kruskal 1960: $\leq$ is a wqo

Systems of inequalities $P_j \subseteq Q_j$

$P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary,

Q_j ... regular expression over variables and languages
recognizable by finite simple semigroups

Finite simple semigroups:

- generalization of finite groups
- Λ, P finite sets, G finite group,
  multiplication on $S = \Lambda \times G \times P$:
  $(\lambda, g, \rho) \cdot (\bar{\lambda}, \bar{g}, \bar{\rho}) = (\lambda, g \cdot g_{\rho\bar{\lambda}} \cdot \bar{g}, \bar{\rho})$

recognizable by a finite group $\iff$
minimal automaton is codeterministic

recognizable by a finite simple semigroup $\iff$
minimal automaton does not contain

$$\begin{tikzcd}
\bullet \arrow[r, "a"] & \bullet \arrow[dr, "b"] & \\
& & \bullet \\
\bullet \arrow[r, "a"'] & \bullet \arrow[ur, "b"'] &
\end{tikzcd}$$

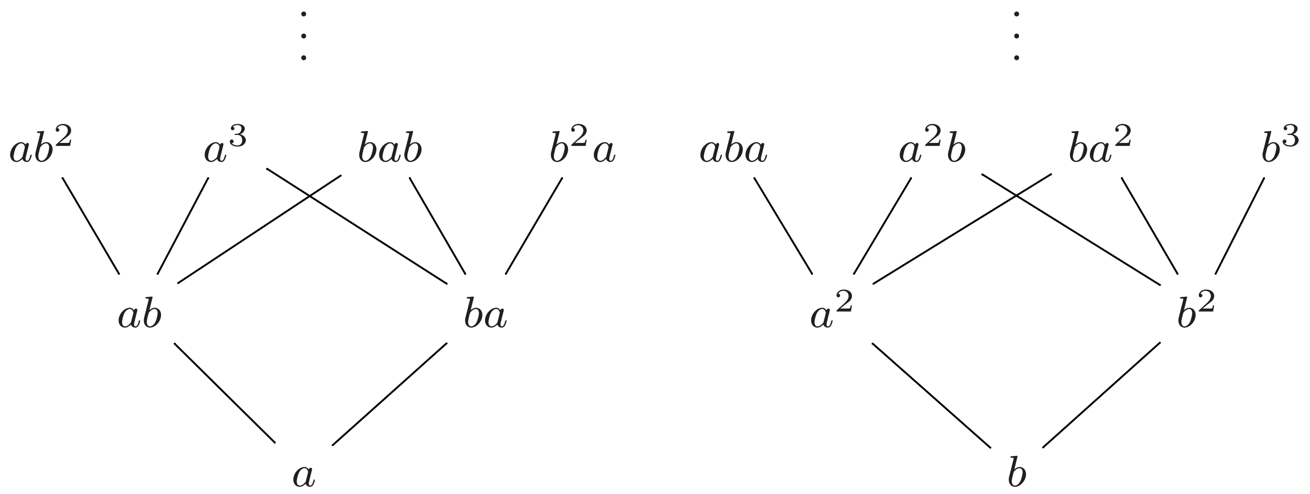
S ... finite simple semigroup

$\sigma\colon A^+ \to S$ homomorphism

well quasi-order $\leq_\sigma$ on A^*:

$$w \leq_\sigma v \iff \begin{array}{l} w = a_1 \cdots a_m,\ a_j \in A, \\ v = v_1 \cdots v_m,\ v_j \in A^+, \\ \sigma(a_j) = \sigma(v_j),\ j = 1, \ldots, m \end{array}$$

Example: $\quad S = \mathbb{Z}_2 \quad \sigma\colon a \mapsto 1,\ b \mapsto 0$



Conclusion:

Well quasi-orders are useful for proving regularity of maximal solutions of language equations.

Possible future research:

1. Regularity of solutions of different kinds of language equations.
2. Existence of algorithms for computing regular solutions.