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The Trace Coding Problem Is Undecidable

Σ ... finite alphabet

I ... symmetric, irreflexive binary relation on Σ

independence relation

$D = (\Sigma \times \Sigma) \setminus I$ ... dependence relation

$(\Sigma, I),\ (\Sigma, D)$ ... (in)dependence graphs

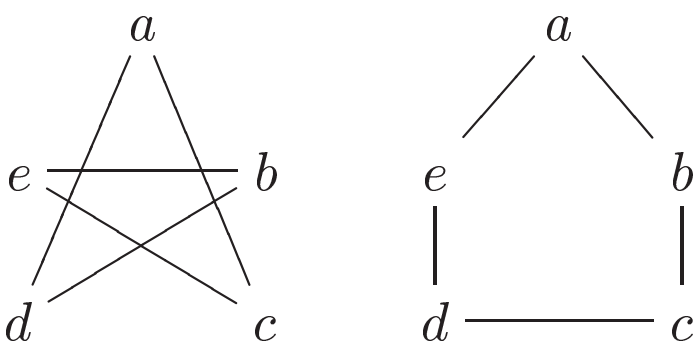
$\sim_I$ ... congruence of the free monoid Σ^*

generated by $\{(ab, ba) \mid (a, b) \in I\}$

$\mathbb{M}(\Sigma, I) = \Sigma^* / \sim_I$ ... trace monoid

elements of $\mathbb{M}(\Sigma, I)$ ... traces

Example:



(Σ, I) (Σ, D)

$\mathbb{M}(\Sigma, I)$ contains e.g. traces $\{abcde\}$ and

$\{acbde, acdbe, acdeb, cabde, cadbe, cadeb, cdabe, cdaeb\}$

Mazurkiewicz 1977:

behaviour of a concurrent system ... set of traces

simulations $\rightsquigarrow$ morphisms of trace monoids

morphism $\varphi : \mathbb{M}(\Sigma, I) \to \mathbb{M}(\Sigma', I')$

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\varphi_0} & (\Sigma')^* \\ \downarrow & & \downarrow \\ \mathbb{M}(\Sigma, I) & \xrightarrow{\varphi} & \mathbb{M}(\Sigma', I') \end{array}$$

$\forall (x, y) \in I : \ \varphi_0(x) \cdot \varphi_0(y) \sim_{I'} \varphi_0(y) \cdot \varphi_0(x)$

injective (uniquely decipherable) morphism ... coding

Theorem: Bruyère, De Felice 1996

φ is a coding $\implies$ φ_0 is a coding.

Trace coding problem: Ochmański 1988

Instance: Independence graphs (Σ, I) and (Σ', I').

Question: Does there exist a coding from the monoid $\mathbb{M}(\Sigma, I)$ to the monoid $\mathbb{M}(\Sigma', I')$?

Special cases:

- Free monoids: can be encoded into $\{a, b\}^*$.
- Free commutative monoids: linear independence.

Bruyère, De Felice 1995:
decidable if $\mathbb{M}(\Sigma, I)$ is a direct product of free monoids, i.e. if (Σ, D) is transitive.

Is a morphism $\varphi : \Sigma^* \to \{a, b\}^* \times \{c, d\}^*$ a coding?
not recursively enumerable

Strong morphisms:

morphism $\varphi : \mathbb{M}(\Sigma, I) \to \mathbb{M}(\Sigma', I')$ strong:

$$\forall (x, y) \in I : \ \mathrm{alph}(\varphi(x)) \times \mathrm{alph}(\varphi(y)) \subseteq I'$$

Theorem: Diekert, Muscholl, Reinhardt 1995

The strong trace coding problem is NP-complete.

Define $H : \Sigma \to 2^{\Sigma'}$ by $H(x) = \mathrm{alph}(\varphi(x))$ for $x \in \Sigma$.

A strong coding $\varphi : \mathbb{M}(\Sigma, I) \to \mathbb{M}(\Sigma', I')$ exists
if and only if a mapping $H : \Sigma \to 2^{\Sigma'}$ exists such that

$$H(x) \times H(y) \subseteq I' \iff (x, y) \in I$$

$$H(x) \times H(y) \subseteq I' \cup \mathrm{id}_{\Sigma'} \implies (x, y) \in I \cup \mathrm{id}_{\Sigma}$$

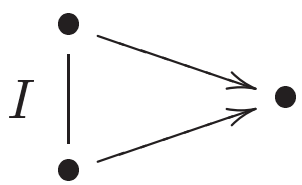
Weak morphisms:

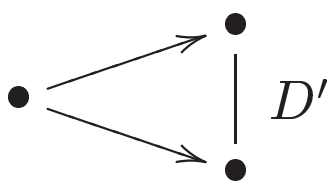
I ... symmetric, ~~ir~~reflexive binary relation on Σ

morphism $\varphi : \mathbb{M}(\Sigma, I) \to \mathbb{M}(\Sigma', I')$ weak:

$$\forall (x, y) \in I : \ \mathrm{alph}(\varphi(x)) \times \mathrm{alph}(\varphi(y)) \subseteq I'$$

Characterization by prohibited pictures:

strong: 

weak: 

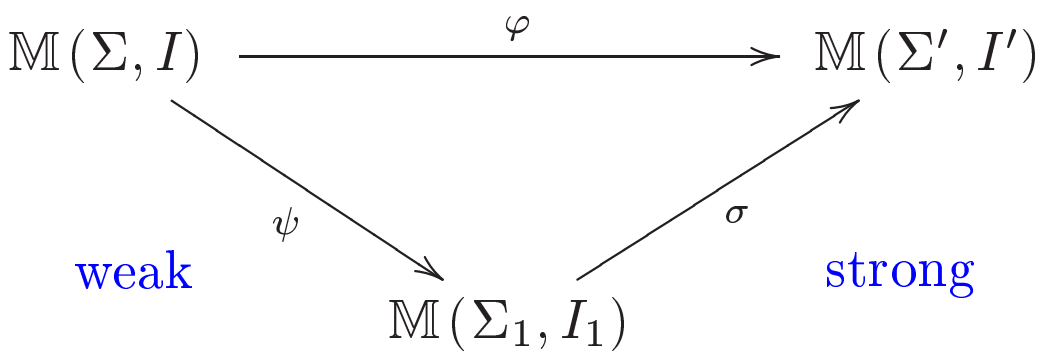
	strong	weak
for $x \in \Sigma$ alph $(\varphi(x))$ contains	arbitrary letters	independent letters
alph $(\varphi(x)) \cap$ alph $(\varphi(y))$ for $(x, y) \in I$, $x \neq y$	$\emptyset$	independent letters

Example: $\Sigma^* \to (\Sigma')^*, \quad |\Sigma| \geq 2$

$$\text{a strong coding exists} \iff |\Sigma'| \geq 2$$

$$\text{a weak coding exists} \iff |\Sigma'| \geq |\Sigma|$$

Decomposition of a trace morphism:



Example:

(Σ', D') : $\quad a - b - c - d - e - f - g$

$abefgacgbfce \sim_{I'} abacbc \cdot efg^2fe \sim_{I'} (abc)^2 \cdot \left(efg^2fe\right)$

Theorem: The trace coding problem can be effectively reduced to the weak trace coding problem.

Theorem: The trace coding problem is decidable if (Σ, D) is acyclic.

Post's correspondence problem

$\downarrow$

existence of weak codings
with prescribed contents of images of letters

$\downarrow$

trace coding problem

Theorem: The (weak) trace coding problem is not recursively enumerable.

Some open questions about the trace coding problem:

- Is the dual problem recursively enumerable?
- $\mathbb{M}(\Sigma, I)$ is a free product of free commutative monoids.
- $\mathbb{M}(\Sigma', I')$ is a direct product of free monoids.
- restrictions to other natural classes of trace morphisms.