Largest Solutions of Left-Linear Language Inequalities

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Outline:

1. Inequalities over regular languages.
2. Non-regular constants.
3. Implicit left-linear inequalities.
4. Regularity of solutions—sketch of the proof.

finite alphabet $A = \{a, b, \dots\}$
A^* ... the monoid of finite words over A
$A^+ = A^* \setminus \{\varepsilon\}$
finite set of variables $\mathcal{V} = \{X_1, \dots, X_n\}$

Explicit polynomial equations:
$X_i = P_i, \quad i = 1, \dots, n$
$P_i \subseteq (A \cup \mathcal{V})^*$ finite
smallest solutions ... context-free languages
Example: $X = aXb \cup \{\varepsilon\} \implies X = \{a^k b^k \mid k \in \mathbb{N}_0\}$

Explicit left-linear equations:
$X_i = L_{0i} \cup X_1 L_{1i} \cup \dots \cup X_n L_{ni}$

Example: $X \cup Y = A^*$
every language is a component of a minimal solution

Implicit left-linear inequalities:

$X_1K_1 \cup \cdots \cup X_nK_n \subseteq L_0 \cup X_1L_1 \cup \cdots \cup X_nL_n$

$K_i, L_i \subseteq A^*$ regular languages

has largest solution:

$$M_kK \subseteq L_0 \cup M_kL \implies (\bigcup M_k)K \subseteq L_0 \cup (\bigcup M_k)L$$

largest solution:

- algorithmically regular (Rabin 1969)
- computation ExpTime-complete (Aiken et al. 1994; Charatonik, Podelski 1998; Baader, Küsters 2001)

$P_j \subseteq L_j$

$L_j \subseteq A^*$ regular, P_j regular expressions

Conway 1971

maximal solutions:

- finitely many
- algorithmically regular

Two-sided concatenation

$XL = LX$

formulated by Conway 1971

MK 2005:

There exists a finite language L such that the largest solution is not recursively enumerable.

Non-regular constants

φ arbitrary substitution,
i.e. $\varphi\colon B^* \to \wp(A^*)$ homomorphism wrt concatenation

$L \subseteq A^*$ regular $\implies$ $\varphi^{-1}(L)$ regular

largest solution of $\varphi(X) \subseteq \overline{L}$ is $\overline{\varphi^{-1}(L)}$

$X_1K_1 \cup \cdots \cup X_nK_n \subseteq L_0 \cup X_1L_1 \cup \cdots \cup X_nL_n$
K_i arbitrary, L_i regular

MK 2004

largest solution:

- regular
- for context-free K_i algorithmically regular

$P_j \subseteq L_j$

$L_j \subseteq A^*$ regular, $P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary

Conway 1971: for given L_j finitely many possible components of maximal solutions, all of them regular

Example: $X \cdot Y \subseteq L$

$\sim$ ... congruence of A^* of finite index recognizing L,

i.e. $u \sim v \implies (u \in L \iff v \in L)$

Every solution contained in a solution recognized by $\sim$:

$M \cdot N \subseteq L \implies (M\sim) \cdot (N\sim) \subseteq L$

$u \in M\sim$, $v \in N\sim$

$\implies u \sim u'$, $u' \in M$, $v \sim v'$, $v' \in N$

$\implies uv \sim u'v' \in L$

$\implies uv \in L$

$XK \subseteq LX$

K arbitrary, L regular

MK 2004

largest solution:

- regular
- for context-free K algorithmically recursive

$P_j \subseteq Q_j$

$P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary

Q_j . . . regular expression over variables and languages
recognizable by finite simple semigroups

L recognizable by a finite simple semigroup $\iff$
minimal automaton of L does not contain

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[dr, "b"] & \\ & & \bullet \\ \bullet \arrow[r, "a"'] & \bullet \arrow[ur, "b"'] & \end{tikzcd}$$

MK 2004: all maximal solutions regular

Solutions closed wrt a well quasi-order

Wqo $\leq$ on A^* contains neither  nor $\bullet\ \bullet\ \bullet \cdots$

Monotone: $u \leq v$ & $\tilde{u} \leq \tilde{v} \implies u\tilde{u} \leq v\tilde{v}$

Theorem: Ehrenfeucht, Haussler, Rozenberg 1983

$L \subseteq A^*$ is regular $\iff$ L is upward closed with respect to a monotone well quasi-order on A^*

Example:

Congruence of finite index is a monotone wqo.

upward closed = recognized by the congruence

Applying well quasi-orders to inequalities:

Right-hand sides determine a wqo such that every solution is contained in an upward closed solution.

Theorem:

For any regular languages $L_0, \ldots, L_n$ all largest solutions of inequalities of the form

$$X_1K_1 \cup \cdots \cup X_nK_n \subseteq L_0 \cup X_1L_1 \cup \cdots \cup X_nL_n$$
$$(K_1, \ldots, K_n \text{ arbitrary languages})$$

are regular and can be algorithmically computed.

Corollary:

For $L_0, \ldots, L_n$ regular and $K_1, \ldots, K_n$ context-free, the largest solution of

$$X_1K_1 \cup \cdots \cup X_nK_n \subseteq L_0 \cup X_1L_1 \cup \cdots \cup X_nL_n$$

can be algorithmically computed.

Sketch of the proof

First step:

Construct a finite automaton recognizing all components of largest solutions.

Second step:

$(N_1, \ldots, N_n) \ldots$ candidate for being a largest solution

new inequality with variables $Y_1, \ldots, Y_n$:

$$N_1 Y_1 \cup \cdots \cup N_n Y_n \subseteq \underbrace{L_0 \cup N_1 L_1 \cup \cdots \cup N_n L_n}_{\text{regular constant}}$$

$(K_1, \ldots, K_n) \ldots$ its largest solution (regular)

$(N_1, \ldots, N_n)$ largest solution of some inequality $\iff$
$(N_1, \ldots, N_n)$ largest solution for these $K_1, \ldots, K_n$

$XK \subseteq a^+ \cup Xb^2$: $\qquad A = \{a, b\}$

largest solution	largest K
$\emptyset$	A^*
A^*	A^*b^2
a^*	$a^*b^2 \cup a^+$
a^+	$a^*b^2 \cup a^*$
$a^+(b^2)^*$	$(b^2)^*$

$K = (b^2)^*$:

$$w \in X \implies w \in XK \implies w \in a^+ \text{ or } wb^{-2} \in X$$

$K = a^*b^2 \cup a^*$:

$$\begin{aligned} w \in X &\implies wa \in XK \implies w \in a^* \\ &\implies w \in XK \implies w \in a^+ \end{aligned}$$

$ab \in K$:

$$w \in X \implies wab \in XK, \text{ contradiction}$$

$XK \subseteq L_0 \cup XL$

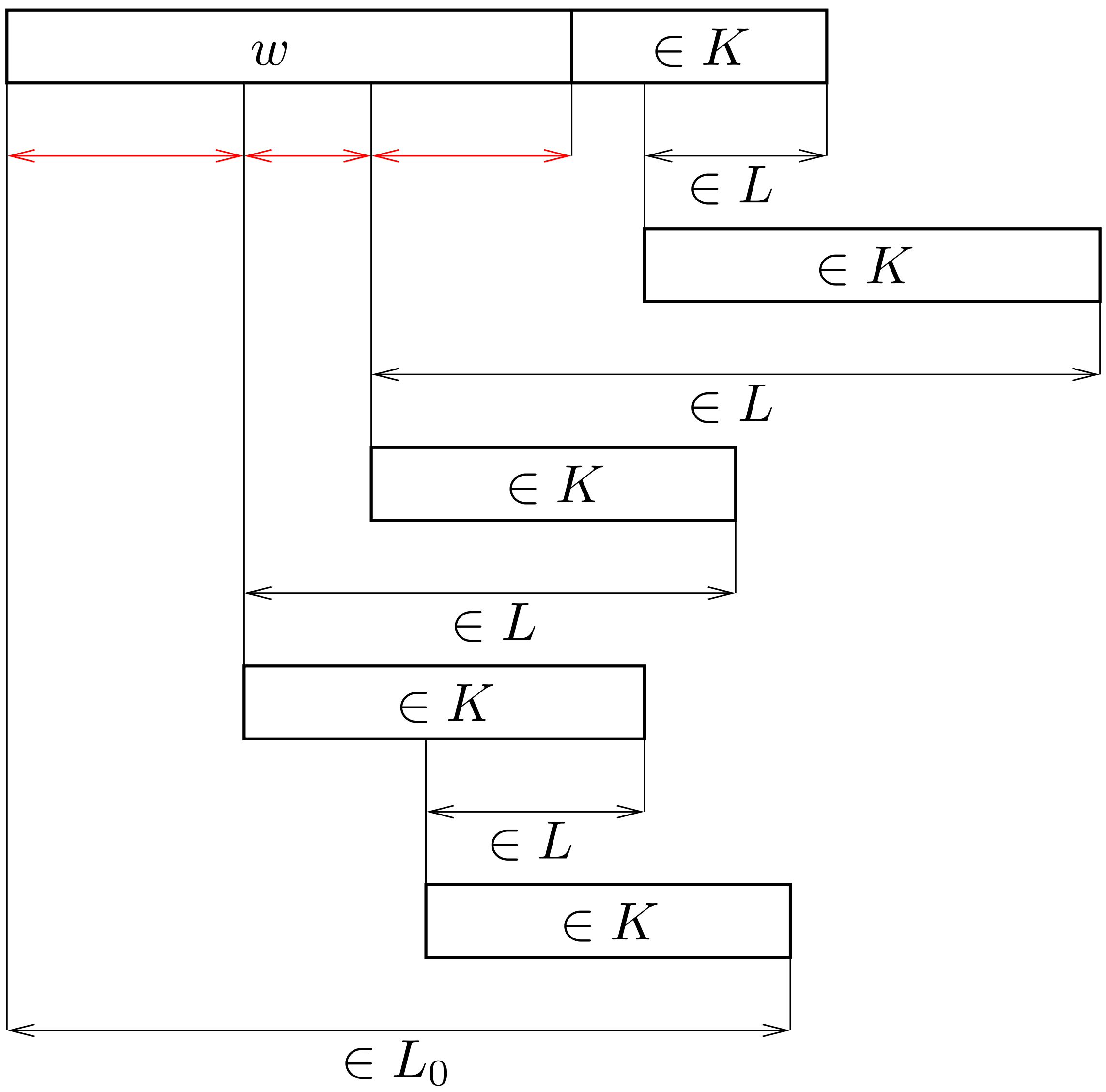
Game: position: $w \in A^*$

attacker: $u \in K$, $w \longrightarrow wu$

defender: $wu \in L_0$ winning position

$v \in L$, $wu = \tilde{w}v$, $wu \longrightarrow \tilde{w}$

largest solution = all winning positions for defender



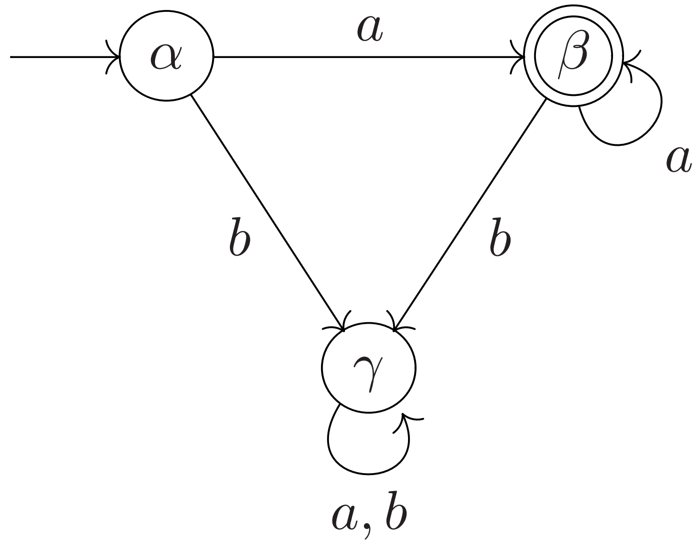
$XK \subseteq a^+ \cup Xb^2$

M ... an arbitrary solution

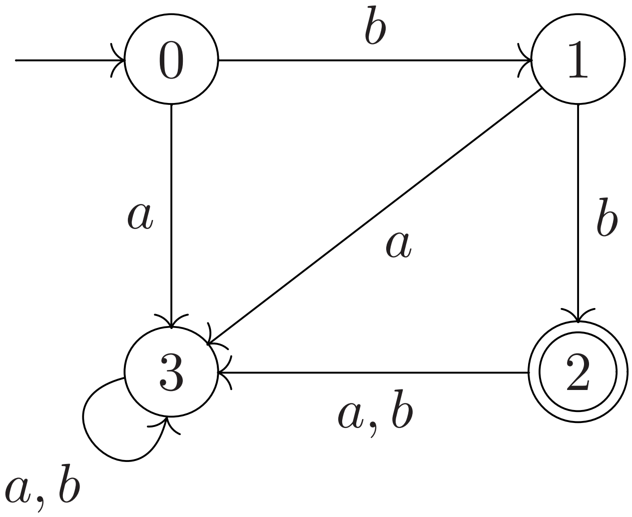
We construct a regular solution $N \supseteq M$.

Minimal automata of a^+ and b^2:

$\mathcal{A}_{a^+}$:



$\mathcal{A}_{b^2}$:



Every word $w \in M$ is characterized by:

1. State reached by w in $\mathcal{A}_{a^+}$.
2. States reached by $v^{-1}w$ in $\mathcal{A}_{b^2}$, for all prefixes v of w belonging to M.

$M = \{ab^2\}$

$\mathrm{Pref}(ab^2) \cap M = \{ab^2\}$

$M^{ab^2} = (\gamma, \emptyset)$

$N^w = (\gamma, \emptyset) \implies w \in A^*bA^*$

$N = A^*bA^*$

$M = \{a, ab^2\}$

$M^a = (\beta, \emptyset)$
$M^{ab^2} = (\gamma, \{2\})$

$N^w = (\beta, \emptyset) \implies w \in a^+$
$N^w = (\gamma, \{2\}) \implies w \in A^*bA^*$
and $\mathrm{Pref}(w) \cap N = \{wb^{-2}, w\}$

$N = a^+ \cup a^+b^2$

$M = \{a^{2k}b^{2l} \mid k > l \geq 0\}$

(solution for $K = \{\varepsilon\}$; then largest solution $a^+(b^2)^*$)

$\mathrm{Pref}(a^{2k}b^{2l}) \cap M = \{a^2, \ldots, a^{2k}, a^{2k}b^2, \ldots, a^{2k}b^{2l}\}$

$M^{a^2} = (\beta, \emptyset)$
$M^{a^{2k}} = (\beta, \{3\})$, for $k \geq 2$
$M^{a^{2k}b^{2l}} = (\gamma, \{2,3\})$, for $k > l \geq 1$

$N^w = (\beta, \emptyset) \implies w \in a^+$
$N^w = (\beta, \{3\}) \implies w \in a^+$ and $\mathrm{Pref}(w) \cap N \neq \{w\}$
$N^w = (\gamma, \{2,3\}) \implies wb^{-1} \notin N$ and $wb^{-2} \in N$
and $\mathrm{Pref}(wb^{-2}) \cap N \neq \{wb^{-2}\}$

$N = a^+a(b^2)^* \cup \{a\}$

$M = \{a^{2k}b^{2l} \mid k > l > 0\}$

$\mathrm{Pref}(a^{2k}b^{2l}) \cap M = \{a^{2k}b^2, \ldots, a^{2k}b^{2l}\}$

$M^{a^{2k}b^2} = (\gamma, \emptyset)$, for $k \geq 2$
$M^{a^{2k}b^4} = (\gamma, \{2\})$, for $k \geq 3$
$M^{a^{2k}b^{2l}} = (\gamma, \{2,3\})$, for $k > l \geq 3$

$N = A^*bA^*$

Automaton recognizing largest solutions

For each word w we consider all possibilities for its prefixes and compute $(M^w, \{M^v \mid v \in \mathrm{Pref}(w) \cap M\})$.

State corresponding to ab^2 is defined by the pairs:

$\big((\gamma, \emptyset), \{(\gamma, \emptyset)\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{ab^2\}$)

$\big((\gamma, \{3\}), \{(\alpha, \emptyset), (\gamma, \{3\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{\varepsilon, ab^2\}$)

$\big((\gamma, \{2\}), \{(\beta, \emptyset), (\gamma, \{2\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{a, ab^2\}$)

$\big((\gamma, \{2, 3\}), \{(\alpha, \emptyset), (\beta, \{3\}), (\gamma, \{2, 3\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{\varepsilon, a, ab^2\}$)

$\big((\gamma, \{1\}), \{(\gamma, \emptyset), (\gamma, \{1\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{ab, ab^2\}$)

$\big((\gamma, \{1, 3\}), \{(\alpha, \emptyset), (\gamma, \{3\}), (\gamma, \{1, 3\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{\varepsilon, ab, ab^2\}$)

$\big((\gamma, \{1, 2\}), \{(\beta, \emptyset), (\gamma, \{1\}), (\gamma, \{1, 2\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{a, ab, ab^2\}$)

$\big((\gamma, \{1, 2, 3\}), \{(\alpha, \emptyset), (\beta, \{3\}), (\gamma, \{1, 3\}), (\gamma, \{1, 2, 3\})\}\big)$
(for $\mathrm{Pref}(ab^2) \cap M = \{\varepsilon, a, ab, ab^2\}$)

Open questions

1. Regularity of solutions of similar systems of inequalities, for instance:

   $KXL \subseteq MX$

   $KX \subseteq LX, \; XM \subseteq XN$

2. Existence of algorithms for computing regular solutions.