Algebra III

Category Theory, Universal Algebra, and Modules

2026

Abstract

This text is based on the Algebra III lectures and accompanying lecture notes of John Bourke during the Winter Semester of the 2025/2026 academic year at Masaryk University. The text was written by Ondřej Skotnica, a student in the class.

Contents

Chapter 1

Category Theory

1.1 Motivation

In math Category Theory studies structures like sets, groups, rings, topological spaces. Each forms a category of "structures of that type". Category theory studies relations between these different areas of mathematics (or categories of mathematics).

$$
\begin{tikzcd}
\text{Sets} \arrow[d, shift left, "\text{free group on a set}"] \arrow[dr, bend left, "\text{polynomial ring}"] & & \text{Pointed spaces} \arrow[r, "\text{forgets}"] \arrow[ddll, dashed, "\text{fundamental group}"] & \text{Spaces} \arrow[dl, "\text{singular complex}"] \\
 & & \text{Chain complexes} & \\
\text{Groups} \arrow[u, shift left, "\text{forgets group structure}"] \arrow[r, shift left] & \text{Rings} \arrow[l, shift left, "\text{forgets}"] & &
\end{tikzcd}
$$

What do these different categories have in common? What structure is preserved when we move between them? The fundamental notion in Category Theory is an *arrow/morphism*:

$$A \longrightarrow B$$

which captures the relationship between two things.

1.2 Categories

Definition 1.1. A category $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}(\mathcal{C})$, $A \in \mathrm{Ob}(\mathcal{C})$ is called an object
- $\forall A, B \in \mathrm{Ob}(\mathcal{C})$a class $\mathcal{C}(A, B)$ (also alled $\mathrm{Hom}_{\mathcal{C}}(A, B)$) of morphisms from A to B, $f \in \underbrace{\mathcal{C}(A, B)}$ is a morphism $f\colon \underbrace{A}_{\text{domain}} \longrightarrow \underbrace{B}_{\text{codomain}}$
- $\forall A, B, C \in \mathrm{Ob}(\mathcal{C})$, a function $\circ\colon \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$ defined by $A \xrightarrow{f} B \xrightarrow{g} C \mapsto A \xrightarrow{gf} C$ called composition.
- $\forall A \in \mathrm{Ob}(\mathcal{C})$, a morphism$1_A \in \mathcal{C}(A, A)$ called the identity on A.
- These satisfy following:

– **Associativity:** $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h \in \mathcal{C}(C,D) \Rightarrow$ $h \circ (g \circ f) = (h \circ g) \circ f$,

– **Left & right unit laws:** $f \in \mathcal{C}(A,B) \Rightarrow 1_B \circ f = f = f \circ 1_A$.

Notation. We often write $gf\colon A \longrightarrow C$ instead of $g \circ f$.

Associativity and unit laws imply that given

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} A_n$$

there is a unique way to compose the arrows $f_n, f_{n-1}, \ldots, f_2, f_1$ independently of the brackets and identities e.g. for $n = 4$

$$((f_4 f_3) f_2) f_1 = (f_4 1_{A_3})((f_3 f_2) f_1).$$

Given a diagram such as

$$\begin{tikzcd}
A \arrow[rr, "f"'] \arrow[d, "h"'] & & B \arrow[d, "g"] \\
C \arrow[r, "i"] & D \arrow[r, "j"] & E
\end{tikzcd}$$

we say the *diagram commutes* if the paths agree, in this case $gf = jih$. In a larger diagram e.g.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "h"'] \arrow[drr, phantom, "="] & & B \arrow[r, "k"] \arrow[d, "g"] \arrow[dr, phantom, "="] & F \arrow[d, "l"] \\
C \arrow[r, "i"'] & D \arrow[r, "j"'] & E \arrow[r, "m"'] & G
\end{tikzcd}$$

the commutativity of subdiagrams implies the commutativity of the outer diagram. This is called *diagram chasing.*

Example 1.1. Category $\mathcal{S}et$, where objects are sets and morphisms $f\colon A \to B$ are set functions. Given $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ the function $gf\colon A \to C$ is defined by $gf(x) = g(f(x))$. As of identity laws we have $1_A(x) = x$.

Example 1.2. $\mathcal{M}on$, the cat of monoids and monoid homomorphisms:

$$\begin{aligned}
&f\colon (A, m_A, e_A) \longrightarrow (B, m_B, e_B),\\
&f(e_A) = e_B,\\
&f\left(m_A(x,y)\right) = m_B(fx, fy)
\end{aligned}$$

Example 1.3. $\mathcal{G}rp$, the category of groups and group homomorphisms.

Example 1.4. $\mathcal{R}ng$, the category of rings and ring homomorphisms.

Example 1.5. K-$\mathcal{V}ect$, the category of K-vector spaces and linear transformations

These are examples of *algebraic categories.* More generally, given (Ω, E) where Ω is signature and E set of equations we can consider the cat (Ω, E)-$\mathcal{A}\ell g$ of (Ω, E)-algebras. This captures all of the above examples. We will return to this example in the second part of this course.

Example 1.6. $\mathcal{T}op$ is the category of topological spaces and continuous functions.

Definition 1.2. A morphism $f\colon A \longrightarrow B \in \mathcal{C}$ is an *isomorphism/iso* if

$$\begin{aligned}&\exists g\colon B \longrightarrow A \in \mathcal{C} \quad \text{s.t.}\\ &gf = 1_A \quad \& \quad fg = 1_B.\end{aligned}$$

Remark. We can express this via the diagram.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \arrow[dr, "1_B"] & \\
& A \arrow[r] & B
\end{tikzcd}$$

We say that g is the inverse of f and write $g = f^{-1}$. This notation is justified by following propositions.

Proposition 1.1. *Let $\mathcal{C}$ be cat and $A, B \in \mathrm{Ob}(\mathcal{C})$, if $f\colon A \longrightarrow B$ is an isomorphism, then its inverse is unique.*

Proof. Suppose $g, h\colon B \longrightarrow A$ are inverses to f.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "1_B"'] \arrow[drr, bend right=60, "h"'] & A \arrow[d, "f"] \arrow[dr, "1_A"] \arrow[dr, bend left=60, "g"] & \\
& B \arrow[r, "h"'] & A
\end{tikzcd}$$

Therefore $g = h$.

The corresponding proof in algebraic notation is

$$g = 1_A \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ 1_B = h.$$

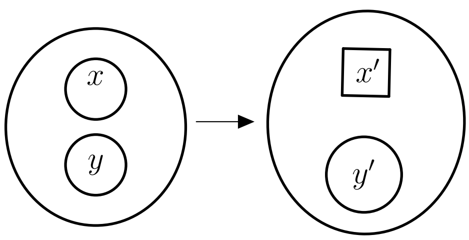
□

Example 1.7. In $\mathcal{S}et$, the isomorphism $f\colon A \longrightarrow B$ are the *bijections.*

In other algebraic cats such as $\mathcal{G}rp$, an isomorphism is sometimes defined as a bijective homomorphism. This coincides with the categorical notion, since if a homomorphism is bijective its inverse (as a function) is also a homomorphism.

Example 1.8. For $\mathcal{T}op$ an isomorphism is a continuous function with a continuous inverse. In $\mathcal{T}op$ exists continuous maps which are not homeomorphisms e.g.

open sets



one not open set
one open set

In summary, the categorical definition of isomorphism captures the correct notion in all of our examples.

Definition 1.3. A category $\mathcal{C}$ is said to be *locally small* if for all $A, B \in \mathcal{C}$ the class $\mathcal{C}(A, B)$ is a *set*.
If, in addition, the class $\mathrm{Ob}(\mathcal{C})$ is a *set*, we say it is *small*.

Example 1.9. $\mathcal{S}et$ is not small, one cannot form "set of all sets". However, it is locally small as the collection $\mathcal{S}et(A, B)$ of functions from A to B forms a set.

Example 1.10. All of the examples considered so far are locally small, but not small.

Example 1.11. $\emptyset$, the Empty category is a small category, since the class of objects is the empty set and the class of morphisms is the empty set.

Example 1.12. $\mathbf{1}$, the Terminal (or Unit) category is category with exactly one object and its obligatory identity morphism, thus it is a small category.

Example 1.13. A category with two objects and one morphism between them. Sometimes denoted as category $\mathbf{2}$.

- $\mathrm{Ob}(\mathbf{2}) = \{0, 1\}$,
- $\mathrm{Hom}_{\mathbf{2}}(0, 1) = \{f\}$ where $f\colon 0 \longrightarrow 1$.

Example 1.14. A preordered set $(X, \leq)$ is a set equipped with binary relation $\leq$ that satisfies the following two properties for all $x, y, z \in X$:

- **Reflexivity:** $x \leq x$,
- **Transitivity:** $x \leq y \;\&\; y \leq z \Rightarrow x \leq z$.

If the relation also satisfies

- **Antisymmetry:** $x \leq y \;\&\; y \leq x \Rightarrow y = x$,

Then it is called partially ordered set or poset. From any preorder $(X, \leq)$ we can define a small category $\mathcal{X}^*$ whose objects are elements of X such that there exists a single morphism $x \to y \iff x \leq y$ and no morphisms from x to y otherwise.

Example 1.15. If $(M, \underbrace{\times}_{mult.}, \underbrace{e}_{unit})$ is a monoid we can form a small category $\sum M$ with one object $*$ and whose morphisms $* \xrightarrow{m} *$ are the elements of M. We compose them as follows:

$$\begin{tikzcd}
* \arrow[r, "m"] \arrow[dr, "n\times m"'] & * \arrow[d, "n"] \\
& *
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r, "e", "\text{unit is } 1_*"'] & *
\end{tikzcd}$$

We can get axioms for a category easily from associativity and unit laws for monoids. If, furthermore, $(M, \times, e)$ is a group (all elements have inverse) then $\sum M$ is a *groupoid*: a category in which all morphisms are isomorphisms.

$$\begin{array}{ccc}
\text{Monoids} & \equiv & \text{1-object small categories} \\
\cup\!| & & \cup\!| \\
\text{Groups} & \equiv & \text{1-object small groupoids.}
\end{array}$$

1.3 Functors

Definition 1.4. Let $\mathcal{A}, \mathcal{B}$ be categories. A *functor* $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

- a function $\mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ which we write as $X \longmapsto FX$,
- for each pair $X, Y \in \mathcal{A}$ a function $\mathcal{A}(X,Y) \longrightarrow \mathcal{B}(FX, FY)$ written as $X \xrightarrow{\alpha} Y \longmapsto FX \xrightarrow{F\alpha} FY$.

This satisfies the axioms:

- given $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \in \mathcal{A}$, we have $F(\beta \circ \alpha) = F\beta \circ F\alpha$.
- given $X \in \mathcal{A}$ we have $F(1_X) = 1_{FX}$.

In other words, a functor sends objects to objects, arrows to arrows (respecting their domain & codomain) & preserves identities and composition.

Example 1.16. Forgetful functors: e.g.

$$U\colon \mathcal{G}rp \longrightarrow \mathcal{S}et$$
$$(X, \underbrace{m}_{mult.}, \underbrace{e}_{unit}) \longmapsto X$$

which sends a group to its underlying set, i.e. forgets the group structure. Similarly it sends a group homomorphism to its underlying function:
$f(X, m_x, e_x) \to (Y, m_y, e_y) \longmapsto f\colon X \to Y$
It clearly preserves composition & identities.

Similarly, there are forgetful functors from cats "sets with structure" to $\mathcal{S}et$ such as $U\colon \mathcal{T}op \to \mathcal{S}et$ or $\mathcal{R}ng, \mathcal{M}on$ etc. Similarly, $U\colon \mathcal{R}ng \to \mathcal{M}on$ which sends $(R, +, \times, 0, 1) \mapsto (R, \times, 1)$. But more of it later.

Example 1.17. $\mathcal{CAT}$ Given functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ we can compose them in obvious way:

$$X \longmapsto G(F(X)) = GF(X)$$
$$X \xrightarrow{\alpha} Y \longmapsto GF(X) \xrightarrow{GF\alpha} GF(Y)$$

Likewise we have identity functor $1_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}\colon\ X \mapsto X$. In this way, we obtain a (large) category $\mathcal{CAT}$ of categories and functors. In fact this is a **2**-*category*.

1.4 Limits & Colimits

Definition 1.5. An object $T \in \mathcal{C}$ is *Terminal* if given any object $X \in \mathcal{C}$ there exists a unique morphism $f\colon X \longrightarrow T$.

Theorem 1.1. *Terminal objects are unique up to isomorphism.*

Proof. Let S, T be Terminal objects. As T is Terminal, there exists unique $S \xrightarrow{f} T$. As S is Terminal, there exists unique $T \xrightarrow{g} S$. As T is Terminal, the two maps

$$T \underset{1_T}{\overset{fg}{\rightrightarrows}} T \qquad \text{are equal.}$$

As S is Terminal, similarly $gf = 1_S$, so that $f\colon S \to T$ is an iso. $\square$

Example 1.18. In $\mathscr{Set}$, a set is Terminal precisely if it has one element, e.g $\{*\}$.

Similarly in $\mathscr{Grp}$, $\mathscr{Mon}$, $\mathscr{Rng}$, $\mathscr{Top}$ etc. The 1-element set has the unique structure or a group, monoid, ring, topological space etc. This makes it the Terminal object in the corresponding category.

Definition 1.6. An object $I \in \mathcal{C}$ is *initial* if given any other $X \in \mathcal{C}$ there exists a unique morphism $I \xrightarrow{f} X$.

Example 1.19. In $\mathscr{Set}$, the initial object is the empty set $\emptyset$ since $\exists! f\colon \emptyset \longrightarrow X$.

Definition of initial object is *dual* to that of Terminal object. All arrows in definition are reversed. One can prove that initial objects are unique up to iso by reversing arrows in proof for Terminal objects. Another approach would be the use of a dual/opposite category $\mathcal{C}^{\text{op}}$.

Definition 1.7. Dual category $\mathcal{C}^{\text{op}}$ to category $\mathcal{C}$ has the same objects as $\mathcal{C}$ and for all objects X, Y: $\mathcal{C}^{\text{op}}(X, Y) = \mathcal{C}(Y, X)$.

comp in $\mathcal{C}$:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

comp in $\mathcal{C}^{\text{op}}$:
$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ & Z \arrow[u, "g"'] \arrow[ul, "f\circ g"] \end{tikzcd}$$

Identities in $\mathcal{C}^{\text{op}}$ are as in $\mathcal{C}$

An object $I \in \mathcal{C}$ is initial if it is Terminal in $\mathcal{C}^{\text{op}}$. Using this we see that uniqueness of initial objects follows (indeed is equivalent to) uniqueness of Terminal objects.

Theorem 1.2. *Initial objects are unique up to unique isomorphism.*

Proof. Let I_1, I_2 be initial in $\mathcal{C}$. So terminal in $\mathcal{C}^{\text{op}}$. Hence $\exists!$ iso $I_1 \to I_2 \in \mathcal{C}^{\text{op}}$ i.e. unique iso in $\mathcal{C}$. $\square$

Every categorical definition, theorem and proof has a *dual* obtained by reversing the arrows. More formally, this is obtained by interpreting the original definition, theorem or proof in $\mathcal{C}^{\text{op}}$ instead of $\mathcal{C}$.

Definition 1.8. Let $A, B \in \mathcal{C}$. We say that a diagram

$$\begin{tikzcd} & A \\ P \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & B \end{tikzcd}$$

is a *product* diagram if it has the *universal property (u.p.)* that, given any other diagram

$$\begin{tikzcd} & A \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$$

$\exists!\ X \xrightarrow{h} P$ s.t. the following diagram commutes.

$$\begin{tikzcd} & & A \\ X \arrow[urr, bend left, "f"] \arrow[r, "h"] \arrow[drr, bend right, "g"'] & P \arrow[ur, "p"] \arrow[dr, "q"] & \\ & & B \end{tikzcd}$$

In this case we call the object P the product of A and B.

Exercise. The product of A and B is *unique* up to *unique isomorphism* (similar to the case of Terminal objects.) For this reason, we often write

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & B \end{tikzcd}$$

and call it "the product".

Example 1.20. In $\mathcal{S}et$, the product is the *cartesian product* $A \times B = \{(a,b) \colon a \in A, b \in B\}$ with

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & B \end{tikzcd} \qquad \begin{tikzcd} & a \\ (a,b) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & b \end{tikzcd}$$

Given $\begin{tikzcd} & A \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$ if $\begin{tikzcd} & & A \\ X \arrow[urr, bend left, "f"] \arrow[r, "h"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "p"'] \arrow[dr, "q"] & \\ & & B \end{tikzcd}$

Then $phx = fx$ and $qhx = gx$ so $hx = (fx, gx)$. In particular, $h \colon X \to A \times B \colon x \mapsto (fx, gx)$ is the unique function s.t. $ph = f$ and $qh = g$.

Similarly, in $\mathcal{G}rp$, $\mathcal{R}ng$ etc. products are the usual cartesian products. The dual notion is that of *coproduct*. Coproducts in $\mathcal{C}$ are products in $\mathcal{C}^{op}$. In elementary terms, the coproduct of A and B is a diagram

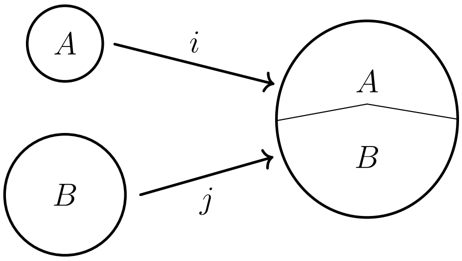
$$\begin{tikzcd} A \arrow[dr, "i"] & \\ & C \\ B \arrow[ur, "j"'] & \end{tikzcd}$$ with the *universal property* that given

$$\begin{tikzcd} A \arrow[dr, "f"] & \\ & X \\ B \arrow[ur, "g"'] & \end{tikzcd} \qquad \exists! C \xrightarrow{h} X \text{ s.t.} \qquad \begin{tikzcd} A \arrow[dr, "i"'] \arrow[drr, bend left, "f"] & & \\ & C \arrow[r, "h"'] & X. \\ B \arrow[ur, "j"] \arrow[urr, bend right, "q"'] & & \end{tikzcd}$$

Again, the coproduct is unique up to unique isomorphism. We often write it as

$$\begin{tikzcd} A \arrow[dr, "i"] & \\ & A+B \\ B \arrow[ur, "j"'] & \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} A \arrow[dr, "i"] & \\ & A \cup B. \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

Example 1.21. In $\mathcal{S}et$, the coproduct is the disjoint union



$A+B = \{(a,1) \colon a \in A\} \cup \{(b,2) \colon b \in B\}$
or various other ways.

In $\mathcal{G}rp$, the coproduct is the *free product* if elements of $G+H$ are "alternating words" in G and H e.g. $g_1 h_1 g_2 h_2 g_3 ...$

Exercise. Coproducts and products are dual concepts. Thinking about $\mathcal{S}et$, can you see a sense in which addition and multiplication of numbers are dual?

Exercise. What are coproducts of vector spaces, abelian groups or top. spaces?

Definition 1.9. Let $A \underset{g}{\overset{f}{\rightrightarrows}} B \in \mathcal{C}$. The *equaliser* comes equipped with an arrow $i \colon E \to A$ such that $E \xrightarrow{i} A \underset{g}{\overset{f}{\rightrightarrows}} B$ commutes (i.e. $fi = gi$). It has the *universal property* that given $X \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that $fh = gh$ then

$$\exists! X \xrightarrow{\bar{h}} E \qquad \text{such that} \qquad \begin{tikzcd} X \arrow[dr, "h"] \arrow[dd, "\bar{h}"'] & \\ \arrow[r, phantom, "//"] & A. \\ E \arrow[ur, "i"'] & \end{tikzcd}$$

As usual, equalisers are unique up to isomorphism.

Example 1.22. In $\mathcal{S}et$, the equaliser is the subset

$$\begin{tikzcd} E = \{x \in A \colon fx = gx\} \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B. \\ X \arrow[u, "\bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

If $fhx = ghx$ then $hx \in E$, so $\bar{h}x = hx$ gives factorization.

In the other *algebraic categories* such as $\mathcal{Grp}$, $\mathcal{Rng}$ etc. equalisers are constructed as in $\mathcal{Set}$. However, in $\mathcal{Grp}$ a special case is quite illuminating.

Example 1.23. In $\mathcal{Grp}$, for each pair of groups G, H we have $G \xrightarrow{0} H\colon x \mapsto 0_H$ sending all elements to zero (i.e. the unit element of H.) Then the equaliser of

$$E = \{x \in G\colon fx = 0\} \xhookrightarrow{\;i\;} A \underset{0}{\overset{f}{\rightrightarrows}} B$$

is the *kernel* of f. Thus *kernels* of group homomorphisms are special cases of *equalisers.*

Coequalisers in $\mathcal{C}$ are equalisers in $\mathcal{C}^{\mathrm{op}}$ in elementary terms, the coequaliser of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an object equipped with a morphism $B \to C$ s.t.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r] \arrow[dr, "h"'] & C \arrow[d, dashed, "\exists!\bar{h}"] \\
 & & D
\end{tikzcd} \qquad \text{commutes.}$$

Coequalisers capture *quotients* and *quotients* in algebraic type of categories are hard to describe explicitly, but we will mention some cases.

Example 1.24. Given an equivalence relation E on a set X, we can view it as a subset

$E = \{(x, y)\colon xEy\} \subseteq X \times X$ and then have $E \underset{t}{\overset{s}{\rightrightarrows}} X\colon (x, y) \underset{t}{\overset{s}{\rightrightarrows}} \begin{matrix} x \\ y \end{matrix}$ in $\mathcal{Set}$.

$E \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{\;p\;} C$ is a set C with property that if xEy then $px = py$ and is the universal such. Indeed, it is $X \xrightarrow{p} \underbrace{X/E}_{\substack{\text{set of} \\ E \text{ classes}}} : x \mapsto \underbrace{[x]_E}_{\substack{\text{equivalence} \\ \text{classes of } x}}$.

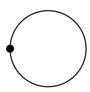
Exercise. Check details of this example.

Example 1.25. Given a normal subgroup $H \leq G$ the coequaliser of $H \underset{0}{\overset{i}{\rightrightarrows}} G$

is by definition the universal $G \to ?$ sending all elements of H to 0. Indeed, it is the quotient $G \xrightarrow{p} G/H\colon g \mapsto gH$ by the normal subgroup: given $G \xrightarrow{f} A$ such that $fx = 0$ $\forall x \in H$, $\exists! G/H \xrightarrow{\bar{f}} A$ such that

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "p"'] & {} \arrow[d, phantom, "||"] & A \\
 & G/H \arrow[ur, "\bar{f}"'] &
\end{tikzcd} \qquad \text{i.e. } \bar{f}(gH) = f(g)$$

Example 1.26. In Topolgy, coequalisers capture quotient spaces for instance, the *circle* is obtained by *gluing together* the two endpoints of the interval $[0, 1]$, it is the coequaliser

$$* \underset{1}{\overset{0}{\rightrightarrows}} [0, 1] \twoheadrightarrow \quad 0 = 1$$ 

Definition 1.10. Let $\mathcal{J}$ be a small cat and $\mathcal{C}$ a cat. A functor $D\colon \mathcal{J} \to \mathcal{C}$ is called a *diagram of shape* $\mathcal{J}$ *in* $\mathcal{C}$.

Example 1.27. $\mathcal{J} = \boxed{0 \underset{v}{\overset{u}{\rightrightarrows}} 1}$ A diagram $D\colon \mathcal{J} \to \mathcal{C}$ is specified by morphism $A \underset{g}{\overset{f}{\rightrightarrows}} B$ (i.e.$D0 = A, D1 = B, Du = f, Dv = g$).

Definition 1.11.
- Given a diagram $D\colon \mathcal{J} \to \mathcal{C}$ a *cone* over D is an object $A \in \mathcal{C}$ together with morphism $A \xrightarrow{fj} Dj$ for each $j \in \mathcal{J}$ such that for all $j \xrightarrow{\alpha} i \in \mathcal{J}$ the triangle

$$\begin{tikzcd} & Di \arrow[dd, "D\alpha"'] \\ A \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\ & Dj \end{tikzcd}$$

commutes.

- A *limit* of D is a *cone* $\left(L \xrightarrow{p_j} Dj \colon j \in \mathcal{J}\right)$ with the universal property that given any other cone $\left(A \xrightarrow{f_j} Dj, j \in \mathcal{J}\right)$ there exists a unique $A \xrightarrow{k} L$ such that

$$\begin{tikzcd} A \arrow[d, "k"'] \arrow[dr, "f_j"] & \\ L \arrow[r, "p_j"'] & Dj \end{tikzcd}$$

commutes for all $j \in \mathcal{J}$

Example 1.28. $\mathcal{J} = \boxed{0 \underset{v}{\overset{u}{\rightrightarrows}} 1}$ A diagram $\mathcal{J} \xrightarrow{D} \mathcal{C}$ is a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ and cone consists of maps $X \xrightarrow{p_0} A$, $X \xrightarrow{p_1} B$ such that

$$\begin{tikzcd} & A \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\ X \arrow[ur, "p_0"] \arrow[dr, "p_1"'] & \\ & B \end{tikzcd}$$

commutes

or, equivalently, a single map $X \xrightarrow{p_0} A$ such that $fp_0 = gp_0$ since p_1 is forced to be this composite. In this way we see that the limit of D is precisely the equaliser of f and g.

Example 1.29. $\mathcal{J} = \boxed{0 \qquad\quad 1}$ $\mathcal{J}$-shaped limits are products.

Example 1.30. $\mathcal{J} = \boxed{}$ The empty cat. $\mathcal{J}$-shaped limits are Terminal objects.

Definition 1.12. The *colimit* of a diagram $D\colon \mathcal{J} \to \mathcal{C}$ is a *cocone*.

$$\begin{tikzcd} Di \arrow[dr, "p_i"] \arrow[dd, "D\alpha"'] & \\ \arrow[r, phantom, "//"] & X \\ Dj \arrow[ur, "p_j"'] & \end{tikzcd} \qquad \text{with the dual u.p.}$$

equivalently, the limit of $D^{\mathrm{op}}\colon \mathcal{J}^{\mathrm{op}} \to \mathcal{C}^{op}$. Details are left to the reader.

Example 1.31. *Pullbacks* are $\mathcal{J}$-limits for

$$\mathcal{J} = \boxed{\begin{tikzcd} & 0 \arrow[d] \\ 2 \arrow[r] & 1 \end{tikzcd}} \qquad \text{i.e. given} \quad \begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

its pullback P comes with a commuting square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "//"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \qquad \text{such that given} \qquad \begin{tikzcd} D \arrow[r, "r"] \arrow[d, "s"'] \arrow[dr, phantom, "//"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

$\exists!\ D \xrightarrow{t} P$ s.t. $pt = r$ and $qt = s$. Dually, *pushouts* are colimits over $\mathcal{J}^{\mathrm{op}}$.

Theorem 1.3. *Limits and colimits are unique up to unique isomorphism.*

Proof. The two cases are dual hence we will do limits. Let $D\colon \mathcal{J} \to \mathcal{C}$ be a diagram and let $(A, p_i\colon A \to Di)_{i\in\mathcal{J}}$ and $(B, q_i\colon B \to D_i)_{i\in\mathcal{J}}$ be limits of D. By their universal properties

$\forall i \exists! k\colon A \longrightarrow B$ such that $\qquad$ $\forall i \exists! l\colon B \longrightarrow A$ such that

$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "p_i"'] & \arrow[d, phantom, "\|"] & B \arrow[dl, "q_i"] \\ & Di & \end{tikzcd} \qquad \& \qquad \begin{tikzcd} B \arrow[rr, "l"] \arrow[dr, "q_i"'] & \arrow[d, phantom, "\|"] & A \arrow[dl, "p_i"] \\ & Di & \end{tikzcd}$$

Then

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[dr, "p_i"'] & B \arrow[r, "l"] \arrow[d, "q_i"] & A \arrow[dl, "p_i"] \\ & Di & \end{tikzcd} \qquad \& \qquad \begin{tikzcd} A \arrow[rr, "1_A"] \arrow[dr, "p_i"'] & \arrow[d, phantom, "\|"] & A \arrow[dl, "p_i"] \\ & Di & \end{tikzcd}$$

So by universal property of A, $l \circ k = 1_A$. Similarly $k \circ l = 1_B \Rightarrow k$ is an iso. $\square$

Before moving away from limits and colimits, we mention infinite products. Given a set X we can view it as a *discrete category* which is category where all arrows are identities. Then a diagram $A\colon X \to \mathcal{C}$ consists of a family of objects $(A_x\colon x \in X)$ and its *limit* $\prod_{x\in X} A_x$ is the (X-indexed) product of the objects $(A_x\colon x \in X)$. This comes equipped with maps $\prod_{x\in X} A_x \xrightarrow{p_x} A_x$ for all $x \in X$ and it is universal amongst such objects.

Example 1.32. In $\mathcal{S}et$, $\prod_{x\in X} A_x = \{(a_x)_{x\in X}\colon a_x \in A_x\}$ As special cases

- $X = \boxed{0 \qquad 1}$, we obtain ordinary (binary) products
- $X = \boxed{\qquad}$,we obtain the Terminal object

Definition 1.13. A category $\mathcal{C}$ is *complete* if it is has *limits* of all diagrams, and cocomplete if it has *colimits* of all diagrams.

Remark. All algebraic categories are both complete and cocomplete. It is not hard to see that they are complete. It is harder to show that they are cocomplete, since colimits in algebraic cats are more complex. We will return to this later.

Can we capture *injectivity & surjectivity* of morphisms in categories?

1.5 Natural Transformation

Definition 1.14. A morphism $f\colon X \to Y$ is *mono/monic/a monomorphism* if

$$\forall Z \overset{g}{\underset{h}{\rightrightarrows}} X$$

satisfying $fg = fh$ then $g = h$. Dually, $f\colon Y \to X$ is *epi/an epimorphism* if

$$\forall X \overset{g}{\underset{h}{\rightrightarrows}} Z$$

satisfying $gf = hf$ then $g = h$.

Example 1.33. In $\mathcal{S}et$, monos are precisely injective functions. Indeed, let X, Y, Z be sets and $z \in Z$ and let $f\colon X \to Y$, $g, h\colon Z \to X$.

$$Z \overset{g}{\underset{h}{\rightrightarrows}} X \xrightarrow{f} Y$$

If f is injective, then $fg = fh \Rightarrow fg(z) = fh(z) \Rightarrow g(z) = h(z)$ for all $z \in Z$. Hence $g = h$. Thus, f is *monic.*
Conversely, suppose f is *mono*, we must show f is injective i.e. $f(x) = f(y) \Rightarrow x = y$. Consider 1 element set $Z = \{*\}$. Let

$$\bar{x}\colon \{*\} \longrightarrow X : * \longmapsto x$$
$$\bar{y}\colon \{*\} \longrightarrow X : * \longmapsto y$$

then, if $f(x) = f(y)$ we get

$$f \circ \bar{x}(*) = f(x) = f(y) = f \circ \bar{y}(*)$$

so as f is *mono* we have $\bar{x} = \bar{y}$ so

$$x = \bar{x}(*) = \bar{y}(*) = y.$$

In all *algebraic categories*, mono $\equiv$ injective homomorphisms. Proof that injective $\Rightarrow$ mono is just as in $\mathcal{S}et$, we will prove mono $\Rightarrow$ injective in section on universal algebra.

Example 1.34. In $\mathcal{S}et$, epis $\equiv$ surjective functions. Let X, Y, Z be sets and let $f\colon X \to Y$, $g, h\colon Y \to Z$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z \end{tikzcd}$$

Suppose f is surjective and that $gf = hf$. Then, let $y \in Y$. As f is surjective $\exists x \in X$ such that $f(x) = y$. Hence for all $y \in Y$

$$g(y) = gf(x) = hf(x) = h(y),$$

thus $g = h$, so f is *epimorpism.*
Conversely, let $X \xrightarrow{f} Y$ be *epi* and let $Z = \{0, 1\}$. We must show f is surjective i.e. $\mathrm{im}(f) = Y$. Suppose f is not surjective, thus $\exists y_0 \in Y$ such that $y_0 \notin \mathrm{im}(f)$. Let $g, h\colon Y \to Z$ define as follows

$$h(y) = 1 \quad \forall y \in y,$$

$$g(y) = \begin{cases} 0 & \text{if } y = y_0, \\ 1 & \text{if } y \neq y_0. \end{cases}$$

Then for any $x \in X$: $gf(x) = 1 = hf(x)$. Since f is *epi*, then $g = h$. Which is contradiction as $g(y_0) \neq h(y_0)$. Hence $y_0 \in \mathrm{im}(f)$ and is surjective.

In *algebraic categories* surjectivity always implies epimoprhims. But the converse is not true in general. In fact, in $\mathcal{R}ng$, the inclusion homomorphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is *epi* but not surjective.

Exercise. Check the above.

Definition 1.15. Let $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ be functors. A *natural transformation* $\eta\colon F \Rightarrow G$ consists of:

- for each $X \in \mathcal{A}$ a morphism $\eta_X\colon FX \to GX$ such that for all $X \xrightarrow{\alpha} Y \in \mathcal{A}$ the square

$$\begin{tikzcd} FX \arrow[r, "\eta_X"] \arrow[d, "F\alpha"'] & GX \arrow[d, "G\alpha"] \\ FY \arrow[r, "\eta_Y"'] & GY \end{tikzcd}$$

commutes.

Notation. We write $\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\eta\Downarrow"] & B \end{tikzcd}$ for a *natural transformation.*

Example 1.35. $\mathcal{G} = \boxed{\begin{tikzcd} 1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 0 \end{tikzcd}}$ Let X be diagram $X\colon \mathcal{G} \to \mathcal{C}$ which consists of

$$\begin{tikzcd} X1 \arrow[r, shift left, "Xs=s_X"] \arrow[r, shift right, "Xt=t_X"'] & X0 \end{tikzcd}$$

and let Y be arbitrary functor $Y\colon \mathcal{G} \to \mathcal{C}$. *Natural transformation* $\eta\colon X \to Y$ consists of

$$\begin{tikzcd} X1 \arrow[r, "\eta_1"] \arrow[d, shift right, "s_X"'] \arrow[d, shift left, "t_X"] & Y1 \arrow[d, shift right, "s_Y"'] \arrow[d, shift left, "t_Y"] \\ X0 \arrow[r, "\eta_0"'] & Y0 \end{tikzcd}$$

making both squares commute.

In particular, if $\mathcal{C} = \mathcal{S}et$, X consists of

- a set $X0$ of *vertices* $x, y, z ...$,
- a set $X1$ of *directed edges* $f, g, h...$ which have *source* $s_X(f)$ and *target* $t_X(f)$.

Let $a, b \in X0$, we write $a \xrightarrow{f} b$ if $s_X(f) = a$ & $t_X(f) = b$. So $X : \mathcal{G} \to \mathcal{S}et$ is exactly a directed graph e.g.

$$\begin{tikzcd} x \arrow[loop left] \arrow[r] \arrow[dr] & y \arrow[d, shift left] \arrow[dr, shift left] \arrow[dr, shift right] \\ & z \arrow[u, shift left] \arrow[r] & w \end{tikzcd}$$

A natural transformation $\eta : X \to Y$ consists of:

- maps $\eta_1 : X1 \to Y1$ and $\eta_0 : X0 \to Y0$ making the following squares commute in a way that preserves *source* and *target*.

$$\begin{tikzcd} X1 \arrow[r, "\eta_1"] \arrow[d, shift right, "s_X"'] \arrow[d, shift left, "t_X"] & Y1 \arrow[d, shift right, "s_Y"'] \arrow[d, shift left, "t_Y"] \\ X0 \arrow[r, "\eta_0"'] & Y0 \end{tikzcd}$$

$$x \longmapsto \eta_0(x)$$
$$x \xrightarrow{f} y \longmapsto \eta_0(x) \xrightarrow{\eta_1(f)} \eta_0(y)$$

I.e. if $x = s_X(f)$ then $\eta_0(x) = \eta_0 s_X(f) = s_Y \eta_1(f)$ and similarly for t. Thus a *natural transformation* is precisely a morphism of directed graphs.

Example 1.36. Let $\mathcal{A}, \mathcal{B}$ be posets viewed as categories where $\exists! a \to b \iff a \leq b$. Given $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ are order-preserving functions viewed as functors, then a *natural transformation* $\eta : F \Rightarrow G$ means $Fx \leq Gx$ for all $x \in \mathcal{A}$. The commutativity condition is redundant since all diagrams commute in $\mathcal{B}$

Example 1.37. Consider

$$\begin{tikzcd} \mathcal{S}et \arrow[r, "F"] \arrow[rr, bend right, "W=UF"'] & \mathcal{M}on \arrow[r, "U"] & \mathcal{S}et \end{tikzcd}$$

Where U is forgetful functor and F is free monoid functor. Let $I_{\mathcal{S}et}$ be an identity functor. We can define a *natural transformation* $\eta : I_{\mathcal{S}et} \Rightarrow W$ whose component at X is

$$X \xrightarrow{\eta_X} WX$$
$$x \longmapsto [x]$$

At $f : X \to Y$ we need

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & WX \arrow[d, "Wf"] \\ Y \arrow[r, "\eta_Y"'] & WY \end{tikzcd}$$

By definition, $Wf([x_1, ..., x_n]) = [f(x_1), ..., f(x_n)]$, so $Wf \circ \eta_X(x) = Wf[x] = [f(x)] = \eta_Y(f(x)) = \eta_Y \circ f(x)$.

Consider categories $\mathcal{A},\mathcal{B}$. Given natural transformations

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=50, "H"'{name=D}] & \mathcal{B} \arrow[from=U, to=M, Rightarrow, "\alpha"'] \arrow[from=M, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

we can compose them (vertically) to obtain a natural transformation

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "H"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, Rightarrow, "\beta\alpha"'] \end{tikzcd}$$

with components $FX \xrightarrow{\alpha_X} GX \xrightarrow{\beta_X} HX$ for each X. The naturality condition is easy to check, thus is left to reader as an exercise. This composition of *natural transformations* is associative (as composition in $\mathcal{B}$ is associative). We also have identity *natural transformation*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, Rightarrow, "1_F"'] \end{tikzcd}$$

with components $1_{FX} \colon FX \longrightarrow FX$. Altogether, we obtain a category $[\mathcal{A},\mathcal{B}]$ called the *Functor category* whose objects are functors $\mathcal{A} \to \mathcal{B}$ and arrows are *natural transformations.*

Example 1.38. $[1 \rightrightarrows 0, \mathcal{S}et]$ is the category of directed graphs.

1.6 Equivalence of Categories

The question arises, when are two categories the same? Since $\mathcal{CAT}$ is a category we can speak of isomorphisms of categories, but this is too strong a notion. Better notion would be equivalence of categories which we will look at in the following section.

Definition 1.16. A natural transformation $\begin{tikzcd} \mathcal{A} \arrow[r, bend left=30, "F"{name=U}] \arrow[r, bend right=30, "G"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, Rightarrow, "\eta"'] \end{tikzcd}$ is a *natural isomorphism* if it is an isomorphism in $[\mathcal{A},\mathcal{B}]$. We write $\eta \colon F \cong G$ for *natural isomorphism.*

Lemma 1.1. *Let $\mathcal{A},\mathcal{B}$ be categories and $F,G \colon \mathcal{A} \rightrightarrows \mathcal{B}$ functors. Let $\eta \colon F \Rightarrow G$ be natural transformation. Then $\eta \colon F \Rightarrow G$ is a natural isomorphism if and only if $\eta_X \colon FX \longrightarrow GX$ is an isomorphism in $\mathcal{B}$.*

Proof. If η is an iso in $[\mathcal{A},\mathcal{B}]$, then exists $\theta \colon G \Rightarrow F$ such that $\theta \circ \eta = 1_F$ and $\eta \circ \theta = 1_G$. By definition of composition and identities in $[\mathcal{A},\mathcal{B}]$, this means that for all objects $a \in \mathcal{A}$ $\theta_a \circ \eta_a = 1_{Fa}$ and $\eta_a \circ \theta_a = 1_{Ga}$. Hence each η_a is isomorphism with inverse θ_a.
Conversely, suppose each η_a is an isomorphism with inverse θ_a. We must show that the $\theta_a \colon Ga \to Fa$ define a natural transformation $\theta \colon G \to F$. At $\alpha \colon a \to b$, where a,b are objects of $\mathcal{A}$, we need that

$$\begin{tikzcd} Fa \arrow[r, "\eta_a"] \arrow[d, "F\alpha"'] \arrow[dr, phantom, "\not\!\!/"] & Ga \arrow[r, "\theta_a"] \arrow[d, "G\alpha"] \arrow[dr, phantom, "\not\!\!/"] & Fa \arrow[d, "F\alpha"] \\ Fb \arrow[r, "\eta_b"'] & Gb \arrow[r, "\theta_b"'] & Fb \end{tikzcd}$$

Since η_a is an iso, then it is also epi ($gf = hf \Rightarrow gff^{-1} = hff^{-1} \Rightarrow g = h$). Thus it suffices to show $F\alpha \circ \theta_a \circ \eta_a = \theta_b \circ G\alpha \circ \eta_a$. But $F\alpha \circ \theta_a \circ \eta_a = F\alpha$ whilst $\theta_b \circ G\alpha \circ \eta_a = \theta_b \circ \eta_b \circ F\alpha = F\alpha$. Where the first equality uses naturality of η $\square$

Definition 1.17. A functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an equivalence of categories if exists functor $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ and natural isomorphisms α, β

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "1_{\mathcal{A}}"'] \arrow[dr, phantom, shift left=3, "\alpha\ \cong"] & \mathcal{B} \arrow[d, "G"] \arrow[dr, "1_{\mathcal{B}}"] \arrow[dr, phantom, shift right=3, "\beta\ \cong"'] & \\
& \mathcal{A} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

Example 1.39. Consider the category $FDVect$ of real finite-dimensional vector spaces and linear transformations. Its subcategory $FDVect_{\mathbb{R}}$ with objects $\mathbb{R}^n$ and linear transformations. Then the inclusion

$$FDVect_{\mathbb{R}} \underset{j}{\hookrightarrow} FDVect$$

is an equivalence of categories. The equivalence inverse $P\colon FDVect \longrightarrow FDVect_{\mathbb{R}}$ sends a vector space V of dimension n to $\mathbb{R}^n$. Then we have an isomorphism

$$\begin{aligned} &\eta_V : jP(V) = \mathbb{R}^n \cong V \\ &b_i = (0, ..., 1, ...0) \longmapsto v_i \end{aligned}$$

where $\{v_1, ..., v_n\}$ is a basis of V. On arrows $f\colon V \longrightarrow W \in FDVect$, we use the choice of basis specified by η_V and $\eta - W$ to define $R(f)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ as

$$\mathbb{R}^n \xrightarrow{\eta_V} V \xrightarrow{f} W \xrightarrow{(\eta_W)^{-1}} \mathbb{R}^m :$$

$$b_i \mapsto v_i \mapsto f(v_i) = \sum_{j=1}^{m} a_{ij} w_j \mapsto \sum_{j=1}^{m} a_{ij} b_j$$

Then
$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\eta_V"] \arrow[d, "jR(f)"'] \arrow[dr, phantom, "="] & V \arrow[d, "f"] \arrow[dd, bend left, "1"] \\
\mathbb{R}^m \arrow[dr, "\eta_W"'] \arrow[r, phantom, shift right=6, "="] & W \arrow[l, "(\eta_W)^{-1}"'] \arrow[d, "1"] \arrow[r, phantom, "="] & {} \\
& W
\end{tikzcd}$$
gives naturality of the isomorphisms.

Similarly $Rj = 1$. Note that defining R in this way depends on the choice of bases.

1.7 Adjoint functors

Definition 1.18. Let $\mathcal{A}, \mathcal{B}$ be categories. An *adjunction* consists of functors

$$\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

with bijections $\mathcal{B}(Fa, b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a, Ub)$ for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and these are *natural in each variable.*

- Naturality in b means: given $Fa \xrightarrow{\alpha} b$ & $b \xrightarrow{\beta} b'$ we have $\varphi_{a,b'}(\beta \circ \alpha) = U\beta \circ \varphi_{a,b}(\alpha)$.
- Naturality in a means: given $Fa \xrightarrow{\alpha} b$ & $a' \xrightarrow{\beta} a$ we have $\varphi_{a',b}(\alpha \circ F\beta) = \varphi_{a,b}(\alpha) \circ \beta$.

Remark. Key point is that if we have an adjunction, maps $Fa \to b$ *bijectively corresponds* to maps $a \to Ub$ in a natural way. In that case, we often write $\frac{Fa \longrightarrow b}{a \longrightarrow Ub}$.

In the examples below, we won't check the naturality conditions; we leave it to the reader as an exercise.

Notation. We say that F is *left adjoint* to U and write $F \dashv U$.

Example 1.40. *Free* $\dashv$ *forgetful* functors.

$$F \dashv U \qquad \begin{tikzcd} \mathcal{V}ect \arrow[r, phantom, "\perp"] & \mathcal{S}et \arrow[l, bend right, "F \text{ free vect. sp.}"'] \end{tikzcd} \quad \mathcal{V}ect \underset{U \text{ forgetful}}{\overset{F \text{ free vect. sp.}}{\leftrightarrows}} \mathcal{S}et$$

$FX = \{\lambda_1 x_1 + ... + \lambda_n x_n \colon x_1, ..., x_n \in X, \lambda_1, ..., \lambda_n \in K\}$, where K is a field and X is basis of considered vector space. At function $f\colon X \to Y$

$$Ff\colon FX \longrightarrow FY\colon \sum_i \lambda_i x_i \longmapsto \sum_i \lambda_i f(x_i)$$

is obtained by *linear extension.* We have a bijection

$$\begin{aligned} r\colon \mathcal{V}ect(FX, A) &\longrightarrow \mathcal{S}et(X, UA) \\ FX \xrightarrow{f} A &\longmapsto X \xrightarrow{r(f)} UA \\ & \qquad x \longmapsto f(x) \end{aligned}$$

This is a bijection whose inverse send $g\colon X \longrightarrow UA$ to the linear maps
$\bar{g}\colon FX \longrightarrow A\colon \sum_i \lambda_i x_i \longmapsto \sum_i g(x_i)$. Indeed, we must show that $r(\bar{g}) = g$ and $f = \overline{r(f)}$. Now

$$r(\overline{g})(x) \overset{\text{def}}{=} \overline{g}(x) = g(x)$$

where second equality holds as x is basis element of FX. On the other hand,

$$\begin{aligned} &\overline{r(f)}(\lambda_1 x_1 +, ..., +\lambda_n x_n) \overset{\text{def}}{=} \lambda_1 rf(x_1) + ... + \lambda_n rf(x_n) \overset{\text{def}}{=} \\ &\overset{\text{def}}{=} \lambda_1 f(x_1) + ... + \lambda_n f(x_n) = f(\lambda_1 x_1 + ... + \lambda_n x_n). \end{aligned}$$

Since f is linear transformation, as required.

Example 1.41. Similarly, we have an adjunction

$$\begin{tikzcd} \mathcal{M}on \arrow[r, shift right, "U \text{ forgetful}"'] \arrow[r, phantom, "\perp"] & \mathcal{S}et \arrow[l, shift right, "F \text{ free}"'] \end{tikzcd}$$

Recall that FX is the free/list monoid, a structure consisting of all finite sequences (strings) of elements from a given set. Given $FX \xrightarrow{f} A$, corresponding map is

$$X \longrightarrow UA\colon x \longmapsto f[x].$$

Given $X \xrightarrow{g} UA$, the corresponding map is

$$\overline{g}\colon FX \longrightarrow A\colon [x_1, ..., x_n] \longmapsto g(x) \cdot ... \cdot g(x_n).$$

Where $\cdot$ denotes multiplication in monoid A. Note that this definition is forced on us since $[x_1, ..., x_n] = [x_1] \cdot ... \cdot [x_n]$ and $\overline{g}$ must preserve multiplication and satisfy $\overline{g}[x] = g(x)$ for $x \in X$ in order to have a bijection.

Similarly, the forgetful functions form $\mathcal{G}rp$ or $\mathcal{R}ng$ to $\mathcal{S}et$ have left adjoints, sending a set to free group or free ring. More generally, *forgetful* functors in universal algebra have left adjoints, we will see this soon.

Example 1.42. The *forgetful* functor $U\colon \mathcal{G}rp \longrightarrow \mathcal{M}on$ has among the left free adjoint (which we recommend the reader to think through) also a right adjoint. Its right adjoint $R\colon \mathcal{M}on \Longrightarrow \mathcal{G}rp$ sends monoid M to subgroup RM of invertible elements in M.

If G is a group and M is a monoid, then a monoid map $U(G) \longrightarrow M$ takes elements of G to invertible elements of M and so factors as $G \longrightarrow R(M)$ through $R(M) \hookrightarrow M$. We obtain $\frac{U(G) \longrightarrow M}{G \longrightarrow R(M)}$ a bijection.

Example 1.43. We have adjoint functors

$$\begin{tikzcd} \mathcal{T}op \arrow[r, "U"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{S}et \arrow[l, bend right, "D \text{ discrete}"'] \arrow[l, bend left, "I \text{ indiscrete}"] \end{tikzcd}$$

- $DX =$ set with all subsets open.
- $IX =$ set with $X, \emptyset$ open.

Then any function $X \longmapsto UA$ is continuous with respect to discrete topology, so there is a bijection $\frac{X \longrightarrow UA}{DX \longrightarrow A}$. Similarly, any function $UX \longrightarrow A$ is continuous with respect to indiscrete topology on A, so again we have bijection $\frac{UX \longrightarrow A}{X \longrightarrow IA}$.

Let $A \in \mathcal{S}et$. We have a functor

$$\begin{aligned} A \times \text{-}\colon \mathcal{S}et &\longrightarrow \mathcal{S}et \\ X &\longmapsto A \times X. \end{aligned}$$

Now there is a bijection between functions $A \times X \xrightarrow{f} Y$ and functions $X \xrightarrow{\bar{f}} \mathcal{S}et(A, Y) = Y^A$ where

$$\begin{aligned} \bar{f}(x) = g \colon A &\longrightarrow Y \\ a &\longmapsto f(x, a). \end{aligned}$$

This process is called *currying*. Therefore we have adjunction

$$\begin{aligned} A \times \text{-} &\dashv (\text{-})^A \\ (\text{-})^A \colon \mathcal{S}et &\longrightarrow \mathcal{S}et \\ X &\longmapsto X^A. \end{aligned}$$

Let's consider adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, leftarrow, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, leftarrow, shift left, "U"] \end{tikzcd}$$

. with bijection $\mathcal{B}(Fa, b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a, Ub)$. Taking $b = Fa$, we obtain

$$\begin{aligned} \mathcal{B}(Fa, Fa) &\longrightarrow \mathcal{A}(a, UFa) \\ Fa \xrightarrow{1_{Fa}} Fa &\longmapsto a \xrightarrow{\eta_a} UFa. \end{aligned}$$

The η_a is called the *unit* of the adjunction. We have

$$\begin{tikzcd} a \arrow[r, "\eta_a = \varphi_{a,Fa(1)}"] \arrow[d, "\alpha"'] \arrow[dr, "\varphi_{a,Fb(F\alpha)}"] & UFa \arrow[d, "UF\alpha"] \\ b \arrow[r, "\eta_b = \varphi_{b,Fb(1)}"'] & UFb \end{tikzcd}$$

where the top right triangle commutes using naturality at

$$\begin{tikzcd} Fa \arrow[r, "1_{Fa}"] \arrow[dr, "F\alpha"'] & Fa \arrow[d, "F\alpha"] \\ & Fb \end{tikzcd}$$

and the bottom left using naturality at

$$\begin{tikzcd} Fa \arrow[d, "F\alpha"'] \arrow[dr, "F\alpha"] & \\ Fb \arrow[r, "1_{Fb}"'] & Fb \end{tikzcd}$$

Therefore $\eta \colon 1_{\mathcal{A}} \Longrightarrow UF$ is a *natural transformation*. In fact, the components $\varphi_{a,b} \colon \mathcal{B}(Fa, b) \longrightarrow \mathcal{A}(a, Ub)$ are determined by η.

$$\varphi_{a,b}(\alpha) = U\alpha \circ \varphi_{a,Fa}(1) = U\alpha \circ \eta_a. \tag{(A)}$$

Theorem 1.4. *Let* $\begin{tikzcd} A \arrow[r, leftarrow, shift left, "F"] & B \arrow[l, leftarrow, shift left, "U"] \end{tikzcd}$ *be functors. Let $a \in \mathcal{A}, b \in \mathcal{B}$ be objects, and let $\alpha \colon Fa \longrightarrow b \in \mathcal{B}$ be morphism. Then there is bijection between adjunctions $(F \dashv U, \varphi)$ and natural transformations $\eta \colon 1_{\mathcal{A}} \Longrightarrow UF$ with the universal property that, given $a \xrightarrow{\alpha} Ub$ there exists a unique $Fa \xrightarrow{\overline{\alpha}} b$ such that*

$$\begin{tikzcd} & UFa \arrow[dr, "U\overline{\alpha}"] \arrow[d, phantom, "\parallel"] & \\ a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"'] & {} & Ub \end{tikzcd} \tag{(1)}$$

Proof. By (A) above, if we have an adjunction then

$$\begin{aligned} \mathcal{B}(Fa,b) &\xrightarrow{\varphi_{a,b}} \mathcal{A}(a,Ub) \\ \overline{\alpha} &\longmapsto U\overline{\alpha}\circ\eta_a \end{aligned}$$

so that (1) just says that this map is a *bijection*. We have already showed that the assignment $(F,U,\varphi)\longmapsto(F,U,\eta)$ is *injective*, so it remains to show that each (F,U,η) satisfying (1) arises from an adjunction.
Given $\eta\colon 1_{\mathcal{A}}\Longrightarrow UF$ satisfying (1) as above, we must show that the maps

$$\begin{aligned} \mathcal{B}(Fa,b) &\xrightarrow{\varphi_{a,b}} \mathcal{A}(a,Ub) \\ Fa\xrightarrow{\alpha} b &\longmapsto a\xrightarrow{\eta_a} UFa\xrightarrow{U\alpha} Ub \end{aligned}$$

form an adjunction. Certainly, by (1), they are *bijections*, so it remains to check naturality, which is straight forward, thus is left to the reader as an exercise. ☐

In fact, even less is required.

Theorem 1.5. *An adjunction is specified by a functor $U\colon\mathcal{B}\longrightarrow\mathcal{A}$ and for each object $a\in\mathcal{A}$ an object $Fa\in\mathcal{B}$ and morphism $a\xrightarrow{\eta_a}UFa$ with the universal property that, for every object $b\in\mathcal{B}$ and every morphism $a\xrightarrow{\alpha}Ub$, there exists a unique morphism $\overline{\alpha}\colon Fa\to b$ such that (1) commutes:*

$$\begin{tikzcd} & UFa \arrow[dr, "U\overline{\alpha}"] \arrow[d, phantom, "="] & \\ a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"'] & & Ub \end{tikzcd}$$

Proof. If $(F\dashv U,\varphi)$ is an adjunction, then the *unit* $\eta\colon 1_{\mathcal{A}}\Longrightarrow UF$ as established in Theorem 1.4 satisfies the specified universal property by definition. Conversely, given the above, we define $F\alpha\colon Fa\to Fb$ as the unique map such that the following square commutes (such a unique map exists by (1)).

$$\begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\not\!\!/"] & UFa \arrow[d, "UF\alpha"] \\ b \arrow[r, "\eta_b"'] & UFb \end{tikzcd} \qquad ((2))$$

Indeed, we are forced to define $F\alpha$ this way in order for η to be natural transformation. Therefore, we will be finished, if we can show F is a functor.
To see that F preserves identities, observe that

$$\begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[d, "1_a"'] \arrow[dr, phantom, "\not\!\!/"] & UFa \arrow[d, "U1_a=1_{Fa}"] \\ a \arrow[r, "\eta_a"'] & UFb \end{tikzcd}$$

commutes, so $F1_a=1_{Fa}$ by (2).

To see that $F\beta \circ F\alpha = F(\beta \circ \alpha)$ consider

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "/\!/"] & UFa \arrow[d, "UF\alpha"] \arrow[dd, bend left, "U(F\beta \circ F\alpha)"] \\
b \arrow[r, "\eta_b"] \arrow[d, "\beta"'] \arrow[dr, phantom, "/\!/"] & UFb \arrow[d, "UF\beta"] \\
c \arrow[r, "\eta_c"] & UFc
\end{tikzcd}$$

Since the outside commutes, by (2) we have $F\beta \circ F\alpha = F(\beta \circ \alpha)$.
Thus F is a functor and $\eta \colon 1_{\mathcal{A}} \Longrightarrow UF$ is a natural transformation. $\square$

Corollary 1.1. *$U \colon \mathcal{B} \longrightarrow \mathcal{A}$ has a left adjoint if*

$$\forall a \in \mathcal{A}, \exists Fa \in \mathcal{B} \;\&\; \eta_a \colon a \longrightarrow UFa$$

with the universal property that, given $a \xrightarrow{\alpha} Ub$ there exists a unique $Fa \xrightarrow{\bar{\alpha}} b$ such that (1) commutes.

Remark. This is often easiest way to check a functor has a left adjoint in practice.

Example 1.44. Consider functor $U \colon \mathcal{Mon} \longrightarrow \mathcal{Set}$, a set $X \in \mathcal{Set}$ and monoid $M \in \mathcal{Mon}$. Let's have

$$X \xrightarrow{\eta_x} UFX$$
$$x \longmapsto [x]$$

where FX is list monoid. Given $f \colon X \longrightarrow UM$, condition (1) says that $\exists!\ FX \longrightarrow M$ such that:

$$\begin{tikzcd}
 & UFX \arrow[dr, "U\bar{f}"] \arrow[d, phantom, "||"] & \\
X \arrow[ur, "\eta_x"] \arrow[rr, "f"'] & {} & UM.
\end{tikzcd}$$

In other words, $\exists!$ monoid homomorphism $FX \xrightarrow{\bar{f}} M$ such that $\bar{f}([x]) = f(x)$. This is the case, in order for $\bar{f}$ to be homomorphism, we must define

$$\bar{f}([x_1, ..., x_n]) = \bar{f}([x_1]) \cdot ... \cdot \bar{f}([x_n]) = f(x_1) \cdot ... \cdot f(x_n)$$
$$\bar{f}[\;] = e_M$$

and this is a monoid homomorphism.

Another important corollary of the above is following theorem.

Theorem 1.6. *Let $U \colon \mathcal{B} \longrightarrow \mathcal{A}$. Then its left adjoint, if it exists, is unique up to natural isomorphism.*

Proof. Suppose F_1, F_2 are left adjoints to U with units $a \xrightarrow{\eta_{1a}} UF_1a$ & $a \xrightarrow{\eta_{2a}} UF_2a$ satisfying (1). Then by the u.p. (1) of F_1a, $\exists!\ K_a \colon F_1a \longrightarrow F_2a$ such that

$$\begin{tikzcd} & UF_1a \arrow[dr, "UK_a"] \arrow[d, phantom, "||"] & \\ a \arrow[ur, "\eta_{1a}"] \arrow[rr, "\eta_{2a}"'] & & UF_2a \end{tikzcd} \quad \text{\& likewise} \quad \begin{tikzcd} & UF_1a \arrow[dr, "UL_a"] \arrow[d, phantom, "||"] & \\ a \arrow[ur, "\eta_{2a}"] \arrow[rr, "\eta_{1a}"'] & & UF_1a \end{tikzcd}$$

To establish that K_a is an isomorphism with inverse L_a, we observe that by the functoriality of U and the definitions of K_a and L_a from the diagrams above:

$$U(L_a \circ K_a) \circ \eta_{1a} = U(L_a) \circ (U(K_a) \circ \eta_{1a}) = U(L_a) \circ \eta_{2a} = \eta_{1a}.$$

Since we also have $U(\mathrm{id}_{F_1a}) \circ \eta_{1a} = \mathrm{id}_{UF_1a} \circ \eta_{1a} = \eta_{1a}$, the uniqueness part of the universal property of η_{1a} implies that $L_a \circ K_a = \mathrm{id}_{F_1a}$. By a symmetric argument for the composition $K_a \circ L_a$ using the universal property of η_{2a}, it follows that $K_a \circ L_a = \mathrm{id}_{F_2a}$. Thus, K_a is an isomorphism.

$$\begin{tikzcd} F_1a \arrow[r, "K_a"] \arrow[d, "F_1\alpha"'] \arrow[dr, phantom, "\not\parallel"] & F_2a \arrow[d, "F_2\alpha"] \\ F_1b \arrow[r, "K_b"'] & F_2b \end{tikzcd}$$

which by u.p. of F_1a is to show that agree the two outer paths of

$$\begin{tikzcd} a \arrow[r, "\eta_{1a}"'] & UF_1a \arrow[r, "UK_a"] \arrow[d, "UF_1\alpha"'] \arrow[dr, phantom, "\not\parallel"] & UF_2a \arrow[d, "UF_2\alpha"] \\ & UF_1b \arrow[r, "UK_b"'] & UF_2b \end{tikzcd}$$

The upper path is

$$UF_2\alpha \circ UK_a \circ \eta_{1a} = UF_2\alpha \circ \eta_{2a} \overset{\text{by nat.}}{=} \eta_{2b} \circ \alpha.$$

The lower path is

$$UK_b \circ UF_1\alpha \circ \eta_{1a} = UK_b \circ \eta_{1b} \circ \alpha = \eta_{2b} \circ \alpha.$$

□

Remark. This result illustrates a fundamental principle of category theory: a functor U (typically a forgetful functor) uniquely determines its left adjoint F (the associated free construction) up to natural isomorphism. This implies that the properties and structure of free objects are entirely characterized by the data provided by the underlying forgetful functor.

The question arises, what do adjoint functors preserve? Let $D\colon \mathcal{J} \longrightarrow \mathcal{A}$ be a diagram. A cone over D consists of maps $L \xrightarrow{p_i} Di$ for $i \in \mathcal{J}$ satisfies

$$\begin{tikzcd} & Di \arrow[dd, "D\alpha"] \\ L \arrow[ur, "p_i"] \arrow[dr, "p_j"'] \arrow[r, phantom, "="] & \\ & Dj \end{tikzcd} \quad \text{for} \quad \begin{tikzcd} i \arrow[dd, "\alpha"] \\ \\ j \end{tikzcd}$$

It is a *limit cone* (or L is the *limit*) if, given a cone $(k_i\colon A \to Di)_{i\in\mathcal{J}}$, $\exists! k\colon A \to L$ such that $p_i \circ k = k_i$ for all i. If $U\colon \mathcal{A} \to \mathcal{B}$ is a functor, it takes the cone $(p_i\colon L \to Di)_{i\in\mathcal{J}}$ to a cone $(Up_i\colon UL \to UDi)_{i\in\mathcal{J}}$.

Definition 1.19. Let $U\colon \mathcal{A} \to \mathcal{B}$ be functor. We say that U *preserves* the limit L of D if the cone $(UL \xrightarrow{Up_i} UDi)_{i\in\mathcal{J}}$ is a limit cone.

Similarly, we can speak of functors preserving colimits.

Theorem 1.7. *Let $F \dashv U$ be an adjunction where $U : \mathcal{A} \to \mathcal{B}$ is the right adjoint, then U preserves all limits that exist in $\mathcal{A}$.*

Proof. Consider limit cone $(L \xrightarrow{p_i} Di)_{i \in \mathcal{G}}$ and

$$\begin{tikzcd} \mathcal{G} \arrow[r, "D"] & \mathcal{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "F"'] \end{tikzcd}.$$

We must show, that $(UL \xrightarrow{Up_i} UDi)_{i\mathcal{G}}$ is a limit cone. Consider cone $(X \xrightarrow{k_i} UDi)_{i \in \mathcal{G}}$. Using bijections $\mathcal{A}(FX, Di) \xrightarrow{\varphi} \mathcal{B}(X, UDi)$ we obtain maps $FX \xrightarrow{\varphi^{-1}k} Di$ and we claim these form a cone to D. We must show

$$\begin{tikzcd} & Di \arrow[dd, "D\alpha"] \\ FX \arrow[ur, "\varphi^{-1}k_i"] \arrow[dr, "\varphi^{-1}k_j"'] \arrow[r, phantom, "//"] & \\ & Dj \end{tikzcd}$$

but this is equivalent to showing images of these maps $FX \rightrightarrows Dj$ under $\mathcal{A}(FX, Dj) \xrightarrow{\varphi} \mathcal{B}(X, UDj)$ are equal. Well $\varphi\varphi^{-1}k_j = k_j$.

$$\varphi(D\alpha \circ \varphi^{-1}k_i) = UD\alpha \circ \varphi\varphi^{-1}k_i = UD\alpha \circ k_i$$

where the first equality holds by naturality of φ, so their images are the two paths of the following diagram, which agree since the k_i's form a cone.

$$\begin{tikzcd} & UDi \arrow[dd, "UD\alpha"] \\ X \arrow[ur, "k_i"] \arrow[dr, "k_j"'] \arrow[r, phantom, "//"] & \\ & UDj \end{tikzcd}$$

Since the maps $\varphi^{-1}k_i : FX \to Di$ form a cone we obtain a unique $l : FX \to L$ such that for all $i \in \mathcal{G}$

$$\begin{tikzcd} FX \arrow[r, "l"] \arrow[dr, "\varphi^{-1}k_i"'] \arrow[dr, phantom, "//", shift left=2] & L \arrow[d, "p_i"] \\ & Di \end{tikzcd}$$

Since $\mathcal{A}(FX, L) \xrightarrow{\varphi} \mathcal{B}(X, UL)$ is a bijection, this equally says that $\exists!\ l : X \to UL$ such that

$$\begin{tikzcd} FX \arrow[r, "\varphi^{-1}\bar{l}"] \arrow[dr, "\varphi^{-1}k_i"'] \arrow[dr, phantom, "//", shift left=2] & L \arrow[d, "p_i"] \\ & Di \end{tikzcd}$$

But, using naturality of φ, these equations hold if and only if for all $i \in \mathcal{J}$ the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\bar{l}"] \arrow[dr, "k_i"'] & UL \arrow[d, "Up_i"] \arrow[dl, phantom, "="] \\ & UDi \end{tikzcd} \qquad ((*))$$

Thus $\exists!\ \bar{l} \colon X \to UL$ such that ($*$). Therefore $\left(UL \xrightarrow{Up_i} UDi\right)_{i\in\mathcal{J}}$. $\square$

For example, $U \colon \mathcal{Grp} \to \mathcal{Set}$ preserves products, equalisers, Terminal object etc. More generally, forgetful functors from algebraic *cats* to $\mathcal{Set}$ preserve all limits. Dually to Theorem 1.7:

Theorem 1.8. *Let $F \dashv U$ be an adjunction where $F \colon \mathcal{B} \to \mathcal{A}$ is the left adjoint, then F preserves all colimits that exist in $\mathcal{B}$.*

Exercise. Prove that forgetful functor $U \colon \mathcal{Grp} \to \mathcal{Set}$ does not have a right adjoint (i.e. is not a left adjoint)

Example 1.45. Let $\mathcal{Field}$ be category of fields and field homomorphisms. Fields are commutative rings satisfying

$$0 \neq 1 \ \&\ x \neq 0 \Rightarrow \exists y \colon xy = 1.$$

These equations involve *negation*, thus we are not able to define fields within the framework of universal algebra. We will show the forgetful functor $U \colon \mathcal{Field} \to \mathcal{Set}$ does not have a left adjoint.
If it did have a left adjoint $F \colon$, then F would send the initial object $\emptyset \in \mathcal{Set}$ to an initial object in $\mathcal{Field}$ (as left adjoints preserve colimits). So it suffices to show $\mathcal{Field}$ does not have an initial object.
Firstly, Let $f \colon R \to S$ be a field homomorphism. Then f is injective. Indeed, since f is, in particular, a homomorphism of rings, its kernel $\ker(f)$ is an ideal of R. It is a fundamental property of any field that its only ideals are the trivial ideal $\{0_R\}$ and the unit ideal R.
By the axioms of field homomorphisms, we have $f(1_R) = 1_S$. Since any field S is non-trivial by definition (satisfying $1_S \neq 0_S$), it follows that $1_R \notin \ker(f)$. This observation forces $\ker(f) \neq R$, and consequently, we must have $\ker(f) = \{0_R\}$. A ring homomorphism with a trivial kernel is injective, which completes the proof.
Let $\mathbb{Z}_p$ be field of integers modulo p, where p is prime number. Let F be field. If F is initial, then for all prime p, q there exist injective maps

$$\begin{tikzcd} & \mathbb{Z}_p \\ F \arrow[ur, hook] \arrow[r, hook] & \mathbb{Z}_q \end{tikzcd}$$

Since these maps are injective, they reflect equations $p \cdot 1 = 0 \ \&\ q \cdot 1 = 0$, so these equations hold in F. But as p, q are coprime, then $\exists n, m \in \mathbb{Z}$ such that $1 = np + mq$ by Bézout's identity. Hence in F

$$1 = 1 \cdot 1 = (np + mq) \cdot 1 = n(p \cdot 1) + m(q \cdot 1) = n0 + m0 = 0.$$

Hence $1 = 0$ in F, so F is not a field.

Proposition 1.2. *Right adjoints preserve monomorphisms.*

Proof. Let $U\colon \mathcal{A} \to \mathcal{B}$ have adjoint F and consider *mono* $a \xrightarrow{f} b \in \mathcal{A}$. Consider

$$x \overset{u}{\underset{v}{\rightrightarrows}} Ua \xrightarrow{Uf} Ub$$

satisfying $Uf \circ u = Uf \circ v$. We must show $u = v$. By naturality of φ we obtain maps

$$Fx \overset{\varphi^{-1}u}{\underset{\varphi^{-1}v}{\rightrightarrows}} a \xrightarrow{f} b$$

and the diagram commutes. Since f is mono $\varphi^{-1}u = \varphi^{-1}v$. Therefore $u = v$, so that Uf is mono as claimed. ☐

Dually

Proposition 1.3. *Left adjoints preserve epimorphisms.*

Proof. Left as an exercise to the reader. ☐

1.8 Full & faithful functors

Definition 1.20. Consider a functor $U\colon \mathcal{A} \to \mathcal{B}$ with functions $U\colon \mathcal{A}(a,b) \xrightarrow{U_{a,b}} \mathcal{B}(Ua, Ub)$. We call U:

- **faithful** if each $U_{a,b}$ is injective,
- **full** if each $U_{a,b}$ is surjective,
- **fully faithful** if each $U_{a,b}$ is bijective.

In elementary terms, *faithful* means that given $\alpha, \beta\colon a \rightrightarrows b$ such that $U\alpha = U\beta$ then $\alpha = \beta$.
Full means given $Ua \xrightarrow{\alpha} Ub$, $\exists a \xrightarrow{\beta} b$ such that $U\beta = \alpha$.
Fully faithful means given $Ua \xrightarrow{\alpha} Ub$, $\exists!\ a \xrightarrow{\beta} b$ such that $U\beta = \alpha$.

Note. When U is fully faithful, there could still be two different maps $a \xrightarrow{\beta} b$ and $c \xrightarrow{\gamma} d$ such that $U\beta = U\gamma = \alpha$ as long as $c \neq a$ or $d \neq b$.

Definition 1.21. A *subcategory* $\mathcal{A}$ of category $\mathcal{B}$ is a subcollection of objects and arrows in $\mathcal{B}$ closed under identities and composition. The $i\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is a functor. If i is fully faithful it is a *full subcategory.*

Example 1.46. $U\colon \mathcal{Grp}, \mathcal{Mon}, \mathcal{Rng} \longrightarrow \mathcal{Set}$ are all faithful. $i\colon \mathcal{Grp} \hookrightarrow \mathcal{Mon}$ is a full subcategory.

Remark. Let $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$ be a full subcategory. A left adjoint to the inclusion functor i is uniquely specified by assigning to each object $b \in \mathcal{B}$ an object $Rb \in \mathcal{A}$ and a morphism $\eta_b\colon b \to i(Rb)$ in $\mathcal{B}$, such that for every object $c \in \mathcal{A}$ and every morphism $f\colon b \to i(c)$ in $\mathcal{B}$, there exists a unique morphism $\bar{f}\colon Rb \to c$ in $\mathcal{A}$ satisfying the condition $i(\bar{f}) \circ \eta_b = f$. This universal property of the unit η_b is expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
Rb \arrow[dr, "\bar{f}"] & \\
b \arrow[u, "\eta_b"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & c
\end{tikzcd}$$

1.9 Horizontal composition

We have already seen vertical composition of natural transformations. Now we turn our sight to a second kind of composition, called *horizontal composition*.

Definition 1.22. Given
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\eta"'] & \mathcal{B} \arrow[r, "H"] & \mathcal{C} \end{tikzcd}$$
we can define
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "HF"{name=U}] \arrow[r, bend right, "HG"'{name=D}] \arrow[Rightarrow, from=U, to=D, "H\eta"'] & \mathcal{C} \end{tikzcd}$$

to be the natural transformation with components at $x \in \mathcal{A}$, $HFX \xrightarrow{H\eta_x} HGX$.
Given the configuration of categories, functors, and natural transformations:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"{name=U1}] \arrow[r, bend right, "G"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "\alpha"'] & \mathcal{B} \arrow[r, bend left, "H"{name=U2}] \arrow[r, bend right, "I"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"'] & \mathcal{C} \end{tikzcd}$$

we have two ways of defining a composite natural transformation from HF to IG. These correspond to the two paths in the following diagram of natural transformations:

$$\begin{tikzcd} HF \arrow[r, Rightarrow, "\beta_F"] \arrow[d, Rightarrow, "H\alpha"'] & IF \arrow[d, Rightarrow, "I\alpha"] \\ HG \arrow[r, Rightarrow, "\beta_G"'] & IG \end{tikzcd}$$

where the first path is $I\alpha \circ \beta_F$ and the second is $\beta_G \circ H\alpha$. These two paths agree by the naturality of β at the morphism $\alpha_x = FX \to GX$. Specifically, for any $X \in \mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} HFX \arrow[r, "\beta_{Fx}"] \arrow[d, "H\alpha_x"'] & IFX \arrow[d, "I\alpha_x"] \\ HGX \arrow[r, "\beta_{Gx}"'] & IGX \end{tikzcd}$$

The resulting natural transformation

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "HF"{name=U}] \arrow[r, bend right, "IG"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\beta*\alpha"'] & \mathcal{C} \end{tikzcd}$$

is called the *horizontal composite* of α and β

Remark. Categories, functors, and natural transformations form a **2-category**.

Chapter 2

Universal Algebra

2.1 Introduction

Universal algebra (also called *general algebra*) studies sets equipped with algebraic operations satisfying prescribed equations. Typical examples include groups, monoids, and rings.

Operations are usually given by functions such as a binary operation $x+y$ or constants such as 0 (nullary operations), and equations are identities like

$$x + 0 = x.$$

Remark. Universal algebra does not directly cover structures such as *fields*. The inverse operation in a field is not a total function, since 0^{-1} is undefined. Moreover, conditions like $x \neq 0$ are inequalities rather than equations, and hence fall outside the purely equational framework.

2.2 Signatures and Algebras

Definition 2.1. A *signature* $\Omega = (\Omega_n)_{n\in\mathbb{N}}$, consists of a set Ω_n for each $n \in \mathbb{N}$.

Remark. Elements of Ω_n are thought of as *n-ary operation symbols*. Therefore 0-ary operations correspond to constants.

Example 2.1.
1. **Monoids.** The signature consists of $\Omega_0 = \{e\}$ and $\Omega_2 = \{m\}$, with $\Omega_n = \varnothing$ for $n \neq 0, 2$.
2. **Groups.** The signature has $\Omega_0 = \{e\}$, $\Omega_1 = \{(-)^{-1}\}$, and $\Omega_2 = \{m\}$.
3. **Rings.** The signature has $\Omega_0 = \{0, 1\}$, $\Omega_1 = \{-\}$, and $\Omega_2 = \{+, \times\}$.

Definition 2.2. Let Ω be a signature. An Ω-*algebra* consists of a set X together with, for each $n \in \mathbb{N}$ and each $m \in \Omega_n$, a function

$$\begin{aligned} m_X \colon X^n &\longrightarrow X : \\ (a_1, ..., a_n) &\longmapsto m_X(a_1, ..., a_n.) \end{aligned}$$

Remark. For $n \in \mathbb{N}$, the set X^n denotes the set of functions $\{1, \ldots, n\} \to X$. In particular, $X^0 = \mathcal{S}et(\emptyset, X) \cong 1$, the singleton set, which is terminal in $\mathcal{S}et$.

If $e \in \Omega_0$, then the corresponding map $e_X \colon 1 \to X$ selects a distinguished element $e_X \in X$. This explains the terminology *constant*.

Notation. An Ω-algebra $(X, (m_X)_{m\in\Omega_n, n\in\mathbb{N}})$ will usually be denoted simply by X.

Example 2.2. Let $\Omega = \{e, m\}$ with e nullary and m binary. An Ω-algebra consists of:

- a binary operation $m_X \colon X^2 \to X \colon (a, b) \mapsto m_X(a, b)$,
- a distinguished element $e_X \colon 1 \to X \colon * \mapsto e_X \in X$.

Such a structure is called a *pointed magma.* No equations (such as associativity) are imposed.

Exercise. Find a signature Ω such that $\mathbb{R}$-vector spaces can be described as Ω-algebras satisfying suitable equations.

Definition 2.3. Let X and Y be Ω-algebras. A *homomorphism* $f \colon X \to Y$ is a function such that for all $n \in \mathbb{N}$ and $m \in \Omega_n$, the diagram

$$\begin{tikzcd} X^n \arrow[r, "f^n"] \arrow[d, "m_X"'] \arrow[dr, phantom, "\sslash"] & Y^n \arrow[d, "m_Y"] \\ X \arrow[r, "f"'] & Y^n \end{tikzcd}$$

commutes. Equivalently,

$$f(m_X(a_1, \dots, a_n)) = m_Y(f(a_1), \dots, f(a_n))$$

for all $a_1, \dots, a_n \in X$.

Remark. This captures the general notions of homomorphisms for monoids, groups, rings, and similar algebraic structures, when we choose the appropriate signature.

Example 2.3. For the monoid signature $\Omega = \{e, \cdot\}$, a homomorphism satisfies

$$f(a \cdot_X b) = f(a) \cdot_Y f(b), \qquad f(e_X) = e_Y.$$

There is a category Ω-$\mathcal{A}\ell g$ whose objects are Ω-algebras and whose morphisms are homomorphisms and it has a forgetful functor

$$U \colon \Omega\text{-}\mathcal{A}\ell g \longrightarrow \mathcal{S}et.$$

Our goal now is to describe its *left adjoint.*

2.3 Terms and Term Algebras

Definition 2.4. Let Ω be a signature and X a set. The set $T_\Omega(X)$ of Ω-*terms in variables* X is defined inductively by

- $T^0_\Omega(X) = X$,
- $T^{i+1}_\Omega(X) = T^i_\Omega(X) \cup \{m(t_1, \dots, t_n) \mid m \in \Omega_n,\ t_j \in T^i_\Omega(X)\}$,
- $T_\Omega(X) = \bigcup_{i\ge 0} T^i_\Omega(X)$.

The *depth* of a term t is the least i such that $t \in T^i_\Omega(X)$.

Example 2.4. Let $\Omega = \{m, e\}$ and $X = \{a, b, c\}$.

- Depth 0: a, b, c
- Depth 1: e, $m(a, b)$, $m(b, a)$, $m(b, c)$
- Depth 2: $m(a, e)$, $m(a, m(b, c))$

Remark. $T_\Omega(X)$ consists of all expressions one can build from the set of formal variables X and operation symbols in Ω

In fact, $T_\Omega(X)$ is an Ω-*algebra*, given $m \in \Omega_n$, we define

$$\begin{aligned} T_\Omega(X)^n &\xrightarrow{m} T_\Omega(X) \colon \\ (t_1, ..., t_n) &\longmapsto m(t_1, ..., t_n) \end{aligned}$$

We also have a function $\eta_X \colon X \to UT_\Omega(X) \colon x \mapsto x$.

Theorem 2.1. *The forgetful functor $U \colon \Omega\text{-}\mathscr{Alg} \to \mathscr{Set}$ has a left adjoint, sending a set X to the term algebra $T_\Omega(X)$.*

Proof. Given $X \xrightarrow{f} UY$ we must show that there exists a unique Ω-*algebra* map $T_\Omega(X) \xrightarrow{\bar{f}} Y$ such that

$$\begin{tikzcd} & UT_\Omega(X) \arrow[dr, "U\bar{f}"] \arrow[d, phantom, "\|"] & \\ X \arrow[ur, "\eta_X"] \arrow[rr, "f"'] & {} & UY. \end{tikzcd}$$

Since the triangle must commute, we must define $\bar{f}x = fx$ for $x \in X$. We define $\bar{f}$ inductively on depth. Depth 0 terms are elements of X, where $\bar{f}x = fx$ as above. Suppose we have defined it on terms of depth $\leq i$. Then consider a term $m(t_1, ..., t_n)$ of depth $i + 1$. Then $t_1, ..., t_n$ are of depth $\leq i$. In order that $\bar{f}$ is a homomorphism, we must set

$$\bar{f}(m(t_1, ..., t_n)) = m_Y(\bar{f}(t_1), ..., \bar{f}(t_n)). \tag{2.1}$$

This completes the definition of $\bar{f}$, and 2.1 shows that it is a homomorphism. Since we were forced to define $\bar{f}$ in this way, it is the unique extension of f to a homomorphism as required. ☐

Example 2.5. Let $\Omega = \{m, e\}$, $X = \{a, b, c\}$ and $X \xrightarrow{f} UY$, we obtain

$$\begin{aligned} T_\Omega(X) &\xrightarrow{\bar{f}} Y \\ a, b, c &\longmapsto fa, fb, fc \\ e &\longmapsto e_Y \\ m(a, m(b, c)) &\longmapsto m_Y(fa, m_Y(fb, fc)) \quad \text{etc.} \end{aligned}$$

2.4 Equations

Definition 2.5. An Ω-*equation* in variables X is a pair $(s, t) \in T_\Omega(X)^2$, usually written as $s = t$.

Example 2.6. For the signature $\Omega = \{e, \cdot\}$, for *magmas*, the following are equations in variables:

- $x \cdot e = x$,
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

2.4.1 Satisfaction of Equations

Let A be an Ω-algebra. Let $a = (a_x)_{x\in X}$ be an X-tuple of elements in A, which we identify with the function $a\colon X \to UA$ defined by $x \mapsto a_x$, where U denotes the forgetful functor. By the universal property of the free Ω-algebra $T_\Omega(X)$, there exists a unique homomorphism $\bar{a}\colon T_\Omega(X) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} UT_\Omega(X) \arrow[dr, "U\bar{a}"] & \\ X \arrow[u, "\eta_X"] \arrow[r, "a"'] \arrow[ur, phantom, "="] & UA \end{tikzcd}$$

Definition 2.6. Let $(s,t) \in T_\Omega(X)$ be an equation and A an Ω-algebra. Then the X-tuple a *satisfies* the equation $s = t$ if $\bar{a}(s) = \bar{a}(t)$. The Ω-algebra A satisfies $s = t$, written $A \models s = t$, if every X-tuple a of A does so.

Example 2.7. Given $x, y, z \xrightarrow{a} A$ a magma, $x, y, z \mapsto a_1, a_2, a_3$, the extension

$$\begin{aligned} T_\Omega(X) &\longrightarrow A \\ (x\cdot y)\cdot z &\longmapsto (a_1 \cdot a_2)\cdot a_3 \\ x\cdot (y\cdot z) &\longmapsto a_1 \cdot (a_2\cdot a_3) \end{aligned}$$

So A satisfies the equation $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ iff A is associative.

Definition 2.7. A *universal algebraic theory* is a pair (Ω, E), where E is a set of Ω-equations. An (Ω, E)-algebra is an Ω-algebra satisfying all equations in E.

Example 2.8.

- Monoids arise from $\Omega = \{\cdot, e\}$ with associativity and unit equations. E.g.

$$E = \{(x\cdot y)\cdot z = x\cdot (y\cdot z),\ x\cdot e = x,\ e\cdot x = x\}$$

- Groups arise by adding a unary inverse operation $\Omega = \{\cdot, e, (-)^{-1}\}$ and inverse equations to E for monoids and E.g.

$$\{x\cdot x^{-1} = e,\ x^{-1}\cdot x = e\} \cup E$$

Remark. Categories such as $\mathcal{Grp}$, $\mathcal{Mon}$, $\mathcal{Rng}$, and $\mathcal{Vect}$ are all of the form (Ω, E)-$\mathcal{Alg}$. This is the basic setting of universal algebra.

Definition 2.8. For (Ω, E) as above, we write (Ω, E)-Alg $\hookrightarrow$ Ω-Alg for the *full subcategory* of (Ω, E)-algebras. This means that the *objects* are (Ω, E)*-algebras* and the *morphisms* are simply the **Ω-algebra homomorphisms** between them.

The inclusion (Ω, E)-Alg $\overset{i}{\hookrightarrow}$ Ω-Alg is a *(fully faithful) functor*.
We obtain a composite forgetful functor to $\mathcal{Set}$, as in the diagram below:

$$\begin{tikzcd} (\Omega, E)\text{-Alg} \arrow[r, hook, "i"] \arrow[dr, "U"'] & \Omega\text{-Alg} \arrow[d, "U_\Omega"] \\ & \mathcal{Set} \end{tikzcd}$$

Later we will show that i and U have left adjoints – in particular, there exist free (Ω, E)-algebras.

Example 2.9.
- All of the algebraic categories earlier considered e.g., $\mathcal{Vect}$, $\mathcal{Grp}$, $\mathcal{Rng}$, $\mathcal{Mon}$, etc. are of the form (Ω, E)-Alg for suitable Ω and E.
- This is the framework and scope of what people typically call *universal algebra.*

Our goal now is to study good properties of categories of the form Ω-Alg & (Ω, E)-Alg. Firstly we look at Ω-Alg, the case of (Ω, E)-Alg follows easily from Ω-Alg. In particular, we will study limits, *kernels* and quotients. For that, we need to understand *subalgebras*, *homomorphic images* & *image factorisation.*

2.4.2 Subalgebras

Definition 2.9. Let $A \in \Omega$-Alg. A *subalgebra* $B \hookrightarrow A$ of A is a subset B of A such that: if $s \in \Omega_n$ & $b_1, \ldots, b_n \in B$, then $s^A(b_1, \ldots, b_n) \in B$.

- In particular, B is then an Ω-algebra and the inclusion $B \hookrightarrow A$ is an *injective homomorphism.*

2.4.3 Homomorphic Images

Definition 2.10. Let $A \in \Omega$-Alg. A *homomorphic image of* A is a surjective homomorphism $f \colon A \twoheadrightarrow B$.

Let $f \colon A \to B \in \Omega$-Alg. Then let $\operatorname{im} f = \{b \in B : \exists a \in A \text{ with } f(a) = b\}$. Then $\operatorname{im} f \hookrightarrow B$ is a *subalgebra* of B.

Indeed, if $s \in \Omega_n$, $b_1 = f(a_1), \ldots, b_n = f(a_n)$, then

$$s(b_1, \ldots, b_n) = s(f(a_1), \ldots, f(a_n)) = f(s(a_1, \ldots, a_n)) \in \operatorname{im} f.$$

2.4.4 Image Factorisation

In particular, we obtain a factorisation in Ω-Alg:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \\ & \operatorname{im} f \arrow[ur, hook, "i"'] & \end{tikzcd}$$

where $e \colon a \mapsto fa$ represents the *homomorphic image* (surjection) and i represents the *subalgebra inclusion.*

Later, we will look at the *First Isomorphism Theorem*, which explains how to view $\operatorname{im} f$ as a quotient.

2.5 Constructions on Ω-algebras

Proposition 2.1. *Ω-$\mathcal{Alg}$ has (infinite) products and equalisers and $U \colon \Omega$-$\mathcal{Alg} \to \mathcal{Set}$ preserves them.*

Proof. Consider a set I and a family $(A_i)_{i\in I}$ of Ω-algebras (i.e. $A\colon I \to (\Omega, E)$-Alg) Their product as sets is the direct product:

$$\prod_{i\in I} A_i = \{\bar{a} = (a_i)_{i\in I} : a_i \in A_i\} \xrightarrow{p_i} A_i$$
$$(a_i)i \in I \longmapsto a_i$$

We want to show that $\prod_{i\in I} A_i$ has the structure of an Ω-algebra such that each p_i is a homomorphism.

This says given $s \in \Omega_n$ and $\bar{a}^1, \ldots, \bar{a}^n$, we have:

$$s(\bar{a}^1, \ldots, \bar{a}^n)_i = s^{A_i}(\bar{a}^1_i, \ldots, \bar{a}^n_i) \in A_i$$

In other words, we are forced to equip $\prod A_i$ with component-wise Ω-algebra structure.

Given $B \in \Omega\text{-}\mathscr{A}\ell g$ and homomorphisms $(f_i : B \to A_i)_{i\in I}$, we have a unique function $f : B \to \prod A_i$ such that $p_i \circ f = f_i$, namely $(fb)_i = f_i(b)$.

We must check f is a homomorphism:

$$f(s(b^1, \ldots, b^n)) = s(fb^1, \ldots, fb^n) \in \prod_{i\in I} A_i$$

Check components at $i \in I$:

$$(f(s(b^1, \ldots, b^n)))_i = f_i(s(b^1, \ldots, b^n)) = s(f_i b^1, \ldots, f_i b^n) = (s(fb^1, \ldots, fb^n))_i$$

since f_i is a homomorphism. $\square$

Given $f, g : A \to B$ in $\Omega\text{-}\mathscr{A}\ell g$. Their equaliser in $\mathscr{S}et$ is $E = \{x \in A : fx = gx\}$.

In fact, E is a subalgebra of A: If $x_1, \ldots, x_n \in E$, such that $fx_i = gx_i$, then:

$$f(s(x_1, \ldots, x_n)) = s(fx_1, \ldots, fx_n) = s(gx_1, \ldots, gx_n) = g(s(x_1, \ldots, x_n))$$

using the assumption that f and g are homomorphisms. In particular, the inclusion $E \hookrightarrow A$ is a homomorphism and $E \hookrightarrow A \rightrightarrows B$ commutes.

For the universal property of the equaliser, consider $h : C \to A$ such that $fh = gh$. By the universal property of the equaliser in $\mathscr{S}et$, we have a unique function $\bar{h} : C \to E$ such that the diagram commutes. Then $\bar{h}(x) = h(x)$, and so $\bar{h}$ is a homomorphism since h is.

2.6 Quotients and Congruences

What about colimits? Key tool will be quotients by congruences. Congruences generalize equivalence relations for sets, normal subgroups for groups, and (2-sided) ideals for rings.

Definition 2.11. Let A be an Ω-algebra. An equivalence relation $E \subseteq A^2$ is called a *congruence* if it is a subalgebra of A^2.

In elementary terms, a congruence is an equivalence relation E such that for $s \in \Omega_n$: $(x_1, y_1) \in E, \ldots, (x_n, y_n) \in E$, then $(s(x_1, \ldots, x_n), s(y_1, \ldots, y_n)) \in E$.

We will write xEy or $x \sim y$ to mean $(x, y) \in E$. If $E \subseteq A^2$ is a congruence, we can form the diagram $E \rightrightarrows A$ in Ω-$\mathscr{A}\ell g$ where $d(x, y) = x$, $c(x, y) = y$. We will form the coequaliser:

$$E \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{p} A/E$$

in Ω-$\mathscr{A}\ell g$.

Elements of A/E are equivalence classes $[a]$ with $p(a) = [a] = \{x : x \sim a\}$. Observe that for p to be a homomorphism we are forced to define:

$$s^{A/E}([a_1], \dots, [a_n]) = [s^A(a_1, \dots, a_n)]$$

Is this operation **well defined**? Suppose $[b_1] = [a_1], \dots, [b_n] = [a_n]$. Then $(b_1, a_1) \in E, \dots, (b_n, a_n) \in E$. Since E is a congruence, we have:

$$(s(b_1, \dots, b_n), s(a_1, \dots, a_n)) \in E$$

so $[s(b_1, \dots, b_n)] = [s(a_1, \dots, a_n)]$ as required.

In particular, A/E is an Ω-algebra and $p : A \to A/E$ is a surjective homomorphism.

Proposition 2.2. *$E \rightrightarrows A \xrightarrow{p} A/E$ is a coequaliser in Ω-$\mathscr{A}\ell g$.*

Proof. Firstly if $(x, y) \in E$ then $pd(x, y) = p(x) = [x] = [y] = pc(x, y)$, so $pd = pc$.

Given $f : A \to B$ with $fd = fc$. This means that if $(x, y) \in E$, i.e., $x \sim y$, then $fx = fy$. Therefore $[x] = [y] \implies fx = fy$. Therefore we can extend f along p:

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & A/E \arrow[d, dashed, "\bar{f}"] \\ & B \end{tikzcd}$$

where $\bar{f}([a]) = f(a)$. Clearly $\bar{f}$ is a homomorphism, since f is. Since p is surjective, $\bar{f}$ is the unique map extending f along p. $\square$

2.7 Kernels and Isomorphism Theorems

Definition 2.12. Let $f : A \to B \in \Omega$-$\mathscr{A}\ell g$. The *kernel* of f is the congruence $K_f = \{(x, y) : fx = fy\} \hookrightarrow A^2$.

It is the equaliser of $f\pi_1$ and $f\pi_2$ and so a subalgebra of A^2.

$$K_f \overset{i}{\hookrightarrow} A^2 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \xrightarrow{f} B$$

Now f satisfies $fd = fc$ (where d, c are projections from K_f).

$$K_f \underset{c=\pi_2 i}{\overset{d=\pi_1 i}{\rightrightarrows}} A$$

Therefore, we get a unique factorisation of f through the coequaliser A/K_f:

$$\begin{tikzcd} a \arrow[dr, mapsto] & A \arrow[rr, "f"] \arrow[dr, "p"'] & & B & fa \\ & & A/K_f \arrow[ur, "\bar{f}"'] & & \\ & & {[a]} \arrow[uurr, mapsto] & & \end{tikzcd}$$

Theorem 2.2 (**First Isomorphism Theorem**). *Given a homomorphism $f\colon A \to B$ in Ω-Alg, the induced map*

$$t\colon A/K_f \to \operatorname{im} f, \quad [a] \mapsto f(a)$$

is an isomorphism in Ω-Alg.

Proof. Since

$$K_f \overset{d}{\underset{c}{\rightrightarrows}} A \xrightarrow{e} \operatorname{im} f$$

commutes (as $(x,y) \in K_f \implies fx = fy$), we get a unique map t from the coequaliser such that

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "e"'] & A/K_f \arrow[d, "t"] \\ & \operatorname{im} f \end{tikzcd} \qquad \text{commutes.}$$

This says $t[a] = fa$.

- For surjectivity: if $b \in \operatorname{im} f$, then $b = fa$ so $t[a] = fa = b$.
- For injectivity: suppose $t[a] = t[b]$. That is, $fa = fb$. Then $(a,b) \in K_f$, so $[a] = [b]$.

□

Corollary 2.1. *If $f : A \twoheadrightarrow B \in \Omega$-$\mathscr{Alg}$ is surjective, then $A/K_f \cong B$. In particular*

$$K_f \overset{d}{\underset{c}{\rightrightarrows}} A \xrightarrow{f} B \qquad \textit{is coequaliser.}$$

Proof. In this case $\operatorname{im} f = B$. Hence $A/K_f \xrightarrow{t} B$ is an iso by the previous result. Now $K_f \overset{d}{\underset{c}{\rightrightarrows}} A \xrightarrow{p} A/K_f$ is coequaliser. As t is an isomorpism and coequalisers are invariant under isomorphism, the following diagram is a coequaliser:

$$\begin{tikzcd} K_f \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A \arrow[r, "p"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, bend right=20, "\parallel"] & A/K_f \arrow[r, "t"] & B \end{tikzcd}$$

is a coequaliser. □

2.8 Generating Congruences and Colimits

Let A be an Ω-algebra. Let $\operatorname{Cong}(A)$ denote the set of congruences on A.

Remark. $\operatorname{Cong}(A)$ is a poset, ordered by inclusion.

Lemma 2.1. *If $(E_i)_{i\in I}$ is a set of congruences, then $\bigcap_{i\in I} E_i$ is a congruence.*

Proof. Let $s \in \Omega_n$ and $(x_1,y_1),\dots,(x_n,y_n) \in \bigcap E_i$. Then

$$(x_1,y_1),\dots,(x_n,y_n) \in \bigcap_{i\in I} E_i$$
$$(s(x_1,\dots,x_n), s(y_1,\dots,y_n)) \in E_i \ \forall i$$
$$(s(x_1,\dots,x_n), s(y_1,\dots,y_n)) \in \bigcap_{i\in I} E_i$$

Similarly, if $(x,y) \in \bigcap_{i\in I} E_i$ and $(y,z) \in \bigcap_{i\in I} E_i$ then

$$(x_1,y_1),...,(x_n,y_n) \in E_i,\ \forall i$$

so as E_i is congruence (and thus transitive)

$$(x,z) \in E_i,\ \forall i$$
$$(x,z) \in \bigcap_{i\in I} E_i$$

Reflexivity and symmetry are similar. □

Lemma 2.2. *Let $X \subseteq A \times A$. Then there exists a smallest congruence E_X containing X.*

Proof. Consider the set $I = \{E \in \mathrm{Cong}(A) : X \subseteq E\}$, which is non-empty as it contains $A \times A$. Form the congruence $E_X = \bigcap_{E\in I} E$. Then $X \subseteq E_X$ as $X \subseteq E$ for all $E \in I$. Also $E_X \subseteq E$ for any E containing X, by construction. □

Proposition 2.3. *Ω-$\mathcal{A}\ell g$ has coequalisers.*

Proof. Consider $f,g : X \rightrightarrows Y$ in Ω-$\mathcal{A}\ell g$. Let $E_{f,g} \subseteq Y \times Y$ be the congruence generated by $\{(fx,gx) : x \in X\}$. Then we have a commutative diagram:

$$\begin{tikzcd}
(fx,gx) & E_{f,g} \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & Y \arrow[r, "p"] & Y/E_{f,g}, \\
x \arrow[u, mapsto] & X \arrow[u] \arrow[ur, shift left, "f"] \arrow[ur, shift right, "g"'] & &
\end{tikzcd}$$

where $d(x,y) = x$ and $c(x,y) = y$. We proved the quotient map is a coequaliser of the congruence. We must show $pf = pg$ is a coequaliser of f,g.

So let $h : Y \to Z$ in Ω-$\mathcal{A}\ell g$ satisfy $hf = hg$. Recall the congruence $\ker h = \{(a,b) : ha = hb\}$ on Y. Since $hfx = hgx$, we have $\{(fx,gx) : x \in X\} \subseteq \ker h$. Since $E_{f,g}$ is the smallest congruence containing these pairs, $E_{f,g} \subseteq \ker h$. That is, given $(u,v) \in E_{f,g}$, we have $hu = hv$.

Therefore, we obtain a unique homomorphism $\bar{h} : Y/E_{f,g} \to Z$ such that $\bar{h}p = h$. In particular, $Y/E_{f,g}$ is also the coequaliser of f and g. □

Example 2.10. We can "present" algebraic structures using coequalisers. Often one speaks of an algebra generated by elements $x_1,\dots,x_n$ subject to equations $s_j = t_j$:

$$\langle x_1,\dots,x_n \mid s_1 = t_1,\dots,s_m = t_m\rangle$$

where $s_i,t_i \in T_\Omega(\{x_1,\dots,x_n\})$. These correspond to functions $s,t : \{1,\dots,m\} \rightrightarrows UF_\Omega(\{x_1,\dots,x_n\})$. Which define homomorphisms $\bar{s},\bar{t} : F_\Omega(m) \rightrightarrows F_\Omega(\{x_1,\dots,x_n\})$, whose coequaliser is the presented algebra.

Proposition 2.4. *The category (Ω,E)-Alg has all coproducts.*

Proof. Consider Ω-algebras $(X_i)_{i\in I}$. We can form the coproduct $\sum_{i\in I} X_i$ as sets, which is the disjoint union, and then the free Ω-algebra on this:

$$\begin{tikzcd}
X_i \arrow[r, "q_i"] \arrow[rr, bend right, "k_i"', "="] & \sum_{i\in I} X_i \arrow[r, "\eta"] & F_\Omega(\sum_{i\in I} X_i)
\end{tikzcd}$$

But the problem is that the k_i's are not (Ω, E)-Alg homomorphisms. To fix this, we consider the congruence E on $F_\Omega(\sum X_i)$ generated by:

$$\{(k_i(s(x_1, \dots, x_n)), s(k_i x_1, \dots, k_i x_n)) : i \in I, n \in \mathbb{N}, s \in \Omega_n, (x_1, \dots, x_n) \in X_i^n\}$$

Then each composite

$$\begin{tikzcd} X_i \arrow[r, "k_i"] \arrow[rr, bend right, "l_i"'] \arrow[rr, phantom, bend right=20, "\shortparallel"] & F_\Omega\left(\sum_{i\in I} X_i\right) \arrow[r, "p"] & F_\Omega\left(\sum_{i\in I} X_i\right)/E \end{tikzcd}$$

is a homomorphism. Indeed:

$$\begin{aligned} l_i(s(x_1, \dots, x_n)) &:= p(k_i(s(x_1, \dots, x_n))) \\ &= p(s(k_i x_1, \dots, k_i x_n)) \\ &= s(p(k_i x_1), \dots, p(k_i x_n)) \\ &= s(l_i x_1, \dots, l_i x_n). \end{aligned}$$

since pairs are in E and p is a homomorphism.

For the universal property, consider $X_i \xrightarrow{f_i} Y \in (\Omega, E)$-Alg, $\forall i \in I$.

$$\begin{tikzcd} X_i \arrow[r, "q_i"] \arrow[drr, "f_i"'] & \sum_{i\in I} X_i \arrow[r, "\eta"] \arrow[dr, "(1)\exists! f"] & F_\Omega\left(\sum_{i\in I} X_i\right) \arrow[r, "p"] \arrow[d, "(2)\exists! g"] & F_\Omega\left(\sum_{i\in I} X_i\right)/E \arrow[dl, "(3)\exists! h"] \\ & & Y & \end{tikzcd}$$

Figure 2.1:

By the u.p. of $\sum X_i$ in category $\mathcal{S}et$, there exists a unique function $f : \sum X_i \to Y$ such that $f \circ q_i = f_i$.

By the u.p. of the free algebra, there exists a unique homomorphism $g : F_\Omega(\sum X_i) \to Y$ such that $g \circ \eta = f$.

Now

$$\begin{aligned} g(k_i(s(x_1, ..., x_n))) &= g(\eta(q_i(s(x_1, ..., x_n)))) = f(q_i(s(x_1, ..., x_n))) = f_i(s(x_1, ..., x_n)) \\ &= s(f_i x_1, ..., f_i x_n) \\ &= s(g k_i x_1, ..., g k_i x_n) \\ &= g(s(k_i x_1, ..., k_i x_n)). \end{aligned}$$

Therefore, g respects the relations in E, so there exists a unique h as in (3) in 2.1 such that $h \circ p = g$. In particular, $h \circ l_i = f_i$.

For uniqueness, consider the following diagram which commutes for all i:

$$\begin{tikzcd} X_i \arrow[r, "q_i"] & \sum_{i\in I} X_i \arrow[r, "\eta"] & F_\Omega(\sum_{i\in I} X_i) \arrow[r, "p"] & F_\Omega(\sum_{i\in I} X_i)/E \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & Y \end{tikzcd}$$

Then by u.p. of $\sum_{i\in I} X_i$ in category $\mathcal{S}et$,

$$r\circ p\circ\eta = s\circ p\circ\eta.$$

But since $r\circ p, s\circ p$ are Ω-alg homomorphisms, the u.p. of the free algebra implies that $r\circ p = s\circ p$. Since p is a quotient map, it is surjective (epimorphism). Therefore $r = s$. $\square$

Corollary 2.2. *Ω-$\mathcal{A}\ell g$ has all colimits. (Since all colimits can be constructed from coproducts and coequalisers).*

2.9 Projectivity

Our goal now is to extend knowledge of Ω-algebras to (Ω, E)-Alg. The key will be to use closure properties of (Ω, E)-Alg $\hookrightarrow$ Ω-Alg Before turning to closure properties of (Ω, E)-Algs, it will be useful to consider *projectivity*

Definition 2.13. An (Ω, E)-Alg A is *projective* if given a surjective map $f\colon B \twoheadrightarrow C \in$ (Ω, E)-Alg and given a map $g\colon A \to C$

$$\begin{tikzcd}
A \arrow[dr, "g"'] & B \arrow[d, two heads, "f"] \\
& C
\end{tikzcd}$$

there exists a map $\bar{g}\colon A \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "\bar{g}"] \arrow[dr, "g"'] & B \arrow[d, two heads, "f"] \\
& C
\end{tikzcd}$$

Proposition 2.5. *Free algebras in Ω-Alg are projective.*

Proof. Given
$$\begin{tikzcd}
F_\Omega X \arrow[dr, "g"'] & B \arrow[d, two heads, "f"] \\
& C
\end{tikzcd}$$
in Ω-Alg consider

$$\begin{tikzcd}
& & U_\Omega B \arrow[d, "U_\Omega f"] \\
X \arrow[rru, bend left, "t=s\circ U_\Omega g\circ\eta"] \arrow[r, "\eta"'] & U_\Omega F_\Omega X \arrow[r, "U_\Omega g"'] & U_\Omega C \arrow[u, bend left, "\exists s"]
\end{tikzcd}$$

where a map $s\colon U_\Omega C \to U_\Omega B$ such that $U_\Omega f\circ s = 1_{U_\Omega C}$ exists by the Axiom of Choice (since f is surjective). Then there exists a unique $\bar{t}\colon F_\Omega X \to B$ such that the following diagram commutes

$$\begin{tikzcd}
& U_\Omega F_\Omega X \arrow[dr, "U_\Omega\bar{t}"] \arrow[d, phantom, "||"] & \\
X \arrow[ur, "\eta_X"] \arrow[rr, "t"'] & {} & U_\Omega B
\end{tikzcd}$$

Then

$$\begin{tikzcd}
 & U_\Omega F_\Omega X \arrow[dr, "U_\Omega \bar{t}"] \arrow[d, phantom, "||"] & \\
X \arrow[ur, "\eta_X"] \arrow[rr, "t"] \arrow[d, "\eta_X"'] & {} \arrow[d, phantom, "\backslash\backslash"] & U_\Omega B \arrow[d, "U_\Omega f"] \\
U_\Omega F_\Omega X \arrow[rr, "U_\Omega g"'] & {} & U_\Omega C
\end{tikzcd}$$

so $f \circ \bar{t} = g$ by the universal property of freeness (uniqueness of the extension). $\square$

2.10 (Ω, E)-Alg – Closure properties

Proposition 2.6. *The full subcategory (Ω, E)-Alg $\hookrightarrow$ Ω-Alg is closed under products, subalgebras, and quotients.*

Proof. Let $(s, t) \in E$. Then $s, t \in F_\Omega(X)$ for some set X. Recall that an algebra A satisfies the equation $s = t$ if and only if

$$\text{each } f\colon F_\Omega(X) \to A \text{ in } \Omega\text{-Alg satisfies } f(s) = f(t).$$

Products: Consider the product $\prod_{i \in I} A_i \xrightarrow{p_i} A_i$ where each $A_i \in (\Omega, E)$-Alg. Let $f\colon F_\Omega(X) \to \prod_{i \in I} A_i$ be a homomorphism. Then for each component i, we have

$$p_i(f(s)) = p_i(f(t))$$

because the composition $p_i \circ f$ maps to $A_i \in (\Omega, E)$-Alg. Since the images agree in each component, it follows that $f(s) = f(t)$. Thus, $\prod_{i \in I} A_i \in (\Omega, E)$-Alg.

Subalgebras: Let $A \in (\Omega, E)$-Alg and let $B \overset{j}{\hookrightarrow} A$ be a subalgebra. Consider a homomorphism $F_\Omega(X) \xrightarrow{f} B$. Since $A \in (\Omega, E)$-Alg, the composition satisfies

$$j(f(s)) = j(f(t)).$$

As j is injective, this implies $f(s) = f(t)$. Hence, $B \in (\Omega, E)$-Alg.

Quotients: Let $p\colon A \twoheadrightarrow B$ be a surjective homomorphism with $A \in (\Omega, E)$-Alg. Consider a homomorphism $F_\Omega(X) \xrightarrow{f} B$. By the projectivity of $F_\Omega(X)$, there exists a lift $\bar{f}$:

$$\begin{tikzcd}
 & A \arrow[dd, two heads, "p"] & \\
F_\Omega(X) \arrow[ur, dashed, "\exists \bar{f}"] \arrow[dr, "f"'] & & \in \Omega\text{-Alg} \\
 & B &
\end{tikzcd}$$

Since $A \in (\Omega, E)$-Alg, we have $\bar{f}(s) = \bar{f}(t)$. Consequently,

$$f(s) = p(\bar{f}(s)) = p(\bar{f}(t)) = f(t).$$

Therefore, $B \in (\Omega, E)$-Alg as required. $\square$

Corollary 2.3. *The full subcategory (Ω, E)-Alg $\hookrightarrow$ Ω-Alg is closed under limits and coequalizers.*

Proof. Since limits can be constructed from products and equalizers, it is sufficient to establish the case for equalizers (as we already know (Ω, E)-Alg is closed under products).

Consider the equalizer diagram in Ω-Alg:

$$C \overset{e}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad \in \Omega\text{-Alg}$$

Here, C is a subalgebra of A. Since (Ω, E)-Alg is closed under subalgebras in Ω-Alg, it is therefore also closed under equalizers. Consequently, the subcategory is closed under all limits.

For coequalizers, given $f, g \colon A \rightrightarrows B$ with $A, B \in (\Omega, E)$-Alg, the coequalizer in Ω-Alg is the quotient

$$B \twoheadrightarrow B/E_{f,g}.$$

Since this map is surjective and (Ω, E)-Alg is closed under homomorphic images (quotients), it follows that $B/E_{f,g} \in (\Omega, E)$-Alg. Thus, the subcategory is closed under coequalizers. □

Remark. A congruence on an algebra $A \in (\Omega, E)$-Alg is, by definition, a congruence on the underlying Ω-Alg.

Observe then that a congruence $E \subseteq A^2$ is automatically an (Ω, E)-Alg as well, as it is a subalgebra of a product of (Ω, E)-Algs.

Also, we have the following result:

Theorem 2.3 (**First Isomorphism Theorem**). *Given a homomorphism $f \colon A \to B$ in (Ω, E)-Alg, the induced map*

$$t \colon A/K_f \to \operatorname{im} f, \quad [a] \mapsto f(a)$$

is an isomorphism in (Ω, E)-Alg.

Proof. As (Ω, E)-Alg is closed under images (which are subalgebras) and quotients, this result follows immediately from the First Isomorphism Theorem for Ω-Alg. □

2.11 Free (Ω, E)-algebras

We would like to prove:

Proposition 2.7. *The inclusion (Ω, E)-Alg $\hookrightarrow$ Ω-Alg has a left adjoint.*

Since (Ω, E)-Alg $\hookrightarrow$ Ω-Alg is closed under products and subalgebras, a more general statement is:

Proposition 2.8. *Let $\mathcal{C} \xrightarrow{j} \Omega$-Alg be a full subcategory closed under products and subalgebras. Then j has a left adjoint.*

In order to prove it, we need to consider, for an $A \in \Omega$-Alg, the collection $\mathbb{Q}(A)$ of surjections $A \twoheadrightarrow B$ where $B \in \Omega$-Alg.

The problem is that $\mathbb{Q}(A)$ is a proper class—just consider maps to each distinct 1-element Ω-algebra. To resolve this, let us say that $(f, B) \sim (g, C)$ if there exists an isomorphism $h\colon B \to C$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, two heads, "f"'] \arrow[dr, two heads, "g"] & \\ B \arrow[rr, "h", "\cong"'] & & C \end{tikzcd}$$

By the First Isomorphism Theorem, $(f, B) \cong (A/\ker f, p_{\ker f})$. Thus, we have a surjective function

$$\mathrm{Cong}(A) \longrightarrow \mathbb{Q}(A)/\sim$$

defined by $E \longmapsto (A \to A/E)$, which is in fact a bijection.

Since $\mathrm{Cong}(A) \subseteq P(A \times A)$ is a set, it follows that $\mathbb{Q}(A)/\sim$ is a set.

Proposition 2.9. *Let $\mathcal{C} \xrightarrow{j} \Omega$-Alg be a full subcategory closed under products and subalgebras. Then j has a left adjoint.*

Proof. Given $X \in \Omega$-Alg, we need to find $RX \in \mathcal{C}$ and a map $\eta_X\colon X \to RX$ satisfying the universal property: For any homomorphism $f\colon X \to Y$ with $Y \in \mathcal{C}$, there exists a unique $\bar{f}\colon RX \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \\ RX \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Since $\mathbb{Q}(X)/\sim$ is a set, consider a representative set of surjections $\{X \xrightarrow{p_i} X_i : i \in I\}$ for each equivalence class, where each $X_i \in \mathcal{C}$. Thus, every surjection $X \twoheadrightarrow Z$ with $Z \in \mathcal{C}$ is equivalent to one of the p_i's.

Consider the canonical map p into the product:

$$\begin{tikzcd} X \arrow[r, "\exists! p"] \arrow[dr, "p_i"'] & \prod_{i \in I} X_i \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

Then $\prod_{i \in I} X_i \in \mathcal{C}$ as $\mathcal{C}$ is closed under products.

Now factor p through its image:

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "\eta"'] & & \prod_{i \in I} X_i \\ & RX = \mathrm{im}(p) \arrow[ur, hook] & \end{tikzcd}$$

Then $RX \in \mathcal{C}$ as $\mathcal{C}$ is closed under subalgebras.

Next, consider an arbitrary map $f\colon X \to Y$ with $Y \in \mathcal{C}$. Factor f as:

$$X \twoheadrightarrow \mathrm{im}\, f \hookrightarrow Y.$$

Then $\operatorname{im} f \in \mathcal{C}$ as it is a subalgebra of $Y \in \mathcal{C}$. Consequently, the surjection $u : X \twoheadrightarrow \operatorname{im} f$ is equivalent to some p_i in our representative set. This implies there is an isomorphism s (as in the diagram below) making the triangle commute.

We can now construct the required map $\bar{f}$:

$$
\begin{tikzcd}
X \arrow[rrrr, "f"] \arrow[drrr, two heads, "u"] \arrow[ddrr, "p_i"] \arrow[dddr, "p"] \arrow[dddd, two heads, "\eta_X"'] & & & & Y \\
& & & \operatorname{im} f \arrow[ur, tail, "v"'] & \\
& & X_i \arrow[ur, "s"'] & & \\
& \prod X_i \arrow[ur, "\pi_i"'] & & & \\
RX \arrow[ur, tail] \arrow[uuuurrrr, bend right, "\bar{f}"'] & & & &
\end{tikzcd}
$$

(Note: The map s exists as the section/projection relationship, and the diagram allows us to define $\bar{f}$ by composing the inclusion $RX \hookrightarrow \prod X_i$, the projection π_i, the isomorphism u, and the inclusion v).

Moreover, $\bar{f}$ is the unique factorization. Since η_X is surjective, it is an epimorphism, which enforces uniqueness. $\square$

Corollary 2.4. *The inclusion (Ω, E)-Alg $\hookrightarrow \Omega$-Alg has a left adjoint.*

Remark. Here is another construction of the adjoint: Let A be an Ω-algebra and $E(X)$ the set of equations in E in variables X. The problem is that given $(s,t) \in E(X)$, there may exist $f\colon F_\Omega(X) \to A$ such that $f(s) \neq f(t)$. So consider the set

$$R = \{(f(s), f(t)) \in A^2 : (s,t) \in E(X), f\colon F_\Omega(X) \to A, X \in \mathcal{S}et\}$$

and let $\overline{R} \subseteq A^2$ be the congruence it generates.

Now form $p\colon A \twoheadrightarrow A/\overline{R}$. Consider $k\colon F_\Omega(X) \to A/\overline{R}$. We must show $k(s) = k(t)$. As p is surjective, we can find a lift $\bar{k}$ (using projectivity of free algebras):

$$
\begin{tikzcd}
F_\Omega X \arrow[d, "\bar{k}"'] \arrow[dr, "k"] \arrow[dr, phantom, "/\!/", near end, shift right=2ex] & \\
A \arrow[r, "p"'] & A/\bar{R}
\end{tikzcd}
$$

But then $(\bar{k}(s), \bar{k}(t)) \in \overline{R}$, so

$$k(s) = p\bar{k}(s) = p\bar{k}(t) = k(t).$$

Hence $A/\overline{R} \in (\Omega, E)$-Alg.

For the universal property, consider $g\colon A \to B$ with $B \in (\Omega, E)$-Alg. Given $(f(s), f(t)) \in R$ for some $f\colon F_\Omega(X) \to B$, we have

$$g(f(s)) = g(f(t))$$

as B satisfies the equations $(s,t) \in E$. Hence $R \subseteq \ker g$, and so $\overline{R} \subseteq \ker g$. Thus we get a unique factorization:

$$\begin{tikzcd} A \arrow[r, two heads, "p"] \arrow[dr, "g"'] & A/\overline{R} \arrow[d, dashed, "\exists! \bar{g}"] \\ & B \end{tikzcd}$$

as required.

Corollary 2.5. *The forgetful functor (Ω, E)-Alg $\to \mathcal{S}et$ has a left adjoint.*

Proof. We have:

$$\begin{tikzcd} (\Omega, E)\text{-Alg} \arrow[r, "i"'] \arrow[rd, "U"'] \arrow[r, phantom, "\perp"] & \Omega\text{-Alg} \arrow[l, bend right, "R"'] \arrow[d, shift right, "U_\Omega"'] \arrow[d, phantom, "\vdash"] \\ & \mathcal{S}et \arrow[u, shift right, "F)\Omega"'] \end{tikzcd}$$

Adjoints compose, so $R \circ F_\Omega \dashv U_\Omega \circ i = U$. Indeed:

$$\begin{aligned} (\Omega, E)\text{-Alg}(RF_\Omega X, A) &\cong \Omega\text{-Alg}(F_\Omega X, iA) \\ &\cong \mathcal{S}et(X, U_\Omega iA) \\ &= \mathcal{S}et(X, UA). \end{aligned}$$

□

Proposition 2.10. *Free (Ω, E)-algebras are projective.*

Proof. The proof is just as for free Ω-algebras; it just uses the universal property of freeness. □

2.12 Cocompleteness

Theorem 2.4. *(Ω, E)-Alg is cocomplete.*

Firstly, we know that (Ω, E)-Alg $\hookrightarrow \Omega$-Alg has a left adjoint (it is a reflective subcategory) and that Ω-Alg is cocomplete (has colimits). So it suffices to prove:

Proposition 2.11. *Let $i \colon \mathcal{A} \to \mathcal{B}$ be fully faithful and have a left adjoint R. If $\mathcal{B}$ is cocomplete, so is $\mathcal{A}$.*

Proof. Consider $D \colon J \to \mathcal{A}$. Form $\operatorname{colim}(iD) \in \mathcal{B}$ where $iD \colon J \to \mathcal{A} \to \mathcal{B}$. There is a bijection (natural in $X \in \mathcal{A}$) between:

1. Cocones $(D_j \to X)_{j \in J}$ in $\mathcal{A}$.
2. Cocones $(iD_j \to iX)_{j \in J}$ in $\mathcal{B}$ (since i is fully faithful).
3. Morphisms $\operatorname{colim}(iD) \to iX$ in $\mathcal{B}$.
4. Morphisms $R(\operatorname{colim}(iD)) \to X$ in $\mathcal{A}$ (as $R \dashv i$).

Note: starting at $X = R(\mathrm{colim}(iD))$ with the identity on $R(\mathrm{colim}(iD))$ gives us a cocone $(D_j \to R(\mathrm{colim}(iD)))_{j\in J}$, through which each other cocone factors uniquely. This shows it is the universal cocone. □

Exercise (Alternative proof of cocompleteness of (Ω, E)-Alg). Since we have shown (Ω, E)-Alg has coequalizers, it suffices to prove it has coproducts. Modify the proof that Ω-Alg has coproducts by using free algebras to prove that (Ω, E)-Alg has coproducts.

2.13 Birkhoff's Theorem & Related Topics

Question: Which full subcategories of Ω-Alg are of the form (Ω, E)-Alg for a set of equations E?
Answer: Well, full subcategories (Ω, E)-Alg are closed under homomorphic images, products, and subalgebras. In fact, we will see that $\mathcal{C} = (\Omega, E)$-Alg if and only if it has these closure properties. This is **Birkhoff's Theorem**.

Recall that an equation is a pair $(s, t) \in (T_\Omega X)^2$ for some set X. When we defined equational theories (Ω, E), we required E to be a *set* of equations

$$E \subseteq \bigcup_X \{(s,t) \in (T_\Omega X)^2\}$$

but X was not fixed! Note that the collection of all equations is a class, not a set. In fact, it suffices to consider equations in a fixed countable set $\omega = \{x_1, x_2, \dots\}$ of variables, and there is only a set of these, since $T_\Omega(\omega)$ is a set.

Proposition 2.12. *Let $\mathcal{C} \subseteq \Omega$-Alg. The following are equivalent:*

1. *$\mathcal{C} = (\Omega, E)$-Alg for E a set of equations in variables ω.*
2. *$\mathcal{C} = (\Omega, E)$-Alg for E a set of equations.*
3. *$\mathcal{C} = (\Omega, E)$-Alg for E a class of equations.*

Firstly, we prove a lemma.

Lemma 2.3. *Let $f\colon X \to Y \in \mathcal{S}et$ be injective, and write $f^* = F_\Omega f\colon F_\Omega X \to F_\Omega Y$ for the induced homomorphism between free Ω-algebras. Let A be an Ω-algebra. Given $s, t \in T_\Omega X$, then*

$$A \models s = t \iff A \models f^*(s) = f^*(t).$$

Proof. Suppose $A \models s = t$. Consider $p\colon F_\Omega Y \to A$. Then the composite

$$F_\Omega X \xrightarrow{f^*} F_\Omega Y \xrightarrow{p} A$$

is a homomorphism, so $p(f^*(s)) = p(f^*(t))$. Therefore $A \models f^*(s) = f^*(t)$.

Conversely, suppose $A \models f^*(s) = f^*(t)$. Must show $A \models s = t$. If A is empty, this is trivial. So assume A is non-empty. Let $v\colon X \to U_\Omega A$ be a valuation. We can find an extension $\exists w\colon Y \to U_\Omega A$ such that the diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "v"'] & & Y \arrow[dl, dashed, "\exists\omega"] \\ & U_\Omega A & \end{tikzcd}$$

(where $w(f(x)) = v(x)$ and $w(y) = a$ for $y \notin \mathrm{im}(f)$, for some fixed $a \in A$). Let $\bar{w}\colon F_\Omega Y \to A$ be the unique map extending w. Then

$$\begin{tikzcd} F_\Omega X \arrow[r, "f^*"] \arrow[dr, "\bar{v}"'] & F_\Omega Y \arrow[d, "\bar{w}"] \\ & A \end{tikzcd}$$

commutes using the universal property of $F_\Omega X$. Therefore $\bar{v}(s) = \bar{w}(f^*(s)) = \bar{w}(f^*(t)) = \bar{v}(t)$. □

Proof of Proposition. Certainly $(1 \Rightarrow 2)$ since it is a special case, and $(2 \Rightarrow 3)$ since a set is a class. So it remains to prove $(3 \Rightarrow 1)$. For this, we show that given $s, t \in T_\Omega X$, there exist $\bar{s}, \bar{t} \in T_\Omega \omega$ such that

$$A \models s = t \iff A \models \bar{s} = \bar{t}.$$

Indeed, then (Ω, E)-Alg $= (\Omega, E')$-Alg for $E' = \{(\bar{s}, \bar{t}) : (s,t) \in E\} \subseteq (T_\Omega \omega)^2$.

Now let $s, t \in T_\Omega X$. We will define a *finite* subset $U \subseteq X$ such that $s, t \in T_\Omega U \subseteq T_\Omega X$. Let $j\colon U \hookrightarrow X$ be the inclusion. Then $j^*\colon T_\Omega U \to T_\Omega X$ is also the inclusion of subsets. Given this, take any injective $i\colon U \to \omega$. Then

$$\begin{aligned} A \models s = t \text{ in vars } X &\iff A \models s = t \text{ in vars } U \quad \text{(by Lemma applied to } j) \\ &\iff A \models i^*(s) = i^*(t) \text{ in vars } \omega \quad \text{(by Lemma applied to } i) \end{aligned}$$

completing the claim. What is U? Well, given $s \in T_\Omega X$ we can inductively define its (finite) set of variables:

- $\mathrm{var}(x) = \{x\}$
- $\mathrm{var}(f(t_1, \ldots, t_n)) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_n) \subseteq X$.

Then $s \in T_\Omega(\mathrm{var}(s))$. So $s, t \in T_\Omega(\mathrm{var}(s) \cup \mathrm{var}(t))$. Let $U = \mathrm{var}(s) \cup \mathrm{var}(t)$. □

As a result, I will allow a class of equations in E.

Closure Properties and HSP

Given a full subcategory $\mathcal{C} \hookrightarrow \Omega$-Alg, we define

$$E(\mathcal{C}) = \{(s,t) : \text{if } A \in \mathcal{C} \text{ then } A \models s = t\},$$

the class of equations satisfied by objects of $\mathcal{C}$. Then $\mathcal{C} \subseteq (\Omega, E(\mathcal{C}))$-Alg obviously, since if $A \in \mathcal{C}$, then A satisfies any equation satisfied by all members of $\mathcal{C}$.

Write:

- $H(\mathcal{C})$ = closure of $\mathcal{C}$ in Ω-Alg under **homomorphic images**.
- $S(\mathcal{C})$ = closure under **subalgebras**.
- $P(\mathcal{C})$ = closure under **products**.

We define $HSP(\mathcal{C})$ as the smallest full subcategory containing $\mathcal{C}$ closed under these three properties.

Note. It exists! It can be constructed as the intersection of all full subcategories of Ω-Alg with these properties (of which Ω-Alg is one).

Theorem 2.5. *For $\mathcal{C} \subseteq \Omega$-Alg as above, $HSP(\mathcal{C}) = (\Omega, E(\mathcal{C}))$-Alg.*

Proof. Since $\mathcal{C} \subseteq (\Omega, E(\mathcal{C}))$-Alg and the latter is closed under H, S, P we have $HSP(\mathcal{C}) \subseteq (\Omega, E(\mathcal{C}))$-Alg. For brevity, we will write $E(\mathcal{C})$-algebra for $(\Omega, E(\mathcal{C}))$-algebra in what follows. We must prove each $E(\mathcal{C})$-algebra belongs to $HSP(\mathcal{C})$.

Free $E(\mathcal{C})$-algebras $F_{\mathcal{E}}X$ exist on any set X, with unit $X \to UF_{\mathcal{E}}X$. For each $E(\mathcal{C})$-algebra A, we have $F_E(UA) \in E(\mathcal{C})$-Alg and a unique homomorphism ϵ_A such that:

$$\begin{tikzcd} UF_EUA \arrow[dr, "U\epsilon_A"] & \\ & UA \\ UA \arrow[uu, "\eta_{UA}"] \arrow[ur, "1_{UA}"'] & \end{tikzcd}$$

Hence ϵ_A is surjective, implying A is a homomorphic image of a free $E(\mathcal{C})$-algebra. So it suffices to prove each free $E(\mathcal{C})$-algebra belongs to $SP(\mathcal{C})$.

To this end, let $F_\Omega X$ be the free Ω-algebra on X and let

$$N = \{(s,t) \in (F_\Omega X)^2 : \mathcal{C} \not\models s = t\}.$$

Then given $(s,t) \notin N$, there exists $A_{s,t} \in \mathcal{C}$ and a map $k_{s,t} \colon F_\Omega X \to A_{s,t}$ such that $k_{s,t}(s) \neq k_{s,t}(t)$. Consider the induced map k to the product:

$$\begin{tikzcd} F_\Omega X \arrow[r, "k"] \arrow[dr, "q_X"'] & \prod_{(s,t)\notin N} A_{s,t} \\ & RX \arrow[u, hook, "i"'] \end{tikzcd}$$

Factor k through its image $\mathrm{im}(k) = RX$ in Ω-Alg. Then $RX \in SP(\mathcal{C}) \subseteq E(\mathcal{C})$-Alg. It remains to prove that RX is the free $E(\mathcal{C})$-algebra on X.

For this, it is enough to show that if A is an $E(\mathcal{C})$-algebra, then each $f \colon F_\Omega X \to A$ factors uniquely through $q_X \colon F_\Omega X \twoheadrightarrow RX$. (Since $(\Omega, E(\mathcal{C}))\text{-Alg}(RX, A) \cong \Omega\text{-Alg}(F_\Omega X, A) \cong \mathcal{S}et(X, U_\Omega A)$).

Consider again the diagram:

$$\begin{tikzcd} F_\Omega X \arrow[r, "k"] & \prod_{(s,t)\notin N} A_{s,t} \\ & RX \arrow[u, hook] \end{tikzcd}$$

Now if $(s,t) \notin N$, then $k(s) \neq k(t)$ because they are unequal in the (s,t)-component (by construction of $k_{s,t}$). On the other hand, if $(s,t) \in E(\mathcal{C})$, then $k(s) = k(t)$ as

$$\prod A_{s,t} \in HSP(\mathcal{C}) \subseteq E(\mathcal{C})\text{-Alg}.$$

Therefore $k(s) = k(t) \iff (s,t) \in E(\mathcal{C})$.

Since $k = i \circ q_X$ and i is injective, therefore $q_X(s) = q_X(t) \iff (s,t) \in E(\mathcal{C})$. Thus $\ker(q_X) = \{(s,t) \in (F_\Omega X)^2 : (s,t) \in E(\mathcal{C})\}$. Since q_X is surjective, by the First Isomorphism Theorem, RX is the quotient of $F_\Omega X$ by its kernel. Therefore given $f : F_\Omega X \to A$ such that $f(s) = f(t)$ for each $(s,t) \in E(\mathcal{C})$, there exists a unique $\bar{f} : RX \to A$:

$$\begin{tikzcd} F_\Omega X \arrow[r, "f"] \arrow[d, "q_X"'] & A \\ RX \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

In particular, if A is an $E(\mathcal{C})$-algebra, then f has this property, so we obtain the unique factorization. □

Theorem 2.6 (**Birkhoff's Theorem**). *A full subcategory $\mathcal{K} \subseteq \Omega$-Alg is of the form (Ω, E)-Alg if and only if $\mathcal{K}$ is closed under homomorphic images, products, and subalgebras.*

Proof. By last week, (Ω, E)-Alg has these closure properties. Conversely, $\mathcal{K} = HSP(\mathcal{K}) = (\Omega, E(\mathcal{K}))$-Alg using the previous theorem. □

Algebraic Functors

By an **algebraic functor** $W : (\Omega', E')\text{-Alg} \to (\Omega, E)\text{-Alg}$, we mean one which commutes with the forgetful functors to $\mathcal{S}et$:

$$\begin{tikzcd} (\Omega', E')\text{-Alg} \arrow[rr, "W"] \arrow[dr, "U'"'] & & (\Omega, E)\text{-Alg} \arrow[dl, "U"] \\ & \mathcal{S}et & \end{tikzcd}$$

Examples: $U : \mathcal{R}ng \to \mathcal{A}b$, $\mathcal{G}rp \to \mathcal{M}on$ (forgetting inverses).

Theorem 2.7. *Each algebraic functor has a left adjoint.*

Proof. A W-reflection of $X \in (\Omega, E)$-Alg is a map $\eta_X : X \to W(X')$ which is universal, in the sense that given $f : X \to WY$, there exists a unique $\bar{f} : X' \to Y$ such that $W\bar{f} \circ \eta_X = f$. This implies natural bijections $(\Omega', E')\text{-Alg}(X', Y) \cong (\Omega, E)\text{-Alg}(X, WY)$.

We must show that each X has a W-reflection. Then W has a left adjoint.

Step 1: Each free algebra has a W-reflection. Indeed, the natural bijection

$$(\Omega', E')\text{-Alg}(F'X, A) \cong \mathcal{S}et(X, U'A) = \mathcal{S}et(X, UWA) \cong (\Omega, E)\text{-Alg}(FX, WA)$$

proves it (where $F'X$ is the free algebra in the domain category and FX in the codomain).

Step 2: If $D : J \to (\Omega, E)$-Alg and each D_i has a W-reflection D_i', then $\operatorname{colim} D$ has a W-reflection. Let $D_i \xrightarrow{\eta_i} W(D_i')$. Then $D' : J \to (\Omega', E')$-Alg is a functor, so can form $\operatorname{colim} D'$, which we claim is the W-reflection of $\operatorname{colim} D$. Have bijections between:

1. Maps $\operatorname{colim} D \to WY$ (definition of colimit).

2. Cocones $(D_i \xrightarrow{f_i} WY)_{i \in J}$.

3. Cocones $(D'_i \xrightarrow{\bar{f}_i} Y)$ (W-reflection of D_i).

4. Maps $\operatorname{colim} D' \to Y$ (definition of colimit).

So (Ω', E')-Alg$(\operatorname{colim} D', Y) \cong (\Omega, E)$-Alg$(\operatorname{colim} D, WY)$, proving the claim.

Step 3: Each algebra is a coequaliser of frees. Let $A \in (\Omega, E)$-Alg. Consider $FUA \xrightarrow{\epsilon_A} A$. Therefore ϵ_A is surjective (as $\epsilon_A(x) = x$ for all $x \in UA$). Thus, A is the coequaliser of the kernel pair of ϵ_A:

$$K \rightrightarrows FUA \xrightarrow{\epsilon_A} A$$

(where $K = FUA \times_A FUA \subseteq FUA \times FUA$). Now consider the free algebra on K, giving $FUK \rightrightarrows FUA \to A$. Since $\epsilon_K \colon FUK \to K$ is surjective (epi), the coequaliser of the composite pair is the same as the coequaliser of the kernel pair, that is, A.

Since each algebra is a colimit (coequaliser) of free algebras (which have reflections), and reflections are preserved by colimits (Step 2), every algebra has a W-reflection. Thus W has a left adjoint. $\square$

Chapter 3

Modules

3.1 R-definitions

Definition 3.1. Let R be a ring. A left R-module $(M, +, 0)$ is an abelian group M with a function

$$\cdot : R \times M \longrightarrow M \tag{3.1}$$

satisfying the following axioms for all $r, s \in R$ and $a, b \in M$:

$$\left.\begin{array}{ll} 1) & r \cdot (a+b) = r \cdot a + r \cdot b \\ 2) & (r+s) \cdot a = r \cdot a + s \cdot a \end{array}\right\} \text{bilinear}$$

$$\left.\begin{array}{ll} 3) & (rs) \cdot a = r \cdot (s \cdot a) \\ 4) & 1 \cdot a = a \end{array}\right\} \text{associative unital action}$$

Definition 3.2. Let M, N be R-modules. A function $f : M \longrightarrow N$ is called a module homomorphism if for all $x, y \in M$ and $r \in R$,

$$f(x+y) = f(x) + f(y),$$
$$f(r \cdot x) = r \cdot f(x).$$

Homomorphisms of R-modules are sometimes called R-linear maps.

3.2 Modules as Algebraic Structures

There is a category R-Mod of left R-modules which has forgetful functors

$$\begin{tikzcd} R\text{–Mod} \arrow[r] \arrow[dr, "U"'] & \mathcal{Ab} \arrow[d] \\ & \text{Set} \end{tikzcd}$$

The theory of modules can be naturally subsumed under the framework of universal algebra. This categorical perspective allows us to inherit general results concerning limits, colimits, and adjoint functors. Modules fit into the framework of universal algebra. Indeed, let

$$\Omega = \{+, 0\} \cup \{r \cdot - : r \in R\}$$

where $+$ is binary operation symbol, 0 is nullary operation symbol and $r \cdot$ - is unary operation symbol. Let E be the set of equations:

$$\begin{aligned}
&(x+y)+z = x+(y+z)\\
&x+y = y+x\\
&x+0 = x\\
&r\cdot(x+y) = r\cdot x + r\cdot y\\
&(r+s)\cdot x = r\cdot x + s\cdot x\\
&(r\cdot s)\cdot x = r\cdot(s\cdot x)\\
&0\cdot x = 0\\
&1\cdot x = x
\end{aligned}$$

Then R-Mod $\cong (\Omega, E)$-Alg. By our earlier results on (Ω, E)-Alg has R-Mod all limits and colimits, and each forgetful functor in

$$\begin{tikzcd}
R\text{–Mod} \arrow[r, "U'"] \arrow[dr, "U"'] & \mathcal{Ab} \arrow[d, "U''"] \\
& \mathrm{Set}
\end{tikzcd}$$

has a left adjoint.

Example 3.1. Ring $R(+,\cdot,0,1)$ itself is an R-module, where $r+s$, 0, $r\cdot s$ are the operations. It is the free R-module on 1 generator. Given M an R-module and $a \in M, \exists!$ linear map $f : R \longrightarrow M$ such that $f(1) = a$, namely $f(r) = r\cdot f(1) = r\cdot a$. That this is a homomorphism just says $(r+s)\cdot a = r\cdot a + s\cdot a$ and $(r\cdot s)\cdot a = r\cdot(s\cdot a)$ which holds since M is an R-module.

Example 3.2. Let $R(+,\cdot,0,1)$ be ring, for any $n \in \mathbb{N}$ is the product R^n with poin-twise structure an R-module.

Example 3.3. Let $R = K$ be a field, then K-module is a vector space over K.

Example 3.4. Let $\mathbb{Z}(+,\cdot,0,1)$ be the ring of integers. A $\mathbb{Z}$-module is an abelian group. Indeed, if M is an abelian group, we are forced to define $\cdot : \mathbb{Z}\times M \longrightarrow M$ as follows:

Since bilinearity implies - $\cdot\, m : \mathbb{Z} \longrightarrow M$ is a homomorphism of abelian groups and since $\mathbb{Z}$ is the free abelian on 1 element, we are forced to define

$$n\cdot m = \underbrace{(1+...+1)}_{\text{n times}}\cdot m = \underbrace{1\cdot m + ... + 1\cdot m}_{\text{n times}} = \underbrace{m+...+m}_{\text{n times}}, \quad n \geq 0.$$

And similarly for negative n.

Example 3.5. Let K be a field, let $K[x]$ be the polynomial ring in 1 variable x. $K[x]$-modules amount to K-vector spaces V together with a K-linear map $T : V \longrightarrow V$. These are important because they capture $(n\times m)$-matrices with polynomial coefficients in $K[x]$-linear maps

$$K[x]^n \longrightarrow K[x]^m,$$

$$\text{e.g.} \quad \begin{pmatrix} x^2+1 & 3 \\ 2x^7 & 9x+4 \end{pmatrix}.$$

Example 3.6. Let G be a group and R be a ring. We can form the **group ring** $R[G]$, defined as the set of all formal R-linear combinations:

$$R[G] = \{\lambda_1 \cdot g_1 + ... + \lambda_k \cdot g_k : \lambda_1, ..., \lambda_k \in R, g_1, ..., g_k \in G\}$$

This structure forms an abelian group under component-wise addition. The multiplication in $R[G]$ is defined by:

$$\left(\sum_{i=1}^{k} \lambda_i \cdot g_i\right) \cdot \left(\sum_{j=1}^{l} \eta_j \cdot h_j\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} (\lambda_i \cdot \eta_j) \cdot (g_i \cdot h_j)$$

where the operation is obtained by extending the multiplication of G bilinearly.

Left $R[G]$-modules are fundamental in algebra and are often called **group representations**. An $R[G]$-module structure on an R-module M is equivalent to providing an action

$$\cdot : G \times M \longrightarrow M$$

satisfying the following axioms for all $g, h \in G$ and $m \in M$:

1) $1_G \cdot m = m$ (where 1_G is the identity of G),

2) $(g \cdot h) \cdot m = g \cdot (h \cdot m)$,

3) $g \cdot (m + n) = g \cdot m + g \cdot n$ and $g \cdot (r \cdot m) = r \cdot (g \cdot m)$ for $r \in R$.

This is of most interest when $R = K$ is a field and M is a **finite-dimensional** vector space, e.g., $M = K^n$. In this case, a $K[G]$-module corresponds to a group homomorphism:

$$\rho : G \longrightarrow \mathrm{GL}(n, K)$$

into the general linear group of invertible $(n \times n)$-matrices over K. Further properties of these structures will be studied in detail in *Algebra IV*.

Example 3.7.

- Let Man be the category of smooth manifolds and smooth functions.

- Let $M \in \mathrm{Ob}(\mathrm{Man})$. We define $C^\infty(M) := \mathrm{Hom}_{\mathfrak{Man}}(M, \mathbb{R})$ as the set of smooth real-valued functions on M. This set forms a ring under point-wise operations:

$$(f + g)(x) := f(x) + g(x)$$
$$(f \cdot g)(x) := f(x) \cdot g(x)$$

  for all $f, g \in C^\infty(M)$ and $x \in M$.

- A vector bundle over M is a triple (E, p, M), where $p\colon E \longrightarrow M$ is a smooth map between manifolds such that each fiber $p^{-1}(x)$ is a finite-dimensional vector space.

- A section s of the bundle p is a smooth map $s\colon M \longrightarrow E$ such that $p \circ s = \mathrm{id}_M$. This can be represented by the following diagram:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ M \arrow[ur, "s"] \arrow[r, "\mathrm{id}_M"'] & M \end{tikzcd}$$

- **Example:** Let $TM \xrightarrow{p} M$ be the tangent bundle of M. A smooth section $s \in \mathrm{Sect}(p)$ is called a smooth vector field.
- Let $\mathrm{Sect}(p)$ be the set of all smooth sections of p. Then $\mathrm{Sect}(p)$ is a $C^\infty(M)$-module. Indeed, given $s, s' \in \mathrm{Sect}(p)$ and $f \in C^\infty(M)$, we can define:

  1) **Addition:** Since $s(x), s'(x) \in p^{-1}(x)$ and $p^{-1}(x)$ is a vector space, we have:

  $$(s + s')(x) \coloneqq s(x) + s'(x) \in p^{-1}(x)$$

  2) **Scalar multiplication:** Using the ring $C^\infty(M)$, we have:

  $$(f \cdot s)(x) \coloneqq f(x) \cdot s(x) \in p^{-1}(x)$$

  Thus, $s + s' \in \mathrm{Sect}(p)$ and $f \cdot s \in \mathrm{Sect}(p)$, which satisfies the axioms of a module.

3.3 Another perspective on R-modules

Let A be an abelian group. The set of homomorphisms $\mathcal{Ab}(A, A)$ forms a ring equipped with:

- **Addition and Zero Element:**

$$\underbrace{\begin{gathered}(f+g)(x) \coloneqq f(x) + g(x)\\ 0(x) \coloneqq 0_A\end{gathered}}_{\text{component-wise structure}}$$

- **Multiplication:**

$$\underbrace{(f \cdot g)(x) \coloneqq (f \circ g)(x) = f(g(x))}_{\text{composition of morphisms}}$$

Proposition 3.1. *There is a bijection between R-module structures on A and ring homomorphisms $R \longrightarrow \mathcal{Ab}(A, A)$.*

Proof. Given an R-module structure defined by the action $\cdot : R \times A \longrightarrow A$, we define a mapping $\varphi\cdot : R \longrightarrow \mathcal{Ab}(A, A)$ by:

$$r \longmapsto r \cdot \text{-} : A \longrightarrow A.$$

Then $r \cdot \text{-} : A \longrightarrow A$ is homomorphism as $r \cdot \text{-}$ is linear. $\varphi \cdot (r + s) = \varphi \cdot (r) + \varphi \cdot (s)$ as $(r + s) \cdot a = r \cdot a + s \cdot a$. Similarly $\varphi\cdot$ preserves multiplication. Conversely, given $\varphi : R \longrightarrow \mathcal{Ab}(A, A)$ we define $R \times A \longrightarrow A$ by:

$$(r, a) \longmapsto \varphi(r) \cdot (a).$$

Which is clearly an inverse part of bijection. $\square$

3.4 Basic Theory of R-modules

We know that the category of R-modules has all limits and colimits and a good theory of kernels and quotients by congruences. These are very simple in the setting of R-modules, just as for abelian groups.

Definition 3.3. Let R be a ring and M be an R-module, a subset $N \subseteq M$ is a *submodule* of M if N is a subgroup of group M that is closed under scalar multiplication by elements of R.

Exercise. Show that specifying a congruence on $M \in R$-Mod is equivalent to specifying a submodule of M.

Definition 3.4. Let $f : M \longrightarrow N$ be a homomorphism in R-Mod. The *kernel* of f is the submodule $\ker f = \{x \in M : f(x) = 0\} \subseteq M$.

Given a submodule $M \subseteq N$, the *quotient* is the set $N/M = \{n + M : n \in N\}$, which is the quotient group equipped with the action $r \cdot (n + M) = r \cdot n + M$. We then have the module homomorphism $p : N \longrightarrow N/M$ defined by $n \longmapsto n + M$.

These constructions correspond to the equalizer and coequaliser in R-Mod:

$$\ker f \hookrightarrow M \underset{f}{\overset{0}{\rightrightarrows}} N \qquad \& \qquad M \overset{0}{\underset{}{\rightrightarrows}} N \longrightarrow N/M$$

Proposition 3.2. *Given $f : M \longrightarrow N$, we have a coequaliser $M \rightrightarrows N \longrightarrow N/\mathrm{im} f$.*

Proof. Here $\mathrm{im} f = \{y \in N : \exists x \in M \text{ s.t. } f(x) = y\}$. By the above, $N \longrightarrow N/\mathrm{im} f$ is a coequaliser, but if $g : N \longrightarrow A$ satisfies $g \circ f = 0$, then $g(\mathrm{im} f) = 0$. Thus, there exists a unique $h : N/\mathrm{im} f \longrightarrow A$ such that $h \circ p = g$. $\square$

Specialising the First Isomorphism Theorem from universal algebra to this setting, we obtain:

Theorem 3.1. *If $f : M \longrightarrow N$ is a morphism in R-Mod, then*

$$M/\ker f \xrightarrow{\sim} \mathrm{im} f : m + \ker f \longmapsto f(m)$$

is an isomorphism. In particular, if f is surjective, then $M/\ker f \cong N$.

This is all routine. What is interesting about R-Mod is what happens for products and coproducts.

Example 3.8. If R is a ring, left ideals of R are precisely the submodules of R.

Example 3.9. If $f : R \longrightarrow S$ is a ring homomorphism, then S becomes an R-module where we define $r \cdot s := f(r) \cdot s$ for $r \in R, s \in S$. In particular, for $I \subseteq R$ an ideal, the diagram

$$I \hookrightarrow R \longrightarrow R/I$$

belongs to R-Mod.

3.5 Initial and Terminal Objects

Proposition 3.3. $0 = \{0\}$ *is both the terminal and initial object in R-Mod.*

Proof. It is clearly terminal. It is initial as we must define $\varphi : 0 \longrightarrow M$ by $\varphi(0) = 0_M$. $\square$

Remark. The zero homomorphism $0 : M \longrightarrow N$ is the composite $M \longrightarrow 0 \longrightarrow N$.

3.6 Products versus Coproducts

Given a set $(M_i)_{i\in I}$ of R-modules, as for any algebraic category, the product $\prod_{i\in I} M_i$ consists of sequences $(a_i)_{i\in I}$ with component-wise module structure.

Definition 3.5. The *direct sum* $\bigoplus_{i\in I} M_i$ is the submodule consisting of those $(a_i)_{i\in I}$ for which $a_i \neq 0$ only for finitely many $i \in I$.

Remark. In particular, if I is a finite set, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$. So for instance, $A \oplus B = A \times B$.

There are R-module homomorphisms $q_j : M_j \longrightarrow \bigoplus_{i\in I} M_i$ defined by $a \longmapsto q_j(a)$ where $(q_j(a))_i = a$ if $i = j$ and 0 otherwise.

Theorem 3.2. *The maps* $(q_i : M_i \longrightarrow \bigoplus_{j\in I} M_j)_{i\in I}$ *exhibit* $\bigoplus_{i\in I} M_i$ *as the coproduct in R-Mod.*

Proof. Consider $(f_i : M_i \longrightarrow N)_{i\in I}$ in R-Mod. We must show that there exists a unique homomorphism f making the following diagram commute:

$$\begin{tikzcd} M_i \arrow[r, "q_i"] \arrow[dr, "f_i"'] & \bigoplus_{j\in I} M_j \arrow[d, dashed, "f"] \\ & N \end{tikzcd}$$

Given $a = (a_i)_{i\in I} \in \bigoplus M_i$, we have $a = q_{i_1}(a_{i_1}) + \cdots + q_{i_n}(a_{i_n})$ where $i_1, \ldots, i_n$ are the indices at which a is non-zero. Then for f to be a homomorphism satisfying $f \circ q_i = f_i$, we must define:

$$f(a) = f_{i_1}(a_{i_1}) + \cdots + f_{i_n}(a_{i_n}).$$

We use that N is commutative to prove that f is a homomorphism. Indeed, given $a, b \in \bigoplus_{i\in I} M_i$, let $i_1, \ldots, i_n, j_1, \ldots, j_m \in I$ be those indices at which a and b are respectively non-zero. Then:

$$\begin{aligned} f(a+b) &= f(\textstyle\sum q_k(a_k + b_k)) \\ &= f_{i_1}(a_{i_1} + b_{i_1}) + \cdots + f_{j_m}(a_{j_m} + b_{j_m}) \\ &= f_{i_1}(a_{i_1}) + f_{i_1}(b_{i_1}) + \cdots + f_{j_m}(a_{j_m}) + f_{j_m}(b_{j_m}) \\ &= (f_{i_1}(a_{i_1}) + \ldots) + (f_{j_1}(b_{j_1}) + \ldots) \\ &= f(a) + f(b). \end{aligned}$$

Also,

$$\begin{aligned} f(r \cdot a) &= f(r(q_{i_1}(a_{i_1}) + \cdots + q_{i_n}(a_{i_n}))) \\ &= f(q_{i_1}(r \cdot a_{i_1}) + \cdots + q_{i_n}(r \cdot a_{i_n})) \\ &= f_{i_1}(r \cdot a_{i_1}) + \cdots + f_{i_n}(r \cdot a_{i_n}) \\ &= r \cdot f_{i_1}(a_{i_1}) + \cdots + r \cdot f_{i_n}(a_{i_n}) \\ &= r \cdot f(a). \end{aligned}$$

□

Remark. In particular, $A \oplus B = A \times B$ is both product and coproduct. Therefore R^n is both the n-fold product and n-fold coproduct in R-Mod. This corresponds to the fact that homomorphisms $R^m \longrightarrow R^n$ correspond to matrices with coefficients in R. Indeed, f as above bijectively corresponds to:

1. $(R^m \xrightarrow{f_i} R^n)_{1 \leq i \leq m}$ (as R^m is a coproduct)
2. $(R \xrightarrow{f_{ij}} R)_{1 \leq i \leq m, 1 \leq j \leq n}$ (as R^n is a product)
3. Elements $(\lambda_{ij} \in R)_{i,j}$ (as R is the free R-module on 1 element)
4. i.e., $(m \times n)$-matrices.

Under this correspondence, composition of homomorphisms corresponds to matrix multiplication.

3.7 Free R-modules

From universal algebra, we have an adjunction:

$$\begin{tikzcd} R\text{-}\mathcal{Mod} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{Set} \arrow[l, shift right, "F"'] \end{tikzcd}$$

and we know that $F(1) = R$, where 1 is a 1-element set. Since each set $X \cong \sum_{x \in X} \{x\} \cong \sum_{x \in X} 1_x$, and the left adjoint F preserves coproducts, we have:

$$F(X) \cong F(\sum_{x \in X} 1) \cong \bigoplus_{x \in X} F(1) \cong \bigoplus_{x \in X} R.$$

That is, $\bigoplus_{x \in X} R$ is the free R-module on X.

We can describe it as follows: recall $\bigoplus_{x \in X} R \leq \prod_{x \in X} R$. Let $X_i \in \prod_{x \in X} R$ denote the sequence whose value at x_i is 1 and 0 otherwise. Then

$$\bigoplus_{x \in X} R = \{r_1 X_1 + \cdots + r_n X_n : x_1, \ldots, x_n \in X, r_1, \ldots, r_n \in R\}$$

is the set of formal R-linear combinations of elements of X with component-wise structure.

3.8 Left versus Right R-modules

A right R-module is an abelian group $(M, +, 0)$ together with an action $M \times R \longrightarrow M$ satisfying:

1. $(a + b) \cdot r = a \cdot r + b \cdot r$
2. $a \cdot (r + s) = a \cdot r + a \cdot s$
3. $a \cdot (r \cdot s) = (a \cdot r) \cdot s$
4. $a \cdot 1 = a$

These form a category Mod-R.

Let R^{op} be the ring with the same structure as R, except with multiplication $r{*}s \coloneqq s{\cdot}r$. (Note: if R is commutative then $R = R^{\text{op}}$ as rings). Then we have an isomorphism R-Mod $\cong R^{\text{op}}$-Mod. If M is a right R-module, the corresponding left R^{op}-module is $(M, +, 0)$ but with $r{*}m \coloneqq m{\cdot}r$. Note that $a{\cdot}(r{\cdot}s) = (a{\cdot}r){\cdot}s$ becomes $(s{*}r){*}a = s{*}(r{*}a)$, which is why we use R^{op} instead.

As a result, right modules over R are left modules over a different ring, which is why it suffices to develop the general theory using just left modules.

3.9 Tensor products of R-modules

Definition 3.6. Let R be a commutative ring and let M, N, L be R-modules. A *bilinear map* is a function

$$F : M \times N \longrightarrow L \tag{3.2}$$

such that for every $m \in M$ and $n \in N$:

- The map $F(m, -) : N \longrightarrow L$ is R-linear.
- The map $F(-, n) : M \longrightarrow L$ is R-linear.

In other words, F is linear in each variable. Explicitly, this means:

$$\begin{aligned} F(m, a+b) &= F(m,a) + F(m,b), \\ F(m, r \cdot a) &= r \cdot F(m,a), \\ F(a+b, n) &= F(a,n) + F(b,n), \\ F(r \cdot a, n) &= r \cdot F(a,n). \end{aligned}$$

Definition 3.7. Let $M, N \in R$-Mod. The *tensor product* $M \otimes N \in R$-Mod comes with a *universal bilinear map*

$$\theta : M \times N \longrightarrow M \otimes N$$

satisfying the following universal property: For any given bilinear map $F : M \times N \longrightarrow L$, there exists a **unique** linear map $\overline{F} : M \otimes N \longrightarrow L$ such that the following triangle commutes:

$$\begin{tikzcd} M \times N \arrow[d, "\theta"'] \arrow[dr, "F"] \arrow[dr, phantom, "=" near start, shift right=3] & \\ M \otimes N \arrow[r, dashed, "\exists!\overline{F}"'] & L \end{tikzcd}$$

Remark. Let $\operatorname{Bil}(M, N; L)$ denote the set of bilinear maps $M \times N \longrightarrow L$. The definition states that we have a bijection:

$$\operatorname{Bil}(M, N; L) \cong R\text{-Mod}(M \otimes N, L)$$

which is natural in L.

We have not yet proved the existence of $M \otimes N$ (which we will do later). Its main properties follow from its defining universal property. We prove them now.

Theorem 3.3. *Let R be a commutative ring.*

1. *The tensor product $M \otimes N$ is unique up to isomorphism.*
2. *The tensor product extends to a functor R-Mod $\times$ R-Mod $\longrightarrow$ R-Mod.*
3. *We have natural isomorphisms of R-modules:*

$$M \otimes N \cong N \otimes M, \quad R \otimes M \cong M, \quad M \otimes R \cong M.$$

Proof. **1. Uniqueness:** Suppose $\theta : M \times N \longrightarrow M \otimes_1 N$ and $\varphi : M \times N \longrightarrow M \otimes_2 N$ are both universal bilinear maps. By their universal properties, there exist unique linear maps $\bar{\varphi} : M \otimes_1 N \longrightarrow M \otimes_2 N$ and $\bar{\theta} : M \otimes_2 N \longrightarrow M \otimes_1 N$ such that the following diagrams commute:

$$\begin{tikzcd} M \otimes_1 N \arrow[r, "\bar{\varphi}"] & M \otimes_2 N \\ M \times N \arrow[u, "\theta"] \arrow[ur, "\varphi"'] & \end{tikzcd} \qquad \& \qquad \begin{tikzcd} M \otimes_2 N \arrow[r, "\bar{\theta}"] & M \otimes_1 N \\ M \times N \arrow[u, "\varphi"] \arrow[ur, "\theta"'] & \end{tikzcd}$$

From the left diagram, we have $\bar{\varphi} \circ \theta = \varphi$. From the right diagram, we have $\bar{\theta} \circ \varphi = \theta$.

Substituting the first into the second, we get $\bar{\theta} \circ (\bar{\varphi} \circ \theta) = \theta$, which implies $(\bar{\theta} \circ \bar{\varphi}) \circ \theta = \theta$.

Observe that the identity map $1_{M \otimes_1 N}$ also satisfies the property $1_{M \otimes_1 N} \circ \theta = \theta$. By the uniqueness part of the universal property for θ (specifically, that the lift of the bilinear map θ is unique), we must have:

$$\bar{\theta} \circ \bar{\varphi} = 1_{M \otimes_1 N}$$

Similarly, by considering the composition $(\bar{\varphi} \circ \bar{\theta}) \circ \varphi = \varphi$, we deduce that $\bar{\varphi} \circ \bar{\theta} = 1_{M \otimes_2 N}$. Thus, $\bar{\varphi}$ and $\bar{\theta}$ are inverse isomorphisms.

2. Functoriality: By R-Mod $\times$ R-Mod, we mean the product category where objects are pairs (A, B) of R-modules and morphisms $(f, g) : (A, B) \longrightarrow (C, D)$ are pairs of linear maps $f : A \longrightarrow C$ and $g : B \longrightarrow D$.

Given (f, g), consider the map $f \times g : A \times B \longrightarrow C \times D$ followed by $\theta_{C,D} : C \times D \longrightarrow C \otimes D$. The composite $\theta_{C,D} \circ (f \times g)$ is bilinear:

$$(\theta \circ (f \times g))(a, -) = \theta(f(a), g(-))$$

This is a composition of two linear maps, hence linear. Similarly for the second variable. Therefore, by the universal property of $A \otimes B$, we obtain a unique linear map $f \otimes g : A \otimes B \longrightarrow C \otimes D$ such that the square commutes:

$$\begin{tikzcd} A \times B \arrow[r, "f \times g"] \arrow[d, "\theta_{A,B}"'] & C \times D \arrow[d, "\theta_{C,D}"] \\ A \otimes B \arrow[r, "f \otimes g"] & C \otimes D \end{tikzcd}$$

Uniqueness implies functoriality.

3. Isomorphisms:

Symmetry ($A \otimes B \cong B \otimes A$): Consider the bilinear map $A \times B \xrightarrow{s_{A,B}} B \times A \xrightarrow{\theta_{B,A}} B \otimes A$, where $s_{A,B}(a, b) = (b, a)$. This induces a unique linear map such that the square commutes:

$$\begin{tikzcd} A \times B \arrow[r, "s_{A,B}"] \arrow[d, "\theta_{A,B}"'] \arrow[dr, phantom, "="] & B \times A \arrow[d, "\theta_{B,A}"] \\ A \otimes B \arrow[r, "\bar{s}_{A,B}"'] & B \otimes A \end{tikzcd}$$

By the universal property, we see that $\overline{s}_{B,A} \circ \overline{s}_{A,B} = 1_{A \otimes B}$, so $\overline{s}_{A,B}$ is an isomorphism.

Unit ($R \otimes A \cong A$): Consider the function $K : R \times A \longrightarrow A$ defined by $(r, a) \longmapsto r \cdot a$. This is clearly bilinear (R-module axiom). Conversely, let $F : R \times A \longrightarrow B$ be bilinear. Then $F(r, a) = F(r \cdot 1, a) = r \cdot F(1, a)$. We obtain a unique factorization through A

$$\begin{tikzcd}
R \times A \arrow[r, "k"] \arrow[d, two heads] & B \\
A \arrow[ur, "F(1,-)"'] &
\end{tikzcd}$$

by

$$F(1, -)\, K(r, a) = F(1, -)\, r \cdot a = F(1, r \cdot a) = r \cdot F(1, a) = F(r, a).$$

so that $R \times A \xrightarrow{k} A$ is the universal bilinear map, thus there exists unique isomorphism $R \otimes A \xrightarrow{l} A$ such that the triangle commutes:

$$\begin{tikzcd}
R \times A \arrow[r, "\theta"] \arrow[dr, "k"'] & R \otimes A \\
& A \arrow[u, "l"']
\end{tikzcd}$$

Finally, we have composite isomorphisms $A \otimes R \cong R \otimes A \cong A$. $\square$

3.10 Construction of the Tensor Product

Given R-modules M, N, consider the free R-module on the set $M \times N$, denoted $F(UM \times UN)$. We have the function $\eta : UM \times UN \longrightarrow UF(UM \times UN)$ mapping $(a, b) \longmapsto (a, b)$. This map is not bilinear. We require precisely the relations E generated by:

$$\begin{aligned}
(a, b + b') &= ((a, b) + (a, b')) \\
(a, r \cdot b) &= r \cdot (a, b) \\
(r \cdot a, b) &= r \cdot (a, b) \\
(a + a', b) &= ((a, b) + (a', b))
\end{aligned}$$

for all $r \in R, a, a' \in M, b, b' \in N$. These do not hold in the free module. Therefore, we consider the submodule $\overline{E}$ generated by these expressions and form the quotient:

$$M \otimes N := F(UM \times UN)/\overline{E}.$$

The composite map $UM \times UN \xrightarrow{\eta} UF(UM \times UN) \twoheadrightarrow U(F(UM \times UN)/\overline{E})$ is bilinear since the quotient forces the equations in E to hold.

Theorem 3.4. *The map*

$$\begin{tikzcd}
UM \times UN \arrow[r, "\eta"] \arrow[rr, bend right, "\theta"'] & UF(UM \times UN) \arrow[r, "p"] & U(F(UM \times UN)/\overline{E})
\end{tikzcd}$$

is the universal bilinear map.

Proof. Consider a bilinear function $H : UM \times UN \longrightarrow UA$. Then there exists a unique linear map $F(UM\times UN) \xrightarrow{\overline{H}} A$ such that $U\overline{H}\circ\eta = H$ i.e. $\overline{H}(a,b) = H(a,b)$ on generators. Then H is bilinear iff $\overline{H}$ forces the equations in E to hold iff $\overline{H}$ factors through quotient map p.

In particular, if H is bilinear, we obtain a unique R-module map $F(UM\times UN)/\overline{E} \xrightarrow{\overline{\overline{H}}} A$ such that the following diagram commutes:

$$\begin{tikzcd} UM \times UN \arrow[rr, "\theta"] \arrow[dr, "H"'] & & U(F(UM \times UN)/\overline{E}) \arrow[dl, "U\overline{\overline{H}}"] \\ & UA & \end{tikzcd}$$

Thus θ is universal bilinear map. □

Remark. Explicitly, $M \otimes N := F(UM \times UN)/\overline{E}$ is generated by elements $a \otimes b := [(a,b)]$ for $a \in M, b \in N$ subject to the relations:

$$\begin{aligned} a \otimes (b+c) &= a \otimes b + a \otimes c \\ a \otimes (rb) &= r(a \otimes b) \\ (a+b) \otimes c &= a \otimes c + b \otimes c \\ (ra) \otimes c &= r(a \otimes c) \end{aligned}$$

Note that not every element of $M \otimes N$ is of the form $a \otimes b$; generally, they are linear combinations $\sum r_i(a_i \otimes b_i)$.

The standard isomorphism $R \otimes A \xrightarrow{\overset{\mathrm{e}}{\sim}}$ sends $r \otimes a \mapsto ra$, since the universal bilinear map $k\colon R \times A \to A$ constructed earlier sends $(r,a) \mapsto r \cdot a$.

3.11 The Internal Hom in R-Mod

Proposition 3.4. *Let $M, N \in R$-Mod. The set R-Mod(M,N) of linear maps is an R-module when we define:*

$$\begin{aligned} (f+g)(a) &= f(a) + g(a) \\ (r \cdot f)(a) &= r \cdot f(a) \end{aligned}$$

Proof. Let's check $f + g \in R\text{-Mod}(M,N)$

$$\begin{aligned} (f+g)(a+b) &= f(a+b) + g(a+b) \\ &= f(a) + f(b) + g(a) + g(b) \\ \text{(as } N \text{ abelian)} \quad &= f(a) + g(a) + f(b) + g(b) \\ &= (f+g)(a) + (f+g)(b). \end{aligned}$$

$$\begin{aligned} (f+g)(ra) &= f(ra) + g(ra) \\ &= rf(a) + rg(a) \\ &= r\left((f+g)(a)\right). \end{aligned}$$

So $f + g \in R\text{-Mod}(M,N)$.

$r \cdot f$ clearly preserves addition. But

$$r \cdot f(sa) = rf(sa) = r \cdot s \cdot f(a) = (rs) \cdot fa = sr \cdot fa = s \cdot (r \cdot f(a))$$

uses that R is commutative. □

Remark. If R is not commutative, R-Mod(M, N) is still an abelian group.

For a fixed $M \in R$-Mod, we define a functor

$$\begin{aligned} R\text{-Mod}(M, -)\colon R\text{-Mod} &\longrightarrow R\text{-Mod}, \\ N &\longmapsto R\text{-Mod}(M, N). \end{aligned}$$

On morphisms, for $f\colon N \to L$ in R-Mod, we define

$$R\text{-Mod}(M, f)\colon R\text{-Mod}(M, N) \longrightarrow R\text{-Mod}(M, L), \qquad \varphi \longmapsto f \circ \varphi.$$

It is straightforward to verify that

$$R\text{-Mod}(M, 1_N) = 1_{R\text{-Mod}(M,N)} \quad \text{and} \quad R\text{-Mod}(M, g \circ f) = R\text{-Mod}(M, g) \circ R\text{-Mod}(M, f),$$

hence R-Mod$(M, -)$ is a functor.

We know that $\otimes : R\text{-Mod} \times R\text{-Mod} \longrightarrow R\text{-Mod}$ defined by $(M, N) \longmapsto M \otimes N$ is a functor. We obtain functors:

$$\begin{aligned} M \otimes - : R\text{-Mod} &\longrightarrow R\text{-Mod} \\ - \otimes N : R\text{-Mod} &\longrightarrow R\text{-Mod} \end{aligned}$$

Specifically, $M \otimes - : R\text{-Mod} \longrightarrow R\text{-Mod}$ sends $N \longmapsto M \otimes N$ and a morphism $f : N \longrightarrow L$ to $1_M \otimes f$.

Proposition 3.5. *We have adjunctions:*

$$- \otimes B \dashv R\text{-}Mod(B, -) \quad \textit{and} \quad B \otimes - \dashv R\text{-}Mod(B, -).$$

Proof. For $- \otimes B \dashv R$-Mod$(B, -)$ to be an adjunction, we need a natural bijection:

$$R\text{-Mod}(A \otimes B, C) \cong R\text{-Mod}(A, R\text{-Mod}(B, C)).$$

This is a composite bijection:

$$R\text{-Mod}(A \otimes B, C) \cong \text{Bilin}(A, B; C) \cong R\text{-Mod}(A, R\text{-Mod}(B, C)).$$

Indeed, given a bilinear map $F : A \times B \longrightarrow C$, the linear map $\overline{F} : A \longrightarrow R$-Mod$(B, C)$ is defined by:

$$a \longmapsto F_a : (b \longmapsto F(a, b)).$$

- The linearity of $F(a, -)$ corresponds to each F_a being linear.
- The linearity of $F(-, b)$ corresponds to $\overline{F}$ being linear.

It is clear that this describes a bijection.

Finally, for $B \otimes - \dashv R$-Mod$(B, -)$, we use symmetry:

$$R\text{-Mod}(B \otimes A, C) \cong R\text{-Mod}(A \otimes B, C) \cong R\text{-Mod}(A, R\text{-Mod}(B, C)).$$

□

Corollary 3.1. *Both $A \otimes -$ and $- \otimes A : R$-Mod $\longrightarrow R$-Mod preserve colimits. In particular:*

1. *They preserve direct sums.*

2. *If $f : M \longrightarrow N$ is a surjective map, then $A \otimes f$ and $f \otimes A$ are surjective.*

Proof. By the previous proposition, they are left adjoints, and left adjoints preserve colimits, hence direct sums.

For surjectivity, let $f : M \twoheadrightarrow N$. Then $\ker f \rightrightarrows M \longrightarrow N$ is a coequalizer. Hence so is:

$$A \otimes \ker f \rightrightarrows A \otimes M \xrightarrow{A \otimes f} A \otimes N$$

where $A \otimes f$ is the map induced by f. By the construction of coequalizers in R-Mod, $A \otimes f$ is surjective. $\square$

The above says:

$$A \otimes \left(\bigoplus_{i \in I} B_i \right) \cong \bigoplus_{i \in I} (A \otimes B_i) \quad \text{and} \quad \left(\bigoplus_{i \in I} A_i \right) \otimes B \cong \bigoplus_{i \in I} (A_i \otimes B).$$

Corollary 3.2. *$R^m \otimes R^n \cong R^{mn}$.*

Proof.

$$\begin{aligned} R^m \otimes R^n &\cong R^m \otimes \left(\bigoplus_n R \right) \\ &\cong \bigoplus_n (R^m \otimes R) \quad \text{(by the above)} \\ &\cong \bigoplus_n R^m \quad (\text{as } A \otimes R \cong A) \\ &\cong \bigoplus_n \left(\bigoplus_m R \right) \\ &\cong R^{n \times m} \end{aligned}$$

$\square$

Example 3.10. If R^n has basis elements $e_i = (0, \ldots, 1, \ldots, 0)$, then $R^{n \times m} \cong R^n \otimes R^m$ has basis $\{e_i \otimes e_j\}_{i=1 \ldots n, j=1 \ldots m}$.

Example 3.11. In $\mathcal{A}b$, consider $\mathbb{Z}_p \otimes \mathbb{Z}_q$ for p, q coprime. In fact, $\mathbb{Z}_p \otimes \mathbb{Z}_q \cong 0$.

Proof. Consider $a \otimes b \in \mathbb{Z}_p \otimes \mathbb{Z}_q$. By Bézout's theorem, there exist $n, m \in \mathbb{Z}$ such that $np + mq = 1$. Then:

$$\begin{aligned} a \otimes b &= (np + mq) \cdot (a \otimes b) \\ &= (np) \cdot (a \otimes b) + (mq) \cdot (a \otimes b) \\ &= n \cdot (pa \otimes b) + m \cdot (a \otimes qb) \\ &= n \cdot (0 \otimes b) + m \cdot (a \otimes 0) \\ &= 0 + 0 = 0. \end{aligned}$$

$\square$

Example 3.12. Recall that if $I \subseteq R$ is an ideal, then I and R/I are R-modules. Consider the submodule

$$IM = \{r_1 m_1 + \cdots + r_n m_n : r_i \in I, m_i \in M, n \in \mathbb{N}\}.$$

Proposition 3.6. *Let M be an R-module. Then $R/I \otimes M \cong M/IM$.*

Proof. $I \hookrightarrow R \twoheadrightarrow R/I$ is a coequalizer in R-Mod. So as $M \otimes -$ preserves colimits, the sequence

$$M \otimes I \xrightarrow{j \otimes \mathrm{id}} M \otimes R \longrightarrow M \otimes R/I \longrightarrow 0$$

is a coequalizer. We have $M \otimes R \cong M$. Then $l \circ (j \otimes \mathrm{id})(r \otimes b) = l(r \otimes b) = r \cdot b \in M$. So $M \otimes R/I$ is the coequalizer of the map $I \otimes M \longrightarrow M$ which sends $a \otimes m \longmapsto a \cdot m$. This equals $M/\mathrm{im}(\alpha) = M/IM$ since $\mathrm{im}(\alpha) = IM$. $\square$

Example 3.13. In $\mathcal{Ab}$, $2\mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$. Indeed, $2\mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2/(2\mathbb{Z})\mathbb{Z}_2 \cong \mathbb{Z}_2/0 \cong \mathbb{Z}_2$.

3.12 Associativity of the Tensor Product

The associativity isomorphism $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$ is a little more subtle. The outline of the proof:

Let $\mathrm{Trilin}(M, N, L; A)$ be the set of trilinear functions $M \times N \times L \longrightarrow A$. We will show there are natural bijections:

1. $\mathrm{Trilin}(M, N, L; A) \cong R\text{-Mod}((M \otimes N) \otimes L, A)$
2. $\mathrm{Trilin}(M, N, L; A) \cong R\text{-Mod}(M \otimes (N \otimes L), A)$

This implies $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$, since then both of these modules classify trilinear functions, via universal trilinear maps $\theta_1 : M \times N \times L \longrightarrow (M \otimes N) \otimes L$ and $\theta_2 : M \times N \times L \longrightarrow M \otimes (N \otimes L)$, and so are isomorphic (in the same way as shown earlier for bilinear maps).

Actually I will just show (1) since (2) is similar. We have:

$$\begin{aligned} \mathrm{Trilin}(M, N, L; A) &\cong \mathrm{Bilin}(M, N; R\text{-Mod}(L, A)) \\ &\cong R\text{-Mod}(M \otimes N, R\text{-Mod}(L, A)) \\ &\cong R\text{-Mod}((M \otimes N) \otimes L, A). \end{aligned}$$

Indeed, given a trilinear map $F : M \times N \times L \longrightarrow A$, the bilinear map is $M \times N \longrightarrow R\text{-Mod}(L, A)$ defined by $(m, n) \longmapsto F(m, n, -)$.

- Each $\overline{F}_{m,n}$ is linear corresponds to $F(m, n, -) : L \longrightarrow A$ being linear.
- Bilinearity of F corresponds to each $F(-, n, l)$ and $F(m, -, l)$ being linear.

It is clear that this describes a bijection.

Remark. A category $\mathcal{C}$ with a "Tensor product" functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and unit $I \in \mathcal{C}$ together with natural isomorphisms:

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C), \quad I \otimes A \cong A, \quad A \cong A \otimes I$$

and $A \otimes B \cong B \otimes A$ satisfying some axioms is called a *symmetric monoidal category.*

It is *closed* if $- \otimes A$ and $A \otimes -$ have right adjoints $[A, -]$. We have (mostly) shown that $(R\text{-Mod}, \otimes, R)$ is a closed symmetric monoidal category.

3.13 Projective and Injective Modules

We have already met projectives in universal algebra. Recall that an (Ω, E)-algebra A is *projective* if given a surjective homomorphism $f : B \twoheadrightarrow C$ and a homomorphism $g : A \longrightarrow C$, there exists a homomorphism $\bar{g} : A \longrightarrow B$ such that the diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, dashed, "\bar{g}"'] \arrow[d, "g"] \\ B \arrow[r, two heads, "f"] & C \end{tikzcd}$$

Projective modules are an instance of the above when (Ω, E)-Alg $= R$-Mod.

We proved already that each free algebra is projective. For the converse, recall that in a category $\mathcal{C}$, we say A is a *retract* of B if there exist morphisms:

$$\begin{tikzcd} A \arrow[r, shift left, "i"] & B \arrow[l, shift left, "p"] \end{tikzcd}$$

such that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "1_A"'] \arrow[dr, phantom, shift left=2, "/\!/"] & B \arrow[d, "p"] \\ & A \end{tikzcd}$$

In this case i is a monomorphism and p is an epimorphism.

Theorem 3.5. *In (Ω, E)-Alg, projectives are precisely the retracts of free algebras.*

Proof. We know that free algebras are projective. Hence it suffices to show that retracts of projective objects are projective.

Let $A \xrightarrow{f} B \xrightarrow{g} A$ satisfy $g \circ f = 1_A$ and assume that B is projective. Let $p\colon M \twoheadrightarrow N$ be surjective and let $k\colon A \to N$ be a morphism.

Since B is projective, there exists $\bar{k}\colon B \to M$ such that

$$p \circ \bar{k} = k \circ g.$$

Taking the composite $\bar{k} \circ f\colon A \to M$, we obtain the commutative diagram

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "1_A"] \arrow[ddr, bend left=60, "k"] & \\ B \arrow[r, "g"] \arrow[d, "\bar{k}"'] \arrow[dr, phantom, "/\!/"] & A \arrow[d, "k"] \\ M \arrow[r, two heads, "p"'] & N \end{tikzcd}$$

Explicitly,

$$p \circ (\bar{k} \circ f) = (p \circ \bar{k}) \circ f = (k \circ g) \circ f = k \circ (g \circ f) = k,$$

hence A is projective.

Conversely, let M be projective. By the universal property of the free algebra $F(UM)$, there exists a unique homomorphism

$$\varepsilon_M \colon F(UM) \to M$$

such that $U\varepsilon_M \circ \eta_{UM} = 1_{UM}$, i.e. the diagram

$$\begin{tikzcd} UFUM \arrow[dr, "U\varepsilon_M"] & \\ UM \arrow[u, "\eta_{UM}"] \arrow[r, "1_{UM}"'] \arrow[ur, phantom, "="] & UM \end{tikzcd}$$

commutes.

Since $U\varepsilon_M = 1_{UM}$, the morphism ε_M is an epimorphism and therefore surjective in (Ω, E)-Alg. As M is projective, there exists a morphism $i \colon M \to F(UM)$ such that

$$\varepsilon_M \circ i = 1_M.$$

Hence M is a retract of the free algebra $F(UM)$. □

In particular, an R-module is projective if and only if it is a retract of a free module. In the additive setting, we can say a bit more.

Here is a useful characterization of retracts in the additive setting.

Proposition 3.7. *In R-Mod, A is a retract of B if and only if A is a direct summand of B (that is, $\exists C$ such that $A \oplus C \cong B$).*

Proof. Suppose $A \oplus C \cong B$. Certainly

$$\begin{aligned} A &\xrightarrow{i_A} A \oplus C, & A \oplus C &\xrightarrow{p_A} A \\ a &\longmapsto (a, 0) & (a, c) &\longmapsto a \end{aligned}$$

satisfying $p_A \circ i_A = 1_A$ so A s a retract of $A \oplus C$. Hence

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[ddrr, "1_A"'] & A \oplus C \arrow[r, "\alpha"] \arrow[dr, "1_{A\oplus C}"'] & B \arrow[d, "\alpha^{-1}"] \\ & & A \oplus C \arrow[d, "p_A"] \\ & & A \end{tikzcd}$$

So A is a retract of B.

Conversely, suppose $A \xrightarrow{i} B \xrightarrow{p} A$ with $p \circ i = 1_A$. Form the quotient

$$\begin{aligned} A \xrightarrow{i} B &\xrightarrow{q} B/\operatorname{im} i \\ b &\longmapsto b + \operatorname{im} i \end{aligned}$$

We will show that

$$\begin{aligned} B &\xrightarrow{\langle p,q\rangle} A \oplus B/\operatorname{im} i \\ x &\longmapsto \langle px, qx\rangle \end{aligned}$$

is an isomorphism. Let $x \in B$. Then

$$x = \underbrace{ipx}_{\in \ker q} + (\underbrace{x - ipx}_{\in \ker p \text{ as } pip = p})$$

Suppose $(px, qx) = (0, 0)$. As $qx = 0$,

$$0 = (x - ipx) + \operatorname{im} i = x + \operatorname{im} i$$

Hence $x \in \operatorname{im} i$ so $x = iy$. But then

$$0 = px = piy = y, \text{ so } x = iy = 0.$$

Hence $\langle p, q \rangle$ is injective. Let $(a, b + \operatorname{im} i \in A \oplus B / \operatorname{im} i$. Consider $ia + (b - ipb) \in B$ Then p sends it to a since

$$pia = a \quad \& \quad pipb = pb$$

And q sends it to

$$qia + qb - qipb = qb = b + \operatorname{im} i$$

Hence $\langle p, q \rangle$ is surjective. ☐

Example 3.14. Consider the ring $R = \mathbb{Z}_6$.

- $\mathbb{Z}_2$ is a $\mathbb{Z}_6$-module (where $m \cdot n = (mn) \pmod 2$).
- $\mathbb{Z}_3$ is a $\mathbb{Z}_6$-module (where $m \cdot n = (mn) \pmod 3$).

$\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module. We have an isomorphism of $\mathbb{Z}_6$-modules:

$$\mathbb{Z}_6 \xrightarrow{\sim} \mathbb{Z}_2 \oplus \mathbb{Z}_3, \quad n \longmapsto (n \pmod 2, n \pmod 3).$$

Since $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are direct summands, they are retracts of $\mathbb{Z}_6$. Thus $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules, but neither is free.

Example 3.15 (Geometry)**.** Let $\mathcal{Man}$ be the category of smooth manifolds and smooth functions. Let $p : E \longrightarrow M$ be a vector bundle over M. Let $\operatorname{Sect}(p)$ be the set of smooth sections of p.

Recall $C^\infty(M)$ is a ring and $\operatorname{Sect}(p)$ is a $C^\infty(M)$-module.

Theorem 3.6 (Serre-Swan)**.** $\operatorname{Sect}(p)$ *is a finitely generated projective module and, if M is connected, then every finitely generated projective module arises in this way.*

3.14 Injective Modules

Dual to projective modules, we have injective ones.

Definition 3.8. A module M is *injective* if for each monomorphism $i : A \hookrightarrow B$ and homomorphism $f : A \longrightarrow M$, there exists $h : B \longrightarrow M$ such that $h \circ i = f$.

$$\begin{tikzcd}
A \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & B \arrow[dl, dashed, "h"] \\
& M &
\end{tikzcd}$$

To understand injective modules in practice, the following is a useful tool.

Theorem 3.7 (**Baer's Criterion**). *An R-module M is injective if and only if it has the extension property with respect to submodules $I \hookrightarrow R$ (i.e., ideals of R).*

Proof. One direction is trivial. Conversely, it suffices to show that M has the extension property with respect to submodule inclusions

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & B \\ & M \end{tikzcd}$$

Consider a map $f : A \longrightarrow M$. Let P be the set of pairs (B_i, f_i) where $A \subseteq B_i \subseteq B$ is a submodule and $f_i : B_i \longrightarrow M$ satisfies

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] \arrow[dr, phantom, shift left=2, "="] & B \arrow[d, "f_i"] \\ & M \end{tikzcd}$$

We define an ordering $(B_i, f_i) \leq (B_j, f_j)$ if $B_i \subseteq B_j$ and $f_j|_{B_i} = f_i$. This is a poset. We claim it has a maximal element, which we will obtain by Zorn's Lemma.

Suppose $\{(B_i, f_i)\}_{i\in I}$ is a chain in this poset. Form $B' = \bigcup_{i\in I} B_i$. This is clearly a submodule. We can define $f^* : \bigcup B_i \longrightarrow M$ by $f^*(x) = f_i(x)$ where $x \in B_i$. This is well-defined: indeed if $x \in B_i$ and $x \in B_j$, then without loss to generality $i \leq j$, so $f_j(x) = f_i(x)$ by definition of the ordering. It is clearly a homomorphism. Therefore, the chain has an upper bound. Hence, by Zorn's Lemma, there exists a maximal element (N, g). as pictured:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & N \arrow[r, hook] \arrow[d, "g"] & B \\ & M & \end{tikzcd}$$

Suppose $N \neq B$. We will show this leads to a contradiction. Let $b \in B \setminus N$ and consider the pullback

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "-\cdot b"'] & R \arrow[d, "-\cdot b"] \\ N \arrow[r] & B \end{tikzcd}$$

which has $I = \{r \in R \colon rb \in N\}$, then we have

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "-\cdot b"'] & R \arrow[d, "-\cdot b"] \\ N \arrow[r] & N + Rb \subseteq B \end{tikzcd}$$

submodule $N + Rb = \{n + Rb \colon n \in N, r \in R\} \subseteq B$.

If we can show $g \colon N \to M$ extends to $N + Rb$, we will contradict the maximality of N. Consider the diagram where h exists by the hypothesis on M (injectivity relative to ideals):

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "\cdot b"'] & R \arrow[d, "\cdot b"] \arrow[dd, bend left, "h"] \\ N \arrow[r, hook] \arrow[dr, "g"'] & N + Rb \\ & M \end{tikzcd}$$

Note that the restriction $h|_I$ satisfies $h(x) = g(xb)$ for all $x \in I$.

Define the extension $l\colon N + Rb \to M$ by:

$$l(n + rb) = g(n) + h(r).$$

To show l is well-defined, suppose $n + rb = n' + r'b$. Then $(r - r')b = n' - n$. Since $n' - n \in N$, we have $(r - r') \in I$. Thus:

$$\begin{aligned} l(n + rb) - l(n' + r'b) &= (g(n) + h(r)) - (g(n') + h(r')) \\ &= g(n - n') + h(r - r') \\ &= g(n - n') + g((r - r')b) \quad (\text{since } r - r' \in I) \\ &= g(n - n' + (r - r')b) \\ &= g(0) = 0. \end{aligned}$$

Thus l extends g to the strictly larger submodule $N + Rb$, contradicting the maximality of (N, g). Hence $N = B$, and therefore f extends to B. This shows that M is injective. ☐

Corollary 3.3. *In $\mathcal{A}b$, A is injective if and only if it is divisible: that is, $\forall a \in A$, $\forall n \in \mathbb{N} \setminus \{0\}$, there exists $b \in A$ such that $nb = a$.*

Proof. Here $R = \mathbb{Z}$ and the ideals are all principal: $n\mathbb{Z}$ for $n \in \mathbb{N}$. For $n = 0$, we have $0 \to \mathbb{Z}$. Now every A has the extension property with respect to $0 \hookrightarrow \mathbb{Z}$, namely $0 \to A$, so this condition is redundant.

Consider $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ for $n \neq 0$. A map $f : n\mathbb{Z} \longrightarrow A$ is specified by $a \in A$ such that $f(n) = a$. To give an extension is to give $g : \mathbb{Z} \longrightarrow A$ with

$$a = f(n) = g(n) = n \cdot g(1).$$

Let $b = g(1)$. Then A is injective iff all such extensions exist, that is, for all a there exists b such that $nb = a$. This is precisely the definition of divisibility. ☐

3.15 Projective and Injective Resolutions

In homological algebra, an important point is that each module M has a projective and injective resolution. The key here is: given M, we can find:

1. A projective P and a surjection $P \twoheadrightarrow M$.
2. An injective I and a monomorphism $M \hookrightarrow I$.

The first we know: take $FUM \xrightarrow{\varepsilon_M} M$, where $F(UM)$ is free (hence projective) and ε_M is surjective.

The second is more complex and is a special case of a construction called the **small object argument**. To understand it, we need to know a little about directed colimits.

Definition 3.9. Let P be a poset. It is said to be *directed* if:

- There exists $a \in P$.
- Given $a, b \in P$, there exists $c \in P$ such that $a \leq c$ and $b \leq c$.

It is λ-directed (where λ is an infinite cardinal) if given $A \subseteq P$ where $|A| < \lambda$, there exists $p \in P$ such that $a \leq p$ for all $a \in A$.

Example 3.16. P is directed $\iff$ P is $\aleph_0$-directed. $\omega = \{0 \to 1 \to 2 \to \dots\}$ is directed.

Definition 3.10. If P is a directed poset, the colimit of $D : P \longrightarrow \mathcal{C}$ is called a *directed colimit.*

Example 3.17. A diagram $A_0 \to A_1 \to A_2 \to \dots$ specifies a chain (of length ω).

3.16 Injectivity & the small object argument

Now we will show that given a module M, there exists an injective module M^* & a mono $M \hookrightarrow M^*$.

Will do this as a special case of a more general construction, called the *Small object argument*, which applies in many other settings.

3.17 Directed colimits

Definition 3.11. If P is a λ-directed poset, the colimit of $D : P \to \mathcal{C}$ is called a λ*-directed colimit.*

Proposition 3.8. *Given $D : P \to \mathcal{S}et$ with P directed, let $\bigcup_{i \in P} Di$ be the coproduct in $\mathcal{S}et$, whose elements we can write as pairs (i, x) where $x \in Di$. Then $\operatorname{col} D = \bigcup_{i \in P} D_i / \sim$ (or $\bigcup Di/ \sim$) where $\sim$ is the equivalence relation:*

$$(i, x) \sim (j, y) \iff \exists (k, z) \text{ with } i, j \leq k$$

and

$$\begin{tikzcd}
x \arrow[ddrr, mapsto] & & \\
Di \arrow[dr, "d_{i,k}"] & & \\
& Dk & z \\
Dj \arrow[ur, "d_{j,k}"'] & & \\
y \arrow[uurr, mapsto] & &
\end{tikzcd}$$

(2) Furthermore $U : R\text{-}Mod \to \mathcal{S}et$ preserves directed colimits.

Proof. (1) That this is an eq-rel follows from fact that P is directed. For transitivity, we use:

Given $[(i,x)] \sim [(j,y)] \sim [(l,z)]$, as depicted: $x \in D_i$, $y \in D_j$, $z \in D_l$. $\exists k$ s.t. $d_{i,k}(x) = d_{j,k}(y)$ in D_k. $\exists m$ s.t. $d_{j,m}(y) = d_{l,m}(z)$ in D_m.

$$\begin{tikzcd}
Di \arrow[dr, "d_{i,k}"] & \\
& Dk \\
Dj \arrow[ur, "d_{j,k}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Dj \arrow[dr, "d_{j,m}"] & \\
& Dm \\
Dl \arrow[ur, "d_{l,m}"'] &
\end{tikzcd}$$

Making these elts equal. By directedness, $\exists n \geq k, m$ so

$$\begin{tikzcd}
Di \arrow[dr, "d_{i,k}"] & & \\
& Dk \arrow[dr, "d_{k,n}"] \arrow[d, phantom, "\|"] & \\
Dj \arrow[ur, "d_{j,k}"] \arrow[dr, "d_{j,m}"'] & {} & Dn \\
& Dm \arrow[ur, "d_{n,m}"'] & \\
Dl \arrow[ur, "d_{l,m}"'] & &
\end{tikzcd}$$

And so $[(x,i)] \sim [(z,l)]$. We have cocone

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & {[(i,x)]} \arrow[dd, phantom, "\|"] \\
& Di \arrow[dr, "q_i"] \arrow[dd, "d_{i,j}"'] & & \\
& & UDi/\sim & {} \\
& Dj \arrow[ur, "q_j"] & & {[(j,d_{i,j}(x))]} \atop \text{using } i,j \leq j \\
d_{i,j}(x) \arrow[urrr, mapsto] & & &
\end{tikzcd}$$

If $(f_i \colon D_i \to X)_{i \in P}$ is a cocone, we must define

$$\bigcup_{i \in P} Di/\sim \xrightarrow{f} X$$
$$[(i,x)] \longmapsto f_i(x)$$

Cocone relations ensure it is well defined. Therefore it is the *colimit* in $\mathcal{S}et$.

(2) Suppose each Di is a R-module & each $d_{i,j} : Di \to Dj$ is a homomorphism. We will define R-module structure on $Di/\sim$. How to define R-module structure?
a) $r \cdot [(i,x)] := [(i,rx)]$
b) Given $[(i_1,x_1)]$, $[(i_2,x_2)]$, $\exists i$ s.t. $i_1, i_2 \leq i$: then define

$$[(i_1,x_1)] + [(i_2,x_2)] = [(i, d_{i_1,i}(x_1) + d_{i_2,i}(x_2))]$$

It is routine to check this defines R-module structure such that each q_i is a R-module homomorphism. and that each $Di/\sim$ is then the colimit in R-Mod. □

Remark. Directed colimits can be constructed in (Ω, E)-Alg as for R-modules. In particular, $U : (\Omega, E)\text{-Alg} \to \mathcal{S}et$ preserves them.

3.18 Finitely presentable & λ-presentable Objects

If $c \in \mathcal{C}$, we can form the functor

$$\mathcal{C} \longrightarrow \mathcal{S}et$$

$$\begin{tikzcd} d \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}(c,d) \arrow[d, "f*"] & c \arrow[r, "\alpha"] \arrow[d, mapsto, shift left=4ex, phantom] & d \\ e & \mathcal{C}(c,e) & c \arrow[r, "f\alpha"'] & e \end{tikzcd}$$

It is called the hom-functor / representable functor.

Definition 3.12. c is *λ-presentable* if $\mathcal{C}(c,-)$ preserves λ-directed colimits. For $\lambda = \omega$, we say c is *finitely presentable.*

Very abstract - here is a more elementary description.

Proposition 3.9. *c is λ-presentable $\iff$ given a λ-directed diagram $D : \mathcal{J} \to \mathcal{C}$ with colimit $Di \xrightarrow{p_i} \operatorname{col} D$,*
(1) given $f : c \to \operatorname{col} D$, then $\exists i \in \mathcal{J}$ & $f_i : c \to Di$ such that f factors through D_i.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[dr, dashed, "f_i"'] & \operatorname{col} D \\ & Di \arrow[u, "p_i"'] \end{tikzcd}$$

(2) given $f, g \colon c \rightrightarrows Di$ such that following diagram commutes:

$$c \underset{g}{\overset{f}{\rightrightarrows}} Di \xrightarrow{p_i} \operatorname{col} D$$

then there exists $j \geq i$ such that following diagram commutes

$$c \underset{g}{\overset{f}{\rightrightarrows}} Di \xrightarrow{p_{i,j}} Dj$$

Proof. Consider the functor

$$\mathcal{C} \coloneqq \mathcal{J} \xrightarrow{D} \mathcal{C} \xrightarrow{\mathcal{C}(c,-)} \mathcal{S}et$$

We have the unique function

$$\begin{tikzcd}
\operatorname{col}\mathcal{C}(c,D-) \arrow[r, phantom, "="] & \bigcup_{i\in\mathcal{J}} \mathcal{C}(c,Di)/\sim \arrow[r, "\lambda"] & \mathcal{C}(c,\operatorname{col}D) \\
& \mathcal{C}(c,Di) \arrow[u, "q_i"] \arrow[ur, "(p_i)*"'] &
\end{tikzcd}$$

commuting with the cocones and $\mathcal{C}(c,-)$ preserves the colimit D iff λ is an isomorphism. Explicitly, $\lambda([(i,\alpha : c \to Di)]) = c \xrightarrow{\alpha} Di \xrightarrow{p_i} \operatorname{col}D$.

Hence λ is surjective iff given $f\colon c \to \operatorname{col}D$ there exists $i \in \mathcal{J}$ such that following diagram commutes:

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[dr, dashed, "f_i"'] \arrow[dr, phantom, "=", shift left=3] & \operatorname{col}D \\
& Di \arrow[u, "p_i"']
\end{tikzcd}$$

λ is injective iff given

$$\begin{tikzcd}
& Di \arrow[dr, "p_i"] & & & & Di \arrow[dr, "d_{i,k}"] & \\
c \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \operatorname{col}D & \exists k \text{ such that} & c \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "="] & & Dk \\
& Dj \arrow[ur, "p_j"'] & & & & Dj \arrow[ur, "d_{j,k}"'] &
\end{tikzcd}$$

This diagram certainly implies (2) if $i = j$. Conversely, if (2) holds, then given

$$\begin{tikzcd}
& Di \arrow[dr, "p_i"] & \\
c \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "="] & & \operatorname{col}D \\
& Dj \arrow[ur, "p_j"'] &
\end{tikzcd}$$

let $k \geq j, i$, we have

$$\begin{tikzcd}
& Di \arrow[dr, "d_{i,k}"] \arrow[drr, bend left, "p_i"] & & \\
c \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "="] & & Dk \arrow[r, "p_k"] & \operatorname{col}D \\
& Dj \arrow[ur, "d_{j,k}"'] \arrow[urr, bend right, "p_j"] & &
\end{tikzcd}$$

and both path $c \rightrightarrows Dk$ become equal in $\operatorname{col}D$. So by (2) there exists $l \geq k$ with

$$d_{k,l} \circ (d_{i,k} \circ \alpha) = d_{k,l} \circ (d_{j,k} \circ \beta)$$

In other words $d_{i,l} \circ \alpha = d_{j,l} \circ \beta$ $\square$

Example 3.18. In $\mathcal{S}et$, the Finitely presentable objects are the finite sets: Each set X is a directed colimit of its finite subsets $p_U\colon U \hookrightarrow X$. Hence if X is finitely presentable, then $1_X : X \to X$ factors through some $p_U \cdot -$ i.e. $X \xrightarrow{g} U \xrightarrow{in} X$.

$$\begin{tikzcd}
X \arrow[rr, "1_X"] \arrow[dr, dashed] & & X \\
& U \arrow[ur, "p_U"'] &
\end{tikzcd}$$

Hence $p_U : U \to X$ is surjective, so $U = X$ and X is Finite.

If X is finite, consider
$$f : X \to \operatorname{col} D = \bigcup_{i \in I} D_i / \sim .$$
Then each $f(x) = [(i_x, a_x)]$ for some i_x. We can choose $i \geq i_x$ for each $x \in X$ and then
$$f(x) = [(i, d_{i_x,i}(a_x))]$$
so f factors through $D_i \xrightarrow{p_i} \operatorname{col} D$. Similarly one can verify the other condition.

Example 3.19. In **Set**, X is λ-presentable $\iff$ $|X| < \lambda$. (Argument similar to that above.)

Example 3.20. In (Ω, E)-Alg, A is λ-presentable $\iff$ it can be presented using fewer than λ generators and equations, which in categorical terms says that there exists sets X, Y of cardinality $< \lambda$ and a coequaliser
$$FY \rightrightarrows FX \to A$$
in (Ω, E)-Alg.

Example 3.21. In R-Mod, this applies. Moreover if M is a module, let $\lambda = \sup(|M|, |R|)$. Let $\lambda < \lambda^+$ be the smallest larger cardinal. Then M is λ^+-presentable.

By preceding point, to prove this we need to show that M is a coequaliser of free modules on sets of card $\leq \lambda$. Indeed, consider $\varepsilon : F_U M \twoheadrightarrow M$ (surjection from free module on elements of M). Then

$$\begin{tikzcd}
FUN \arrow[r, two heads] \arrow[rr, bend left, "t"] & N = \ker(\varepsilon_M) \arrow[r, hook] & FUM \arrow[r, two heads, "\varepsilon_M"] & M
\end{tikzcd}$$

has $M \cong FUM/\operatorname{im} t$ so M is the coequaliser.

$$\begin{tikzcd}
FUN \arrow[r, shift left, "t"] \arrow[r, shift right, "0"'] & FUM \arrow[r, two heads, "\varepsilon_M"] & M
\end{tikzcd}$$

Moreover $|N| \leq |F_U M| \leq \lambda$ since elements of $F_U M$ of form
$$r_1 m_1 + \cdots + r_n m_n$$
for $r \in R, m \in M$.

Hence it is a presentation of M using $\leq \lambda$ generators & $\leq \lambda$ equations. Therefore M is λ^+-presentable by the previous point.

Exercise. Consider

$$\begin{tikzcd} (\Omega, E)\text{-Alg} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "U"'] & \mathcal{S}et \arrow[l, bend right, "F"'] \end{tikzcd}$$

Prove that if A is the coequaliser

$$FX \overset{u}{\underset{v}{\rightrightarrows}} FY \longrightarrow A$$

where $|X|, |Y| < \lambda$, then A is λ-presentable $\in (\Omega, E)$-Alg.
(Hint: use (2) above, that U preserves directed colimits and adjointness).

3.19 Injectivity and the small object argument

Consider a class J of morpisms $j \colon A \to B$ in category $\mathcal{C}$. We say $X \in \mathcal{C}$ is J-injective if given

$$\begin{tikzcd} A \arrow[r, "j\in J"] \arrow[dr, "f"'] & B \\ & X \end{tikzcd}$$

there exists $g \colon B \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "j\in J"] \arrow[dr, "f"'] & B \arrow[d, dashed, "g"] \arrow[dl, phantom, "/\!/"] \\ & X \end{tikzcd}$$

Example 3.22. In R-Mod take $J = \{\text{all monos}\}$. Then X is J-injective iff X is injective. Or by *Baer's criterion*, we can take $J = \{\text{all submodules of } R\}$ and J-injective iff injective.
goal: given X, construct $X \to X^{\#}$ with $X^{\#}$ J-injective.

3.19.1 The small object argument

Theorem 3.8. *Let $\mathcal{C}$ be a cocomplete locally small category and J a set of maps in $\mathcal{C}$ between λ-presentable objects. Then given $X \in \mathcal{C}$, there exists a morphism $p : X \longrightarrow X^{\#}$ where $X^{\#} \in \mathrm{Inj}(J)$, such that p is a λ-directed composite of pushouts of coproducts of maps in J.*

Lemma 3.1 (Construction)**.** *Consider $X \in \mathcal{C}$. We construct $X \xrightarrow{p_X} RX$ such that given*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j\in J"'] & X \\ B \end{tikzcd}$$

There exists f' such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j\in J"'] \arrow[dr, phantom, "/\!/"] & X \arrow[d, "p_X"] \\ B \arrow[r, "f'"'] & RX \end{tikzcd}$$

To do this, let

$$S = \{(j, f) : j : A \to B \in J, F : A \to X \in \mathcal{C}\}$$

This is a set since J is a set and $\mathcal{C}$ is locally small, so $\mathcal{C}(A, X)$ is a set. We have

$$\begin{aligned} S &\longrightarrow \mathcal{C} \\ (j, f) &\longmapsto A \end{aligned}$$

a form its colimit, the coproduct with coproduct inclusions

$$A \xrightarrow{i_{0(j,f)}} \sum_{(j,f)\in S} A$$
$$B \xrightarrow{i_{1(j,f)}} \sum_{(j,f)\in B} B$$

By the universal property of the coproduct, there is a unique arrow making the as below making the square commute for all j:

$$\begin{tikzcd} A \arrow[r, "i_{0(j,f)}"] \arrow[d, "j"'] & \sum_{(j,f)\in S} A \arrow[d, "\sum_{(j,f)\in S} j"] \\ B \arrow[r, "i_{1(j,f)}"'] & \sum_{(j,f)\in S} B \end{tikzcd}$$

We also have $\varepsilon : \sum_{(j,f)\in S} A \longrightarrow X$, the unique arrow such that

$$A \xrightarrow{\quad i_{(j,f)} \quad} \sum_{(j,f)\in S} A \xrightarrow{\quad \varepsilon \quad} X = f \colon A \to X$$

for all $(j, f) \in S$.

Now we form the pushout as in the right square below:

$$\begin{tikzcd} A \arrow[r, "i_{0(j,f)}"] \arrow[d, "j"'] & \sum_{(j,f)\in S} A \arrow[r, "\varepsilon"] \arrow[d, "\sum_{(j,f)\in S} j"'] & X \arrow[d, "p_X"] \\ B \arrow[r, "i_{1(j,f)}"'] & \sum_{(j,f)\in S} B \arrow[r, "q"] \arrow[ur, phantom, "\ulcorner", very near end] & RX \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} A \arrow[r, "i_{0(j,f)}"'] \arrow[d, "j"'] \arrow[rr, bend left, "f"] & \sum_{(j,f)\in S} A \arrow[r, "\varepsilon"] \arrow[d, "\sum_{(j,f)\in S} j"] & X \arrow[d, "p_X"] \\ B \arrow[r, "i_{1(j,f)}"'] & \sum_{(j,f)\in S} B \arrow[r, "q"] & RX \end{tikzcd}$$

commutes as required.

3.20 Ordinals

Let Ord be the (large) poset of ordinals $\alpha < \beta$. It is totally ordered and looks like:

$$0, 1, 2, \dots, \omega, \omega+1, \dots, \omega+\omega, \dots$$

Ordinals are either:

- 0
- Successors $\alpha + 1$
- Limit ordinals like ω.

Let Ord be the (large) poset of ordinals and let $\mathrm{Ord}_{<\alpha}$ be the poset of ordinals $< \alpha$. A functor $X : \mathrm{Ord} \to \mathcal{C}$ involves

$$X_0 \to X_1 \to X_2 \to \dots \to X_\omega \to X_{\omega+1} \to \dots$$

and we will call it an *ordinal chain*. A functor $Ord_{<\alpha} \to \mathcal{C}$ will be a α-chain. Given an α-chain $Ord_{<\alpha}$ with colimit X_α a cocone

$$\begin{tikzcd}
X_0 \arrow[r, "x_{0,1}"] \arrow[drrr, "x_0"'] & X_1 \arrow[r, "x_{1,2}"] \arrow[drr, "x_1"] & \dots \arrow[r] & X_\beta \arrow[r, "x_{\beta,\beta+1}"] \arrow[d, "x_\beta"] & \dots & \dots \arrow[dll, dotted] \\
 & & & X_\alpha & &
\end{tikzcd}$$

We call $x_0 \colon X_0 \to X_\alpha$ the α-composite of the morphisms in the α-chain.

Proof of theorem 3.8. We construct an ordinal chain as follows by transfinite induction:

- $X_0 = X$.
- At ordinal $n+1$, set $X_{n+1} = RX_n$ with $X_n \xrightarrow{p_n^{n+1}} X_{n+1} = X_n \xrightarrow{p_{X_n}} RX_n$.
- At limit ordinal α, set $X_\alpha = \mathrm{col}_{\beta<\lambda}(X_\beta)$, i.e., the colimit of

$$\begin{aligned} Ord_{<\alpha} &\xrightarrow{x_-} \mathcal{C} \\ \beta &\longmapsto X_\beta \end{aligned}$$

  With maps $X_\beta \xrightarrow{p_\beta^\alpha} X_\alpha$ the colimit inclusions.

We set $X^\# = X_\lambda$ with $p : X \to X^\# = p_0^\lambda : X \to X_\lambda$.

Now consider a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & X^\# \\
B &
\end{tikzcd}$$

where $j \in J$. Since λ is regular, $\mathrm{Ord}_{<\lambda}$ is λ-directed. Since A is λ-presentable, $f : A \to X_\lambda$ factors through some X_α for $\alpha < \lambda$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "f'"] & X_\lambda \\ & X_\alpha \arrow[u, "p^\lambda_\alpha"'] \end{tikzcd}$$

But then we have the situation:

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "j"'] & X_\alpha \arrow[d, "p^\lambda_{X_\alpha}"] \\ B \arrow[r, dashed, "f''"] & X_{\alpha+1} = RX_\alpha \end{tikzcd}$$

by the lemma. Then as required we obtain:

$$\begin{tikzcd} A \arrow[r, "f'"'] \arrow[d, "j"'] \arrow[rr, bend left=50, "="] \arrow[dr, phantom, "="] & X_\alpha \arrow[d, "p^{\alpha+1}_\alpha"'] \arrow[dr, "p^\lambda_\alpha"] & \\ B \arrow[r, "f''"'] & X_{\alpha+1} \arrow[r, "p^\lambda_{\alpha+1}"'] \arrow[ur, phantom, "="] & X_\lambda \end{tikzcd}$$

$\square$

Note. If the morphisms in J are between finitely presentable objects, we have an ω-chain

$$X \to RX \to R^2X \to \cdots \to R^nX \to \dots$$

whose colimit is $X^\#$, as below

$$\begin{tikzcd} X \arrow[r] \arrow[drrrr, bend right, "p=p_0"'] & RX \arrow[r] \arrow[drrr, "p_1"'] & R^2X \arrow[r] \arrow[drr, "p_2"] & \dots \arrow[r] & R^nX \arrow[r, dotted] \arrow[d, "p_n"] & \dots \\ & & & & X^\# & \end{tikzcd}$$

Lemma 3.2 (Stability of monos)**.** *In R-Mod,*

a) Coproducts of monos are mono.

b) Pushouts of monos are mono.

c) Directed colimits of monos are mono.

(Precise meaning of these terms will be made clear in the proof).

Proof. a) Let I be a set and consider monos $f_i \colon A_i \to B_i$ for $i \in I$. We will show that following is mono:

$$\bigoplus_{i\in I} A_i \xrightarrow{\bigoplus_{i\in I} f_i} \bigoplus_{i\in I} B_i$$

Consider $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$. Then

$$\bigoplus_{i\in I} f_i(a_i)_{i\in I}$$

If it equals 0, then $f_i a_i = 0$ for all $i \in I$ but as f_i is mono, then $a_i = 0$ for all $i \in I$, so $(a_i)_{i\in I} = 0$. Hence $\ker\left(\bigoplus_{i\in I} f_i\right) = 0$ so $\bigoplus_{i\in I} f_i$ is mono.

b) Consider

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd} \qquad \text{with pushouts} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "j_B"] \\ C \arrow[r, "j_C"'] & D \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

We will show that, if f is mono, the so is j_C. The pushout can be constructed as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i_B"] & \\ C \arrow[r, "i_C"'] & B \bigoplus C \arrow[r] & B \bigoplus C/\operatorname{im}(i_B f - i_C g) \end{tikzcd}$$

Now

$$(i_B f - i_C g)a = \{(fa, 0) = (0, ga) = (fa, -ga)$$

so

$$\operatorname{im}(i_B f - i_C g) = \{(fa, -ga) \colon a \in A\} \coloneqq I$$

Thus we must show that

$$\begin{aligned} C &\xrightarrow{i_C} B \bigoplus C \xrightarrow{p} B \bigoplus C/I \\ c &\longmapsto (0, c) \longmapsto (0, c) + I \end{aligned}$$

is monomorphism. But if $(0, c) + I = 0$, then $(0, c) \in I$. So $(0, c) = (fa, -ga)$ but then $fa = 0$. Since f is monomorphism, $a = 0$ so that $c = -ga = 0$ as required.

c) Let $D \colon \mathcal{J} \to R$-Mod have $\mathcal{J}$ filtered and suppose that each $D\alpha \colon Di \to Dj$ is monomorphism. We will show that each cocone component $p_i \colon Di \to \operatorname{col} D$ is monomorphism. Indeed, $\operatorname{col} D = \bigoplus_{i\in I} Di/\sim$ as earlier constructed and $p_i(x) = [(i, x)]$. If

$$[(i, x) = 0 = [(i, 0)]$$

then by definition of $\sim$, exists $i \xrightarrow{\alpha} j$ such that $D\alpha(x) = D\alpha(0) = 0$, but $D\alpha$ is monomorphism, so $x = 0$ as required. $\square$

Theorem 3.9. *Given $M \in R$-Mod, there is and injective R-mod $M^{\#}$ and a monomorphism $M \xrightarrow{p} M^{\#}$.*

Proof. By the *Baer's criterion*, M is injective iff $M \in \operatorname{Inj}(J)$ for

$$J = \{I \hookrightarrow R \colon I \text{ a submodule of } R\}.$$

Let $|R| = \lambda$ so $|I| \leq |R| \leq \lambda$. Now consider $\lambda < \lambda^+$, the smallest cardinal larger than λ. Then each R-module of cardinality less or equal than λ is λ^+-presentable. Therefore, by *the small object argument*, we obtain an injective R-mod $M^\#$ and $M \xrightarrow{p} M^\#$, where p is a λ-composite of coproducts of monomorphisms. By the "Stability of monos" lemma, therefore p is mono since each $I \hookrightarrow R \in J$ is monomorphism. $\square$