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FREE PARTIALLY COMMUTATIVE SEMIGROUPS IN BAND VARIETIES

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Abstract. We consider the notion of free partially commutative semigroups in semigroup varieties. We give a transparent solution for the word problem for these semigroups in 10 varieties of bands revealing their trace structure. Then we use it to describe all equations solvable in these semigroups. All algorithms are shown to have low polynomial time complexity.

1. Introduction

Free partially commutative semigroups (often called trace semigroups) are a natural generalization of free semigroups. Introduced in a combinatorial context by Cartier and Foata [1], later they were proposed by Mazurkiewicz [4] as a model for concurrent systems. Afterwards, the trace theory developed rapidly and many classical results on free semigroups were generalized to this context (see [2] for an introduction and an overview of the theory).

We consider free partially commutative semigroups in semigroup varieties. Our intention is to obtain descriptions for these semigroups, discover analogies with the standard trace semigroups, possibly gain some more general results (for instance decision algorithms for some parts of their elementary theories). There does not seem to be many varieties where these problems are trivial. In this paper we solve some of them for 10 small varieties of bands. We give solutions for the word problem and for the problem of solvability of equations for these semigroups that reveal their connections with standard traces. Of course, as all bands are locally finite, instead of solvability of problems we are looking for their solvability in polynomial time.

The paper is organized as follows. In Section 2 we give necessary definitions and present some useful consequences of the results in [5], [6] (for details the reader is referred to these works). Sections 3 and 4 are devoted to solving the word problem and the problem of solvability of equations respectively. Finally, in Section 5 we deal with complexity issues of our solutions.

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2. Preliminaries

For a set A, let us denote its cardinality by $|A|$. Let Σ be an arbitrary alphabet. The free semigroup (semigroup of words) over Σ will be denoted Σ^+, the free monoid $\Sigma^* = \Sigma^+ \cup \{\lambda\}$, where λ is the empty word. For $u \in \Sigma^*$, let $\mathsf{c}(u)$ be the set of all letters occurring in u (the *content*). Throughout this paper, we shall use the following operations on words. For $u \in \Sigma^*$, $v \in \Sigma^+$, $a \in \mathsf{c}(v)$ and $i \in \{0, \dots, |\mathsf{c}(v)|\}$,

$\overrightarrow{u}$ is the subsequence of u consisting of the first occurrences of all letters from $\mathsf{c}(u)$,
$\mathsf{h}(v)$ is the first letter of v (the *head*),
$\mathsf{0}(v)$ is the longest prefix of v in $|\mathsf{c}(v)| - 1$ letters,
$\mathsf{0}^i(v)$ is the word resulting from v after applying $\mathsf{0}$ i-times,
$\mathsf{0}^a(v)$ is the longest prefix of v containing no occurrence of a.

Operations $\overleftarrow{u}$, $\mathsf{t}(v)$ (the *tail*) and $\mathsf{1}(v)$ are defined dually.

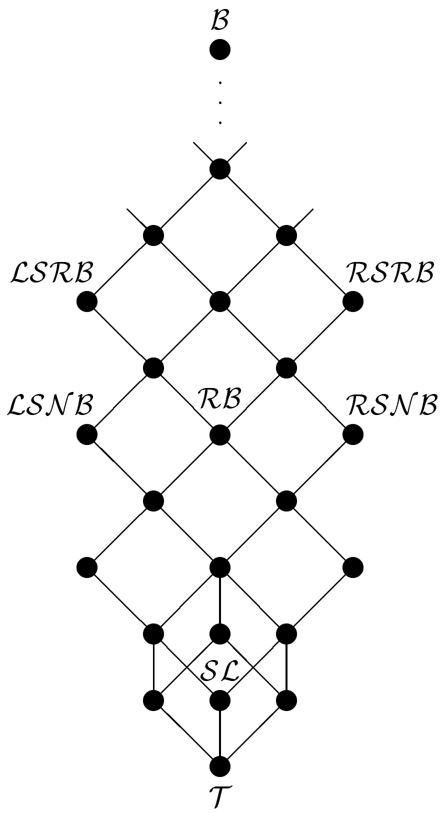

Figure 1. The lattice of all varieties of bands.

Let $\Upsilon \subseteq \Sigma$. The projection morphism from Σ^* to Υ^* will be denoted π_Υ. For $L \subseteq \Sigma^*$, let $\pi_\Upsilon(L) = \{\pi_\Upsilon(u) \mid u \in L\}$.

Let $\mathcal{U}$ be a variety of semigroups. We denote by $\simeq_{\mathcal{U}}$ the fully invariant congruence on Σ^+ corresponding to $\mathcal{U}$. Let $R \subseteq \Sigma^+ \times \Sigma^+$ be an arbitrary binary relation. By $\mathbb{P}(\Sigma, R, \mathcal{U})$ we denote the semigroup presented in $\mathcal{U}$ by the relation R, i.e. $\mathbb{P}(\Sigma, R, \mathcal{U}) = \Sigma^+/\sim_{\mathcal{U}}^R$, where $\sim_{\mathcal{U}}^R$ is the congruence on Σ^+ generated by $\simeq_{\mathcal{U}} \cup R$. A solution of the *word problem* for $\mathbb{P}(\Sigma, R, \mathcal{U})$ is an algorithm which decides, for given $u, v \in \Sigma^+$, whether $u \sim_{\mathcal{U}}^R v$. For the rest of this section, let $R \subseteq \Sigma^+ \times \Sigma^+$ be a content-preserving relation, that is if $(u, v) \in R$, then $\mathsf{c}(u) = \mathsf{c}(v)$.

In this paper we shall meet the following varieties of semigroups:

The variety of

- all semigroups $\mathcal{S}$
- all bands $\mathcal{B} = \mathrm{Mod}\left(x^2 = x\right)$
- semilattices $\mathcal{SL} = \mathrm{Mod}\left(x^2 = x, xy = yx\right)$
- regular bands $\mathcal{RB} = \mathrm{Mod}\left(x^2 = x, xyzx = xyxzx\right)$
- left seminormal bands $\mathcal{LSNB} = \mathrm{Mod}\left(x^2 = x, xyz = xyzxz\right)$
- left semiregular bands $\mathcal{LSRB} = \mathrm{Mod}\left(x^2 = x, xyz = xyzxzyz\right)$

and for the last two also their duals $\mathcal{RSNB}$ and $\mathcal{RSRB}$. A simple picture of the lattice of band varieties is on Figure 1 (for a complete description of this lattice see for example Fennemore [3]). In our considerations we shall deal only with *regular varieties* of bands. These are the varieties corresponding to content-preserving fully invariant congruences, i.e. those lying in the interval $[\mathcal{SL}, \mathcal{B}]$. This interval possesses a uniform structure that was transparently described by Polák [5], [6]. For every variety, he introduced some invariants based on the operations c, h, t, $\mathsf{0}$ and $\mathsf{1}$ which describe the corresponding fully invariant congruence. In the case of the varieties $\mathcal{B}$, $\mathcal{LSNB}$ and $\mathcal{LSRB}$, it looks as follows. For $u, v \in \Sigma^+$,

$$(1)\quad u \simeq_{\mathcal{B}} v \iff \mathsf{c}(u) = \mathsf{c}(v)\ \&\ \mathsf{0}(u) \simeq_{\mathcal{B}} \mathsf{0}(v)\ \&\ \mathsf{1}(u) \simeq_{\mathcal{B}} \mathsf{1}(v)\,,$$

$$(2)\quad u \simeq_{\mathcal{LSNB}} v \iff \overrightarrow{u} = \overrightarrow{v}\ \&\ \mathsf{t}\left(\mathsf{0}^i(u)\right) = \mathsf{t}\left(\mathsf{0}^i(v)\right),\ i = 0, \ldots, |\mathsf{c}(u)| - 1,$$

$$(3)\quad u \simeq_{\mathcal{LSRB}} v \iff \overrightarrow{u} = \overrightarrow{v}\ \&\ \overleftarrow{\mathsf{0}^i(u)} = \overleftarrow{\mathsf{0}^i(v)},\ i = 0, \ldots, |\mathsf{c}(u)| - 1.$$

Using Polák's description, it is easy to derive the following fact. Suppose $\mathcal{U}$ and $\mathcal{V}$ are regular varieties of bands such that $\mathcal{U} \neq \mathcal{U} \vee \mathcal{V} \neq \mathcal{V}$. Then from the structure of the lattice ([6] Figure 3) one can see that one of the varieties (say $\mathcal{U}$) satisfies exactly the same invariants starting by applying $\mathsf{0}$ as $\mathcal{U} \vee \mathcal{V}$ and the other (that is $\mathcal{V}$) satisfies the same invariants starting with 1. Thus we get

Lemma 1. *Let $\mathcal{U}$, $\mathcal{V}$ be regular band varieties satisfying all assumptions made in the previous paragraph. Let $u, v \in \Sigma^+$, $\mathsf{c}(u) = \mathsf{c}(v)$. If $u \sim_{\mathcal{U}}^R u'$ and $v \sim_{\mathcal{V}}^R v'$, then $uv \sim_{\mathcal{U} \vee \mathcal{V}}^R u'v'$.* □

Let X be an arbitrary set of variables such that $\Sigma \cap X = \emptyset$, $u, v \in (X \cup \Sigma)^+$. We recall that a morphism $\varphi : (X \cup \Sigma)^+ \to \Sigma^+$ is said to be a *solution* of an equation $u = v$ in $\mathbb{P}(\Sigma, R, \mathcal{U})$ if $\varphi(a) = a$ for $a \in \Sigma$ and $\varphi(u) \sim^R_{\mathcal{U}} \varphi(v)$. Let us denote by u_0 (u_1) the longest prefix (suffix) of u containing only elements of Σ and by $u[v]$ the word resulting from u by substituting v for all occurrences of every variable. The following lemma describes all equations solvable in the semigroup presented in the join of two regular band varieties $\mathcal{U}$ and $\mathcal{V}$ provided that we already know which equations are solvable in the semigroups given by the same presentation in $\mathcal{U}$ and $\mathcal{V}$.

Lemma 2. *Let $\mathcal{U}$ and $\mathcal{V}$ be regular varieties of bands, $u, v \in (X \cup \Sigma)^+$. Then the equation $u = v$ is solvable in $\mathbb{P}(\Sigma, R, \mathcal{U} \vee \mathcal{V})$ if and only if it is solvable in both $\mathbb{P}(\Sigma, R, \mathcal{U})$ and $\mathbb{P}(\Sigma, R, \mathcal{V})$. In particular (if we take $u, v \in \Sigma^+$), this means $\sim^R_{\mathcal{U}\vee\mathcal{V}} = \sim^R_{\mathcal{U}} \cap \sim^R_{\mathcal{V}}$.*

Proof. The "only if" part of the proof is clear. The "if" part is non-trivial only when $\mathcal{U} \neq \mathcal{U} \vee \mathcal{V} \neq \mathcal{V}$. As we have seen before, in this case we can assume that Lemma 1 is valid. So, suppose that $u = v$ is solvable in $\mathbb{P}(\Sigma, R, \mathcal{U})$ and $\mathbb{P}(\Sigma, R, \mathcal{V})$. By substituting solutions into the equation, we get

$$u_0 u' u_1 \sim^R_{\mathcal{U}} v_0 v' v_1, \, u_0 \bar{u} u_1 \sim^R_{\mathcal{V}} v_0 \bar{v} v_1 \tag{4}$$

for some $u', v', \bar{u}, \bar{v} \in \Sigma^+$, where $\mathsf{c}(u_0 u' u_1) \supseteq \mathsf{c}(u) \cap \Sigma \subseteq \mathsf{c}(u_0 \bar{u} u_1)$. Now we have to distinguish three cases (up to symmetry).

1. If $\mathsf{c}(u) \cap X \neq \emptyset \neq \mathsf{c}(v) \cap X$,
   then we can take any w such that $\mathsf{c}(w) = \mathsf{c}(u_0 u' \bar{u} u_1)$ and we get
   $$\begin{aligned} u[u_0 u' u_1 w u_0 \bar{u} u_1] &\simeq_{\mathcal{B}} u_0 u' u_1 w u_0 \bar{u} u_1 && \text{by (1)} \\ &\sim^R_{\mathcal{U}\vee\mathcal{V}} v_0 v' v_1 w v_0 \bar{v} v_1 && \text{by (4) and Lemma 1} \\ &\simeq_{\mathcal{B}} v_0 v_0 v' v_1 w v_0 \bar{v} v_1 v_1 && \\ &\sim^R_{\mathcal{U}\vee\mathcal{V}} v_0 u_0 u' u_1 w u_0 \bar{u} u_1 v_1 && \text{by (4) and Lemma 1} \\ &\simeq_{\mathcal{B}} v[u_0 u' u_1 w u_0 \bar{u} u_1] && \text{by (1),} \end{aligned}$$
   i.e. $\varphi(x) = u_0 u' u_1 w u_0 \bar{u} u_1$ for all $x \in X$ is a solution in $\mathbb{P}(\Sigma, R, \mathcal{U} \vee \mathcal{V})$.
2. If $\mathsf{c}(u) \cap X \neq \emptyset = \mathsf{c}(v) \cap X$,
   then (4) becomes $u_0 u' u_1 \sim^R_{\mathcal{U}} v$, $u_0 \bar{u} u_1 \sim^R_{\mathcal{V}} v$ and we get
   $$u\left[u_0 u' u_1 u_0 \bar{u} u_1\right] \simeq_{\mathcal{B}} u_0 u' u_1 u_0 \bar{u} u_1 \sim^R_{\mathcal{U}\vee\mathcal{V}} v^2 \simeq_{\mathcal{B}} v.$$
3. If $\mathsf{c}(u) \cap X = \emptyset = \mathsf{c}(v) \cap X$,
   then (4) becomes $u \sim^R_{\mathcal{U}} v$, $u \sim^R_{\mathcal{V}} v$ and we get $u \simeq_{\mathcal{B}} uu \sim^R_{\mathcal{U}\vee\mathcal{V}} vv \simeq_{\mathcal{B}} v$. □

For equations with variables on one side only, we can decide their solvability as soon as we have a solution of the word problem.

Lemma 3. *Let $\mathcal{U}$ be a variety of bands, $u \in (X \cup \Sigma)^+$, $v \in \Sigma^+$, $\mathsf{c}(u) \cap X \neq \emptyset$. Then $u = v$ is solvable in $\mathbb{P}(\Sigma, R, \mathcal{U})$ if and only if $\mathsf{c}(u) \cap \Sigma \subseteq \mathsf{c}(v)$ and $u_0 v u_1 \sim^R_{\mathcal{U}} v$.*

Proof. "$\Longrightarrow$" By substituting a solution into $u = v$, we obtain $u' \in \Sigma^+$ satisfying $u_0 u' u_1 \sim_{\mathcal{U}}^{R} v$. Then $u_0 v u_1 \sim_{\mathcal{U}}^{R} u_0 u_0 u' u_1 u_1 \simeq_{\mathcal{B}} u_0 u' u_1 \sim_{\mathcal{U}}^{R} v$.
"$\Longleftarrow$" $u\,[u_0 v u_1] \simeq_{\mathcal{B}} u_0 v u_1 \sim_{\mathcal{U}}^{R} v$ is a solution. $\square$

Let $D \subseteq \Sigma \times \Sigma$ be a reflexive symmetric binary relation on Σ. We shall call it *dependence relation*. Its complement $(\Sigma \times \Sigma) \setminus D$ will be denoted I and called *independence relation*. Sometimes we shall also work with D and I as graphs on the set of vertices Σ. The set of all letters dependent (independent) on $a \in \Sigma$ is $D(a) = \{b \in \Sigma \mid aDb\}$ ($I(a) = \{b \in \Sigma \mid aIb\}$ respectively).

From now on, let us suppose we have some fixed Σ and D. We define the *free partially commutative semigroup* (or *trace semigroup*) $\mathbb{T}(\Sigma, D, \mathcal{U})$ in the variety $\mathcal{U}$ of semigroups as $\mathbb{P}(\Sigma, \{(ab, ba) \mid (a,b) \in I\}, \mathcal{U})$. The corresponding congruence on Σ^+ will be denoted $\sim_{\mathcal{U}}$. In the case $\mathcal{U} = \mathcal{S}$, $\mathbb{T}(\Sigma, D, \mathcal{S})$ coincides with the usual notion of the trace semigroup and we shall adopt the standard notation $\equiv$ for $\sim_{\mathcal{S}}$. The equivalence class (the *trace*) of a word $u \in \Sigma^*$ will be denoted $u\equiv$. The set of all possible initial letters $\{\mathsf{h}(w) \mid w \in u\equiv\}$ of the trace $u\equiv$ will be denoted $\mathrm{init}(u)$.

From the local finiteness of bands and from the solution of the word problem for free objects in band varieties, it follows that for an arbitrary fixed semigroup finitely presented in some proper subvariety of $\mathcal{B}$ there is an algorithm solving the word problem linear in the length of input words. But it is not clear whether there is a uniform algorithm for some substantial class of presentations which is also polynomial in the number of generators $|\Sigma|$ and the sum of the lengths of the words in the relation R. In the next section we find such an algorithm for the word problem for trace semigroups in all subvarieties of $\mathcal{LSRB} \vee \mathcal{RSRB}$ (see Figure 1). In order to do this, first observe that this is very simple for the subvarieties of $\mathcal{RB}$. Further, we can use Lemma 2 to deal with the joins of the varieties we are already through. Thus, thanks to a left-right symmetry of the lattice, it suffices to solve the problem for the varieties $\mathcal{LSNB}$ and $\mathcal{LSRB}$.

3. Word problem

Let us start with $\mathcal{LSNB}$. First, we have to define a notion that will be used to decide which tails in the condition (2) should be considered when comparing u and v in a trace semigroup.

Definition 1. Let $u \in \Sigma^+$. The mapping $T_u : \{0, \dots, |\mathsf{c}(u)| - 1\} \to 2^{\Sigma}$ is defined as follows. For $i \in \{0, \dots, |\mathsf{c}(u)| - 1\}$, if

$$\mathsf{c}\left(\mathsf{0}^i\left(\overrightarrow{u}\right)\right) \times \mathrm{init}\left(\mathsf{1}^{|\mathsf{c}(u)|-i}\left(\overrightarrow{u}\right)\right) \subseteq D, \tag{5}$$

then $T_u(i)$ is the connected component containing $\mathsf{t}\left(\mathsf{0}^i(u)\right)$ of the graph defined on the set $\mathsf{c}\left(\mathsf{0}^i(u)\right)$ by the relation I. Otherwise $T_u(i)$ equals $\emptyset$.

Remark 1. Notice that in the condition (5) the word $\overrightarrow{u}$ is just split into two parts $\overrightarrow{u} = 0^i(\overrightarrow{u})\, 1^{|\mathsf{c}(u)|-i}(\overrightarrow{u})$ according to i. As we shall see in the proof of the following proposition, the condition ensures us that every commutation in $\overrightarrow{u}$ is performed inside $0^i(\overrightarrow{u})$ or $1^{|\mathsf{c}(u)|-i}(\overrightarrow{u})$. That is also why the validity of (5) for u and v is equivalent provided that $\overrightarrow{u} \equiv \overrightarrow{v}$.

Proposition 1. *Let $u, v \in \Sigma^+$. $u \sim_{\mathcal{LSNB}} v$ holds if and only if $\overrightarrow{u} \equiv \overrightarrow{v}$ and $T_u = T_v$.*

In our considerations we shall implicitly use the following observation. If u is changed to $\bar{u}$ by rewriting ab to ba with aIb, then either $\overrightarrow{u} = \overrightarrow{\bar{u}}$ or a and b are both the first occurrences of the corresponding letters in u and $\overrightarrow{\bar{u}}$ results from $\overrightarrow{u}$ by rewriting ab to ba. In particular, $u \equiv \bar{u}$ implies $\overrightarrow{u} \equiv \overrightarrow{\bar{u}}$.

Proof. "$\Longrightarrow$" $\overrightarrow{u} \equiv \overrightarrow{v}$ is clear. To get $T_u = T_v$, it is enough to show that if i satisfies (5) for u, then it satisfies (5) for v too and $T_u(i) = T_v(i)$. We prove by induction that every $\bar{u}$ obtained from u by rewriting using $\simeq_{\mathcal{LSNB}}$ and $\equiv$ arbitrary many times satisfies the conditions (5) and

$$1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{\bar{u}}\right) \equiv 1^{|\mathsf{c}(u)|-i}(\overrightarrow{u}) \tag{6}$$

$$\mathsf{c}\left(0^i\left(\overrightarrow{\bar{u}}\right)\right) = \mathsf{c}\left(0^i(\overrightarrow{u})\right) \tag{7}$$

$$\operatorname{init}\left(1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{\bar{u}}\right)\right) = \operatorname{init}\left(1^{|\mathsf{c}(u)|-i}(\overrightarrow{u})\right) \tag{8}$$

$$T_{\bar{u}}(i) = T_u(i)\,. \tag{9}$$

All conditions are clearly preserved by $\simeq_{\mathcal{LSNB}}$. So, let $\bar{u}$ satisfy the conditions and let w be obtained from $\bar{u}$ by commuting two neighbouring independent letters. Then (5) for $\bar{u}$ implies $\left(\mathsf{t}\left(0^i\left(\overrightarrow{\bar{u}}\right)\right), \mathsf{h}\left(1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{\bar{u}}\right)\right)\right) \in D$, thus every commutation in $\overrightarrow{\bar{u}}$ must be performed inside $0^i\left(\overrightarrow{\bar{u}}\right)$ or $1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{\bar{u}}\right)$. That is why w satisfies (6) and consequently also (7), (8) and (5). Employing (5) for $\bar{u}$ once again, $\mathsf{t}\left(0^i(\bar{u})\right)$ can be commuted only with some independent letter from $\mathsf{c}\left(0^i(\bar{u})\right)$. Hence, (9) for $\bar{u}$ together with (7) for w give $T_w(i) = T_u(i)$.

"$\Longleftarrow$" Let $\overrightarrow{u} \equiv \overrightarrow{v}$ and $T_u = T_v$. In order to prove $u \sim_{\mathcal{LSNB}} v$, we shall rewrite u and v many times using $\simeq_{\mathcal{LSNB}}$ and $\equiv$ until we get the same word. For simplicity, the modified words will be still denoted by u and v. Notice that by the "only if" part of the proposition, our assumptions will remain valid all the time. Moreover, after every step, the words will be of the form $a_m b_m \cdots a_1 b_1$ with $a_i, b_i \in \Sigma$ for $i = 1, \ldots, m$, where $0^i(a_m b_m \cdots a_1 b_1) = a_m b_m \cdots a_{i+1} b_{i+1}$ for $i = 0, \ldots, m$ and $m = |\mathsf{c}(a_m b_m \cdots a_1 b_1)|$. The first step is $u \simeq_{\mathcal{LSNB}} a_{|\mathsf{c}(u)|} b_{|\mathsf{c}(u)|} \cdots a_1 b_1$, $v \simeq_{\mathcal{LSNB}} a_{\sigma|\mathsf{c}(u)|} c_{|\mathsf{c}(u)|} \cdots a_{\sigma 1} c_1$ for some permutation $\sigma \in \mathbb{S}_{|\mathsf{c}(u)|}$. We proceed in three phases.

First, we show $v \sim_{\mathcal{LSNB}} a_{|\mathsf{c}(u)|} d_{|\mathsf{c}(u)|} \cdots a_1 d_1$ for some set of letters $d_i \in \Sigma$, $i = 1, \ldots, |\mathsf{c}(u)|$. Since $\overrightarrow{u} \equiv \overrightarrow{v}$, it is sufficient to show that we can exchange neighbouring a_i, a_j if $a_i I a_j$. But

$$\begin{aligned} \cdots a_i e a_j \cdots \simeq_{\mathcal{LSNB}} \cdots a_i e a_j a_i \cdots \equiv \cdots a_i e a_i a_j \cdots \simeq_{\mathcal{LSNB}} \\ \simeq_{\mathcal{LSNB}} \cdots a_i a_j \cdots \equiv \cdots a_j a_i \cdots \simeq_{\mathcal{B}} \cdots a_j a_j a_i \cdots , \end{aligned} \tag{10}$$

where e is a letter currently standing between a_i and a_j and for the rewritings using $\simeq_{\mathcal{LSNB}}$ we have exploited the fact that v is of the special form (further we shall do this implicitly).

After the first phase, we have $\overrightarrow{u} = \overrightarrow{v}$. In the second phase, we replace, for all i not satisfying (5), b_{i+1} and d_{i+1} by the same letter. We do this from left to right (i.e. starting with the largest i). Since (5) is not valid, there exist $a \in \operatorname{init}\left(1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{u}\right)\right)$ and $b \in \mathsf{c}\left(0^i\left(\overrightarrow{u}\right)\right)$ such that aIb. Using (10), we can simulate the commutations needed to move a onto $\mathsf{h}\left(1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{u}\right)\right)$ to achieve the same result on u. As (10) performs only local changes of u, it has no influence on the changes made in the previous steps to the left from $\mathsf{h}\left(1^{|\mathsf{c}(u)|-i}\left(\overrightarrow{u}\right)\right)$. Now we can do

$$\cdots b \cdots ea \cdots \simeq_{\mathcal{LSNB}} \cdots b \cdots eab \cdots \equiv \cdots b \cdots eba \cdots \simeq_{\mathcal{LSNB}} \cdots b \cdots ba \cdots ,$$

where e is a letter currently on the former position of b_{i+1}. Finally, we use (10) again to reverse the changes made to $\overrightarrow{u}$. If we do the same for v (with the same selected letters), we get the letter b instead of both b_{i+1} and d_{i+1}.

Despite the changes already made to u and v, we shall keep the same notation. So, it remains to replace all d_{i+1}, for i satisfying (5), with b_{i+1}. Since they lie in the same connected component of I in $\mathsf{c}\left(0^i\left(v\right)\right)$, it is sufficient to be able to replace $\mathsf{t}\left(0^i\left(v\right)\right)$ (let us call it e) by an arbitrary letter $b \in \mathsf{c}\left(0^i\left(v\right)\right)$ such that bIe. This can be done as follows

$$\cdots b \cdots e \cdots \simeq_{\mathcal{LSNB}} \cdots b \cdots be \cdots \equiv \cdots b \cdots eb \cdots \simeq_{\mathcal{LSNB}} \cdots b \cdots b \cdots . \quad \square$$

In the case of the variety $\mathcal{LSRB}$, the situation is a bit more complicated. Here it is already insufficient to compare sequences of the first occurrences in the corresponding trace monoid as in Proposition 1 since some commutations may be blocked by letters which should be preserved due to invariants $\overleftarrow{0^i\left(u\right)}$ as shown in the example.

Example 1. Let $\Sigma = \{a, b, c, d\}$ and $D = (\Sigma \times \Sigma) \setminus \{(c, d), (d, c)\}$. If we take $u = abcad$, then $\overrightarrow{u} = abcd$, but it can be shown that the first occurrences of c and d in u can never be commuted.

Nevertheless, it turns out that if we go through the lower level invariants first and use them to redefine our independence relation, we obtain the right condition for sequences of the first occurrences. The definition of the new independence relation follows.

Definition 2. Let $u \in \Sigma^+$ be a word. We define the independence relation $I_u \subseteq (\mathsf{c}(u) \times \mathsf{c}(u))$ as follows. For $a, b \in \mathsf{c}(u)$,

$$aI_ub \iff aIb \,\&\, \pi_{D(a)}\left(\left(\pi_{D(b)}\left(\overleftarrow{\eth^{b}(u)}\right)\right)_{\equiv}\right) \cap \pi_{D(b)}\left(\left(\pi_{D(a)}\left(\overleftarrow{\eth^{a}(u)}\right)\right)_{\equiv}\right) \neq \emptyset.$$

The corresponding trace equivalence will be denoted $\equiv_u$.

Proposition 2. *Let $u, v \in \Sigma^+$. Then $u \sim_{\mathcal{LSRB}} v$ holds if and only if $\overleftarrow{u} \equiv \overleftarrow{v}$, $\forall a \in \mathsf{c}(u) : \pi_{D(a)}\left(\overleftarrow{\eth^{a}(u)}\right) \equiv \pi_{D(a)}\left(\overleftarrow{\eth^{a}(v)}\right)$ and $\overrightarrow{u} \equiv_u \overrightarrow{v}$.*

Remark 2. Notice that the conditions of the proposition are really symmetric in u and v since $\equiv_u = \equiv_v$ as soon as the second condition is satisfied.

Remark 3. If we apply this proposition to the word from Example 1, we get

$$\pi_{D(c)}\left(\left(\pi_{D(d)}\left(\overleftarrow{\eth^{d}(u)}\right)\right)_{\equiv}\right) = \{ba\}, \quad \pi_{D(d)}\left(\left(\pi_{D(c)}\left(\overleftarrow{\eth^{c}(u)}\right)\right)_{\equiv}\right) = \{ab\},$$

which have no common element. Thus cD_ud and their first occurrences can never be commuted.

Proof. Throughout this proof, we shall write $\tilde{a}$ instead of a if we know that it is the first occurrence of a in the word.

"$\Longrightarrow$" The first two conditions are easy to verify. As for the last one, it suffices to show that if $\tilde{a}$ and $\tilde{b}$ were commuted by $\equiv$ during modifications of u, then aI_ub. In order to do this, let us suppose $u \sim_{\mathcal{LSRB}} w = \cdots \tilde{a}\tilde{b} \cdots$ and aIb. Then $\overleftarrow{\eth^{b}(w)} = \overleftarrow{\eth^{a}(w)}a$. Thus $\pi_{D(b)}\left(\overleftarrow{\eth^{a}(w)}\right) = \pi_{D(b)}\left(\overleftarrow{\eth^{b}(w)}\right) \equiv \pi_{D(b)}\left(\overleftarrow{\eth^{b}(u)}\right)$ and since $\pi_{D(a)}\left(\overleftarrow{\eth^{a}(w)}\right) \equiv \pi_{D(a)}\left(\overleftarrow{\eth^{a}(u)}\right)$,

$$\pi_{D(a)}\left(\pi_{D(b)}\left(\overleftarrow{\eth^{a}(w)}\right)\right) = \pi_{D(b)}\left(\pi_{D(a)}\left(\overleftarrow{\eth^{a}(w)}\right)\right)$$

is a desired element of the intersection.

"$\Longleftarrow$" This part of the proof will be done entirely in the same manner as in the proof of Proposition 1, one should only use an appropriate normal form. In this case, after every step, our words have to be a product of the blocks $\overleftarrow{\eth^{a}(u)}\tilde{a}$ ($\overleftarrow{\eth^{a}(v)}\tilde{a}$ respectively), for all $a \in \mathsf{c}(u)$, and $\overleftarrow{u}$ ($\overleftarrow{v}$ respectively). In the first step, the words can be easily modified into this form using $\simeq_{\mathcal{LSRB}}$. Now we rewrite u and v in two phases.

First, we make $\overrightarrow{u}$ equal to $\overrightarrow{v}$. Since $\overrightarrow{u} \equiv_u \overrightarrow{v}$, it is sufficient to be able to exchange the letters $\tilde{a}$ and $\tilde{b}$ in the word $u = \cdots \overleftarrow{\eth^{a}(u)}\tilde{a}\overleftarrow{\eth^{b}(u)}\tilde{b} \cdots$ when aI_ub. If we have an occurrence of a letter $c \in \Sigma$ inside $\overleftarrow{\eth^{a}(u)}$ and cIa, we can do

$$\cdots c \cdots \tilde{a} \cdots \simeq_{\mathcal{LSRB}} \cdots c \cdots \tilde{a}c \cdots \equiv \cdots c \cdots c\tilde{a} \cdots \simeq_{\mathcal{LSRB}} \cdots c\tilde{a} \cdots . \tag{11}$$

Using (11) many times (also for b), we modify u into

$$\cdots \pi_{D(a)}\left(\overleftarrow{\mathsf{0}^a(u)}\right) \pi_{I(a)}\left(\overleftarrow{\mathsf{0}^a(u)}\right) \tilde{a} \pi_{D(b)}\left(\overleftarrow{\mathsf{0}^b(u)}\right) \pi_{I(b)}\left(\overleftarrow{\mathsf{0}^b(u)}\right) \tilde{b} \cdots .$$

By the definition of I_u, $\pi_{D(a)}\left(\overleftarrow{\mathsf{0}^a(u)}\right)$ and $\pi_{D(b)}\left(\overleftarrow{\mathsf{0}^b(u)}\right)$ can be rewritten using $\equiv$ to reach $\pi_{D(a)}\left(\pi_{D(b)}\left(\overleftarrow{\mathsf{0}^b(u)}\right)\right) = \pi_{D(b)}\left(\pi_{D(a)}\left(\overleftarrow{\mathsf{0}^a(u)}\right)\right)$. Finally, we perform (11) in the reverse direction to insert letters from $\pi_{I(a)}\left(\overleftarrow{\mathsf{0}^a(u)}\right)$ into $\pi_{D(a)}\left(\overleftarrow{\mathsf{0}^a(u)}\right)$ and from $\pi_{I(b)}\left(\overleftarrow{\mathsf{0}^b(u)}\right)$ into $\pi_{D(b)}\left(\overleftarrow{\mathsf{0}^b(u)}\right)$ in order to gain $\overleftarrow{\mathsf{0}^b(u)} = \overleftarrow{\mathsf{0}^a(u)}a$. Then

$$\cdots \overleftarrow{\mathsf{0}^a(u)}\tilde{a}\overleftarrow{\mathsf{0}^a(u)}a\tilde{b} \cdots \simeq_{\mathcal{B}} \cdots \overleftarrow{\mathsf{0}^a(u)}\tilde{a}\tilde{b} \cdots \equiv \cdots \overleftarrow{\mathsf{0}^a(u)}\tilde{b}\tilde{a} \cdots$$

and the first occurrences of a and b are exchanged.

Now it remains to reach equalities $\overleftarrow{\mathsf{0}^a(u)} = \overleftarrow{\mathsf{0}^a(v)}$ and $\overleftarrow{u} = \overleftarrow{v}$. But due to the second condition of the proposition, it is enough to move all letters from $\mathsf{c}(\mathsf{0}^a(u)) \cap I(a)$ using (11) to the end of $\mathsf{0}^a(u)$ ($\mathsf{0}^a(v)$ respectively) in the same order in u and v. $\square$

4. Equations

In this section we give a description of the equations solvable in trace semigroups in the varieties $\mathcal{LSNB}$ and $\mathcal{LSRB}$. Since the problem of solvability of equations without variables is just the word problem discussed in the previous section and the case of equations with variables on one side only was solved in Lemma 3, it remains to deal with equations having variables on both sides. For this purpose, we take $u, v \in (\Sigma \cup X)^+$ such that $\mathsf{c}(u) \cap X \neq \emptyset \neq \mathsf{c}(v) \cap X$.

First, let us suppose that $u = v$ has a solution in $\mathbb{T}(\Sigma, D, \mathcal{U})$, where $\mathcal{U}$ is a variety of bands. After substituting a solution into the equation, we get $u_0 u' u_1 \sim_{\mathcal{U}} v_0 v' v_1$ for some $u', v' \in \Sigma^+$. Then

$$\begin{aligned} u_0 v_0 v' v_1 u_1 &\sim_{\mathcal{U}} u_0 u_0 u' u_1 u_1 \simeq_{\mathcal{B}} u_0 u' u_1 \sim_{\mathcal{U}} \\ &\sim_{\mathcal{U}} v_0 v' v_1 \simeq_{\mathcal{B}} v_0 v_0 v' v_1 v_1 \sim_{\mathcal{U}} v_0 u_0 u' u_1 v_1 . \end{aligned} \tag{12}$$

This time we start by considering the bigger variety $\mathcal{LSRB}$.

Proposition 3. *Let $u, v \in (\Sigma \cup X)^+$, $\mathsf{c}(u) \cap X \neq \emptyset \neq \mathsf{c}(v) \cap X$. Then $u = v$ is solvable in $\mathbb{T}(\Sigma, D, \mathcal{LSRB})$ if and only if $u_0 v_0 \sim_{\mathcal{LSRB}} v_0 u_0 v_0$ and $\overleftarrow{u_1 v_1} \equiv \overleftarrow{v_1 u_1}$.*

Proof. "$\Longrightarrow$" We get $u_0 v_0 \sim_{\mathcal{LSRB}} v_0 u_0 v_0$ by verifying the conditions of Proposition 2 for these two words. To obtain the second and third conditions of Proposition 2 for $u_0 v_0$ and $v_0 u_0 v_0$ and the last condition of Proposition 3, it is sufficient to apply Proposition 2 to (12).

"$\Longleftarrow$" Let w be a product of all letters from $(\mathsf{c}(u) \cup \mathsf{c}(v)) \cap \Sigma$. Then we have the following solution of $u = v$

$$u[u_0v_0wu_1v_1] \simeq_{\mathcal{B}} u_0v_0wu_1v_1u_1 \sim_{\mathcal{LSRB}} v_0u_0v_0wu_1u_1v_1 \simeq_{\mathcal{B}} v[u_0v_0wu_1v_1],$$

where the middle relation follows from the assumptions by Proposition 2. □

From the "if" part of the previous proof it follows that if there is a solution of the equation in $\mathbb{T}(\Sigma, D, \mathcal{LSRB})$, then there is also a solution using just the letters contained already in u and v. But this is not true for the variety $\mathcal{LSNB}$ as shown by the next example.

Example 2. Let $\Sigma = \{a, b, c, d\}$, $D = (\Sigma \times \Sigma) \setminus \{(a,d), (d,a), (c,d), (d,c)\}$. It is easy to see that the equation $abcx = abacx$ has no solution over $\{a, b, c\}$ since this set generates a free structure in $\mathcal{LSNB}$. In spite of that, there is a solution $\varphi(x) = d$.

The reason for the difference is that for the variety $\mathcal{LSRB}$ all conditions of Proposition 2 are evaluated locally, but in the case of $\mathcal{LSNB}$ we can alter the validity of the condition (5) by adding some letters to an arbitrary place in the entire word. As we shall see, if this can be done in the right way to get a solution, then it can be done even by only one letter. The following definition will be used to determine the values of i for which $T_u(i)$ and $T_v(i)$ should be changed.

Definition 3. Let $s, t \in \Sigma^*$ be such that $\overrightarrow{st} \equiv \overrightarrow{ts}$. We define the set of natural numbers

$$B_{s,t} = \{i \in \{1, \dots, |\mathsf{c}(st)| - 1\} \mid T_{st}(i) \neq T_{ts}(i)\}.$$

Remark 4. Under the assumptions of the definition, if $T_{st}(i) \neq T_{ts}(i)$, then by Remark 1 i satisfies (5) for both st and ts and $T_{st}(i) \neq \emptyset \neq T_{ts}(i)$.

Proposition 4. *Let $u, v \in (\Sigma \cup X)^+$, $\mathsf{c}(u) \cap X \neq \emptyset \neq \mathsf{c}(v) \cap X$. Then $u = v$ is solvable in $\mathbb{T}(\Sigma, D, \mathcal{LSNB})$ if and only if the following conditions are satisfied.*

1. $\overrightarrow{u_0v_0} \equiv \overrightarrow{v_0u_0}$.
2. *If u_1, v_1 are both non-empty, then $\mathsf{t}(u)$ and $\mathsf{t}(v)$ lie in the same connected component of I in Σ.*
3. *If $B_{u_0,v_0} \neq \emptyset$, then there exist $a \in \Sigma$ and $b \in \mathsf{c}\left(0^{\max B_{u_0,v_0}}(\overrightarrow{u_0v_0})\right)$ such that aIb and $a\, I\, \mathsf{c}\left(1^{|\mathsf{c}(u_0v_0)| - \max B_{u_0,v_0}}(\overrightarrow{u_0v_0})\right)$.*

Proof. "$\Longrightarrow$" The conditions 1, 2 follow immediately from (12) and Proposition 1. If $B_{u_0,v_0} \neq \emptyset$, let us denote $j = \max B_{u_0,v_0} + |\mathsf{c}(u_0v_0v'v_1u_1) \setminus \mathsf{c}(u_0v_0)|$. If j satisfies (5) for the first word in (12), then using Proposition 1 we get the validity of (5) also for the last word in (12). Therefore

$$T_{u_0v_0v'v_1u_1}(j) = T_{u_0v_0}(\max B_{u_0,v_0}) \neq T_{v_0u_0}(\max B_{u_0,v_0}) = T_{v_0u_0u'u_1v_1}(j),$$

which is a contradiction with Proposition 1 in the case of the equivalence (12).

Thus (5) is not valid and there are $a \in \operatorname{init}\left(1^{|\mathsf{c}(u_0v_0)|-\max B_{u_0,v_0}}\left(\overrightarrow{u_0v_0v'v_1u_1}\right)\right)$ and $b \in \mathsf{c}\left(0^j\left(\overrightarrow{u_0v_0v'v_1u_1}\right)\right)$ such that aIb. As $\max B_{u_0,v_0}$ satisfies (5) for the word u_0v_0, $a \notin \mathsf{c}(u_0v_0)$ and the rest is clear.

"$\Longleftarrow$" If $B_{u_0,v_0} \neq \emptyset$, define w to be a product of all letters from the set $(\mathsf{c}(u) \cup \mathsf{c}(v)) \cap \Sigma$. Further, define w' to be a product of all letters on some path from $\mathsf{t}(u)$ to $\mathsf{t}(v)$ in the graph of I if u_1, v_1 are non-empty, else consider w' to be the empty word. Then by Proposition 1 and the condition 2

$$\begin{aligned} u\left[u_0v_0aww'u_1v_1\right] &\simeq_{\mathcal{B}} u_0v_0aww'u_1v_1u_1 \sim_{\mathcal{LSNB}} u_0v_0aww'u_1v_1, \\ v\left[u_0v_0aww'u_1v_1\right] &\simeq_{\mathcal{B}} v_0u_0v_0aww'u_1v_1. \end{aligned}$$

Denote the last words on these lines r and s. The final task is to show (using Proposition 1) that $r \sim_{\mathcal{LSNB}} s$, since then the previous substitution is a required solution. From the condition 1 we get $\overrightarrow{r} \equiv \overrightarrow{s}$. Let $j = |\mathsf{c}(r) \setminus \mathsf{c}(u_0v_0)|$. $T_r(i) = T_s(i)$ for $i < j$ is clear. From 3 it can be derived $a \notin \mathsf{c}(u_0v_0)$ and using 3 again one gets that (5) is not valid for $j \leq i \leq j + \max B_{u_0,v_0}$, thus $T_r(i) = \emptyset = T_s(i)$. $T_r(i) = T_s(i)$ for $i > j + \max B_{u_0,v_0}$ follows from $T_{u_0v_0}(i-j) = T_{v_0u_0}(i-j)$ and $\overrightarrow{r} \equiv \overrightarrow{s}$ by Remark 1.

If $B_{u_0,v_0} = \emptyset$, the proof is similar but much simpler. □

5. Conclusion

Our main task was not only to give a transparent description but also such that would yield an algorithm for verification polynomial in the size of its input, i.e. in the length of input words and in the number of letters and variables. So, consider Σ finite and denote by k the length of the word uv, $l = |\mathsf{c}(uv)|$ and $m = |\Sigma|$. Then we can use standard algorithms for working with traces (for the condition in Definition 2, it is sufficient, for instance, to modify the algorithm for computing lexicographical normal forms) to gain the following upper estimations for the time complexity of our algorithms.

The word problem for $\mathcal{LSNB}$:	$O\left(k + l^3\right)$
The word problem for $\mathcal{LSRB}$:	$O\left(l\left(k + l^3\right)\right)$
The solvability of equations for $\mathcal{LSNB}$:	$O\left(k + l^3 + m^2\right)$
The solvability of equations for $\mathcal{LSRB}$:	$O\left(l\left(k + l^3\right)\right)$

When we look higher in the lattice of band varieties, the behaviour of commutations with respect to invariants becomes much more complex threatening the possibility of getting invariant-based solutions of the word problem. Unfortunately, we still do not know about any other semigroup variety where this problem is not trivial but has a transparent solution.

References

[1] P. Cartier, D. Foata. *Problèmes combinatoires de commutation et réarrangements*. Lecture Notes in Mathematics 85, Springer-Verlag, Berlin, 1969.

[2] V. Diekert, G. Rozenberg, editors. *The Book of Traces*. World Scientific, Singapore, 1995.

[3] Ch. Fennemore. All varieties of bands. *Semigroup Forum*, **1**:172–179, 1970.

[4] A. Mazurkiewicz. Concurrent program schemes and their interpretations. DAIMI Report PB 78, Aarhus University, Aarhus, 1977.

[5] L. Polák. On varieties of completely regular semigroups II. *Semigroup Forum*, **36**:253–284, 1987.

[6] L. Polák. On varieties of completely regular semigroups III. *Semigroup Forum*, **37**:1–30, 1988.

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