Largest Solutions of Left-Linear Language Inequalities

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Abstract

We prove that largest solutions of systems of implicit one-sided linear language inequalities are regular provided all constants occurring on their right-hand sides are regular languages.

1 Introduction

It is one of the basic facts about regular languages that they are precisely languages definable by right-linear (or left-linear) grammars, and that languages generated by such grammars are nothing but components of smallest solutions of systems of explicit one-sided linear equations. Moreover, one can easily see that regular languages can be characterized also as components of largest solutions of such systems. So it is natural to ask whether the same is true also for some more general classes of systems of language equations or inequalities.

In this paper we consider systems of implicit left-linear inequalities, that is, inequalities of the form

$$K_0 \cup X_1 K_1 \cup \cdots \cup X_n K_n \subseteq L_0 \cup X_1 L_1 \cup \cdots \cup X_n L_n \,,$$

where $K_0, \ldots, K_n$ and $L_0, \ldots, L_n$ are constant languages over a given alphabet. Such systems in general do not possess smallest solutions; for instance for the inequality $X \cup Y \subseteq A^*$, every language over A is a component of one of its minimal solutions. But whenever such a system is solvable, it possesses the largest solution. For systems of one-sided linear inequalities where all constant languages occurring in the system are regular, solvability can be decided and the largest solution of the system, which is always regular, can be algorithmically computed due to Rabin's results on MSO logic over infinite trees [9]; actually,

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both problems turned out to be ExpTime-complete [1, 3, 2]. The aim of this paper is to show that largest solutions of these systems are in fact regular even when regularity of constant languages is required only on right-hand sides of inequalities.

Recently, a number of results on regularity of maximal solutions has been obtained also for some language equations and inequalities where concatenation with constants from both sides occurs (see [5] for a survey). These solutions are sometimes non-recursively enumerable even for very simple equations over regular languages, like commutation equations [7, 8]. But interestingly, for every kind of equations of the form $E = F$, where E and F are regular expressions of a certain type over variables and constant regular languages, for which maximal solutions of arbitrary finite systems are known to be regular, such a result can be directly generalized to the case of corresponding inequalities of the form $E \subseteq F$, where constant languages are required to be regular only on right-hand sides. This is true for inequalities whose right-hand sides are constant regular languages [4] (see [5]) as well as for those inequalities where all constants on the right are recognized by finite simple semigroups [6]. In this paper we prove that this happens also in the case of systems of one-sided linear inequalities.

Theorem. *Let m and n be positive integers and let Σ be the system of language inequalities*

$$K_{0j} \cup X_1K_{1j} \cup \cdots \cup X_nK_{nj} \subseteq L_{0j} \cup X_1L_{1j} \cup \cdots \cup X_nL_{nj}\,, \quad j = 1, \ldots, m\,,$$

where $L_{0j}, \ldots, L_{nj}$ are regular languages and $K_{0j}, \ldots, K_{nj}$ are arbitrary languages over an alphabet A. If the system Σ possesses a solution, then the largest solution of Σ over A is regular. Moreover, for given right-hand sides, there are only finitely many languages which can occur as components of largest solutions of such systems and these languages can be algorithmically calculated.

The proof of the theorem is given in Section 3. Let us first illustrate the result with an example and then proceed to introduce the necessary notation.

Example 1. Let us consider inequalities

$$X_1K_1 \cup X_2K_2 \subseteq \{a\} \cup X_1a \cup X_2b\,, \tag{1}$$

where K_1 and K_2 are arbitrary languages over the alphabet $A = \{a, b\}$. The theorem states that there are only finitely many different largest solutions of these inequalities over A. Actually, it is not difficult to verify that the largest solution of (1) is

$$(X_1, X_2) = \begin{cases} (A^*, A^*) & \text{if } \varepsilon \notin K_1 \cup K_2\,, \\ (A^+, A^*) & \text{if } \varepsilon \in K_1 \text{ and } \varepsilon \notin K_2\,, \\ (aA^*, aA^*) & \text{if } K_1 \cap b^* \neq \emptyset \text{ and } \varepsilon \in K_2\,, \\ (A^*, A^*aA^*) & \text{if } K_1 \cap b^* = \emptyset \text{ and } \varepsilon \in K_2\,, \end{cases}$$

where ε denotes the empty word.

2 Notation

Throughout the paper we consider a finite alphabet A. As usual, we write A^* (A^+) for the set of all (non-empty, respectively) finite words over A with the operation of concatenation; words will be denoted by s through z. If $w = uv$ for some words $u, v, w \in A^*$, then the words u and v are called a *prefix* and a *suffix* of w, respectively, and we write $v = u^{-1}w$. The set of all prefixes of a word w is denoted $\mathrm{Pref}(w)$. For languages $K, L \subseteq A^*$ and a word $u \in A^*$ we use the usual notation $u^{-1}L = \{v \in A^* \mid uv \in L\}$ and $K^{-1}L = \{v \in A^* \mid \exists u \in K : uv \in L\}$.

For a deterministic automaton $\mathcal{A} = (S, \delta, q^0, F)$ over A, any set of its states $Q \subseteq S$ and a word $w \in A^*$, we write $\delta(Q, w)$ for the set of states $\{\delta(q, w) \mid q \in Q\}$.

An equivalence relation $\sim$ on A^* is a *right congruence* if it satisfies

$$\forall u, v \in A^* \ \forall a \in A : u \sim v \implies ua \sim va \, .$$

Right congruences of A^* of finite index directly correspond to finite automata; the state set of the automaton determined by $\sim$ is $A^*/{\sim}$ and the transition function is defined by the rule $\delta(u{\sim}, a) = (ua){\sim}$ for $u \in A^*$ and $a \in A$. A language is recognized by an automaton if and only if it is a union of certain classes of the corresponding right congruence.

A solution of a system of language inequalities in variables $X_1, \ldots, X_n$ over an alphabet A is an n-tuple $(M_i)_{i=1}^n$ of languages over A such that by substituting M_i for X_i every inequality becomes a valid inclusion. Solutions of a given system are partially ordered by componentwise inclusion, i.e. $(M_i)_{i=1}^n \leq (N_i)_{i=1}^n$ if and only if $M_i \subseteq N_i$ for $i = 1, \ldots, n$, and we say that $(M_i)_{i=1}^n$ is contained in $(N_i)_{i=1}^n$. Because concatenation distributes over infinite union, it is easy to see that all solvable systems Σ considered in this paper possess the largest solution with respect to this ordering, which can be obtained as the componentwise union of all solutions.

3 Proof of the Theorem

To simplify notation, let us first encode the system Σ into one inequality. We choose new distinct letters $a_j \notin A$ for every $j \in \{1, \ldots, m\}$ and define languages

$$K_i = \bigcup_{j=1}^{m} K_{ij} a_j \quad \text{and} \quad L_i = \bigcup_{j=1}^{m} L_{ij} a_j , \quad \text{for } i = 0, \ldots, n \, .$$

Then all languages L_i, for $i \in \{0, \ldots, n\}$, are regular, and it is easy to see that the system Σ has exactly the same solutions over A as the inequality

$$K_0 \cup X_1 K_1 \cup \cdots \cup X_n K_n \subseteq L_0 \cup X_1 L_1 \cup \cdots \cup X_n L_n \, . \tag{2}$$

Therefore, in what follows we will consider inequality (2) instead of the original system.

For $i \in \{0, \ldots, n\}$, let $\mathcal{A}_i = (S_i, \delta_i, q_i^0, F_i)$ be a finite deterministic automaton accepting the language L_i, and such that the initial state q_i^0 cannot be reached by reading a non-empty word. We are going to construct a right congruence of finite index on A^* according to automata $\mathcal{A}_i$, and prove that every solution of (2) is contained in a regular solution of (2), and every component of this regular solution is a union of certain classes of this right congruence. Therefore, in particular, for every solvable system Σ, its largest solution over A is regular.

First notice that in order to test whether a given word u belongs to the i-th component of the largest solution of (2), we have to append to u some word x from one of the languages K_i and then ask whether either the word ux belongs to L_0 or we can remove from ux a suffix belonging to a certain language L_k in such a way that the remaining prefix $\bar{u}$ of ux belongs to the k-th component of the solution. In the latter case, the same test can be applied to the resulting word $\bar{u}$. Continuing this way, each test produces a new word $\bar{u}$ which is either a prefix of u or it is constructed from such a prefix v of u by appending a certain word w obtained by successive addition of elements of some languages $K_{\hat{\imath}}$ and removal of elements of some $L_{\hat{k}}$ without further modifying this prefix v; i.e. such a word w belongs to some language of the form

$$((\cdots(((K_{i_0} \cdot L_{i_1}^{-1}) \cdot K_{i_1}) \cdot L_{i_2}^{-1}) \cdots) \cdot K_{i_{l-1}}) \cdot L_{i_l}^{-1} \,,$$

where i_0 is the component of the solution containing v. The proof of the theorem is based on the fact that the way in which possible words w are constructed depends only on the component of the solution where v belongs to, and not on the actual word v. And when we are eventually going to remove not only a suffix of w but the whole word w together with a suffix of v, we only need to know for every language $L_{\hat{k}}$ the behaviour (with respect to this language) of all suffixes of v whose removal produces a prefix of u belonging to the $\hat{k}$-th component of the solution. In other words, we are interested only in the states of $\mathcal{A}_{\hat{k}}$ reached by these suffixes of v.

Therefore we consider for every word u all possible guesses for a solution of (2), and for each such guess we classify states of automata $\mathcal{A}_k$ obtained by reading differences between prefixes of u belonging to corresponding components of this guessed solution. Let $P = (P_i)_{i=1}^n \in (\wp(A^*))^n$, where P_i are arbitrary languages over A, which together represent a guess of a solution of (2). In order to obtain a required classification for the word u, we have to describe for every prefix v of u all possibilities for removing a suffix from v. Given an arbitrary word v, we consider for every $k \in \{1, \ldots, n\}$ the set

$$P_k^v = \{\delta_k(q_k^0, w^{-1}v) \mid w \in P_k \cap \mathrm{Pref}(v)\} \,,$$

which consists of all states of $\mathcal{A}_k$ reachable from the initial state using some suffix of v whose removal produces a word from P_k. Putting these sets together as an n-tuple and adding to it the state of $\mathcal{A}_0$ corresponding to the word v itself, we define

$$P^v = (\delta_0(q_0^0, v), (P_k^v)_{k=1}^n) \in S_0 \times \prod_{k=1}^n \wp(S_k) \,.$$

The $(n+1)$-tuple P^v characterizes all possibilities available for removing one suffix from v (when the language L_0 is used, the whole word v has to be removed).

Finally, we collect the values P^v for all prefixes v of u. This has to be done separately for prefixes belonging to different languages P_l, and so for $l = 1, \dots, n$ we define

$$\hat{P}_l^u = \{P^v \mid v \in P_l \cap \mathrm{Pref}(u)\} \subseteq S_0 \times \prod_{k=1}^{n} \wp(S_k)\,.$$

Assume we have an arbitrary solution $M = (M_i)_{i=1}^n \in (\wp(A^*))^n$ of (2) over A and let

$$N_i = \big\{u \in A^* \mid \exists P \in (\wp(A^*))^n \colon u \in P_i \,\&\, \forall l \in \{1, \dots, n\} \colon \hat{P}_l^u \subseteq \{M^v \mid v \in M_l\}\big\}$$

for all $i \in \{1, \dots, n\}$. In other words, we put a word u into N_i if there exists a guess P of a solution of (2) such that for every prefix of u belonging to P_l there exists a word in M_l offering the same possibilities for removing a suffix. Notice that when we test membership of u in N_i, instead of P we can equivalently consider the n-tuple $(P_l \cap \mathrm{Pref}(u))_{l=1}^n \in (\wp(\mathrm{Pref}(u)))^n$. Further, observe that if P is a witness for $u \in N_i$ and w is any prefix of u belonging to P_k for some $k \in \{1, \dots, n\}$, then P can also be used as a witness for w (because the inclusion $\mathrm{Pref}(w) \subseteq \mathrm{Pref}(u)$ means that $\hat{P}_l^w \subseteq \hat{P}_l^u$ trivially holds for every $l \in \{1, \dots, n\}$), and therefore $w \in N_k$.

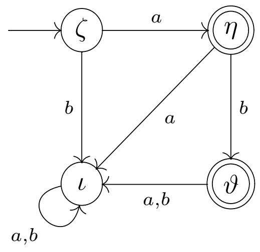
Clearly $M_i \subseteq N_i$ for all $i \in \{1, \dots, n\}$ since the solution M itself serves as a witness P for every $u \in M_i$. We are going to prove that $(N_i)_{i=1}^n$ is a regular solution of (2). First, let us give an example of the construction of N_i.

Example 2. Let K_1 and K_2 be arbitrary languages over the alphabet $A = \{a, b\}$ and consider the inequality

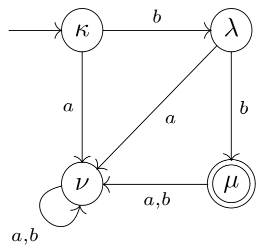
$$X_1K_1 \cup X_2K_2 \subseteq \{a, ab\} \cup X_1\{a, ab\} \cup X_2b^2\,. \tag{3}$$

Let us assume that the languages $M_1 = a^+$ and $M_2 = \{a^gb^h \mid g > h \geq 0\}$ form a (non-regular) solution of (3). We are going to calculate the corresponding regular solution (N_1, N_2) defined using the following automata accepting the languages $L_0 = L_1 = \{a, ab\}$ and $L_2 = \{b^2\}$:

$\mathcal{A}_0 = \mathcal{A}_1$:



$\mathcal{A}_2$:



First we have to compute M^u for every $u \in M_1 \cup M_2$:

$$\begin{aligned}
M^a &= (\eta, \{\zeta\}, \{\kappa\})\,, \\
M^{a^2} &= (\iota, \{\zeta, \eta\}, \{\kappa, \nu\})\,, \\
M^{a^g} &= (\iota, \{\zeta, \eta, \iota\}, \{\kappa, \nu\})\,, && \text{for } g \geq 3\,, \\
M^{a^g b} &= (\iota, \{\vartheta, \iota\}, \{\kappa, \lambda, \nu\})\,, && \text{for } g \geq 2\,, \\
M^{a^g b^h} &= (\iota, \{\iota\}, \{\kappa, \lambda, \mu, \nu\})\,, && \text{for } g > h \geq 2\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\{M^v \mid v \in M_1\} = \big\{(\eta, \{\zeta\}, \{\kappa\}), (\iota, \{\zeta, \eta\}, \{\kappa, \nu\}), (\iota, \{\zeta, \eta, \iota\}, \{\kappa, \nu\})\big\} \text{ and} \\
&\{M^v \mid v \in M_2\} = \{M^v \mid v \in M_1\} \cup \big\{(\iota, \{\vartheta, \iota\}, \{\kappa, \lambda, \nu\}), (\iota, \{\iota\}, \{\kappa, \lambda, \mu, \nu\})\big\}\,.
\end{aligned}$$

We will show that $N_1 = a^+$ and $N_2 = a^+ab^* \cup \{a\}$.

Let us first verify that this pair $(Q_1, Q_2) = (a^+, a^+ab^* \cup \{a\})$ serves as a witness for all elements of Q_1 and Q_2. Because Q coincides with M on a^+, it is easy to see that $\hat{Q}_1^u \subseteq \{M^v \mid v \in M_1\}$ and $\hat{Q}_2^u \subseteq \{M^v \mid v \in M_2\}$ for all $u \in a^+$. And for the remaining words in Q_2 we have

$$\begin{aligned}
Q^{a^g b} &= (\iota, \{\vartheta, \iota\}, \{\kappa, \lambda, \nu\})\,, && \text{for } g \geq 2\,, \\
Q^{a^g b^h} &= (\iota, \{\iota\}, \{\kappa, \lambda, \mu, \nu\})\,, && \text{for } g \geq 2 \text{ and } h \geq 2\,,
\end{aligned}$$

which proves $\hat{Q}_2^u \subseteq \{M^v \mid v \in M_2\}$ also for all $u \in Q_2 \setminus a^+$.

It remains to show that there exist no witnesses for words which do not belong to (Q_1, Q_2). First assume that P is a witness for a certain word $u \in N_1$. Then for every $x \in P_1 \cap \mathrm{Pref}(u)$ the triple P^x belongs to $\{M^v \mid v \in M_1\}$. Therefore either $\delta_0(\zeta, x) = \eta$ or $\eta \in P_1^x$. The former condition is equivalent to $x = a$, and in the latter case we obtain $xa^{-1} \in P_1 \cap \mathrm{Pref}(u)$. This implies that $P_1 \cap \mathrm{Pref}(u) \subseteq a^+$, in particular, it means $u \in a^+ = Q_1$.

Now take any word $u \in N_2$ and let P be a witness for u. Because for every $x \in P_2 \cap \mathrm{Pref}(u)$ we have $P^x \in \{M^v \mid v \in M_2\}$, one of the following conditions holds: either $\delta_0(\zeta, x) = \eta$ or $\eta \in P_1^x$ or $\vartheta \in P_1^x$ or $\mu, \nu \in P_2^x$. From these conditions we can deduce $x = a$ or $xa^{-1} \in P_1 \cap \mathrm{Pref}(u) \subseteq N_1 = a^+$ or $x(ab)^{-1} \in P_1 \cap \mathrm{Pref}(u) \subseteq a^+$ or $x(b^2)^{-1} \in P_2 \cap \mathrm{Pref}(u) \cap AA^+$, respectively. Consequently, $P_2 \cap \mathrm{Pref}(u) \subseteq a^+ab^* \cup \{a\} = Q_2$, and we immediately obtain $u \in Q_2$ as required.

Let us now return to the proof of the theorem.

Lemma 1. *The n-tuple $N = (N_i)_{i=1}^n$ is a solution of* (2).

Proof. Let us consider any $i \in \{1, \ldots, n\}$ and words $u \in N_i$ and $x \in K_i$. We have to prove that $ux \in L_0 \cup N_1L_1 \cup \cdots \cup N_nL_n$. By the definition of N_i there exists $P \in (\wp(A^*))^n$ such that $u \in P_i$ and $\hat{P}_l^u \subseteq \{M^v \mid v \in M_l\}$ for every $l \in \{1, \ldots, n\}$. In particular, taking $l = i$ we can use the fact $P^u \in \hat{P}_i^u$ to obtain a word $v \in M_i$ satisfying $P^u = M^v$. Then, since M is a solution, either $vx \in L_0$ or $vx \in M_k L_k$ for a certain $k \in \{1, \ldots, n\}$. If the former case arises, we immediately get $ux \in L_0$ because $\delta_0(q_0^0, u) = \delta_0(q_0^0, v)$. In the latter case, the prefix of vx belonging to M_k is either a prefix of v or it contains v as its prefix.

First assume that $v = yz$, where $y \in M_k$ and $zx \in L_k$. Because we have $P_k^u = M_k^v$, there exists $w \in P_k \cap \mathrm{Pref}(u)$ such that $\delta_k(q_k^0, w^{-1}u) = \delta_k(q_k^0, z)$. This in particular means that P can be used to verify $w \in N_k$ and that $w^{-1}ux \in L_k$ holds. Hence, we have $ux \in N_k L_k$ as required.

Now assume that $x = yz$, where $vy \in M_k$ and $z \in L_k$. In order to prove that $uy \in N_k$, we construct a witness $Q = (Q_l)_{l=1}^n$ by the rule

$$Q_l = (P_l \cap \mathrm{Pref}(u)) \cup (u \cdot (v^{-1}(M_l \cap \mathrm{Pref}(vy)))) \,,$$

i.e. we take the same prefixes of u as in the witness P and add prefixes of uy longer than u corresponding to prefixes of vy longer than v which belong to the solution M. Observe that for every $p \in \{1, \ldots, n\}$ the equivalences

$$u \in P_p \iff q_p^0 \in P_p^u \iff q_p^0 \in M_p^v \iff v \in M_p$$

follow from $P^u = M^v$ and from the fact that the initial state can be reached only by the empty word. Thus we have

$$\forall p \in \{1, \ldots, n\} \colon Q_p \cap \mathrm{Pref}(u) = P_p \cap \mathrm{Pref}(u) \tag{4}$$

and so in particular $Q^u = P^u = M^v$.

It is clear that $uy \in Q_k$ because vy belongs to M_k. Let $l \in \{1, \ldots, n\}$ and consider an arbitrary word $w \in Q_l = Q_l \cap \mathrm{Pref}(uy)$. To verify that Q is a required witness, we have to find a word $s \in M_l$ satisfying $Q^w = M^s$. If $w \in P_l \cap \mathrm{Pref}(u)$ then $Q^w = P^w$ due to (4), and so the existence of such a word s follows from properties of P. Otherwise we have $w = u\hat{w}$ where $v\hat{w} \in M_l$. Then if we put $s = v\hat{w}$, we obtain $Q^w = M^s$ since $\delta_0(q_0^0, u) = \delta_0(q_0^0, v)$ and

$$\begin{aligned} Q_p^w &= \delta_p(Q_p^u, \hat{w}) \cup \{\delta_p(q_p^0, t^{-1}\hat{w}) \mid ut \in Q_p \cap \mathrm{Pref}(w)\} \\ &= \delta_p(M_p^v, \hat{w}) \cup \{\delta_p(q_p^0, t^{-1}\hat{w}) \mid vt \in M_p \cap \mathrm{Pref}(s)\} = M_p^s \end{aligned}$$

for every $p \in \{1, \ldots, n\}$. Hence, we have $ux \in N_k L_k$ and therefore N is a solution of inequality (2). □

Lemma 2. *For every $i \in \{1, \ldots, n\}$, the language N_i is regular.*

Proof. We prove this lemma by constructing a right congruence $\sim$ of A^* of finite index such that N_i is a union of some classes of $\sim$. For each $u \in A^*$ we consider the set

$$\tilde{u} = \{(P^u, (\hat{P}_l^u)_{l=1}^n) \mid P \in (\wp(A^*))^n\}$$

and define the equivalence relation $\sim$ for all $u, v \in A^*$ by the rule

$$u \sim v \iff \tilde{u} = \tilde{v} \,.$$

It is clear that $\sim$ has finite index as $\tilde{u}$ is a subset of a finite set

$$S_0 \times \prod_{k=1}^{n} \wp(S_k) \times \left(\wp\left(S_0 \times \prod_{k=1}^{n} \wp(S_k)\right)\right)^n .$$

If we have $u \in N_i$ and $u \sim v$, then there exists a witness P for u, and by the definition of $\sim$ we can find $Q \in (\wp(A^*))^n$ satisfying $P^u = Q^v$ and $\hat{P}_l^u = \hat{Q}_l^v$ for $l = 1, \ldots, n$. Since the fact $u \in P_i$ implies $q_i^0 \in P_i^u = Q_i^v$ and consequently $v \in Q_i$, one can immediately see that Q is a witness for v, and therefore $v \in N_i$ as required. Hence, the language N_i is a union of certain classes of $\sim$.

To complete the proof, we have to show that $\sim$ is a right congruence. Assume $u \sim v$ holds and let $a \in A$ be any letter. Take an arbitrary $P \in (\wp(A^*))^n$ and let us verify

$$(P^{ua}, (\hat{P}_l^{ua})_{l=1}^n) \in \widetilde{va} , \tag{5}$$

which is sufficient to conclude that $ua \sim va$ due to symmetry.

Because $u \sim v$, there exists $Q \in (\wp(A^*))^n$ such that $P^u = Q^v$ and $\hat{P}_l^u = \hat{Q}_l^v$ for $l = 1, \ldots, n$. Then we can define $R \in (\wp(A^*))^n$ using the rule

$$R_i = \begin{cases} Q_i \setminus \{va\} & \text{if } ua \notin P_i , \\ Q_i \cup \{va\} & \text{if } ua \in P_i . \end{cases}$$

Now we establish (5) by verifying $P^{ua} = R^{va}$ and $\hat{P}_l^{ua} = \hat{R}_l^{va}$ for $l = 1, \ldots, n$. The former equality is true because $\delta_0(q_0^0, ua) = \delta_0(q_0^0, va)$ is a consequence of $\delta_0(q_0^0, u) = \delta_0(q_0^0, v)$ and because for every $k \in \{1, \ldots, n\}$ we have

$$P_k^{ua} = \delta_k(P_k^u, a) = \delta_k(Q_k^v, a) = R_k^{va}$$

when $ua \notin P_k$ and

$$P_k^{ua} = \delta_k(P_k^u, a) \cup \{q_k^0\} = \delta_k(Q_k^v, a) \cup \{q_k^0\} = R_k^{va}$$

when $ua \in P_k$. For the latter equality $\hat{P}_l^{ua} = \hat{R}_l^{va}$ we also distinguish two cases. For $ua \notin P_l$ we calculate

$$\hat{P}_l^{ua} = \hat{P}_l^u = \hat{Q}_l^v = \hat{R}_l^v = \hat{R}_l^{va}$$

and for $ua \in P_l$ we obtain

$$\hat{P}_l^{ua} = \hat{P}_l^u \cup \{P^{ua}\} = \hat{Q}_l^v \cup \{R^{va}\} = \hat{R}_l^v \cup \{R^{va}\} = \hat{R}_l^{va}$$

using the former equality. □

Let us recall that for an arbitrary solution M of the system Σ over A we have found an n-tuple N of languages over A, which is a regular solution of Σ due to Lemmas 1 and 2, and contains M. Therefore every solution of Σ is contained in a regular one. In particular, if we take for M the largest solution of Σ, we can see that it is itself regular.

Finally, notice that in order to calculate the set $\tilde{u}$, it is sufficient to take into account only $P \in (\wp(\mathrm{Pref}(u)))^n$, and so the right congruence $\sim$ can be calculated algorithmically. This gives us a description of possible components of the largest solution since we know from the proof of Lemma 2 that every component of the largest solution is recognized by the automaton corresponding to $\sim$.

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