Codescent objects in 2-dimensional universal algebra

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0.2 Notation

We will fix most of our notation in the main text as we introduce it. Here we fix notation for some basic 2-categorical concepts.

- Cat is the 2-category of small categories.
- CAT is the 2-category of Set-enriched categories. It contains locally small categories, functors and natural transformations.
- 2-CAT is the 2-category of Cat-enriched categories. It contains locally small 2-categories, 2-functors and 2-natural transformations.
- Given a 2-category $\mathcal{C}$ its underlying category, obtained by forgetting 2-cells, will be denoted by $\mathcal{UC}$.
- Given 2-categories $\mathcal{C}$ and $\mathcal{D}$, $[\mathcal{C}, \mathcal{D}]$ is the 2-category of 2-functors, 2-natural transformations and modifications.
- $Ps(\mathcal{C}, \mathcal{D})$ is the 2-category of 2-functors, pseudonatural transformations and modifications. $Lax(\mathcal{C}, \mathcal{D})$ and $Oplax(\mathcal{C}, \mathcal{D})$ have the same objects and 2-cells as $Ps(\mathcal{C}, \mathcal{D})$ but 1-cells are respectively lax and oplax transformations.
- $Hom(\mathcal{C}, \mathcal{D})$ is the 2-category of pseudofunctors, pseudonatural transformations and modifications.
- $\mathcal{C}^{op}$ is the 2-category with the same objects as $\mathcal{C}$, 1-cells pointing in the opposite direction and 2-cells with orientation unchanged from $\mathcal{C}$.

Chapter 1

Overview

This thesis is concerned with 2-categories and the importance of a 2-categorical colimit known as the codescent object. The notion of codescent object, or more precisely its dual, the descent object, was first named in [54] though appeared un-named a decade earlier in [49]. It is a colimit associated to a (truncated) simplicial object which appears in several contexts in 2-category theory.
The simplest such case arises in the 2-category Cat. An object of Cat is a small category A. Such a small category may equally be presented as a diagram of sets:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

its internal presentation. Each set may be viewed as a discrete category; thus the above diagram of sets may be viewed as a diagram in Cat. The codescent object of this diagram in Cat is exactly the category A, exhibited by a "codescent morphism", the bijective on objects inclusion $\epsilon : A_0 \longrightarrow A$:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description] \arrow[r, "\epsilon"] & A
\end{tikzcd}$$

and a natural transformation $\epsilon \circ d \Longrightarrow \epsilon \circ c$. Thus each small category is the codescent object of its internal presentation.
The association by which one obtains the internal presentation of a small category A is a specific instance of the "higher kernel" construction [55], a finite limit construction. Corresponding to an arbitrary functor $f : A \longrightarrow B$ there exists an internal category in Cat, its higher kernel, as on the left of the diagram below:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description] \arrow[r, "f"] & B
\end{tikzcd}$$

This in particular constitutes a truncated simplicial object. Taking its codescent object C gives a factorisation as indicated in the diagram:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description] \arrow[rr, "f"] \arrow[dr, "f_1"'] & & B \\
& & & C \arrow[ur, "f_2"'] &
\end{tikzcd}$$

The resulting factorisation of f is the well known factorisation of a functor as bijective on objects followed by fully faithful. Thus the factorisation of a functor through the codescent object of its higher kernel coincides with the factorisation of a functor as bijective on objects followed by fully faithful. Indeed the bijections on objects are precisely the codescent morphisms, each exhibiting its codomain as the codescent object of its higher kernel.

In order to study this 2-dimensional situation we might draw an analogy with a better known 1-dimensional situation. The bijective on objects/fully faithful factorisation of a functor bears a resemblence to the factorisation of a function between sets, through its image, as surjective followed by injective. This factorisation of functions is captured by the notion of a regular category [23]; it is obtained by factoring an arrow in a category through the coequaliser of its kernel pair.

It is evident that Set is a regular category and well known that the category of algebras $\mathrm{Set}^{\mathrm{T}}$ for any monad upon it is also regular. An exactness property holds for Set and lifts to any category of algebras $\mathrm{Set}^{\mathrm{T}}$ upon it. Namely each equivalence relation is effective: the kernel pair of its coequaliser. These properties of Set and $\mathrm{Set}^{\mathrm{T}}$, regularity together with the effectiveness of equivalence relations, are those defining the notion of an exact category [4]. Each category of algebras has a further property: the free algebras form a projective cover of $\mathrm{Set}^{\mathrm{T}}$.

In this thesis we will study 2-dimensional analogues of these notions, beginning with Cat itself. There is a 2-dimensional analogue of equivalence relation, that of a catead, introduced in [10]. Cat satisfies an exactness property: each catead is the higher kernel of its codescent object. In Chapters 2 through 5 we are concerned with base 2-categories, with a focus upon this exactness property. From Chapter 6 onwards we turn to 2-dimensional universal algebra. We give an overview of those chapters now.

- In the introductory chapter, Chapter 2, we give an exposition of those concepts of 2-category theory required for the further reading of the thesis, beginning with 2-categorical limits and colimits, and with a detailed treatment of codescent objects. We describe the (Bijective on objects/fully faithful)-factorisation system on Cat and its internal analogue. In the case of Cat we describe the universal property of this factorisation system and finally show that cateads are effective in Cat, using the language of double categories.

- In Chapter 3 we consider 2-categories of the form $Cat(\mathcal{E})$ for $\mathcal{E}$ a category with pullbacks. The main aim of the chapter is to prove that each such 2-category has codescent objects of cateads and that cateads are effective, and furthermore that any 2-functors of the form $Cat(F)$ preserves codescent objects of cateads. Along the way we develop some of the theory of representable 2-categories [21].

- In Chapter 4 we establish further 2-categorical properties of those 2-categories of the form $Cat(\mathcal{E})$. This involves the introduction of a 2-categorical notion of projective cover. We obtain a list of 2-categorical properties characterising those 2-categories of the form $Cat(\mathcal{E})$ up to 2-equivalence. This result is extended to a biequivalence of 2-categories, thereby obtaining an additional characterisation of those 2-functors of the form $Cat(F)$ for a pullback preserving functor F.

- Much of the work in the previous chapters concerns internal categories in categories with pullbacks and a 2-functor $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$, where $\mathrm{Cat}_{\mathrm{pb}}$ the 2-category whose objects and 1-cells are categories with pullbacks and pullback preserving functors respectively. In Chapter 5 we prove that the underlying category of $\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed.

This concludes our treatment of “base 2-categories” and our interest turns to 2-categories of algebras upon such a base. Unlike the 1-dimensional setting there are several 2-categories of algebras in which one may be interested, for instance T-Alg_{s},T-Alg and T-Alg_{l} with strict/pseudo/lax morphisms respectively. The original appearance of the descent object was precisely in these terms. In [49] Street observed that for T a 2-monad on a 2-category $\mathcal{A}$ and strict algebras A and B there exists a truncated cosimplicial diagram:

$$\begin{tikzcd}
\text{T-Alg}_{\text{l}}(A,B) \arrow[r] & \mathcal{A}(A,B) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{A}(TA,B) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{A}(T^2A,B)
\end{tikzcd}$$

as on the right above, whose descent object is precisely $\text{T-Alg}_{\text{l}}(A,B)$. This is the reason behind the importance of codescent objects in 2-dimensional monad theory. Such issues are further discussed in Chapter 6.

- In Chapter 6 we give an exposition of those aspects of 2-dimensional monad theory which will play a role in the remainder of the thesis. Our focus is upon the importance of codescent objects in two-dimensional monad theory, via the notions of lax and pseudo-morphism classifiers, the notion of flexibility, and uses of the (Bijective on objects/fully faithful)-factorisation system in that subject.
- In Chapters 3 and 4 we considered 2-functors of the form $Cat(F)$, those in the image of the 2-functor $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$. In the short Chapter 7 we consider 2-monads which arise in this manner and the simple nature of pseudo-morphism classifiers for them. In the case of 2-monads arising from cartesian monads we show that in good cases the lax morphism classifier also admits a simple description.
- In Chapter 8 we move beyond those 2-functors of the form $Cat(F)$ to the study of a broader class of 2-functor. We begin by considering those 2-functors based upon $Cat(\mathcal{E})$ which are the left Kan extension of their restriction along the embedding $\mathcal{E} \longrightarrow Cat(\mathcal{E})$, and characterise such 2-functors in terms of the codescent objects they preserve. We restrict our attention to the case where $\mathcal{E}$ is a locally finitely presentable and study left Kan extensions along the composite embedding $\mathcal{E}_f \longrightarrow \mathcal{E} \longrightarrow Cat(\mathcal{E})$ characterising such 2-functors as the ones which preserve codescent objects of cateads and filtered colimits, the strongly finitary 2-functors. We use these results to prove that Cat is the free completion of the category of finite sets Set_f under codescent objects of reflexive coherence data and filtered colimits, and furthermore under any class of sifted colimits containing those two. We conclude by studying examples of sifted colimits in Cat.
- In Chapter 4 we introduced a 2-categorical notion of a projective cover and showed that the discrete internal categories form a projective cover of $Cat(\mathcal{E})$. In Chapter 9 we consider projectives in $\mathrm{T\text{-}Alg_s}$ for a strongly finitary 2-monad T on $Cat(\mathcal{E})$ and show the free algebras on discrete internal categories are projective. We ask: To what extent do they form a projective cover of $\mathrm{T\text{-}Alg_s}$? We define the notion of "pie algebra" and prove that the pie algebras are precisely those algebras covered. We use this characterisation to recover the characterisation of "pie weights" of Power and Robinson [45].
- In Chapter 10 we consider codescent objects in T-Alg for several classes of 2-monad. The case of most importance concerns strongly finitary 2-monads on Cat. For such 2-monads we prove that T-Alg has codescent objects of cateads and that cateads are effective in T-Alg.
- Chapter 11 contains some concluding remarks.
- Chapter 12 is an appendix.

Chapter 2

Introduction

The aim of this largely expository chapter is to introduce those 2-categorical concepts which will play a role in this thesis. We begin by reviewing 2-categorical limits and colimits, describing the relevant examples, and give a detailed treatment of codescent objects of strict coherence data. We study properties in 2-categories and properties of colimiting morphisms. We consider (enhanced) factorisation systems on 2-categories and consider the (bijective on objects/fully faithful) factorisation system on Cat. We recall the notion of internal category and the internal (bijective on objects/fully faithful) factorisation. We consider higher kernels and their relationship with codescent objects and bijections on objects. Finally we discuss cateads and describe an exactness property of Cat.

2.1 Limits and colimits in 2-categories

We begin by discussing limits and colimits in 2-categories with much of our exposition based upon [29]. There are a variety of interesting limit constructions in 2-category theory. The fundamental notion is that of a weight, which is a 2-functor $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ where $\mathcal{J}$ is a small 2-category, and Cat the 2-category of small categories.

Definition 2.1. Consider a 2-category $\mathcal{C}$, a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a 2-functor $F : \mathcal{J} \longrightarrow \mathcal{C}$. The limit of F weighted by W is an object $A \in \mathcal{C}$ together with an isomorphism $\mathcal{C}(-, A) \cong [\mathcal{J}, \mathrm{Cat}](W, \mathcal{C}(-, F-))$ in $[\mathcal{C}^{op}, \mathrm{Cat}]$.

By the Yoneda Lemma for 2-categories [30] weighted limits are determined up to isomorphism.

Remark 2.2. To give such an isomorphism amounts, again by Yoneda, to giving a 2-natural transformation, a "cone"[1], $\eta : W \longrightarrow \mathcal{C}(A, F-)$ in $[\mathcal{J}, \mathrm{Cat}]$ satisfying the following:

1. Given a cone $\theta : W \longrightarrow \mathcal{C}(B, F-)$ there exists a unique 1-cell $\theta' : B \longrightarrow A$ such that the composite:

$$\begin{tikzcd} W \arrow[r, "\eta"] & \mathcal{C}(A, F-) \arrow[r, "\mathcal{C}(\theta'{,}1)"] & \mathcal{C}(B, F-) \end{tikzcd}$$

   equals θ.

2. Given a pair of cones $\theta, \phi : W \rightrightarrows \mathcal{C}(B, F-)$ and a modification $\mu : \theta \Longrightarrow \phi$ there exists a unique 2-cell $\mu' : \theta' \Longrightarrow \phi'$ such that the composite:

$$\begin{tikzcd} W \arrow[r, "\eta"] & \mathcal{C}(A, F-) \arrow[r, bend left, "\mathcal{C}(\theta'{,}1)"] \arrow[r, bend right, "\mathcal{C}(\phi'{,}1)"'] \arrow[r, phantom, "\Downarrow \mathcal{C}(\mu'{,}1)"] & \mathcal{C}(B, F-) \end{tikzcd}$$

   equals μ.

These 2 conditions are respectively referred to as the 1-dimensional and 2-dimensional universal properties of the cone. A cone $W \longrightarrow \mathcal{C}(A, F-)$ with both universal properties will often be referred to as the universal, or limiting cone, and is said to exhibit A as the limit of F weighted by W.

One may also consider pseudo, lax and oplax limits; these are defined as above but with $Ps(\mathcal{J}, \mathrm{Cat})$, $Lax(\mathcal{J}, \mathrm{Cat})$ and $Oplax(\mathcal{J}, \mathrm{Cat})$ respectively replacing $[\mathcal{J}, \mathrm{Cat}]$ in the definition. In fact each of these are special cases of the notion of weighted limit introduced in Definition 2.1. This fact follows from the existence of left 2-adjoints to the inclusion of $[\mathcal{J}, \mathrm{Cat}]$ into $Ps(\mathcal{J}, \mathrm{Cat})$, $Lax(\mathcal{J}, \mathrm{Cat})$ and $Oplax(\mathcal{J}, \mathrm{Cat})$ as first described by Street [46]. A weighted limit is determined up to isomorphism by its universal property. A genuinely distinct notion where this is not the case is that of bilimit. This is defined exactly as in Definition 2.1. but with $Ps(\mathcal{J}, \mathrm{Cat})$ and $Ps(\mathcal{C}^{op}, \mathrm{Cat})$ replacing $[\mathcal{J}, \mathrm{Cat}]$ and $[\mathcal{C}^{op}, \mathrm{Cat}]$ and with isomorphisms replaced by equivalences. Isomorphisms are equivalences; thus the pseudo-limit of a diagram, if it exists, is in particular its bilimit. Consequently any 2-category with pseudo-limits admits all bilimits. Bilimits will be of secondary interest to us, with the exception of Chapter 9 where we briefly consider another notion, that of "cone bilimit", which lies between weighted limit and bilimit. The notion of colimit relative to a weight is defined in a dual manner:

Definition 2.3. Consider a 2-category $\mathcal{C}$, a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a 2-functor $F : \mathcal{J}^{op} \longrightarrow \mathcal{C}$. The colimit of F weighted by W is an object $A \in \mathcal{C}$ together with an isomorphism $\mathcal{C}(A, -) \cong [\mathcal{J}, \mathrm{Cat}](W, \mathcal{C}(F-, -))$ in $[\mathcal{C}, \mathrm{Cat}]$.

---

[1] The 2-natural transformations that we refer to as "cones" are called "cylinders" by Kelly in [30].

Thus any weight W gives rise to a notion of limit and colimit weighted by it. All the above remarks concerning weighted limits apply also to weighted colimits. In the cases of interest to us the indexing 2-category $\mathcal{J}$ of the weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ will be very small; typically with a finite or countable set of objects, and finite hom-categories. Furthermore the weights of interest to us will take values amongst only a select few categories, and so we name these now.

- We write $\mathbf{n}$ for the category with objects $\{0, 1, \ldots, n-1\}$ and a single morphism $i \longrightarrow i + j$ for $j \geq 0$. For example $\mathbf{2}$ and $\mathbf{3}$ are the preordered categories $0 \longrightarrow 1$ and $0 \longrightarrow 1 \longrightarrow 2$.
- We write $\mathrm{I}(\mathbf{n})$ for the unique category with objects $\{0, 1, \ldots, n-1\}$ and a single morphism between any two. Thus each morphism of $\mathrm{I}(\mathbf{n})$ is invertible and indeed $\mathrm{I}(\mathbf{n})$ is the image of $\mathbf{n}$ under the reflection of $R : \mathrm{Cat} \longrightarrow \mathrm{Gpd}$, the 2-category of groupoids.
  For example $\mathrm{I}(\mathbf{2})$ is the category with an inverse pair $0 \leftrightarrows 1$.

We now define some specific examples of 2-categorical limits and colimits. The usual limits of one dimensional category theory each have a 2-categorical analogue and we first consider these.

- **Conical Limits in 2-categories**
  In ordinary category theory a type of limit is specified by an indexing category $\mathcal{J}$. We associate to $\mathcal{J}$ the weight $\Delta(\mathbf{1}) : \mathcal{J} \longrightarrow \mathrm{Cat}$ which is constant at the terminal object $\mathbf{1}$ of Cat, and define the conical limit of a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ to be the limit of F weighted by $\Delta(\mathbf{1})$. There is no reason why $\mathcal{J}$ should not be a 2-category and F a 2-functor, thus conical limits and colimits are defined for 2-categories $\mathcal{J}$ too.
  When we speak of products, pullbacks or equalisers in a 2-category we speak of them in this sense. For example given a pair of objects $A, B \in \mathcal{C}$ their product $A \times B$ has a universal cone consisting of projections:

$$\begin{tikzcd} & A \\ A \times B \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & B \end{tikzcd}$$

  The usual one dimensional universal property of products corresponds to the defining isomorphism of Definition 2.1 being pointwise bijective on objects. That the correspondence is fully faithful means that there is also a two dimensional aspect to the universal property. This asserts that given a diagram:

$$\begin{tikzcd} & A \\ C \arrow[ur, bend left, "f_1"{name=F1}] \arrow[ur, "g_1"'{name=G1}] \arrow[dr, "f_2"{name=F2}] \arrow[dr, bend right, "g_2"'{name=G2}] \arrow[from=F1, to=G1, Rightarrow, "\theta_1"'] \arrow[from=F2, to=G2, Rightarrow, "\theta_2"'] & \\ & B \end{tikzcd}$$

  we not only have the usual induced factorisations $(f_1, f_2) : C \longrightarrow A \times B$ and $(g_1, g_2) : C \longrightarrow A \times B$ but additionally a 2-cell:

$$\begin{tikzcd} C \arrow[r, bend left, "{(f_1,f_2)}"{name=U}] \arrow[r, bend right, "{(g_1,g_2)}"'{name=D}] \arrow[from=U, to=D, Rightarrow, "{(\theta_1,\theta_2)}"] & A \times B \end{tikzcd}$$

  unique in that it yields θ_1 and θ_2 upon postcomposition with p and q respectively. The one dimensional universal property tells us one dimensional information. For example the pair (p, q) is jointly monic. The 2-dimensional aspect tells us two dimensional information. For example the pair (p, q) is jointly faithful.

- **Cotensors and tensors: Powers and copowers**
  Given a category X we have an associated weight $X : \mathbf{1} \longrightarrow \mathrm{Cat}$ which assigns to the single object of $\mathbf{1}$ the category X. Given an object A of a 2-category $\mathcal{C}$ its cotensor with X, or power, A^X is the limit of $A : 1 \longrightarrow \mathcal{C}$ weighted by X. It is defined by an isomorphism of categories $\mathcal{C}(B, A^X) \cong \mathrm{Cat}(X, C(B, A))$ 2-natural in B. The universal cone is then a functor $X \longrightarrow \mathcal{C}(A^X, A)$. Of particular interest is the case of cotensors with $\mathbf{2}$. The universal cone $\mathbf{2} \longrightarrow \mathcal{C}(A^{\mathbf{2}}, A)$ is then a "universal 2-cell":

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

  The 1-dimensional univeral property of this 2-cell is that any other 2-cell with codomain A factors uniquely through it, via a unique 1-cell. In Cat given a category A, A^X is the functor category $[X, A]$. In particular $A^{\mathbf{2}} = [\mathbf{2}, A]$, the usual arrow category of A.

Definition 2.4. Consider a 2-functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a diagram $F : \mathcal{J} \longrightarrow \mathcal{C}$ with limit and universal cone $\eta : W \longrightarrow \mathcal{C}(A, F-)$. The 2-functor G induces a cone in $\mathcal{D}$:

$$W \xrightarrow{\eta} \mathcal{C}(A, F-) \xrightarrow{G_{A,F-}} \mathcal{D}(GA, GF-)$$

We say that G preserves this limit if this cone exhibits GA as the limit of GF weighted by W.

In order to verify that a certain cone exhibits an object as a limit or colimit one needs to verify that it satisfies the one and two dimensional aspects of the universal property. The following useful proposition shows that the two dimensional aspect may often be obtained for free.

Proposition 2.5. Consider a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$.

1. Suppose that $\mathcal{C}$ has cotensors with $\mathbf{2}$ and consider a diagram $F : \mathcal{J}^{op} \longrightarrow \mathcal{C}$. If a cone $W \longrightarrow \mathcal{C}(F-, A)$ satisfies the 1-dimensional universal property of the colimit, then it also satisfies the 2-dimensional universal property, and so exhibits A as the colimit.
2. Suppose that $\mathcal{C}$ has tensors with $\mathbf{2}$ and consider a diagram $F : \mathcal{J} \longrightarrow \mathcal{C}$. If a cone $W \longrightarrow \mathcal{C}(A, F-)$ satisfies the 1-dimensional universal property, then it also satisfies the 2-dimensional universal property, and so exhibits A as the limit.

Proof. The proof is given in Section 2 of [29]. □

We now run through other pairs of 2-categorical limits and colimits which will play a role in this thesis.

- **Inserters and coinserters**
  These are defined by the following weight:

$$\begin{tikzcd} \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \arrow[r, mapsto, "W"] & \mathbf{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \mathbf{2} \end{tikzcd}$$

  where on the left we have the category with two distinct objects and a parallel pair. On the right the two morphisms are labelled by their images in $\mathbf{2}$. A functor from the category on the left, or its opposite, to a 2-category $\mathcal{C}$ consists of a parallel pair:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

in $\mathcal{C}$. Cones and cocones respectively consist of an object, 1-cell and 2-cell (C, j, η) as on the left and right below:

Cones:
$$\begin{tikzcd}
& A \arrow[dr, "f"] \arrow[dd, phantom, "\eta\Downarrow"] & \\
C \arrow[ur, "j"] \arrow[dr, "j"'] & & B \\
& A \arrow[ur, "g"'] &
\end{tikzcd}$$

Cocones:
$$\begin{tikzcd}
& B \arrow[dr, "j"] \arrow[dd, phantom, "\eta\Downarrow"] & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\
& B \arrow[ur, "j"'] &
\end{tikzcd}$$

The above cone is the inserter of f and g if:

1. Given another cone (D, h, α) there exists a unique arrow $k : D \longrightarrow C$ such that $j \circ k = h$ and $\eta \circ k = \alpha$.
2. Given a pair of cones (D, h_1, α_1) and (D, h_2, α_2) together with a 2-cell $\rho : h_1 \Longrightarrow h_2$ such that the square:
$$\begin{tikzcd}
fh_1 \arrow[r, Rightarrow, "f\rho"] \arrow[d, Rightarrow, "\alpha_1"'] & fh_2 \arrow[d, Rightarrow, "\alpha_2"] \\
gh_1 \arrow[r, Rightarrow, "g\rho"'] & gh_2
\end{tikzcd}$$
commutes, there exists a unique 2-cell $\rho' : k_1 \Longrightarrow k_2$ between the induced factorisations such that $j\rho' = \rho$.

The defining properties of the coinserter are dual and of course may be deduced from the defining weight.

Example 2.6. An ordinary graph $G = G_1 \overset{d}{\underset{c}{\rightrightarrows}} G_0$ may be seen as a graph in Cat by viewing the sets G_1 and G_0 as discrete categories. The coinserter of this graph is the free category FG upon it. Recall that FG has the same objects as G_0 and morphisms composable strings $[\alpha_1\alpha_2 \ldots \alpha_n]$ of arrows of the graph. In the universal cocone:

$$\begin{tikzcd}
& G_0 \arrow[dr, "j"] \arrow[dd, phantom, "\eta\Downarrow"] & \\
G_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & FG \\
& G_0 \arrow[ur, "j"'] &
\end{tikzcd}$$

the functor $j : G_0 \longrightarrow FG$ is bijective on objects whilst η is the natural transformation sending an arrow α of G to $[\alpha] : d(\alpha) \longrightarrow c(\alpha)$.

- **Equifiers and coequifiers**
  These are defined by the weight:

$$\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\,\Downarrow"] & \cdot \arrow[r, mapsto, "W"] & \mathbf{1} \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{2}
\end{tikzcd}$$

where the 2-category on the left hand side has a pair of objects, a pair of parallel morphisms, and a pair of parallel 2-cells. The 2-cell on the right is the unique such since $\mathbf{2}$ is a preorder. The weight W identifies the parallel 2-cells on the left.
Given a parallel pair of 2-cells:

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow\,\Downarrow\beta"] & B
\end{tikzcd}$$

a cone is specified by an object C and 1-cell $h : C \longrightarrow A$ such that $\alpha h = \beta h$[2]. The equifier of the pair α and β is the universal such object and 1-cell "equifying" the pair α and β. The exact details here may of course be worked out from the weight.
The coequifier of α and β is given by an arrow out of B, $h : B \longrightarrow C$ such that $h\alpha = h\beta$. It is the universal 1-cell which "coequifies" the 2-cells α and β.

- **Pie limits**
  Those limits which may be constructed using only products, inserters and equifiers form an important class of limit known as pie limits [45]. Dually those colimits constructible from coproducts, coinserters and coequifiers are called pie colimits. Cotensors and tensors are pie. Indeed all weighted limits and colimits discussed in this chapter, with the exception of general conical limits, are pie. Pie limits and colimits will be considered in detail in Chapter 9. Their importance may be illustrated by the fact that any 2-category admitting pie limits admits all pseudo, lax and oplax limits. In particular, as such a 2-category admits all pseudo-limits, it admits all bilimits.

Proposition 2.7. Any 2-category which has products, inserters and equifiers admits all pseudo, lax and oplax limits. Consequently any such 2-category admits all bilimits.

Proof. A proof is given in [29]. □

- **Comma objects**
  The defining weight is:

$$\begin{tikzcd} & \cdot \arrow[d] & & & & \mathbf{1} \arrow[d, "0"] \\ \cdot \arrow[r] & \cdot & {} \arrow[r, mapsto, "W"] & {} & \mathbf{1} \arrow[r, "1"'] & \mathbf{2} \end{tikzcd}$$

  where the category on the left has three distinct objects.
  Given an opspan:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

  a cone consists of an object, a pair of 1-cells and a 2-cell:

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

  The comma object is the universal such cone, and is denoted $f|g$. If $A = B = C$ and $f = g = 1_C$ the comma object $1_C|1_C$ is equally the cotensor of C with $\mathbf{2}$. The comma object derives its name from the case of Cat, where the comma object is the familiar comma category.

- **Inverters and coinverters**
  The defining weight is:

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \cdot & {} \arrow[r, mapsto, "W"] & {} & \mathbf{1} \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{I}(\mathbf{2}) \end{tikzcd}$$

---

[2] The cone actually contains more data than that specified here, but this is the essential content, in the same sense that the limiting cone of an equaliser diagram:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & B \end{tikzcd}$$

contains not only a morphism $E \longrightarrow A$ but a morphism $E \longrightarrow B$. The latter of these becomes redundant upon specifying that $\alpha e = \beta e$.

where the 2-category on the left has a pair of objects, two distinct parallel 1-cells and a 2-cell between them. The 2-cell on the right is the unique such, and is an isomorphism since $\mathrm{I}(\mathbf{2})$ is a groupoid.
Given a 2-cell:

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

its inverter consists of an object and 1-cell with codomain A, $h : C \longrightarrow A$ such that αh is invertible. The coinverter of α is given by an object and 1-cell $h : B \longrightarrow D$ with domain B, now universal in inverting α "from the right".

Example 2.8. Categories of fractions are examples of coinverters [28]. Given a category A, a calculus of fractions [19] on A consists of a collection of arrows Σ therein satisfying several axioms. The corresponding category of fractions $A[\Sigma^{-1}]$ comes equipped with a projection $p : A \longrightarrow A[\Sigma^{-1}]$. The projection has the property that if $f \in \Sigma$ then $p(f)$ is invertible, and is universal amongst those functors coinverting the arrows of Σ. We may view Σ as a discrete category with objects the elements of Σ. Then we have the evident domain and codomain functors and a natural transformation:

$$\begin{tikzcd} \Sigma \arrow[r, bend left, "d"{name=U}] \arrow[r, bend right, "c"'{name=D}] & A \arrow[from=U, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

assigning to an element of Σ the corresponding arrow of A. The universal property of $p : A \longrightarrow A[\Sigma^{-1}]$ described asserts precisely that it satisfies the 1 dimensional universal property of the coinverter of the above natural transformation; the 2-dimensional universal property coming for free by Proposition 2.5.

Remark 2.9. Given a 2-cell:

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

its coinverter may be constructed from coinserters and coequifiers as follows. Firstly form the coinserter:

$$\begin{tikzcd} & B \arrow[dr, "j"] & \\ A \arrow[ur, "g"] \arrow[dr, "f"'] & \eta\Downarrow & C \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

We then have two parallel pairs of 2-cells: $(1, \eta \circ j\alpha : jf \Longrightarrow jf)$ and $(1, j\alpha \circ \eta : jg \Longrightarrow jg)$. Forming the coequifier $h : C \longrightarrow D$ of the first pair ensures that $h\eta \circ hj\alpha = 1$ so that $hj\alpha$ has a left inverse. Now forming the coequifier $k : D \longrightarrow E$ of the pair $(1, hj\alpha \circ h\eta : jg \Longrightarrow jg)$ ensures that $khj\alpha$ has a right inverse and is thus invertible. It is straightforward to verify that E satisfies the universal property of the coinverter.

Remark 2.10. Each weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ has an associated "iso" weight:

$$\mathcal{J} \xrightarrow{W} \mathrm{Cat} \xrightarrow{R} \mathrm{Gpd} \xrightarrow{\iota} \mathrm{Cat}$$

where R is the reflection of the inclusion $\iota : \mathrm{Gpd} \longrightarrow \mathrm{Cat}$. These new weights describe related limits of importance. The distinction between a weight and its iso-version then is that every 2-cell appearing in a cone for the iso-weight is invertible.
For instance the weight for inserters becomes that for "iso-inserters". The 2-cell "inserted" in the universal cone for the iso-inserter is necessarily invertible; the cone now has its universal property only with respect to such cones, whilst the 2-dimensional universal property remains the same as that for the inserter. The

case of iso-coinserter is dual.
Associated to the notion of equifier we similarly have that of iso-equifier and iso-coequifier, also referred to as equi and coequi-inverters by Street [51]. Again the iso-equifier has universal cone a 1-cell $h : C \longrightarrow A$ which now not only equifies the 2-cells α and β but additionally the resulting 2-cell $\alpha h = \beta h$ is invertible; furthermore this cone only has its universal property with respect to those 1-cells which both "equify" and "invert". Applying the same construction to the weights for comma objects gives iso-comma objects. Inverters remain unchanged by this construction since each category in the image of their defining weight is already a groupoid.

In order to introduce descent and codescent objects we should first recall the simplicial category. Our notation follows Verity [57].

Definition 2.11. Δ_+ is the skeletal category of finite ordinals and order preserving maps. It has:

- Objects: ordinal numbers: $[n] = \{0 < 1 < 2 \ldots < n\}$, for each $n \geq -1$.
- 1-cells: order preserving functions.

Notation 2.12. For $[n] \in \Delta_+$ and $j \in [n]$ we denote the following maps:

- $\delta^n_j : [n-1] \longrightarrow [n]$ the unique injective order preserving function whose fibre at j is the empty set.
- $\sigma^n_j : [n+1] \longrightarrow [n]$ the unique surjective order preserving function whose fibre at j has 2 elements.

As the superscripts for these arrows are determined by their codomains, we may omit them when the codomain of the arrow is visible.

Proposition 2.13. The category Δ_+ is generated by the "face" and "degeneracy" maps, $\delta^n_j : [n-1] \longrightarrow [n]$ and $\sigma^n_j : [n+1] \longrightarrow [n]$ subject to the "simplicial identites":

- For $j < i \in [n+1]$: $\quad \delta^{n+1}_i \delta^n_j = \delta^{n+1}_j \delta^n_{i-1}$.
- For $j \leq i \in [n-1]$: $\quad \sigma^{n-1}_i \sigma^n_j = \sigma^{n-1}_j \sigma^n_{i+1}$.
- For all $j \in [n]$ and $i \in [n-1]$:

$$\sigma^{n-1}_i \delta^n_j = \begin{cases} \delta^{n-1}_j \rho^{n-2}_{i-1} & \text{if } j < i \\ 1_{[n]} & \text{if } j = i \text{ or } j = i+1 \\ \delta^{n-1}_{j-1} \sigma^{n-2}_i & \text{if } j > i+1 \end{cases}$$

Proof. See [42]. ☐

Proposition 2.14. Δ_+ admits the structure of a strict monoidal category $(\Delta_+, \oplus, [-1])$. It is the free strict monoidal category containing a monoid.

Proof. See [42]. ☐

Definition 2.15. Δ is the full subcategory of Δ_+ with objects all finite ordinals excepting $[-1]$.

Definition 2.16. Δ_2 is the full subcategory of Δ with objects [0],[1] and [2].

Definition 2.17. Δ_2^- is the subcategory of Δ_2 with the same objects and all arrows with the exception that $\Delta_2^-([2],[1]) = \emptyset$.

2.2 Codescent objects

- **Codescent objects of strict coherence data**

  There are a variety of notions of codescent object. There are both "strict" and "weak" versions, and for each of these a "reflexive" version. The first appearance in the literature of descent/codescent objects was in [49] and they were first named in [54].

  Identifying each finite ordinal $[n-1]$, for $n \geq 1$, with the category $\mathbf{n}$ identifies Δ as a full subcategory of Cat. Thus we have an embedding $\Delta \longrightarrow \mathrm{Cat}$. Restricting this embedding to the subcategories Δ_2 and Δ_2^- of Δ give rise to functors $\Delta_2 \longrightarrow \mathrm{Cat}$ and $\Delta_2^- \longrightarrow \mathrm{Cat}$.

  The functor $\Delta_2^- \longrightarrow \mathrm{Cat}$ is the weight for strict descent and codescent objects:

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & {[1]} \arrow[l, "\sigma_0" description] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_0"'] & {[2]} & {} \arrow[r, mapsto, "W"] & {} & \mathbf{1} \arrow[r, shift left=2, "\delta_1=0"] \arrow[r, shift right=2, "\delta_0=1"'] & \mathbf{2} \arrow[l, "\sigma_0" description] \arrow[r, shift left=2, "\delta_2"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_0"'] & \mathbf{3}
\end{tikzcd}$$

  On the left above are drawn the generating arrows of Δ_2^-; we use the same labels to denote their images in Cat. We will focus here upon codescent objects, descent objects in a 2-category $\mathcal{A}$ of course being codescent objects in $\mathcal{A}^{op}$. A functor $(\Delta_2^-)^{op} \longrightarrow \mathcal{A}$ consists of a diagram:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

  satisfying the dual identities to the simplicial identities: precisely $dp = dm, cq = cm, cp = dq$ and $di = ci = 1$. Such a diagram is referred to as "strict coherence data". A cocone consists of an object, 1-cell and 2-cell (A, f, α):

$$\begin{tikzcd}
 & A_0 \arrow[dr, "f"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & A \\
 & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

  satisfying the equations:

  1.

$$\begin{tikzcd}
 & A_1 \arrow[r, "d"] & A_0 \arrow[dr, "f"] & \\
A_2 \arrow[ur, "p"] \arrow[r, "m"] \arrow[dr, "q"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & A \\
 & A_1 \arrow[r, "c"'] & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & A_1 \arrow[rr, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & A_0 \arrow[dr, "f"] & \\
A_2 \arrow[ur, "p"] \arrow[dr, "q"'] & & A_0 \arrow[rr, "f"] & & A \\
 & A_1 \arrow[ur, "d"] \arrow[rr, "c"'] & \Downarrow\alpha & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

  2.

$$\begin{tikzcd}
 & & A_0 \arrow[dr, "f"] & & & & \\
A_0 \arrow[urr, bend left, "1"] \arrow[r, "i"] \arrow[drr, bend right, "1"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & A & = & A_0 \arrow[r, "f"] & A \\
 & & A_0 \arrow[ur, "f"'] & & & &
\end{tikzcd}$$

  where the single 1-cell f on the right hand side of equation (2) represents the identity 2-cell $1 : f \Longrightarrow f$. We sometimes refer to equations (1) and (2) as the multiplicative and unital equations for a codescent cocone. This cocone exhibits A as the codescent object of the strict coherence data if:

  1. Given another cocone (B, g, β) there exists a unique 1-cell $k : A \longrightarrow B$ such that $kf = g$ and $k\alpha = \beta$.

2. Suppose we are given a pair of cocones (B, g_1, β_1) and (B, g_2, β_2), and a 2-cell $\theta : g_1 \Longrightarrow g_2$ such that the square:

$$\begin{tikzcd}
g_1 d \arrow[r, Rightarrow, "\beta_1"] \arrow[d, Rightarrow, "\theta d"'] & g_1 c \arrow[d, Rightarrow, "\theta c"] \\
g_2 d \arrow[r, Rightarrow, "\beta_2"'] & g_2 c
\end{tikzcd}$$

   commutes. Then there exists a unique 2-cell $\theta' : k_1 \Longrightarrow k_2$ between the induced factorisations such that $\theta' f = \theta$.

Remark 2.18. Codescent objects and isocodescent objects may be formed from coinserters and coequifiers as follows. Given coherence data as above firstly form the coinserter:

$$\begin{tikzcd}
& A_0 \arrow[dr, "f"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \alpha & A \\
& A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

The two equations (1) and (2) above for a codescent cocone will not hold but do constitute two pairs of parallel 2-cells. Forming the coequifiers of the parallel pair of equation (1) gives a 1-cell $h : A \longrightarrow B$. Now forming the coequifier of the parallel 2-cells of equation (2) postcomposed with h gives the codescent object. One can form the isocodescent object by firstly forming the codescent object. Its universal cocone comes equipped with a 2-cell whose coinverter is the isocodescent object. We have already seen, in Remark 2.9, that coinverters may be formed from coinserters and coequifiers; thus isocodescent objects may be formed from coinserters and coequifiers too.

- **Codescent objects of strict reflexive coherence data**
  The defining weight is the inclusion $\Delta_2 \longrightarrow \mathrm{Cat}$. A functor $\Delta_2^{op} \longrightarrow \mathcal{C}$ is called strict reflexive coherence data and consists of a diagram:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=6, "p"] \arrow[r, "m" description] \arrow[r, shift right=6, "q"'] & A_1 \arrow[l, shift right=3, "l" description] \arrow[l, shift left=3, "r" description] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

  The data distinguishing strict reflexive coherence data from strict coherence data is the pair of morphisms $r, l : A_1 \longrightarrow A_2$. The additional equations that must be satisfied are the additional identities: $li = ri$, $pl = id$, $ml = ql = 1$, $qr = ic$ and $mr = pr = 1$.
  It is straightforward to verify that the weighted colimit of this diagram is just the codescent object of the underlying strict coherence data.

Remark 2.19. The codescent object of strict reflexive coherence data is simply the codescent object of the underlying strict coherence data. As our interest in such diagrams is primarily in computing their codescent objects we should justify our interest in the reflexive case. The analogous 1-dimensional situation is the relationship between graphs and reflexive graphs. Whilst the colimit of either such diagram in a category is the coequaliser of its underlying graph, coequalisers of reflexive graphs are better behaved. For example in the category of sets finite products commute with coequalisers of reflexive graphs [24]; but not with coequalisers in general. A consequence of this is that in a category of algebras for a finitary monad on Set, coequalisers of reflexive pairs are computed at the level of underlying sets. Similarly, in Cat finite products commute with codescent objects of strict reflexive coherence data though not strict coherence data in general; so that again it is important to distinguish the reflexive case. This will play an important role in Chapter 8, in which we exhibit Cat as a free completion under codescent objects of strict reflexive coherence data and filtered colimits.

Definition 2.20. We say that strict coherence data $\Delta_2^{-op} \longrightarrow \mathcal{C}$ is *reflexive* if it underlies a functor $\Delta_2^{op} \longrightarrow \mathcal{C}$.

Example 2.21. In the case of Cat weighted colimits may be calculated via left Kan extension along the Yoneda embedding. Consider the inclusion $\iota : \Delta_2 \longrightarrow \mathrm{Cat}$ (the weight for reflexive codescent objects) and the left Kan extension:

$$\begin{tikzcd}
{[\Delta_2^{op}, \mathrm{Cat}]} \arrow[dr, "col_\iota(-)"] & \\
\Delta_2 \arrow[u, "y"] \arrow[r, "\iota"'] & \mathrm{Cat}
\end{tikzcd}$$

An object of $[\Delta_2^{op}, \mathrm{Cat}]$ is reflexive coherence data; the left Kan extension then acts by computing its codescent object. The left Kan extension has right 2-adjoint $\mathrm{Cat}(\iota-, 1) : \mathrm{Cat} \longrightarrow [\Delta_2^{op}, \mathrm{Cat}]$. We have an adjunction of categories:

$$\begin{tikzcd}
\mathcal{U}\mathrm{Cat} \arrow[r, shift right, "ob"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "j"']
\end{tikzcd}$$

where $j : \mathrm{Set} \longrightarrow \mathcal{U}\mathrm{Cat}$ views each set as a discrete category and ob takes the underlying set of a category. Lifting this adjunction to functor categories yields a composite adjunction:

$$\begin{tikzcd}
\mathcal{U}\mathrm{Cat} \arrow[r, shift right, "\mathrm{Cat}(\iota-{,}1)"'] \arrow[r, phantom, "\perp"] & {[\Delta_2^{op}, \mathcal{U}\mathrm{Cat}]} \arrow[l, shift right, "col_\iota(-)"'] \arrow[r, shift right, "ob^*"'] \arrow[r, phantom, "\perp"] & {[\Delta_2^{op}, \mathrm{Set}]} \arrow[l, shift right, "\iota^*"']
\end{tikzcd}$$

where the composite right adjoint computes the nerve of a category, restricted to Δ_2^{op}. The objects of $[\Delta_2^{op}, \mathrm{Set}]$ constitute reflexive coherence data in Cat, when the sets are viewed as discrete categories. The functor ι^* views them in precisely this manner; thus the composite left adjoint computes the codescent objects of the coherence data of $[\Delta_2^{op}, \mathrm{Set}]$ viewed as coherence data in Cat. The right adjoint is fully faithful so that the counit of the adjunction is an isomorphism. In particular given a small category $A \in \mathrm{Cat}$ its image under the right adjoint is the corresponding "internal category" in Set:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=4, "p"] \arrow[r, "m" description] \arrow[r, shift right=4, "q"'] & A_1 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

where A_0 and A_1 are respectively the set of objects and arrows of A whilst A_2 is the set of composable pairs of arrows of A. The functions d and c are the domain and codomain functions; whilst i and m respectively assign to an object and composable pair the corresponding identity morphism and composite. The projections p and q assign to a composable pair its back and front arrow respectively whilst the two unlabelled arrows are uniquely induced by the pullback property of the set of composable pairs A_2. As they are not relevant with regards computation of the codescent object they are unlabelled. As the counit is an isomorphism we see that the category A is the codescent object of the above diagram of sets viewed as discrete categories. Explicitly the exhibiting cocone:

$$\begin{tikzcd}
& A_0 \arrow[dr, "\epsilon"] \arrow[dd, phantom, "\Downarrow\eta"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & A \\
& A_0 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

has $\epsilon : A_0 \longrightarrow A$ the bijective on objects inclusion of A_0 into A. The natural transformation η assigns to object of $\alpha \in A_1$ the corresponding arrow $\alpha : d\alpha \longrightarrow c\alpha$. The two axioms for this 2-cell to be a cocone correspond to the statements that $i : A_0 \longrightarrow A_1$ and $m : A_2 \longrightarrow A_1$ assign to an object its identity morphism, and to a composable pair its composite and one may easily verify the universal property directly.
Observe then that each category is "presented" as the codescent object of an internal category, its nerve.

Example 2.22. The embedding $j : \Delta \longrightarrow \mathrm{Cat}$ induces an adjunction

$$\begin{tikzcd}
\mathcal{U}\mathrm{Cat} \arrow[r, leftarrow, shift left=2] \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "\mathrm{Cat}(j-,1)"'] & {[\Delta^{op}, \mathrm{Set}]}
\end{tikzcd}$$

where the right adjoint assigns to a category its nerve; the left adjoint assigns to a simplicial set its "fundamental category". As the nerve functor is "2-coskeletal" [36] it may be recovered from the restricted nerve functor $\mathcal{U}\mathrm{Cat} \longrightarrow [\Delta_2^{op}, \mathrm{Set}]$ (the composite right adjoint of Example 2.21). This is achieved by post-composing the restricted nerve with the right Kan extension functor $Ran : [\Delta_2^{op}, \mathrm{Set}] \longrightarrow [\Delta^{op}, \mathrm{Set}]$, which is induced by restriction along the inclusion $\Delta_2 \longrightarrow \Delta$. Right Kan extension is right adjoint to restriction $Res : [\Delta^{op}, \mathrm{Set}] \longrightarrow [\Delta_2^{op}, \mathrm{Set}]$; thus we have the composite adjunction:

$$\begin{tikzcd}
\mathcal{U}\mathrm{Cat} \arrow[r, leftarrow, shift left=2, "col_\iota(-)"] \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "\mathrm{Cat}(\iota-,1)"'] \arrow[rrr, bend right, "\mathrm{Cat}(j-,1)"'] & {[\Delta_2^{op}, \mathcal{U}\mathrm{Cat}]} \arrow[r, leftarrow, shift left=2, "\iota^*"] \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "ob^*"'] & {[\Delta_2^{op}, \mathrm{Set}]} \arrow[r, leftarrow, shift left=2, "Res"] \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "Ran"'] & {[\Delta^{op}, \mathrm{Set}]}
\end{tikzcd}$$

This formula for the left adjoint to the nerve asserts precisely that the fundamental category of a simplicial set may be computed by taking the underlying coherence data and computing its codescent object. This description of the left adjoint is easily seen to correspond to the presentation of Gabriel and Zisman [19], of the fundamental category of a simplicial set.

- **Eilenberg-Moore and Kleisli Objects**
  We briefly remark upon Eilenberg-Moore and Kleisli objects which will play a lesser role in this thesis than those limits considered thus far. The strict monoidal category $(\Delta_+, \oplus, [-1])$ is the free strict monoidal category containing a monoid, so that its suspension as a 2-category, $\Sigma(\Delta_+)$, is the free 2-category containing a monad. In other words a 2-functor $\Sigma(\Delta_+) \longrightarrow \mathcal{C}$ is precisely a monad in $\mathcal{C}$. We have the constant 2-functor at $\mathbf{1}$, $\Delta(\mathbf{1}) : \Sigma(\Delta_+) \longrightarrow \mathrm{Cat}$. The lax limit of a monad, $\Sigma(\Delta_+) \longrightarrow \mathcal{C}$, weighted by $\Delta(\mathbf{1})$ is its Eilenberg-Moore object; the lax colimit is the Kleisli object. In the case of Cat these are respectively the well known Eilenberg-Moore and Kleisli categories associated to a monad.

Notation 2.23. In each of the cases of coequifiers, coinverters, coinserters, codescent objects and Kleisli objects the universal cocone contains a single 1-cell.
We refer to that 1-cell as the coequifier/coinverter/coinserter/codescent/Kleisli morphism.

2.3 Properties of objects and arrows in 2-categories

Definition 2.24. Let P and Q respectively be classes of objects and arrows of Cat.

1. A 2-category $\mathcal{A}$ is said to be locally P if each hom category $\mathcal{A}(A, B)$ lies in P.
2. A 2-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be locally Q if each hom functor $F_{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$ lies in Q.

Definition 2.25. Consider classes P and Q as in the preceding definition.

1. We say that P is representable if it has the following property: $C \in P$ iff $\forall A \in \mathrm{Cat} : \mathrm{Cat}(A, C) \in P$.
2. We say that Q is representable if it has the following property: $F \in Q$ iff $\forall A \in \mathrm{Cat} : \mathrm{Cat}(A, F) \in Q$.

Remark 2.26. Examples of representable classes of categories include the classes of discrete categories, groupoids and equivalence relations. A non-example is the class of 1-object categories. Examples of representable classes of functors include the classes of fully faithful functors, faithful functors and conservative functors. A non-example is the class of full functors.

Definition 2.27. Consider a representable class of categories P and an object A in a 2-category $\mathcal{A}$.

1. The object A is said to be P if each for each $C \in \mathcal{A}$ the category $\mathcal{A}(C, A)$ lies in P.
2. The object A is said to be co-P if each for each $C \in \mathcal{A}$ the category $\mathcal{A}(A, C)$ lies in P.

Consider a representable class Q and an arrow $f : A \longrightarrow B$ in a 2-category $\mathcal{C}$.

1. The arrow $f : A \longrightarrow B$ is said to be Q if for each $C \in \mathcal{C}$ the functor $\mathcal{C}(C, f) : \mathcal{C}(C, A) \longrightarrow \mathcal{C}(C, B)$ lies in Q.
2. The arrow $f : A \longrightarrow B$ is said to be co-Q if for each $C \in \mathcal{C}$ the functor $\mathcal{C}(f, C) : C(B, C) \longrightarrow C(A, C)$ lies in Q.

Notation 2.28. In accordance with Definition 2.27 we fix some notation. Let $f : A \longrightarrow B$ be an arrow of a 2-category $\mathcal{C}$.

- f is said to be fully faithful if $\mathcal{C}(C, f)$ is so, and co-fully faithful if each $\mathcal{C}(f, C)$ is so, for each $C \in \mathcal{C}$.
- f is said to be faithful if $\mathcal{C}(C, f)$ is so, and co-faithful if $\mathcal{C}(f, C)$ is so, for each $C \in \mathcal{C}$.
- f is said to be conservative if $\mathcal{C}(C, f)$ is so. It is said, following [13], to be liberal if $\mathcal{C}(f, C)$ is conservative, for each $C \in \mathcal{C}$.

Lemma 2.29. Coequifier and coinverter morphisms are epi and co-fully faithful. Coinserter, codescent and Kleisli morphisms are co-faithful.

Proof. Each of these statements follows immediately from the universal property of the weighted colimit in question. ☐

Remark 2.30. The properties of Lemma 2.29 are useful in 2-categorical algebra, just as in ordinary category theory, knowing that coequalizer morphisms (regular epis) are epimorphisms is useful. On the other hand it is often useful to have, in specific 2-categories such as Cat, more concrete information than these properties provide. One way of obtaining such information is via factorisation systems, which we introduce next. For such an application of factorisation systems see Corollary 2.44 below.

2.4 Orthogonal and enhanced factorisation systems

In this section we consider some of the notions of orthogonality which appear in 2-category theory. The first of these, that of Definition 2.31 below, is of a kind definable in any enriched category and occurs in the work of Day [15]. The "strong orthogonality" of Definition 2.33 was first considered by Street and Walters [50] and is specific to 2-category theory.

Definition 2.31. Let $\mathcal{C}$ be a 2-category and E and M classes of 1-cells of $\mathcal{C}$. We say that E is orthogonal to M if given a pair of morphisms $e : A \longrightarrow B \in E$ and $m : C \longrightarrow D \in M$ the commutative diagram of categories and functors:

$$\begin{tikzcd}
\mathcal{C}(B, C) \arrow[r, "e^*"] \arrow[d, "m_*"'] & \mathcal{C}(A, C) \arrow[d, "m_*"] \\
\mathcal{C}(B, D) \arrow[r, "e^*"'] & \mathcal{C}(A, D)
\end{tikzcd}$$

is a pullback in Cat, where e^* and m_* are given by composition with e and m.

Remark 2.32. 1. The assertion that the above square is a pullback at the level of underlying sets is precisely the statement that given a commuting square with $e \in E$ and $m \in M$:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "r"'] & B \arrow[d, "s"] \arrow[dl, dotted, "\exists! h"'] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

there exists a unique diagonal 1-cell $h : B \longrightarrow C$ rendering both triangles commutative.

2. The two dimensional aspect of the pullback in Cat asserts that given a commutative diagram as on the left below (considering the outside of the square only) with e and m as before:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, bend right=60, "r_1"'] \arrow[d, "r_2"] \arrow[d, phantom, shift right=4ex, "\overset{\theta_1}{\Rightarrow}"] & B \arrow[dl, bend right=20, "\exists! h_1"'] \arrow[dl, bend left=20, "\exists! h_2"] \arrow[dl, phantom, "\overset{\exists!\phi}{\Downarrow}"] \arrow[d, "s_1"'] \arrow[d, bend left=60, "s_2"] \arrow[d, phantom, shift left=4ex, "\overset{\theta_2}{\Rightarrow}"] \\
C \arrow[r, "m"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "\hat{\theta_1}"'] & B \arrow[d, "\hat{\theta_2}"] \arrow[dl, dotted, "\exists!\hat{\phi}"'] \\
C^{\mathbf{2}} \arrow[r, "m^{\mathbf{2}}"'] & D^{\mathbf{2}}
\end{tikzcd}$$

there exist not only unique factorisations h_1 and h_2 but a unique diagonal 2-cell ϕ as drawn, such that $m\phi = \theta_2$ and $\phi e = \theta_1$.

If $\mathcal{C}$ has cotensors with $\mathbf{2}$ the 2-cells θ_1 and θ_2 correspond uniquely to 1-cells $\hat{\theta_1} : A \longrightarrow C^{\mathbf{2}}$ and $\hat{\theta_2} : B \longrightarrow D^{\mathbf{2}}$ rendering the right square commutative. To give a diagonal 2-cell ϕ is equally to give a diagonal 1-cell $\hat{\phi}$ as in the diagram on the right above. Therefore E is orthogonal to M, in this 2-categorical sense, if we have the unique factorisations of the first part of the Remark and furthermore the arrows of M are closed in $\mathcal{C}^{\mathbf{2}}$ under cotensors with $\mathbf{2}$.

This is a convenient fact: it is often clear that a class of arrows is closed under cotensors with $\mathbf{2}$, for instance fully faithful arrows are so. In such cases one only needs to verify the 1-dimensional condition of the first part of the Remark.

Definition 2.33. We say that E is strongly orthogonal to M if:

1. E is orthogonal to M.
2. Given $e \in E$ and $m \in M$ and a 2-cell isomorphism

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "r"'] \arrow[dr, phantom, "\cong_{\alpha}"] & B \arrow[d, "s"] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

there exist a unique pair $(h : B \longrightarrow C, \beta : s \cong mh)$ with β an isomorphism:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "r"'] & B \arrow[d, "s"] \arrow[dl, "h"'] \arrow[dl, phantom, shift left=3ex, "\cong_{\beta}"] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

such that the left triangle commutes and $\beta e = \alpha$.

Proposition 2.34. In any 2-category $\mathcal{C}$:

- Coequifier morphisms are strongly orthogonal to faithful arrows.
- Coinverter morphisms are strongly orthogonal to conservative arrows.

- Coinserter, Kleisli and codescent morphisms are strongly orthogonal to fully faithful arrows.

Since fully faithful arrows are both faithful and conservative it is immediate that coequifier and coinverter morphisms are also strongly orthogonal to fully faithful arrows.

Proof. We will only consider the cases of coequifier and coinserter morphisms, the other cases are similar with the case of coinverters being remarked upon in [50]. Recall from Remark 2.32(2) that the two dimensional aspect of orthogonality of classes E and M follows from the one dimensional aspect if $\mathcal{C}$ has cotensors with $\mathbf{2}$ and the arrows of M are closed under cotensors with $\mathbf{2}$. In any 2-category admitting cotensors with $\mathbf{2}$ all of the above classes: faithful, conservative and fully faithfuls, are closed under cotensors with $\mathbf{2}$. Since the 2-categories which appear in this thesis all have cotensors with $\mathbf{2}$ we will therefore not consider the two dimensional aspect of orthogonality, as it immediately follows from the one-dimensional aspect in such cases, though all of the above statements are true in full generality and we only act as such to quicken our proof. Consequently we need only consider one dimensional orthogonality and the strong orthogonality condition.
Consider the condition of strong orthogonality: Given a square with e and m in the appropriate classes as in diagram (1) below:

(1)
$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "r"'] \arrow[dr, phantom, "\cong_\alpha"] & B \arrow[d, "s"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

(2)
$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "r"'] & B \arrow[d, "s"] \arrow[dl, "h"'] \arrow[d, phantom, shift right=4ex, "\cong_\beta"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

we must construct a unique pair $(h : B \longrightarrow C, \beta : s \cong mh)$ with β an isomorphism such that $he = r$ and $\beta e = \alpha$ as in (2) above. If e "co-reflects identities" (meaning that given a 2-cell ϕ such that ϕe is an identity then ϕ is an identity) then this condition subsumes one dimensional orthogonality, by taking the 2-cell α to be an identity.
Certainly coequifier morphisms co-reflect identities. For suppose $e : A \longrightarrow B$ is a coequifier morphism. It is in particular epi and co-fully faithful. Therefore given $\theta : f \Longrightarrow g \in \mathcal{C}(B, E)$ such that θe is an identity we have $fe = ge$ so that $f = g$ by epi-ness of e. Then $1_f e = \theta e$ so that $1_f = \theta$ as e is co-faithful. Similarly coinserter morphisms co-reflect identities. To see this let e be the coinserter morphism of a graph $d, c : X \rightrightarrows A$, say with universal cocone $(B, e, \eta : ed \Longrightarrow ec)$. Suppose that precomposing $\theta : f \Longrightarrow g$ with e yields an identity. We claim this implies that $f = g$. By the universal property of the coinserter B, f and g are determined by precomposition with e and η. Now $\theta e = 1$ by assumption so that $fe = ge$. Since θe is an identity the composite:

$$\begin{tikzcd} & A \arrow[dr, "e"] & & \\ X \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & B \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] \arrow[from=F, to=G, Rightarrow, "\theta"] & E \\ & A \arrow[ur, "e"'] & & \end{tikzcd}$$

equals each of $f\eta$ and $g\eta$. Therefore $f = g$. As the coinserter morphism e is co-faithful we now deduce that θ is an identity.

1. Consider the case then of coequifiers. We suppose that in diagram (1) e is a coequifier morphism and m faithful. Let e be the coequifier of a pair of 2-cells $\theta, \phi : d \Longrightarrow c \in \mathcal{C}(E, A)$. We claim that $mr\theta = mr\phi$. We have the equation:

$$mr\theta = \begin{tikzcd} & & C \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\alpha^{-1}"] & \\ E \arrow[r, bend left, "d"{name=D}] \arrow[r, bend right, "c"'{name=C}] \arrow[from=D, to=C, Rightarrow, "\theta"] & A \arrow[ur, "r"] \arrow[r, "e"] \arrow[dr, "r"'] & B \arrow[r, "s"] \arrow[d, phantom, "\Downarrow\alpha"] & D \\ & & C \arrow[ur, "m"'] & \end{tikzcd} = \begin{tikzcd} & & C \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\alpha^{-1}"] & \\ E \arrow[r, bend left, "d"{name=D}] \arrow[r, bend right, "c"'{name=C}] \arrow[from=D, to=C, Rightarrow, "\phi"] & A \arrow[ur, "r"] \arrow[r, "e"] \arrow[dr, "r"'] & B \arrow[r, "s"] \arrow[d, phantom, "\Downarrow\alpha"] & D \\ & & C \arrow[ur, "m"'] & \end{tikzcd} = mr\phi$$

The first and last equality hold upon cancelling inverses. The middle equality holds because $e\phi = e\theta$ by assumption. Since $mr\theta = mr\phi$ and m is faithful we have $r\theta = r\phi$. Therefore by the universal property of the coequifier B there exists a unique arrow $h : B \longrightarrow C$ rendering commutative the left triangle of diagram (2) above. Now we have a 2-cell $\alpha : se \cong mr = mhe$. As the coequifier morphism e is co-fully faithful it follows that there exists a unique 2-cell $\beta : s \cong mh$ such that $\beta e = \alpha$. Furthermore β is invertible since any co-fully faithful arrow co-reflects isomorphisms (is liberal). Therefore coequifier morphisms are strongly orthogonal to faithful ones.

2. Suppose now that e is the coinserter morphism of the same graph considered before. Consider the composite 2-cell:

$$\begin{tikzcd}
 & & A \arrow[dr, "e"] \arrow[rr, "r"] & & C \arrow[dr, "m"] & \\
(3) & X \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & B \arrow[rr, "s"] \arrow[ur, phantom, "\Downarrow\alpha^{-1}"] \arrow[dr, phantom, "\Downarrow\alpha"] & & D \\
 & & A \arrow[ur, "e"] \arrow[rr, "r"'] & & C \arrow[ur, "m"'] &
\end{tikzcd}$$

As m is fully faithful there exists a unique 2-cell $\theta : rd \Longrightarrow rc$ such that $m\theta$ equals the above composite (3). By the universal property of the coinserter B there is consequently a unique 1-cell $h : B \longrightarrow C$ rendering commutative the leftmost triangle of Diagram (2) and such that $h\eta = \theta$.
Now we have the 2-cell isomorphism $\alpha : se \Longrightarrow mr = mhe$. The square:

$$\begin{tikzcd}
sed \arrow[r, Rightarrow, "\alpha d"] \arrow[d, Rightarrow, "s\eta"'] & mhed \arrow[d, Rightarrow, "mh\eta"] \\
sec \arrow[r, Rightarrow, "\alpha c"'] & mhec
\end{tikzcd}$$

commutes since $mh\eta = m\theta$ equals:

$$\begin{tikzcd}
mrd \arrow[r, Rightarrow, "\alpha^{-1}d"] & sed \arrow[r, Rightarrow, "s\eta"] & sec \arrow[r, Rightarrow, "\alpha c"] & mrc
\end{tikzcd}$$

the composite 2-cell of diagram (3). Therefore by the 2-dimensional universal property of the coinserter B there exists a unique 2-cell $\beta : s \Longrightarrow mh$ such that $\beta e = \alpha$. Furthermore β is invertible as we could have equally used the 2-cell $\alpha^{-1} : mhe = mr \Longrightarrow se$ to construct a 2-cell $mh \Longrightarrow s$, its inverse, in an identical manner.
With regards uniqueness suppose that we had a second pair $(h_2 : B \longrightarrow C, \beta_2 : s \cong mh_2)$ satisfying the equations: $h_2 e = r$ and $\beta_2 e = \alpha$. Since e is co-faithful it suffices to show that $h_2 = h$. In order to show that $h_2 = h$ it suffices to show that both 1-cells agree upon composition with the coinserter morphism $e : A \longrightarrow B$ and the exhibiting 2-cell η. Certainly we have $h_2 e = r = he$. To show that $h\eta = h_2\eta$ it suffices, since m is faithful, to show that $mh\eta = mh_2\eta$. Now $mh_2\eta$ equals the composite:

$$\begin{tikzcd}
 & A \arrow[dr, "e"] \arrow[rr, "r"] & & C \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\beta_2^{-1}"] & \\
X \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & B \arrow[ur, "h_2"] \arrow[rr, "s"] \arrow[dr, "h_2"'] & {} \arrow[d, phantom, "\Downarrow\beta_2"] & D \\
 & A \arrow[ur, "e"] \arrow[rr, "r"'] & & C \arrow[ur, "m"'] &
\end{tikzcd}$$

However as $\beta_2 e = \alpha$ this composite agrees with the composite 2-cell of diagram (3). The same is true of $mh\eta$ and so we have $h_2 = h$.

□

Remark 2.35. It is worth remarking that having established the truth of Proposition 2.34 in an arbitrary 2-category $\mathcal{C}$ one may use the various 2-categorical dualities to obtain new results. For instance the statement

“Coequifier morphisms are orthogonal to faithful arrows” becomes, upon reversing 1-cells, the statement “Co-faithful arrows are orthogonal to equifier morphisms”. The strong orthogonality condition however is not self dual. Upon reversing 1-cells the opposite of strong orthogonality asserts the existence of a unique diagonal and 2-cell:

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "r"'] \arrow[dr, phantom, "\cong_{\beta}" near start] & B \arrow[d, "s"] \arrow[dl, "h"] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

Thus we might say “Co-faithful arrows are (co)strongly orthogonal to equifier morphisms“ (for lack of an alternative). One may also reverse 2-cells to obtain new results.

Definition 2.36. An orthogonal factorisation system on a 2-category $\mathcal{C}$ consists of classes E and M of 1-cells of $\mathcal{C}$ such that:

1. Each isomorphism of $\mathcal{C}$ belongs to both E and M.
2. E and M are closed under composition.
3. Each arrow $f : A \longrightarrow B$ may be factored as $A \xrightarrow{e} C \xrightarrow{m} B$ where $e \in E$ and $m \in M$.
4. E is orthogonal to M.

If in addition E is strongly orthogonal to M then we say that this is an enhanced factorisation system [44].

Proposition 2.37. Let (E, M) be a factorisation system on a 2-category $\mathcal{C}$. Suppose that $f \in \mathcal{C}$ is orthogonal to each morphism of M. Then $f \in E$.

Proof. We may factor $f : A \longrightarrow B$ as $e : A \longrightarrow C \in E$ followed by $m : C \longrightarrow B \in M$. This gives a commuting square:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] & B \arrow[d, "1"] \arrow[dl, dotted, "\exists ! h"'] \\
C \arrow[r, "m"'] & B
\end{tikzcd}$$

A unique diagonal map $h : B \longrightarrow C$ exists since f orthogonal to $m \in M$ and, by uniqueness, it is easily seen to provide an inverse to m. As m is an isomorphism it belongs to E. Consequently $f = em \in E$ as E is closed under composition. ☐

Remark 2.38. The notion of orthogonal factorisation system on a 2-category is a straightforward generalisation of an orthogonal factorisation system on a category. An orthogonal factorisation system (E, M) on a category C is defined just as in Definition 2.36, but the class of arrows E is only required to be orthogonal to the class M in the sense of Property 1 of Remark 2.32. Consequently given an orthogonal factorisation system on a 2-category the same classes of arrows determine an orthogonal factorisation system on its underlying category.

Proposition 2.39. Let $\mathcal{C}$ be a 2-category with cotensors with $\mathbf{2}$ and (E, M) be an orthogonal factorisation system on its underlying category. If the arrows of M are closed under cotensoring with $\mathbf{2}$ then (E, M) constitutes an orthogonal factorisation system on the 2-category $\mathcal{C}$.

Proof. This follows from Remark 2.32(2) in which we showed that if M is closed under cotensoring with $\mathbf{2}$ then orthogonality in the 2-categorical sense follows from orthogonality in the one dimensional sense. ☐

Remark 2.40. The factorisation systems of interest to us, namely the (bijective on objects/fully faithful factorisation) system on Cat and its internal analogue, arise from fibrations at the level of underlying categories. We next recall the notion of a fibration, and how a fibration gives rise to a factorisation system.

Definition 2.41. Consider a functor $u : A \longrightarrow B$ and a pair $(\alpha : b \longrightarrow ua, a)$ where $\alpha \in B$ and $a \in A$. A cartesian lift of $(\alpha : b \longrightarrow ua, a)$ is an arrow $\beta : c \longrightarrow a$ of A such that:

1. $u(\beta) = \alpha$.
2. Given any arrow $r : d \longrightarrow a$ of A and an arrow $k : ud \longrightarrow b$ of B such that the triangle on the left commutes:

$$\begin{tikzcd}
ud \arrow[d, "k"'] \arrow[dr, "ur"] & & & d \arrow[d, "\exists! h"'] \arrow[dr, "r"] & \\
b \arrow[r, "\alpha"'] & ua & & c \arrow[r, "\beta"'] & a
\end{tikzcd}$$

   there exists a unique arrow $h : d \longrightarrow c$ such that the triangle on the right commutes and such that $uh = k$.

This functor u is a fibration if each such pair has a cartesian lift.

Proposition 2.42. Let A and B be categories and $u : A \longrightarrow B$ a fibration. Then there exists an (E, M) factorisation system on A where $E = \{\alpha : a \longrightarrow b \in A : u(\alpha) \text{ is invertible}\}$ and $M = \{\alpha : a \longrightarrow b \in A : \alpha \text{ is a cartesian morphism for the fibration } u\}$.

Proof. Certainly both classes E and M are closed under isomorphisms and composition. This is obvious in the case of E whilst proofs of the corresponding facts for cartesian morphisms may be found in [9]. We must check that each arrow of A may be factored as an E followed by an M, and that the classes are orthogonal. With regards the former consider $\alpha : a \longrightarrow b \in A$. We have $(u\alpha : ua \longrightarrow ub, b)$ with cartesian lift $\alpha_2 : c \longrightarrow b \in M$ which satisfies $u\alpha_2 = u\alpha$. As α_2 is cartesian there consequently exists a unique arrow $\alpha_1 : a \longrightarrow c$ such that $\alpha_2\alpha_1 = \alpha$ and $u\alpha_1 = 1_{ua}$. Therefore $\alpha_1 \in E$ and we have the required factorisation.
Consider a commutative square as in diagram (1) below with $e \in E$ and $m \in M$:

$$\begin{tikzcd}
(1) & a \arrow[r, "e"] \arrow[d, "r"'] & b \arrow[d, "s"] & (2) & & ub \arrow[dl, "ur(ue)^{-1}"'] \arrow[d, "us"] & (3) & a \arrow[r, "e"] \arrow[d, "r"'] & b \arrow[d, "s"] \arrow[dl, dotted, "\exists! k"'] & (4) & a \arrow[d, shift right, "ke"'] \arrow[d, shift left, "r"] \arrow[dr, "se"] & & (5) & & b \arrow[dl, shift right, "l"'] \arrow[dl, shift left, "k"] \arrow[d, "s"] \\
& c \arrow[r, "m"'] & d & & uc \arrow[r, "um"'] & ud & & c \arrow[r, "m"'] & d & & c \arrow[r, "m"'] & d & & c \arrow[r, "m"'] & d
\end{tikzcd}$$

As ue is invertible we have a commutative triangle as in diagram (2). As m is cartesian there exists a unique arrow $k : b \longrightarrow c$ rendering commutative the right triangle of diagram (3) and such that $uk = ur(ue)^{-1}$. We must check that the left triangle of diagram (3) also commutes. This is to show that the two vertical arrows of diagram (4) agree. Certainly both vertical arrows render the triangle (4) commutative. As m is cartesian it consequently suffices to verify that both have the same image under u. We have $u(ke) = (uk)(ue) = (ur)(ue)^{-1}(ue) = ur$. Therefore $ke = r$ as required so that both triangles of diagram (3) are commutative. We must check that k is the unique diagonal rendering commutative both triangles of (3). If there were another such diagonal l we would have a commuting pair of triangles as in diagram (5) additionally satisfying $ke = le$. Since m is cartesian it suffices to check that $ul = uk$. Now $(ul)(ue) = (uk)(ue)$. As ue is invertible this implies that $ul = uk$ as required. ☐

Example 2.43. The (bijective on objects/fully faithful) factorisation system on Cat
Any functor $u : C \longrightarrow D$ may be factored as a bijective on objects functor followed by a fully faithful one. The intermediate category E has the same objects as C, whilst a morphism $a \longrightarrow b \in E$ is a triple $\langle a, f : ua \longrightarrow ub, b \rangle$, where $f \in D$. Composition of arrows in E is inherited from composition in D so that we obtain a factorisation of u:

$$C \xrightarrow{u_1} E \xrightarrow{u_2} D$$

In this factorisation u_1 is identity on objects, whilst the action of u_2 on objects of E is the same as the action of u on objects of C. Given an arrow $g : a \longrightarrow b$, u_1 and u_2 act as:

$$\begin{tikzcd} a \arrow[r, "g"] & b \arrow[r, mapsto, "u_1"] & \langle a, ua \arrow[r, "ug"] & ub, b\rangle \arrow[r, mapsto, "u_2"] & ua \arrow[r, "ug"] & ub \end{tikzcd}$$

The functor u_1 is, in particular, bijective on objects, whilst u_2 is clearly fully faithful. It is straightforward to verify that bijections on objects are orthogonal to fully faithful functors and certainly each isomorphism is both bijective on objects and fully faithful, and we will in any case give a more general proof of these claims in Corollary 2.62. Thus (Bijective on objects functors, fully faithful functors) form an orthogonal factorisation system on Cat. In fact this is an enhanced factorisation system as shown in [50]. This factorisation system is that associated to the fibration $ob : \mathcal{U}\text{Cat} \longrightarrow \text{Set}$, which forgets the arrows of a small category, in the manner described in Proposition 2.42.

Corollary 2.44. In Cat coinserter, coequifier, coinverter, codescent and Kleisli morphisms are bijective on objects.

Proof. These types of arrow are orthogonal to fully faithful morphisms by Proposition 2.34. As bijections on objects and fully faithful functors form an orthogonal factorisation system on Cat it follows, by Proposition 2.37, that each such arrow is bijective on objects. ☐

Remark 2.45. To say (E, M) is an orthogonal factorisation system on a 2-category $\mathcal{C}$ is equally to say that (M, E) is an orthogonal factorisation system on $\mathcal{C}^{op}$. As noted in Remark 2.35 the strong orthogonality condition is not self-dual in this sense; we might call co-enhanced an orthogonal factorisation system on $\mathcal{C}$ for which the corresponding factorisation system on $\mathcal{C}^{op}$ is enhanced: those factorisation systems satisfying the co-strong orthogonality condition. Though such factorisation systems will not be considered here, natural examples do exist: the (Final functor/Discrete fibration)-factorisation system on Cat [47] is easily seen to be one.

2.5 Internal categories and bijections on objects

In this section we describe the internal analogue of the (Bijective on objects/fully faithful) factorisation system on Cat. We begin by recalling the well known notion of internal category [16], and the 2-category of internal categories, and use the opportunity to fix our notation for internal categories.

Definition 2.46. Let $\mathcal{E}$ be a category with pullbacks. The data for an internal category X in $\mathcal{E}$ is strict coherence data:

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "i_x" description] \end{tikzcd}$$

The objects $X_0/X_1/X_2$ are referred to respectively as the object of objects/arrows/composable pairs. The arrows d_x, i_x, c_x are the domain, identity and codomain maps for X whilst p_x, q_x, m_x are the first and second projections and composition maps respectively. This object of $[\Delta_2^{-op}, \mathcal{E}]$ must satisfying the following properties:

1. The square:

$$\begin{tikzcd} X_2 \arrow[r, "p_x"] \arrow[d, "q_x"'] & X_1 \arrow[d, "c_x"] \\ X_1 \arrow[r, "d_x"'] & X_0 \end{tikzcd}$$

   is a pullback.

2. The induced arrows $(i_x, 1), (1, i_x) : X_1 \rightrightarrows X_2$ satisfy the identity axioms: $m_x \circ (i_x, 1) = 1 = m_x \circ (1, i_x)$.

3. Consider the object of composable triples, the pullback:

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d] & X_2 \arrow[d, "p_x"] \\ X_2 \arrow[r, "q_x"'] & X_1 \end{tikzcd}$$

and the induced arrows $(m_x, 1), (1, m_x) : X_3 \rightrightarrows X_2$. These satisfy associativity: $m_x \circ (m_x, 1) = m_x \circ (1, m_x)$.

Remark 2.47. The above definition presents the minimal data required to define an internal category in a category with pullbacks. Evidently more maps exist as a consequence of the pullbacks; for instance the arrows $(i_x, 1), (1, i_x) : X_1 \rightrightarrows X_2$ extend X uniquely to an object of $[\Delta_2^{op}, \mathcal{E}]$ and consequently we may equally speak of X as an object of $[\Delta_2^{op}, \mathcal{E}]$.

Remark 2.48. An internal category in Set is precisely a small category.

Definition 2.49. An internal functor $f : X \longrightarrow Y$ is a morphism of $[\Delta_2^{op}, \mathcal{E}]$ between the internal categories X and Y. Thus it consists of three arrows $\{f_i : X_i \longrightarrow Y_i, i = 0, 1, 2\}$ although the arrow $f_2 : X_2 \longrightarrow Y_2$ into the pullback is determined by f_0 and f_1. An internal natural transformation:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

is given by a 1-cell $\overline{\alpha} : X_0 \longrightarrow Y_1$, its "arrow component", such that $d_y\overline{\alpha} = f_0$, $c_y\overline{\alpha} = g_0$ and such that the "naturality square":

$$\begin{tikzcd} X_1 \arrow[r, "(\overline{\alpha}\circ d_x, g_1)"] \arrow[d, "(f_1, \overline{\alpha}\circ c_x)"'] & Y_2 \arrow[d, "m_y"] \\ Y_2 \arrow[r, "m_y"'] & Y_1 \end{tikzcd}$$

commutes.

Remark 2.50. Internal categories in $\mathcal{E}$, internal functors and internal natural transformations form a 2-category $Cat(\mathcal{E})$. Its underlying category structure is inherited from that of the functor category $[\Delta_2^{op}, \mathcal{E}]$. Given vertically composable internal natural transformations $\alpha : f \Longrightarrow g$ and $\beta : g \Longrightarrow h$ of $Cat(\mathcal{E})(X, Y)$ their vertical composite $\beta \circ \alpha : f \Longrightarrow h$ has arrow component $\overline{\beta \circ \alpha} : X_0 \longrightarrow Y_1$ the composite:

$$X_0 \xrightarrow{(\overline{\alpha}, \overline{\beta})} Y_2 \xrightarrow{m_y} Y_1$$

where $(\overline{\alpha}, \overline{\beta}) : X_0 \longrightarrow Y_2$ is the unique arrow into the pullback determined by the commutativity of $c_y\overline{\alpha} = g_0 = d_y\overline{\beta}$.
Given an internal functor $k : W \longrightarrow X$ the internal natural transformation $\alpha k : fk \Longrightarrow gk$ has arrow component:

$$\overline{\alpha k} = W_0 \xrightarrow{k_0} X_0 \xrightarrow{\overline{\alpha}} Y_1$$

If we have $k : Y \longrightarrow Z$ then $k\alpha : kf \Longrightarrow kg$ has arrow component:

$$\overline{k\alpha} = X_0 \xrightarrow{\overline{\alpha}} Y_1 \xrightarrow{k_1} Z_1$$

The identity natural transformation $1_f : f \Longrightarrow f$ has arrow component:

$$\overline{1_f} = X_0 \xrightarrow{f_0} Y_0 \xrightarrow{i_y} Y_1$$

and this latter fact evidently justifies our notational distinction between an internal natural transformation and its arrow component.

Remark 2.51. In the case of Set, we have $Cat(\text{Set}) = \text{Cat}$ the 2-category of small categories.

Remark 2.52. Given a 2-category $\mathcal{C}$ we may of course consider an internal category X in it, the only distinction being that we expect X_2 to be a pullback in the 2-categorical sense. If the 2-category $\mathcal{C}$ has pullbacks (in the 2-categorical sense) then every 1-pullback is a 2-pullback in any case, since the one dimensional universal property is sufficient to recognise the pullback up to isomorphism. Consequently if $\mathcal{C}$ has 2-pullbacks we have $Cat(\mathcal{C}) = Cat(\mathcal{UC})$.

Remark 2.53. It is important to observe that if $\mathcal{C}$ is a 2-category there are two possibilities as to what the 2-cells of $Cat(\mathcal{C})$ might be. The first possibility, described in the Remark 2.52 and the one that will be pursued in this thesis, is to set $Cat(\mathcal{C}) = Cat(\mathcal{UC})$, thereby ignoring the 2-cells of $\mathcal{C}$. The second possibility would be to define $Cat(\mathcal{C})$ to be the full sub 2-category of $[\Delta_2^{op}, \mathcal{C}]$ with objects the internal categories in $\mathcal{C}$. These 2 possibilities are quite distinct. For instance given a category with pullbacks $\mathcal{E}$ we may view it as a locally discrete 2-category and upon doing so the approach we will take gives $Cat(\mathcal{E})$ the 2-category of categories internal to $\mathcal{E}$ as described in Remark 2.50. However under the second approach "$Cat(\mathcal{E})$" would be a full sub 2-category of the locally discrete $[\Delta_2^{op}, \mathcal{E}]$, and thus locally discrete itself.

Remark 2.54. We now consider the internal analogue of the (Bijective on objects/fully faithful)-factorisation system on Cat. The notion of an internally fully faithful functor presented here is slightly different to that which typically appears in the literature, for instance in [11]. It is usually assumed that the base category is finitely complete, in which case the notion of an internally fully faithful functor simplifies slightly, as described in Remark 2.58. Again the fibration $ob : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ giving rise to the factorisation system is well known, though again only to the author's knowledge considered in the case that $\mathcal{E}$ is finitely complete, in which case ob has a fully faithful right adjoint and arguments are used to prove it is a fibration [11] which are inapplicable here. Although we presume the following definitions and results concerning this factorisation system to be well known we consequently give a full treatment of all aspects involved.

Definition 2.55. Let $\mathcal{E}$ be a category with pullbacks.

1. An internal functor $f : X \longrightarrow Y$ is said to be internally bijective on objects if $f_0 : X_0 \longrightarrow Y_0$ is an isomorphism.

2. $f : X \longrightarrow Y$ is said to be internally fully faithful if:

$$\begin{tikzcd}
 & X_0 \arrow[dr, "f_0"] & \\
 & & Y_0 \\
X_1 \arrow[uur, "d_x"] \arrow[r, "f_1"] \arrow[ddr, "c_x"'] & Y_1 \arrow[ur, "d_y"] \arrow[dr, "c_y"'] & \\
 & & Y_0 \\
 & X_0 \arrow[ur, "f_0"'] &
\end{tikzcd}$$

exhibits X_1 as the limit in $\mathcal{E}$ of the diagram:

$$\begin{tikzcd}
X_0 \arrow[dr, "f_0"] & \\
 & Y_0 \\
Y_1 \arrow[ur, "d_y"] \arrow[dr, "c_y"'] & \\
 & Y_0 \\
X_0 \arrow[ur, "f_0"'] &
\end{tikzcd}$$

Remark 2.56. When $\mathcal{E}$ =Set (so that $Cat(\mathcal{E}) = \text{Cat}$) it is clear that the internal bijections on objects are precisely the bijective on objects functors. Furthermore a functor $f : X \longrightarrow Y$ is internally fully faithful precisely when the arrows X_1 of X may be identified with triples $(a, \alpha : fa \longrightarrow fb, b)$ where $a, b \in X$ and $\alpha \in Y$. Therefore the internally fully faithful functors in Cat are precisely the fully faithful ones, in the ordinary sense.

Remark 2.57. Consider the diagram of Definition 2.55(2) of which we consider the limit. We are considering categories with pullbacks and should remark that this limit is constructible in any category with pullbacks.

Observe that the diagram consists of a pair of opspans. Forming the pullbacks of these separately induces a pair of spans (the pullback projections), thus four arrows in total. The middle two arrows then have common codomain Y_1 and so constitute an opspan. The pullback of this is the limit of the diagram.

Remark 2.58. If the category $\mathcal{E}$ additionally has products then the limit of Definition 2.55(2) reduces to the pullback square on the left below:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & Y_1 \arrow[d, "{(d_y,c_y)}"] & & X_1 \arrow[r, "f_1"] \arrow[d, "{(d_x,c_x)}"'] & Y_1 \arrow[d, "{(d_y,c_y)}"] \\
(X_0)^2 \arrow[r, "(f_0)^2"'] & (Y_0)^2 & & (X_0)^2 \arrow[r, "(f_0)^2"'] & (Y_0)^2
\end{tikzcd}$$

whilst an internal functor $f : X \longrightarrow Y$ is internally fully faithful precisely if the square on the right above is a pullback.

Proposition 2.59. The internally fully faithful functors are precisely the fully faithful arrows in $Cat(\mathcal{E})$.

Proof. It is straightforward to verify directly that any internally fully faithful functor is actually fully faithful. Conversely given a fully faithful internal functor $f : X \longrightarrow Y$ (a fully faithful arrow of $Cat(\mathcal{E})$) it suffices to check that for each $A \in \mathcal{E}$ the diagram:

$$\begin{tikzcd}
 & \mathcal{E}(A,X_0) \arrow[dr, "f_0^*"] & \\
 & & \mathcal{E}(A,Y_0) \\
\mathcal{E}(A,X_1) \arrow[uur, "d_x^*"] \arrow[r, "f_1^*"] \arrow[ddr, "c_x^*"'] & \mathcal{E}(A,Y_1) \arrow[ur, "d_y^*"] \arrow[dr, "c_y^*"'] & \\
 & & \mathcal{E}(A,Y_0) \\
 & \mathcal{E}(A,X_0) \arrow[ur, "f_0^*"'] &
\end{tikzcd}$$

is limiting in Set. Now since $\mathcal{E}(A,-)$ preserves pullbacks it takes internal categories in $\mathcal{E}$ to internal categories in Set so that, abusing notation, we have an internal functor $\mathcal{E}(A,f) : \mathcal{E}(A,X) \longrightarrow \mathcal{E}(A,Y)$ in Set. We need only show that this is a fully faithful functor in $Cat(\text{Set}) = \text{Cat}$, since the above diagram will then be limiting by Remark 2.56. What we actually describe here, and will consider in much greater detail in the next chapter, is the action of a 2-functor $Cat(\mathcal{E}(A,-)) : Cat(\mathcal{E}) \longrightarrow Cat(\text{Set})$ on an arrow $f : X \longrightarrow Y$. It thereby suffices to show that this 2-functor preserves fully faithfulness. This will follow from Corollary 3.53 of Chapter 3.

(An alternative approach is to observe that $Cat(\mathcal{E}(A,-))$ is representable, represented by the canonical discrete internal category $[A]$ which has $[A]_i = A$ for $i = 0, 1, 2$. Any representable preserves fully faithfulness, since that concept is representable and so the result follows.) □

Notation 2.60. We observed in Proposition 2.59 that the internally fully faithful functors are precisely the fully faithfuls in $Cat(\mathcal{E})$ and we will refer to them as fully faithful without abuse of notation. Furthermore we will refer to the internally bijective on objects functors simply as being bijective on objects where the context is clear.

Proposition 2.61. Let $\mathcal{E}$ have pullbacks. The forgetful functor $ob : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ assigning to an internal category X its object of objects X_0 is a fibration. The morphisms inverted by ob are precisely the bijections on objects, whilst the cartesian morphisms are the fully faithfuls.

Proof. Given an internal category X and a morphism of $\mathcal{E}$, $f : A \longrightarrow X_0$, we must describe its cartesian lifting. This is to be an internal functor $k : B \longrightarrow X$, in particular satisfying $k_0 = f$.
We set $B_0 = A$ and $k_0 = f$. Define B_1 to be the limit of the pair of opspans on the right of Diagram (1) below:

$$\begin{tikzcd}
 & & A \arrow[dr, "f"] & \\
 & & & X_0 \\
(1) & B_1 \arrow[uur, "d_b"] \arrow[r, "k_1"] \arrow[ddr, "c_b"'] & X_1 \arrow[ur, "d_x"] \arrow[dr, "c_x"'] & \\
 & & & X_0 \\
 & & A \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A \arrow[dr, "f"] & \\
 & & & X_0 \\
(2) & B_2 \arrow[uur, "d_b p_b"] \arrow[r, "m_x k_2"] \arrow[ddr, "c_b q_b"'] & X_1 \arrow[ur, "d_x"] \arrow[dr, "c_x"'] & \\
 & & & X_0 \\
 & & A \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A \arrow[dr, "f"] & \\
 & & & X_0 \\
(3) & Y_1 \arrow[uur, "h d_y"] \arrow[r, "g_1"] \arrow[ddr, "h c_y"'] & X_1 \arrow[ur, "d_x"] \arrow[dr, "c_x"'] & \\
 & & & X_0 \\
 & & A \arrow[ur, "f"'] &
\end{tikzcd}$$

Note that this limit exists by Remark 2.57. Define the morphisms d_b, c_b and k_1 as the cone projections in that diagram as indicated. It is immediate that the induced internal functor $k : B \longrightarrow X$, upon its full description, will be fully faithful. The commuting squares of Diagram (1) show that we have so far constructed a morphism of graphs. The identity map $i_b : B_0 \longrightarrow B_1$ is induced by the triple $(1 : A \longrightarrow A, i_x f : A \longrightarrow X_1, 1 : A \longrightarrow A)$ which constitutes a cone to the above diagram and it follows at once that we have a reflexive graph morphism. The object B_2 and the morphisms p_b and q_b are now predetermined as the pullback and pullback projections of the opspan (c_b, d_b). Furthermore so is k_2. It is the unique arrow $k_2 : B_2 \longrightarrow X_2$ into the pullback X_2 induced by the equality: $c_x k_1 p_b = f c_b p_b = f d_b q_b = d_x k_1 q_b$. It remains to describe the composition map $m_b : B_2 \longrightarrow B_1$ for B. Having completely defined the proposed internal functor k observe that the map m_b must now satisfy $m_x k_2 = k_1 m_b$. Now we have a commutative diagram (2). As B_1 is the limit there consequently exists a unique such map $m_b : B_2 \longrightarrow B_1$ satisfying the required equations: $k_1 m_b = m_x k_2$, $d_b m_b = d_b p_b$ and $c_b m_b = c_b q_b$. Having defined all of the structure maps of the internal category A one must check that they indeed provide an internal category structure, but this is routine.
In total what we have shown so far is that given $f : A \longrightarrow X_0$ there exists a unique (up to isomorphism) fully faithful $k : B \longrightarrow X$ such that $ob(k) = f$.
In order to show that k is the cartesian lifting of the pair $(X, f : A \longrightarrow X_0)$ consider an internal functor $g : Y \longrightarrow X$ and an arrow $h : Y_0 \longrightarrow A$ of $\mathcal{E}$ such that $fh = g_0$. We must show there exists a unique extension of h to an internal functor $l : Y \longrightarrow B$ such that $kl = g$ and $l_0 = h$. Consider the commutative diagram (3). This induces a unique arrow $l_1 : Y_1 \longrightarrow B_1$ into the limit such that $k_1 l_1 = g_1, d_b l_1 = h d_y$ and $c_b l_1 = h c_y$. In other words there exists a unique extension of h to a graph morphism satisfying the required conditions. As l_2 is uniquely determined by l_1 it remains only to verify that this determines an internal functor, namely that identites and composition are preserved. This is the case.
Clearly the morphisms inverted by ob are precisely the bijections on objects. Furthermore consider a fully faithful $r : X \longrightarrow Y$ and the pair $(r_0 : X_0 \longrightarrow Y_0, Y)$. We have already observed that such a pair admits a unique, up to isomorphism, extension to a fully faithful arrow of $Cat(\mathcal{E})$ with the same action on objects, its cartesian lifting. Since $r : X \longrightarrow Y$ is fully faithful it follows that r is isomorphic to the cartesian lifting. As cartesian liftings are determined only up to isomorphism it follows that r is the cartesian lifting itself. Therefore the fully faithful arrows are precisely the cartesian lifts.

□

Corollary 2.62. (Bijections on objects/fully faithfuls) form an orthogonal factorisation system on the 2-category $Cat(\mathcal{E})$ whenever $\mathcal{E}$ is a category with pullbacks.

Proof. In light of Proposition 2.61 it now follows from Proposition 2.42 that the bijections on objects and fully faithfuls form a orthogonal factorisation system on the underlying category of $Cat(\mathcal{E})$. In order to show this extends to a factorisation system on $Cat(\mathcal{E})$ as a 2-category it will suffice by Proposition 2.39, upon showing that $Cat(\mathcal{E})$ has cotensors with $\mathbf{2}$, to show that the fully faithful arrows are closed under cotensors with $\mathbf{2}$. It is easy to see that fully faithful arrows are closed under cotensors with $\mathbf{2}$ in any 2-category. That $Cat(\mathcal{E})$ has cotensors with $\mathbf{2}$ is well known, and is proven in Chapter 3. This completes the proof. ☐

2.6 Codescent objects of higher kernels and bijections on objects

Definition 2.63. Let $\mathcal{C}$ be a 2-category with pullbacks and comma objects. Each arrow $f : A \longrightarrow B$ of $\mathcal{C}$ has an internal category associated to it, $K(f)$:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description]
\end{tikzcd}$$

called its "higher kernel" [55]. Here $f|f$ is the comma object of $f : A \longrightarrow B$ with universal cone as on the left below:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
f|f|f \arrow[r, "p"] \arrow[d, "q"'] & f|f \arrow[d, "c"] \\
f|f \arrow[r, "d"'] & A
\end{tikzcd}$$

whilst $f|f|f$ denotes the pullback on the right above. The composition map $m : f|f|f \longrightarrow f|f$ is the unique arrow into the comma object such that:

$$\begin{tikzcd}
 & f|f \arrow[r, "d"] & A \arrow[dr, "f"] & \\
f|f|f \arrow[ur, "p"] \arrow[r, "m"] \arrow[dr, "q"'] & f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & B \\
 & f|f \arrow[r, "d"'] & f|f \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & f|f \arrow[rr, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & A \arrow[dr, "f"] & \\
f|f|f \arrow[ur, "p"] \arrow[dr, "q"'] & & A \arrow[rr, "f"] & & B \\
 & f|f \arrow[ur, "d"] \arrow[rr, "c"'] & \Downarrow\alpha & A \arrow[ur, "f"'] &
\end{tikzcd}$$

whilst the identity map $i : A \longrightarrow f|f$ is the unique arrow into the comma object such that:

$$\begin{tikzcd}
 & & A \arrow[dr, "f"] & & & & \\
A \arrow[urr, bend left, "1"] \arrow[r, "i"] \arrow[drr, bend right, "1"'] & f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & B & = & A \arrow[r, "f"] & B \\
 & & a_0 \arrow[ur, "f"'] & & & &
\end{tikzcd}$$

It is routine to verify that the higher kernel is an internal category by using the universal property of the comma object $f|f$.

Remark 2.64. Given an arrow $f : A \longrightarrow B$ in a 2-category $\mathcal{C}$ with comma objects and pullbacks consider its higher kernel $K(f)$:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description]
\end{tikzcd}$$

and the corresponding universal cone for the comma object:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

The triple (B, f, α) evidently forms a codescent cocone (by virtue of the defining equations for m and i of Definition 2.63) from the higher kernel. Suppose that the codescent object $QK(f)$ of the higher kernel exists and denote its universal cocone $(QK(f), f_1, \beta)$. Then we obtain a unique arrow $f_2 : QK(f) \longrightarrow B$:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description] \arrow[rr, "f"] \arrow[dr, "f_1"'] & & B \\
& & & QK(f) \arrow[ur, "f_2"'] &
\end{tikzcd}$$

such that $f_2 f_1 = f$ and $f_2 \beta = \alpha$. In particular if $\mathcal{C}$ has higher kernels and codescent objects of them then each arrow of $\mathcal{C}$ may be factored through the codescent object of its higher kernel, via the corresponding codescent morphism.

Remark 2.65. Consider the factorisation of the preceding Remark in the case that $\mathcal{C}$ is a locally discrete 2-category. In that case a comma object in $\mathcal{C}$ is simply a pullback square: the 2-cell in the universal cone is forced to be an identity. Consequently the higher kernel of an arrow in $\mathcal{C}$ is simply its kernel pair, iterated one step further than usual. Similarly the codescent object of strict coherence data in $\mathcal{C}$ is simply the coequaliser of its underlying graph: the codescent morphism now being forced to coequalise the arrows of the graph, the equations between 2-cells for a codescent cocone holding immediately as all 2-cells are identities. Therefore in the case $\mathcal{C}$ is a locally discrete 2-category the factorisation of an arrow of $\mathcal{C}$ through the higher kernel of its codescent object agrees with its regular factorisation through the coequaliser of its kernel pair [23].

Example 2.66. Given a functor $f : A \longrightarrow B \in \mathrm{Cat}$ the higher kernel of f admits a simple explicit description. The comma category $f|f$ has objects:

$$(a, fa \xrightarrow{\alpha} fb, b)$$

where $a, b \in \mathcal{A}$. Given $a \in \mathcal{A}$ we have:

$$a \qquad \xmapsto{\;i\;} \qquad (a, fa \xrightarrow{1} fa, a)$$

The pullback $f|f|f$ has objects:

$$(a, fa \xrightarrow{\alpha} fb, b, fb \xrightarrow{\beta} fc, c)$$

where $a, b, c \in \mathcal{A}$ and composition at the level of objects is given by:

$$(a, fa \xrightarrow{\alpha} fb, b, fb \xrightarrow{\beta} fc, c) \qquad \xmapsto{\;m\;} \qquad (a, fa \xrightarrow{\beta \circ \alpha} fc, c) \quad .$$

The extension of i and m to morphisms of their respective domains is evident.

Proposition 2.67. In Cat the factorisation of a functor $f : A \longrightarrow B$ through the codescent object of its higher kernel agrees with its (bijective on objects/fully faithful) factorisation.

Proof. Our proof here essentially follows [55]. Consider the higher kernel of $f : A \longrightarrow B$:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description]
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & B \\
& A \arrow[ur, "f"'] &
\end{tikzcd}$$

together with its comma cone on the right above. Consider the factorisation of f as bijective on objects followed by fully faithful as on the left below:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_1"'] & & B \\
& C \arrow[ur, "f_2"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dr, "f_1"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\theta & C \\
& A \arrow[ur, "f_1"'] &
\end{tikzcd}$$

Since f_2 is fully faithful the natural transformation α factors uniquely through f_2 to give a natural transformation θ as on the right above. Now the triple (f, α, B) is a codescent cocone from the higher kernel. The equations for a codescent cocone only concern equalities of 2-cells. Therefore the triple (f_1, θ, C) is also a codescent cocone, since its composite with the faithful functor f_2 is a codescent cocone.
We now show that (f_1, θ, C) is the universal cocone. To see this let us consider its explicit description in more detail. As described in Example 2.66 the category C has the same objects as A and arrows given by triples $(a, \alpha : fa \longrightarrow fb, b)$ where $a, b \in A$ and $\alpha \in B$ and with composition inherited from the category B. In other words the arrows of C are precisely the objects of the comma category $f|f$. Given an object $(a, \alpha : fa \longrightarrow fb, b)$ of $f|f$ its image under $f_1 d$ and $f_1 c$ are a and b respectively whilst $\theta(a, \alpha : fa \longrightarrow fb, b) = (a, \alpha : fa \longrightarrow fb, b)$. Now given any other cocone $(D, g, \phi : gd \Longrightarrow gc)$ we define $k : C \longrightarrow D$ to agree with g on objects (as f_1 is the identity on objects). Arrows of C are simply objects of $f|f$ so that given such an arrow $(a, \alpha : fa \longrightarrow fb, b)$ we define $k(a, \alpha : fa \longrightarrow fb, b) = \phi(a, \alpha : fa \longrightarrow fb, b)$. This ensures that both $kf_1 = g$ and $k\theta = \phi$ and upon showing that k is a functor it is evidently the unique one satisfying these equations. That k preserves composition and identities now follows from the multiplicative and unital equations for a codescent cocone. Therefore C is the codescent object $QK(f)$ of the higher kernel of f, with codescent morphism f_1 and exhibiting 2-cell θ. Since postcomposing the natural transformation θ with the fully faithful functor $f_2 : C \longrightarrow D$ gives the natural transformation associated to the comma object $f|f$ we see that this is precisely the factorisation of f through the codescent object of its higher kernel described in Remark 2.64. ☐

Corollary 2.68. In Cat the codescent morphisms are precisely the bijections on objects. In particular each codescent morphism exhibits its codomain as the codescent object of its higher kernel.

Proof. We have already seen, in Corollary 2.44, that codescent morphisms in Cat must be bijective on objects. Conversely consider a bijective on objects functor $f : A \longrightarrow B$ and its factorisation $f = f_2 f_1$ through the codescent object of its higher kernel. By Proposition 2.67 this is equally the factorisation of f as bijective on objects followed by fully faithful. As f is itself bijective on objects it follows that f_2 is both bijective on objects and fully faithful: an isomorphism. Now f_1 is the codescent morphism of the higher kernel of f and f_2 is an isomorphism. Consequently $f = f_2 f_1$ is the codescent morphism exhibiting its codomain as the codescent object of its higher kernel. ☐

Definition 2.69. Let $\mathcal{C}$ be a 2-category admitting higher kernels. We say that a codescent morphism is effective if it exhibits its codomain as the codescent object of its higher kernel.

Corollary 2.70. Codescent morphisms are effective in Cat.

Proof. This is immediate from Corollary 2.68. ☐

2.7 Two sided discrete fibrations and cateads

Definition 2.71. A span in Cat:

$$\begin{tikzcd}
 & A \arrow[dl, "p"'] \arrow[dr, "q"] & \\
B & & C
\end{tikzcd}$$

(also written $B \xrightarrow{(p,q)} C$) is said to be a two sided discrete fibration (or discrete fibration from B to C) if it satisfies the following three conditions:

1. *Unique p-lifts:* Given $a \in A$ and $\alpha : b \longrightarrow pa$ of B an arrow β with codomain a such that $p(\beta) = \alpha$ and such that $q(\beta)$ is an identity arrow is said to be a *p-lift* of the pair (α, a). Each pair (α, b) must have a unique p-lift, which we denote by $\alpha^p : b^p \longrightarrow a$.

2. *Unique q-lifts:* Given $a \in A$ and $\alpha : qa \longrightarrow c$ of C an arrow β with domain a such that $q(\beta) = \alpha$ and such that $p\beta$ is an identity is referred to as a *q-lift* of the pair (a, α). Each pair (a, α) must have a unique q-lift, which we denote by $\alpha^q : a \longrightarrow c^q$.

3. *The bimodule condition:* Given an arrow $\alpha : a \longrightarrow b \in A$ we may consider the p-lift $(p\alpha)^p : a^p \longrightarrow b$ of the pair $(p\alpha : pa \longrightarrow pb, b)$ and the q-lift $(q\alpha)^q : a \longrightarrow b^q$ of the pair $(a, q\alpha : qa \longrightarrow qb)$. We then require that $b^q = a^p$ and that the composite $(p\alpha)^p(q\alpha)^q : a \longrightarrow b$ is equal to $\alpha : a \longrightarrow b$.

Remark 2.72. Two sided discrete fibrations $(p, q) : B \longrightarrow C$ correspond to functors $B^{op} \times C \longrightarrow \mathrm{Set}$ (variously known as bimodules [40], distributors and profunctors). We have an equivalence of categories $DFib(B, C) \simeq [B^{op} \times C, \mathrm{Set}]$ where $DFib(B, C)$ is the full subcategory of $Span(B, C)$ with objects the discrete fibrations from B to C. Given a two sided discrete fibration $(p, q) : B \longrightarrow C$, as in Definition 2.71, the corresponding "profunctor" $F_{p,q} : B^{op} \times C \longrightarrow \mathrm{Set}$ is defined on objects by $F_{p,q}(b, c) = \{x \in A : px = b \text{ and } qx = c\}$, the two sided fibre at the pair (b, c). Given $\alpha : b_1 \longrightarrow b_2 \in B$ and $c \in C$ we have an action $F_{p,q}(\alpha, 1) : F_{p,q}(b_2, c) \longrightarrow F_{p,q}(b_1, c)$ which assigns to an element $x \in F_{p,q}(b_2, c)$ the domain of the p-lift of the pair $(\alpha : b_1 \longrightarrow b_2 = px, x)$. Uniqueness of p-lifts ensures that $F_{p,q}$ functorial in B^{op} for fixed $c \in C$. Similarly the q-lifts provide a functorial action of $F_{p,q}$ on C for fixed values of B. The bimodule condition asserts that the actions, which have been defined by fixing values of B and C separately, are compatible; thus ensuring that $F_{p,q}$ is a bimodule. Conversely given a bimodule one obtains the corresponding two sided discrete fibration by forming a 2-sided category of elements, as described in [58].

Remark 2.73. A relation on a set X may be thought of either as a function $R : X \times X \longrightarrow 2$ to the set with two elements or as a jointly monic pair $d, c : R \rightrightarrows X$; the latter description allowing a representable generalisation to categories other than Set. Bimodules $A^{op} \times A \longrightarrow \mathrm{Set}$ may be considered a 2-dimensional analogue of the notion of relation [40]. By virtue of the equivalence between bimodules and two sided discrete fibrations described in Remark 2.72, a two sided discrete fibration $d, c : R \rightrightarrows A$ may equally be thought of as a 2-dimensional relation. This latter description gives a notion of "relation" internal to the 2-category Cat, one which may be generalised representably to other 2-categories.

Definition 2.74. A span:

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[dr, "q"] & \\ B & & C \end{tikzcd}$$

in a 2-category $\mathcal{C}$ is said to be a two sided discrete fibration if for each $D \in \mathcal{C}$ the span:

$$\mathcal{C}(D, B) \xrightarrow{(\mathcal{C}(D,p), \mathcal{C}(D,q))} \mathcal{C}(D, C)$$

is a two sided discrete fibration in Cat.

Remark 2.75. Consider a two sided discrete fibration $B \xrightarrow{(p,q)} C$ in a 2-category $\mathcal{C}$ as above. By a p-lift we now mean a p-lift for the two sided discrete fibration:

$$\mathcal{C}(D, B) \xrightarrow{(\mathcal{C}(D,p), \mathcal{C}(D,q))} \mathcal{C}(D, C)$$

for some $D \in \mathcal{C}$. A p-lift is therefore associated to a 2-cell such as on the left below:

$$\begin{tikzcd} D \arrow[r, "r"] \arrow[dr, "s"'] \arrow[dr, phantom, shift left=3, "\Nearrow \alpha"] & A \arrow[d, "p"] \\ & B \end{tikzcd} \qquad\qquad \begin{tikzcd} D \arrow[r, bend left, "r"] \arrow[r, bend right, "s^p"'] \arrow[r, phantom, "\alpha^p \Uparrow"] & A \end{tikzcd}$$

and consists of a 2-cell as on the right above with property that $p\alpha^p = \alpha$ and such that $q\alpha^p$ is an identity. Similar remarks apply to q-lifts.

Proposition 2.76. Consider a two sided discrete fibration as in Definition 2.74 and a 2-cell:

$$\begin{tikzcd} D \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "g"', ""{name=L, above}] & A \arrow[Rightarrow, from=U, to=L, "\alpha"] \end{tikzcd}$$

such that $p\alpha$ and $q\alpha$ are identities. Then α is an identity 2-cell. In other words any two sided discrete fibration reflects identities.

Proof. The 2-cell α is a p-lift for $(1 : pf = pg, g)$. However the identity 2-cell on g is also a p-lift for this 2-cell. As p-lifts are unique it follows that $\alpha = 1_g$. ☐

Example 2.77. Consider an opspan:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

in a 2-category $\mathcal{C}$. If its comma object $f|g$ exists, with universal cone:

$$\begin{tikzcd} f|g \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Downarrow\alpha"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

then the span $A \xrightarrow{(p,q)} B$ is a two sided discrete fibration in $\mathcal{C}$. For details see [58].

Definition 2.78. Let $\mathcal{C}$ be a 2-category with pullbacks. A catead [10] in $\mathcal{C}$ consists of an internal category:

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "i_x" description] \end{tikzcd}$$

for which the span formed by its domain and codomain maps $X_0 \xrightarrow{(d_x,c_x)} X_0$ is a two sided discrete fibration.

Definition 2.79. Let $\mathcal{C}$ be a 2-category with pullbacks. Following [10] we denote by $Kat(\mathcal{C})$ the full sub 2-category of $Cat(\mathcal{C}) = Cat(\mathcal{UC})$ whose objects are cateads.

Proposition 2.80. Let $\mathcal{C}$ be a 2-category with comma objects and pullbacks. Given an arrow $f : A \longrightarrow B$ of $\mathcal{C}$ its higher kernel is a catead in $\mathcal{C}$.

Proof. Each higher kernel is an internal category in $\mathcal{C}$. As the domain and codomain maps for the higher kernel are the projections from the comma object $d, c : f|f \longrightarrow A$ they form a two sided discrete fibration by Example 2.77. ☐

2.8 An exactness property of Cat

In the preceding sections we have seen that each arrow in a 2-category $\mathcal{C}$, with higher kernels and codescent objects, may be factored through the codescent object of its higher kernel. We drew an analogy between that factorisation and the regular factorisation of an arrow of a category through the kernel pair of its coequaliser. In any category a kernel pair is an equivalence relation. In Set and in any exact category [4] each equivalence relation is the kernel pair of its coequaliser. We have observed that each higher kernel is a catead. In this section we complete the analogy by showing that, in Cat, each catead is the higher kernel of its codescent object. This notion of exactness was first considered by Bourn and Penon in [10], though without reference to codescent objects, whilst closely related notions of exactness have been considered by Street in [52] and [53]. The exactness property of Cat of Proposition 2.83 was shown in [10] to hold in greater generality, though double categorical methods may be used to quickly deal with the case of Cat, which is how we approach Proposition 2.83 below.

Remark 2.81. Let $\mathcal{C}$ be a 2-category with codescent objects of cateads. Given a catead X in $\mathcal{C}$:

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "i_x" description]
\end{tikzcd}$$

consider then its codescent object QX with universal cocone (QX, f, α):

$$\begin{tikzcd}
 & X_0 \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\
X_1 \arrow[ur, "d_x"] \arrow[dr, "c_x"'] & & QX \\
 & X_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

If the comma cone (X_1, d_x, c_x, α) exhibits X_1 as the comma object $f|f$ of $f : X_0 \longrightarrow QX$ then the two defining equations for the codescent cocone equally exhibit the morphisms m_x and i_x as the unit and composition maps defining the higher kernel of f, so that X is the higher kernel of its codescent morphism $f : X_0 \longrightarrow QX$.

Definition 2.82. Let $\mathcal{C}$ be a 2-category with pullbacks, comma objects and codescent objects of cateads. We say that "each catead is the higher kernel of its codescent object" if the cocone exhibiting its codescent object equally exhibits the catead as the higher kernel of its codescent morphism, as described in Remark 2.81. In this situation we equally say, following [10], that "cateads are effective in $\mathcal{C}$ ".

Proposition 2.83. Cateads are effective in Cat.

In order to prove this statement it will be convenient to use the graphical notation of double categories.

Definition 2.84. A double category A:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

is an internal category in Cat [16].

Remark 2.85. We briefly recall the double category notation. A double category A, as above, has objects, vertical and horizontal arrows, and squares.

- The objects of the double category above are the objects of A_0, which we denote by $x, y, z \ldots$.
- The vertical arrows are the arrows of A_0. These will be written with a subscript of 0, for instance $\alpha_0 : x \longrightarrow y$. The vertical arrows may be composed vertically down the page and this gives rise to the vertical category $V(A)$ which is precisely the category A_0.
- Objects α_1, β_1 of A_1 may be thought of as arrows $\alpha_1 : x \longrightarrow y$ where the image of α_1 under the "domain" and "codomain" maps d and c is x and y respectively. Each of these will have a subscript of 1; and they will be the horizontal arrows of the double category. The horizontal arrows have a composition induced by the multiplication $m : A_2 \longrightarrow A_1$. Thus the double category has a "horizontal category" $H(A)$ with arrows the horizontal morphisms.

- An arrow of A_1, $\phi : \alpha_1 \longrightarrow \beta_1$ is then a "square":

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\theta\Downarrow"] & x \arrow[d, "\beta_0"] \\
y \arrow[r, "\beta_1"'] & z
\end{tikzcd}$$

  with the image of θ under d and c respectively the left and right vertical arrows.
  Composition in A_1 is represented by vertical pasting of squares down the page. The composition induced by the multiplication $m : A_2 \longrightarrow A_1$ for the internal category is given by horizontal pasting of squares from left to right.
  The identity squares for vertical composition and horizontal composition are represented by

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & x \arrow[d, equal] \\
w \arrow[r, "\alpha_1"'] & x
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
w \arrow[r, equal] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "="] & w \arrow[d, "\alpha_0"] \\
y \arrow[r, equal] & y
\end{tikzcd}$$

Remark 2.86. The double category:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

is a catead in Cat if the pair (d, c) forms a two sided discrete fibration. In the double category language the three axioms for a two sided discrete fibration become:

1. Unique d-lifts:
   Each diagram:

$$\begin{tikzcd}
w \arrow[d, "\alpha_0"'] & \\
x \arrow[r, "\beta_1"'] & y
\end{tikzcd}
\qquad \text{induces a unique square:} \qquad
\begin{tikzcd}
w \arrow[r] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\alpha_0^{\mathbf{d}}\Downarrow"] & y \arrow[d, equal] \\
x \arrow[r, "\beta_1"'] & y
\end{tikzcd}$$

   as drawn.

2. Unique c-lifts:
   Each diagram:

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] & x \arrow[d, "\beta_0"] \\
 & z
\end{tikzcd}
\qquad \text{induces a unique square:} \qquad
\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0^{\mathbf{c}}\Downarrow"] & x \arrow[d, "\beta_0"] \\
w \arrow[r] & z
\end{tikzcd}$$

3. Bimodule condition:
   For any square θ we have the equality:

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\theta\Downarrow"] & x \arrow[d, "\beta_0"] \\
y \arrow[r, "\beta_1"'] & z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, equal] \arrow[dr, phantom, "\beta_0^{\mathbf{c}}\Downarrow"] & x \arrow[d, "\beta_0"] \\
w \arrow[r] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\alpha_0^{\mathbf{d}}\Downarrow"] & z \arrow[d, equal] \\
y \arrow[r, "\beta_1"'] & z
\end{tikzcd}$$

The bimodule condition in particular implies that any square θ may be recovered from its four sides by taking d and c-lifts. Consequently a catead is, in this sense, a locally preordered double category. Thus there is no need to label squares.

Remark 2.87. In the following proof we will show that the codescent object of the catead A is its horizontal category $H(A)$ with exhibiting codescent morphism $f : A_0 = V(A) \longrightarrow H(A)$ an identity on objects functor. Such functors appear in the double category literature and are known as "holonomies" [12], often associated with "connections". The codescent morphism we construct will be an instance of a holonomy associated to a connection, and the approach is closely related to the study of Fiore of the relationship between connections and "foldings" [18].

Proof. Given a catead as above we will show that its codescent object is the horizontal category $H(A)$. The codescent morphism will then be a functor $f : A_0 = V(A) \longrightarrow H(A)$ from the vertical category to the horizontal category. The objects of $V(A)$ and $H(A)$ are the objects of A_0. Thus we define f to be the identity on objects.

An arrow of $\alpha_0 : x \longrightarrow y$ of A_0 is a vertical arrow. We define a horizontal arrow $f(\alpha_0) : x \longrightarrow y$ as the vertical domain of the d-lift:

$$\begin{tikzcd}
x \arrow[r, "f(\alpha_0)"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & y \arrow[d, equal] \\
y \arrow[r, equal] & y
\end{tikzcd}
\qquad \text{induced by} \qquad
\begin{tikzcd}
x \arrow[d, "\alpha_0"'] & \\
y \arrow[r, equal] & y
\end{tikzcd}$$

The bimodule condition at the square:

$$\begin{tikzcd}
x \arrow[r, equal] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "="] & x \arrow[d, "\alpha_0"] \\
y \arrow[r, equal] & y
\end{tikzcd}$$

shows that equally $f(\alpha_0)$ is the vertical codomain of the c-lift:

$$\begin{tikzcd}
x \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & x \arrow[d, "\alpha_0"] \\
x \arrow[r, "f(\alpha_0)"'] & y
\end{tikzcd}
\qquad \text{induced by} \qquad
\begin{tikzcd}
x \arrow[r, equal] & x \arrow[d, "\alpha_0"] \\
 & y
\end{tikzcd}$$

The squares:

$$\begin{tikzcd}
x \arrow[r, "f(\alpha_0)"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & y \arrow[r, "f(\beta_0)"] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & z \arrow[d, equal] \\
z \arrow[r, equal] \arrow[d, "\beta_0"'] \arrow[dr, phantom, "="] & z \arrow[r, "f(\beta_0)"] \arrow[d, "\beta_0"'] \arrow[dr, phantom, "\Downarrow"] & z \arrow[d, equal] \\
z \arrow[r, equal] & z \arrow[r, equal] & z
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
x \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & x \arrow[d, equal] \\
x \arrow[r, equal] & x
\end{tikzcd}$$

respectively show, via the uniqueness of d-lifts, that $f(\beta_0\alpha_0) = f(\beta_0)f(\alpha_0)$, and that $f(1_x) = 1_{f(x)}$. Thus f is a functor.

We need to describe a natural transformation:

$$\begin{tikzcd}
 & A_0 \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\eta"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & H(A) \\
 & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

Given an object of A_1, a horizontal morphism $\alpha_0 : x \longrightarrow y$, we have $fd(\alpha_0) = x$ and $fc(\alpha_0) = y$. We define $\eta_{\alpha_0} : fd(\alpha_0) \longrightarrow fc(\alpha_0)$ to be the morphism $\alpha_0 : fd(\alpha_0) = x \longrightarrow y = fc(\alpha_0)$ itself. We must verify naturality.

Thus we must show that given an arrow of A_1, a square:

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & x \arrow[d, "\beta_0"] \\
y \arrow[r, "\beta_1"'] & z
\end{tikzcd}$$

that we have the equality in the horizontal category $H(A)$:

$$w \xrightarrow{f(\alpha_0)} y \xrightarrow{\beta_1} z \quad = \quad w \xrightarrow{\alpha_1} x \xrightarrow{f(\beta_0)} z$$

The squares:

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & x \arrow[r, "f(\beta_0)"] \arrow[d, "\beta_0"] \arrow[dr, phantom, "\Downarrow"] & z \arrow[d, equal] \\
y \arrow[r, "\beta_1"'] & z \arrow[r, equal] & z
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
w \arrow[r, "f(\alpha_0)"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & y \arrow[r, "\beta_1"] \arrow[d, equal] \arrow[dr, phantom, "\|"] & z \arrow[d, equal] \\
y \arrow[r, equal] & y \arrow[r, "\beta_1"'] & z
\end{tikzcd}$$

exhibit the pair of 1-cells in question as the vertical domain of the unique d-lift of:

$$\begin{tikzcd}
w \arrow[d, "\alpha_0"'] & \\
y \arrow[r, "\beta_1"'] & z
\end{tikzcd}$$

Consequently they agree and so η is natural. We must verify that the triple $(H(A), f, \eta)$ constitutes a codescent cocone. Consider an object of A_2: a horizontally composable pair: $(\alpha_1 : x \longrightarrow y, \beta_1 : y \longrightarrow z)$. Its images under $p, q : A_2 \longrightarrow A_1$ are the first and second arrows α_1 and β_1 respectively. Thus:

$$\eta p(\alpha_1 : x \longrightarrow y, \beta_1 : y \longrightarrow z) = \eta_{\alpha_1} = \alpha_1$$

and:

$$\eta q(\alpha_1 : x \longrightarrow y, \beta_1 : y \longrightarrow z) = \eta_{\beta_1} = \beta_1$$

The image of $(\alpha_1 : x \longrightarrow y, \beta_1 : y \longrightarrow z)$ under ηm is

$$\eta m(\alpha_1 : x \longrightarrow y, \beta_1 : y \longrightarrow z) = m(\alpha_1, \alpha_2)$$

The multiplicative equation for a codescent cocone then asserts that in the horizontal category $H(A)$ we have:

$$x \xrightarrow{\alpha_1} y \xrightarrow{\beta_1} z = x \xrightarrow{m(\alpha_1, \alpha_2)} z$$

This equality holds since horizontal composition is defined by the multiplication $m : A_2 \longrightarrow A_1$. The unital equation for a codescent cocone is the assertion that $\eta i(x) : x \longrightarrow x \in H(A)$ is an identity morphism for each $xinA_0$. Of course $i : A_0 \longrightarrow A_1$ is just the map assigning to an object of A_0 the horizontal identity on it and so $\eta i(x) = i(x)$ is indeed the identity in $H(A)$. Consequently (f, η) is indeed a codescent cocone to the catead. It remains to check its universal property.
Consider a cocone (g, ϕ):

$$\begin{tikzcd}
& A_0 \arrow[dr, "g"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\phi & B \\
& A_0 \arrow[ur, "g"'] &
\end{tikzcd}$$

We must show that there exists a unique $k : H(A) \longrightarrow B$ such that $kf = g$ and $k\eta = \phi$. Since f is the identity on objects the requirement that $kf = g$ implies that we must define $kx = gx$ for each object $x \in H(A)$. Now

consider an arrow of $H(A)$, a horizontal arrow $\alpha_1 : x \longrightarrow y$. This is equally an object of A_1. The equation $k\eta = \phi$ states that, at $\alpha_1 \in A_1$, we have $k(\eta_{\alpha_1}) : gx \longrightarrow gy = \phi_{\alpha_1} : gx \longrightarrow gy$. As $\eta_{\alpha_1} = \alpha_1 : x \longrightarrow y$ this asserts that we must define $k(\alpha_1) : x \longrightarrow y = \phi_{\alpha_1} : gx \longrightarrow gy$. Consequently k is unique in satisfying these constraints. It remains to show that k is functorial.
For this we should show that:

1. Given $(\alpha_1 : x \longrightarrow y, \beta_1 : y \longrightarrow z)$ of $H(A)$ then:

$$gx \xrightarrow{\phi_{\alpha_1}} gy \xrightarrow{\phi_{\beta_1}} gz = gx \xrightarrow{\phi_{m(\alpha_1,\alpha_2)}} gz.$$

2. Given $x \in H(A)$ then

$$gx \xrightarrow{\phi_{i_x}} gx = gx \xrightarrow{1_{gx}} gx.$$

These express precisely the multiplicative and unital conditions for the triple (B, g, ϕ) to be a cocone. Therefore k is functorial. Consequently the cocone $(H(A), f, \eta)$ satisfies the 1-dimensional universal property of a codescent cone. The 2-dimensional universal property is immediate, by Proposition 2.5, as Cat has cotensors with **2**. Therefore this cocone exhibits $H(A)$ as the codescent object of the catead A.
It remains to show that the catead:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

is the higher kernel of the codescent morphism $f : A_0 \longrightarrow H(A)$. We must show that the comma cone (A_1, d, η, c):

$$\begin{tikzcd}
 & A_0 \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\eta"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & H(A) \\
 & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

exhibits A_1 as the comma category $f|f$. Now by definition the comma category $f|f$ has objects: triples $(x, \alpha_1 : fx \longrightarrow fy, y)$ where $x, y \in A_0$ and $\alpha_1 \in H(A)$. As f is the identity on objects we have $(x, \alpha_1 : fx \longrightarrow fy, y) = (x, \alpha_1 : x \longrightarrow y, y)$. The cone (d, η, c) induces, by the universal property of the comma category $f|f$, a functor $A_1 \longrightarrow f|f$ which acts on a morphism of A_1 as below:

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & x \arrow[d, "\beta_0"] & {} \arrow[d, phantom, "\longmapsto"] & (w, \arrow[d, "\alpha_0"'] & w \arrow[rr, "\alpha_1"] \arrow[d, "f(\alpha_0)" description] & & x, \arrow[d, "f(\beta_0)" description] & x) \arrow[d, "\beta_0"] \\
y \arrow[r, "\beta_1"'] & z & {} & (y, & y \arrow[rr, "\beta_1"'] & & z, & z)
\end{tikzcd}$$

where the inside square on the right is a commutative square in $H(A)$. It suffices to show that this functor $A_1 \longrightarrow f|f$ is an isomorphism of categories. It is clearly bijective on objects. To see it is faithful recall that the square on the left is the unique such square with those four boundary arrows, as discussed at the end of Remark 2.86. As the functor $A_1 \longrightarrow f|f$ retains full information about all four of these, faithfulness follows immediately. To show it is full suppose that we are given horizontal morphisms $\alpha_1 : w \longrightarrow x, \beta_1 : y \longrightarrow z$ and vertical morphisms $\alpha_0 : w \longrightarrow y$ and $\beta_0 : x \longrightarrow z$ such that $f(\beta_0)\alpha_1 = \beta_1 f(\alpha_0)$ as on the right above. We must construct a square as on the left with those four bounding arrows. It is the composite:

$$\begin{tikzcd}
w \arrow[r, "\alpha_1"] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & x \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & x \arrow[d, "\beta_0"] \\
w \arrow[r, "\alpha_1"'] \arrow[d, equal] & x \arrow[r, "f(\beta_0)"'] \arrow[d, phantom, "\parallel"] & z \arrow[d, equal] \\
w \arrow[r, "f(\alpha_0)"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow"] & y \arrow[r, "\beta_1"] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & z \arrow[d, equal] \\
y \arrow[r, equal] & y \arrow[r, "\beta_1"'] & z
\end{tikzcd}$$

The top left and bottom right squares are vertical identities, as is the middle square (using the equation $f(\beta_0)\alpha_1 = \beta_1 f(\alpha_0)$). The top right and bottom left squares respectively exhibit $f(\beta_0)$ and $f(\alpha_0)$ via their construction as c and d-lifts. Thus the comparison $A_1 \longrightarrow f|f$ is full, and so an isomorphism. $\square$

Chapter 3

Internal categories and cateads

In Proposition 2.83 of Chapter 2 we proved that each catead in Cat is the higher kernel of its codescent object. The main aim of this chapter is to extend this result to 2-categories of internal categories; those of the form $Cat(\mathcal{E})$ for $\mathcal{E}$ a category with pullbacks. We won't take the direct approach that we took in the case of Cat $= Cat(\text{Set})$; the detailed calculations required in that result become more difficult to manage when we cannot chase elements, but will take a more global approach which we now motivate by way of analogy with a better known situation.

An important notion in category theory is that of an "exact category" [9]. An exact category has, in particular, coequalisers of equivalence relations, kernel pairs and each equivalence relation is the kernel pair of its coequaliser; this latter condition also referred to as "equivalence relations are effective". We may express the fact that a category $\mathcal{E}$ has coequalisers relative to the diagonal functor $\Delta : \mathcal{E} \longrightarrow Graph(\mathcal{E})$ which sends an object of $\mathcal{E}$ to the constant graph upon it. $\mathcal{E}$ has coequalisers precisely when the diagonal has a left adjoint:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & Graph(\mathcal{E}) \arrow[l, bend right, "Q"']
\end{tikzcd}$$

The left adjoint Q assigns to a graph in $\mathcal{E}$ its coequaliser. On the other hand, if we are interested only in coequalisers of equivalence relations we may consider, upon observing that each constant graph is an equivalence relation, the factored diagonal $\Delta : \mathcal{E} \longrightarrow ERel(\mathcal{E})$ with codomain the category of equivalence relations in $\mathcal{E}$. As $ERel(\mathcal{E})$ is a full subcategory of $Graph(\mathcal{E})$ containing the constant graphs, we see that $\mathcal{E}$ has coequalisers of equivalence relations precisely if there exists a left adjoint:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & ERel(\mathcal{E}) \arrow[l, bend right, "Q"']
\end{tikzcd}$$

The effectiveness of equivalence relations in $\mathcal{E}$ may be described relative to this adjunction upon viewing the situation 2-categorically. If $\mathcal{E}$ has pullbacks then each equivalence relation in $\mathcal{E}$ is, in a unique way, an internal category in $\mathcal{E}$, so that $ERel(\mathcal{E})$ may be viewed as a full sub 2-category of the 2-category of categories internal to $\mathcal{E}$. Viewing $ERel(\mathcal{E})$ in such a manner, and $\mathcal{E}$ as a locally discrete 2-category, the above adjunction now becomes an adjunction in 2-CAT, a 2-adjunction.

Given an equivalence relation X in $\mathcal{E}$ the unit of the adjunction at X is an internal functor $X \longrightarrow \Delta Q(X)$:

$$\begin{tikzcd}
X_1 \arrow[r, "ed=ec"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & QX \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] \\
X_0 \arrow[r, "e"] & QX
\end{tikzcd}$$

whose objects map is the coequaliser of d and c, and whose arrow component is the arrow witnessing the equality $ed = ec$. Since $ERel(\mathcal{E})$ is a full sub 2-category of $Cat(\mathcal{E})$ this unit component is fully faithful in $ERel(\mathcal{E})$ precisely if it is fully faithful in $Cat(\mathcal{E})$. This is the case precisely if the diagram on the left below:

$$\begin{tikzcd}
 & X_0 \arrow[dr, "e"] & \\
 & & QX \\
X_1 \arrow[uur, "d"] \arrow[r, "ed=ec"] \arrow[ddr, "c"'] & QX \arrow[ur, "1"] \arrow[dr, "1"'] & \\
 & & QX \\
 & X_0 \arrow[ur, "e"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & X_0 \arrow[dr, "e"] & \\
X_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & QX \\
 & X_0 \arrow[ur, "e"'] &
\end{tikzcd}$$

exhibits X_1 as the limit, as described in Chapter 2. This is equally to say, upon collapsing the identity morphisms, that the square on the right above is a pullback. But this is precisely to say that X is the kernel pair of its coequaliser. Therefore $\mathcal{E}$ has coequalisers of equivalence relations and they are effective precisely if the unit of the 2-adjunction is pointwise fully faithful.

Our approach in this chapter will be much alike the situation just described for exact categories and we outline it now:

- In Section 1 we study "diagonal 2-functors". For $\mathcal{C}$ a 2-category with sufficient cotensors we consider the diagonal $\Delta : \mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ determined by the weight for reflexive codescent objects; a left adjoint to this 2-functor exists if $\mathcal{C}$ has codescent objects of reflexive coherence data. We consider exactness and naturality properties of the diagonal. We factor the underlying functor of the diagonal to obtain new functors: $\mathcal{UC} \longrightarrow \mathcal{U}Cat(\mathcal{UC})$ and $\mathcal{UC} \longrightarrow \mathcal{U}Kat(\mathcal{C})$ and show that a left adjoint to the latter exists precisely when $\mathcal{C}$ has codescent objects of cateads.

- In Section 2 we consider representable 2-categories and their relationship with 2-categories of internal categories. We extend the factored diagonal of the previous section $\mathcal{UC} \longrightarrow \mathcal{U}Cat(\mathcal{UC})$ to a 2-functor $\mathcal{C} \longrightarrow Cat(\mathcal{UC})$ for each representable 2-category and show that this gives a pseudonatural transformation $\Delta' : 1_{\text{Rep}} \Longrightarrow Cat(\mathcal{U}-)$. We prove that this is the unit of a biadjunction:

  $$\begin{tikzcd}
  \mathrm{Cat}_{\mathrm{pb}} \arrow[r, shift right=1.5ex, "Cat(-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Rep} \arrow[l, shift right=1.5ex, "\mathcal{U}"']
  \end{tikzcd}$$

  between the 2-category Rep of representable 2-categories and the 2-category $\mathrm{Cat}_{\mathrm{pb}}$ of categories with pullbacks. We conclude this section by remarking upon some similar biadjunctions and their comonadicity.

- In Section 3 we consider two sided discrete fibrations and cateads in representable 2-categories, showing that the former and thus the latter admit a finite limit characterisation in such 2-categories. We use this to show that $Kat(\mathcal{C})$ is a representable 2-category if $\mathcal{C}$ is one, and extend to a 2-functor $Kat(-) : \text{Rep} \longrightarrow \text{Rep}$. We factor the pseudonatural transformation $\Delta' : 1_{\text{Rep}} \Longrightarrow Cat(\mathcal{U}-)$ through $\hat{\Delta} : 1_{\text{Rep}} \Longrightarrow Kat(-)$ and show that for each representable 2-category $\mathcal{C}$ the 2-functor $\hat{\Delta}_{\mathcal{C}} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ has a left 2-adjoint with pointwise fully faithful unit precisely if $\mathcal{C}$ has codescent objects of cateads and they are effective.

- In Section 4 we show that when $\mathcal{C} = Cat(\mathcal{E})$, for a category $\mathcal{E}$ with pullbacks, the 2-functor $\hat{\Delta}_{\mathcal{C}} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ has a left adjoint with pointwise fully faithful unit; thus $Cat(\mathcal{E})$ has codescent objects of cateads and cateads are effective. Our approach relies upon the biadjunction of Section 2 and is closely related to the theory of KZ-doctrines. We characterise the codescent morphisms in $Cat(\mathcal{E})$ as the bijections on objects and show that each 2-functor of the form $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$, for $F : A \longrightarrow B \in \mathrm{Cat}_{\mathrm{pb}}$, preserves codescent objects of cateads.

The results of this chapter are the building blocks for much of the later work in this thesis. We now describe which aspects of this chapter are original work, and which are not. One of the main results of the Chapter is Theorem 3.65 which states that if a category $\mathcal{E}$ has pullbacks then $Cat(\mathcal{E})$ has codescent objects of cateads and they are effective. The author presented this, and some results of Chapters 4 and 10, at the Annual Category Theory Conference CT2009 in Cape Town. After that presentation he learnt from D. Bourn and J. Penon about their unpublished manuscript [10] in which they proved, amongst other things, essentially the same result. Let us be precise then about the relationship. Theorem 3.65 may be rephrased using Proposition 3.47 as the following assertion.

- If $\mathcal{C}$ is a 2-category of the form $Cat(\mathcal{E})$ for $\mathcal{E}$ with *pullbacks* then the diagonal $\hat{\Delta} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ has a left 2-adjoint with pointwise fully faithful unit.

It is in terms of this latter statement that we obtain Theorem 3.65 later in the chapter. The author learned of this approach from [10] where it is proven in the Appendix that:

- If $\mathcal{C}$ is a 2-category of the form $Cat(\mathcal{E})$ for $\mathcal{E}$ with *finite limits* then the diagonal $\hat{\Delta} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ has a left 2-adjoint with pointwise fully faithful unit.

The only distinction in the two above statements is between pullbacks and finite limits and this is of minor significance. Therefore the author was significantly influenced in his presentation of the results of this chapter by [10]. Our terminology "catead", previously "congruence", follows their terminology "catéade". Proposition 3.60 in this chapter is due entirely to [10], where it appears as Proposition 1.6(2), the author having previously been unaware of it. Indeed, in the presentation of this chapter, that result is of fundamental importance, being required to "construct" the left adjoint of Proposition 3.47 in the case of $Cat(\mathcal{E})$, a result the author had previously obtained only by lengthy internal calculations.

On the other hand our presentation differs from that of Bourn and Penon. In [10] they do not describe how the left 2-adjoint to the diagonal above takes the codescent object of a catead; thus none of the explicit descriptions of how our results relate to codescent objects may be found in [10].

Having stated the main results of this chapter which were previously proven, albeit in very slightly different contexts, by other authors, let me state which results of the other sections are, or are not, original and in particular which results I consider to be original and of significant interest.

The first section on diagonal 2-functors is essentially standard enriched category theory. Though the author has not seen it in print before we do not claim great originality here, this section not containing any individual results considered significant with the exception that it clarifies some later constructions.

The second section is concerned with the relationship between categories with pullbacks and representable 2-categories. The notion of representable 2-category is due to Gray, first appearing in summary form in [21]. Gray's book "Formal Category Theory: Adjointness for 2-categories" [22] was intended to be the first part of a series of three books, the second and third of which were to be concerned with 2-categories of internal categories and representable 2-categories respectively, as described in his introduction to [22]. These later volumes were not published and to the author's knowledge the only further development of the subject of representable 2-categories was by Street in [48], in which he considers the 2-functor $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ for a representable 2-category $\mathcal{C}$. Our Proposition 3.25(1) is thus proven by Street in Proposition 2 of [48], this also, in the case of a finitely complete 2-category $\mathcal{C}$, being contained in the later manuscript of Bourn and Penon [10]. The 2-category of representable 2-categories Rep and its relationship with the 2-category of categories with pullbacks $\mathrm{Cat}_{\mathrm{pb}}$ is the main topic of Section 2 of this chapter. The author has not seen the 2-category Rep studied elsewhere and our main result of that section is Theorem 3.33 in which we exhibit the biadjunction:

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{pb}} \arrow[r, phantom, "\perp"] \arrow[r, bend right, swap, "Cat(-)"] & \mathrm{Rep} \arrow[l, bend right, swap, "\mathcal{U}"]
\end{tikzcd}$$

which exhibits the close relationship between categories with pullbacks and representable 2-categories, as connected by the representable 2-category $Cat(\mathcal{E})$ of categories internal to a category with pullbacks $\mathcal{E}$. The author considers this biadjunction an important result about representable 2-categories and it may not be found elsewhere.

In Section 3 the further theory of representable 2-categories developed has not to the author's knowledge appeared elsewhere, the main results of interest being the finite limit characterisation of two sided discrete fibrations of Proposition 3.40 and Corollary 3.41.

In Section 4 one result which has no analogue elsewhere and which is of importance in later chapters is Theorem 3.66, in which it is proven that any 2-functor of the form $Cat(F)$ for $F \in \mathrm{Cat}_{\mathrm{pb}}$ preserves codescent objects of cateads.

3.1 Diagonal 2-functors

In ordinary category theory given a small category $\mathcal{J}$ and a category $\mathcal{C}$ we always have the diagonal functor $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}, \mathcal{C}]$ which sends $X \in \mathcal{C}$ to the constant functor Δ_X defined by $\Delta_X(j) = X$ for each $j \in \mathcal{J}$. If $\mathcal{C}$ has $\mathcal{J}$-colimits, these provide Δ with a left adjoint $col : [\mathcal{J}, \mathcal{C}] \longrightarrow \mathcal{C}$.

If $\mathcal{J}$ and $\mathcal{C}$ are instead 2-categories the diagonal, as above but with the obvious extension to 2-cells, again exists, but only bears the same relation to colimits in the case of conical colimits. We will describe a suitable generalisation of the "diagonal functor" which takes into account our interest, at the 2-dimensional level, in more general weights, in particular the weight for codescent objects. This 2-functor will exist only if $\mathcal{C}$ has sufficient cotensors.

Remark 3.1. Everything we say in this section holds more generally for categories enriched over a symmetric monoidal closed category $\mathcal{V}$. We are interested only in 2-categories: categories enriched over the cartesian closed category $\mathcal{V} = \text{Cat}$, and consequently will only deal with this case.

Consider a small 2-category $\mathcal{J}$. If a 2-category $\mathcal{C}$ has W-limits for all weights $W : \mathcal{J} \longrightarrow \text{Cat}$ then these organise into a 2-functor:

$$lim : [\mathcal{J}, \text{Cat}]^{op} \times [\mathcal{J}, \mathcal{C}] \longrightarrow \mathcal{C}$$

where the value of lim at (W, F) is the limit of F weighted by W, as described in Chapter 3 of [30]. Cotensors are those limits defined by weights $1 \longrightarrow \text{Cat}$. Fixing $\mathcal{J} = 1$ we obtain, whenever $\mathcal{C}$ has cotensors, a 2-functor $lim : \text{Cat}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$. We won't be dealing with 2-categories with all cotensors, but only those cotensors in the image of a fixed weight $W : \mathcal{J} \longrightarrow \text{Cat}$. Let Cat_W denote the full sub 2-category of Cat with objects: Wj for $j \in \mathcal{J}$. If $\mathcal{C}$ has cotensors with those categories then we have similarly a 2-functor:

$$lim : \text{Cat}_W^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$

Consider the composite:

$$\mathcal{J}^{op} \times \mathcal{C} \xrightarrow{W^{op} \times 1} \text{Cat}_W^{op} \times \mathcal{C} \xrightarrow{lim} \mathcal{C}$$

We may consider its transpose, using the partial closedness of 2-CAT [1], a 2-functor $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$.

Definition 3.2. Consider a weight $W : \mathcal{J} \longrightarrow \text{Cat}$ and a 2-category $\mathcal{C}$ with cotensors with all categories in the image of W. The diagonal 2-functor $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ associated to W is the transpose of the 2-functor:

$$\mathcal{J}^{op} \times \mathcal{C} \xrightarrow{W^{op} \times 1} \text{Cat}_W^{op} \times \mathcal{C} \xrightarrow{lim} \mathcal{C}$$

described above.

Remark 3.3. For $X \in \mathcal{C}$, $\Delta(X) : \mathcal{J}^{op} \longrightarrow \text{Cat}$ acts as $\Delta(X)(j) = X^{Wj}$. Given a 1-cell $f : j \longrightarrow k \in \mathcal{J}$ the arrow $\Delta(X)(f) = X^{Wf} : X^{Wk} \longrightarrow X^{Wj}$ is the unique arrow rendering commutative the square:

$$\begin{tikzcd}
Wj \arrow[r, "\eta_j"] \arrow[d, "Wf"'] & \mathcal{C}(X^{Wj}, X) \arrow[d, "\mathcal{C}(X^{Wf}, 1)"] \\
Wk \arrow[r, "\eta_k"'] & \mathcal{C}(X^{Wk}, X)
\end{tikzcd}$$

Given a 2-cell $\alpha \in \mathcal{J}(j,k)(f,g)$ the 2-cell $\Delta(X)(\alpha) = X^{W\alpha} : X^{Wf} \Longrightarrow X^{Wg}$ is the unique one such that:

$$\mathcal{C}(X^{W\alpha}, 1) \circ \eta_j = \eta_k \circ W\alpha$$

---

[1] By partial closedness we mean that for a small 2-category $\mathcal{J}$ we have an isomorphism 2-CAT$(\mathcal{J} \times \mathcal{A}, \mathcal{C}) \cong$ 2-CAT$(\mathcal{A}, [\mathcal{J}, \mathcal{C}])$ naturally in $\mathcal{A}$ and $\mathcal{C}$. If $\mathcal{J}$ is not small the 2-category $[\mathcal{J}, \mathcal{C}]$ is not locally small and so we cannot speak of 2-CAT as being fully cartesian closed.

The following Proposition and the Corollary succeeding it explain the relationship between the diagonal and W-colimits in $\mathcal{C}$.

Proposition 3.4. Let $\mathcal{C}$ have cotensors with all categories in the image of $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ so that the diagonal exists. Then we have an isomorphism of categories:

$$[\mathcal{J}^{op}, \mathcal{C}](F, \Delta(X)) \cong [\mathcal{J}, \mathrm{Cat}](W, \mathcal{C}(F-, X))$$

2-natural in X.

Proof. We will use the calculus of ends as described in Chapters 2 and 3 of [30]. We have isomorphisms 2-natural in X:

$$[\mathcal{J}^{op}, \mathcal{C}](F, \Delta(X)) \cong \int_j \mathcal{C}(Fj, X^{Wj}) \cong \int_j \mathrm{Cat}(Wj, \mathcal{C}(Fj, X)) \cong [\mathcal{J}, \mathrm{Cat}](W, \mathcal{C}(F-, X))$$

The first isomorphism uses the description of the enriched functor category via ends and the definition of $\Delta(X)$. The second isomorphism uses the universal property of cotensors whilst the third isomorphism returns from end to functor category notation. ☐

Corollary 3.5. Let $\mathcal{C}$ and W be as in Proposition 3.4. The colimit of $F : \mathcal{J}^{op} \longrightarrow \mathcal{C}$ exists if and only if the 2-functor $[\mathcal{J}^{op}, \mathcal{C}](F, \Delta(-))$ is representable; the representation giving the colimit. In particular $\mathcal{C}$ has W-colimits if and only $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ has a left 2-adjoint and the left adjoint is then precisely the colimit 2-functor $col_W : [\mathcal{J}^{op}, \mathcal{C}] \longrightarrow \mathcal{C}$.

Proof. By definition, to give a representation of $[\mathcal{J}, \mathrm{Cat}](W, \mathcal{C}(F-, 1))$ is to give the colimit of F weighted by W. The isomorphism of Proposition 3.4 ensures that to give such a representation is equally to give a representation of $[\mathcal{J}^{op}, \mathcal{C}](F, \Delta(-))$ and the first part of the result follows. Consequently $\mathcal{C}$ has all W-colimits if and only if $[\mathcal{J}^{op}, \mathcal{C}](F, \Delta(-))$ is representable for each F. This is the case precisely when $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ has a left 2-adjoint. ☐

Remark 3.6. The weight of primary interest to us is the weight for codescent objects of strict reflexive coherence data. This is the embedding $\iota : \Delta_2 \longrightarrow \mathrm{Cat}$ described in Chapter 2. The image of $\iota : \Delta_2 \longrightarrow \mathrm{Cat}$ consists of the categories **1**, **2** and **3**. If a 2-category $\mathcal{C}$ has cotensors with these then the diagonal $\Delta : \mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ exists, and, by Corollary 3.5, has a left 2-adjoint if and only if $\mathcal{C}$ has codescent objects of strict reflexive coherence data.
In fact less is needed. Recall from Proposition 2.5 that in the case of a 2-category, such as $\mathcal{C}$, admitting cotensors with **2**, the 2-dimensional universal property of a colimit follows from its 1-dimensional aspect. This amounts, in the present case, to the fact that the 2-category $\mathcal{C}$ has codescent objects of strict reflexive coherence data, if and only if the underlying functor $\mathcal{U}\Delta : \mathcal{U}\mathcal{C} \longrightarrow \mathcal{U}[\Delta_2^{op}, \mathcal{C}]$ has a left adjoint. This observation equally follows from Proposition 3.1 of [8] where it proved that a 2-functor which preserves cotensors with **2** has a left 2-adjoint if and only if its underlying functor has a left adjoint. We prove that the diagonal $\Delta : \mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ preserves cotensors with **2** in the next proposition.

Proposition 3.7. Let $\mathcal{C}$ and $\mathcal{J}$ be as before.

1. Suppose that $\mathcal{C}$ has all H-limits for some weight H. Then $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ preserves them.
2. Suppose that $\mathcal{J}^{op}$ has some limit and that it is preserved by $W^{op} : \mathcal{J}^{op} \longrightarrow \mathrm{Cat}^{op}$. Then it is preserved by each 2-functor $\Delta(X) : \mathcal{J}^{op} \longrightarrow \mathcal{C}$.

Proof. 1. If $\mathcal{C}$ has H-limits then so does $[\mathcal{J}^{op}, \mathcal{C}]$ and they are pointwise. In particular they are preserved, and jointly reflected, by the evaluation 2-functors $ev_j : [\mathcal{J}^{op}, \mathcal{C}] \longrightarrow \mathcal{C}$. To show that $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ preserves H-weighted limits it consequently suffices to show that the composite $ev_j \circ \Delta$ does so. Now $ev_j \circ \Delta(X) = X^{Wj}$. Moreover $ev_j \circ \Delta = (-)^{Wj}$ the canonical 2-functor sending an object X of $\mathcal{C}$ to its cotensor with Wj. Since limits commute with limits in any 2-category it follows that $ev_j \circ \Delta$ preserves H-limits and so Δ does too.

2. Like any 2-functor $\Delta(X) : \mathcal{J}^{op} \longrightarrow \mathcal{C}$ preserves any limit preserved by its composite with each representable $\mathcal{C}(Y, -) : \mathcal{C} \longrightarrow \mathrm{Cat}$ for $Y \in \mathcal{C}$. $\mathcal{C}(Y, -) \circ \Delta(X) = \mathcal{C}(Y, \Delta(X))$ which, by the universal property of cotensors, is naturally isomorphic to $\mathrm{Cat}(W-, \mathcal{C}(Y, X))$. This latter 2-functor is the composite:

$$\begin{tikzcd} \mathcal{J}^{op} \arrow[r, "W^{op}"] & \mathrm{Cat}^{op} \arrow[r, "\mathrm{Cat}(-{,}\mathcal{C}(Y{,}X))"] & \mathrm{Cat} \end{tikzcd}$$

which preserves any limit preserved by W^{op}, as required.

□

Example 3.8. We now consider the example of interest again, namely the diagonal 2-functor corresponding to the embedding $\Delta_2 \longrightarrow \mathrm{Cat}$. This is obtained by restriction of the embedding $\Delta \longrightarrow \mathrm{Cat}$ and we will study the restriction by means of the full embedding. In fact it will be useful to consider the simplicial category as a 2-category. Recall that each object $[n] \in \Delta$ is a totally ordered set, in particular a preorder. Consequently each homset $\Delta([n], [m])$ becomes a preorder, inheriting its order pointwise from the ordinal $[m]$. Precisely, given $f, g \in \Delta([n], [m])$ we have $f \leq g$ if $fx \leq gx \quad \forall x \in [n]$. Consequently the simplicial category underlies a (locally preordered) 2-category $\overline{\Delta}$ [51], and the embedding of Δ into Cat extends to a 2-fully faithful embedding $\overline{\Delta} \longrightarrow \mathrm{Cat}$. We now consider colimits in $\overline{\Delta}$ which are preserved by its inclusion into Cat. In particular the 2-cell:

$$\begin{tikzcd} {[0]} \arrow[r, bend left, "\delta_1"{name=U}] \arrow[r, bend right, "\delta_0"'{name=D}] & {[1]} \arrow[from=U, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

exhibits [1] as the tensor of [0] with the category **2**; and this colimit is preserved by the embedding of $\overline{\Delta} \longrightarrow \mathrm{Cat}$. Furthermore certain pushouts exist in $\overline{\Delta}$. For each $[n] \in \overline{\Delta}$ the square:

$$\begin{tikzcd} {[n]} \arrow[r, "\delta_0"] \arrow[d, "\delta_{n+1}"'] & {[n+1]} \arrow[d, "\delta_{n+2}"] \\ {[n+1]} \arrow[r, "\delta_0"'] & {[n+2]} \end{tikzcd}$$

is a pushout preserved by the embedding. As each of these pushouts and tensors is preserved by the embedding $\iota : \overline{\Delta} \longrightarrow \mathrm{Cat}$ the corresponding pullbacks and cotensors are preserved by its opposite, $\iota : \overline{\Delta}^{op} \longrightarrow \mathrm{Cat}^{op}$.

Consider then a 2-category $\mathcal{C}$ with cotensors with each category **n**, and the induced diagonal 2-functors: $\Delta : \mathcal{C} \longrightarrow [\overline{\Delta}^{op}, \mathcal{C}]$ and $\Delta : \mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$. The second may be obtained from postcomposition with the 2-functor $[\overline{\Delta}^{op}, \mathcal{C}] \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ induced by restricting along the inclusion $\Delta_2 \longrightarrow \Delta \longrightarrow \overline{\Delta}$ where Δ_2 and Δ are viewed as locally discrete 2-categories. Consider $\Delta : \mathcal{C} \longrightarrow [\overline{\Delta}^{op}, \mathcal{C}]$, an object $X \in \mathcal{C}$, and $\Delta(X) : \overline{\Delta}^{op} \longrightarrow \mathcal{C}$. By Proposition 3.7(2) this 2-functor preserves each limit in $\overline{\Delta}^{op}$ preserved by ι^{op}, thus the above cotensor and pullbacks in $\overline{\Delta}^{op}$. In particular each square:

$$\begin{tikzcd} X^{\mathbf{n+2}} \arrow[r, "X^{(\delta_{n+2})}"] \arrow[d, "X^{(\delta_0)}"'] & X^{\mathbf{n+1}} \arrow[d, "X^{(\delta_0)}"] \\ X^{\mathbf{n+1}} \arrow[r, "X^{(\delta_{n+1})}"'] & X^{\mathbf{n}} \end{tikzcd}$$

is a pullback so that $\Delta(X)$ is an internal category in $\mathcal{C}$ determined by its restriction to Δ_2^{-op}, which we write as:

$$\Delta(X) \qquad = \qquad \begin{tikzcd} X^{\mathbf{3}} \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X^{\mathbf{2}} \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X \arrow[l, "i_x" description] \end{tikzcd}$$

where d_x, c_x, i_x to be thought of as the domain, codomain and identity maps of the internal category, the maps p_x, q_x respectively the projections taking the first and second arrows of a composable pair, and m_x the composition map.
Since $\Delta(X) : \overline{\Delta}^{op} \longrightarrow \mathcal{C}$ preserves the cotensor described above we additionally have a 2-cell:

$$\begin{tikzcd}
X^{\mathbf{2}} \arrow[r, bend left, "d_x"] \arrow[r, bend right, "c_x"'] \arrow[r, phantom, "\Downarrow \eta_x"] & X
\end{tikzcd}$$

the universal 2-cell exhibiting X^2 as the cotensor of X with $\mathbf{2}$ in $\mathcal{C}$. Since $\overline{\Delta}$ is a locally preordered 2-category it is immediate that the morphism $m_x : X^3 \longrightarrow X^2$ satisfies the equation:

$$d_xp_x = d_xm_x \xRightarrow{\eta_x m_x} c_xm_x = c_xp_x \quad \text{equals} \quad d_xp_x \xRightarrow{\eta_x p_x} c_xp_x = d_xq_x \xRightarrow{\eta_x q_x} c_xq_x \quad .$$

By the universal property of $X^{\mathbf{2}}$ it is necessarily the unique such. Furthermore we have $\eta_x i_x = 1$. Again by the universal property of $X^{\mathbf{2}}$ this equation defines i_x.
Now restricting our attention to the diagonal: $\Delta(X) : \Delta_2^{op} \longrightarrow \mathcal{C}$ the 2-cell above is no longer part of the data, though it is implicitly there, as $X^{\mathbf{2}}$ is still the cotensor of X with $\mathbf{2}$ in $\mathcal{C}$, and d_x and c_x the projections from the cotensor.

Remark 3.9. The pullback squares of the previous example:

$$\begin{tikzcd}
X^{\mathbf{n+2}} \arrow[r, "X^{(\delta_{n+2})}"] \arrow[d, "X^{(\delta_0)}"'] & X^{\mathbf{n+1}} \arrow[d, "X^{(\delta_0)}"] \\
X^{\mathbf{n+1}} \arrow[r, "X^{(\delta_{n+1})}"'] & X^{\mathbf{n}}
\end{tikzcd}$$

exhibit, inductively, each cotensor $X^{\mathbf{n}}$ for $n > 2$ as constructible by pullback from the objects $X^{\mathbf{2}}$, X and the cotensor projections $d_x = X^{(\delta_1)}$ and $c_x = X^{(\delta_0)}$. Thus any 2-category with cotensors with $\mathbf{2}$ and pullbacks has cotensors with $\mathbf{n}$ for each n. Such 2-categories are known as representable 2-categories [21] and will be studied in the next section.

Remark 3.10. Suppose again that a 2-category $\mathcal{C}$ has cotensors with each ordinal $\mathbf{n}$. We showed in Example 3.8 that each 2-functor $\Delta(X)$ in the image of the diagonal $\mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ is an internal category in $\mathcal{C}$:

$$\begin{tikzcd}
\Delta(X) \quad = & X^{\mathbf{3}} \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X^{\mathbf{2}} \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X \arrow[l, "i_x" description]
\end{tikzcd}$$

Equally it is an internal category in the underlying category $\mathcal{UC}$ of $\mathcal{C}$. As $\mathcal{U}Cat(\mathcal{UC})$ is a full subcategory of $\mathcal{U}[\Delta_2^{op}, \mathcal{C}]$ it follows that the underlying functor of the diagonal factors through $\mathcal{U}Cat(\mathcal{UC})$:

$$\mathcal{UC} \xrightarrow{\mathcal{U}\Delta'} \mathcal{U}Cat(\mathcal{UC}) \xrightarrow{\iota} \mathcal{U}[\Delta_2^{op}, \mathcal{C}]$$

(The implication of the above notation is that we can extend the first component of the factorisation to a 2-functor $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ and we will indeed show this to be the case. This is not immediate as the 2-cells of $Cat(\mathcal{UC})$ and $[\Delta_2^{op}, \mathcal{C}]$ are different. However they do, in special cases, bear a relationship, which we we will use to define and study $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ via the more straightforward embedding $\mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$.)
Furthermore $\Delta(X)$ is a catead in $\mathcal{C}$ as the domain and codomain maps $d_x, c_x : X^{\mathbf{2}} \rightrightarrows X$ are the projections from the cotensor with $\mathbf{2}$, and thus a two sided discrete fibration by Example 2.77. In particular the defining equations for the arrows m_x and i_x described in Example 3.8 exhibit $\Delta(X)$ as the higher kernel of $1 : X \longrightarrow X$. The diagonal then factors further to a functor:

$$\mathcal{UC} \xrightarrow{\mathcal{U}\hat{\Delta}} \mathcal{U}Kat(\mathcal{C})$$

Corollary 3.11. Let $\mathcal{C}$ be a 2-category admitting cotensors with each category $\mathbf{n}$. The functor:

$$\mathcal{U}\hat{\Delta} : \mathcal{U}\mathcal{C} \xrightarrow{\mathcal{U}\hat{\Delta}} \mathcal{U}Kat(\mathcal{C})$$

obtained by factorising the diagonal functor $\mathcal{U}\Delta : \mathcal{U}\mathcal{C} \longrightarrow \mathcal{U}Kat(\mathcal{C})$ has a left adjoint if and only if $\mathcal{C}$ has codescent objects of cateads.

Proof. Consider then a catead $X \in \mathcal{U}Kat(\mathcal{C})$ and $A \in \mathcal{C}$. We have $\mathcal{U}Kat(\mathcal{C})(X, \hat{\Delta}(A)) = \mathcal{U}[\Delta_2^{op}, \mathcal{C}](X, \Delta(A))$ natural in A. Therefore $\mathcal{U}Kat(\mathcal{C})(X, \hat{\Delta}(-)) = \mathcal{U}[\Delta_2^{op}, \mathcal{C}](X, \Delta(-))$. This second functor is representable precisely when $\mathcal{C}$ has codescent objects of cateads by Corollary 3.5. Therefore $\mathcal{U}Kat(\mathcal{C})(X, \hat{\Delta}(-))$ is representable precisely for each X precisely if $\mathcal{C}$ has codescent objects of cateads. ☐

Our final concern with diagonal 2-functors regards their naturality.

Definition 3.12. Let 2-CAT$_W$ denote the locally full sub 2-category of 2-CAT with objects those 2-categories admitting cotensors with each category Wj, and morphisms: 2-functors preserving such cotensors.

Remark 3.13. Given a 2-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ of 2-CAT$_W$ we have the canonical 2-natural transformation:

$$\begin{tikzcd}
\mathrm{Cat}_W \times \mathcal{C} \arrow[r, "lim_{\mathcal{C}}"] \arrow[d, "1\times F"'] & \mathcal{C} \arrow[d, "F"] \\
\mathrm{Cat}_W \times \mathcal{D} \arrow[r, "lim_{\mathcal{D}}"'] \arrow[ur, phantom, "\Downarrow lim_F"] & \mathcal{D}
\end{tikzcd}$$

whose component at the pair (A, X) is the canonical comparison $lim_F(X, A) : F(X^A) \longrightarrow X^{FA}$ in $\mathcal{D}$. This comparison is an isomorphism since, as a morphism of 2-CAT$_W$, F preserves cotensors with each object of Cat$_W$. As 2-CAT is partially closed as a 2-category the composite 2-cell on the left below:

$$\begin{tikzcd}
\mathcal{J}^{op} \times \mathcal{C} \arrow[r, "W\times 1"] \arrow[d, "1\times F"'] & \mathrm{Cat}_W^{op} \times \mathcal{C} \arrow[r, "lim_{\mathcal{C}}"] \arrow[d, "1\times F"'] & \mathcal{C} \arrow[d, "F"] & \text{induces:} & \mathcal{C} \arrow[r, "\Delta_{\mathcal{C}}"] \arrow[d, "F"'] & {[\mathcal{J}^{op}, \mathcal{C}]} \arrow[d, "{[\mathcal{J}, F]}"] \\
\mathcal{J}^{op} \times \mathcal{D} \arrow[r, "W\times 1"'] & \mathrm{Cat}_W^{op} \times \mathcal{D} \arrow[r, "lim_{\mathcal{D}}"'] \arrow[ur, phantom, "\Downarrow lim_F"] & \mathcal{D} & & \mathcal{D} \arrow[r, "\Delta_{\mathcal{D}}"'] \arrow[ur, phantom, "\Downarrow \Delta_F"] & {[\mathcal{J}^{op}, \mathcal{D}]}
\end{tikzcd}$$

upon transposition.

Proposition 3.14. $[\mathcal{J}^{op}, -] :$ 2-CAT$_W \longrightarrow$ 2-CAT$_W$ gives an endo 2-functor of 2-CAT$_W$ and the diagonal 2-functors $\Delta_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ together with the isomorphisms Δ_F for $F \in$ 2-CAT$_W$ combine to give a pseudonatural transformation $\Delta : 1_{\text{2-CAT}_W} \Longrightarrow [\mathcal{J}^{op}, -]$.

Proof. For $\mathcal{C} \in$ 2-CAT$_W$ the 2-category $[\mathcal{J}^{op}, \mathcal{C}]$ has cotensors with each category Wj, pointwise. By the pointwise nature of cotensors in the functor 2-category we see that given $F \in$ 2-CAT$_W$ the 2-functor $[\mathcal{J}^{op}, F]$ preserves such cotensors too, and so is a morphism of 2-CAT$_W$. Therefore $[\mathcal{J}^{op}, -] :$ 2-CAT $\longrightarrow$ 2-CAT restricts to a 2-functor $[\mathcal{J}^{op}, -] :$ 2-CAT$_W \longrightarrow$ 2-CAT$_W$.
By Proposition 3.7(1) the diagonal $\Delta_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{J}^{op}, \mathcal{C}]$ preserves cotensors with Wj and so lies in 2-CAT$_W$. Moreover the components Δ_F are isomorphisms as required for pseudonaturality. The components lim_F described in Remark 3.13 obey naturality and compatibility constraints, which upon transposition, become exactly those required to exhibit Δ as a pseudonatural transformation. ☐

3.2 Representable 2-categories and categories with pullbacks

We are interested in $Cat(\mathcal{E})$ for $\mathcal{E}$ a category with pullbacks.

Definition 3.15. $\mathrm{Cat}_{\mathrm{pb}}$ is the sub 2-category of CAT whose objects are categories with pullbacks, 1-cells: pullback preserving functors and 2-cells: arbitrary natural transformations.

Remark 3.16. Given a morphism $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ we obtain a 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ which acts pointwise. In other words given an object $X \in Cat(\mathcal{A})$ the internal category $Cat(F)X$ has:

$$Cat(F)(X)_i = FX_i \quad \text{for} \quad i = 0, 1, 2 \quad .$$

In full detail:

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "i_x" description] \arrow[r, mapsto, "Cat(F)"] & FX_2 \arrow[r, shift left=2, "Fp_x"] \arrow[r, "Fm_x" description] \arrow[r, shift right=2, "Fq_x"'] & FX_1 \arrow[r, shift left=2, "Fd_x"] \arrow[r, shift right=2, "Fc_x"'] & FX_0 \arrow[l, "Fi_x" description]
\end{tikzcd}$$

That $Cat(F)X$ so defined is an internal category is due to the fact that F preserves pullbacks.
Given an internal functor $f : X \longrightarrow Y$ the internal functor $Cat(F)f : Cat(F)X \longrightarrow Cat(F)Y$ is defined by

$$Cat(F)(f)_i = Ff_i \quad \text{for} \quad i = 0, 1, 2 \quad .$$

An internal natural transformation $\alpha : f \Longrightarrow g$ in $Cat(\mathcal{A})(X, Y)$ is specified by an arrow $\overline{\alpha} : X_0 \longrightarrow Y_1$. $Cat(F)\overline{\alpha} : Cat(F)f \Longrightarrow Cat(F)g$ is the internal natural transformation with arrow component:

$$F\overline{\alpha} : FX_0 \longrightarrow FY_1 \quad .$$

That $Cat(F)$ takes internal functors and internal natural transformations in $\mathcal{A}$ to the same in $\mathcal{B}$ again clearly follows from the fact that F preserves pullbacks.
Given a natural transformation $\alpha : F \Longrightarrow G$ of $\mathrm{Cat}_{\mathrm{pb}}(\mathcal{A}, \mathcal{B})$ we obtain a 2-natural transformation $Cat(\alpha) : Cat(F) \Longrightarrow Cat(G)$. Given X of $Cat(\mathcal{A})$, the component $Cat(\alpha)_X : Cat(F)X \longrightarrow Cat(G)X$ is the internal functor:

$$\begin{tikzcd}
FX_2 \arrow[r, shift left=2, "Fp_x"] \arrow[r, "Fm_x" description] \arrow[r, shift right=2, "Fq_x"'] \arrow[d, "\alpha_{X_2}"'] & FX_1 \arrow[r, shift left=2, "Fd_x"] \arrow[r, shift right=2, "Fc_x"'] \arrow[d, "\alpha_{X_1}"'] & FX_0 \arrow[l, "Fi_x" description] \arrow[d, "\alpha_{X_0}"] \\
GX_2 \arrow[r, shift left=2, "Gp_x"] \arrow[r, "Gm_x" description] \arrow[r, shift right=2, "Gq_x"'] & GX_1 \arrow[r, shift left=2, "Gd_x"] \arrow[r, shift right=2, "Gc_x"'] & GX_0 \arrow[l, "Gi_x" description]
\end{tikzcd}$$

It is routine to verify that this makes $Cat(\alpha)$ into a 2-natural transformation, and furthermore that we obtain, so defined, a 2-functor $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow$2-CAT. Of course the 2-categories in the image of $Cat(-)$ have more structure than that of arbitrary 2-categories. They are, to begin with, representable 2-categories, which is the perspective we take for the remainder of this chapter.

Definition 3.17. A 2-category is representable [21] if it has cotensors with **2** and pullbacks. Rep is the sub 2-category of 2-CAT with objects: representable 2-categories, 1-cells: 2-functors which preserve cotensors with **2** and pullbacks and 2-cells: arbitrary 2-natural transformations.

Proposition 3.18. Given a category $\mathcal{E}$ with pullbacks the 2-category $Cat(\mathcal{E})$ has pullbacks. Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ the induced 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves pullbacks.

Proof. Pullbacks in $Cat(\mathcal{E})$ are easily seen to be pointwise; in other words the fully faithful inclusion of the underlying category $\mathcal{U}(Cat(\mathcal{E})) \longrightarrow [\Delta_2^{op}, \mathcal{E}]$ creates them. Of course there is a 2-dimensional aspect of the universal property of pullbacks in $Cat(\mathcal{E})$ which must be verified but this is routine. Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ the induced 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$, which acts pointwise, consequently preserves pullbacks. ☐

Proposition 3.19. Let $\mathcal{E}$ be a category with pullbacks.

1. Given $X \in Cat(\mathcal{E})$ its cotensor with $\mathbf{2}$, $X^{\mathbf{2}}$ exists. One may choose the cotensor and its universal 2-cell:

$$\begin{tikzcd} X^{\mathbf{2}} \arrow[r, bend left, "d_X"] \arrow[r, bend right, "c_X"'] \arrow[r, phantom, "\Downarrow\eta"] & X \end{tikzcd}$$

   to satisfy the following properties: $X^{\mathbf{2}}_0 = X_1$, $(d_X)_0 = d_x$, $(c_X)_0 = c_x$ and $\overline{\eta} : X_0^2 = X_1 \longrightarrow X_1$ is the identity on X_1.

2. A 2-cell in $Cat(\mathcal{E})$:

$$\begin{tikzcd} Y \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & X \end{tikzcd}$$

   exhibits Y as the cotensor of X with $\mathbf{2}$ if and only if:

   - $\overline{\theta} : Y_0 \longrightarrow X_1$ is an isomorphism.
   - The naturality square:

$$\begin{tikzcd} Y_1 \arrow[r, "(\overline{\theta}\circ d_y, g_1)"] \arrow[d, "(f_1, \overline{\theta}\circ c_y)"'] & X_2 \arrow[d, "m_x"] \\ X_2 \arrow[r, "m_x"'] & X_1 \end{tikzcd}$$

     is a pullback square.

3. Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ the induced 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves cotensors with $\mathbf{2}$.

Proof. Here we justify these claims in the case of $\mathrm{Cat} = Cat(\mathrm{Set})$. The proof for an arbitrary category $\mathcal{E}$ with pullbacks amounts to showing that the construction, in the case of Set, may be carried out more generally for $\mathcal{E}$ with pullbacks. We give a proof of the general case in Appendix 12.1.

1. Consider the case of $\mathrm{Cat} = Cat(\mathrm{Set})$. Then X is just a small category. The cotensor $X^{\mathbf{2}}$ exists and is the familiar arrow category of X. The objects of the arrow category $X^{\mathbf{2}}$ are simply the arrows of X. Thus $X^{\mathbf{2}}_0 = X_1$. Consider the universal 2-cell:

$$\begin{tikzcd} X^{\mathbf{2}} \arrow[r, bend left, "d_X"] \arrow[r, bend right, "c_X"'] \arrow[r, phantom, "\Downarrow\eta"] & X \end{tikzcd}$$

   Given an arrow $\alpha : a \longrightarrow b$ of X, an object of the arrow category, we have $d_X\alpha = a$ and $c_X\alpha = b$. Thus $(d_X)_0$ and $(c_X)_0$ are indeed the domain and codomain functions d_x and c_x of the category X. Furthermore the universal 2-cell η is given by:

$$\eta(\alpha : a \longrightarrow b) = d_X(\alpha) = a \xrightarrow{\alpha} b = c_X(\alpha) \quad .$$

   Thus the arrow map of the natural transformation $\overline{\eta} : X_0^2 = X_1 \longrightarrow X_1$ is indeed the identity. Therefore the claimed conditions of the first part of the proposition are satisfied in the case $\mathcal{E} = \mathrm{Set}$.

2. Morphisms of the arrow category (elements of the set $X^{\mathbf{2}}_1$) are commutative squares in X. Therefore we have a pullback square:

$$\begin{tikzcd} X^{\mathbf{2}}_1 \arrow[r] \arrow[d] & X_2 \arrow[d, "m_x"] \\ X_2 \arrow[r, "m_x"'] & X_1 \end{tikzcd}$$

which witnesses the fact that a commutative square is given by pairs of "composable pairs of arrows of X" (elements of X_2) that agree upon composition. It is easily verified that the pullback square is the naturality square as claimed in Part 2 of the proposition. Consequently in the case of Cat the conditions of Part 2 are verified. We next show that the properties of Part 2 characterise cotensors with $\mathbf{2}$ in Cat. Given a natural transformation:

$$\begin{tikzcd} Y \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & X \end{tikzcd}$$

as above there exists a unique functor $h : Y \longrightarrow X^{\mathbf{2}}$ satisfying $d_X h = f, c_X h = g$ and $\eta h = \theta$. Given $a \in Y$ we have $h(a) = \theta_a : fa \longrightarrow ga$. Consequently $h_0 : X_0 \longrightarrow X_0^{\mathbf{2}} = X_1$ equals $\overline{\theta} : X_0 \longrightarrow Y_1$. Therefore h_0 is an isomorphism if and only if $\overline{\theta}$ is an isomorphism. Given $\alpha : a \longrightarrow b$, $h(\alpha)$ is the commutative square on the left below:

$$\begin{tikzcd} fa \arrow[r, "\theta_a"] \arrow[d, "f\alpha"'] & ga \arrow[d, "g\alpha"] \\ fb \arrow[r, "\theta_b"'] & gb \end{tikzcd} \qquad \begin{tikzcd} Y_1 \arrow[r, "(\overline{\theta}\circ d_y, g_1)"] \arrow[d, "(f_1, \overline{\theta}\circ c_y)"'] & X_2 \arrow[d, "m_x"] \\ X_2 \arrow[r, "m_x"'] & X_1 \end{tikzcd}$$

Therefore $h_1 : Y_1 \longrightarrow X_1^{\mathbf{2}}$ is the unique map into the pullback $X_1^{\mathbf{2}}$ determined by the naturality square of θ drawn on the right above. Consequently h_1 is an isomorphism if and only if that naturality square is a pullback. The functor h is an isomorphism if and only if both h_0 and h_1 are. Now the 2-cell θ exhibits Y as the cotensors of X with $\mathbf{2}$ if and only if h is an isomorphism which is the case precisely the two conditions of Part 2 are verified.

3. Having shown that the properties of Part 2 are sufficient to characterise cotensors with $\mathbf{2}$ it is straightforward to verify Part 3 of the proposition. Consider any 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ for $F : A \longrightarrow B$ of $\mathrm{Cat}_{\mathrm{pb}}$. $Cat(F)$ acts pointwise and F preserves pullbacks; therefore the 2-functor $Cat(F)$ preserves the characterising properties of Part 2.

☐

Corollary 3.20. Given $\mathcal{E}$ of $\mathrm{Cat}_{\mathrm{pb}}$ the 2-category $Cat(\mathcal{E})$ is a representable 2-category. Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ the 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ is a morphism of Rep. Therefore we have a 2-functor:

$$Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep} \quad .$$

Proof. This combines the results of Propositions 3.18 and 3.19. ☐

Remark 3.21. Consider the forgetful 2-functor $\mathcal{U}$: 2-CAT$\longrightarrow$CAT which forgets 2-cells. Each representable 2-category $\mathcal{A}$ has pullbacks, and so its underlying category $\mathcal{U}\mathcal{A}$ has pullbacks too. Each morphism $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Rep preserves pullbacks and so the underlying functor $\mathcal{U}F : \mathcal{U}\mathcal{A} \longrightarrow \mathcal{U}\mathcal{B}$ preserves pullbacks. Therefore the forgetful 2-functor $\mathcal{U}$ restricts to a forgetful 2-functor $\mathcal{U} : \mathrm{Rep} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$. The main aim of the remainder of this section is to show that we have a biadjunction [2] of 2-categories:

$$\begin{tikzcd} \mathrm{Cat}_{\mathrm{pb}} \arrow[r, bend right, "Cat(-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Rep} \arrow[l, bend right, "\mathcal{U}"'] \end{tikzcd}$$

The counit is easy to describe. For each category $\mathcal{E}$ with pullbacks we have the objects functor $ob_{\mathcal{E}} : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ which assigns to an internal category X its objects of objects X_0 and acts upon an internal

---

[2] We recall the precise definition of a biadjunction in Definition 3.32.

functor $f : X \longrightarrow Y$ as $ob_{\mathcal{E}}(f) = f_0$. As pullbacks in $Cat(\mathcal{E})$ are pointwise in $\mathcal{E}$ it is clear that this functor preserves pullbacks and so constitutes a morphism of $\mathrm{Cat}_{\mathrm{pb}}$. Furthermore it is straightforward to verify that these components are 2-natural in $\mathcal{E}$. Therefore we have a 2-natural transformation:

$$ob : \mathcal{U}Cat(-) \Longrightarrow 1_{\mathrm{Cat}_{\mathrm{pb}}}$$

which will be the counit of the biadjunction. We now go about describing the unit of the biadjunction, a pseudonatural transformation $1_{\mathrm{Rep}} \Longrightarrow Cat(\mathcal{U}-)$, whose components will be the 2-functors $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ suggested in Remark 3.10.

Remark 3.22. By Remark 3.9 each representable 2-category $\mathcal{C}$ has cotensors with each category of the form $\mathbf{n}$. We saw in Example 3.8 that for such $\mathcal{C}$ the diagonal $\Delta : \mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ exists, and that its underlying functor takes its values in $\mathcal{U}Cat(\mathcal{U}\mathcal{C})$, thus giving a factorisation:

$$\begin{tikzcd}
\mathcal{U}\mathcal{C} \arrow[r, "\mathcal{U}\Delta'"] \arrow[dr, "\mathcal{U}\Delta"'] & \mathcal{U}Cat(\mathcal{U}\mathcal{C}) \arrow[d, "\iota"] \\
 & \mathcal{U}[\Delta_2^{op}, \mathcal{C}]
\end{tikzcd}$$

We have yet to describe the action of $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ on 2-cells. Given a 2-cell of $\mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & Y \arrow[from=U, to=D, phantom, "\Downarrow \alpha"]
\end{tikzcd}$$

we define the internal natural transformation:

$$\begin{tikzcd}
\Delta'(X) \arrow[r, bend left, "\Delta'(f)"{name=U}] \arrow[r, bend right, "\Delta'(g)"'{name=D}] & \Delta'(Y) \arrow[from=U, to=D, phantom, "\Downarrow \Delta'(\alpha)"]
\end{tikzcd}$$

as below:

$$\begin{tikzcd}
X^{\mathbf{3}} \arrow[r, shift left, "f^{\mathbf{3}}"] \arrow[r, shift right, "g^{\mathbf{3}}"'] \arrow[d, shift right=2, "p_x"'] \arrow[d, "m_x" description] \arrow[d, shift left=2, "q_x"] & Y^{\mathbf{3}} \arrow[d, shift right=2, "p_y"'] \arrow[d, "m_y" description] \arrow[d, shift left=2, "q_y"] \\
X^{\mathbf{2}} \arrow[r, shift left, "f^{\mathbf{2}}"] \arrow[r, shift right, "g^{\mathbf{2}}"'] \arrow[d, shift right=2, "d_x"'] \arrow[d, shift left=2, "c_x"] & Y^{\mathbf{2}} \arrow[d, shift right=2, "d_y"'] \arrow[d, shift left=2, "c_y"] \\
X \arrow[u, "i_x" description] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[ur, "\overline{\Delta'(\alpha)}" description] & Y \arrow[u, "i_y" description]
\end{tikzcd}$$

to have its arrow component $\overline{\Delta'(\alpha)} : X \longrightarrow Y^{\mathbf{2}}$ the unique 1-cell into the cotensor $Y^{\mathbf{2}}$ such that postcomposition with the universal 2-cell $\eta_y : d_y \Longrightarrow c_y$ returns α.
We still have to check that this indeed gives an internal natural transformation and that, so defined, makes $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ 2-functorial. Instead of proving these facts immediately we first examine the relationship between 2-cells of $Cat(\mathcal{U}\mathcal{C})$ and $[\Delta_2^{op}, \mathcal{C}]$. This will enable us to deduce the 2-functoriality of Δ' from that of Δ. Furthermore this approach will enable us to deduce that Δ' is a morphism of Rep from the corresponding fact for Δ.

Lemma 3.23. Given an internal category $X \in Cat(\mathcal{U}\mathcal{C})$ and an object A of $\mathcal{C}$ we have an isomorphism of categories: $t_{X,A} : Cat(\mathcal{U}\mathcal{C})(X, \Delta'(A)) \cong [\Delta_2^{op}, \mathcal{C}](X, \Delta(A))$ which is the identity on objects. This isomorphism is natural (in the 1-categorical sense) in X and A.

Proof. Certainly the categories $Cat(\mathcal{UC})(X, \Delta'(A))$ and $[\Delta_2^{op}, \mathcal{C}](X, \Delta(A))$ have the same objects so that it remains to describe the relationship between 1-cells: 2-cells of the respective 2-categories $Cat(\mathcal{UC})$ and $[\Delta_2^{op}, \mathcal{C}]$. Given an internal natural transformation $\theta : f \Longrightarrow g$ as on the left below:

$$(1) \qquad \begin{tikzcd}
X_2 \arrow[r, shift left, "f_2"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right=2, "p_x"'] \arrow[d, "m_x"] \arrow[d, shift left=2, "q_x"] & A^{\mathbf{3}} \arrow[d, shift right=2, "p_a"'] \arrow[d, "m_a"] \arrow[d, shift left=2, "q_a"] \\
X_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, shift right=2, "d_x"'] \arrow[d, shift left=2, "c_x"] & A^{\mathbf{2}} \arrow[d, shift right=2, "d_a"'] \arrow[d, shift left=2, "c_a"] \\
X_0 \arrow[u, "i_x"] \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[ur, "\overline{\theta}"] & A \arrow[u, "i_a"]
\end{tikzcd} \qquad \text{we describe:} \qquad (2) \qquad \begin{tikzcd}
X_2 \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\theta_2\Downarrow"] \arrow[d, shift right=2, "p_x"'] \arrow[d, "m_x"] \arrow[d, shift left=2, "q_x"] & A^{\mathbf{3}} \arrow[d, shift right=2, "p_a"'] \arrow[d, "m_a"] \arrow[d, shift left=2, "q_a"] \\
X_1 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\theta_1\Downarrow"] \arrow[d, shift right=2, "d_x"'] \arrow[d, shift left=2, "c_x"] & A^{\mathbf{2}} \arrow[d, shift right=2, "d_a"'] \arrow[d, shift left=2, "c_a"] \\
X_0 \arrow[u, "i_x"] \arrow[r, bend left, "f_0"] \arrow[r, bend right, "g_0"'] \arrow[r, phantom, "\theta_0\Downarrow"] & A \arrow[u, "i_a"]
\end{tikzcd}$$

the corresponding 2-cell, $t(\theta) = (\theta_0, \theta_1, \theta_2)$, in $[\Delta_2^{op}, \mathcal{C}]$. Let $\eta_a : d_a \Longrightarrow c_a$ denote the universal 2-cell defining the cotensor $A^{\mathbf{2}}$. The 1-cell $\overline{\theta} : X_0 \longrightarrow A^{\mathbf{2}}$ corresponds uniquely to the 2-cell $\eta_a\overline{\theta} : f_0 = d_a\overline{\theta} \Longrightarrow c_a\overline{\theta} = g_0$ and we define $\theta_0 = \eta_a\overline{\theta}$.

To give a 2-cell between the arrow components $\theta_1 : f_1 \Longrightarrow g_1$ is to give a pair of 2-cells $d_a f_1 \Longrightarrow d_a g_1$ and $c_a f_1 \Longrightarrow c_a g_1$ such that the square:

$$\begin{tikzcd}
d_a f_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\eta_a f_1"'] & d_a g_1 \arrow[d, Rightarrow, "\eta_a g_1"] \\
c_a f_1 \arrow[r, Rightarrow] & c_a g_1
\end{tikzcd}$$

commutes. We have the pair of 2-cells $d_a f_1 = f_0 d_x \xRightarrow{\eta_a\overline{\theta}d_x} g_0 d_x = d_a g_1$ and $c_a f_1 = f_0 c_x \xRightarrow{\eta_a\overline{\theta}c_x} g_0 c_x = c_a g_1$. To show that the square commutes with these two arrows filled in, first consider the top-right path:

$$d_a f_1 \xRightarrow{\eta_a\overline{\theta}d_x} d_a g_1 \xRightarrow{\eta_a g_1} c_a g_1$$

It equals the pasting composite on the left below:

$$\begin{tikzcd}
& X_0 \arrow[r, "\overline{\theta}"] & A^{\mathbf{2}} \arrow[dr, bend left=60, "d_a"] \arrow[dr, "c_a"'] \arrow[dr, phantom, "\Downarrow\eta_a"] & \\
X_1 \arrow[ur, "d_x"] \arrow[drr, "g_1"'] & & & A \\
& & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[ur, bend right=60, "c_a"'] \arrow[ur, phantom, "\Downarrow\eta_a"] &
\end{tikzcd} = \begin{tikzcd}
& & A^{\mathbf{2}} \arrow[dr, bend left=60, "d_a"] \arrow[dr, "c_a"'] \arrow[dr, phantom, "\Downarrow\eta_a"] & \\
X_1 \arrow[r, "{(\overline{\theta}d_x, g_1)}"] & A^{\mathbf{3}} \arrow[ur, "p_a"] \arrow[dr, "q_a"'] & & A \\
& & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[ur, bend right=60, "c_a"'] \arrow[ur, phantom, "\Downarrow\eta_a"] &
\end{tikzcd} = \begin{tikzcd}
X_1 \arrow[r, "{(\overline{\theta}d_x, g_1)}"] & A^{\mathbf{3}} \arrow[r, "m_a"] & A^{\mathbf{2}} \arrow[r, bend left, "d_a"] \arrow[r, bend right, "c_a"'] \arrow[r, phantom, "\Downarrow\eta_a"] & A
\end{tikzcd}$$

where the first equation replaces the arrows $\overline{\theta}d_x$ and g_1 by the corresponding map into the pullback; the second equation holds by definition of the map m_a as described in Example 3.8.

Similarly the left-bottom path of the square

$$d_a f_1 \xRightarrow{\eta_a f_1} c_a f_1 \xRightarrow{\eta_a\overline{\theta}c_x} c_a g_1$$

equals the pasting composite on the left below:

$$\begin{tikzcd}
& & A^{\mathbf{2}} \arrow[dr, bend left=60, "d_a"] \arrow[dr, "c_a"'] \arrow[dr, phantom, "\Downarrow\eta_a"] & \\
X_1 \arrow[urr, "f_1"] \arrow[dr, "c_x"'] & & & A \\
& X_0 \arrow[r, "\overline{\theta}"'] & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[ur, bend right=60, "c_a"'] \arrow[ur, phantom, "\Downarrow\eta_a"] &
\end{tikzcd} = \begin{tikzcd}
& & A^{\mathbf{2}} \arrow[dr, bend left=60, "d_a"] \arrow[dr, "c_a"'] \arrow[dr, phantom, "\Downarrow\eta_a"] & \\
X_1 \arrow[r, "{(f_1, \overline{\theta}c_x)}"] & A^{\mathbf{3}} \arrow[ur, "p_a"] \arrow[dr, "q_a"'] & & A \\
& & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[ur, bend right=60, "c_a"'] \arrow[ur, phantom, "\Downarrow\eta_a"] &
\end{tikzcd} = \begin{tikzcd}
X_1 \arrow[r, "{(f_1, \overline{\theta}c_x)}"] & A^{\mathbf{3}} \arrow[r, "m_a"] & A^{\mathbf{2}} \arrow[r, bend left, "d_a"] \arrow[r, bend right, "c_a"'] \arrow[r, phantom, "\Downarrow\eta_a"] & A
\end{tikzcd}$$

The two rightmost pasting diagrams agree by naturality of the internal natural transformation θ. By the 2-dimensional universal property of $A^{\mathbf{2}}$ we therefore obtain a unique 2-cell $\theta_1 : f_1 \Longrightarrow g_1$ satisfying that $d_a\theta_1 = \theta_0 d_x$ and $c_a\theta_1 = \theta_0 d_x$.

For the pair θ_0 and θ_1 to be part of a 2-cell $t(\theta) = (\theta_0, \theta_1, \theta_2)$ of $[\Delta_2^{op}, \mathcal{C}]$ we must additionally check that $i_a\theta_0 = \theta_1 i_x$. In order to see that these 2-cells are equal it suffices to show that they agree upon postcomposition with the jointly faithful projections from the cotensor $A^{\mathbf{2}}$, d_a and c_a. We have $d_a i_a \theta_0 = \theta_0 = \theta_0 d_x i_x = d_a\theta_1 i_x$ and similarly $c_a i_a \theta_0 = \theta_0 = \theta_0 c_x i_x = c_a \theta_1 i_x$. Therefore $i_a\theta_0 = \theta_1 i_x$ as required.

It remains to construct the 2-cell θ_2. For θ_2 to be part of a 2-cell $t(\theta) = (\theta_0, \theta_1, \theta_2)$ in $[\Delta_2^{op}, \mathcal{C}]$ it must satisfy $p_a\theta_2 = \theta_1 p_x$ and $q_a\theta_2 = \theta_1 q_x$. The maps p_a and q_a are the projections from the pullback $A^{\mathbf{3}}$ along $c_a, d_a : A^{\mathbf{2}} \rightrightarrows A$. The commutativity of $c_a\theta_1 p_x = \theta_0 c_x p_x = \theta_0 d_x q_x = d_a \theta_1 q_x$ induces, by the 2-dimensional universal property of the pullback, a unique 2-cell $\theta_2 : f_2 \Longrightarrow g_2$ satisfying these constraints.

It remains to check that we have an equality of 2-cells: $m_a\theta_2 = \theta_1 m_x$. Both of these 2-cells have codomain object $A^{\mathbf{2}}$. Therefore it suffices to show that they agree upon postcomposition with the jointly faithful pair d_a and c_a. Now $d_a m_a \theta_2 = d_a p_a \theta_2 = d_a \theta_1 p_x = \theta_0 d_x p_x = \theta_0 d_x m_x = d_a \theta_1 m_x$. Similarly $c_a m_a \theta_2 = c_a q_a \theta_2 = c_a \theta_1 q_x = \theta_0 c_x m_x = \theta_0 c_x m_x = c_a \theta_1 m_x$. Therefore the triple $t\theta = (\theta_0, \theta_1, \theta_2) : f \Longrightarrow g$ indeed defines a 2-cell in $[\Delta_2^{op}, \mathcal{C}]$.

Our argument above gives rise to the following observation. Though a 2-cell of $[\Delta_2^{op}, \mathcal{C}]$ from X to $\Delta(A)$, as in (2), consists of a triple of 2-cells in $\mathcal{C}$, $(\theta_0, \theta_1, \theta_2)$, the 2-cells θ_1 and θ_2 are determined uniquely by θ_0 if they exist. For as the 1-cells (d_a, c_a) are jointly faithful there can be but a single 2-cell θ_1 satisfying the necessary relationship with θ_0: ($d_a\theta_1 = \theta_0 d_x$ and $c_a\theta_1 = \theta_0 c_x$.) Similarly θ_2 is uniquely determined if it exists as the pair of pullback projections (p_a, q_a) are jointly faithful. In other words the forgetful 2-functor $ob : [\Delta_2^{op}, \mathcal{C}] \longrightarrow \mathcal{C}$ is faithful when applied to the hom category $ob_{X,\Delta(A)} : [\Delta_2^{op}, \mathcal{C}](X, \Delta(A)) \longrightarrow \mathcal{C}(X_0, A)$ (whenever X is an internal category), a fact which will expediate our proof of the functoriality of t.

In order to prove that t is functorial consider a pair of internal natural transformations: $\theta : f \Longrightarrow g$ and $\phi : g \Longrightarrow h$. We must show that $t(\phi)t(\theta) = (\phi_0\theta_0, \phi_1\theta_1, \phi_2\theta_2)$ equals $t(\phi \circ \theta) = ((\phi \circ \theta)_0, (\phi \circ \theta)_1, (\phi \circ \theta)_2)$ where $\phi \circ \theta$ denotes composition of the internal natural transformations in $Cat(\mathcal{UC})$. Since $ob_{X,\Delta(A)} : [\Delta_2^{op}, \mathcal{C}](X, \Delta(A)) \longrightarrow \mathcal{C}(X_0, A)$ is faithful it suffices to show that $(\phi \circ \theta)_0 = \phi_0\theta_0$.

The composite internal natural transformation $\phi \circ \theta : f \Longrightarrow h$ has arrow component the composite:

$$X_0 \xrightarrow{(\overline{\theta}, \overline{\phi})} A^{\mathbf{3}} \xrightarrow{m_a} A^{\mathbf{2}}$$

so that $(\phi \circ \theta)_0$ is, by the definition of t, the left composite 2-cell below:

$$\begin{tikzcd}
X_0 \arrow[r, "(\overline{\theta}, \overline{\phi})"] & A^{\mathbf{3}} \arrow[r, "m_a"] & A^{\mathbf{2}} \arrow[r, bend left, "d_a"] \arrow[r, bend right, "c_a"'] \arrow[r, phantom, "\Downarrow \eta_a"] & A
\end{tikzcd}
=
\begin{tikzcd}
 & & A^{\mathbf{2}} \arrow[dr, "c_a"'] \arrow[dr, bend left=50, "d_a"] \arrow[dr, phantom, bend left=25, "\Downarrow \eta_a"] & \\
X_0 \arrow[r, "(\overline{\theta}, \overline{\phi})"] & A^{\mathbf{3}} \arrow[ur, "p_a"] \arrow[dr, "q_a"'] & & A \\
 & & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[ur, bend right=50, "c_a"'] \arrow[ur, phantom, bend right=25, "\Downarrow \eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}
 & A^{\mathbf{2}} \arrow[dr, "c_a"'] \arrow[dr, bend left=50, "d_a"] \arrow[dr, phantom, bend left=25, "\Downarrow \eta_a"] & \\
X_0 \arrow[ur, "\overline{\theta}"] \arrow[dr, "\overline{\phi}"'] & & A \\
 & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[ur, bend right=50, "c_a"'] \arrow[ur, phantom, bend right=25, "\Downarrow \eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}
X_0 \arrow[r, bend left=60, "f_0"] \arrow[r, "g_0" description] \arrow[r, bend right=60, "h_0"'] \arrow[r, phantom, shift left=3, "\Downarrow \theta_0"] \arrow[r, phantom, shift right=3, "\Downarrow \phi_0"] & A
\end{tikzcd}$$

The first equation holds by the definition of m_a. The second is clear and the third equation holds by definition of θ_0 and ϕ_0. Therefore t is functorial.

The functor t is of course the identity on objects. It is evidently faithful since given a pair of internal natural transformations $\theta, \phi : f \Longrightarrow g$ we have $\theta_0 = \eta_a\overline{\theta}$ and $\phi_0 = \eta_a\overline{\phi}$. But if these 2-cells are equal, then by the universal property of the universal 2-cell η_a it must be that $\overline{\theta} = \overline{\phi}$ and so $\theta = \phi$. In order to show that t is an isomorphism as claimed it remains to show it is full.

Starting then with a 2-cell $(\theta_0, \theta_1, \theta_2) : f \Longrightarrow g$ of $[\Delta_2^{op}, \mathcal{C}](X, \Delta(A))$ as in (2) above it remains to describe its preimage under t. We have already seen that such a 2-cell of $[\Delta_2^{op}, \mathcal{C}]$ is determined by its component 2-cell θ_0, so that it suffices to describe an internal natural transformation $\phi : f \Longrightarrow g$ such that $\phi_0 = \theta_0$. The universal property of the cotensor $A^{\mathbf{2}}$ induces a unique 1-cell $\overline{\phi} : X_0 \longrightarrow A^{\mathbf{2}}$ such that $d_a\overline{\phi} = f_0$, $c_a\overline{\phi} = g_0$ and $\eta_a\overline{\phi} = \theta_0$. It is clear then by the definition of t that upon showing that $\overline{\phi} : X_0 \longrightarrow A^{\mathbf{2}}$ indeed defines an internal natural transformation $\phi : f \Longrightarrow g$ we will have $\phi_0 = \theta_0$ as required. The equations $d_a\overline{\phi} = f_0$ and $c_a\overline{\phi} = g_0$ show that ϕ has the correct domain and codomain for an internal natural transformation: $\phi : f \Longrightarrow g$. It remains to show that the naturality square:

$$\begin{tikzcd}
X_1 \arrow[r, "(\overline{\phi}d_x{,}g_1)"] \arrow[d, "(f_1{,}\overline{\phi}c_x)"'] & A^{\mathbf{3}} \arrow[d, "m_a"] \\
A^{\mathbf{3}} \arrow[r, "m_a"'] & A^{\mathbf{2}}
\end{tikzcd}$$

is commutative. By the universal property of $A^{\mathbf{2}}$ it suffices to show that both paths around the square agree upon postcomposition with the universal 2-cell η_a. By the definition of m_a postcomposed with η_a the top right path upon postcomposition with η_a equals the leftmost composite below:

$$\begin{tikzcd}[column sep=small]
 & & A^{\mathbf{2}} \arrow[rd, bend left=50, "d_a"] \arrow[rd, "c_a"'] \arrow[rd, phantom, bend left=25, "\Downarrow\eta_a"] & \\
X_1 \arrow[r, "(\overline{\phi}d_x{,}g_1)"] & A^{\mathbf{3}} \arrow[ru, "p_a"] \arrow[rd, "q_a"'] & & A \\
 & & A^{\mathbf{2}} \arrow[ru, "d_a"] \arrow[ru, bend right=50, "c_a"'] \arrow[ru, phantom, bend right=25, "\Downarrow\eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & A^{\mathbf{2}} \arrow[rd, bend left=50, "d_a"] \arrow[rd, "c_a"'] \arrow[rd, phantom, bend left=25, "\Downarrow\eta_a"] & \\
X_1 \arrow[ru, "\overline{\phi}d_x"] \arrow[rd, "g_1"'] & & A \\
 & A^{\mathbf{2}} \arrow[ru, "d_a"] \arrow[ru, bend right=50, "c_a"'] \arrow[ru, phantom, bend right=25, "\Downarrow\eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & X_0 \arrow[rd, bend left=50, "f_0"] \arrow[rd, "g_0"'] \arrow[rd, phantom, bend left=25, "\Downarrow\theta_0"] & \\
X_1 \arrow[ru, "d_x"] \arrow[rd, "g_1"'] & & A \\
 & A^{\mathbf{2}} \arrow[ru, "d_a"] \arrow[ru, bend right=50, "c_a"'] \arrow[ru, phantom, bend right=25, "\Downarrow\eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & A^{\mathbf{2}} \arrow[rd, "d_a"] & \\
X_1 \arrow[ru, bend left=50, "f_1"] \arrow[ru, "g_1"'] \arrow[ru, phantom, bend left=25, "\Downarrow\theta_1"] \arrow[rd, "g_1"'] & & A \\
 & A^{\mathbf{2}} \arrow[ru, "d_a"] \arrow[ru, bend right=50, "c_a"'] \arrow[ru, phantom, bend right=25, "\Downarrow\eta_a"] &
\end{tikzcd}$$

The first equality is clear. The second holds since $\eta_a \circ \overline{\phi} = \theta_0$. The equation $d_a\theta_1 = \theta_0 d_x$ holds as $(\theta_0, \theta_1, \theta_2)$ is a 2-cell of $[\Delta_2^{op}, \mathcal{C}]$. This gives the third equality.
Similarly postcomposing the left and bottom path of the square by η_a gives the left composite 2-cell below:

$$\begin{tikzcd}[column sep=small]
 & & A^{\mathbf{2}} \arrow[rd, bend left=50, "d_a"] \arrow[rd, "c_a"'] \arrow[rd, phantom, bend left=25, "\Downarrow\eta_a"] & \\
X_1 \arrow[r, "(f_1{,}\overline{\phi}c_x)"] & A^{\mathbf{3}} \arrow[ru, "p_a"] \arrow[rd, "q_a"'] & & A \\
 & & A^{\mathbf{2}} \arrow[ru, "d_a"] \arrow[ru, bend right=50, "c_a"'] \arrow[ru, phantom, bend right=25, "\Downarrow\eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & A^{\mathbf{2}} \arrow[rd, bend left=50, "d_a"] \arrow[rd, "c_a"'] \arrow[rd, phantom, bend left=25, "\Downarrow\eta_a"] & \\
X_1 \arrow[ru, "f_1"] \arrow[rd, "\overline{\phi}c_x"'] & & A \\
 & A^{\mathbf{2}} \arrow[ru, "d_a"] \arrow[ru, bend right=50, "c_a"'] \arrow[ru, phantom, bend right=25, "\Downarrow\eta_a"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & A^{\mathbf{2}} \arrow[rd, bend left=50, "d_a"] \arrow[rd, "c_a"'] \arrow[rd, phantom, bend left=25, "\Downarrow\eta_a"] & \\
X_1 \arrow[ru, "f_1"] \arrow[rd, "c_x"'] & & A \\
 & X_0 \arrow[ru, "f_0"] \arrow[ru, bend right=50, "g_0"'] \arrow[ru, phantom, bend right=25, "\Downarrow\theta_0"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & A^{\mathbf{2}} \arrow[rd, bend left=50, "d_a"] \arrow[rd, "c_a"'] \arrow[rd, phantom, bend left=25, "\Downarrow\eta_a"] & \\
X_1 \arrow[ru, "f_1"] \arrow[rd, "f_1"] \arrow[rd, bend right=50, "g_1"'] \arrow[rd, phantom, bend right=25, "\Downarrow\theta_1"] & & A \\
 & A^{\mathbf{2}} \arrow[ru, "c_a"'] &
\end{tikzcd}$$

The first and second equalities above hold just as they do in the preceding string of equations. The final equality now holds upon applying the equation $\theta_0 c_x = c_a\theta_1$. The final two composite 2-cells of both this string and the string above it equal the horizontal composite:

$$\begin{tikzcd}
X_1 \arrow[r, bend left=40, "f_1"] \arrow[r, bend right=40, "g_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & A^{\mathbf{2}} \arrow[r, bend left=40, "d_a"] \arrow[r, bend right=40, "c_a"'] \arrow[r, phantom, "\Downarrow\eta_a"] & A
\end{tikzcd}$$

Therefore $\phi : f \Longrightarrow g$ is an internal natural transformation and in particular t is an isomorphism of categories. It is straightforward to prove that this isomorphism is natural in X and A. ☐

Remark 3.24. In Remark 3.22 we defined $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ on 2-cells of $\mathcal{C}$ but did not show that the image of a 2-cell in $\mathcal{C}$ actually defined an internal natural transformation. This follows from Lemma 3.23 upon observing that the action of Δ' on 2-cells is described by the composite functor:

$$\mathcal{C}(X,Y) \xrightarrow{\Delta_{X,Y}} [\Delta_2^{op}, \mathcal{C}](\Delta(X), \Delta(Y)) \xrightarrow{t^{-1}_{\Delta(X),Y}} Cat(\mathcal{U}\mathcal{C})(\Delta'(X), \Delta'(Y))$$

This evidently shows that Δ' preserves vertical composition of 2-cells. In the following proposition we show that Δ' is a 2-functor. This enables us to extend the naturality of the isomorphisms t in the second variable and thus study limit preservation properties of Δ' via those of Δ as given in Proposition 3.7(1).

Proposition 3.25. Let $\mathcal{C}$ be a representable 2-category.

1. $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ is a 2-functor.
2. For a fixed internal category X the isomorphisms $t_{X,A} : Cat(\mathcal{UC})(X, \Delta'(A)) \cong [\Delta_2^{op}, \mathcal{C}](X, \Delta(A))$ are 2-natural in $A \in \mathcal{C}$.
3. Suppose that $\mathcal{C}$ has W-limits for some weight W. Then $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ preserves them. In particular it preserves cotensors with **2** and pullbacks and so is a morphism of Rep.

Proof. 1. We have already seen that $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ has an underlying functor and preserves vertical composition of 2-cells by Remark 3.24. Therefore it remains to show that Δ' preserves whiskering of 2-cells on both the left and right.
To show whiskering on the left is preserved is to show that given an arbitrary 1-cell $f : A \longrightarrow B$ of $\mathcal{C}$, the left square below commutes for all $C \in \mathcal{C}$:

$$\begin{tikzcd}
\mathcal{C}(B,C) \arrow[r, "\Delta'_{B,C}"] \arrow[d, "f^*"'] & Cat(\mathcal{UC})(\Delta'(B), \Delta'(C)) \arrow[r, "t_{\Delta'(B),C}"] \arrow[d, "(\Delta'(f))^*"] & {[\Delta_2^{op}, \mathcal{C}](\Delta(B), \Delta(C))} \arrow[d, "(\Delta(f))^*"] \\
\mathcal{C}(A,C) \arrow[r, "\Delta'_{A,C}"'] & Cat(\mathcal{UC})(\Delta'(A), \Delta'(C)) \arrow[r, "t_{\Delta'(A),C}"'] & {[\Delta_2^{op}, \mathcal{C}](\Delta(A), \Delta(C))}
\end{tikzcd}$$

Since the lower right arrow $t_{\Delta'(A),C}$ is an isomorphism of categories, it suffices to check the left square commutes upon postcomposition with it. The right hand square commutes since by Lemma 3.23 t is natural in the first variable. Thus it suffices to verify that the outer square commutes. But this equals:

$$\begin{tikzcd}
\mathcal{C}(B,C) \arrow[r, "\Delta_{B,C}"] \arrow[d, "f^*"'] & {[\Delta_2^{op}, \mathcal{C}](\Delta(B), \Delta(C))} \arrow[d, "(\Delta(f))^*"] \\
\mathcal{C}(A,C) \arrow[r, "\Delta_{A,C}"'] & {[\Delta_2^{op}, \mathcal{C}](\Delta(A), \Delta(C))}
\end{tikzcd}$$

which commutes as, being a 2-functor, Δ preserves whiskering on the left by f. Similarly naturality of t in the second variable implies that Δ' preserves whiskering on the right, and so Δ' is indeed 2-functorial.

2. This is straightforward. All that prevented us observing the 2-naturality of t in the second variable before was that we had not shown Δ' to be a 2-functor.

3. Consider a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a 2-functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ with weighted limit $A \in \mathcal{C}$. Consider its limiting cone: $\eta : W \Longrightarrow \mathcal{C}(A, F-)$. We must show that the composite cone:

$$W \xRightarrow{\eta} \mathcal{C}(A, F-) \xRightarrow{\Delta'_{A,F-}} Cat(\mathcal{UC})(\Delta'(A), \Delta' F-)$$

is a limiting cone in $Cat(\mathcal{UC})$. This is to show that for each X of $Cat(\mathcal{UC})$ the induced functor:

$$Cat(\mathcal{UC})(X, \Delta'(A)) \longrightarrow [\mathcal{J}, \mathrm{Cat}](W, Cat(\mathcal{UC})(X, \Delta' F-))$$

is an isomorphism for each internal category X. For fixed $X \in Cat(\mathcal{UC})$ we have a 2-natural isomorphism of 2-functors $t_{X,-} : Cat(\mathcal{UC})(X, \Delta' -) \cong [\Delta_2^{op}, \mathcal{C}](X, \Delta -)$, since t is 2-natural in the second variable. This yields the composite:

$$W \xRightarrow{\eta} \mathcal{C}(A, F-) \xRightarrow{\Delta'_{A,F-}} Cat(\mathcal{UC})(\Delta'(A), \Delta' F-) \xRightarrow{t_{\Delta' A,-}} [\Delta_2^{op}, \mathcal{C}](\Delta(A), \Delta F-)$$

which equals:

$$W \xRightarrow{\eta} \mathcal{C}(A, F-) \xRightarrow{\Delta_{A,F-}} [\Delta_2^{op}, \mathcal{C}](\Delta(A), \Delta F-) \quad .$$

Now $\Delta : \mathcal{C} \longrightarrow [\Delta_2^{op}, \mathcal{C}]$ preserves W-limits (by Proposition 3.7(1)). Therefore this cone exhibits $\Delta(A)$ as the limit of ΔF in $[\Delta_2^{op}, \mathcal{C}]$. Thus the induced functor:

$$[\Delta_2^{op}, \mathcal{C}](X, \Delta(A)) \longrightarrow [\mathcal{J}, \mathrm{Cat}](W, [\Delta_2^{op}, \mathcal{C}](X, \Delta F-))$$

is an isomorphism. By 2-naturality of the isomorphisms $t_{X,-}$ we have a commuting square:

$$\begin{tikzcd}
Cat(\mathcal{UC})(X, \Delta'(A)) \arrow[r] \arrow[d, "t_{X,A}"'] & {[\mathcal{J}, \mathrm{Cat}](W, Cat(\mathcal{UC})(X, \Delta' F-))} \arrow[d, "{[\mathcal{J},\mathrm{Cat}](W, t_{X,F-})}"] \\
{[\Delta_2^{op}, \mathcal{C}](X, \Delta(A))} \arrow[r] & {[\mathcal{J}, \mathrm{Cat}](W, [\Delta_2^{op}, \mathcal{C}](X, \Delta F-))}
\end{tikzcd}$$

The two vertical arrows are isomorphisms as each component of t is an isomorphism. We have already seen that the bottom horizontal arrow is an isomorphism. Thus the top arrow is also an isomorphism of categories as required.

By assumption $\mathcal{C}$ is a representable 2-category and so has cotensors with **2** and pullbacks. Therefore $\mathcal{UC}$ has pullbacks so that $Cat(\mathcal{UC})$ is a representable 2-category by Corollary 3.20. As $\mathcal{C}$ has pullbacks and cotensors with **2**, Δ' preserves them. Therefore $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ is a morphism of Rep.

□

Remark 3.26. We have described 2-functors $\mathcal{U} : \mathrm{Rep} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$ and $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$ and, in the preceding proposition, for each representable 2-category $\mathcal{C}$ a morphism of Rep: $\Delta'_{\mathcal{C}} = \Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$. In the following proposition we extend these morphisms of Rep to a pseudonatural transformation $\Delta' : 1_{\mathrm{Rep}} \Longrightarrow Cat(\mathcal{U}-)$.

Proposition 3.27. The morphisms of Rep, $\Delta'_{\mathcal{C}} = \Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$, constitute the arrow components of a pseudonatural transformation $\Delta' : 1_{\mathrm{Rep}} \Longrightarrow Cat(\mathcal{U}-)$.

Proof. In Proposition 3.14 we constructed the pseudonatural transformation $\Delta : 1_{2\text{-CAT}_W} \Longrightarrow [\mathcal{J}^{op}, -]$ corresponding to a weight $W : \mathcal{J} \longrightarrow Cat$. In the case of the weight of interest, the inclusion $\iota : \Delta_2 \longrightarrow \mathrm{Cat}$, this becomes a pseudonatural transformation $\Delta : 1_{2\text{-CAT}_\iota} \Longrightarrow [\Delta_2^{op}, -]$. The objects of 2-CAT$_\iota$ are the 2-categories that have cotensors with the categories **2** and **3** and the morphisms are 2-functors preserving such cotensors. Each representable 2-category has such cotensors by Remark 3.9, and it follows by the same Remark that each morphism of Rep preserves them. It is clear then that this restricts to a pseudonatural transformation $\Delta : 1_{\mathrm{Rep}} \Longrightarrow [\Delta_2^{op}, -]$. For a morphism $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Rep the component of Δ_F is the 2-natural isomorphism on the left below:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta_{\mathcal{C}}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow \Delta_F"] & {[\Delta_2^{op}, \mathcal{C}]} \arrow[d, "{[\Delta_2, F]}"] \\
\mathcal{D} \arrow[r, "\Delta_{\mathcal{D}}"'] & {[\Delta_2^{op}, \mathcal{D}]}
\end{tikzcd} \quad \text{which induces:} \quad \begin{tikzcd}
\mathcal{UC} \arrow[r, "\mathcal{U}\Delta_{\mathcal{C}}"] \arrow[d, "\mathcal{U}F"'] \arrow[dr, phantom, "\Downarrow \mathcal{U}\Delta_F"] & \mathcal{U}{[\Delta_2^{op}, \mathcal{C}]} \arrow[d, "{\mathcal{U}[\Delta_2, F]}"] \\
\mathcal{UD} \arrow[r, "\mathcal{U}\Delta_{\mathcal{D}}"'] & \mathcal{U}{[\Delta_2^{op}, \mathcal{D}]}
\end{tikzcd}$$

the latter 2-cell, the natural isomorphism underlying Δ_F.

This factors through the full subcategory $\mathcal{U}Cat(\mathcal{UD})$ giving a natural isomorphism $\mathcal{U}\Delta'_F$ such that:

$$\begin{tikzcd}
\mathcal{UC} \arrow[r, "\mathcal{U}\Delta'_{\mathcal{C}}"] \arrow[d, "\mathcal{U}F"'] \arrow[dr, phantom, "\mathcal{U}\Delta'_F \Downarrow"] & \mathcal{U}Cat(\mathcal{UC}) \arrow[r, "\iota"] \arrow[d, "\mathcal{U}Cat(\mathcal{U}F)"] & \mathcal{U}{[\Delta_2^{op}, \mathcal{C}]} \arrow[d, "{\mathcal{U}[\Delta_2^{op}, F]}"] \\
\mathcal{UD} \arrow[r, "\mathcal{U}\Delta'_{\mathcal{D}}"'] & \mathcal{U}Cat(\mathcal{UD}) \arrow[r, "\iota"] & \mathcal{U}{[\Delta_2^{op}, \mathcal{D}]}
\end{tikzcd} = \begin{tikzcd}
\mathcal{UC} \arrow[r, "\mathcal{U}\Delta_{\mathcal{C}}"] \arrow[d, "\mathcal{U}F"'] \arrow[dr, phantom, "\Downarrow \mathcal{U}\Delta_F"] & \mathcal{U}{[\Delta_2^{op}, \mathcal{C}]} \arrow[d, "{\mathcal{U}[\Delta_2, F]}"] \\
\mathcal{UD} \arrow[r, "\mathcal{U}\Delta_{\mathcal{D}}"'] & \mathcal{U}{[\Delta_2^{op}, \mathcal{D}]}
\end{tikzcd}$$

For an object A of $\mathcal{C}$ this has component the internal functor:

$$\begin{tikzcd}
F(A^{\mathbf{3}}) \arrow[r, "(\Delta'_F)_A)_2"] \arrow[d, shift right=2, "Fp_a"'] \arrow[d] \arrow[d, shift left=2, "Fq_a"] & (FA)^{\mathbf{3}} \arrow[d, shift right=2, "p_{Fa}"'] \arrow[d] \arrow[d, shift left=2, "q_{Fa}"] \\
F(A^{\mathbf{2}}) \arrow[r, "(\Delta'_F)_A)_1"] \arrow[d, shift right=2, "Fd_a"'] \arrow[d, shift left=2, "Fc_a"] & (FA)^{\mathbf{2}} \arrow[d, shift right=2, "d_{Fa}"'] \arrow[d, shift left=2, "c_{Fa}"] \\
FA \arrow[r, "(\Delta'_F)_A)_0=1"'] \arrow[u] & FA \arrow[u]
\end{tikzcd}$$

where the isomorphism $(\Delta'_F)_A)_1 : F(A^{\mathbf{2}}) \longrightarrow FA^{\mathbf{2}}$ is the unique arrow such that $\eta_{FA}(\Delta'_F)_A)_1 = F\eta_A$ and the isomorphism $(\Delta'_F)_A)_2 : F(A^{\mathbf{3}}) \longrightarrow FA^{\mathbf{3}}$ the unique arrow commuting with the pullback projections.
It is straightforward to verify, using this explicit description of the components of Δ'_F that the components are indeed 2-natural, thus underlying a 2-natural isomorphism:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta'_{\mathcal{C}}"] \arrow[d, "F"'] \arrow[dr, phantom, "\Delta'_F \Downarrow"] & Cat(\mathcal{U}\mathcal{C}) \arrow[d, "Cat(\mathcal{U}F)"] \\
\mathcal{D} \arrow[r, "\Delta'_{\mathcal{D}}"'] & Cat(\mathcal{U}\mathcal{D})
\end{tikzcd}$$

which constitutes a 2-cell of Rep. The components $\mathcal{U}\Delta_F$ obey the pasting equations for pseudonaturality since the Δ'_F do. Therefore the 2-cells $\mathcal{U}(\Delta'_F)$ do too, as the inclusions $\mathcal{U}Cat(\mathcal{C}) \longrightarrow \mathcal{U}[\Delta_2^{op}, \mathcal{C}]$ are each fully faithful and thus reflect equations between 2-cells. As the forgetful 2-functor $\mathcal{U}$ is itself locally faithful it follows similarly that the 2-cells $\Delta' F$ of Rep indeed constitute the 2-cell components of a pseudonatural transformation $\Delta' : 1_{\mathrm{Rep}} \Longrightarrow Cat(\mathcal{U}-)$. ☐

Remark 3.28. We have described what will be the unit $\Delta' : 1_{\mathrm{Rep}} \Longrightarrow Cat(\mathcal{U}-)$ and the counit ob : $\mathcal{U}Cat(-) \Longrightarrow 1_{\mathrm{Cat}_{\mathrm{pb}}}$ of the proposed biadjunction:

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{pb}} \arrow[r, shift right=2, "Cat(-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Rep} \arrow[l, shift right=2, "\mathcal{U}"']
\end{tikzcd}$$

and these will constitute all of the required data. To establish the biadjunction it remains only to verify certain equations which will follow easily from those of Lemma 3.31 below. In order to prove that Lemma we first fix a notational convention, and in Remark 3.30 make some canonical choices for the values of $\Delta' : Cat(\mathcal{E}) \longrightarrow Cat(\mathcal{U}Cat(\mathcal{E}))$.

Notation 3.29. Consider $\Delta' : Cat(\mathcal{E}) \longrightarrow Cat(\mathcal{U}Cat(\mathcal{E}))$ for a category $\mathcal{E}$ with pullbacks. Given $X \in Cat(\mathcal{E})$ we have $\Delta'(X) \in Cat(\mathcal{U}Cat(\mathcal{E}))$ the internal category in $Cat(\mathcal{E})$:

$$\begin{tikzcd}
X^{\mathbf{3}} \arrow[r, shift left=3, "p_X"] \arrow[r, "m_X" description] \arrow[r, shift right=3, "q_X"'] & X^{\mathbf{2}} \arrow[r, shift left=3, "d_X"] \arrow[r, shift right=3, "c_X"'] & X \arrow[l, "i_X" description]
\end{tikzcd}$$

In this case we will use capital letters on the subscripts: $p_X, d_X \ldots$ to avoid confusion with those arrows of $\mathcal{E}$: $p_x, q_x \ldots$ that define the internal category X.
(Observe that this is in contrast with the general situation of $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ for an arbitrary representable 2-category $\mathcal{C}$ that we have so far considered. In that case no confusion is likely so we continue to use lowercase subscripts.)

Remark 3.30. Consider again the case of $\Delta' : Cat(\mathcal{E}) \longrightarrow Cat(\mathcal{U}Cat(\mathcal{E}))$ for a category $\mathcal{E}$ with pullbacks and $\Delta'(X)$ for $X \in Cat(\mathcal{E})$:

$$\begin{tikzcd}
X^{\mathbf{3}} \arrow[r, shift left=2, "p_X"] \arrow[r, "m_X" description] \arrow[r, shift right=2, "q_X"'] & X^{\mathbf{2}} \arrow[r, shift left=2, "d_X"] \arrow[r, shift right=2, "c_X"'] & X \arrow[l, "i_X" description]
\end{tikzcd}$$

By Proposition 3.19(1) there is a canonical choice for the value of the cotensor $X^{\mathbf{2}}$ and the universal 2-cell:

$$\begin{tikzcd}
X^{\mathbf{2}} \arrow[r, bend left, "d_X"] \arrow[r, bend right, "c_X"'] \arrow[r, phantom, "\Downarrow\eta_X"] & X
\end{tikzcd}$$

Specifically we may choose that $X_0^{\mathbf{2}} = X_1$, $(d_X)_0 = d_x$, $(c_X)_0 = c_x$ and that the arrow component $\overline{\eta_X} : X_0^{\mathbf{2}} = X_1 \longrightarrow X_1$ is the identity 1-cell on X_1.
We have pullback squares in $Cat(\mathcal{E})$ and $\mathcal{E}$ respectively:

$$\begin{tikzcd}
X^{\mathbf{3}} \arrow[r, "p_X"] \arrow[d, "q_X"'] & X^{\mathbf{2}} \arrow[d, "c_X"] \\
X^{\mathbf{2}} \arrow[r, "d_X"'] & X_0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_2 \arrow[r, "p_x"] \arrow[d, "q_x"'] & X_1 \arrow[d, "c_x"] \\
X_1 \arrow[r, "d_x"'] & X_0
\end{tikzcd}$$

Having chosen that $X_0^{\mathbf{2}} = X_1$, $(d_X)_0 = d_x$, $(c_X)_0 = c_x$ it follows, by the pointwise nature of pullbacks in $Cat(\mathcal{E})$, that we may specify that $X_0^{\mathbf{3}} = X_2$, $(p_X)_0 = p_x$ and that $(q_X)_0 = q_x$. In other words the image of the square on the left under $ob : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ is precisely the square on the right. We henceforth suppose that these choices have been made for $\Delta' : Cat(\mathcal{E}) \longrightarrow Cat(\mathcal{U}Cat(\mathcal{E}))$.

Lemma 3.31.

1. The 2-functor:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[r, "\Delta'_{Cat(\mathcal{E})}"] & Cat(\mathcal{U}Cat(\mathcal{E})) \arrow[r, "Cat(ob_{\mathcal{E}})"] & Cat(\mathcal{E})
\end{tikzcd}$$

   is the identity.

2. Consider $F : \mathcal{A} \longrightarrow \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$ and $\mathcal{C} \in \mathrm{Rep}$. The composite 2-cells in Rep:

(1)

$$\begin{tikzcd}
Cat(\mathcal{A}) \arrow[r, "\Delta'_{Cat(\mathcal{A})}"] \arrow[d, "Cat(F)"'] \arrow[dr, phantom, "\Downarrow\Delta'_{Cat(F)}"] & Cat(\mathcal{U}Cat(\mathcal{A})) \arrow[d, "Cat(\mathcal{U}Cat(F))"] & \\
Cat(\mathcal{B}) \arrow[r, "\Delta'_{Cat(\mathcal{B})}"'] & Cat(\mathcal{U}Cat(\mathcal{B})) \arrow[r, "Cat(ob_{\mathcal{U}\mathcal{B}})"'] & Cat(\mathcal{B})
\end{tikzcd}$$

   and

(2)

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta'_{\mathcal{C}}"] \arrow[d, "\Delta'_{\mathcal{C}}"'] \arrow[dr, phantom, "\Downarrow\Delta'_{\Delta'_{\mathcal{C}}}"] & Cat(\mathcal{U}\mathcal{C}) \arrow[d, "Cat(\mathcal{U}\Delta'_{\mathcal{C}})"] & \\
Cat(\mathcal{U}\mathcal{C}) \arrow[r, "\Delta'_{Cat(\mathcal{U}\mathcal{C})}"'] & Cat(\mathcal{U}Cat(\mathcal{U}\mathcal{C})) \arrow[r, "Cat(ob_{\mathcal{U}\mathcal{C}})"'] & Cat(\mathcal{U}\mathcal{C})
\end{tikzcd}$$

   are identity 2-cells.

Proof. 1. The image of an internal category X in $\mathcal{E}$ under the 2-functor:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[r, "\Delta'_{Cat(\mathcal{E})}"] & Cat(\mathcal{U}Cat(\mathcal{E})) \arrow[r, "Cat(ob_{\mathcal{E}})"] & Cat(\mathcal{E})
\end{tikzcd}$$

is the internal category on the left below:

$$\begin{tikzcd}
X_0^{\mathbf{3}} \arrow[r, shift left=2, "(p_X)_0"] \arrow[r, "(m_X)_0" description] \arrow[r, shift right=2, "(q_X)_0"'] & X_0^{\mathbf{2}} \arrow[r, shift left=2, "(d_X)_0"] \arrow[r, shift right=2, "(c_X)_0"'] & X_0 \arrow[l, "(i_X)_0" description] & \text{which equals:} & X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "(m_X)_0" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "(i_X)_0" description]
\end{tikzcd}$$

by Remark 3.30. To show that $Cat(ob_{\mathcal{E}})\Delta'$ is the identity on objects it suffices then to show that $(i_X)_0 = i_x$ and $(m_X)_0 = m_x$. The morphism $i_X : X \longrightarrow X^{\mathbf{2}}$ is, as described in Example 3.8, the unique such arrow which yields the identity 2-cell on X upon postcomposition with the universal 2-cell η_X. The arrow component of the internal natural transformation $\eta_X \circ i_X$ is the 1-cell:

$$X_0 \xrightarrow{(i_X)_0} X_0^{\mathbf{2}} \xrightarrow{\overline{\eta_X}} X_1$$

whilst the identity natural transformation on X has arrow component $X_0 \xrightarrow{i_x} X_1$. Since $\overline{\eta_X}$ is, by Remark 3.30, the identity on X_1 we therefore have $(i_X)_0 = i_x$.
As described in Example 3.8 $m_X : X^{\mathbf{3}} \longrightarrow X^{\mathbf{2}}$ is the unique such 1-cell such that:

$$\begin{tikzcd}
 & & & & & X^{\mathbf{2}} \arrow[dr, "c_X" description] \arrow[dr, bend left=50, "d_X"] \arrow[dr, phantom, bend left=25, "\Downarrow\eta_X"] & \\
X^{\mathbf{3}} \arrow[r, "m_X"] & X^{\mathbf{2}} \arrow[r, bend left=50, "d_X"] \arrow[r, bend right=50, "c_X"'] \arrow[r, phantom, "\Downarrow\eta_X"] & X & = & X^{\mathbf{3}} \arrow[ur, "p_X"] \arrow[dr, "q_X"'] & & X \\
 & & & & & X^{\mathbf{2}} \arrow[ur, "d_X" description] \arrow[ur, bend right=50, "c_X"'] \arrow[ur, phantom, bend right=25, "\Downarrow\eta_X"] &
\end{tikzcd}$$

The internal natural transformation on the left has arrow component $\overline{\eta_X}(m_X)_0 : X_2 \longrightarrow X_1$ which equals $(m_X)_0 : X_2 \longrightarrow X_1$ since $\overline{\eta_X}$ is the identity on X_1. The top and bottom internal natural transformations on the right hand side $\eta_X p_X$ and $\eta_X q_X$ respectively have arrow components $p_x : X_2 \longrightarrow X_1$ and $q_x : X_2 \longrightarrow X_1$ using the fact that $\overline{\eta_X}$ is the identity on X_1 again, and that p_X and q_X respectively have object maps p_x and q_x. The vertical composite of these internal natural transformations is then the arrow:

$$X_2 \xrightarrow{(p_x,q_x)} X_2 \xrightarrow{m_x} X_1$$

where $(p_x, q_x) : X_2 \longrightarrow X_2$ is the unique arrow into the pullback X_2 induced by the commutativity of $c_x p_x = d_x q_x$. As p_x and q_x are the projections from the pullback X_2 this of course implies $(p_x, q_x) : X_2 \longrightarrow X_2$ is the identity on X_2 so that the arrow component of the internal natural transformation for the pasting diagram on the right is just $m_x : X_2 \longrightarrow X_1$. Equating the two above diagrams we then have $(m_X)_0 : X_2 \longrightarrow X_1 = m_x : X_2 \longrightarrow X_1$. This shows that $Cat(ob_{\mathcal{E}})\Delta'$ is the identity on objects.
Given an internal functor $f : X \longrightarrow Y$ consider its image under Δ':

$$\begin{tikzcd}
X^{\mathbf{3}} \arrow[r, shift left=2, "p_X"] \arrow[r, "m_X" description] \arrow[r, shift right=2, "q_X"'] \arrow[d, "f^{\mathbf{3}}"] & X^{\mathbf{2}} \arrow[r, shift left=2, "d_X"] \arrow[r, shift right=2, "c_X"'] \arrow[d, "f^{\mathbf{2}}"] & X \arrow[l, "i_X" description] \arrow[d, "f"] \\
Y^{\mathbf{3}} \arrow[r, shift left=2, "p_Y"] \arrow[r, "m_Y" description] \arrow[r, shift right=2, "q_Y"'] & Y^{\mathbf{2}} \arrow[r, shift left=2, "d_Y"] \arrow[r, shift right=2, "c_Y"'] & Y \arrow[l, "i_Y" description]
\end{tikzcd}$$

We need to show that $f_0^{\mathbf{2}} = f_1$ and $f_0^{\mathbf{3}} = f_2$. The 1-cell $f^{\mathbf{2}}$ is the unique one such that $\eta_Y f^{\mathbf{2}} = f\eta_X$. Taking arrow components of these internal natural transformations gives the equation $\overline{\eta_Y} f_0^{\mathbf{2}} = f_1 \overline{\eta_X}$. Since both $\overline{\eta_X}$ and $\overline{\eta_Y}$ are identities we obtain $f_0^{\mathbf{2}} = f_1$. Now $f^{\mathbf{3}}$ is the unique arrow such that $p_Y f^{\mathbf{3}} = f^{\mathbf{2}} p_X$ and $q_Y f^{\mathbf{3}} = f^{\mathbf{2}} q_X$. Taking the corresponding object maps we obtain $p_y f_0^{\mathbf{3}} = f_1 p_x$ and $q_y f_0^{\mathbf{3}} = f_1 q_x$. But p_x and q_x are themselves pullback projections and f_2 is the unique map satisfying these equations; thus $f_0^{\mathbf{3}} = f_2$.

Given an internal natural transformation $\theta : f \Longrightarrow g$ of $Cat(\mathcal{E})(X, Y)$ the internal natural transformation $\Delta'(\theta)$ in $Cat(\mathcal{U}Cat(\mathcal{E}))$ has arrow component $\overline{\Delta'(\theta)} : X \longrightarrow Y^{\mathbf{2}}$ the unique internal functor such that $\eta_Y \overline{\Delta'(\theta)} = \theta$. Taking arrow components of these internal natural transformations gives $\overline{\eta_Y}\,\overline{\Delta'(\theta)}_0 = \overline{\theta}$. As $\overline{\eta_Y}$ is the identity this gives $\overline{\Delta'(\theta)}_0 = \overline{\theta}$ so that $\Delta'(\theta)_0 = \theta$ as required.

2. We will first prove the following claim which we will show subsumes the claims that the 2-cells (1) and (2) above are identites.

   - Consider a 2-functor $F : \mathcal{C} \longrightarrow Cat(\mathcal{E})$ of Rep. Given A of $\mathcal{C}$ consider the corresponding internal category in $\mathcal{C}$:

     $$\Delta'(A) \quad = \quad \begin{tikzcd} A^{\mathbf{3}} \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] & A^{\mathbf{2}} \arrow[r, shift left=2, "d_a"] \arrow[r, shift right=2, "c_a"'] & A \arrow[l, "i_a" description] \end{tikzcd}$$

     with universal 2-cell $\eta_a : d_a \Longrightarrow c_a$. Suppose that, for each $A \in \mathcal{C}$ we have the equality in $\mathcal{E}$:

     $$\begin{tikzcd} (FA^{\mathbf{3}})_0 \arrow[r, shift left, "(Fp_a)_0"] \arrow[r, shift right, "(Fq_a)_0"'] & (FA^{\mathbf{2}})_0 \arrow[r, shift left, "(Fd_a)_0"] \arrow[r, shift right, "(Fc_a)_0"'] & (FA)_0 \end{tikzcd} \quad = \quad \begin{tikzcd} (FA)_2 \arrow[r, shift left, "p_{Fa}"] \arrow[r, shift right, "q_{Fa}"'] & (FA)_1 \arrow[r, shift left, "d_{Fa}"] \arrow[r, shift right, "c_{Fa}"'] & (FA)_0 \end{tikzcd}$$

     and that the arrow component $\overline{F\eta_a} : (FA^{\mathbf{2}})_0 = FA_1 \longrightarrow FA_1$ is the identity on FA_1. Then the 2-cell $Cat(ob_{\mathcal{E}})\Delta'_F$ is an identity.
     For consider the component of Δ'_F at A, the internal functor in $Cat(\mathcal{U}Cat(\mathcal{E}))$ as described in Proposition 3.27:

     $$(1) \qquad \begin{tikzcd} F(A^{\mathbf{3}}) \arrow[r, shift left=2, "Fp_a"] \arrow[r, "Fm_a" description] \arrow[r, shift right=2, "Fq_a"'] \arrow[d, "k_2"] & F(A^{\mathbf{2}}) \arrow[r, shift left=2, "Fd_a"] \arrow[r, shift right=2, "Fc_a"'] \arrow[d, "k_1"] & FA \arrow[l, "Fi_a" description] \arrow[d, "1"] \\ (FA)^{\mathbf{3}} \arrow[r, shift left=2, "p_{Fa}"] \arrow[r, "m_{Fa}" description] \arrow[r, shift right=2, "q_{Fa}"'] & (FA)^{\mathbf{2}} \arrow[r, shift left=2, "d_{Fa}"] \arrow[r, shift right=2, "c_{Fa}"'] & FA \arrow[l, "i_{Fa}" description] \end{tikzcd}$$

     where $k_1 : F(A^{\mathbf{2}}) \longrightarrow (FA)^{\mathbf{2}}$ is the unique arrow of $Cat(\mathcal{E})$ such that $\eta_{Fa}k_1 = F\eta_a$ and $k_2 : F(A^{\mathbf{3}}) \longrightarrow (FA)^{\mathbf{3}}$ the unique arrow such that $k_1Fp_a = p_{Fa}k_2$ and $k_1Fq_a = q_{Fa}k_2$. We need to show that the object maps of the internal functors k_1 and k_2 are identities.
     Equating arrow components of the internal natural transformations on either side of the equation $\eta_{Fa}k_1 = F\eta_a$ gives $\overline{\eta_{FA}}(k_1)_0 = \overline{F\eta_a}$. Now by our choice of cotensors in $Cat(\mathcal{E})$ we have $\overline{\eta_{FA}}$ is an identity and by assumption $\overline{F\eta_a}$ is an identity too. Therefore $(k_1)_0$ is an identity.
     Taking object components of the internal functors on either side of the equations: $k_1Fp_a = p_{Fa}k_2$ and $k_1Fq_a = q_{Fa}k_2$ give, using our assumptions, the equation $(p_{Fa})_0(k_2)_0 = (k_1)_0(Fp_a)_0 = (Fp_a)_0 = (p_{Fa})_0$ and similarly $(q_{Fa})_0(k_2)_0 = (q_{Fa})_0$. But the pair $(p_{Fa})_0, (q_{Fa})_0$ are themselves the pullback projections of c_{Fa} and d_{Fa} and therefore jointly monic. This implies that $(k_2)_0$ is an identity.

In order to prove that the 2-cells (1) and (2) are identities it suffices to verify that the 2-functors $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ and $\Delta' : C \longrightarrow Cat(\mathcal{U}\mathcal{C})$ verify the above conditions.
With regards the first of these consider $X \in Cat(\mathcal{A})$. Then:

$$(Cat(F)X^{\mathbf{2}})_0 = F(X_0^{\mathbf{2}}) = FX_1 = (Cat(F)X)_0^{\mathbf{2}}$$

where the first equation holds by the pointwise definition of $Cat(F)$, and the second and third hold respectively by our choice of cotensors in $Cat(\mathcal{A})$ and $Cat(\mathcal{B})$. The internal natural transformation $\eta_X : d_X \Longrightarrow c_X$ has arrow component $\overline{\eta_X}$ the identity on X_1. Now $\overline{Cat(F)\eta_X} = F\overline{\eta_X}$ by definition of $Cat(F)$ on 2-cells so that $\overline{Cat(F)\eta_X}$ is an identity as required. Similarly we can use the pointwise definition of $Cat(F)$ and the choices of cotensor and pullbacks in $Cat(\mathcal{A})$ and $Cat(\mathcal{B})$ to deduce the other required conditions. Therefore $Cat(ob_{\mathcal{E}})\Delta'_{Cat(F)}$ is an identity 2-cell as required. With regards $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ consider $A \in \mathcal{C}$ and $\Delta'(A)$:

$$\begin{tikzcd}
A^{\mathbf{3}} \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] & A^{\mathbf{2}} \arrow[r, shift left=2, "d_a"] \arrow[r, shift right=2, "c_a"'] & A \arrow[l, "i_a" description]
\end{tikzcd}$$

with universal 2-cell $\eta_a : d_a \Longrightarrow c_a$. Then we have:

$$\begin{tikzcd}
 & & & (A^{\mathbf{2}})^{\mathbf{3}} \arrow[r, shift left, "(d_a)^{\mathbf{3}}"] \arrow[r, shift right, "(c_a)^{\mathbf{3}}"'] \arrow[d, shift right=2, "p_{a^{\mathbf{2}}}"'] \arrow[d] \arrow[d, shift left=2, "q_{a^{\mathbf{2}}}"] & A^{\mathbf{3}} \arrow[d, shift right=2, "p_a"'] \arrow[d] \arrow[d, shift left=2, "q_a"] \\
\Delta'(A^{\mathbf{2}}) \arrow[r, bend left, "\Delta'(d_a)"] \arrow[r, bend right, "\Delta'(c_a)"'] \arrow[r, phantom, "\Downarrow\Delta'(\eta_a)"] & \Delta'(A) & = & (A^{\mathbf{2}})^{\mathbf{2}} \arrow[r, shift left, "(d_a)^{\mathbf{2}}"] \arrow[r, shift right, "(c_a)^{\mathbf{2}}"'] \arrow[d, shift right=2, "d_{a^{\mathbf{2}}}"'] \arrow[d, shift left=2, "c_{a^{\mathbf{2}}}"] & A^{\mathbf{2}} \arrow[d, shift right=2, "d_a"'] \arrow[d, shift left=2, "c_a"] \\
 & & & A^{\mathbf{2}} \arrow[u] \arrow[r, shift left, "d_a"] \arrow[r, shift right, "c_a"'] \arrow[ur, "\overline{\Delta'\eta_a}" description] & A \arrow[u]
\end{tikzcd}$$

Clearly then $\Delta'(A^{\mathbf{2}})_0 = A^{\mathbf{2}}$ whilst $\overline{\Delta'(\eta_a)}$ is the unique arrow $A^{\mathbf{2}} \longrightarrow A^{\mathbf{2}}$ such that $\eta_a\overline{\Delta'(\eta_a)} = \eta_a$, namely the identity. It is straightforward to verify that $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{UC})$ satisfies the remaining conditions above and so $Cat(ob_{\mathcal{UC}})\Delta'_{\Delta'_{\mathcal{C}}}$ is an identity 2-cell.

□

Definition 3.32. Consider 2-categories $\mathcal{C}$ and $\mathcal{D}$ and 2-functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$. We say F is the left biadjoint [3] of G when there exist pseudonatural transformations $\epsilon : FG \Longrightarrow 1$ and $\eta : 1 \Longrightarrow GF$, and invertible modifications $\theta : G\epsilon \circ \eta G \longrightarrow 1_G$ and $\phi : 1_F \longrightarrow \epsilon F \circ F\eta$ such that for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$ the 2-cells:

$$\begin{tikzcd}
 & & FGD \arrow[dr, "\epsilon_D"] & \\
(1) \quad FGD \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[r, "F\eta_{GD}"] & FGFGD \arrow[ur, "\epsilon_{FGD}"] \arrow[dr, "FG\epsilon_D"'] \arrow[u, phantom, "\phi_{GD}\Downarrow"] \arrow[d, phantom, "F\theta_D\Downarrow"] \arrow[rr, phantom, "\epsilon_{\epsilon_D}\Downarrow"] & & D \\
 & & FGD \arrow[ur, "\epsilon_D"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & GFC \arrow[dr, "GF\eta_C"] \arrow[drr, bend left, "1"] & & \\
(2) \quad C \arrow[ur, "\eta_C"] \arrow[dr, "\eta_C"'] \arrow[rr, phantom, "\eta_{\eta_C}\Downarrow"] & & GFGFC \arrow[r, "G\epsilon_{FC}"] \arrow[u, phantom, "\Downarrow G\phi_C"] \arrow[d, phantom, "\Downarrow\theta_{GC}"] & GFC \\
 & GFC \arrow[ur, "\eta_{GFC}"'] \arrow[urr, bend right, "1"'] & &
\end{tikzcd}$$

are identities.

Theorem 3.33. We have a biadjunction:

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{pb}} \arrow[r, shift right=2, "Cat(-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Rep} \arrow[l, shift right=2, "\mathcal{U}"']
\end{tikzcd}$$

with $\mathcal{U} : \mathrm{Rep} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$ the left biadjoint of $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$.

Proof. We have already described the pseudonatural transformations $\Delta' : 1_{\mathrm{Rep}} \Longrightarrow Cat(\mathcal{U}-)$ and $ob : \mathcal{U}Cat(-) \Longrightarrow 1_{\mathrm{Cat}_{\mathrm{pb}}}$ the latter being indeed 2-natural. We need to describe invertible modifications

$$Cat(ob) \circ \Delta'_{Cat(-)} \longrightarrow 1_{Cat(-)} \quad \text{and} \quad 1_{\mathcal{U}} \longrightarrow ob_{\mathcal{U}} \circ \mathcal{U}\Delta' \quad .$$

---

[3] The general notion of biadjunction [51] involves pseudofunctors as opposed to 2-functors. However in the case of the present biadjunction we have genuine 2-functors, and give the definition at this level of generality; the diagrams (1) and (2) simplifying somewhat in the absence of pseudofunctoriality constraints.

In fact we will see that $Cat(ob) \circ \Delta'_{Cat(-)} = 1_{Cat(-)}$ and $1_{\mathcal{U}} = ob_{\mathcal{U}} \circ \mathcal{U}\Delta'$ so that we may take the identity modifications.

In Lemma 3.31(1) we proved that for each $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$ the composite 2-functor: $Cat(ob_{\mathcal{E}}) \circ \Delta'_{Cat(\mathcal{E})} : Cat(\mathcal{E}) \longrightarrow Cat(\mathcal{E})$ is the identity. To show that $Cat(ob) \circ \Delta'_{Cat(-)} = 1_{Cat(-)}$ it remains to show that for $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ we have:

$$\begin{tikzcd}
Cat(\mathcal{A}) \arrow[r, "\Delta'_{Cat(\mathcal{A})}"] \arrow[d, "Cat(F)"'] & Cat(\mathcal{U}Cat(\mathcal{A})) \arrow[r, "Cat(ob_{\mathcal{A}})"] \arrow[d, "Cat(\mathcal{U}Cat(F))"] & Cat(\mathcal{B}) \arrow[d, "Cat(F)"] & {} \arrow[d, phantom, "="] & Cat(\mathcal{A}) \arrow[r, "1"] \arrow[d, "Cat(F)"'] & Cat(\mathcal{A}) \arrow[d, "Cat(F)"] \\
Cat(\mathcal{B}) \arrow[r, "\Delta'_{Cat(\mathcal{B})}"'] & Cat(\mathcal{U}Cat(\mathcal{B})) \arrow[r, "Cat(ob_{\mathcal{B}})"'] & Cat(\mathcal{B}) & {} & Cat(\mathcal{B}) \arrow[r, "1"] & Cat(\mathcal{B})
\arrow[from=1-1, to=2-2, phantom, "\Downarrow\Delta'_{Cat(F)}"]
\end{tikzcd}$$

which amounts to showing that the 2-cell $Cat(ob_{\mathcal{B}}) \circ \Delta'_{Cat(F)}$ is an identity. We proved this to be the case in Lemma 3.31(2).

Next we show that $1_{\mathcal{U}} = ob_{\mathcal{U}} \circ \mathcal{U}\Delta'$. Given a representable 2-category $\mathcal{C}$ and an arrow $f : A \longrightarrow B \in \mathcal{C}$ its image under $ob_{\mathcal{UC}} \circ \mathcal{U}\Delta'_{\mathcal{C}}$ is depicted below:

$$\begin{tikzcd}
 & & A^{\mathbf{3}} \arrow[r, "f^{\mathbf{3}}"] \arrow[d, shift right=2, "p_a"'] \arrow[d, "m_a" description] \arrow[d, shift left=2, "q_a"] & B^{\mathbf{3}} \arrow[d, shift right=2, "p_b"'] \arrow[d, "m_b" description] \arrow[d, shift left=2, "q_b"] & & \\
A \arrow[r, "f"] & B \arrow[r, mapsto, "\mathcal{U}(\Delta'_{\mathcal{C}})"] & A^{\mathbf{2}} \arrow[r, "f^{\mathbf{2}}"] \arrow[d, shift right=2, "d_a"'] \arrow[d, shift left=2, "c_a"] & B^{\mathbf{2}} \arrow[r, mapsto, "ob_{\mathcal{UC}}"] \arrow[d, shift right=2, "d_b"'] \arrow[d, shift left=2, "c_b"] & A \arrow[r, "f"] & B \\
 & & A \arrow[r, "f"] \arrow[u, "i_a" description] & B \arrow[u, "i_b" description] & &
\end{tikzcd}$$

Therefore for each representable 2-category $\mathcal{C}$ the composite $ob_{\mathcal{UC}} \circ \mathcal{U}\Delta'_{\mathcal{C}}$ is the identity on $\mathcal{C}$. To show that $1_{\mathcal{U}} = ob_{\mathcal{U}} \circ \mathcal{U}\Delta'$ it remains to show that given $F : \mathcal{C} \longrightarrow \mathcal{D}$ of Rep we have the equality:

$$\begin{tikzcd}
\mathcal{UC} \arrow[r, "\mathcal{U}\Delta'_{\mathcal{C}}"] \arrow[d, "\mathcal{U}F"'] & \mathcal{U}Cat(\mathcal{UC}) \arrow[r, "ob_{\mathcal{UC}}"] \arrow[d, "\mathcal{U}Cat(\mathcal{U}F)"] & \mathcal{UC} \arrow[d, "\mathcal{U}F"] & {} \arrow[d, phantom, "="] & \mathcal{UC} \arrow[r, "1"] \arrow[d, "\mathcal{U}F"'] & \mathcal{UC} \arrow[d, "\mathcal{U}F"] \\
\mathcal{UD} \arrow[r, "\mathcal{U}\Delta'_{\mathcal{D}}"'] & \mathcal{U}Cat(\mathcal{UD}) \arrow[r, "ob_{\mathcal{UD}}"'] & \mathcal{UD} & {} & \mathcal{UD} \arrow[r, "1"] & \mathcal{UD}
\arrow[from=1-1, to=2-2, phantom, "\Downarrow\mathcal{U}(\Delta'_F)"]
\end{tikzcd}$$

This is clear from the description of Δ'_F at an object $A \in \mathcal{C}$ given in diagram (1) of the proof of Lemma 3.31(2).

Both modifications are consequently identities. We must show that the composite 2-cells (1) and (2) of Definition 3.32 are identities. With regards (1) this is clear, since each modification component is an identity and the counit is 2-natural. Consider diagram (2) of Definition 3.32. Since both modifications are identities we need only show that given given $\mathcal{C} \in$ Rep the 2-cell:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta'_{\mathcal{C}}"] \arrow[d, "\Delta'_{\mathcal{C}}"'] & Cat(\mathcal{UC}) \arrow[d, "Cat(\mathcal{U}\Delta'_{\mathcal{C}})"] & \\
Cat(\mathcal{UC}) \arrow[r, "\Delta'_{Cat(\mathcal{UC})}"'] & Cat(\mathcal{U}Cat(\mathcal{UC})) \arrow[r, "Cat(ob_{\mathcal{UC}})"'] & Cat(\mathcal{UC})
\arrow[from=1-1, to=2-2, phantom, "\Downarrow\Delta'_{\Delta'_{\mathcal{C}}}"]
\end{tikzcd}$$

is an identity. We proved this to be the case in Lemma 3.31(2). ☐

Remark 3.34. It is worth remarking upon some other biadjunctions which follow from the above one and their comonadic natures. These remarks will not be relevant anywhere else in this thesis and so we treat them briefly. Let Φ be a class of diagram types, small categories, containing at least the diagram shape for pullbacks, namely the category consisting of a single opspan:

$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Let CAT_Φ denote the 2-category of categories with Φ-limits, functors which preserve Φ-limits and all natural transformations. Let 2-CAT_Φ denote the 2-category of 2-categories with conical Φ-limits, now in the 2-categorical sense, and cotensors with $\mathbf{2}$, whose morphisms are 2-functors preserving Φ-limits and cotensors with $\mathbf{2}$, and all 2-natural transformations. We have a biadjunction:

$$\begin{tikzcd} \mathrm{CAT}_\Phi \arrow[r, bend right, "Cat(-)"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}_\Phi \arrow[l, bend right, "\mathcal{U}"'] \end{tikzcd}$$

We have observed this in the minimal case where Φ consists of just one category, the opspan above. Then $\mathrm{CAT}_\Phi = \mathrm{Cat}_{\mathrm{pb}}$ and 2-$\mathrm{CAT}_\Phi = \mathrm{Rep}$ and we recover the biadjunction of Theorem 3.33. In order to establish the more general case claimed here, observe firstly that if $\mathcal{E}$ has Φ-limits then so does $Cat(\mathcal{E})$, pointwise, together with cotensors with $\mathbf{2}$ as $\mathcal{E}$ has pullbacks. If $F \in \mathrm{CAT}_\Phi$ it preserves pullbacks and so $Cat(F)$ preserves cotensors with $\mathbf{2}$ as before, and preserves all Φ-limits, these being pointwise. Thus $Cat(F) \in$ 2-CAT_Φ and we have obtain the 2-functor $Cat(-) : \mathrm{CAT}_\Phi \longrightarrow 2\text{-}\mathrm{CAT}_\Phi$. Trivially we have the forgetful 2-functor $2\text{-}\mathrm{CAT}_\Phi \longrightarrow \mathrm{CAT}_\Phi$. Certainly each objects functor $ob_\mathcal{E} : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ preserves Φ-limits, as they are pointwise in $Cat(\mathcal{E})$. By Proposition 3.25(3) the 2-functor $\Delta'_\mathcal{C} : \mathcal{C} \longrightarrow \mathrm{Cat}(\mathcal{UC})$ preserves any limits that $\mathcal{C}$ has. Consequently if $\mathcal{C} \in$ 2-CAT_Φ we have $\Delta'_\mathcal{C} \in$ 2-CAT_Φ too and Δ' restricts to a pseudonatural transformation $1_{2\text{-}\mathrm{CAT}_\Phi} \Longrightarrow Cat(\mathcal{U}-)$. Therefore all of the data for the biadjunction lifts to the present situation. The equations for a biadjunction hold just as before so that indeed the biadjunction lifts as claimed.
One case of interest is when $\Phi = \{\text{All finite categories}\}$. Then the biadjunction becomes:

$$\begin{tikzcd} FinComp(\mathrm{CAT}) \arrow[r, bend right, "Cat(-)"'] \arrow[r, phantom, "\perp"] & FinComp(2\text{-}\mathrm{CAT}) \arrow[l, bend right, "\mathcal{U}"'] \end{tikzcd}$$

where $FinComp$(CAT) and $FinComp$(2-CAT) are respectively the 2-categories whose objects are finitely complete categories and finitely complete 2-categories. If we take $\Phi = \{\text{All small categories}\}$ the biadjunction similarly becomes:

$$\begin{tikzcd} Comp(\mathrm{CAT}) \arrow[r, bend right, "Cat(-)"'] \arrow[r, phantom, "\perp"] & Comp(2\text{-}\mathrm{CAT}) \arrow[l, bend right, "\mathcal{U}"'] \end{tikzcd}$$

where $Comp$(CAT) and $Comp$(2-CAT) now have objects: small complete categories and 2-categories respectively. Returning to the general biadjunction:

$$\begin{tikzcd} \mathrm{CAT}_\Phi \arrow[r, bend right, "Cat(-)"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}_\Phi \arrow[l, bend right, "\mathcal{U}"'] \end{tikzcd}$$

we turn our attention to the comonadicity of $\mathcal{U}$. In Theorem 3.6 of [41] the authors' prove that a right biadjoint 2-functor $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{B}$ is bicategorically monadic (meaning that $\mathcal{A}$ is biequivalent, via the canonical comparison, to the 2-category Ps-T-Alg of pseudo-algebras for the induced pseudomonad) if $\mathcal{U}$ reflects adjoint equivalences and if $\mathcal{A}$ has "pseudo-coequalizers of $\mathcal{U}$-absolute codescent objects" and $\mathcal{U}$ preserves them.

Their terminology "pseudo-coequalizer" does not mean a pseudo-colimit, but only a pseudo-colimit up to equivalence, a bicolimit. In particular if $\mathcal{A}$ admits all bi-colimits and $\mathcal{U}$ preserves them, as well as reflecting adjoint equivalences, then $\mathcal{U}$ will be bicategorically monadic. Dually if $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{B}$ is a left biadjoint 2-functor, $\mathcal{A}$ has bilimits and $\mathcal{U}$ preserves these as well as reflecting adjoint equivalences then $\mathcal{U}$ is bicategorically comonadic. Consider then the left biadjoint 2-functor $\mathcal{U} : \text{2-CAT}_\Phi \longrightarrow \text{CAT}_\Phi$ of interest. It is straightforward to verify directly that both 2-CAT$_\Phi$ and CAT$_\Phi$ have pie-limits: products, inserters and equifiers, in each case formed at the level of underlying categories. Therefore 2-CAT$_\Phi$ has all pie limits and $\mathcal{U}$ preserves them. In particular $\mathcal{U}$ preserves all pseudo-limits, which are specific instances of pie limits, as described in [29]. As the pseudo-limit of a diagram is in particular its bilimit it follows that 2-CAT$_\Phi$ has all bilimits and $\mathcal{U}$ preserves them. In order to show that $\mathcal{U}$ reflects adjoint equivalences what we must show is that given $F : \mathcal{A} \longrightarrow \mathcal{B} \in \text{2-CAT}_\Phi$ such that $\mathcal{U}(F) \in \text{CAT}_\Phi$ has a left adjoint equivalence inverse G, then G underlies a 2-functor $H : \mathcal{B} \longrightarrow \mathcal{A}$ such that H is the left adjoint equivalence inverse of F in 2-CAT$_\Phi$ and indeed that the data for the adjoint equivalence in 2-CAT$_\Phi$ lies, via $\mathcal{U}$, precisely over the adjoint equivalence in CAT$_\Phi$. As $F \in \text{2-CAT}_\Phi$ preserves cotensors with **2**, Proposition 3.1 of [8] ensures that such a left 2-adjoint H does exist, living over the adjoint equivalence in the manner required. According to that result the natural transformations constituting the unit and counit of the adjunction become the 2-natural transformation of the adjunction between F and H in 2-CAT. Thus the natural isomorphisms of the adjoint equivalence between 2-natural isomorphisms and we have an adjoint equivalence in 2-CAT. Since any 2-equivalence preserves all limits we see that $H \in \text{2-CAT}_\Phi$ as required. Therefore $\mathcal{U}$ reflects adjoint equivalences and the forgetful 2-functor is bicategorically comonadic in the sense of [41].

3.3 Two sided discrete fibrations and cateads in representable 2-categories

In this section we consider cateads in representable 2-categories, in particular showing that for a representable 2-category $\mathcal{C}$ the full sub 2-category of $Cat(\mathcal{UC})$, $Kat(\mathcal{C})$, containing the cateads in $\mathcal{C}$ is a representable 2-category. Furthermore we extend this description to obtain an endo 2-functor $Kat(-) : \text{Rep} \longrightarrow \text{Rep}$ by restricting $Cat(\mathcal{U}-) : \text{Rep} \longrightarrow \text{Rep}$. The distinction between cateads and internal categories is concerned with two sided discrete fibrations; thus an understanding of two sided discrete fibrations in representable 2-categories will be of key importance to these results. In order to understand two sided discrete fibrations in representable 2-categories we use comma objects. These exist in any representable 2-category.

Proposition 3.35. Any representable 2-category has comma objects. Consider 2-categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}$ representable, and a 2-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which preserves cotensors with **2** and pullbacks. Then F preserves comma objects. In particular any morphism of Rep preserves comma objects.

Proof. Each representable 2-category $\mathcal{C}$ has cotensors with **2** and pullbacks. To prove the proposition it will suffice to show that comma objects may be constructed from cotensors with **2** and pullbacks.
The following construction of the comma object $f|g$ of an opspan in $\mathcal{C}$:

$$\begin{tikzcd}
 & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

via cotensors with **2** and pullbacks is standard knowledge in the case of Cat [42] and well known in an arbitrary 2-category [48]. As indicated in the following diagram:

$$\begin{tikzcd}
f|g \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[d] & C^{\mathbf{2}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & C \arrow[d, "1"] \\
B \arrow[r, "g"'] & C \arrow[r, "1"'] & C
\end{tikzcd}$$

the comma object may be constructed by first forming $C^{\mathbf{2}}$ with its universal 2-cell as in the bottom right square. Then form the other three squares by pullback. The resulting projections and 2-cell then exhibit $f|g$ as the comma object of the opspan, regardless of the order in which one forms pullbacks. $\square$

Remark 3.36. Recall the definition of a two sided discrete fibration $A \xrightarrow{(p,q)} B$ in Cat given in Definition 2.71. The defining properties are unique p-lifts, unique q-lifts and the bimodule condition. It is well known that given a 2-sided discrete fibration as above the functor p is a fibration and the functor q an opfibration [58]. An alternative description of two sided discrete fibrations in such terms, as presented in the next proposition, will be convenient for our purposes.

Proposition 3.37. A span $A \xrightarrow{(p,q)} B$ in Cat is a two sided discrete fibration if and only if it has unique p-lifts, unique q-lifts and moreover the p-lifts and q-lifts are respectively cartesian and opcartesian morphisms for p and q respectively. In particular p is then an opfibration and q a fibration.

Proof. The forward implication is proven in Theorem 2.11 of [58]. Conversely suppose that the span has unique p-lifts and q-lifts, and that these are respectively cartesian and opcartesian morphisms for the fibrations p and q. We must show that the bimodule condition is satisfied. Given a morphism $\alpha : a \longrightarrow b$ of A consider the p-lift $(p\alpha)^p : a^p \longrightarrow b$ of the pair $(p\alpha : pa \longrightarrow pb, b)$. The image under p of the p-lift $(p\alpha)^p : a^p \longrightarrow b$ is precisely $p\alpha : pa \longrightarrow pb$. Consequently, as the p-lift is a cartesian morphism, we have a unique factorisation of $\alpha : a \longrightarrow b$:

$$\begin{tikzcd}
a \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & a^p \arrow[d, "(p\alpha)^p"] \\
 & b
\end{tikzcd}$$

such that $p\beta$ is the identity on pa. Now $q(p\alpha)^p = 1_{qa}$ by the definition of p-lifts, so that taking the image of the above triangle under q we see $q\beta = q\alpha$. Consequently $\beta : a \longrightarrow a^p$ is a q-lift of the pair $(a, p\alpha : pa \longrightarrow pb)$ and by assumption the unique such. Thus we necessarily have that $\beta = (q\alpha)^q$. Substituting $(q\alpha)^q$ for β in the above triangle now verifies the bimodule condition. $\square$

Remark 3.38. We now use this second description of two sided discrete fibrations to give a "finite limit" characterisation of such spans in a 2-category with sufficient limits. Firstly we recall a well known finite limit characterisation of discrete fibrations and opfibrations; a special case of the more general characterisations of fibrations and opfibrations of [48] and [22].
Given an arrow $p : A \longrightarrow B$ of a 2-category with comma objects consider the comma objects $B|p = 1_B|p$ and $p|B = p|1_B$ with their respective limiting cones:

$$\begin{tikzcd}
B|p \arrow[r] \arrow[dr] & A \arrow[d, "p"] \arrow[dl, phantom, "\Nearrow"] & & p|B \arrow[r] \arrow[dr] & A \arrow[d, "p"] \arrow[dl, phantom, "\Swarrow"] \\
 & B & \text{and} & & B
\end{tikzcd}$$

Consider the cotensor $A^{\mathbf{2}}$ and its universal 2-cell:

$$\begin{tikzcd}
A^{\mathbf{2}} \arrow[r, bend left, "d_a"{name=U}] \arrow[r, bend right, "c_a"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\eta_a"]
\end{tikzcd}$$

The 2-cell $p\eta_a : pd_a \Longrightarrow pc_a$ obtained by postcomposing η_a by p induces, by the universal properties of the comma objects $B|p$ and $p|B$, a pair of 1-cells $p_* : A^{\mathbf{2}} \longrightarrow B|p$ and $p^* : A^{\mathbf{2}} \longrightarrow p|B$ which respectively recover the 2-cell $p\eta_a$ upon postcomposition with the universal 2-cells exhibiting $B|p$ and $p|B$ as comma objects.
It is well known that the arrow $p : A \longrightarrow B$ is a discrete fibration precisely when the induced 1-cell $p_* : A^{\mathbf{2}} \longrightarrow B|p$ is an isomorphism, and a discrete opfibration precisely if the induced 1-cell $p^* : A^{\mathbf{2}} \longrightarrow p|B$ is an isomorphism. A useful and immediate consequence of this fact is that any comma object preserving 2-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ from $\mathcal{C}$ a 2-category with comma objects preserves discrete fibrations and opfibrations. We now consider the two sided analogue of this characterisation.

Remark 3.39. Consider the case of a span in a 2-category $\mathcal{C}$:

$$\begin{tikzcd}
& A \arrow[dl, "p"'] \arrow[dr, "q"] & \\
B & & C
\end{tikzcd}$$

If $\mathcal{C}$ has sufficient limits we may form the following diagram:

$$\text{(1)} \qquad \begin{tikzcd}
& & I_p \arrow[dll] \arrow[ddr, "\iota_p"'] & & I_q \arrow[ddl, "\iota_q"] \arrow[drr] & & \\
B \arrow[ddr, "i_b"'] & & & & & & C \arrow[ddl, "i_c"] \\
& & & A^{\mathbf{2}} \arrow[dll, "p^{\mathbf{2}}"'] \arrow[drr, "q^{\mathbf{2}}"] \arrow[ddl, "p_*"'] \arrow[ddr, "q^*"] & & & \\
& B^{\mathbf{2}} & & & & C^{\mathbf{2}} & \\
& & B|p & & q|C & &
\end{tikzcd}$$

in which the middle span $(p^{\mathbf{2}}, q^{\mathbf{2}}) : B^{\mathbf{2}} \longrightarrow C^{\mathbf{2}}$ is obtained from the span $(p,q) : B \longrightarrow C$ by taking cotensors with $\mathbf{2}$. The morphisms i_b and i_c are respectively the unique ones with the property that postcomposition with the universal 2-cells for $B^{\mathbf{2}}$ and $C^{\mathbf{2}}$ yield identity 2-cells on B and C. Both squares are pullbacks. The morphisms p_* and q^* are those constructed in Remark 3.38. In that Remark we noted that p is a discrete fibration if and only if p_* is an isomorphism, and that q is an opfibration if and only if q^* is an isomorphism. One can then see the relationship with the characterisation of the next proposition.

Proposition 3.40. Suppose that $(p,q) : B \longrightarrow C$ is a span in a 2-category $\mathcal{C}$ and that $\mathcal{C}$ is sufficiently complete so as to admit the construction of diagram (1) of Remark 3.39. Then the span (p,q) is a two sided discrete fibration in $\mathcal{C}$ if and only the composite 1-cells:

$$I_q \xrightarrow{\iota_q} A^{\mathbf{2}} \xrightarrow{p_*} B|p$$

and

$$I_p \xrightarrow{\iota_p} A^{\mathbf{2}} \xrightarrow{q^*} q|C$$

are both isomorphisms.

Proof. By definition the span (p,q) is a 2-sided discrete fibration if and only if for each object D of $\mathcal{C}$ the image in Cat of the span under $\mathcal{C}(D,-)$, $(\mathcal{C}(D,p), \mathcal{C}(D,q))$, is a two sided discrete fibration in Cat. As $\mathcal{C}(D,-)$ preserves all limits that exist in $\mathcal{C}$ the image of the composite:

$$I_q \xrightarrow{\iota_q} A^{\mathbf{2}} \xrightarrow{p_*} B|p$$

is:

$$I_{\mathcal{C}(D,q)} \xrightarrow{\iota_{\mathcal{C}(D,q)}} \mathcal{C}(D,A)^{\mathbf{2}} \xrightarrow{\mathcal{C}(D,p)_*} \mathcal{C}(D,B)|\mathcal{C}(D,p)$$

the functor obtained via the same construction applied to the span in Cat. The representables $\mathcal{C}(D,-)$ jointly reflect isomorphisms. Consequently the composite in $\mathcal{C}$ is an isomorphism if and only if this functor in Cat is one. Similarly the second composite in $\mathcal{C}$ is an isomorphism if and only:

$$I_{\mathcal{C}(D,p)} \xrightarrow{\iota_{\mathcal{C}(D,p)}} \mathcal{C}(D,A)^{\mathbf{2}} \xrightarrow{\mathcal{C}(D,q)^*} \mathcal{C}(D,q)|\mathcal{C}(D,C)$$

is an isomorphism. Therefore it suffices to verify the proposition when $\mathcal{C}$ =Cat. Suppose that this is the case. I_q is then the full subcategory of the arrow category $A^{\mathbf{2}}$ with objects: morphisms $\alpha : a_1 \longrightarrow a_2 \in A$ such that $q\alpha$ is an identity arrow. In other words the objects of I_q are precisely p-lifts; $\alpha : a_1 \longrightarrow a_2$ is a p-lift of $(p\alpha : pa_1 \longrightarrow pa_2, a_2)$.

The comma object $B|p$ is the comma category with objects: pairs $(\alpha : b \longrightarrow pa, a)$ and morphisms: pairs of arrows, one each in A and B, as in the following diagram:

$$\begin{tikzcd}
(b_1 \arrow[r, "\alpha_1"] \arrow[d, "r"'] & p(a_1) \arrow[d, "ps"'] & ,a_1) \arrow[d, "s"'] \\
(b_2 \arrow[r, "\alpha_2"] & p(a_2) & ,a_2)
\end{tikzcd}$$

such that the square is commutative. The functor $p_*\iota_q : I_q \longrightarrow B|p$ acts on objects by sending a p-lift $\alpha : a \longrightarrow b$ to $(p\alpha : pa \longrightarrow pb, b)$ with the evident action on morphisms. Consequently we have unique p-lifts if and only $p_*\iota_q : I_q \longrightarrow B|p$ is bijective on objects.

We now show that each p-lift is a cartesian morphism for p if and only if $p_*\iota_q : I_q \longrightarrow B|p$ is fully faithful. Suppose firstly that each p-lift is cartesian. Then consider a pair of p-lifts $\alpha : a_1 \longrightarrow a_2$ and $\beta : b_1 \longrightarrow b_2$ in I_q and a morphism:

$$\begin{tikzcd}
(p(a_1) \arrow[r, "p\alpha"] \arrow[d, "r"'] & p(a_2) \arrow[d, "ps"'] & ,a_2) \arrow[d, "s"'] \\
(p(b_1) \arrow[r, "p\beta"] & p(b_2) & ,b_2)
\end{tikzcd}$$

between their respective images in $B|p$. As $\beta : b_1 \longrightarrow b_2$ is a cartesian morphism there exists a unique 1-cell $r' : a_1 \longrightarrow b_1$ such that $pr' = r$ and such that:

$$\begin{tikzcd}
& a_1 \arrow[dl, "r'"'] \arrow[d, "s\alpha"] \\
b_1 \arrow[r, "\beta"'] & b_2
\end{tikzcd}$$

commutes. This shows that the commutative square:

$$\begin{tikzcd}
a_1 \arrow[r, "\alpha"] \arrow[d, "r'"'] & a_2 \arrow[d, "s"] \\
b_1 \arrow[r, "\beta"'] & b_2
\end{tikzcd}$$

is the unique morphism in I_q with image under $p_*\iota_q$ the above morphism (r, s) in $B|p$. Thus $p_*\iota_q$ is fully faithful.

Conversely suppose $p_*\iota_q$ is fully faithful and let $\alpha : a_1 \longrightarrow a_2 \in I_q$. We must show this p-lift is a cartesian morphism for p. Consider a morphism $\beta : b \longrightarrow a_2$ whose image in B factors through $p\alpha$:

$$\begin{tikzcd}
& p(b) \arrow[dl, "r"'] \arrow[d, "p\beta"] \\
p(a_1) \arrow[r, "p\alpha"'] & p(a_2)
\end{tikzcd}$$

via some arrow $r : p(b) \longrightarrow p(a_1)$. This corresponds to a morphism of $B|p$:

$$\begin{tikzcd}
(p(b) \arrow[r, "1"] \arrow[d, "r"'] & p(b), \arrow[d, "p\beta"'] & b) \arrow[d, "\beta"'] \\
(p(a_1) \arrow[r, "p\alpha"] & p(a_2), & a_2)
\end{tikzcd}$$

The element $(1 : p(b) \longrightarrow p(b), b)$ of $B|p$ is the image under $p_* \iota_q$ of $1 : b \longrightarrow b \in I_q$. By fully faithfulness this arrow in $B|p$ has a unique preimage:

$$\begin{tikzcd}
b \arrow[r, "1"] \arrow[d, "r'"'] & b \arrow[d, "\beta"] \\
a_1 \arrow[r, "\alpha"'] & a_2
\end{tikzcd}$$

in I_q. The arrow $r' : b \longrightarrow a_1$ is therefore the unique one satisfying $\alpha r' = \beta$ and $pr' = r$. Thus the p-lift $\alpha : a_1 \longrightarrow a_2$ is indeed cartesian.
Consequently unique p-lifts exist and are cartesian for p if and only if $p_* \iota_q : I_q \longrightarrow B|p$ is both bijective on objects and fully faithful; an isomorphism of categories.
In a dual manner we can show that $q^* \iota_p$ is an isomorphism if and only unique q-lifts exist and are opcartesian for q. ☐

Corollary 3.41. Let $\mathcal{C}$ be a representable 2-category.

1. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a 2-functor which preserves cotensors with **2** and pullbacks. Then F preserves two sided discrete fibrations. In particular any morphism of Rep preserves two sided discrete fibrations.

2. Consider a family $\{F_i\}_{i \in I} : C \longrightarrow D$ of 2-functors which preserve cotensors with **2** and pullbacks. If they jointly reflect isomorphisms then they jointly reflect two sided discrete fibrations.

Proof.

1. Consider the span $(p, q) : B \longrightarrow C$ in $\mathcal{C}$ of Proposition 3.40. Since $\mathcal{C}$ is representable it has sufficient limits (comma objects and pullbacks) to construct the morphisms $p_* \iota_q : I_q \longrightarrow B|p$ and $q^* \iota_p : I_p \longrightarrow q|C$. Now F preserves pullbacks and cotensors with **2** and so comma objects by Proposition 3.35. Therefore F preserves the construction of the two maps in question. In other words their respective images under F are the maps $(Fp)_* \iota_{(Fq)} : I_{(Fq)} \longrightarrow FB|Fp$ and $(Fq)^* \iota_{(Fp)} : I_{(Fp)} \longrightarrow Fq|FC$ corresponding to the span (Fp, Fq) in $\mathcal{D}$. Since any 2-functor preserves isomorphisms both of these maps are isomorphisms. Therefore by Proposition 3.40 the span (Fp, Fq) is a 2-sided discrete fibration in $\mathcal{D}$.

2. By the first part of the proposition each $F_i : \mathcal{C} \longrightarrow \mathcal{D}$ preserves two sided discrete fibrations. As these are characterised by certain maps being isomorphisms any such family jointly reflects two sided discrete fibrations.

☐

Definition 3.42. Let $\mathcal{C}$ be a representable 2-category. Recall that $Kat(\mathcal{C})$ is the full sub 2-category of $Cat(\mathcal{UC})$ whose objects are the cateads in $\mathcal{C}$. We denote the inclusion by $j_{\mathcal{C}} : Kat(\mathcal{C}) \longrightarrow Cat(\mathcal{UC})$.

Example 3.43. Any category $\mathcal{E}$ may be viewed as a locally discrete 2-category. As all 2-cells are identities any span in $\mathcal{E}$ is trivially a two sided discrete fibration. Consequently a catead in $\mathcal{E}$ is just an internal category in $\mathcal{E}$ so that $Kat(\mathcal{E}) = Cat(\mathcal{E})$.

Proposition 3.44. Let $\mathcal{C}$ be a representable 2-category. Then $Kat(\mathcal{C})$ is a representable 2-category and the inclusion $j_c : Kat(\mathcal{C}) \longrightarrow Cat(\mathcal{UC})$ a morphism of Rep. In other words $Kat(\mathcal{C})$ is closed in $Cat(\mathcal{UC})$ under pullbacks and cotensors with **2**.

Proof. Firstly consider the case of pullbacks. Consider a pullback diagram in $Cat(\mathcal{U}\mathcal{C})$:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

with each of X, Y and Z cateads. The internal category P is a catead precisely if its domain and codomain maps $d_p, c_p : P_1 \rightrightarrows P_0$ form a two sided discrete fibration in $\mathcal{C}$. As pullbacks in $Cat(\mathcal{U}\mathcal{C})$ are pointwise in $\mathcal{C}$ it suffices therefore to verify that two sided discrete fibrations commute with pullbacks in $\mathcal{C}$. As $\mathcal{C}$ is a representable 2-category, two sided discrete fibrations in $\mathcal{C}$ are characterised in terms of limits (Proposition 3.40). As limits commute with limits in any 2-category, it follows that pullbacks commute with two sided discrete fibrations in $\mathcal{C}$ and therefore P is a catead if each of X, Y and Z are cateads. Consequently $Kat(\mathcal{C})$ is closed in $Cat(\mathcal{U}\mathcal{C})$ under pullbacks.

Consider a catead $X \in Kat(\mathcal{C})$ and its cotensor with $\mathbf{2}$, $X^{\mathbf{2}}$ in $Cat(\mathcal{U}\mathcal{C})$. We must show that $X^{\mathbf{2}}$ is a catead, which is to say, that the domain and codomain maps $d_{\mathbf{2}}, c_{\mathbf{2}} : X_1^{\mathbf{2}} \rightrightarrows X_0^{\mathbf{2}}$ form a two sided discrete fibration in $\mathcal{C}$. This is, by definition, the case if for each $A \in \mathcal{C}$ the functors $\mathcal{C}(A, d_{\mathbf{2}}), \mathcal{C}(A, c_{\mathbf{2}}) : \mathcal{C}(A, X_1^{\mathbf{2}}) \rightrightarrows \mathcal{C}(A, X_0^{\mathbf{2}})$ form a two sided discrete fibration in Cat. Now the representable $\mathcal{C}(A, -) : \mathcal{C} \longrightarrow \text{Cat}$ is a morphism of Rep so that we have the morphism of Rep: $Cat(\mathcal{U}\mathcal{C}(A, -)) : Cat(\mathcal{U}\mathcal{C}) \longrightarrow Cat(\mathcal{U}\text{Cat})$ and the above pair of functors are precisely the domain and codomain maps of the internal category in Cat, $Cat(\mathcal{U}\mathcal{C}(A, -))X^{\mathbf{2}}$ which we write as $Cat(\mathcal{U}\mathcal{C}(A, X^{\mathbf{2}}))$. Since $Cat(\mathcal{U}\mathcal{C}(A, -)) : Cat(\mathcal{U}\mathcal{C}) \longrightarrow Cat(\mathcal{U}\text{Cat})$ acts as $\mathcal{C}(A, -)$ pointwise, it takes cateads in $\mathcal{C}$ to cateads in Cat, so that $Cat(\mathcal{U}\mathcal{C}(A, X))$ is a catead in Cat. Furthermore, being a morphism of Rep it preserves cotensors with $\mathbf{2}$ so that $Cat(\mathcal{U}\mathcal{C}(A, X^{\mathbf{2}})) = Cat(\mathcal{U}\mathcal{C}(A, X))^{\mathbf{2}}$. Consequently it suffices to verify that $Kat(\text{Cat})$ is closed in $Cat(\mathcal{U}\text{Cat})$ under cotensors with $\mathbf{2}$.

In the case of Cat each catead is a higher kernel by Proposition 2.83. Suppose then that X is the higher kernel:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description]
\end{tikzcd}$$

of some functor $f : A \longrightarrow B$ in Cat and let the 2-cell:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow \eta"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}$$

be the one exhibiting $f|f$ as the comma object of f. The universal property of the arrow category $B^{\mathbf{2}}$ uniquely induces a functor $k : f|f \longrightarrow B^{\mathbf{2}}$ defined on objects of $f|f$ by $k(a, \alpha : fa \longrightarrow fb, b) = (\alpha : fa \longrightarrow fb)$. Using the explicit construction of cotensors with $\mathbf{2}$ in $Cat(\mathcal{U}\text{Cat})$ described in Proposition 3.19 it is straightforward to check that $X^{\mathbf{2}}$ is the higher kernel of $k : f|f \longrightarrow B^{\mathbf{2}}$, and therefore a catead. Therefore $Kat(\mathcal{C})$ is closed in $Cat(\mathcal{U}\mathcal{C})$ under cotensors with $\mathbf{2}$. □

Proposition 3.45.

1. We have a 2-functor $Kat(-) : \text{Rep} \longrightarrow \text{Rep}$ defined in such a manner that the inclusions $j_{\mathcal{C}} : Kat(\mathcal{C}) \longrightarrow Cat(\mathcal{U}\mathcal{C})$ are the components of a 2-natural transformation $j : Kat(-) \Longrightarrow Cat(\mathcal{U}-)$.

2. Each 2-functor $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ takes its image in $Kat(\mathcal{C})$. Let $\hat{\Delta}_{\mathcal{C}} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ denote the resulting factored 2-functor, so that we have $j_{\mathcal{C}}\hat{\Delta}_{\mathcal{C}} = \Delta'_{\mathcal{C}}$. The components $\hat{\Delta}$ become the arrow components of a pseudonatural transformation $\hat{\Delta} : 1_{\text{Rep}} \Longrightarrow Kat(-)$ so that we have a factorisation of the pseudonatural transformation Δ':

$$\begin{tikzcd}
1_{\text{Rep}} \arrow[r, Rightarrow, "\hat{\Delta}"] \arrow[dr, Rightarrow, "\Delta'"'] & Kat(-) \arrow[d, Rightarrow, "j"] \\
 & Cat(\mathcal{U}-)
\end{tikzcd}$$

Proof. 1. Any morphism of Rep preserves discrete fibrations by Corollary 3.41(1); thus given $F : \mathcal{C} \longrightarrow \mathcal{D}$ of Rep the 2-functor $Cat(\mathcal{U}F) : Cat(\mathcal{U}\mathcal{C}) \longrightarrow Cat(\mathcal{U}\mathcal{D})$ restricts to a 2-functor $Kat(F) : Kat(\mathcal{C}) \longrightarrow Kat(\mathcal{D})$ so that we have a commuting square:

$$\begin{tikzcd}
Kat(\mathcal{C}) \arrow[r, "j_{\mathcal{C}}"] \arrow[d, "Kat(F)"'] & Cat(\mathcal{U}\mathcal{C}) \arrow[d, "Cat(\mathcal{U}F)"] \\
Kat(\mathcal{D}) \arrow[r, "j_{\mathcal{D}}"'] & Cat(\mathcal{U}\mathcal{D})
\end{tikzcd}$$

As $Kat(\mathcal{C})$ is a full sub(2)category of $Cat(\mathcal{U}\mathcal{C})$ the extension of $Kat(-)$ to 2-cells of Rep is clear, as furthermore is the 2-naturality of the components j_c.

2. As observed in Example 3.10 each internal category in the image of $\Delta' : \mathcal{C} \longrightarrow Cat(\mathcal{U}\mathcal{C})$ is a catead so that $\Delta'_{\mathcal{C}}$ factors through the full sub 2-category $Kat(\mathcal{C})$ to give a 2-functor $\hat{\Delta}_{\mathcal{C}} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ as described in the statement of this proposition. This 2-functor is indeed a morphism of Rep since $Kat(\mathcal{C})$ is closed in $Cat(\mathcal{U}\mathcal{C})$ under cotensors with $\mathbf{2}$ and pullbacks by Proposition 3.44. As $Kat(\mathcal{C})$ is a full sub 2-category of $Cat(\mathcal{U}\mathcal{C})$ the pseudonaturality components evidently restrict too, and so $\hat{\Delta}$ becomes a pseudonatural transformation with $j\hat{\Delta} = \Delta'$.

□

Lemma 3.46. Let $\mathcal{C}$ be a representable 2-category and consider $f : X \longrightarrow \hat{\Delta}(A) \in Kat(\mathcal{C})$. Consider the diagram below:

$$\begin{tikzcd}
X_1 \arrow[rr, "d_x"] \arrow[dr, "f_1"] \arrow[dd, "c_x"'] & & X_0 \arrow[d, "f_0"] \\
& A^{\mathbf{2}} \arrow[r, "d_a"] \arrow[d, "c_a"'] & A \arrow[d, "1"] \arrow[dl, Rightarrow, "\eta_a"'] \\
X_0 \arrow[r, "f_0"'] & A \arrow[r, "1"'] & A
\end{tikzcd}$$

The following are equivalent.

1. f is fully faithful.
2. The comma cone $(X_1, d_x, \eta_a f_1, c_x)$ exhibits X_1 as the comma object of f_0.
3. The comma cone $(X_1, d_x, \eta_a f_1, c_x)$ exhibits X as the higher kernel of f_0.

Proof. ($1 \iff 2$) As $Kat(\mathcal{C})$ is a full sub 2-category of $Cat(\mathcal{U}\mathcal{C})$ f is fully faithful in $Kat(\mathcal{C})$ if and only if it is fully faithful in $Cat(\mathcal{U}\mathcal{C})$. Suppose then that f is fully faithful. By Proposition 2.59 f is fully faithful precisely if the triple of maps (d_x, f_1, c_x) exhibits X_1 as the limit of the double opspan in the centre of the diagram (as described in Definition 2.55). But the limit of that diagram is equally the triple pullback of Proposition 3.35 and thus the comma cone $(X_1, d_x, \eta_a f_1, c_x)$ exhibits X_1 as the comma object of f_0.
Conversely suppose that X_1 is the comma object. We must check that the triple (d_x, f_1, c_x) exhibits X_1 as the limit of the double opspan. Given then an object B and a triple of maps $(r : B \longrightarrow X_0, s : B \longrightarrow A^{\mathbf{2}}, t : B \longrightarrow X_0)$ constituting a cone to the double opspan we must factor them uniquely through X_1. This triple corresponds uniquely to the triple $(r, \eta_a s : fr \Longrightarrow ft, t)$ by the universal property of $A^{\mathbf{2}}$. By the universal property of the comma object we then obtain a unique arrow $h : B \longrightarrow X_1$ such that $d_x h = r$, $\eta_a f_1 h = \eta_a s$ and $c_x h = t$. By the universal property of $A^{\mathbf{2}}$ it follows then that $f_1 h = s$ so that we have the required factorisation. If another factorisation h_2 existed then we would have $d_x h_2 = d_x h, c_x h_2 = c_x h$ and $\eta_a f_1 h = \eta_a f_1 h_2$ but then we would have $h = h_2$ by the universal property of the comma object. Therefore the factorisation h is the unique such. The two dimensional universal property of the limit is straightforward to verify.
($2 \iff 3$) By definition (3) implies (2) so that it suffices now to show, assuming (2), that X is the higher

kernel of f_0. It remains to verify that the structure maps m_x and i_x for X are those specified in the construction of the higher kernel. In particular we must show that $(\eta_a f_1)i_x = 1_{f_0}$. By definition of i_a we have $\eta_a i_a = 1$. Therefore $(\eta_a f_1)i_x = \eta_a i_a f_0 = 1_{f_0}$. Similar reason shows that m_x is the correct structure map. Thus (2) implies (3). $\square$

Proposition 3.47. Let $\mathcal{C}$ be a representable 2-category. The 2-functor $\hat{\Delta} : \mathcal{C} \longrightarrow Kat(\mathcal{C})$ has a left 2-adjoint if and only if $\mathcal{C}$ has codescent objects of cateads. Furthermore cateads are effective in $\mathcal{C}$ precisely when the unit of the adjunction is pointwise fully faithful.

Proof. The 2-functor $\hat{\Delta}$ is a morphism of Rep. In particular it preserve cotensors with **2**. By Proposition 3.1 of [8] it consequently has a left 2-adjoint if and only if its underlying functor has a left adjoint. We saw in Corollary 3.11 that its underlying functor has a left adjoint if and only if $\mathcal{C}$ has codescent objects of cateads. This completes the first part of the result. Suppose then that we have a 2-adjunction:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift right=2, "\hat{\Delta}"'] \arrow[r, phantom, "\perp"] & Kat(\mathcal{C}) \arrow[l, shift right=2, "Q"']
\end{tikzcd}$$

Given a catead X consider the unit at X, an internal functor: $f : X \longrightarrow \hat{\Delta}Q(X)$:

$$\begin{tikzcd}
 & X_2 \arrow[r, "f_2"] \arrow[d, shift right=2, "p_x"'] \arrow[d, "m_x" description] \arrow[d, shift left=2, "q_x"] & QX^{\mathbf{3}} \arrow[d, shift right=2, "p_{QX}"'] \arrow[d] \arrow[d, shift left=2, "q_{QX}"] \\
(1) & X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d_x"'] \arrow[d, shift left=2, "c_x"] & QX^{\mathbf{2}} \arrow[d, shift right=2, "d_{QX}"'] \arrow[d, shift left=2, "c_{QX}"] \\
 & X_0 \arrow[r, "f_0"] \arrow[u, "i_x" description] & QX \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X_0 \arrow[rd, "f_0"] & \\
(2) \; X_1 \arrow[rru, bend left, "d_x"] \arrow[rrd, bend right, "c_x"'] \arrow[r, "f_1"'] & QX^{\mathbf{2}} \arrow[rr, bend left, "d_{QX}"] \arrow[rr, bend right, "c_{QX}"'] \arrow[rr, phantom, "\Downarrow \eta_{QX}"] & & QX \\
 & & X_0 \arrow[ru, "f_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_1 \arrow[rr, "d_x"] \arrow[rd, "f_1"] \arrow[dd, "c_x"'] & & X_0 \arrow[d, "f_0"] \\
(3) & & QX^{\mathbf{2}} \arrow[r, "d_{QX}"] \arrow[d, "c_{QX}"'] & QX \arrow[d, "1"] \arrow[ld, Rightarrow, "\eta_{QX}"'] \\
 & X_0 \arrow[r, "f_0"'] & QX \arrow[r, "1"'] & QX
\end{tikzcd}$$

Then QX is the codescent object of X with codescent morphism $f_0 : X_0 \longrightarrow QX$ and exhibiting 2-cell as in diagram (2). To say that cateads are effective is to say that for each X the comma cone of diagram (2) exhibits X_1 as the comma object of the codescent morphism $f_0 : X_0 \longrightarrow QX$. The third diagram equals the second and so it follows from Lemma 3.46 that this is the case precisely if f is fully faithful. $\square$

Notation 3.48. We will abbreviate the 2-functors $Kat(-) : \text{Rep} : \longrightarrow \text{Rep}$ and $Cat(-) : \text{Cat}_{\text{pb}} \longrightarrow \text{Rep}$ as $K : \text{Rep} \longrightarrow \text{Rep}$ and $C : \text{Cat}_{\text{pb}} \longrightarrow \text{Rep}$ where expedient.

Remark 3.49. In Proposition 3.45 we defined $K : \text{Rep} \longrightarrow \text{Rep}$ by factoring $\Delta' : 1_{\text{Rep}} \Longrightarrow C\mathcal{U}$ through its image; thus obtaining a 2-functor K and a factorisation of Δ' as:

$$1_{\text{Rep}} \xRightarrow{\hat{\Delta}} K \xRightarrow{j} C\mathcal{U}$$

with j 2-natural. Since for each $\mathcal{A} \in \text{Rep}$ the inclusion $j_{\mathcal{A}} : K\mathcal{A} \longrightarrow C\mathcal{U}\mathcal{A}$ exhibits $\mathcal{A}$ as closed under cotensors with **2** and pullbacks (by Proposition 3.44) we may define the component $\Delta'_{K\mathcal{A}} : K\mathcal{A} \longrightarrow C\mathcal{U}\mathcal{A}$ by restricting $\Delta'_{C\mathcal{U}\mathcal{A}} : C\mathcal{U}\mathcal{A} \longrightarrow C\mathcal{U}C\mathcal{U}\mathcal{A}$ so that we have a commutative diagram:

$$\begin{tikzcd}
K\mathcal{A} \arrow[r, "\Delta'_{\mathcal{A}}"] \arrow[d, "j_{\mathcal{A}}"'] \arrow[rd, phantom, "\Downarrow \Delta'_{j_{\mathcal{A}}} = 1"] & C\mathcal{U}K\mathcal{A} \arrow[d, "C\mathcal{U}j_{\mathcal{A}}"] \\
C\mathcal{U}\mathcal{A} \arrow[r, "\Delta'_{C\mathcal{U}\mathcal{A}}"'] & C\mathcal{U}C\mathcal{U}\mathcal{A}
\end{tikzcd}$$

We suppose in what follows that this choice has been made. Upon this choice being made the following equalities are easily seen to hold, and are direct consequences of the equalities of Lemma 3.31.

Lemma 3.50. 1. For a representable 2-category $\mathcal{A}$ we have the equality:

$$K\mathcal{A} \xrightarrow{\hat{\Delta}_{K\mathcal{A}}} KK\mathcal{A} \xrightarrow{j_{K\mathcal{A}}} \mathcal{CU}K\mathcal{A} \xrightarrow{\mathcal{CU}j_{\mathcal{A}}} \mathcal{CUCUA} \xrightarrow{Cob_{\mathcal{UA}}} \mathcal{CUA} \quad = \quad K\mathcal{A} \xrightarrow{j_{\mathcal{A}}} \mathcal{CUA} \quad .$$

2. Consider $F : \mathcal{A} \longrightarrow \mathcal{B} \in \text{Rep}$. The composite 2-cells in Rep:

(1)
$$\begin{tikzcd}
K\mathcal{A} \arrow[r, "\hat{\Delta}_{K\mathcal{A}}"] \arrow[d, "KF"'] \arrow[dr, phantom, "\Downarrow\hat{\Delta}_{KF}"] & KK\mathcal{A} \arrow[d, "KKF"] & & & \\
K\mathcal{B} \arrow[r, "\hat{\Delta}_{K\mathcal{B}}"'] & KK\mathcal{B} \arrow[r, "j_{K\mathcal{B}}"'] & \mathcal{CU}K\mathcal{B} \arrow[r, "\mathcal{CU}j_{\mathcal{B}}"'] & \mathcal{CUCUB} \arrow[r, "Cob_{\mathcal{UB}}"'] & \mathcal{CUB}
\end{tikzcd}$$

and

(2)
$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] \arrow[d, "\hat{\Delta}_{\mathcal{A}}"'] \arrow[dr, phantom, "\Downarrow\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}}"] & K\mathcal{A} \arrow[d, "K\hat{\Delta}_{\mathcal{A}}"] & & & \\
K\mathcal{A} \arrow[r, "\hat{\Delta}_{K\mathcal{A}}"'] & KK\mathcal{A} \arrow[r, "j_{K\mathcal{A}}"'] & \mathcal{CU}K\mathcal{A} \arrow[r, "\mathcal{CU}j_{\mathcal{A}}"'] & \mathcal{CUCUA} \arrow[r, "Cob_{\mathcal{UA}}"'] & \mathcal{CUA}
\end{tikzcd}$$

are identity 2-cells.

Proof. 1. We have a commuting diagram:

$$\begin{tikzcd}
K\mathcal{A} \arrow[r, "\hat{\Delta}_{K\mathcal{A}}"] \arrow[rr, bend right, "\Delta'_{K\mathcal{A}}"] \arrow[dr, "j_{\mathcal{A}}"'] & KK\mathcal{A} \arrow[r, "j_{K\mathcal{A}}"] & \mathcal{CU}K\mathcal{A} \arrow[r, "\mathcal{CU}j_{\mathcal{A}}"] & \mathcal{CUCUA} \arrow[r, "Cob_{\mathcal{UA}}"] & \mathcal{CUA} \\
& \mathcal{CUA} \arrow[urr, "\Delta'_{\mathcal{CUA}}"] \arrow[urrr, bend right, "1"'] & & &
\end{tikzcd}$$

the top left triangle commuting as $j\hat{\Delta} = \Delta'$. The centre square commutes by Remark 3.49. The rightmost triangle commutes by Lemma 3.31 Part 1.

2. We have $j \circ \hat{\Delta} = \Delta'$ so that the composite (1) equals the left composite below:

$$\begin{tikzcd}
K\mathcal{A} \arrow[r, "\Delta'_{K\mathcal{A}}"] \arrow[d, "KF"'] \arrow[dr, phantom, "\Downarrow\Delta'_{KF}"] & \mathcal{CU}K\mathcal{A} \arrow[r, "\mathcal{CU}j_{\mathcal{A}}"] \arrow[d, "\mathcal{CU}KF"] & \mathcal{CUCUA} \arrow[d, "\mathcal{CUCU}F"] & \\
K\mathcal{B} \arrow[r, "\Delta'_{K\mathcal{B}}"'] & \mathcal{CU}K\mathcal{B} \arrow[r, "\mathcal{CU}j_{\mathcal{B}}"'] & \mathcal{CUCUB} \arrow[r, "Cob_{\mathcal{UB}}"'] & \mathcal{CUB}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
K\mathcal{A} \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "KF"'] & \mathcal{CUA} \arrow[r, "\Delta'_{\mathcal{CUA}}"] \arrow[d, "\mathcal{CU}F"'] \arrow[dr, phantom, "\Downarrow\Delta'_{\mathcal{CU}F}"] & \mathcal{CUCUA} \arrow[d, "\mathcal{CUCU}F"] & \\
K\mathcal{B} \arrow[r, "j_{\mathcal{B}}"'] & \mathcal{CUB} \arrow[r, "\Delta'_{\mathcal{CUB}}"'] & \mathcal{CUCUB} \arrow[r, "Cob_{\mathcal{UB}}"'] & \mathcal{CUB}
\end{tikzcd}$$

Now consider the component of Δ' at the arrow $\mathcal{CU}F \circ j_{\mathcal{A}} = j_{\mathcal{B}} \circ KF : K\mathcal{A} \longrightarrow \mathcal{CUB}$. We have $\Delta'_{\mathcal{CU}F\circ j_{\mathcal{A}}} = \Delta'_{\mathcal{CU}F} j_{\mathcal{A}} \circ \mathcal{CUCU}F\Delta'_{j_{\mathcal{A}}} = \Delta'_{\mathcal{CU}F} j_{\mathcal{A}}$. The first equation is by pseudonaturality of Δ'. The second uses the fact that $\Delta'_{j_{\mathcal{A}}}$ is an identity 2-cell, as described in Remark 3.49. Similarly we have $\Delta'_{j_{\mathcal{B}}\circ KF} = \Delta'_{j_{\mathcal{B}}} KF \circ \mathcal{CU}j_{\mathcal{B}}\Delta'_{KF} = \mathcal{CU}j_{\mathcal{B}}\Delta'_{KF}$, this time using that $\Delta'_{j_{\mathcal{B}}}$ is an identity 2-cell. Therefore the composite 2-cell on the left above equals that on the right. Now $Cob_{\mathcal{UB}}\Delta'_{\mathcal{CU}F}$ is an identity 2-cell by Lemma 3.31 Part 2.

Using that $j \circ \hat{\Delta} = \Delta'$ we see that the composite (2) equals the left hand side below:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Delta'_{K\mathcal{A}}"] \arrow[d, "\hat{\Delta}_{\mathcal{A}}"'] \arrow[dr, phantom, "\Downarrow \Delta'_{\hat{\Delta}_{\mathcal{A}}}"] & CU\mathcal{A} \arrow[d, "CU\hat{\Delta}_{\mathcal{A}}"] & & \\
K\mathcal{A} \arrow[r, "\Delta'_{K\mathcal{A}}"'] \arrow[dr, "j_{\mathcal{A}}"'] & CUK\mathcal{A} \arrow[r, "CUj_{\mathcal{A}}"'] & CUCU\mathcal{A} \arrow[r, "Cob_{U\mathcal{A}}"'] & CU\mathcal{A} \\
 & CU\mathcal{A} \arrow[ur, "\Delta'_{CU\mathcal{A}}"'] & &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A} \arrow[r, "\Delta'_{\mathcal{A}}"] \arrow[d, "\Delta'_{\mathcal{A}}"'] \arrow[dr, phantom, "\Downarrow \Delta'_{\Delta'_{\mathcal{A}}}"] & CU\mathcal{A} \arrow[d, "CU\Delta'_{\mathcal{A}}"] & \\
CU\mathcal{A} \arrow[r, "\Delta'_{CU\mathcal{A}}"'] & CUCU\mathcal{A} \arrow[r, "Cob_{U\mathcal{A}}"'] & CU\mathcal{A}
\end{tikzcd}$$

Observe the lower square on the left hand side composite above. This is the component of Δ' at j_A which is an identity by Remark 3.49. Therefore we may compose these pseudonaturality components to obtain $\Delta'_{\Delta'_{\mathcal{A}}}$ since $\Delta'_{\mathcal{A}} = j_{\mathcal{A}} \circ \hat{\Delta}_{\mathcal{A}}$. Consequently the left hand side equals the right hand side. The right hand composite is an identity by Lemma 3.31 Part 2.

□

Remark 3.51. In Section 4 of this chapter we will construct the left adjoint of Proposition 3.47 above in the case $\mathcal{C} = Cat(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks and prove the unit is pointwise fully faithful as claimed. Since this will involve consideration of fully faithful morphisms in representable 2-categories, specifically $Kat(Cat(\mathcal{E}))$, it will be useful to have a finite limit characterisation of fully faithful morphisms in such 2-categories, which we provide in the following proposition and its corollary.

Proposition 3.52. Let $\mathcal{C}$ be a 2-category and $f : A \longrightarrow B$ an arrow of $\mathcal{C}$. Suppose that the comma object $f|f$ and cotensor with $\mathbf{2}$, $A^{\mathbf{2}}$ exist. Consider as below the universal cone for the comma object, and the universal 2-cell for $A^{\mathbf{2}}$ postcomposed with f:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[rr, phantom, "\Downarrow \eta"] & & B \\
 & A \arrow[ur, "f"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A^{\mathbf{2}} \arrow[r, bend left, "d_a"] \arrow[r, bend right, "c_a"'] \arrow[r, phantom, "\Downarrow \eta_a"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

By the universal property of $f|f$ there exists a unique 1-cell: $k : A^{\mathbf{2}} \longrightarrow f|f$ such that: $d_a = d \circ k$, $c_a = c \circ k$ and $\eta \circ k = f \circ \eta_a$.
The morphism $k : A^{\mathbf{2}} \longrightarrow f|f$ is an isomorphism if and only if $f : A \longrightarrow B$ is fully faithful.

Proof. The notion of fully faithful arrow is representable so that it suffices, as for the case of two sided discrete fibrations described in detail in Proposition 3.40, to verify the claim in the case of Cat.
In the case of Cat the morphism $k : A^{\mathbf{2}} \longrightarrow f|f$ acts on an object of $A^{\mathbf{2}}$ by $k(\alpha : a \longrightarrow b) = (a, f\alpha : fa \longrightarrow fb, b)$ and on an arrow by:

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "r"'] & b \arrow[d, "s"] & & (a, \arrow[d, "r"'] & fa \arrow[r, "f\alpha"] \arrow[d, "fr"] & fb, \arrow[d, "fs"'] & b) \arrow[d, "s"] \\
c \arrow[r, "\beta"'] & d \arrow[r, phantom, ""] & {} \arrow[u, phantom] & (c, & fc \arrow[r, "f\beta"] & fd, & d)
\end{tikzcd}
\qquad \overset{k}{\longmapsto}$$

Now k is clearly bijective on objects if and only if f is fully faithful. If k is an isomorphism it is certainly bijective on objects so that f is fully faithful. Conversely if f is fully faithful then k is bijective on objects. Certainly, as is clear from the above diagram, k is always faithful. Furthermore given a morphism of $f|f$ as on the right above we have $f(s \circ \alpha) = f\alpha \circ fs = f\beta \circ fr = f(\beta \circ r)$ so that by faithfulness of f the square on the left commutes and k is full. Thus k is an isomorphism. □

Corollary 3.53. Let $\mathcal{C}$ be a 2-category with comma objects.

1. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a comma object preserving 2-functor. Then F preserves fully faithful arrows. In particular every morphism of Rep preserves fully faithful arrows.

2. Consider a family $\{F_i\}_{i\in I} : \mathcal{C} \longrightarrow \mathcal{D}$ of comma object preserving 2-functors. If they jointly reflect isomorphisms then they jointly reflect fully faithful arrows.

Proof. The details here are essentially the same as for the case of two sided discrete fibrations described in more detail, in Corollary 3.41.

1. Having characterised fully faithful arrows in terms of comma objects (bearing in mind that cotensors with **2** are comma objects) it is clear that if $\mathcal{C}$ has comma objects the above characterisation applies, and furthermore that any comma object preserving 2-functor preserves fully faithfuls.
2. By the first part of the proposition each $F_i : \mathcal{C} \longrightarrow \mathcal{D}$ preserves fully faithful arrows. As fully faithful arrows are characterised by certain maps being isomorphisms any such family jointly reflects fully faithfulness.

□

3.4 Cateads effective in $Cat(\mathcal{E})$

In the preceding section we proved that $Kat(-) : \mathrm{Rep} \longrightarrow \mathrm{Rep}$ is a 2-functor and exhibited the pseudonatural transformation $\hat{\Delta} : 1 \Longrightarrow Kat$. The main work of this section is to prove that we have a 2-adjunction:

$$\begin{tikzcd} Cat(\mathcal{E}) \arrow[r, shift right=1.5ex, "\hat{\Delta}"'] \arrow[r, phantom, "\perp"] & Kat(Cat(\mathcal{E})) \arrow[l, shift right=1.5ex, "Q"'] \end{tikzcd}$$

whenever $\mathcal{E}$ is a category with pullbacks. The left 2-adjoint will be the composite:

$$\begin{tikzcd} Kat(Cat(\mathcal{E})) \arrow[r, "j_{Cat(\mathcal{E})}"] & Cat(\mathcal{U}Cat(\mathcal{E})) \arrow[r, "Cat(ob_{\mathcal{E}})"] & Cat(\mathcal{E}) \end{tikzcd}$$

It is clear, and will be justified again, that precomposing this 2-functor by $\hat{\Delta}$ gives the identity on $Cat(\mathcal{E})$; thus we need only construct the unit of the adjunction and verify the triangle equations. Essentially three steps are required to achieve this goal and we begin with a summary of those steps.

1. For each representable 2-category $\mathcal{A}$ we prove that $\mathcal{U}\hat{\Delta} : \mathcal{U}\mathcal{A} \longrightarrow \mathcal{U}K\mathcal{A}$, is the left adjoint of the objects functor, which assigns to a catead X its object of objects X_0 (Proposition 3.60).
2. This enables us to construct a modification:

$$\begin{tikzcd} K \arrow[r, bend left, "K\hat{\Delta}", ""{name=U, below}] \arrow[r, bend right, "\hat{\Delta}K"', ""{name=D, above}] & K^2 \arrow[from=U, to=D, Rightarrow, "\lambda"] \end{tikzcd}$$

   satisfying certain special properties (Proposition 3.63).

3. We use the modification to the construct the unit of the adjunction. We then apply a straightforward analogue (Proposition 3.58) of the most basic result of the theory of KZ-doctrines [32], Proposition 3.56, to deduce the triangle equations.

As the approach is based on an idea from the theory of KZ-doctrines we begin by describing what this is, thereafter proceeding in the order described.

Remark 3.54. There are several notions of KZ-doctrine, allowing varying degrees of pseudonaturality. There is the notion introduced by Kock [32] and the weaker KZ-doctrines of Marmolejo [43]. The present situation will not fit into either framework exactly, and though related to both, is most easily seen to be related to the KZ-doctrines of Kock. We begin by recalling the relevant definition of KZ-doctrine and the most basic result about them.

Definition 3.55. A KZ-doctrine [32] on a 2-category $\mathcal{C}$ consists of a 2-functor $T : \mathcal{C} \longrightarrow \mathcal{C}$, 2-natural transformations $y : 1 \Longrightarrow T$ and $m : T^2 \Longrightarrow T$, and a modification:

$$\begin{tikzcd} T \arrow[r, bend left, "Ty"] \arrow[r, bend right, "yT"'] \arrow[r, phantom, "\Downarrow\lambda"] & T^2 \end{tikzcd}$$

satisfying the following axioms:

1. $m \circ Ty = m \circ yT = 1$.
2. For each A of $\mathcal{C}$ the 2-cell:

$$\begin{tikzcd} A \arrow[r, "y_A"] & TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A \end{tikzcd} \quad \text{equals the identity 2-cell:} \quad \begin{tikzcd} A \arrow[r, "y_A"] \arrow[d, "y_A"'] & TA \arrow[d, "Ty_A"] \\ TA \arrow[r, "y_{TA}"'] & T^2A \end{tikzcd}$$

3. For each A of $\mathcal{C}$ the 2-cells:

$$\begin{tikzcd} TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A \arrow[r, "m_A"] & TA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_{TA}"] & T^2A \arrow[r, "Tm_A"] & TA \arrow[r, "m_A"] & A \end{tikzcd}$$

   are identities.

The basic result about KZ-doctrines is:

Proposition 3.56. For each A of $\mathcal{C}$ we have an adjunction:

$$\begin{tikzcd} TA \arrow[r, shift right, "y_{TA}"'] \arrow[r, phantom, "\perp"] & T^2A \arrow[l, shift right, "m_A"'] \end{tikzcd}$$

with identity counit.

Proof. See [32]. ☐

Remark 3.57. In the present case the 2-functor of interest is $K : \text{Rep} \longrightarrow \text{Rep}$. We have already described a pseudonatural transformation $\hat{\Delta} : 1 \Longrightarrow K$ but will not give a transformation $K^2 \Longrightarrow K$. We do however have, for each category $\mathcal{E}$ with pullbacks, the composite morphism of Rep:

$$\begin{tikzcd} KC\mathcal{E} \arrow[r, "j_{C\mathcal{E}}"] & C\mathcal{U}C\mathcal{E} \arrow[r, "Cob_{\mathcal{E}}"] & C\mathcal{E} \end{tikzcd}$$

which we will prove to be the left adjoint of $\hat{\Delta}_{C\mathcal{E}} : C\mathcal{E} \longrightarrow KC\mathcal{E}$. The appropriate context for proving that this is the left adjoint is described in the following proposition.

Proposition 3.58. Consider a 2-category $\mathcal{C}$, a 2-functor $T : \mathcal{C} \longrightarrow \mathcal{C}$ equipped with a pseudonatural transformation $y : 1 \Longrightarrow T$ and modification:

$$\begin{tikzcd} T \arrow[r, bend left, "Ty"] \arrow[r, bend right, "yT"'] \arrow[r, phantom, "\Downarrow\lambda"] & T^2 \end{tikzcd}$$

such that for each A of $\mathcal{C}$ we have:

$$\begin{tikzcd}
A \arrow[r, "y_A"] & TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "y_A"'] \arrow[dr, phantom, "y_{y_A}\Downarrow"] & TA \arrow[d, "Ty_A"] \\
TA \arrow[r, "y_{TA}"'] & T^2A
\end{tikzcd}$$

Consider a morphism $k : TA \longrightarrow A$ which satisfies the following equations:

1. $k \circ y_A$ is the identity on A.
2. The 2-cell:

$$\begin{tikzcd}
TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] & T^2A \arrow[r, "Tk"] & TA \arrow[r, "k"] & A
\end{tikzcd}$$

   is an identity.

3. The 2-cell:

$$\begin{tikzcd}
TA \arrow[r, "y_{TA}"] \arrow[d, "k"'] \arrow[dr, phantom, "y_k\Downarrow"] & T^2A \arrow[d, "Tk"] & \\
A \arrow[r, "y_A"'] & TA \arrow[r, "k"'] & A
\end{tikzcd}$$

   is an identity.

Then we have an adjunction

$$\begin{tikzcd}
A \arrow[r, shift right=2, "y_A"'] \arrow[r, phantom, "\perp"] & TA \arrow[l, shift right=2, "k"']
\end{tikzcd}$$

with identity counit, and unit given by:

$$\begin{tikzcd}
TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] \arrow[rr, bend left=60, "1"] \arrow[dr, "k"'] & T^2A \arrow[r, "Tk"'] \arrow[d, phantom, "\Downarrow y_k"] & TA \\
 & A \arrow[ur, bend right, "y_A"'] &
\end{tikzcd}$$

Proof. By assumption (1) we have $k \circ y_A = 1_A$ and so take the counit to be the identity. In order to verify the triangle equations for the adjunction it now suffices to show that the above 2-cell, the proposed unit, becomes an identity 2-cell upon precomposition with $y_A : A \longrightarrow TA$ and postcomposition with $k : TA \longrightarrow A$. Consider firstly the case of postcomposition with $k : TA \longrightarrow A$. This gives the 2-cell:

$$\begin{tikzcd}
TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] \arrow[rr, bend left=60, "1"] \arrow[dr, "k"'] & T^2A \arrow[r, "Tk"'] \arrow[d, phantom, "\Downarrow y_k"] & TA \arrow[r, "k"'] & A \\
 & A \arrow[ur, bend right, "y_A"'] & &
\end{tikzcd}$$

which is clearly an identity since $k \circ y_k$ is an identity 2-cell by (3) and $k \circ Tk \circ \lambda_A$ is an identity 2-cell by (2). Precomposition of the unit with y_A yields

$$\begin{tikzcd}
A \arrow[r, "y_A"'] & TA \arrow[r, bend left, "Ty_A"] \arrow[r, bend right, "y_{TA}"'] \arrow[r, phantom, "\Downarrow\lambda_A"] \arrow[rr, bend left=70, "1"] \arrow[dr, "k"'] & T^2A \arrow[r, "Tk"'] \arrow[d, phantom, "\Downarrow y_k"] & TA \\
& & A \arrow[ur, bend right, "y_A"'] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "y_A"'] \arrow[dr, phantom, "y_{y_A}\Downarrow"] & TA \arrow[d, "Ty_A"] \\
TA \arrow[r, "y_{TA}"'] \arrow[d, "k"'] \arrow[dr, phantom, "y_k\Downarrow"] & T^2A \arrow[d, "Tk"] \\
A \arrow[r, "y_A"'] & TA
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "ky_A"'] \arrow[dr, phantom, "y_{ky_A}\Downarrow"] & TA \arrow[d, "T(ky_A)"] \\
A \arrow[r, "y_A"'] & TA
\end{tikzcd}$$

the first equality holding since $\lambda_A \circ y_A = y_{y_A}$ by assumption. The second equation holds as y is a pseudonatural transformation. As $k \circ y_A$ is the identity on A, we have that $y_{ky_A} = y_{1_A}$. This is an identity as y is pseudonatural; thus the final 2-cell above is an identity. ☐

Remark 3.59. In order to apply Proposition 3.58 we must construct a suitable modification:

$$\begin{tikzcd}
K \arrow[r, bend left, "K\hat{\Delta}"] \arrow[r, bend right, "\hat{\Delta}K"'] \arrow[r, phantom, "\Downarrow\lambda"] & K^2
\end{tikzcd}$$

The existence of such a modification will follow from the following result of [10].

Proposition 3.60 (Bourn, Penon)**.** 1. Given a 2-category $\mathcal{A} \in$ Rep the functor underlying $\hat{\Delta}$, $\mathcal{U}\hat{\Delta} : \mathcal{UA} \longrightarrow \mathcal{UKA}$, is the left adjoint of the objects functor, the composite:

$$\mathcal{UKA} \xrightarrow{\mathcal{U}j_{\mathcal{A}}} \mathcal{UCUA} \xrightarrow{ob_{\mathcal{UA}}} \mathcal{UA} \quad .$$

Furthermore the adjunction lies in $\mathrm{Cat}_{\mathrm{pb}}$.

2. Given $\mathcal{A}$ and $\mathcal{B}$ of Rep denote the respective counits of the adjunction of the previous part by $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$. Given a 2-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Rep we have the equality:

$$\begin{tikzcd}
& \mathcal{UA} \arrow[dr, "\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] \arrow[d, phantom, "\Downarrow\epsilon_{\mathcal{A}}"] & \\
\mathcal{UKA} \arrow[ur, "ob_{\mathcal{UA}}\mathcal{U}j_{\mathcal{A}}"] \arrow[rr, "1"'] \arrow[d, "\mathcal{U}KF"'] & & \mathcal{UKA} \arrow[d, "\mathcal{U}KF"] \\
\mathcal{UKB} \arrow[rr, "1"] & & \mathcal{UKB}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{UA} \arrow[dr, "\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] \arrow[dd, "\mathcal{U}F"] & \\
\mathcal{UKA} \arrow[ur, "ob_{\mathcal{UA}}\mathcal{U}j_{\mathcal{A}}"] \arrow[dd, "\mathcal{U}KF"'] & \mathcal{U}\hat{\Delta}_F\Downarrow & \mathcal{UKA} \arrow[dd, "\mathcal{U}KF"] \\
& \mathcal{UB} \arrow[dr, "\mathcal{U}\hat{\Delta}_{\mathcal{B}}"] \arrow[d, phantom, "\Downarrow\epsilon_{\mathcal{B}}"] & \\
\mathcal{UKB} \arrow[ur, "ob_{\mathcal{UB}}\mathcal{U}j_{\mathcal{B}}"] \arrow[rr, "1"'] & & \mathcal{UKB}
\end{tikzcd}$$

Remark 3.61. Before proving this proposition we observe that it is a generalisation of a better known adjunction. As described in Example 3.43 if we view a category $\mathcal{E}$ as a locally discrete 2-category then $Kat(\mathcal{E}) = Cat(\mathcal{E})$. If $\mathcal{E}$ has pullbacks then it becomes a representable 2-category: we have $A^{\mathbf{2}} = A$ for $A \in \mathcal{E}$ since all 2-cells are identites. Then $\mathcal{U}\Delta' : \mathcal{E} \longrightarrow \mathcal{U}Kat(\mathcal{E}) = \mathcal{U}Cat(\mathcal{E})$ becomes the functor which assigns to an object $A \in \mathcal{E}$ the canonical "discrete internal category" upon it:

$$\begin{tikzcd}
A \arrow[r, shift left=2, "1"] \arrow[r, "1" description] \arrow[r, shift right=2, "1"'] & A \arrow[r, shift left=2, "1"] \arrow[r, leftarrow, "1" description] \arrow[r, shift right=2, "1"'] & A
\end{tikzcd}$$

The right adjoint proposed in Proposition 3.60 is just the objects functor: $ob_{\mathcal{E}} : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$. In this case the adjunction is well known and easily verified.

Proof. 1. For the purposes of the first part of this proposition only, we denote the composite $ob_{\mathcal{U}\mathcal{A}}\mathcal{U}j_{\mathcal{A}} : \mathcal{U}K\mathcal{A} \longrightarrow \mathcal{U}\mathcal{A}$ simply by $ob : \mathcal{U}K\mathcal{A} \longrightarrow \mathcal{U}\mathcal{A}$. Given a catead X we have $ob(X) = X_0$ and given $f : X \longrightarrow Y$ of $\mathcal{U}K\mathcal{A}$ we have $ob(f) = f_0 : X_0 \longrightarrow Y_0$. Given $f : A \longrightarrow B$ of $\mathcal{A}$ we have $\hat{\Delta}(f)_0 = f$; thus we certainly have $1_{\mathcal{U}\mathcal{A}} = ob \circ \mathcal{U}\hat{\Delta}$. We therefore take the unit of the adjunction to be the identity, so that it suffices to describe the counit $\epsilon : \mathcal{U}\hat{\Delta} \circ ob \Longrightarrow 1_{\mathcal{U}K\mathcal{A}}$ and verify the triangle equations. As the unit is the identity these reduce to $ob \circ \epsilon = 1$ and $\epsilon_{\hat{\Delta}} = 1$.

Given a catead X we need to describe an internal functor $(\epsilon_x) : \hat{\Delta}(X_0) \longrightarrow X$ of $\mathcal{U}Kat(\mathcal{A})$. Since $\hat{\Delta}(X_0)_0 = X_0$ we take this to be the identity on objects: $(\epsilon_x)_0 = 1_{X_0}$. Thus we will describe an identity on objects internal functor:

$$\begin{tikzcd}
(X_0)^{\mathbf{3}} \arrow[r, "(\epsilon_X)_2"] \arrow[d, shift right=2, "p"'] \arrow[d, "m"] \arrow[d, shift left=2, "q"] & X_2 \arrow[d, shift right=2, "p_x"'] \arrow[d, "m_x"] \arrow[d, shift left=2, "q_x"] \\
(X_0)^{\mathbf{2}} \arrow[r, "(\epsilon_X)_1 = \epsilon_x"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & X_1 \arrow[d, shift right=2, "d_x"'] \arrow[d, shift left=2, "c_x"] \\
X_0 \arrow[r, "1"] \arrow[u, "i"] & X_0 \arrow[u, "i_x"]
\end{tikzcd}$$

where we simply write $p, q, \ldots$ for the defining arrows of the catead $\hat{\Delta}(X_0)$ to avoid confusion. To give the data for an internal functor it then remains to define the arrow map $(\epsilon_X)_1 = \epsilon_x$. In order to construct this map we consider the universal 2-cell $\eta : d \Longrightarrow c$ exhibiting $(X_0)^{\mathbf{2}}$ as the cotensor of X_0 with $\mathbf{2}$. This equals the composite:

$$\begin{tikzcd}
(X_0)^{\mathbf{2}} \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\eta"] \arrow[drr, bend right, "i_x c"'] & X_0 \arrow[rr, "1"] \arrow[dr, "i_x"] & & X_0 \\
& & X_1 \arrow[ur, "d_x"'] &
\end{tikzcd}$$

Using the lifting property of the fibration d_x there exists a unique d_x-lift (ϵ_x, θ):

$$\begin{tikzcd}
(X_0)^{\mathbf{2}} \arrow[r, bend left, "\epsilon_x"] \arrow[r, bend right, "i_x c"'] \arrow[r, phantom, "\Downarrow\theta"] & X_1
\end{tikzcd}$$

such that postcomposition with d_x yields the original 2-cell η and such that postcomposition with c_x gives an identity 2-cell. In particular we then have $d_x \circ \epsilon_x = d$ and $c_x \circ \epsilon_x = c_x \circ i_x \circ c = c$. In the diagram of the internal functor above we have thus constructed the 1-cell ϵ_x and shown that we have a morphism of the underlying graphs.

We will show that we have a morphism of reflexive graphs by showing the 2-cell $\theta \circ i$ is an identity; its domain and codomain $\epsilon_x \circ i$ and $i_x \circ c \circ i = i_x$ will then agree. In order to verify that $\theta \circ i$ is an identity it suffices to show that it becomes one upon postcomposition with each of d_x and c_x since two sided discrete fibrations reflect identities (Proposition 2.76). Now certainly postcomposition with c_x gives an identity since $c_x \circ \theta$ is an identity 2-cell. Postcomposition with d_x gives $d_x \circ \theta \circ i = \eta \circ i$ which is an identity by definition of i. Therefore we have a reflexive graph morphism.

The morphism of graphs induces the canonical arrow $(\epsilon_x p, \epsilon_x q) : (X_0)^3 \longrightarrow X_2$ into the pullback X_2: the unique one which becomes $\epsilon_x p$ and $\epsilon_x q$ upon postcomposition with p_x and q_x respectively so that this of course must equal $(\epsilon_x)_2$. It remains to show that the square:

$$\begin{tikzcd}
(X_0)^3 \arrow[r, "{(\epsilon_x p, \epsilon_x q)}"] \arrow[d, "m"'] & X_2 \arrow[d, "m_x"] \\
(X_0)^{\mathbf{2}} \arrow[r, "\epsilon_x"'] & X_1
\end{tikzcd}$$

commutes. We will show they are both equally the unique lifting of a certain 2-cell along the 2-sided discrete fibration. We now go about constructing the requisite 2-cells to witness the claim. Consider the two cells:

$$(1)\quad \begin{tikzcd}
(X_0)^3 \arrow[r, "p"] \arrow[d, "q"'] & (X_0)^{\mathbf{2}} \arrow[r, "\epsilon_x"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\theta"] & X_1 \\
(X_0)^{\mathbf{2}} \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\eta"] & X_0 \arrow[ur, bend right, "i_x"'] &
\end{tikzcd} \quad \text{and} \quad (2)\quad \begin{tikzcd}
(X_0)^3 \arrow[r, "q"] & (X_0)^{\mathbf{2}} \arrow[rr, "\epsilon_x"] \arrow[dr, "c"'] & \Downarrow\theta & X_1 \\
& & X_0 \arrow[ur, "i_x"'] &
\end{tikzcd}$$

Postcomposing the left 2-cell with $c_x : X_1 \longrightarrow X_0$ yields $\eta \circ q$ since $c_x \circ \theta$ is an identity 2-cell. Postcomposing the right 2-cell with $d_x : X_1 \longrightarrow X_0$ equally gives $\eta \circ q$, since $d_x \circ \theta = \eta$. By the 2-dimensional aspect of the universal property of the pullback:

$$\begin{tikzcd}
X_2 \arrow[r, "p"] \arrow[d, "q"'] & X_1 \arrow[d, "c"] \\
X_1 \arrow[r, "d"'] & X_0
\end{tikzcd}$$

we obtain a unique 2-cell:

$$\begin{tikzcd}
X_0^3 \arrow[r, bend left, "{(\epsilon_x p, \epsilon_x q)}"] \arrow[r, bend right, "{(i_x cq, i_x cq)}"'] \arrow[r, phantom, "\Downarrow\phi"] & X_2
\end{tikzcd}$$

such that postcomposition with p_x and q_x respectively give the left and right 2-cells (1) and (2) above. Now we have a pair of 2-cells with common codomain:

$$(3)\quad \begin{tikzcd}
X_0^3 \arrow[r, bend left, "{(\epsilon_x p, \epsilon_x q)}"] \arrow[r, bend right, "{(i_x cq, i_x cq)}"'] \arrow[r, phantom, "\Downarrow\phi"] \arrow[d, "q"'] & X_2 \arrow[d, "m_x"] \\
X_0^{\mathbf{2}} \arrow[r, "i_x c"'] & X_1
\end{tikzcd} \quad \text{and} \quad (4)\quad \begin{tikzcd}
(X_0)^3 \arrow[r, "m"] \arrow[d, "q"'] & (X_0)^{\mathbf{2}} \arrow[r, "\epsilon_x"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\theta"] & X_1 \\
(X_0)^{\mathbf{2}} \arrow[r, "c"'] & X_0 \arrow[ur, bend right, "i_x"'] &
\end{tikzcd}$$

where the commutativity of the square on the left hand side follows from the identity axiom for X to be an internal category. Our goal is to show that the 1-cells which form the domains of the 2-cells (3) and (4) agree. To do so it will suffice to show that these 2-cells agree upon postcomposition with d_x and become identity 2-cells upon postcomposition with c_x, by virtue of the fact that this pair form a two-sided discrete fibration. We see that upon postcomposition with d_x (3) becomes $d_x \circ m_x \circ \phi = d_x \circ p_x \circ \phi$. By definition of ϕ this equals the left composite below:

$$\begin{tikzcd}
(X_0)^3 \arrow[r, "p"] \arrow[d, "q"'] & (X_0)^{\mathbf{2}} \arrow[r, "\epsilon_x"] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\theta"] & X_1 \arrow[d, bend left, "d_x"] \\
(X_0)^{\mathbf{2}} \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\eta"] & X_0 \arrow[ur, bend right, "i_x"'] \arrow[r, "1"'] & X_0
\end{tikzcd} \quad \text{which equals} \quad \begin{tikzcd}
(X_0)^3 \arrow[r, "p"] \arrow[d, "q"'] & (X_0)^{\mathbf{2}} \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] \arrow[d, phantom, "\overset{\Leftarrow}{\eta}"] \\
(X_0)^{\mathbf{2}} \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\eta"] & X_0
\end{tikzcd}$$

the latter equality holding because $d_x \circ \epsilon_x = d$ and $d_x \circ \theta = \eta$. On the other hand postcomposing (4) with d_x gives $d_x \circ \theta \circ m = \eta \circ m$ which equals the composite above. Thus (3) and (4) agree upon postcomposition with d_x.

Postcomposing (3) with c_x gives $c_x \circ m_x \circ \phi = c_x \circ q_x \circ \phi$ which equals (2) postcomposed with c_x. This is an identity as $c_x \circ \theta$ is an identity. Postcomposing (4) with c_x again gives an identity for the same reason. Consequently we have an internal functor.

It is straightforward to verify, by the uniqueness of the construction of the morphism ϵ_x that this is natural in X, so that we have a natural transformation ϵ as required.

The triangle equation $ob \circ \epsilon = 1$ is immediate since for a catead X we have, by definition, $(\epsilon_X)_0 = 1_{X_0}$. It remains to to show that given A in $\mathcal{A}$, the component $\epsilon_{\hat{\Delta}(A)}$ is the identity. Since its objects map is certainly the identity we need only show that the arrow component is the identity.

Recall its construction above. The arrow map $\epsilon_{\hat{\Delta}(A)} : A^{\mathbf{2}} \longrightarrow A^{\mathbf{2}}$ is constructed together with a 2-cell:

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left, "\epsilon_a"{name=U}] \arrow[r, bend right, "i_a c_a"'{name=D}] & A^{\mathbf{2}} \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

This 2-cell is the unique one with codomain $i_a c_a$ which postcomposes with c_a to give an identity, and which postcomposed by d_a equals the universal 2-cell:

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left, "d_a"{name=U}] \arrow[r, bend right, "c_a"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "\eta_a"] \end{tikzcd}$$

Consequently it suffices to construct a 2-cell:

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left, "1"{name=U}] \arrow[r, bend right, "i_a c_a"'{name=D}] & A^{\mathbf{2}} \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

which satisfies the equations which uniquely characterise θ. To give such a 2-cell amounts, by the 2-dimensional universal property of $A^{\mathbf{2}}$ to giving a pair of 2-cells $d_a 1 \Longrightarrow d_a i_a c_a = c_a$ and $c_a \Longrightarrow c_a i_a c_a$ such that the square:

$$\begin{tikzcd} d_a \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\eta_a"'] & d_a i_a c_a \arrow[d, Rightarrow, "\eta_a i_a c_a"] \\ c_a \arrow[r, Rightarrow] & c_a i_a c_a \end{tikzcd}$$

is commutative. Now the 2-cell on the right hand side is the identity 2-cell on c_a since $\eta_a i_a$ is, by definition of i_a, the identity. Thus taking the top 2-cell to be $\eta_a : d_a \Longrightarrow c_a = d_a i_a c_a$ and the bottom one to be the identity 2-cell on c_a we obtain the commutative square:

$$\begin{tikzcd} d_a \arrow[r, Rightarrow, "\eta_a"] \arrow[d, Rightarrow, "\eta_a"'] & c_a \arrow[d, Rightarrow, "1"] \\ c_a \arrow[r, Rightarrow, "1"'] & c_a \end{tikzcd}$$

which induces a 2-cell $\phi : 1 \Longrightarrow i_a c_a$ such that $d_a \phi = \eta_a$ and $c_a \phi = 1$ as required.[4] Therefore $\epsilon_{\hat{\Delta}(A)}$ is the identity.

---

[4] In fact c_a is left adjoint to i_a the 2-cell $\phi : 1 \Longrightarrow i_a c_a$ constructed here is the unit of that adjunction. Though we do not consider this perspective here this adjunction, and many involved in the internal category structure, comes from the 2-categorical structure of Δ as a 2-category [51]. The internal category $\hat{\Delta}(A)$ is a "Kock-Zoberlein category".

2. This routinely follows from the definition of the respective counits which were constructed via the same lifting property.

☐

Corollary 3.62. For each representable 2-category $\mathcal{A}$, $\hat{\Delta} : \mathcal{A} \longrightarrow K\mathcal{A}$ is 2-fully faithful.

Proof. Its underlying functor is certainly fully faithful as, by Proposition 3.60, it has a right adjoint and the unit of the adjunction is an isomorphism. It suffices to check then that $\hat{\Delta}_{\mathcal{A}}$ is locally fully faithful. But given $\alpha : f \Longrightarrow g \in \mathcal{A}(X,Y)$ the induced internal natural transformation $\hat{\Delta}_{\mathcal{A}}(\alpha)$ has arrow component the unique arrow $X \longrightarrow Y^{\mathbf{2}}$ corresponding to it upon postcomposition with η_y. Therefore it is clearly locally fully faithful. ☐

Proposition 3.63. There exists a modification:

$$\begin{tikzcd}
K \arrow[r, bend left, "K\hat{\Delta}"] \arrow[r, bend right, "\hat{\Delta}K"'] \arrow[r, phantom, "\Downarrow\lambda"] & K^2
\end{tikzcd}$$

satisfying $\lambda_{\mathcal{A}} \circ \hat{\Delta}_{\mathcal{A}} = \hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}}$ for each representable 2-category $\mathcal{A}$.

Proof. The data for such a modification λ will consist of 2-cells:

$$\begin{tikzcd}
K\mathcal{A} \arrow[r, bend left, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[r, bend right, "\hat{\Delta}_{K\mathcal{A}}"'] \arrow[r, phantom, "\Downarrow\lambda_{\mathcal{A}}"] & K^2\mathcal{A}
\end{tikzcd}$$

one such for each representable 2-category $\mathcal{A}$. For each $\mathcal{A}$ we have the counit of the adjunction of Proposition 3.60(1):

$$\begin{tikzcd}
\mathcal{U}K\mathcal{A} \arrow[r, "\mathcal{U}j_{\mathcal{A}}"] \arrow[drr, "1"'] & \mathcal{U}C\mathcal{U}\mathcal{A} \arrow[r, "ob_{\mathcal{U}\mathcal{A}}"] \arrow[dr, phantom, "\Swarrow\epsilon_{\mathcal{A}}"] & \mathcal{U}\mathcal{A} \arrow[d, "\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] \\
& & \mathcal{U}K\mathcal{A}
\end{tikzcd}$$

which is a 2-cell in $\mathrm{Cat}_{\mathrm{pb}}$. Taking the image of this 2-cell under $C : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$ gives a 2-cell:

(1)

$$\begin{tikzcd}
& & & & KK\mathcal{A} \arrow[ddl, bend left, "j_{K\mathcal{A}}"] \\
& K\mathcal{A} \arrow[urrr, bend left=10, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "\Delta'_{K\mathcal{A}}"] \arrow[ddl, bend right, "\hat{\Delta}_{K\mathcal{A}}"'] & C\mathcal{U}\mathcal{A} \arrow[d, "\Delta'_{C\mathcal{U}\mathcal{A}}"'] \arrow[dr, "1"] & & \\
& C\mathcal{U}K\mathcal{A} \arrow[r, "C\mathcal{U}j_{\mathcal{A}}"] \arrow[drr, "1"'] & C\mathcal{U}C\mathcal{U}\mathcal{A} \arrow[r, "Cob_{\mathcal{U}\mathcal{A}}"] \arrow[dr, phantom, "\Swarrow C\epsilon_{\mathcal{A}}"] & C\mathcal{U}\mathcal{A} \arrow[d, "C\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] & \\
KK\mathcal{A} \arrow[rrr, bend right=10, "j_{K\mathcal{A}}"'] & & & C\mathcal{U}K\mathcal{A} &
\end{tikzcd}$$

The top left square commutes by Remark 3.49. The triangle to its right commutes by Lemma 3.31(1). The region to the right above that triangle commutes by naturality of $j : K \Longrightarrow C\mathcal{U}$. The bottom left square commutes since $j \circ \hat{\Delta} = \Delta'$. As the inclusion $j_{K\mathcal{A}} : KK\mathcal{A} \longrightarrow C\mathcal{U}K\mathcal{A}$ is 2-fully faithful this 2-cell factors uniquely through $j_{K\mathcal{A}}$ to give a 2-cell:

$$\begin{tikzcd}
K\mathcal{A} \arrow[r, bend left, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[r, bend right, "\hat{\Delta}_{K\mathcal{A}}"'] \arrow[r, phantom, "\Downarrow\lambda_{\mathcal{A}}"] & K^2\mathcal{A}
\end{tikzcd}$$

as required. We claim these 2-cells form a modification. We must show that for each $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Rep the following 2-cells agree:

$$(2) \quad \begin{tikzcd}
K\mathcal{A} \arrow[r, bend left, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[r, bend right, "\hat{\Delta}_{K\mathcal{A}}"'] \arrow[r, phantom, "\Downarrow\lambda_{\mathcal{A}}"] \arrow[d, "KF"'] & KK\mathcal{A} \arrow[d, "KKF"] \\
K\mathcal{B} \arrow[r, "\hat{\Delta}_{K\mathcal{A}}"'] \arrow[ur, phantom, "\Downarrow\hat{\Delta}_{KF}"] & KK\mathcal{B}
\end{tikzcd} \quad = \quad (3) \quad \begin{tikzcd}
K\mathcal{A} \arrow[r, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[d, "KF"'] \arrow[dr, phantom, "\Downarrow K\hat{\Delta}_{F}"] & KK\mathcal{A} \arrow[d, "KKF"] \\
K\mathcal{B} \arrow[r, bend left, "K\hat{\Delta}_{\mathcal{B}}"] \arrow[r, bend right, "\hat{\Delta}_{K\mathcal{B}}"'] \arrow[r, phantom, "\Downarrow\lambda_{\mathcal{B}}"] & KK\mathcal{B}
\end{tikzcd}$$

It suffices to verify that both sides agree upon postcomposition with the fully faithful 2-functor $j_{K\mathcal{B}} : KK\mathcal{B} \longrightarrow C\mathcal{U}K\mathcal{B}$. Upon composition with this 2-functor (2) becomes the diagram below:

$$\begin{tikzcd}
 & & & C\mathcal{U}\mathcal{A} \arrow[dr, "C\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] & \\
 & & C\mathcal{U}C\mathcal{U}\mathcal{A} \arrow[ur, "Cob_{\mathcal{U}\mathcal{A}}"] & \Downarrow C\epsilon_{\mathcal{A}} & \\
K\mathcal{A} \arrow[r, "\Delta'_{K\mathcal{A}}"] \arrow[d, "KF"'] \arrow[dr, phantom, "\Delta'_{KF}\Downarrow"] & C\mathcal{U}K\mathcal{A} \arrow[ur, "C\mathcal{U}j_{\mathcal{A}}"] \arrow[rrr, "1"'] \arrow[d, "C\mathcal{U}KF"] & & & C\mathcal{U}K\mathcal{A} \arrow[d, "C\mathcal{U}KF"] \\
K\mathcal{B} \arrow[r, "\Delta'_{K\mathcal{B}}"'] & C\mathcal{U}K\mathcal{B} \arrow[rrr, "1"] & & & C\mathcal{U}K\mathcal{B}
\end{tikzcd}$$

To see this observe firstly that $j_{K\mathcal{B}} \circ KKF = C\mathcal{U}KF \circ j_{K\mathcal{A}}$ by naturality of j. Now $C\mathcal{U}KF \circ (j_{K\mathcal{A}} \circ \lambda_{\mathcal{A}}) = C\mathcal{U}KF \circ (C(\epsilon_{\mathcal{A}}) \circ \Delta'_{K\mathcal{A}})$ by definition of the 2-cell $\lambda_{\mathcal{A}}$. Therefore $j_{K\mathcal{B}} \circ KKF \circ \lambda_{\mathcal{A}} = C\mathcal{U}KF \circ (C(\epsilon_{\mathcal{A}}) \circ \Delta'_{K\mathcal{A}})$. Certainly $j_{K\mathcal{B}} \circ \hat{\Delta}_{KF} = \hat{\Delta}_{KF}$ and consequently (2) does indeed equal the above diagram. Recall the 2-cell equation in Cat_{pb} of Proposition 3.60(2). Applying $C : \text{Cat}_{\text{pb}} \longrightarrow \text{Rep}$ to that equation enables us to rewrite the above composite as:

$$\begin{tikzcd}
 & & & C\mathcal{U}\mathcal{A} \arrow[dd, "C\mathcal{U}F"] \arrow[ddr, "C\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] & \\
 & & C\mathcal{U}C\mathcal{U}\mathcal{A} \arrow[ur, "Cob_{\mathcal{U}\mathcal{A}}"] \arrow[dd, "C\mathcal{U}C\mathcal{U}F"] & & \\
K\mathcal{A} \arrow[r, "\Delta'_{K\mathcal{A}}"] \arrow[dd, "KF"'] \arrow[ddr, phantom, "\Delta'_{KF}\Downarrow"] & C\mathcal{U}K\mathcal{A} \arrow[ur, "C\mathcal{U}j_{\mathcal{A}}"] \arrow[dd, "C\mathcal{U}KF"] & & C\mathcal{U}\mathcal{B} \arrow[r, phantom, "C\mathcal{U}\hat{\Delta}_{F}\Downarrow"] \arrow[ddr, "C\mathcal{U}\hat{\Delta}_{\mathcal{B}}"] & C\mathcal{U}K\mathcal{A} \arrow[dd, "C\mathcal{U}KF"] \\
 & & C\mathcal{U}C\mathcal{U}\mathcal{B} \arrow[ur, "Cob_{\mathcal{U}\mathcal{B}}"] & \Downarrow C\epsilon_{\mathcal{B}} & \\
K\mathcal{B} \arrow[r, "\Delta'_{K\mathcal{B}}"'] & C\mathcal{U}K\mathcal{B} \arrow[ur, "C\mathcal{U}j_{\mathcal{B}}"] \arrow[rrr, "1"'] & & & C\mathcal{U}K\mathcal{B}
\end{tikzcd}$$

which equals:

$$\begin{tikzcd}
 & & & \mathcal{CUA} \arrow[d, "\mathcal{CU}F"] \arrow[dr, "\mathcal{CU}\hat{\Delta}_{\mathcal{A}}"] & \\
K\mathcal{A} \arrow[urrr, bend left, "j_{\mathcal{A}}"] \arrow[dd, "KF"'] & & & \mathcal{CUB} \arrow[ddr, "\mathcal{CU}\hat{\Delta}_{\mathcal{B}}"'] \arrow[r, phantom, "\mathcal{CU}\hat{\Delta}_F \Downarrow"] & \mathcal{CU}K\mathcal{A} \arrow[dd, "\mathcal{CU}KF"] \\
 & & \mathcal{CUCUB} \arrow[ur, "Cob_{\mathcal{UB}}"] & \Downarrow C\epsilon_{\mathcal{B}} & \\
K\mathcal{B} \arrow[uurrr, bend left, "j_{\mathcal{B}}"] \arrow[r, "\Delta'_{K\mathcal{B}}"'] & \mathcal{CU}K\mathcal{B} \arrow[ur, "\mathcal{CU}j_{\mathcal{B}}"] \arrow[rrr, "1"'] & & & \mathcal{CU}K\mathcal{B}
\end{tikzcd}$$

This latter equality holds upon using the equation $j \circ \hat{\Delta} = \Delta'$ and then applying Parts (1) and (2) of Lemma 3.50. Now $\mathcal{CU}\hat{\Delta}_F \circ j_{\mathcal{A}} = j_{K\mathcal{B}} \circ K\hat{\Delta}_{KF}$ by naturality of j whilst $(C\epsilon_{\mathcal{B}} \circ \Delta'_{K\mathcal{B}}) \circ KF = (j_{K\mathcal{B}} \circ \lambda_{\mathcal{B}}) \circ KF$ by definition of $\lambda_{\mathcal{B}}$. Therefore this composite equals (3) postcomposed with $j_{K\mathcal{B}}$ as required and so λ is a modification.
It remains to verify that for each representable 2-category $\mathcal{A}$ we have the equality:

$$(4) \quad \begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] & K\mathcal{A} \arrow[rr, bend left, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[rr, bend right, "\hat{\Delta}_{K\mathcal{A}}"'] \arrow[rr, phantom, "\Downarrow\lambda_{\mathcal{A}}"] & & KK\mathcal{A}
\end{tikzcd} \quad = \quad (5) \quad \begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] \arrow[d, "\hat{\Delta}_{\mathcal{A}}"'] \arrow[dr, phantom, "\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}}\Downarrow"] & K\mathcal{A} \arrow[d, "K\hat{\Delta}_{\mathcal{A}}"] \\
K\mathcal{A} \arrow[r, "\hat{\Delta}_{K\mathcal{A}}"'] & KK\mathcal{A}
\end{tikzcd}$$

For this it suffices to verify that (4) and (5) agree upon postcomposition with the fully faithful inclusion $j_{K\mathcal{A}} : KK\mathcal{A} \longrightarrow \mathcal{CU}K\mathcal{A}$. Postcomposing (4) with $j_{K\mathcal{A}}$ gives the composite:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] & K\mathcal{A} \arrow[r, "\Delta'_{K\mathcal{A}}"] & \mathcal{CU}K\mathcal{A} \arrow[r, "\mathcal{CU}j_{\mathcal{A}}"] \arrow[drr, "1"'] & \mathcal{CUCUA} \arrow[r, "Cob_{\mathcal{UA}}"] & \mathcal{CUA} \arrow[d, "\mathcal{CU}\hat{\Delta}_{\mathcal{A}}"] \arrow[dl, phantom, "C\epsilon_{\mathcal{A}} \Swarrow"] \\
 & & & & \mathcal{CU}K\mathcal{A}
\end{tikzcd}$$

which may be rewritten as:

$$\begin{tikzcd}
 & K\mathcal{A} \arrow[dr, "\hat{\Delta}_{K\mathcal{A}}"] \arrow[d, phantom, "\Downarrow(\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}})^{-1}"] & & & & \\
\mathcal{A} \arrow[ur, "\hat{\Delta}_{\mathcal{A}}"] \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] \arrow[dr, "\hat{\Delta}_{\mathcal{A}}"'] & K\mathcal{A} \arrow[r, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[d, phantom, "\Downarrow\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}}"] & KK\mathcal{A} \arrow[r, "j_{K\mathcal{A}}"] & \mathcal{CU}K\mathcal{A} \arrow[r, "\mathcal{CU}j_{\mathcal{A}}"] \arrow[drr, "1"'] & \mathcal{CU}K\mathcal{A} \arrow[r, "Cob_{\mathcal{UA}}"] & \mathcal{CUA} \arrow[d, "\mathcal{CU}\hat{\Delta}_{\mathcal{A}}"] \arrow[dl, phantom, "C\epsilon_{\mathcal{A}} \Swarrow"] \\
 & K\mathcal{A} \arrow[ur, "\hat{\Delta}_{K\mathcal{A}}"'] & & & & \mathcal{CUCUA}
\end{tikzcd}$$

using that $j_{K\mathcal{A}} \circ \hat{\Delta}_{K\mathcal{A}} = \Delta'_{K\mathcal{A}}$ and adding in a mutually inverse pair of 2-cells. This equals the vertical composite of the following 2-cells:

$$\begin{tikzcd}
& K\mathcal{A} \arrow[dr, "\hat{\Delta}_{K\mathcal{A}}"] \arrow[d, phantom, "\Downarrow(\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}})^{-1}"] & & & & \\
\mathcal{A} \arrow[ur, "\hat{\Delta}_{\mathcal{A}}"] \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] & K\mathcal{A} \arrow[r, "K(\hat{\Delta}_{\mathcal{A}})"] & KK\mathcal{A} \arrow[r, "j_{K\mathcal{A}}"] & C\mathcal{U}K\mathcal{A} \arrow[r, "C\mathcal{U}j_{\mathcal{A}}"] & C\mathcal{U}C\mathcal{U}\mathcal{A} \arrow[r, "Cob_{\mathcal{U}\mathcal{A}}"] & C\mathcal{U}\mathcal{A} \arrow[d, "C\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] \\
& & & & & C\mathcal{U}K\mathcal{A}
\end{tikzcd}$$

$$\circ$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] & K\mathcal{A} \arrow[r, "K\hat{\Delta}_{\mathcal{A}}"] & KK\mathcal{A} \arrow[r, "j_{K\mathcal{A}}"] & C\mathcal{U}K\mathcal{A} \arrow[r, "C\mathcal{U}j_{\mathcal{A}}"] \arrow[drr, "1"'] & C\mathcal{U}K\mathcal{A} \arrow[r, "Cob_{\mathcal{U}\mathcal{A}}"] \arrow[dr, phantom, "C\epsilon_{\mathcal{A}} \Swarrow"] & C\mathcal{U}\mathcal{A} \arrow[d, "C\mathcal{U}\hat{\Delta}_{\mathcal{A}}"] \\
& & & & & C\mathcal{U}K\mathcal{A}
\end{tikzcd}$$

$$\circ$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] \arrow[dr, "\hat{\Delta}_{\mathcal{A}}"'] & K\mathcal{A} \arrow[r, "K\hat{\Delta}_{\mathcal{A}}"] \arrow[d, phantom, "\Downarrow\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}}"] & KK\mathcal{A} \arrow[r, "j_{K\mathcal{A}}"] & C\mathcal{U}K\mathcal{A} \arrow[drr, "1"'] & & \\
& K\mathcal{A} \arrow[ur, "\hat{\Delta}_{K\mathcal{A}}"'] & & & & C\mathcal{U}K\mathcal{A}
\end{tikzcd}$$

The lowest of the three vertically composable 2-cells is exactly (5) postcomposed with $j_{K\mathcal{A}} : KK\mathcal{A} \longrightarrow C\mathcal{U}K\mathcal{A}$. Thus it will suffice to show that the top two of the three are identity 2-cells. Consider the top of these three. We will show that $Cob_{\mathcal{U}\mathcal{A}} \circ C\mathcal{U}j_{\mathcal{A}} \circ j_{K\mathcal{A}} \circ (\hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}})^{-1}$ is an identity 2-cell. This is equivalent to its inverse, $Cob_{\mathcal{U}\mathcal{A}} \circ C\mathcal{U}j_{\mathcal{A}} \circ j_{K\mathcal{A}} \circ \hat{\Delta}_{\hat{\Delta}_{\mathcal{A}}}$, being an identity, which was shown in Lemma 3.50 Part 2.

In order to show that the middle of the three is an identity it will suffice to show that $C\epsilon_{\mathcal{A}} \circ j_{K\mathcal{A}} \circ K\hat{\Delta}_{\mathcal{A}}$ is an identity. By 2-naturality of j, and 2-functoriality of C, this equals $C\epsilon_{\mathcal{A}} \circ C\mathcal{U}\hat{\Delta}_{\mathcal{A}} \circ j_{\mathcal{A}} = C(\epsilon_{\mathcal{A}} \circ \mathcal{U}\hat{\Delta}_{\mathcal{A}}) \circ j_{\mathcal{A}}$. Now $\epsilon_{\mathcal{A}} \circ \mathcal{U}\hat{\Delta}_{\mathcal{A}}$ is an identity as this is one of the triangle equations for the adjunction of Proposition 3.60(1). Therefore the composite 2-cell is an identity completing the proof. $\square$

Proposition 3.64. Let $\mathcal{E}$ be a category with pullbacks.

1. The 2-functor $\hat{\Delta}_{C\mathcal{E}} : C\mathcal{E} \longrightarrow KC\mathcal{E}$ has left 2-adjoint the composite:

$$KC\mathcal{E} \xrightarrow{j_{C\mathcal{E}}} C\mathcal{U}C\mathcal{E} \xrightarrow{Cob_{\mathcal{E}}} C\mathcal{E} \quad .$$

2. For each $\mathcal{E}$ of $\mathrm{Cat}_{\mathrm{pb}}$ let $Q_{\mathcal{E}} : KC\mathcal{E} \longrightarrow C\mathcal{E}$ denote the composite left 2-adjoint of the first part of the proposition and $\rho_{\mathcal{E}}$ the unit of that adjunction. Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ we have the equality of 2-cells:

$$(1) \quad \begin{tikzcd}
KC\mathcal{A} \arrow[rr, "1"] \arrow[dd, "KCF"'] \arrow[dr, "Q_{\mathcal{A}}"'] & \arrow[d, phantom, "\Downarrow\rho_{\mathcal{A}}"] & KC\mathcal{A} \arrow[dd, "KCF"] \\
& C\mathcal{A} \arrow[ur, "\hat{\Delta}_{C\mathcal{A}}"'] \arrow[dd, "CF"] \arrow[dr, phantom, "\hat{\Delta}_{CF}\Downarrow"] & \\
KC\mathcal{B} \arrow[dr, "Q_{\mathcal{B}}"'] & & KC\mathcal{B} \\
& C\mathcal{B} \arrow[ur, "\hat{\Delta}_{C\mathcal{B}}"'] &
\end{tikzcd}
\quad = \quad (2) \quad
\begin{tikzcd}
KC\mathcal{A} \arrow[rr, "1"] \arrow[d, "KCF"'] & & KC\mathcal{A} \arrow[d, "KCF"] \\
KC\mathcal{B} \arrow[rr, "1"] \arrow[dr, "Q_{\mathcal{B}}"'] & \arrow[d, phantom, "\Downarrow\rho_{\mathcal{B}}"] & KC\mathcal{B} \\
& C\mathcal{B} \arrow[ur, "\hat{\Delta}_{C\mathcal{B}}"'] &
\end{tikzcd}$$

Proof. 1. Having established in Proposition 3.63 the existence of a suitable modification λ it suffices to show that the composite:

$$\begin{tikzcd}
KC\mathcal{E} \arrow[r, "j_{C\mathcal{E}}"] & C\mathcal{U}C\mathcal{E} \arrow[r, "Cob_{\mathcal{E}}"] & C\mathcal{E}
\end{tikzcd}$$

satisfies the properties (1), (2) and (3) of Proposition 3.58.
Property (1) asserts that:

$$\begin{tikzcd}
C\mathcal{E} \arrow[r, "\hat{\Delta}_{C\mathcal{E}}"] & KC\mathcal{E} \arrow[r, "j_{C\mathcal{E}}"] & C\mathcal{U}C\mathcal{E} \arrow[r, "Cob_{\mathcal{E}}"] & C\mathcal{E}
\end{tikzcd}$$

is the identity on $C\mathcal{E}$. The composite of the two leftmost 1-cells is simply $\Delta'_{C\mathcal{E}} : C\mathcal{E} \longrightarrow C\mathcal{U}C\mathcal{E}$ and the claim now follows from Lemma 3.31(1).
Property (2) asserts that the composite:

$$\begin{tikzcd}
KC\mathcal{E} \arrow[r, bend left, "K\hat{\Delta}_{C\mathcal{E}}"] \arrow[r, bend right, "\hat{\Delta}_{KC\mathcal{E}}"'] \arrow[r, phantom, "\Downarrow\lambda_{C\mathcal{E}}"] & KKC\mathcal{E} \arrow[r, "K(Cob_{\mathcal{E}} \circ j_{C\mathcal{E}})"] & KC\mathcal{E} \arrow[r, "Cob_{\mathcal{E}} \circ j_{C\mathcal{E}}"] & C\mathcal{E}
\end{tikzcd}$$

is an identity 2-cell. The composite 1-cells to the right of the 2-cell may be rewritten as:

$$\begin{tikzcd}
KKC\mathcal{E} \arrow[rr, "Kj_{C\mathcal{E}}"] \arrow[dr, "j_{KC\mathcal{E}}"] & & KC\mathcal{U}C\mathcal{E} \arrow[rr, "KCob_{\mathcal{E}}"] \arrow[ddrr, phantom] \arrow[drr, "j_{C\mathcal{U}C\mathcal{E}}"] & & KC\mathcal{E} \arrow[rr, "j_{C\mathcal{E}}"] & & C\mathcal{U}C\mathcal{E} \arrow[rr, "Cob_{\mathcal{E}}"] & & C\mathcal{E} \\
& C\mathcal{U}KC\mathcal{E} \arrow[rrr, "C\mathcal{U}j_{C\mathcal{E}}"] \arrow[rrrrrr, bend right=15, "C(ob_{\mathcal{U}C\mathcal{E}} \circ \mathcal{U}j_{C\mathcal{E}})"'] & & & C\mathcal{U}C\mathcal{U}C\mathcal{E} \arrow[urr, "C\mathcal{U}Cob_{\mathcal{E}}"] \arrow[rrr, "Cob_{\mathcal{U}C\mathcal{E}}"] & & & C\mathcal{U}C\mathcal{E} \arrow[ur, "Cob_{\mathcal{E}}"] &
\end{tikzcd}$$

where the two left squares are rewritten using naturality of $j : K \Longrightarrow C\mathcal{U}$ and the right square by naturality of $ob : \mathcal{U}C \Longrightarrow 1$. Now by definition of $\lambda_{C\mathcal{E}}$ we have $j_{KC\mathcal{E}} \circ \lambda_{C\mathcal{E}} = C\epsilon_{C\mathcal{E}} \circ \Delta'_{KC\mathcal{E}}$. It suffices therefore, traversing the lower path, to verify that $C(ob_{\mathcal{U}C\mathcal{E}} \circ \mathcal{U}j_{C\mathcal{E}}) \circ C\epsilon_{C\mathcal{E}}$ is an identity. But $ob_{\mathcal{U}C\mathcal{E}} \circ \mathcal{U}j_{C\mathcal{E}} \circ \epsilon_{C\mathcal{E}}$ is an identity; this equation being one of the triangle equations for the adjunction of Proposition 3.60(1) in the case of $C\mathcal{E}$.
It remains to verify that Property (3) is satisfied. We must show that the 2-cell:

$$\begin{tikzcd}
KC\mathcal{E} \arrow[r, "\hat{\Delta}_{KC\mathcal{E}}"] \arrow[d, "Cob_{\mathcal{E}} \circ j_{C\mathcal{E}}"'] \arrow[dr, phantom, "\hat{\Delta}_{Cob_{\mathcal{E}} \circ j_{C\mathcal{E}}} \Downarrow"] & KKC\mathcal{E} \arrow[d, "K(Cob_{\mathcal{E}} \circ j_{C\mathcal{E}})"] & & \\
C\mathcal{E} \arrow[r, "\hat{\Delta}_{C\mathcal{E}}"'] & KC\mathcal{E} \arrow[r, "j_{C\mathcal{E}}"'] & C\mathcal{U}C\mathcal{E} \arrow[r, "Cob_{\mathcal{E}}"'] & C\mathcal{E}
\end{tikzcd}$$

is an identity. As $j \circ \hat{\Delta} = \Delta'$ this is equal to the left composite below:

$$\begin{tikzcd}
KC\mathcal{E} \arrow[r, "\Delta'_{KC\mathcal{E}}"] \arrow[d, "Cob_{\mathcal{E}} \circ j_{C\mathcal{E}}"'] \arrow[dr, phantom, "\Delta'_{C(ob_{\mathcal{E}}) \circ j_{C\mathcal{E}}} \Downarrow"] & C\mathcal{U}KC\mathcal{E} \arrow[d, "C\mathcal{U}(Cob_{\mathcal{E}} \circ j_{C\mathcal{E}})"] & \\
C\mathcal{E} \arrow[r, "\Delta'_{C\mathcal{E}}"'] & C\mathcal{U}C\mathcal{E} \arrow[r, "C(ob_{\mathcal{E}})"'] & C\mathcal{E}
\end{tikzcd}
=
\begin{tikzcd}
KC\mathcal{E} \arrow[r, "\Delta'_{KC\mathcal{E}}"] \arrow[d, "j_{C\mathcal{E}}"'] \arrow[dr, phantom, "\Delta'_{j_{C\mathcal{E}}} \Downarrow"] & C\mathcal{U}KC\mathcal{E} \arrow[d, "C\mathcal{U}j_{C\mathcal{E}}"] & \\
C\mathcal{U}C\mathcal{E} \arrow[r, "C\mathcal{U}j_{C\mathcal{E}}"] \arrow[d, "Cob_{\mathcal{E}}"'] \arrow[dr, phantom, "\Delta'_{Cob_{\mathcal{E}}} \Downarrow"] & C\mathcal{U}C\mathcal{U}C\mathcal{E} \arrow[d, "C\mathcal{U}Cob_{\mathcal{E}}"] & \\
C\mathcal{E} \arrow[r, "\Delta'_{C\mathcal{E}}"'] & C\mathcal{U}C\mathcal{E} \arrow[r, "Cob_{\mathcal{E}}"'] & C\mathcal{E}
\end{tikzcd}$$

which equals the right composite by pseudonaturality of $\Delta' : 1 \Longrightarrow C\mathcal{U}$. Now $\Delta'_{j_{C\mathcal{E}}}$ is an identity by Remark 3.49 whilst $Cob_{\mathcal{E}} \circ \Delta'_{Cob_{\mathcal{E}}}$ is an identity by Lemma 3.31(2). Thus the rightmost composite is an identity 2-cell.

2. Firstly observe that the commuting square on the left of the composite (1) of the proposition does indeed commute; that is: $C(F)Q_{\mathcal{A}} = Q_{\mathcal{B}}KC(F)$. For $Q_{\mathcal{B}} = Cob_{\mathcal{B}}j_{C\mathcal{B}}$ and similarly for $\mathcal{A}$. Thus "Q" is the composite 2-natural transformation $Cob \circ j_C : KC \Longrightarrow C$ in $\mathrm{Cat}_{\mathrm{pb}}$: the commutativity of the square is now an instance of its naturality.
By definition of the unit $\rho_{\mathcal{A}}$ as given in Proposition 3.58, (1) equals the left composite below:

$$\begin{tikzcd}
KC\mathcal{A} \arrow[rr, bend left, "K\hat{\Delta}_{C\mathcal{A}}"] \arrow[rr, bend right, "\hat{\Delta}_{KC\mathcal{A}}"'] \arrow[d, "KCF"'] \arrow[drr, bend right, "Q_{\mathcal{A}}"'] & \Downarrow\lambda_{C\mathcal{A}} & KKC\mathcal{A} \arrow[r, "KQ_{\mathcal{A}}"] \arrow[d, phantom, "\hat{\Delta}_{Q_{\mathcal{A}}}\Downarrow"] & KC\mathcal{A} \arrow[d, "KCF"] \\
KC\mathcal{B} \arrow[drr, "Q_{\mathcal{B}}"'] & & C\mathcal{A} \arrow[ur, "\hat{\Delta}_{C\mathcal{A}}"'] \arrow[d, "CF"] \arrow[r, phantom, "\hat{\Delta}_{CF}\Downarrow"] & KC\mathcal{B} \\
 & & C\mathcal{B} \arrow[ur, "\hat{\Delta}_{C\mathcal{B}}"'] &
\end{tikzcd}
=
\begin{tikzcd}
KC\mathcal{A} \arrow[r, bend left, "K\hat{\Delta}_{C\mathcal{A}}"] \arrow[r, bend right, "\hat{\Delta}_{KC\mathcal{A}}"'] \arrow[r, phantom, "\Downarrow\lambda_{C\mathcal{A}}"] \arrow[d, "KCF"'] & KKC\mathcal{A} \arrow[d, "KKCF"] \\
KC\mathcal{B} \arrow[r, "\hat{\Delta}_{KC\mathcal{B}}"'] \arrow[d, "Q_{\mathcal{B}}"'] \arrow[ur, phantom, "\Downarrow\hat{\Delta}_{KCF}"] & KKC\mathcal{B} \arrow[d, "KQ_{\mathcal{B}}"] \\
C\mathcal{B} \arrow[r, "\hat{\Delta}_{C\mathcal{B}}"'] \arrow[ur, phantom, "\Downarrow\hat{\Delta}_{Q_{\mathcal{B}}}"] & KC\mathcal{B}
\end{tikzcd}$$

$$=
\begin{tikzcd}
KC\mathcal{A} \arrow[r, "K\hat{\Delta}_{C\mathcal{A}}"] \arrow[d, "KCF"'] & KKC\mathcal{A} \arrow[d, "KKCF"] \\
KC\mathcal{B} \arrow[r, bend left, "K\hat{\Delta}_{C\mathcal{B}}"] \arrow[r, bend right, "\hat{\Delta}_{KC\mathcal{B}}"'] \arrow[r, phantom, "\Downarrow\lambda_{C\mathcal{B}}"] \arrow[d, "Q_{\mathcal{B}}"'] \arrow[ur, phantom, "\Downarrow K\hat{\Delta}_{CF}"] & KKC\mathcal{B} \arrow[d, "KQ_{\mathcal{B}}"] \\
C\mathcal{B} \arrow[r, "\hat{\Delta}_{C\mathcal{B}}"'] \arrow[ur, phantom, "\Downarrow\hat{\Delta}_{Q_{\mathcal{B}}}"] & KC\mathcal{B}
\end{tikzcd}$$

Now $Q_{\mathcal{B}} \circ KC(F) = C(F) \circ Q_A$, and taking components of $\hat{\Delta}$ at this composite, and using its pseudonaturality gives the first equation above. The second equation holds as λ is a modification. Now $KQ_{\mathcal{B}}K\hat{\Delta}_{CF} = K(Q_{\mathcal{B}}\hat{\Delta}_{CF})$ and $Q_{\mathcal{B}}\hat{\Delta}_{CF} = Cob_{\mathcal{B}}j_{C\mathcal{B}}\hat{\Delta}_{CF} = Cob_{\mathcal{B}}\Delta'_{CF}$ which is an identity by Lemma 3.31(1). Therefore $KQ_{\mathcal{B}}K\hat{\Delta}_{CF}$ is an identity and the final composite equals (2) by definition of $\rho_{\mathcal{B}}$.

□

Theorem 3.65. Let $\mathcal{E}$ be a category with pullbacks. Then $Cat(\mathcal{E})$ has codescent objects of cateads and cateads are effective.

Proof. We saw in Proposition 3.64(1) that $\hat{\Delta}_{C\mathcal{E}} : C\mathcal{E} \longrightarrow KC\mathcal{E}$ has a left 2-adjoint $Q_{\mathcal{E}} = Cob_{\mathcal{E}} \circ j_{C\mathcal{E}}$, so that by Proposition 3.47 $C\mathcal{E}$ has codescent objects of cateads. To show that cateads are effective is to show, by that same proposition, that the unit of the adjunction is pointwise fully faithful. We know from Proposition 2.83 that cateads are effective in Cat $= C$(Set) and will use this to deduce the case for general $\mathcal{E}$. For each object A of $\mathcal{E}$ we have the representable $\mathcal{E}(A, -) = \hat{A} : \mathcal{E} \longrightarrow \mathrm{Set}$ which induces the morphism of Rep: $KC\hat{A} : KC\mathcal{E} \longrightarrow KC(\mathrm{Set})$. We will show that these 2-functors jointly reflect isomorphisms (for A in $\mathcal{E}$). Now an internal functor $f : X \longrightarrow Y$ between internal categories is an isomorphism precisely when its arrow component $f_1 : X_1 \longrightarrow Y_1$ is an isomorphism (as any such internal functor is clearly an isomorphism on objects upon identifying them with identity morphisms). Consider a morphism of $f : X \longrightarrow Y$ of $KC\mathcal{E}$. As $KC\mathcal{E}$ is a full sub 2-category of $C\mathcal{U}C\mathcal{E}$ the morphism f is an isomorphism precisely if it is an isomorphism in $C\mathcal{U}C\mathcal{E}$

which we have just observed is the case precisely if $f_1 : X_1 \longrightarrow Y_1$ is an isomorphism in $C(\mathcal{E})$. Using the same logic again this internal functor f_1 is an isomorphism precisely if its arrow component $(f_1)_1 : (X_1)_1 \longrightarrow (Y_1)_1$ is an isomorphism in $\mathcal{E}$. Suppose that for each A the catead morphism $KC\hat{A}(f)$ is an isomorphism. Now we have $((KC\hat{A}(f))_1)_1 = \mathcal{E}(A, (f_1)_1) : \mathcal{E}(A, (X_1)_1) \longrightarrow \mathcal{E}(A, (Y_1)_1)$ and this arrow is an isomorphism by assumption. Since the representables $\hat{A}$ jointly reflect isomorphisms it follows that $(f_1)_1$ is an isomorphism. Therefore f is an isomorphism so that the 2-functors $KC\hat{A}$ jointly reflect isomorphisms as claimed.
Since each $KC\hat{A} : KC\mathcal{E} \longrightarrow KC(\mathrm{Set})$ is a morphism of Rep it preserves fully faithful arrows by Corollary 3.53(1). Since they jointly reflects isomorphisms it follows from the same corollary that they jointly reflect fully faithful arrows. Now given a catead $X \in KC\mathcal{E}$ we must show that the component of the unit at X, $\rho_{\mathcal{E}}(X) : X \longrightarrow \hat{\Delta}_{C\mathcal{E}} Q_{\mathcal{E}}(X)$, is fully faithful. It suffices therefore to show that its image under $KC\hat{A}$ is so for each A. Applying Proposition 3.64(2) in the case of the representable $\hat{A}$ we have:

$$KC\hat{A}(X) \xrightarrow{KC\hat{A}(\rho_{\mathcal{E}}(X))} KC(\hat{A})\hat{\Delta}_{C\mathcal{E}} Q_{\mathcal{E}}(X) \cong \hat{\Delta}_{C(\mathrm{Set})} C\hat{A} Q_{\mathcal{E}}(X) = \hat{\Delta}_{C(\mathrm{Set})} Q_{\mathrm{Set}} KC\hat{A}(X)$$

$$= \quad KC\hat{A}(X) \xrightarrow{\rho_{\mathrm{Set}}(KC\hat{A}(X))} \hat{\Delta}_{C(\mathrm{Set})} Q_{\mathrm{Set}} KC\hat{A}(X)$$

where the unlabelled isomorphism is the component of $\hat{\Delta}_{C\hat{A}}$ at $Q_{\mathcal{E}}X$. This latter arrow is fully faithful since cateads are effective in $C(\mathrm{Set}) = \mathrm{Cat}$. Therefore $KC\hat{A}(\rho_{\mathcal{E}}(X))$ is fully faithful since its composite with an isomorphism is so. We deduce that $\rho_{\mathcal{E}}(X)$ is fully faithful as required. ☐

Theorem 3.66. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of $\mathrm{Cat}_{\mathrm{pb}}$. Then $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves codescent objects of cateads.

Proof. This is the essential content of Proposition 3.64(2). We have seen already that $Q_{\mathcal{B}} \circ KC(F) = C(F) \circ Q_A$ which is to say that $C(F)$ preserves codescent "objects" of cateads but we must verify that the universal cocones are also preserved. Given a catead $X \in KC\mathcal{A}$ the equality of the 2-natural transformations of that Proposition, at X, asserts the equality of the morphisms of $KC\mathcal{B}$:

$$\begin{tikzcd}
CFX_2 \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & CF(Q_{\mathcal{A}}X)^{\mathbf{3}} \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & (CFQ_{\mathcal{A}}X)^{\mathbf{3}} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & CFX_2 \arrow[r] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & (Q_{\mathcal{B}}KCFX)^{\mathbf{3}} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
CFX_1 \arrow[r, "CF(\rho_{\mathcal{A}}(X)_1)"] \arrow[d, shift right=2, "CFd_x"'] \arrow[d, shift left=2, "CFc_x"] & CF(Q_{\mathcal{A}}X)^{\mathbf{2}} \arrow[r, "k_1"] \arrow[d, shift right=2, "CFd_{(Q_{\mathcal{A}}X)}"'] \arrow[d, shift left=2, "CFc_{(Q_{\mathcal{A}}X)}"] & (CFQ_{\mathcal{A}}X)^{\mathbf{2}} \arrow[d, shift right=2, "d_{(CFQ_{\mathcal{A}}X)}"'] \arrow[d, shift left=2, "c_{(CFQ_{\mathcal{A}}X)}"] & = & CFX_1 \arrow[r, "\rho_{\mathcal{B}}(KCFX)_1"] \arrow[d, shift right=2, "CFd_x"'] \arrow[d, shift left=2, "CFc_x"] & (Q_{\mathcal{B}}KCFX)^{\mathbf{2}} \arrow[d, shift right=2, "d_{(Q_{\mathcal{B}}KCFX)}"'] \arrow[d, shift left=2, "c_{(Q_{\mathcal{B}}KCFX)}"] \\
CFX_0 \arrow[r, "CF(\rho_{\mathcal{A}}(X)_0)"'] \arrow[u] & CFQ_{\mathcal{A}}X \arrow[r, "1"'] \arrow[u] & CFQ_{\mathcal{A}}X \arrow[u] & & CFX_0 \arrow[r, "\rho_{\mathcal{B}}(KCFX)_0"'] \arrow[u] & Q_{\mathcal{B}}KCFX \arrow[u]
\end{tikzcd}$$

where we have only labelled the relevant arrows, and written k_1 for the evident comparison isomorphism. On the left hand side of both diagrams we have the catead $KCFX$. The cocones:

$$(Q_{\mathcal{A}}X, \rho_{\mathcal{A}}(X)_0, \eta_{Q_{\mathcal{A}}X} \circ \rho_{\mathcal{A}}(X)_1) \quad \text{and} \quad (Q_{\mathcal{B}}KCFX, \rho_{\mathcal{B}}(KCFX)_0, \eta_{(Q_{\mathcal{B}}KCFX)} \circ \rho_{\mathcal{B}}(KCFX)_1)$$

associated to the unit $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ at X and $KCFX$ exhibit Q_AX and Q_BKCFX as the codescent objects of the cateads X and $KCFX$ respectively, as described in Proposition 3.47. In order to show that CF preserves codescent objects of cateads we must show therefore that the image under CF of the first cocone:

$$(CFQ_{\mathcal{A}}X, CF\rho_{\mathcal{A}}(X)_0, CF\eta_{Q_{\mathcal{A}}X} \circ CF\rho_{\mathcal{A}}(X)_1)$$

exhibits $CFQ_{\mathcal{A}}X$ as the codescent object of $KCFX$. In fact it equals the second cocone:

$$(Q_{\mathcal{B}}KCFX, \rho_{\mathcal{B}}(KCFX)_0, \eta_{(Q_{\mathcal{B}}KCFX)} \circ \rho_{\mathcal{B}}(KCFX)_1)$$

exactly since we have $CFQ_{\mathcal{A}}X = Q_{\mathcal{B}}KCFX, CF\rho_{\mathcal{A}}(X)_0 = \rho_{\mathcal{B}}(KCFX)_0$ and

$$CF\eta_{Q_{\mathcal{A}}X} \circ CF\rho_{\mathcal{A}}(X)_1) = \eta_{(Q_{\mathcal{B}}KCFX)} \circ k_1 \circ CF\rho_{\mathcal{A}}(X)_1) = \eta_{(Q_{\mathcal{B}}KCFX)} \circ \rho_{\mathcal{B}}(KCFX)_1$$

Therefore CF preserves codescent objects of cateads. ☐

Theorem 3.67. Let $\mathcal{E}$ be a category with pullbacks. The factorisation of an internal functor through the codescent morphism of its higher kernel agrees with its factorisation as bijective on objects followed by fully faithful.

Proof. We proved this to be true in the case of $\mathcal{E} = \text{Set}$ in Proposition 2.67. We begin the proof in the same manner. Given an internal functor $f : X \longrightarrow Y$ consider its higher kernel in $Cat(\mathcal{E})$:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X \arrow[l, "i" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & & Y \\
 & X \arrow[ur, "f"'] &
\end{tikzcd}$$

together with its comma cone on the right above. Consider the factorisation of f as internally bijective on objects followed by fully faithful as on the left below:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "r"'] & & Z \\
 & Y \arrow[ur, "s"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dr, "r"] \arrow[dd, phantom, "\Downarrow\theta"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & & Z \\
 & X \arrow[ur, "r"'] &
\end{tikzcd}$$

Since s is fully faithful the natural transformation α factors uniquely through it to give a natural transformation θ as on the right above. Now the triple (Y, f, α) is a codescent cocone from the higher kernel. As s is faithful it follows that the triple (Z, r, θ) is a codescent cocone. As in Proposition 2.67 it will suffice to show that it is the universal such cocone.

We know this to be true in the case of $\mathcal{E} = \text{Set}$ and will deduce the general case using the jointly conservative representables $\mathcal{E}(A, -) : \mathcal{E} \longrightarrow Set$. As $\mathcal{E}(A, -) = \hat{A}$ preserves pullbacks we may consider $C\hat{A} : C\mathcal{E} \longrightarrow C(\text{Set})$. As a morphism of Rep each such 2-functor preserves comma objects and pullbacks and therefore higher kernels. Furthermore by Theorem 3.66 each such 2-functor preserves codescent objects of cateads. We would like to show that the 2-functors $C\hat{A}$ jointly reflect codescent objects of cateads. As each preserves them, and $C\mathcal{E}$ has codescent objects of cateads it will suffice to show that the 2-functors $C\hat{A}$ jointly reflect isomorphisms.

This is clear: An internal functor $g : C \longrightarrow D$ is an isomorphism precisely when g_1 is an isomorphism. Now $(C\hat{A}(g))_1 = \mathcal{E}(A, g_1)$. Therefore if each $C\hat{A}(g)$ is an isomorphism then $\mathcal{E}(A, g_1)$ must be an isomorphism. As the representables $\mathcal{E}(A, -)$ reflect isomorphisms g_1, and thus g, must be an isomorphism. Therefore the 2-functors $C\hat{A}$ jointly reflect codescent objects of cateads. It consequently suffices to check that the cocone $(C\hat{A}Z, C\hat{A}r, C\hat{A}\theta)$ is the universal cocone in $Cat(\text{Set}) = \text{Cat}$. Since $r : X \longrightarrow Z$ is bijective on objects so is $C\hat{A}r$, its object map simply being $\mathcal{E}(A, r_0)$. Now $s : Z \longrightarrow Y$ is fully faithful. As $C\hat{A}$ is a morphism of Rep it preserves fully faithfulness by Corollary 3.53(1). Therefore $C\hat{A}s$ is fully faithful and the factorisation:

$$\begin{tikzcd}
C\hat{A}(X) \arrow[rr, "C\hat{A}(f)"] \arrow[dr, "C\hat{A}(r)"'] & & C\hat{A}(Y) \\
 & C\hat{A}(Z) \arrow[ur, "C\hat{A}(s)"'] &
\end{tikzcd}$$

is the (bijective on objects/fully faithful) factorisation of f in Cat. Since $C\hat{A}$ preserves higher kernels the cocone $(C\hat{A}r, C\hat{A}\theta, C\hat{A}Z)$ is indeed the unique cocone induced by the (bijective on objects/fully faithful) factorisation of $C\hat{A}f$ in Cat. Therefore it follows from Proposition 2.67 that this cocone is the universal one. $\square$

Theorem 3.68. Let $\mathcal{E}$ be a category with pullbacks. The codescent morphisms in $Cat(\mathcal{E})$ are precisely the internal bijections on objects and codescent morphisms are effective.

Proof. (Bijections on objects/fully faithfuls) form an orthogonal factorisation system on $Cat(\mathcal{E})$. By Proposition 2.34 each codescent morphism is orthogonal to fully faithful morphisms. Therefore each codescent morphism is bijective on objects by Proposition 2.37. Conversely given a bijective on objects internal functor $f : X \twoheadrightarrow Y$ consider its factorisation through the codescent object of its higher kernel as $f = rs$. By Theorem 3.67 this agrees with its bijective on objects/fully faithful factorisation and so s is an isomorphism. Therefore f exhibits its codomain as the codescent object of its higher kernel. ☐

Corollary 3.69. Let $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$. The (Bijective on objects/fully faithful)-factorisation system on $Cat(\mathcal{E})$ is enhanced.

Proof. We need only show that the bijections on objects are strongly orthogonal to the fully faithful internal functors. Of course it is not difficult to prove this directly, but having shown in Theorem 3.68 that the bijections on objects are precisely the codescent morphisms observe that we can apply Proposition 2.34 to deduce the result immediately. ☐

Chapter 4

A characterisation of 2-categories of internal categories

In Chapter 3 we established various properties of 2-categories of the form $Cat(\mathcal{E})$ where $\mathcal{E}$ is a category with pullbacks. Namely if $\mathcal{E}$ has pullbacks then:

1. $Cat(\mathcal{E})$ is a representable 2-category; that is $Cat(\mathcal{E})$ has pullbacks and cotensors with **2**.
2. $Cat(\mathcal{E})$ has codescent objects of cateads and they are effective.
3. Codescent morphisms are effective in $Cat(\mathcal{E})$.

Furthermore we showed that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a pullback preserving functor then:

1. $Cat(F)$ preserves pullbacks and cotensors with **2**.
2. $Cat(F)$ preserves codescent objects of cateads.

In this chapter we extend this list of 2-categorical properties of 2-categories of the form $Cat(\mathcal{E})$ to a set of axioms characterising such 2-categories up to 2-equivalence, and furthermore identify those 2-functors of the form $Cat(F)$ up to 2-natural isomorphism. The two key properties will be the notion of *discreteness* and a 2-categorical notion of *projectivity*. The final result will be expressed in terms of a biequivalence between $\mathrm{Cat}_{\mathrm{pb}}$ and a sub 2-category of Rep.

One aspect of the present chapter was considered by Bourn and Penon in [10]. Namely they consider the notion of discreteness. Under certain conditions related to those of Chapter 2 and the concept of "aneade" not relevant to us, they obtain, for a 2-category $\mathcal{A}$, a right adjoint to the inclusion $Disc(\mathcal{A}) \longrightarrow \mathcal{U}\mathcal{A}$. This enables them to obtain a 2-adjunction:

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right=1.2ex] \arrow[r, phantom, "\perp"] & Cat(Disc(\mathcal{A})) \arrow[l, shift right=1.2ex] \end{tikzcd}$$

much as used in our proof of Theorem 4.18. Their interests are not however related to ours and they do not consider the main aspects of the present chapter, namely the characterisation of Theorem 4.18, codescent morphisms and projectivity, and the biequivalence of Theorem 4.28.

4.1 Discrete objects

Definition 4.1. An object A of a 2-category $\mathcal{A}$ is discrete if for each $B \in \mathcal{A}$ the category $\mathcal{A}(B, A)$ is discrete. Thus discreteness is defined representably. In elementary terms, each 2-cell with codomain A is an identity 2-cell.

Lemma 4.2. Let $\mathcal{A}$ be any 2-category and $A \in \mathcal{A}$.

1. A is discrete if and only if the identity 2-cell:

$$\begin{tikzcd} A \arrow[r, bend left, "1"{name=U}] \arrow[r, bend right, "1"'{name=D}] & A \arrow[Rightarrow, from=U, to=D, "1"] \end{tikzcd}$$

   exhibits A as $A^{\mathbf{2}}$.

2. If $A^{\mathbf{2}}$ exists then A is discrete if and only if the diagonal map $i_a : A \longrightarrow A^{\mathbf{2}}$ is an isomorphism.

Proof. 1. It is clear that the identity 2-cell on A above is the universal such if and only if all 2-cells with codomain A are themselves identities.

2. The diagonal $i_a : A \longrightarrow A^{\mathbf{2}}$ is that induced by the identity 2-cell on A. Consequently i_a is an isomorphism if and only if the identity 2-cell on A is the universal such 2-cell.

□

Corollary 4.3. Let $\mathcal{A}$ be any 2-category and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a 2-functor preserving cotensors with $\mathbf{2}$.

1. Then F preserves discreteness. In particular any morphism of Rep preserves discreteness.

2. If $\mathcal{A}$ has cotensors with $\mathbf{2}$ and F additionally reflects isomorphisms then F reflects discreteness.

Proof. 1. Discrete objects were characterised in terms of cotensors with $\mathbf{2}$ in Lemma 4.2(1), this characterisation not requiring the existence of cotensors in $\mathcal{A}$. The result follows.

2. $\mathcal{A}$ has cotensors with $\mathbf{2}$ and F preserves them. As F reflects isomorphisms it therefore reflects cotensors with $\mathbf{2}$. Therefore using the characterisation of Lemma 4.2(1) (or equally part (2) of that lemma) we see that F reflects discreteness.

□

Corollary 4.4. Let $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$. An internal category $X \in Cat(\mathcal{E})$ is discrete if and only if $i_x : X_0 \longrightarrow X_1$ is an isomorphism.

Proof. By Lemma 4.2 X is discrete if and only if the identity 2-cell on X:

$$\begin{tikzcd} X \arrow[r, bend left, "1"{name=U}] \arrow[r, bend right, "1"'{name=D}] & X \arrow[Rightarrow, from=U, to=D, "1"] \end{tikzcd}$$

exhibits X as $X^{\mathbf{2}}$. This identity internal natural transformation has arrow component $i_x : X_0 \longrightarrow X_1$. Now by Proposition 3.19(2) this internal natural transformation exhibits X as $X^{\mathbf{2}}$ if and only if both its arrow component $i_x : X_0 \longrightarrow X_1$ is an isomorphism and the naturality square:

$$\begin{tikzcd} X_1 \arrow[r, "{(i_x d_x, 1)}"] \arrow[d, "{(1, i_x c_x)}"'] & X_2 \arrow[d, "m_x"] \\ X_2 \arrow[r, "m_x"'] & X_1 \end{tikzcd}$$

is a pullback; thus X is discrete if and only if these conditions are verified. If both conditions are satisfied then $i_x : X_0 \longrightarrow X_1$ is certainly an isomorphism.
Conversely supposing i_x is an isomorphism we see that its inverse is necessarily $d_x = c_x$ since it splits both of these. Thus the pullback maps p_x and q_x are equal and both isomorphisms and since $d_x p_x = d_x m_x$ we see that $m_x = p_x = q_x$ too. As m_x is an isomorphism the commutativity of the naturality square above implies that $(i_x d_x, 1) = (1, i_x c_x)$. Furthermore the map $(i_x d_x, 1) : X_1 \longrightarrow X_2$ is itself an isomorphism since $q_x(i_x d_x, 1) = 1$ and q_x is an isomorphism. As m_x is itself an isomorphism any commutative square, such as the one above, with top and left arrows equal and isomorphisms necessarily exhibits the square as a pullback. Therefore the naturality square is a pullback. ☐

Definition 4.5. For a 2-category $\mathcal{A}$ we denote by $Disc(\mathcal{A})$ the full sub 2-category of $\mathcal{A}$ containing the discrete objects.

Remark 4.6. Observe that although we have defined $Disc(\mathcal{A})$ as a 2-category it is a locally discrete 2-category, for all 2-cells between discrete objects are necessarily identites. Thus we may equally think of $Disc(\mathcal{A})$ as simply a category without any loss of information.

Remark 4.7. Consider an internal category X in $\mathcal{A}$ with each of X_0, X_1 discrete. The domain and codomain maps $d_x, c_x : X_1 \longrightarrow X_0$ automatically form a two-sided discrete fibration; the lifting properties for a two sided discrete fibration automatically verified in the absence of any non-trivial 2-cells. Therefore X is automatically a catead.

Remark 4.8. Consider a category $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$. Given $X \in Cat(\mathcal{E})$ Corollary 4.4 shows that $X \in Disc(Cat(\mathcal{E}))$ if and only if its identity map $i_x : X_0 \longrightarrow X_1$ is an isomorphism. Using this fact it is straightforward to see that $\mathcal{E}$ is equivalent to $Disc(Cat(\mathcal{E}))$ and we describe this equivalence explicitly now. There is a well known adjunction:

$$\begin{tikzcd}
\mathcal{U}Cat(\mathcal{E}) \arrow[r, shift right=1.5ex, "ob"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right=1.5ex, "{[-]}"']
\end{tikzcd}$$

The right adjoint assigns to an internal category X its object of objects X_0; the left adjoint assigning to an object A of $\mathcal{E}$ the canonical discrete internal category:

$$[A] = \begin{tikzcd}
A \arrow[r, shift left=1.5ex, "1"] \arrow[r, "1" description] \arrow[r, shift right=1.5ex, "1"'] & A \arrow[r, shift left=1.5ex, "1"] \arrow[r, shift right=1.5ex, "1"'] & A \arrow[l, "1" description]
\end{tikzcd}$$

with both functors defined on morphisms in the obvious manner. The functor $[-]$ is fully faithful and the unit of the adjunction is the identity. The counit at X is given by the internal functor:

$$\begin{tikzcd}
X_0 \arrow[r, "{(i_x,i_x)}"] \arrow[d, shift right=1.5ex, "1"'] \arrow[d, "1" description] \arrow[d, shift left=1.5ex, "1"] & X_2 \arrow[d, shift right=1.5ex, "p_x"'] \arrow[d, "m_x" description] \arrow[d, shift left=1.5ex, "q_x"] \\
X_0 \arrow[r, "i_x"] \arrow[d, shift right=1.5ex, "1"'] \arrow[d, shift left=1.5ex, "1"] & X_1 \arrow[d, shift right=1.5ex, "d_x"'] \arrow[d, shift left=1.5ex, "c_x"] \\
X_0 \arrow[r, "1"] \arrow[u, "1" description] & X_0 \arrow[u, "i_x" description]
\end{tikzcd}$$

Now any adjunction restricts to an adjoint equivalence between the full subcategories whose objects are those at which the unit and counit components are respectively isomorphisms. In this case the unit is always an isomorphism; the counit is an isomorphism precisely if X is discrete. Therefore the left adjoint $[-] : \mathcal{E} \longrightarrow \mathcal{U}Cat(\mathcal{E})$ restricts to an equivalence $[-] : \mathcal{E} \longrightarrow Disc(Cat(\mathcal{E}))$.

We will require an understanding of the extent to which the discrete objects in a 2-category are closed under limits.

Proposition 4.9. Let $\mathcal{A}$ be any 2-category.

1. Consider a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a diagram $F : \mathcal{J} \longrightarrow \mathcal{A}$. Suppose that the limit of F weighted by W, $lim(F)$, exists. Then $lim(F)$ is discrete if and only if for each $C \in \mathcal{A}$ the category $[\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(C, F-))$ is discrete.

2. $Disc(\mathcal{A})$ is closed in $\mathcal{A}$ under limits.

3. Consider an opspan in $\mathcal{A}$:

$$\begin{tikzcd}
B \arrow[dr, "f"] & \\
& D \\
C \arrow[ur, "g"'] &
\end{tikzcd}$$

with B and C discrete. If the comma object $f|g$ exists then it is also discrete.

Proof. 1. By the definition of limit we have for each $C \in \mathcal{A}$ an isomorphism of categories: $\mathcal{A}(C, lim(F)) \cong [\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(C, F-))$. As discreteness is an isomorphism invariant of categories the result is immediate.

2. Suppose then that the diagram $F : \mathcal{J} \longrightarrow \mathcal{A}$ takes its values in $Disc(\mathcal{A})$. By the first part of the proposition it suffices to show that $[\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(C, F-))$ is a discrete category, the limit must then be discrete if it exists. An arrow of this category is a modification $\theta : \alpha \Longrightarrow \beta$, consisting of natural transformations:

$$\begin{tikzcd}
Wi \arrow[r, bend left, "\alpha_i"] \arrow[r, bend right, "\beta_i"'] \arrow[r, phantom, "\Downarrow\theta_i"] & \mathcal{A}(C, Fi)
\end{tikzcd}$$

one for each $i \in \mathcal{J}$. Each natural transformation θ_i is determined by its components: arrows of $\mathcal{A}(C, Fi)$. But for each $i \in \mathcal{J}$ the category $\mathcal{A}(C, Fi)$ is discrete by assumption. Therefore θ is an identity modification. Consequently $[\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(C, F-))$ is discrete.

3. Let $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ be the weight for comma objects described in Chapter 2. Identifying the above opspan with the corresponding diagram $F : \mathcal{J} \longrightarrow \mathcal{A}$ we must show, using the first part of the present proposition, that for each $A \in \mathcal{A}$ the category $[\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(A, F-))$ is discrete. An arrow of this category is a diagram of the form:

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow s"] \arrow[dr, "0"'] & & \mathcal{A}(A,B) \arrow[dr, "f_*"] & \\
& \mathbf{2} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow r"] & & \mathcal{A}(A,D) \\
\mathbf{1} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow t"] \arrow[ur, "1"] & & \mathcal{A}(A,C) \arrow[ur, "g_*"'] &
\end{tikzcd}$$

serially commutative. Now $\mathcal{A}(A, B)$ and $\mathcal{A}(A, C)$ are discrete by assumption. Therefore s and t are both identities. We must show that r is also an identity. Now $\mathbf{2}$ has two objects 0 and 1. We have $r(0) = fs$ and $r(1) = gt$. Therefore both $r(0)$ and $r(1)$ are identities and so r itself is an identity. Therefore $[\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(A, F-))$ is discrete. Consequently the limit, the comma object in this case, is discrete if it exists.

□

Proposition 4.10. We have a 2-functor $Disc(-) : \mathrm{Rep} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$, sending a representable 2-category to its category of discrete objects, and acting on 1 and 2-cells by restriction.

Proof. Any representable 2-category $\mathcal{A}$ has pullbacks. By Proposition 4.9(2) $Disc(\mathcal{A})$ is closed under limits in $\mathcal{A}$ and therefore has pullbacks. Thus $Disc(\mathcal{A}) \in \mathrm{Cat}_{\mathrm{pb}}$.
Any 2-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Rep preserves cotensors with **2**. By Corollary 4.3(1) any morphism of Rep preserves discreteness so that we obtain a functor $Disc(F) : Disc(\mathcal{A}) \longrightarrow Disc(\mathcal{B})$ by restriction. This preserves pullbacks since F does so. The extension of $Disc(-)$ to 2-cells of Rep is by restriction again. ☐

Notation 4.11. If $\mathcal{A}$ is a representable 2-category it follows that the category $Disc(\mathcal{A})$ has pullbacks and so we may consider $Cat(Disc(\mathcal{A}))$ the 2-category of categories internal to $Disc(\mathcal{A})$. An object of $Cat(Disc(\mathcal{A}))$ is an internal category in $\mathcal{A}$ each component of which is discrete. Consequently we refer to the objects of $Cat(Disc(\mathcal{A}))$ as *pointwise discrete categories* in $\mathcal{A}$.

Remark 4.12. Each pointwise discrete category in $\mathcal{A}$ is a catead by Remark 4.7.

4.2 Projectives

In this section we define a 2-categorical notion of projectivity analogous to the notion of "regular projective" in ordinary category theory. Recall that an object a of a category A is regular projective if the representable $A(a,-) : A \longrightarrow Set$ preserves regular epis. As regular epimorphisms in Set are precisely the surjective functions this may be rephrased as follows:
Given a regular epi $f : b \longrightarrow c$ the function $A(a,f) : A(a,b) \longrightarrow A(a,c)$ is surjective. In other words for each arrow $g : a \longrightarrow c$ there exists a factorisation:

$$\begin{tikzcd}
a \arrow[d, "\exists"'] \arrow[dr, "g"] & \\
b \arrow[r, "f"'] & c
\end{tikzcd}$$

We now consider the corresponding 2-categorical notion obtained upon replacing the regular epis by codescent morphisms.

Definition 4.13. An object A of a 2-category $\mathcal{A}$ is said to be projective if the representable $\mathcal{A}(A,-) : \mathcal{A} \longrightarrow \mathrm{Cat}$ preserves codescent morphisms.

Remark 4.14. Codescent morphisms in Cat are precisely the bijections on objects by Corollary 2.68. Therefore an object $A \in \mathcal{A}$ is projective if and only if given a codescent morphism $f : B \longrightarrow C \in \mathcal{A}$ the functor $\mathcal{A}(A,f) : \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C)$ is bijective on objects. In other words for each arrow $g : A \longrightarrow C$ there exists a *unique* factorisation:

$$\begin{tikzcd}
A \arrow[d, "\exists!"'] \arrow[dr, "g"] & \\
B \arrow[r, "f"'] & C
\end{tikzcd}$$

Definition 4.15. Let $\mathcal{A}$ be a 2-category. A full subcategory $\mathcal{P}$ of the underlying category $\mathcal{U}\mathcal{A}$ is said to be a projective cover of $\mathcal{A}$ if:

1. Each object of $\mathcal{P}$ is projective.
2. For each object $A \in \mathcal{A}$ there exists an object $P \in \mathcal{P}$ and a codescent morphism $P \longrightarrow A$.

We say that P covers A and that $\mathcal{P}$ covers $\mathcal{A}$.

Proposition 4.16. Let $\mathcal{A}$ be a 2-category.

1. Consider $A \in \mathcal{A}$ and suppose that P_1, P_2 are projectives which cover A via codescent morphisms $f : P_1 \longrightarrow A$ and $g : P_2 \longrightarrow A$. Then there exists a unique arrow $h : P_1 \longrightarrow P_2$ such that $gh = f$ and furthermore h is an isomorphism.

2. Suppose that $\mathcal{A}$ has a projective cover $\mathcal{P}$. Then it has a maximal projective cover $\overline{\mathcal{P}}$ which is the unique replete projective cover of $\mathcal{A}$. The inclusion $\mathcal{P} \longrightarrow \overline{\mathcal{P}}$ is an equivalence of categories; in particular any two projective covers are equivalent.

3. Let $\mathcal{P}$ be a full subcategory of $\mathcal{UA}$ with each object projective. Then $\mathcal{P}$ is a projective cover of $\mathcal{A}$ if and only if the inclusion $\iota : \mathcal{P} \longrightarrow \mathcal{UA}$ has a right adjoint whose counit components are codescent morphisms.

Proof. 1. Consider the codescent morphisms $f : P_1 \longrightarrow A$ and $g : P_2 \longrightarrow A$. As P_1 is projective and g a codescent morphism there exists a unique arrow $P_1 \longrightarrow P_2$ such that the triangle below commutes:

$$\begin{tikzcd}
& P_1 \arrow[dr, "f"] \arrow[dl, shift left, "\exists!"] & \\
P_2 \arrow[ur, shift left, "\exists!"] \arrow[rr, "g"'] & & A
\end{tikzcd}$$

Similarly there exists a unique arrow $P_2 \longrightarrow P_1$ making the triangle commute. That these are inverse follows from the uniqueness of factorisations.

2. Let $\mathcal{P}$ be a projective cover of $\mathcal{A}$. Then $\mathcal{A}$ has a maximal projective cover $\overline{\mathcal{P}}$ with objects all projectives in $\mathcal{A}$. As projectivity is an isomorphism invariant $\overline{\mathcal{P}}$ is replete. The inclusion $\iota : \mathcal{P} \longrightarrow \overline{\mathcal{P}}$ is fully faithful as both $\mathcal{P}$ and $\overline{\mathcal{P}}$ are by definition full subcategories of $\mathcal{UA}$. Given an object Q of $\overline{\mathcal{P}}$ the identity morphism $1 : Q \longrightarrow Q$ is a codescent morphism. For observe that Q is the codescent object of the coherence data:

$$\begin{tikzcd}
Q \arrow[r, shift left=2, "1"] \arrow[r, "1" description] \arrow[r, shift right=2, "1"'] & Q \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & Q \arrow[l, "1" description]
\end{tikzcd}$$

with universal cocone $(Q, 1, 1)$. As Q is projective the identity morphism $1 : Q \longrightarrow Q$ is a cover of Q. Since $\mathcal{P}$ is a projective cover there exists some $P \in \mathcal{P}$ covering Q. Part 1 of the present proposition thus implies P and Q are isomorphic. Therefore the inclusion $\iota : \mathcal{P} \longrightarrow \overline{\mathcal{P}}$ is essentially surjective and so an equivalence of categories. As equivalence of categories is an equivalence relation we see that any two projective covers are equivalent.

3. The inclusion $\mathcal{P} \longrightarrow \mathcal{UA}$ has a right adjoint if and only if for each $A \in \mathcal{A}$ there exists some $P \in \mathcal{P}$ and morphism $P \longrightarrow A$ with the universal property that any other morphism $Q \longrightarrow A$ with $Q \in \mathcal{P}$ factors uniquely through it.
Suppose that $\mathcal{P}$ is a projective cover of $\mathcal{A}$. Then there exists such a $P \in \mathcal{P}$, namely the cover of A, and the covering codescent morphism $P \longrightarrow A$ has the required universal property. Therefore the inclusion has a right adjoint $R : \mathcal{UA} \longrightarrow \mathcal{P}$ with $R(A) = P$ and the counit component at A is precisely the covering codescent morphism $P \longrightarrow A$.
Conversely suppose that the inclusion has such a right adjoint. Then the codescent morphisms of the counit exhibit $\mathcal{P}$ as a projective cover of $\mathcal{A}$.

□

Proposition 4.17. Let $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$. Then each discrete object in $Cat(\mathcal{E})$ is projective and $Disc(Cat(\mathcal{E}))$ is a projective cover of $Cat(\mathcal{E})$. Furthermore it is the maximal such.

Proof. Each discrete object in $Cat(\mathcal{E})$ is isomorphic to one of the form $[A]$ for some $A \in \mathcal{E}$ by Remark 4.8. Projectivity is an isomorphism invariant and so it suffices to verify that $[A]$ is projective for each $A \in \mathcal{E}$. By Theorem 3.68 the codescent morphisms in $Cat(\mathcal{E})$ are precisely the bijections on objects. Consider such an internal functor $f : X \longrightarrow Y$, so that $f_0 : X_0 \longrightarrow Y_0$ an isomorphism, and an internal functor $g : [A] \longrightarrow Y$.

We must show that there exists a unique arrow $[A] \longrightarrow X$ rendering the triangle:

$$\begin{tikzcd}[A] \arrow[d, "\exists!"'] \arrow[dr, "g"] & \\ X \arrow[r, "f"'] & Y\end{tikzcd}$$

commutative. Transposing across the adjunction:

$$\begin{tikzcd}\mathcal{U}Cat(\mathcal{E}) \arrow[r, bend right, "ob_{\mathcal{E}}"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend right, "{[-]}"']\end{tikzcd}$$

we see this is equally to give a factorisation in $\mathcal{E}$:

$$\begin{tikzcd}A \arrow[d, "\exists!"'] \arrow[dr, "g_0"] & \\ X_0 \arrow[r, "f_0"'] & Y_0\end{tikzcd}$$

As f_0 is an isomorphism the arrow $f_0^{-1}g_0 : A \longrightarrow X_0$ gives a unique factorisation in $\mathcal{E}$. Transposing back across the adjunction gives the unique factorisation in $Cat(\mathcal{E})$. Thus each discrete object is projective.
At an internal category $X \in Cat(\mathcal{E})$ the counit component of the above adjunction $[X_0] \longrightarrow X$, as described in Remark 4.8, is internally bijective on objects and so a codescent morphism. Thus $Disc(Cat(\mathcal{E}))$ is a projective cover of $Cat(\mathcal{E})$. As discreteness is an isomorphism invariant $Disc(Cat(\mathcal{E}))$ is replete and so the maximal projective cover of $Cat(\mathcal{E})$ by Proposition 4.16(2). ☐

4.3 A characterisation of $Cat(\mathcal{E})$

Theorem 4.18. Let $\mathcal{A}$ be a 2-category. Then $\mathcal{A}$ is 2-equivalent to $Cat(\mathcal{E})$ for some category $\mathcal{E}$ with pullbacks if and only if:

1. $\mathcal{A}$ is a representable 2-category.
2. $\mathcal{A}$ has codescent objects of cateads and they are effective.
3. Codescent morphisms are effective in $\mathcal{A}$.
4. Discrete objects in $\mathcal{A}$ are projective.
5. For each object $A \in \mathcal{A}$ there exists a discrete object P and a codescent morphism $P \longrightarrow A$.

In particular $\mathcal{A} \simeq Cat(Disc(\mathcal{A}))$.

Proof. Firstly consider $Cat(\mathcal{E})$ for some $\mathcal{E}$ with pullbacks. In Chapter 3 we verified the first three conditions of the theorem. By Proposition 4.17 the discrete objects are a projective cover of $Cat(\mathcal{E})$ thereby verifying the fourth and fifth conditions. Furthermore by Remark 4.8 we have an equivalence of categories $\mathcal{E} \simeq Disc(Cat(\mathcal{E}))$. Both $\mathcal{E}$ and $Disc(Cat(\mathcal{E}))$ have pullbacks and any equivalence of categories preserves pullbacks. Therefore the equivalence lies in Cat_{pb}. Any 2-functor preserves equivalences. Therefore the 2-functor $Cat(-) : \text{Cat}_{\text{pb}} \longrightarrow \text{Rep}$ takes the equivalence $\mathcal{E} \simeq Disc(Cat(\mathcal{E}))$ in Cat_{pb} to an equivalence $Cat(\mathcal{E}) \simeq Cat(Disc(Cat(\mathcal{E})))$ in Rep; a 2-equivalence of 2-categories.
Conversely let $\mathcal{A}$ be a 2-category satisfying the above conditions. (4) and (5) together assert that $Disc(\mathcal{A})$ is a projective cover of $\mathcal{A}$. By Proposition 4.16(3) we have an adjunction:

$$\begin{tikzcd}\mathcal{U}\mathcal{A} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & Disc(\mathcal{A}) \arrow[l, bend right, "\iota"']\end{tikzcd}$$

with counit a pointwise codescent morphism. We denote the counit by $\mu : \iota R \Longrightarrow 1$. Now $\mathcal{A}$ is a representable 2-category so that $\mathcal{UA}$ has pullbacks. By Proposition 4.9(2) $Disc(\mathcal{A})$ also has pullbacks and the inclusion ι preserves them. The right adjoint R necessarily preserves pullbacks; thus the adjunction lies in $\mathrm{Cat}_{\mathrm{pb}}$. Since the inclusion ι is injective on objects and fully faithful we may choose the unit to be the identity. Applying $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$ to it gives an adjunction in Rep:

$$\begin{tikzcd}
Cat(\mathcal{UA}) \arrow[r, shift right, "Cat(R)"'] \arrow[r, phantom, "\perp"] & Cat(Disc(\mathcal{A})) \arrow[l, shift right, "Cat(\iota)"']
\end{tikzcd}$$

where the left adjoint is again just the inclusion and the counit is now the map $Cat(\mu)$. The objects of $Cat(Disc(\mathcal{A}))$ are of course the pointwise discrete categories in $\mathcal{A}$. By Remark 4.12 these are cateads in $\mathcal{A}$. Thus the inclusion $Cat(\iota) : Cat(Disc(\mathcal{A})) \longrightarrow Cat(\mathcal{UA})$ factors through its image in $Kat(\mathcal{A})$ via inclusions:

$$\begin{tikzcd}
Cat(Disc(\mathcal{A})) \arrow[r, "\iota"] & Kat(\mathcal{A}) \arrow[r, "j"] & Cat(\mathcal{UA})
\end{tikzcd}$$

Both inclusions are again morphisms of Rep since $Kat(\mathcal{A})$ is closed in $Cat(\mathcal{UA})$ upon pullbacks and cotensors with **2** by Proposition 3.44. Having factored the inclusion through its image, we rephrase to obtain another 2-adjunction in Rep:

$$\begin{tikzcd}
Kat(\mathcal{A}) \arrow[r, shift right, "Cat(R)j"'] \arrow[r, phantom, "\perp"] & Cat(Disc(\mathcal{A})) \arrow[l, shift right, "\iota"']
\end{tikzcd}$$

Properties (1) and (2) of the Theorem ensure, by Proposition 3.47, that we have the 2-functor $\hat{\Delta} : \mathcal{A} \longrightarrow Kat(\mathcal{A})$ in Rep and that it is has a left 2-adjoint $Q : Kat(\mathcal{A}) \longrightarrow \mathcal{A}$ given by taking codescent objects and furthermore the unit $\rho : 1 \Longrightarrow \hat{\Delta}Q$ of that adjunction is pointwise fully faithful. The counit is an isomorphism as $\hat{\Delta}$ is 2-fully faithful by Corollary 3.62. Thus we have a composite 2-adjunction:

$$\begin{tikzcd}
A \arrow[r, shift right, "\hat{\Delta}"'] \arrow[r, phantom, "\perp"] & Kat(\mathcal{A}) \arrow[l, shift right, "Q"'] \arrow[r, shift right, "Cat(R)j"'] \arrow[r, phantom, "\perp"] & Cat(Disc(\mathcal{A})) \arrow[l, shift right, "\iota"']
\end{tikzcd}$$

which we will show to be an adjoint 2-equivalence. We must show then that the unit and counit of the composite adjunction are isomorphisms. In the following, for simplicity of notation, we omit to mention the inclusion $j : Kat(\mathcal{A}) \longrightarrow Cat(\mathcal{UA})$ which simply views each catead as a internal category in $\mathcal{A}$. Since each internal category in $\mathcal{A}$ which appears is a catead this will be unnecessary. We continue to mention the inclusion $\iota : Cat(Disc(\mathcal{A})) \longrightarrow Kat(\mathcal{A})$ as not all cateads which appear are pointwise discrete categories. The unit and counit are respectively the composites:

$$1 = Cat(R)\iota \xRightarrow{Cat(R)\rho_\iota} Cat(R)\hat{\Delta}Q\iota \qquad \text{and} \qquad Q\iota Cat(R)\hat{\Delta} \xRightarrow{QCat(\mu)_{\hat{\Delta}}} Q\hat{\Delta} \cong 1$$

using that the unit of the right 2-adjunction is the identity and the counit of the left one an isomorphism. Consequently it suffices to show that both:

$$(1) \qquad Cat(R)\iota \xRightarrow{Cat(R)\rho_\iota} Cat(R)\hat{\Delta}Q\iota \qquad \text{and} \qquad (2) \qquad Q\iota Cat(R)\hat{\Delta} \xRightarrow{QCat(\mu)_{\hat{\Delta}}} Q\hat{\Delta}$$

are isomorphisms.

Consider (1). We must show that given a pointwise discrete category X the internal functor:

$$Cat(R)\rho_{\iota(X)} : Cat(R)\iota(X) \longrightarrow Cat(R)\hat{\Delta}Q\iota(X)$$

is invertible. It suffices to show that this internal functor is both bijective on objects and fully faithful, since these form orthogonal classes of a factorisation system on $Cat(Disc(\mathcal{A}))$. Now $\rho_{\iota(X)}$ is fully faithful as ρ is pointwise fully faithful. As $Cat(R)$ is a morphism of Rep it preserves fully faithful arrows by Corollary 3.53. Thus $Cat(R)\rho_{\iota(X)}$ is indeed fully faithful. It remains to show that it is bijective on objects. To see this we need to consider the action of R, and thus $Cat(R)$, in more detail. Consider again then the adjunction:

$$\begin{tikzcd}
\mathcal{U}\mathcal{A} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & Disc(\mathcal{A}) \arrow[l, bend right, "\iota"']
\end{tikzcd}$$

Given $A \in \mathcal{A}$ $R(A)$ is the discrete, equally projective, object of $\mathcal{A}$ covering A by the codescent morphism $\mu_A : R(A) \longrightarrow A$. Given $f : A \longrightarrow B$ of $\mathcal{A}$ its image under R is the unique arrow rendering commutative the square on the left below:

$$\begin{tikzcd}
R(A) \arrow[r, "\mu_A"] \arrow[d, "\exists ! R(f)"'] & A \arrow[d, "f"] \\
R(B) \arrow[r, "\mu_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[ld, "\exists ! R(f)"'] \arrow[d, "f"] \\
R(B) \arrow[r, "\mu_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_0 \arrow[ld, "\exists !"'] \arrow[d, "\rho_{\iota(X)_0}"] \\
R(QX) \arrow[r, "\mu_{Q\iota X}"'] & QX
\end{tikzcd}$$

induced by the projectivity of $R(A)$ and the codescent morphism μ_B. We chose that $R\iota = 1$ with identity unit from which it follows that the counit at each object in $Disc(\mathcal{A})$ is also the identity (each discrete object is covered by the identity morphism upon it). Therefore it follows that given $f : A \longrightarrow B$ with A discrete, the arrow $R(f) : R(A) \longrightarrow R(B)$ is the unique one such that the middle triangle above commutes. Now $Cat(R)$ simply acts as R pointwise. Since X is a pointwise discrete category we therefore have $Cat(R)X = X$ and in particular $(Cat(R)\rho_{\iota(X)})_0 = R(\rho_{\iota(X)_0}) : R(X_0) \longrightarrow R(Q\iota(X))$ is the unique arrow such that the rightmost triangle above commutes. But as described in Proposition 3.47 $\rho_{\iota(X)_0} : X_0 \longrightarrow Q\iota X$ is the codescent morphism exhibiting $Q\iota X$ as the codescent object of the catead X. Now as X is a pointwise discrete category X_0 is discrete and thus projective. Consequently by Proposition 4.16(1) $(Cat(R)\rho_{\iota(X)})_0$ is an isomorphism which is to say that $Cat(R)\rho_{\iota(X)}$ is bijective on objects. Therefore it is both bijective on objects and fully faithful. As these form orthogonal classes it is an isomorphism.

Consider (2). We must show for each object $A \in \mathcal{A}$ that:

$$QCat(\mu)_{\hat{\Delta}(A)} : Q\iota Cat(R)\hat{\Delta}(A) \longrightarrow Q\hat{\Delta}(A)$$

is an isomorphism. In other words that Q inverts the internal functor $Cat(\mu)_{\hat{\Delta}(A)} : \iota Cat(R)\hat{\Delta}(A) \longrightarrow \hat{\Delta}(A)$. This is the internal functor below:

$$(3) \qquad \begin{tikzcd}
R(A^{\mathbf{3}}) \arrow[r, "\mu_{A^{\mathbf{3}}}"] \arrow[d, shift right=2, "R(p_a)"'] \arrow[d] \arrow[d, shift left=2, "R(q_a)"] & A^{\mathbf{3}} \arrow[d, shift right=2, "p_a"'] \arrow[d, "m_a" description] \arrow[d, shift left=2, "q_a"] \\
R(A^{\mathbf{2}}) \arrow[r, "\mu_{A^{\mathbf{2}}}"] \arrow[d, shift right=2, "R(d_a)"'] \arrow[d, shift left=2, "R(c_a)"] & A^{\mathbf{2}} \arrow[d, shift right=2, "d_a"'] \arrow[d, shift left=2, "c_a"] \\
R(A) \arrow[r, "\mu_A"] \arrow[u] & A \arrow[u, "i_a" description]
\end{tikzcd}$$

with components the covering codescent morphisms. In particular $\mu_A : R(A) \longrightarrow A$ is a codescent morphism and we claim furthermore that $\iota Cat(R)\hat{\Delta}(A)$ is the higher kernel of μ_A exhibited by the comma cone:

$$\begin{tikzcd}
 & R(A) \arrow[rrd, "\mu_A"] & & \\
R(A^{\mathbf{2}}) \arrow[ru, "R(d_a)"] \arrow[rd, "R(c_a)"'] \arrow[r, "\mu_{A^{\mathbf{2}}}"] & A^{\mathbf{2}} \arrow[rr, bend left, "d_a"] \arrow[rr, bend right, "c_a"'] \arrow[rr, phantom, "\Downarrow \eta_a"] & & A \\
 & R(A) \arrow[rru, "\mu_A"'] & &
\end{tikzcd}$$

By Lemma 3.46 this amounts to showing that the above internal functor is fully faithful, which is to say that the diagram on the left below:

$$\begin{tikzcd}
 & R(A) \arrow[dr, "\mu_A"] & \\
 & & A \\
R(A^{\mathbf{2}}) \arrow[uur, "R(d_a)"] \arrow[r, "\mu_{A^{\mathbf{2}}}"] \arrow[ddr, "R(c_a)"'] & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[dr, "c_a"'] & \\
 & & A \\
 & R(A) \arrow[ur, "\mu_A"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & R(A) \arrow[dr, "\mu_A"] & \\
 & & A \\
\mu_A|\mu_A \arrow[uur, "f"] \arrow[r, "g"] \arrow[ddr, "h"'] & A^{\mathbf{2}} \arrow[ur, "d_a"] \arrow[dr, "c_a"'] & \\
 & & A \\
 & R(A) \arrow[ur, "\mu_A"'] &
\end{tikzcd}$$

exhibits $R(A^{\mathbf{2}})$ as the limit of the double opspan. The actual limit of that double opspan is the comma object $\mu_A|\mu_A$, drawn on the right. By Proposition 4.9(3) the comma object is discrete since $R(A)$ is, and therefore projective. The above cone on the left induces a unique arrow $k : R(A^{\mathbf{2}}) \longrightarrow \mu_A|\mu_A$ into the limit which recovers that cone upon precomposing the limiting cone by k. It will suffice to show that k is invertible. Since $\mu_{A^{\mathbf{2}}} : R(A^{\mathbf{2}}) \longrightarrow A^{\mathbf{2}}$ is a codescent morphism and $\mu_A|\mu_A$ projective there exists a unique arrow $l : \mu_A|\mu_A \longrightarrow R(A^{\mathbf{2}})$ such that the triangle:

$$\begin{tikzcd}
\mu_A|\mu_A \arrow[d, "l"'] \arrow[dr, "g"] & \\
R(A^{\mathbf{2}}) \arrow[r, "\mu_{A^{\mathbf{2}}}"'] & A^{\mathbf{2}}
\end{tikzcd}$$

commutes. We will show that l is inverse to k. To see that $kl : \mu_A|\mu_A \longrightarrow \mu_A|\mu_A$ is the identity it suffices to show that postcomposing with the limiting cone we have $fkl = f$, $gkl = g$ and $hkl = h$. By definition of k we have $fk = R(d_a)$, $gk = \mu_{A^{\mathbf{2}}}$ and $hk = R(c_a)$. By definition of l we then have $gkl = \mu_{A^{\mathbf{2}}}l = g$ as desired. Now fkl is then the morphism $R(d_a)l : \mu_A|\mu_A \longrightarrow R(A)$. Since $\mu_A : R(A) \longrightarrow A$ is a codescent morphism and $\mu_A|\mu_A$ projective the functor $\mathcal{A}(\mu_A|\mu_A, \mu_A) : \mathcal{A}(\mu_A|\mu_A, R(A)) \longrightarrow \mathcal{A}(\mu_A|\mu_A, A)$ is bijective on objects. Thus to show that $R(d_a)l = f$ it suffices to show that $\mu_A R(d_a)l = \mu_A f$. But $\mu_A f = d_a g$ whilst $\mu_A R(d_a)l = d_a \mu_{A^{\mathbf{2}}} l = d_a g$. Similarly $\mu_A hkl = \mu_A l$ and so $hkl = l$. Thus we have $kl = 1$. We need finally to show that $lk : R(A^{\mathbf{2}}) \longrightarrow R(A^{\mathbf{2}})$ is the identity. Since $R(A^{\mathbf{2}})$ is projective and $\mu_{A^{\mathbf{2}}} : R(A^{\mathbf{2}}) \longrightarrow R(A)$ a codescent morphism it suffices to show that $\mu_{A^{\mathbf{2}}} lk = \mu_{A^{\mathbf{2}}}$. We have $\mu_{A^{\mathbf{2}}} lk = gk = \mu_{A^{\mathbf{2}}}$ first using the definition of l and then that of k. Therefore k is invertible.

Consequently $R(A^{\mathbf{2}}) = \mu_A|\mu_A$ with the claimed comma cone above, and this comma cone exhibits $\iota Cat(R)\hat{\Delta}(A)$ as the higher kernel of $\mu_A : R(A) \longrightarrow A$. Now μ_A is a codescent morphism. As codescent morphisms are effective in $\mathcal{A}$, the triple $(A, \mu_A, \eta_a \mu_{A^{\mathbf{2}}})$ exhibits A as the codescent object of $Cat(R)\hat{\Delta}(A)$. $\hat{\Delta}$ is fully faithful so that $A \cong Q\hat{\Delta}(A)$ and furthermore A is the canonical codescent object of $\hat{\Delta}(A)$ with exhibiting cocone $(A, 1_a, \eta_a)$. Now Q assigns to the catead morphism $Cat(\mu)_{\hat{\Delta}(A)}$ of diagram (3) the unique map between the codescent objects taking the respective codescent cones onto one another. Taking these canonical choices of codescent objects and codescent cocones $(A, \mu_A, \eta_a \mu_{A^{\mathbf{2}}})$ and $(A, 1_a, \eta_a)$ the induced map $s : A \longrightarrow A$ between the codescent objects is the unique one satisfying $s \circ \mu_A = 1 \circ \mu_A$ and $s \circ \eta_a \circ \mu_{A^{\mathbf{2}}} = \eta_a \circ \mu_{A^{\mathbf{2}}}$ so that s is the identity on A, in particular an isomorphism. Whilst Q is only defined up isomorphism the fact that the unique induced map $Q(Cat(\mu)_{\hat{\Delta}(A)})$ between the codescent objects is an isomorphism is independent of the particular choice taken by Q. Therefore $QCat(\mu)_{\hat{\Delta}(A)}$ is an isomorphism.

Consequently $\mathcal{A}$ is 2-equivalent to $Cat(Disc(\mathcal{A}))$. ☐

Example 4.19. In Theorem 4.18 we characterised up to 2-equivalence those 2-categories of the form $Cat(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks. We did so by means of constructing an adjoint 2-equivalence. We now examine that adjoint 2-equivalence in the case of $Cat(\mathcal{E})$ itself and the canonical presentation of an internal category as a codescent object witnessed by it. Given a category $\mathcal{E}$ with pullbacks we have the adjoint 2-equivalence:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "\hat{\Delta}_{Cat(\mathcal{E})}"'] & Kat(Cat(\mathcal{E})) \arrow[l, shift right=2, "Q_{Cat(\mathcal{E})}"'] \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "Cat(R)j_{Cat(\mathcal{E})}"'] & Cat(Disc(Cat(\mathcal{E}))) \arrow[l, shift right=2, "\iota"']
\end{tikzcd}$$

constructed in Theorem 4.18. The functor $R : \mathcal{U}Cat(\mathcal{E}) \longrightarrow Disc(Cat(\mathcal{E}))$ sending an internal category to its projective cover is the composite:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[r, "ob_{\mathcal{E}}"] & E \arrow[r, "{[-]_{\mathcal{E}}}"] & Disc(Cat(\mathcal{E}))
\end{tikzcd}$$

which assigns to an internal category X the canonical discrete internal category $[X_0]$. Therefore the right 2-adjoint of the equivalence is the composite:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[r, "\hat{\Delta}_{Cat(\mathcal{E})}"] & Kat(Cat(\mathcal{E})) \arrow[r, "j_{Cat(\mathcal{E})}"] & Cat(\mathcal{U}Cat(\mathcal{E})) \arrow[r, "Cat(ob_{\mathcal{E}})"] & Cat(\mathcal{E}) \arrow[r, "{Cat([-]_{\mathcal{E}})}"] & Cat(Disc(Cat(\mathcal{E}))
\end{tikzcd}$$

Now $j_{Cat(\mathcal{E})} \circ \hat{\Delta}_{Cat(\mathcal{E})} = \Delta'_{Cat(\mathcal{E})}$ by definition of $\hat{\Delta}$ and the composite $Cat(ob_{\mathcal{E}}) \circ \Delta'_{Cat(\mathcal{E})}$ is the identity 2-functor on $Cat(\mathcal{E})$ by Lemma 3.31(1). Therefore the composite right 2-adjoint is simply $Cat([-]_{\mathcal{E}}) : Cat(\mathcal{E}) \longrightarrow Cat(Disc(Cat(\mathcal{E})))$.
Applied at an internal category X we have:

$$\begin{tikzcd}
Cat([-]_{\mathcal{E}})X = {[X_2]} \arrow[r, shift left=3, "{[p_x]}"] \arrow[r, "{[m_x]}" description] \arrow[r, shift right=3, "{[q_x]}"'] & {[X_1]} \arrow[r, shift left=3, "{[d_x]}"] \arrow[r, shift right=3, "{[c_x]}"'] & {[X_0]} \arrow[l, "{[i_x]}" description]
\end{tikzcd}$$

The left 2-adjoint also admits a simple description. We saw in Proposition 3.64(1) that the left 2-adjoint to $\hat{\Delta} : Cat(\mathcal{E}) \longrightarrow Kat(Cat(\mathcal{E}))$ is the composite:

$$\begin{tikzcd}
Kat(Cat(\mathcal{E})) \arrow[r, "j_{Cat(\mathcal{E})}"] & Cat(\mathcal{U}Cat(\mathcal{E})) \arrow[r, "Cat(ob_{\mathcal{E}})"] & Cat(\mathcal{E})
\end{tikzcd}$$

Therefore writing $ob_{\mathcal{E}} : Disc(Cat(\mathcal{E})) \longrightarrow \mathcal{E}$ for the restriction of $ob_{\mathcal{E}} : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ to the discrete internal categories in $\mathcal{E}$ we see that the composite left 2-adjoint is simply $Cat(ob_{\mathcal{E}}) : Cat(Disc(Cat(\mathcal{E})) \longrightarrow Cat(\mathcal{E})$. Consequently the above adjoint 2-equivalence simply reduces to:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "{Cat([-]_{\mathcal{E}})}"'] & Cat(Disc(Cat(\mathcal{E})) \arrow[l, shift right=2, "Cat(ob_{\mathcal{E}})"']
\end{tikzcd}$$

As the unit is an isomorphism this asserts that X is the codescent object of the above pointwise discrete category in $Cat(\mathcal{E})$. The exhibiting codescent cocone admits a simple description. We denote the codescent cocone by $([X_0], \epsilon_x, \theta_x)$. The codescent morphism $\epsilon_x : [X_0] \longrightarrow X$ is the counit of the adjunction:

$$\begin{tikzcd}
\mathcal{U}Cat(\mathcal{E}) \arrow[r, phantom, "\perp"] \arrow[r, shift right=2, "ob_{\mathcal{E}}"'] & \mathcal{E} \arrow[l, shift right=2, "{[-]}"']
\end{tikzcd}$$

described explicitly in Remark 4.8. The exhibiting internal natural transformation:

$$\begin{tikzcd}
 & {[X_0]} \arrow[dr, "\epsilon_x"] \arrow[dd, phantom, "\Downarrow\theta_x"] & \\
{[X_1]} \arrow[ur, "{[d_x]}"] \arrow[dr, "{[c_x]}"'] & & X \\
 & {[X_0]} \arrow[ur, "\epsilon_x"'] &
\end{tikzcd}$$

has arrow component $\overline{\theta_x} = 1 : [X_1]_0 = X_1 \longrightarrow X_1$ the identity morphism.
That the counit is an isomorphism of course asserts that each pointwise discrete category in $Cat(\mathcal{E})$ is isomorphic to $Cat([-]_{\mathcal{E}})X$ for some $X \in Cat(\mathcal{E})$.

4.4 2-functors of the form $Cat(F)$

In the previous section we characterised those 2-categories of the form $Cat(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks up to 2-equivalence. In this section we characterise those 2-functors of the form $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$, for a pullback preserving functor $F : \mathcal{A} \longrightarrow \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$, by means of a biequivalence of 2-categories. We begin by summarising what we have learnt thus far about such 2-functors.
In Chapter 3 we proved that given $F : \mathcal{A} \longrightarrow \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$:

- $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves cotensors with $\mathbf{2}$ and pullbacks (is a morphism of Rep).
- $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves codescent objects of cateads.

These two properties are equivalent to a set of properties which at first glance appear weaker, as shown by Part 2 of Corollary 4.21 below.

Proposition 4.20. Let $\mathcal{A}, \mathcal{B} \in$ Rep have codescent objects of cateads and suppose that both cateads and codescent morphisms are effective in each. Consider a 2-functor $H : \mathcal{A} \longrightarrow \mathcal{B}$.

1. If H preserves codescent objects of cateads then H preserves codescent morphisms.
2. If H is a morphism of Rep and preserves codescent morphisms then it preserves codescent objects of cateads.
3. If H is a morphism of Rep then H preserves codescent objects of cateads if and only if it preserves codescent morphisms.

Proof. 1. Consider a codescent morphism $f : X \longrightarrow Y$ of $\mathcal{A}$. As codescent morphisms are effective f is the codescent morphism of its higher kernel:

$$\begin{tikzcd} f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X \arrow[l, "i" description] \arrow[r, "f"] & Y \end{tikzcd}$$

Each higher kernel is a catead. Therefore H preserves this codescent object; in particular $Hf : HX \longrightarrow HY$ is the codescent morphism exhibiting HY as the codescent object of the image of the higher kernel under H.

2. Cateads are effective in $\mathcal{A}$ by assumption and so each catead is the higher kernel of its codescent object. Consider then a catead X in $\mathcal{A}$ together with its codescent object and universal cocone:

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "i" description] \end{tikzcd}
\qquad
\begin{tikzcd} & X_0 \arrow[dr, "f"] & \\ X_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \eta & Q(X) \\ & X_0 \arrow[ur, "f"'] & \end{tikzcd}$$

The catead is the higher kernel of f exhibited by the comma cone (X_1, d, c, η). As H preserves comma objects and pullbacks (as does any morphism of Rep) it preserves higher kernels. Thus H preserves the above higher kernel so that the comma cone $(HX_1, Hd, Hc, H\eta)$ on the right below:

$$\begin{tikzcd} HX_2 \arrow[r, shift left=2, "Hp"] \arrow[r, "Hm" description] \arrow[r, shift right=2, "Hq"'] & HX_1 \arrow[r, shift left=2, "Hd"] \arrow[r, shift right=2, "Hc"'] & HX_0 \arrow[l, "Hi" description] \end{tikzcd}$$

$$\begin{tikzcd} & HX_0 \arrow[dr, "Hf"] & \\ HX_1 \arrow[ur, "Hd"] \arrow[dr, "Hc"'] & \Downarrow H\eta & HQ(X) \\ & HX_0 \arrow[ur, "Hf"'] & \end{tikzcd}$$

exhibits the catead HX as the higher kernel of Hf. Hf is a codescent morphism by assumption since f is one. As codescent morphisms are effective in $\mathcal{B}$ the codescent cocone $(Q(X), Hf, H\eta)$ exhibits QX as the codescent object of its higher kernel HX. Thus H preserves codescent objects of cateads.

3. This combines the results of the previous two parts.

□

Corollary 4.21. Consider $\mathcal{A}, \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$ and $H : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$.

1. If H preserves codescent objects of cateads then H preserves bijections on objects.
2. If $H \in \mathrm{Rep}$ then H preserves codescent objects of cateads if and only it preserves codescent morphisms (bijections on objects).

Proof. Observe firstly that by Theorem 4.18 both $Cat(\mathcal{A})$ and $Cat(\mathcal{B})$ satisfy the hypotheses of Proposition 4.20 and that by Theorem 3.68 codescent morphisms in $Cat(\mathcal{A})$ and $Cat(\mathcal{B})$ are precisely the bijections on objects.

1. The result now follows immediately from Proposition 4.20(1).
2. The result follows immediately from Proposition 4.20(3).

□

Remark 4.22. Given $F : \mathcal{A} \longrightarrow \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$ the 2-functor $Cat(F) \in \mathrm{Rep}$ preserves codescent objects of cateads and thus codescent morphisms by Corollary 4.21(1). (Alternatively note that it is obvious $Cat(F)$ preserves codescent morphisms as they are just the internal bijections on objects and $Cat(F)$ acts pointwise.) We will show these two properties suffice to characterise the 2-functors of this form. We will prove this claim by means of a biequivalence of 2-categories and we recall that notion, and the basic facts about biequivalences now.

Definition 4.23. A pair of pseudofunctors

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right=2, "G"'] & \mathcal{B} \arrow[l, shift right=2, "F"'] \end{tikzcd}$$

form a biequivalence of 2-categories if there exist equivalences $1_{\mathcal{A}} \simeq FG$ and $1_{\mathcal{B}} \simeq GF$ in the 2-categories $Hom(\mathcal{A}, \mathcal{A})$ and $Hom(\mathcal{B}, \mathcal{B})$ respectively.

Remark 4.24. The following characterisation of biequivalences of 2-categories is standard and may be found for instance in [51].

Proposition 4.25.

1. A pair of pseudofunctors, as above, form a biequivalence if and only if there exist pseudonatural transformations $\epsilon : 1_{\mathcal{A}} \Longrightarrow FG$ and $1_{\mathcal{B}} \Longrightarrow GF$ whose components $\epsilon_A : A \longrightarrow FGA$ and $\eta_B : B \longrightarrow GFB$ are equivalences in $\mathcal{A}$ and $\mathcal{B}$ respectively.
2. A pseudofunctor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a biequivalence of 2-categories (has a biequivalence inverse) if and only if:

(a) F is bi-essentially surjective on objects: For each $B \in \mathcal{B}$ there exists some $A \in \mathcal{A}$ and an equivalence $FA \cong B$ in $\mathcal{B}$.

(b) F is locally an equivalence: Given $A_1, A_2 \in \mathcal{A}$ the functor $F_{A_1,A_2} : \mathcal{A}(A_1, A_2) \longrightarrow \mathcal{B}(FA_1, FA_2)$ is an equivalence of categories.

Its biequivalence inverse is unique up to equivalence in $Hom(\mathcal{B}, \mathcal{A})$.

Proof. 1. Given 2-categories $\mathcal{C}$ and $\mathcal{D}$ the evaluation 2-functors $ev_C : Hom(\mathcal{C}, \mathcal{D}) \longrightarrow \mathcal{D}$ jointly reflect the "property" of being an equivalence. Thus pseudonatural transformations $FG \Longrightarrow 1_{\mathcal{A}}$ and $GF \Longrightarrow 1_{\mathcal{B}}$ which are "pointwise equivalences" are necessarily equivalences in the respective 2-categories $Hom(\mathcal{A}, \mathcal{A})$ and $Hom(B, B)$.

2. This is a 2-categorical analogue of the characterisation of equivalences of categories (equivalences in the 2-category Cat). As F is bi-essentially surjective one can define the object function of a pseudofunctor G in the opposite direction. As F is only locally an equivalence one can only extend G to a pseudofunctor, even if F is a 2-functor, in contrast to the 1-categorical case, in which the functoriality of the equivalence inverse arises from the fact that F is fully faithful (locally an isomorphism).

☐

Definition 4.26. We denote by $\mathbf{B}$ the following locally full sub 2-category of Rep.

- Objects: those 2-categories satisfying the conditions (1-5) of Theorem 4.18.
- 1-cells: 2-functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ of Rep which preserve codescent morphisms.
- 2-cells: 2-natural transformations.

Remark 4.27. Each 2-category of the form $Cat(\mathcal{E})$ lies in $\mathbf{B}$ by Theorem 4.18. Given $F \in \mathrm{Cat}_{\mathrm{pb}}$ we have $Cat(F) \in \mathbf{B}$ by Remark 4.22. Thus $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$ takes its image in $\mathbf{B}$ and we obtain $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathbf{B}$. Restricting $Disc(-) : \mathrm{Rep} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$ to the sub 2-category $\mathbf{B}$ we obtain $Disc(-) : \mathbf{B} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$. We show in the following theorem that these 2-functors form a biequivalence.

Theorem 4.28. We have a biequivalence of 2-categories:

$$\begin{tikzcd}
\mathrm{Cat}_{\mathrm{pb}} \arrow[r, shift right, "Cat(-)"'] & \mathbf{B} \arrow[l, shift right, "Disc(-)"']
\end{tikzcd}$$

Proof. For each $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$ we have the equivalence of categories $[-]_{\mathcal{E}} : \mathcal{E} \longrightarrow Disc(Cat(\mathcal{E}))$ described in Remark 4.8 which assigns to an object of $\mathcal{E}$ the canonical discrete internal category in $\mathcal{E}$ upon it. This is clearly 2-natural in $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$; therefore it is a pseudonatural equivalence $[-] : 1 \Longrightarrow Disc(Cat(-))$ by Proposition 4.25(1). It remains therefore to describe a pseudonatural equivalence $1 \Longrightarrow Cat(Disc(-))$. We take the arrow components, at $\mathcal{A} \in \mathbf{B}$, to be those right adjoint 2-equivalences:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\hat{\Delta}_{\mathcal{A}}"] & Kat(\mathcal{A}) \arrow[r, "Cat(R_{\mathcal{A}})j_{\mathcal{A}}"] & Cat(Disc(\mathcal{A}))
\end{tikzcd}$$

constructed in Theorem 4.18, where we now reindex by the objects of $\mathbf{B}$. Each 2-equivalence certainly preserves all limits and colimits. Thus each such 2-equivalence is a morphism of $\mathbf{B}$ as required. [1] We need to extend these components to a pseudonatural equivalence. We have for each $F : \mathcal{A} \longrightarrow \mathcal{B}$ the components

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[1] We have not any need to consider whether the two components of each 2-equivalence themselves lie in $\mathbf{B}$, and shall not do so.

of $\hat{\Delta} : 1 \Longrightarrow K$ and will combine these with 2-natural isomorphisms as in the right square of the diagram below:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\hat{\Delta}_{\mathcal{A}}"] \arrow[d, "F"'] & & Kat(\mathcal{A}) \arrow[rr, "Cat(R_{\mathcal{A}})j_{\mathcal{A}}"] \arrow[d, "Kat(F)"] & & Cat(Disc(\mathcal{A})) \arrow[d, "Cat(Disc(F))"] \\
\mathcal{B} \arrow[rr, "\hat{\Delta}_{\mathcal{B}}"'] \arrow[urr, phantom, "\Downarrow \hat{\Delta}_F"] & & Kat(\mathcal{B}) \arrow[rr, "Cat(R_{\mathcal{B}})j_{\mathcal{B}}"'] \arrow[urr, phantom, "\Downarrow ?"] & & Cat(Disc(\mathcal{B}))
\end{tikzcd}$$

to give the required pseudonatural transformation. Consider the adjunction (for $\mathcal{A}$ and equally all other objects of $\mathbf{B}$):

$$\begin{tikzcd}
Kat(\mathcal{A}) \arrow[r, bend right, "Cat(R_{\mathcal{A}})j_{\mathcal{A}}"'] \arrow[r, phantom, "\perp"] & Cat(Disc(\mathcal{A})) \arrow[l, bend right, "\iota_{\mathcal{A}}"']
\end{tikzcd}$$

At a catead X this has counit component $Cat(\mu_{\mathcal{A}})X : \iota_{\mathcal{A}} Cat(R_{\mathcal{A}})j_{\mathcal{A}}X \longrightarrow X$ where we omit to mention the inclusions. $Cat(R_{\mathcal{A}})j_{\mathcal{A}}X$ is the pointwise discrete category with components $(Cat(R_{\mathcal{A}})X)_i = R_{\mathcal{A}}X_i$ and the components of the internal functor constituting that counit component are the covering codescent morphisms: $\mu_{\mathcal{A}X_i} : R_{\mathcal{A}}X_i \longrightarrow X_i$. We name the counit globally as $\epsilon_{\mathcal{A}}$:

$$\begin{tikzcd}
Kat(\mathcal{A}) \arrow[r, "Cat(R_{\mathcal{A}})j_{\mathcal{A}}"] \arrow[d, "1"'] & Cat(Disc(\mathcal{A})) \arrow[dl, "\iota_{\mathcal{A}}"] \arrow[dl, phantom, bend right=40, "\overset{\Longleftarrow}{\epsilon_{\mathcal{A}}}"] \\
Kat(\mathcal{A}) &
\end{tikzcd}$$

as opposed to $Cat(\mu_{\mathcal{A}})$ since, though it acts as such, it is not in the image of $Cat(-)$. Declaring, as in the case of Theorem 4.18, that the unit of this adjunction is an identity, and equally for every object of $\mathcal{B}$, we obtain the composite 2-cell of $\mathbf{B}$:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\hat{\Delta}_{\mathcal{A}}"] \arrow[ddd, "F"'] & & Kat(\mathcal{A}) \arrow[rr, "Cat(R_{\mathcal{A}})j_{\mathcal{A}}"] \arrow[dd, "1"'] & \overset{\Longleftarrow}{\epsilon_{\mathcal{A}}} & Cat(Disc(\mathcal{A})) \arrow[d, "Cat(Disc(F))"] \arrow[ddll, "\iota_{\mathcal{A}}"] \\
 & & & & Cat(Disc(\mathcal{B})) \arrow[dd, "1"] \arrow[ddll, "\iota_{\mathcal{B}}"] \\
 & \Downarrow \hat{\Delta}_F & Kat(\mathcal{A}) \arrow[d, "Kat(F)"'] & & \\
\mathcal{B} \arrow[rr, "\hat{\Delta}_{\mathcal{B}}"'] & & Kat(\mathcal{B}) \arrow[rr, "Cat(R_{\mathcal{B}})j_{\mathcal{B}}"'] & & Cat(Disc(\mathcal{B}))
\end{tikzcd}$$

The middle square on the right clearly commutes. The lower triangle commutes as the unit of the adjunction is an identity. It is easy to see that the composite 2-cell is the component of a lax natural transformation. For if we are also given $G : \mathcal{B} \longrightarrow \mathcal{C} \in \mathbf{B}$ and stack the composite components for F and G vertically then we obtain a diagram of four squares, with the components of $\hat{\Delta}$ at F and G stacked vertically on the left. These compose by pseudonaturality of $\hat{\Delta}$. The vertically stacked squares on the right will compose since the triangle equation $\epsilon_{\mathcal{B}}\iota_{\mathcal{B}} = 1$ will cancel out the 2-cell $\epsilon_{\mathcal{B}}$ appearing in the middle of the right stack. Naturality of the composite at 2-cells of $\mathbf{B}$ follows from the 2-naturality of the inclusions $\iota_{\mathcal{A}} : Cat(Disc(\mathcal{A})) \longrightarrow Kat(\mathcal{A})$ and the corresponding naturality condition for $\hat{\Delta}$. Certainly the composite 2-cell above is an identity if we take its component at $1 : \mathcal{A} \longrightarrow \mathcal{A}$: this fact is witnessed on the right hand side by the triangle equation $Cat(R_{\mathcal{A}})j_{\mathcal{A}}\epsilon_{\mathcal{A}} = 1$ and on the left by the pseudonaturality of $\hat{\Delta}$. Consequently these squares do indeed give a lax natural transformation and it remains to show that it is pseudonatural. We will show that the composite

2-cell comprising the right hand square above is an isomorphism. Consider again the catead $X \in Kat(\mathcal{A})$ and the image of the counit component $\epsilon_{\mathcal{A}}X$ under the image of $Kat(F)$ as drawn on the left below.

$$\begin{tikzcd}
FR_{\mathcal{A}}X_2 \arrow[r, "F\mu_{\mathcal{A}X_2}"] \arrow[d, shift right=2, "FR_{\mathcal{A}}p_x"'] \arrow[d] \arrow[d, shift left=2, "FR_{\mathcal{A}}q_x"] & FX_2 \arrow[d, shift right=2, "Fp_x"'] \arrow[d] \arrow[d, shift left=2, "Fq_x"] \\
FR_{\mathcal{A}}X_1 \arrow[r, "F\mu_{\mathcal{A}X_1}"] \arrow[d, shift right=2, "FR_{\mathcal{A}}d_x"'] \arrow[d, shift left=2, "FR_{\mathcal{A}}c_x"] & FX_1 \arrow[d, shift right=2, "Fd_x"'] \arrow[d, shift left=2, "Fc_x"] \\
FR_{\mathcal{A}}X_0 \arrow[r, "F\mu_{\mathcal{A}X_0}"] \arrow[u] & FX_0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{\mathcal{B}}FX_2 \arrow[r, "\mu_{\mathcal{B}FX_2}"] \arrow[d, shift right=2, "R_{\mathcal{B}}Fp_x"'] \arrow[d] \arrow[d, shift left=2, "R_{\mathcal{B}}Fq_x"] & FX_2 \arrow[d, shift right=2, "Fp_x"'] \arrow[d] \arrow[d, shift left=2, "Fq_x"] \\
R_{\mathcal{B}}FX_1 \arrow[r, "\mu_{\mathcal{B}FX_1}"] \arrow[d, shift right=2, "R_{\mathcal{B}}Fd_x"'] \arrow[d, shift left=2, "R_{\mathcal{B}}Fc_x"] & FX_1 \arrow[d, shift right=2, "Fd_x"'] \arrow[d, shift left=2, "Fc_x"] \\
R_{\mathcal{B}}FX_0 \arrow[r, "\mu_{\mathcal{B}FX_0}"] \arrow[u] & FX_0 \arrow[u]
\end{tikzcd}$$

On the right we have the corresponding counit component at $Kat(F)X \in Kat(\mathcal{B})$, the domain of that internal functor precisely the image of $Kat(F)X$ under $Cat(R_{\mathcal{B}})j_{\mathcal{B}} : Kat(\mathcal{B}) \longrightarrow Cat(Disc(\mathcal{B}))$. The internal categories which constitute the domains of both internal functors are pointwise discrete categories in $\mathcal{B}$. Therefore they are unaltered by the coreflection $Cat(R_{\mathcal{B}})j_{\mathcal{B}} : Kat(\mathcal{B}) \longrightarrow Cat(Disc(\mathcal{B}))$ whilst the image of the internal functor on the left under the coreflection has components for $i = 0, 1, 2$ the unique arrows:

$$\begin{tikzcd}
FR_{\mathcal{A}}X_i \arrow[d, "\exists!"'] \arrow[dr, "F\mu_{\mathcal{A}X_i}"] & \\
R_{\mathcal{B}}FX_i \arrow[r, "\mu_{\mathcal{B}FX_i}"'] & FX_i
\end{tikzcd}$$

induced by the projectivity of $FR_{\mathcal{A}}X_i$ and the lower covering codescent morphism. Now F preserves codescent morphisms. Therefore each component $F\mu_{\mathcal{A}X_i} : FR_{\mathcal{A}}X_i \longrightarrow FX_i$ is a covering codescent morphism in $\mathcal{B}$. Therefore, by Proposition 4.16(1), the unique arrow is an isomorphism for each of $i = 0, 1, 2$. Consequently the internal functor $Cat(R_{\mathcal{B}})j_{\mathcal{B}}Kat(F)\epsilon_{\mathcal{A}}(X)$ is an isomorphism so that the lax natural transformation is indeed pseudonatural. As its arrow components are equivalences it gives a pseudonatural equivalence $1_{\mathbf{B}} \Longrightarrow Cat(Disc(-))$ by Proposition 4.25(1). ☐

Corollary 4.29. Let $\mathcal{A}, \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$. A 2-functor $H : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ is, up to 2-natural isomorphism, of the form $Cat(F)$ for some pullback preserving functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ if and only if it is a morphism of Rep and preserves bijections on objects. In particular $H \cong Cat(ob_{\mathcal{B}} \circ Disc(H) \circ [-]_{\mathcal{A}})$.

Proof. Given $F \in \mathrm{Cat}_{\mathrm{pb}}$ the 2-functor $Cat(F)$ is a morphism of Rep and preserves bijections on objects by Remark 4.22.
Conversely the biequivalence $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathbf{B}$ of Theorem 4.28 is locally an equivalence by Proposition 4.25(2). Therefore $Cat(-)_{\mathcal{A},\mathcal{B}} : \mathrm{Cat}_{\mathrm{pb}}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathbf{B}(Cat(\mathcal{A}), Cat(\mathcal{B}))$ is essentially surjective on 1-cells. A morphism $H : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ of $\mathbf{B}$ is precisely a 2-functor of Rep which preserves bijections on objects as these are the codescent morphisms by Theorem 3.68. Consequently given such a 2-functor there exists some $F : \mathcal{A} \longrightarrow \mathcal{B} \in \mathrm{Cat}_{\mathrm{pb}}$ with $Cat(F) \cong H$ as required.
Consider the arrow component of the pseudonatural equivalence $1 \simeq Cat(Disc(-))$ of Theorem 4.28 at $Cat(\mathcal{E})$. This is the composite right adjoint of the 2-equivalence $Cat(R)j\hat{\Delta} : Cat(\mathcal{E}) \longrightarrow Cat(Disc(Cat(\mathcal{E})))$, which as described in Example 4.19 is simply $Cat([-]_{\mathcal{E}})$, with equivalence inverse given by $Cat(ob_{\mathcal{E}})$. Consider the composite 2-natural isomorphism:

$$\begin{tikzcd}
Cat(\mathcal{A}) \arrow[rr, "H"] \arrow[d, "Cat({[-]_{\mathcal{A}}})"'] & & Cat(\mathcal{B}) \arrow[r, "1"] \arrow[d, "Cat({[-]_{\mathcal{B}}})"'] & Cat(\mathcal{B}) \\
Cat(Disc(Cat(\mathcal{A}))) \arrow[rr, "Cat(Disc(H))"'] \arrow[urr, phantom, "\cong"] & & Cat(Disc(Cat(\mathcal{B}))) \arrow[ur, "Cat(ob_{\mathcal{B}})"'] &
\end{tikzcd}$$

where the 2-cell isomorphism filling the left square is the component of the pseudonatural equivalence $1 \simeq Cat(Disc(-))$ at H whilst the right triangle commutes exactly. This 2-natural isomorphism witnesses the claim. □

We conclude this chapter with a few examples and non-examples.

Example 4.30. It is worth explaining how Set, viewed as a locally discrete 2-category, fails to satisfy the axioms of Theorem 4.18 which characterise those 2-categories of the form $Cat(\mathcal{E})$. Certainly Set is representable, as it has pullbacks, and in the absence of any non-identity 2-cells, trivial cotensors with **2**. As all 2-cells are identities the codescent object of coherence data is just the coequaliser of the underlying graph; the higher kernel of a function is simply its kernel pair (extended to an internal category). A catead in Set is simply an internal category in Set, a small category, the two sided discrete fibration condition immediately satisfied in the absence of any non-trivial 2-cells. If cateads were effective in Set then each internal category would be the kernel pair of the coequaliser of its underlying graph. Of course this is not true. Each kernel pair is an equivalence relation and there are many more small categories than there are equivalence relations. Another account on which Set fails is that discrete objects are not typically projective. Each object of Set is certainly discrete as Set is a locally discrete 2-category. However codescent morphisms in Set are just the regular epis or surjective functions. If discrete objects were projective then for each set X and surjective function $f : Y \longrightarrow Z$ the induced function $\mathrm{Set}(X, f) : \mathrm{Set}(X, Y) \longrightarrow \mathrm{Set}(X, Z)$ would be bijective. This is certainly not the case.

Example 4.31. Consider $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$ and $X \in \mathcal{E}$. The representable $Cat(\mathcal{E})([X], -) : Cat(\mathcal{E}) \longrightarrow Cat(\mathrm{Set}) = \mathrm{Cat}$ preserves codescent morphisms since $[X]$ is discrete and discrete objects are projective in $Cat(\mathcal{E})$. Furthermore it preserves all limits and thus is a morphism of Rep. We therefore have, by Corollary 4.29, a 2-natural isomorphism $Cat(\mathcal{E})([X], -) \cong Cat(ob \circ Disc(Cat(\mathcal{E})([X], -)) \circ [-])$. Given an internal category $Y \in Cat(\mathcal{E})$ we have $(Cat(ob \circ Disc(Cat(\mathcal{E})([X], -)) \circ [-](Y))_i = Cat(\mathcal{E})([X], [Y_i])_0 \cong \mathcal{E}(X, Y_i)$ the latter isomorphism holding by virtue of the adjunction:

$$\begin{tikzcd}
\mathcal{U}Cat(\mathrm{Set}) \arrow[r, bend right, "ob"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "{[-]}"']
\end{tikzcd}$$

Therefore we see that $Cat(\mathcal{E})([X], -) \cong Cat(\mathcal{E}(X, -))$.

Example 4.32. It is well known and straightforward to verify directly that if $\mathcal{J}$ is a small category then we have $[\mathcal{J}, \mathrm{Cat}] \cong Cat(\mathcal{J}, \mathrm{Set})$. On the other hand this is never true if $\mathcal{J}$ is a 2-category which is not locally discrete. We now explain why not.

For a small 2-category $\mathcal{J}$ consider $[\mathcal{J}, \mathrm{Cat}]$. Limits and colimits are pointwise therein; consequently all limit, colimit and exactness properties of Cat carry over immediately to $[\mathcal{J}, \mathrm{Cat}]$, regardless of whether or not $\mathcal{J}$ is locally discrete. If $[\mathcal{J}, \mathrm{Cat}]$ is not of the form $Cat(\mathcal{E})$ it must be the case, by Theorem 4.18, that the discrete objects do not form a projective cover. Now the discrete objects in $[\mathcal{J}, \mathrm{Cat}]$ are just the pointwise discrete ones: presheaves whose values are discrete categories. Similarly codescent morphisms are the pointwise codescent morphisms: the pointwise bijections on objects. Suppose now that $\mathcal{J}$ has a non-identity 2-cell:

$$\begin{tikzcd}
i \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] & j \arrow[from=F, to=G, Rightarrow, phantom, "\Downarrow\alpha"]
\end{tikzcd}$$

and consider the representable $\mathcal{J}(i, -) : \mathcal{J} \longrightarrow \mathrm{Cat}$. Suppose that we had a discrete presheaf $F : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a pointwise bijection on objects $\theta : F \Longrightarrow \mathcal{J}(i, -)$. Since each Fj is discrete $F\alpha$ must be an identity 2-cell. 2-naturality of θ at the 2-cell α asserts that $\mathcal{J}(i, \alpha) \circ \theta_i = \theta_j \circ F(\alpha)$. Thus $\mathcal{J}(i, \alpha) \circ \theta_i$ must be an identity 2-cell. As $\theta_i : Fi \longrightarrow \mathcal{J}(i, i)$ is bijective on objects this implies that $\mathcal{J}(i, \alpha)$ is an identity 2-cell in Cat. But the component of $\mathcal{J}(i, \alpha)$ at $1_i \in \mathcal{J}(i, i)$ is precisely α and so this is impossible. Therefore the discrete objects in $[\mathcal{J}, Cat]$ cannot form a projective cover.

Corollary 4.33. Let $\mathcal{J}$ be a small category and consider the limit 2-functor $lim : [\mathcal{J}, \text{Cat}] \longrightarrow \text{Cat}$. This preserves codescent objects of cateads.

Proof. In Example 4.32 we observed in particular that $[\mathcal{J}, \text{Cat}]$ is representable, has codescent objects of cateads and that both cateads and codescent morphisms are effective, these properties inherited from the corresponding properties for Cat. As limits commute with limits $lim : [\mathcal{J}, \text{Cat}] \longrightarrow \text{Cat}$ is a morphism of Rep. Therefore it suffices by Proposition 4.20(2) to show that lim preserves codescent morphisms. In the present case this amounts to showing that lim takes pointwise bijections on objects to bijections on objects. Now $ob : \mathcal{U}\text{Cat} \longrightarrow \text{Set}$ preserves $\mathcal{J}$-limits so that we have a commuting diagram:

$$\begin{tikzcd}
{[\mathcal{J}, \mathcal{U}\text{Cat}]} \arrow[r, "\mathcal{U}(lim)"] \arrow[d, "{[\mathcal{J}, ob]}"'] & \mathcal{U}\text{Cat} \arrow[d, "ob"] \\
{[\mathcal{J}, \text{Set}]} \arrow[r, "lim"] & \text{Set}
\end{tikzcd}$$

The morphisms inverted by ob are precisely the bijections on objects. Therefore given a pointwise bijective on objects natural transformation $\eta : F \longrightarrow G$ it suffices to show it is inverted by $ob \circ \mathcal{U}(lim)$. But $[\mathcal{J}, ob]$ inverts pointwise bijections on objects and so $lim \circ [\mathcal{J}, ob] = ob \circ \mathcal{U}(lim)$ inverts them too. Thus $lim : [\mathcal{J}, \text{Cat}] \longrightarrow \text{Cat}$ preserves codescent morphisms and so codescent objects of cateads. □

Chapter 5

$\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed

In Chapters 3 and 4 we have considered the 2-category $\mathrm{Cat}_{\mathrm{pb}}$ as a base for 2-categories of internal categories. In this chapter we show that its underlying category is cartesian closed, the morphisms of the internal hom given by cartesian natural transformations. We then show that the 2-category with objects and 1-cells the same as $\mathrm{Cat}_{\mathrm{pb}}$, but 2-cells the cartesian natural transformations, is a cartesian closed 2-category.

A related category is proven to be cartesian closed by Taylor in [56] using a similar technique. In that paper the author considers a category **SDom** of stable domains, which are posets with connected meets and directed joins, such that the former distribute over the latter, and with morphisms the order preserving maps which preserve each aspect of the described structure. The category **SDom** is shown to be cartesian closed in Theorem 4.4.3 of that paper. Given stable domains A and B the set $\mathbf{SDom}(A, B)$ becomes a stable domain upon being equipped with the "Berry order" and this gives the relevant internal hom. The "Berry order" may be described as follows. If categories with pullbacks A and B are posets then the category $\mathrm{Cat}_{\mathrm{pb}}(A, B)$ is a poset and the ordering induced by the cartesian natural transformations is precisely the "Berry order". Taylor also informed the author that he was aware those techniques carried over to the case of the present chapter.

Notation 5.1. In this chapter we aim to show that the underlying category of $\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed. In order to do so we will need to place some size restrictions upon its objects, which until now we have allowed to be locally small categories with pullbacks. For this chapter *only* $\mathrm{Cat}_{\mathrm{pb}}$ refers to the *category* of *small* categories with pullbacks and pullback preserving functors. As usual the objects of Cat are small categories but as with $\mathrm{Cat}_{\mathrm{pb}}$ we temporarily ignore its 2-cells. For this chapter *only* we therefore denote by Cat the *category* of small categories and functors so that $\mathrm{Cat}_{\mathrm{pb}}$ is now a subcategory of Cat. [1]

Remark 5.2. Our aim is to show that $\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed. Firstly we must show that $\mathrm{Cat}_{\mathrm{pb}}$ has products. Given A,B of $\mathrm{Cat}_{\mathrm{pb}}$, the cartesian product in Cat, $A \times B$, has pullbacks, constructed pointwise. As pullbacks in $A \times B$ are pointwise, the projections from $A \times B$ preserve them. It is straightforward to check the universal property is satisfied so that the cartesian product in $\mathrm{Cat}_{\mathrm{pb}}$ is just the ordinary product of categories. Furthermore $\mathrm{Cat}_{\mathrm{pb}}$ evidently has terminal object $\mathbf{1}$, since the category $\mathbf{1}$ has pullbacks, and any functor with codomain $\mathbf{1}$ necessarily preserves pullbacks.
To show that $\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed is to provide a right adjoint to each functor $- \times A : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$ for $A \in \mathrm{Cat}_{\mathrm{pb}}$; the right adjoint is then denoted by $[A,-]_{\mathrm{pb}}$ and referred to as the "internal hom".

Definition 5.3. Consider functors $F, G : A \rightrightarrows B$ and a natural transformation $r : F \Longrightarrow G$. The natural transformation is said to be cartesian if for each morphism $f : a \longrightarrow b$ of A, the naturality square:

$$\begin{tikzcd}
Fa \arrow[r, "r_a"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\lrcorner", very near start] & Ga \arrow[d, "Gf"] \\
Fb \arrow[r, "r_b"] & Gb
\end{tikzcd}$$

is a pullback.

Remark 5.4. It is clear that the identity natural transformation on any functor is cartesian; as furthermore is the vertical composite of a pair of cartesian natural transformations $r : F \Longrightarrow G$ and $s : G \Longrightarrow H$. Therefore one obtains a category whose objects are functors and morphisms: cartesian natural transformations.

Definition 5.5. Given $A, B \in \mathrm{Cat}_{\mathrm{pb}}$ let $[A,B]_{\mathrm{pb}}$ denote the category whose objects are pullback preserving functors and whose morphisms are the cartesian natural transformations.

Remark 5.6. In order to show that $[A,B]_{\mathrm{pb}}$ provides the internal hom for the cartesian closed structure we must first check that it is actually an object of $\mathrm{Cat}_{\mathrm{pb}}$.

Proposition 5.7. $[A,B]_{\mathrm{pb}}$ has pullbacks.

Proof. Given pullback preserving functors F,G and H from A to B, and a pair of cartesian natural transformations $t : F \Longrightarrow H$ and $u : G \Longrightarrow H$ and we must construct the pullback in $[A,B]_{\mathrm{pb}}$. As B has pullbacks, pullbacks in the ordinary functor category $[A,B]$ exist and are constructed pointwise. Consider the pullback in $[A,B]$:

$$\begin{tikzcd}
P \arrow[r, Rightarrow, "r"] \arrow[d, Rightarrow, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, Rightarrow, "t"] \\
G \arrow[r, Rightarrow, "u"] & H \ .
\end{tikzcd}$$

We will show that this is the pullback in $[A,B]_{\mathrm{pb}}$. To do so we must firstly show that this square lives in $[A,B]_{\mathrm{pb}}$, which is to say that P preserves pullbacks and that r and s are cartesian.

---

[1] An alternative approach would be to show that $\mathrm{Cat}_{\mathrm{pb}}$, with objects locally small categories with pullbacks as before, is partially closed.

- P preserves pullbacks:
  Given a pullback square in A:

$$\begin{tikzcd}
a \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[d, "\phi"] \\
b \arrow[r, "\theta"] & d
\end{tikzcd}$$

  we must show that the square:

$$\begin{tikzcd}
Pa \arrow[r, "P\beta"] \arrow[d, "P\alpha"'] & Pc \arrow[d, "P\phi"] \\
Pb \arrow[r, "P\theta"] & Pd
\end{tikzcd}$$

  is a pullback too. In the composite square:

$$\begin{tikzcd}
Pa \arrow[r, "r_a"] \arrow[d, "P\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & Fa \arrow[r, "F\beta"] \arrow[d, "F\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & Fc \arrow[d, "F\phi"] \\
Pb \arrow[r, "r_b"'] & Fb \arrow[r, "F\theta"'] & Fd
\end{tikzcd}$$

  both smaller squares are pullbacks; the left square as r is cartesian, the right hand square as F preserves pullbacks. Thus the composite square is a pullback. The equations $F\beta \circ r_a = r_c \circ P\beta$ and $F\theta \circ r_b = r_d \circ P\theta$ hold by naturality of r and so we may rewrite the above square as:

$$\begin{tikzcd}
Pa \arrow[r, "P\beta"] \arrow[d, "P\alpha"'] & Pc \arrow[r, "r_c"] \arrow[d, "P\phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & Fc \arrow[d, "F\phi"] \\
Pb \arrow[r, "P\theta"'] & Pd \arrow[r, "r_d"'] & Fd
\end{tikzcd} .$$

  The right hand square in the composite is a pullback as r is cartesian. As the composite square is a pullback, the left hand square must be a pullback too.

- r,s are cartesian: We shall consider the case of r.
  Given a morphism $\alpha : a \longrightarrow b$ of A we must show that the square

$$\begin{tikzcd}
Pa \arrow[r, "r_a"] \arrow[d, "P\alpha"'] & Fa \arrow[d, "F\alpha"] \\
Pb \arrow[r, "r_b"] & Fb
\end{tikzcd}$$

  is a pullback. Both squares in the composite

$$\begin{tikzcd}
Pa \arrow[r, "s_a"] \arrow[d, "r_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & Ga \arrow[r, "G\alpha"] \arrow[d, "u_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gb \arrow[d, "u_b"] \\
Fa \arrow[r, "t_a"'] & Ha \arrow[r, "H\alpha"'] & Hb
\end{tikzcd}$$

  are pullbacks: the left hand square because pullbacks in the functor category are constructed pointwise; the right hand square because u is cartesian. Therefore the composite square is a pullback. By naturality of t and s we may rewrite this composite as:

$$\begin{tikzcd}
Pa \arrow[r, "P\alpha"] \arrow[d, "r_a"'] & Pb \arrow[r, "s_b"] \arrow[d, "r_b"'] \arrow[dr, phantom, "\lrcorner", very near start] & Gb \arrow[d, "u_b"] \\
Fa \arrow[r, "F\alpha"'] & Fb \arrow[r, "t_b"'] & Hb
\end{tikzcd}$$

The right hand square is a pullback, and so as the composite is, it follows that the left hand square is a pullback.

Therefore the square:

$$\begin{tikzcd}
P \arrow[r, Rightarrow, "r"] \arrow[d, Rightarrow, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, Rightarrow, "t"] \\
G \arrow[r, Rightarrow, "u"] & H
\end{tikzcd}$$

indeed lies in $[A,B]_{\mathrm{pb}}$. Its universal property is easily checked upon noting that if r_1 and r_2 are vertically composable natural transformations such that r_2 and $r_2 \circ r_1$ are both cartesian, then r_1 is cartesian; this fact following from the pasting laws for pullback squares. ☐

Remark 5.8. Given a morphism $F : B \longrightarrow C$ of $\mathrm{Cat}_{\mathrm{pb}}$, there is an induced pullback preserving functor $[A,F]_{\mathrm{pb}} : [A,B]_{\mathrm{pb}} \longrightarrow [A,C]_{\mathrm{pb}}$ given by composition with F, and so we obtain an endofunctor $[A,-]_{\mathrm{pb}} : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Cat}_{\mathrm{pb}}$. We shall show that $[A,-]_{\mathrm{pb}}$ is right adjoint to $- \times A$ by providing a unit and counit for the adjunction. These may be lifted to $\mathrm{Cat}_{\mathrm{pb}}$ directly from the case of Cat. In the case of the cartesian closedness of Cat, the unit and counit are given by evaluation: $ev_B : [A,B] \times A \longrightarrow B$ and coevaluation: $coev_B : B \longrightarrow [A, B \times A]$. That these lift directly to the case of $\mathrm{Cat}_{\mathrm{pb}}$ is the content of the following lemma.

Proposition 5.9. Consider $A, B \in \mathrm{Cat}_{\mathrm{pb}}$.

1. The restriction of $ev_B : [A,B] \times A \longrightarrow B$ to $[A,B]_{\mathrm{pb}} \times A$ preserves pullbacks and thus is a morphism of $\mathrm{Cat}_{\mathrm{pb}}$. Accordingly we define the counit components via restriction as $ev_B : [A,B]_{\mathrm{pb}} \times A \longrightarrow B$ and these are natural in B.
2. The functor $coev_B : B \longrightarrow [A, B \times A]$ preserves pullbacks and takes its image in $[A, B \times A]_{\mathrm{pb}}$. Accordingly we define the unit components via this factorization as $coev_B : B \longrightarrow [A, B \times A]_{\mathrm{pb}}$ and these are natural in B.

Proof. 1. Let:

$$\begin{tikzcd}
(P,a) \arrow[r, Rightarrow, "{(r,\alpha)}"] \arrow[d, Rightarrow, "{(s,\beta)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (F,b) \arrow[d, Rightarrow, "{(t,\theta)}"] \\
(G,c) \arrow[r, Rightarrow, "{(u,\phi)}"] & (H,d)
\end{tikzcd}$$

be a pullback diagram in $[A,B]_{\mathrm{pb}} \times A$ (corresponding to a pullback square in each of $[A,B]_{\mathrm{pb}}$ and A). The image of this pullback square under ev_B is the outer square of:

$$\begin{tikzcd}
Pa \arrow[r, "r_a"] \arrow[d, "s_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & Fa \arrow[r, "F\alpha"] \arrow[d, "t_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & Fb \arrow[d, "t_b"] \\
Ga \arrow[r, "u_a"] \arrow[d, "G\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & Ha \arrow[r, "H\alpha"] \arrow[d, "H\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & Hb \arrow[d, "H\theta"] \\
Gc \arrow[r, "u_c"'] & Hc \arrow[r, "H\phi"'] & Hd
\end{tikzcd}$$

The top left square is a pullback as it is a component of the pullback square in $[A,B]_{\mathrm{pb}}$. The bottom right square is a pullback as it is the image of the pullback square in A, under the pullback preserving functor H. The top right and bottom left squares are pullbacks as both t and u are cartesian. Consequently the outer square is a pullback square as desired. It is straightforward to verify that the counit components so defined constitute a natural transformation $ev : [A,-]_{\mathrm{pb}} \times A \longrightarrow 1_{\mathrm{Cat}_{\mathrm{pb}}}$.

2. To see that $B \xrightarrow{coev_B} [A, B \times A]$ preserves pullbacks, note that we have the product in Cat:

$$[A, B \times A] \cong [A,B] \times [A,A]$$

and that the following diagram commutes:

$$\begin{tikzcd}
& & {[A,B]} \\
B \arrow[urr, "\Delta"] \arrow[r, "coev_B"] \arrow[drr, "\widehat{1_A}"'] & {[A, B \times A]} \arrow[ur] \arrow[dr] & \\
& & {[A,A]}
\end{tikzcd}$$

where $\Delta(b)$ is the constant functor at b for each object b of B, $\widehat{1_A}(b) = 1_A$ is the identity functor on A, whilst the unlabelled arrows are the projections from the product. Both Δ and $\widehat{1_A}$ clearly preserve pullbacks, so $coev_B = (\Delta, \widehat{1_A})$ preserves pullbacks.

To see that the image of $coev_B$ lies in $[A, B \times A]_{\mathrm{pb}}$, we must show firstly that given an object $b \in B$, the functor $coev_B(b) = (\Delta, \widehat{1_A})(b) = (\Delta(b), 1_A)$ preserves pullbacks. Certainly the constant functor $\Delta(b)$ at b preserves pullbacks, as does 1_A, so that $coev_B(b) = (\Delta(b), 1_A)$ preserves pullbacks. Given a morphism $\alpha : a \longrightarrow b$ of B, we must verify that the natural transformation $coev_B(\alpha)$ is cartesian. Now $coev_B(\alpha) = (\Delta(\alpha), 1_{1_A})$, and as both $\Delta(\alpha)$ and 1_{1_A} are cartesian, it follows that $coev_B(\alpha)$ is.

It is straightforward to verify that the unit components so defined constitute a natural transformation $coev : 1_{\mathrm{Cat}_{\mathrm{pb}}} \longrightarrow [A, - \times A]_{\mathrm{pb}}$ as required.

☐

Theorem 5.10. The internal hom, unit and counit defined thus far give $\mathrm{Cat}_{\mathrm{pb}}$ the structure of a cartesian closed category.

Proof. It remains to verify the triangle equations for the unit and counit. Being defined exactly as in the case of Cat (where the triangle equations hold) they certainly hold in $\mathrm{Cat}_{\mathrm{pb}}$. Therefore $\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed. ☐

Remark 5.11. As $\mathrm{Cat}_{\mathrm{pb}}$ is cartesian closed we may consider categories enriched over it. In particular $\mathrm{Cat}_{\mathrm{pb}}$ obtains the structure of a $\mathrm{Cat}_{\mathrm{pb}}$ category itself, $\mathbf{Cat_{pb}}$, obtained by setting $\mathbf{Cat_{pb}}(A,B) = [A,B]_{\mathrm{pb}}$ for $A, B \in \mathrm{Cat}_{\mathrm{pb}}$. Indeed $\mathbf{Cat_{pb}}$ is a cartesian closed $\mathrm{Cat}_{\mathrm{pb}}$ category. The forgetful functor $U : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Cat}$ is finite product preserving and so, as described in [17], induces a 2-functor: $U_* : \mathrm{Cat}_{\mathrm{pb}}\text{–CAT} \longrightarrow \mathrm{Cat}\text{–CAT}$. Given a $\mathrm{Cat}_{\mathrm{pb}}$-category A, U_*A has the same objects as A whilst $U_*A(a,b) = U(A(a,b))$. In particular $U_*\mathbf{Cat_{pb}}$ is the 2-category of categories with pullbacks, pullback preserving functors and cartesian transformations. We conclude by showing that this is a cartesian closed 2-category.

Corollary 5.12. The 2-category of categories with pullbacks, pullback preserving functors and cartesian natural transformations, $U_*\mathbf{Cat_{pb}}$, is a cartesian closed 2-category.

Proof. For objects A, B, C of $\mathrm{Cat}_{\mathrm{pb}}$ we have

$$U_*\mathbf{Cat_{pb}}(A \times B, C) = U(\mathbf{Cat_{pb}}(A \times B, C)) = U([A \times B, C]_{pb}) \cong U([A,[B,C]_{pb}]_{pb}) = U_*\mathbf{Cat_{pb}}(A, [B,C]_{pb})$$

naturally in A and C. ☐

Chapter 6

Introduction to 2-dimensional monad theory

In this purely expository chapter we recall those facts about 2-dimensional monad theory which we will require in further chapters. 2-monads on Cat, or doctrines, were introduced by Lawvere in [39] as a means to studying categories with equational structure. The subject has been significantly developed by the Australian school of category theory, beginning with the papers of Kelly and Street [25],[26] in the early 1970's, as part of a general program of the study of 2-categories. One continuing thread of this research has been the study of the common non-strict 2-categorical notions, by means of the better behaved strict ones.

The study of the non-strict via the strict may be summarised as the study of left adjoints to the inclusions of the 2-category with strict algebras and strict morphisms T-Alg_{s} into various 2-categories with weaker notions of algebras and algebra morphism such as T-Alg and Ps-T-Alg. Fundamental early papers in this regard are those of Street [46],[49] which construct such adjoints in significant special cases and use them to study pseudo and lax limits by means of genuine weighted limits.

The subject reached maturity with the publication of [8] in which, in particular, the left adjoint to the inclusion of T-Alg_{s} into T-Alg was constructed in the case that T-Alg_{s} is cocomplete. That construction was clarified by the later work of Lack [34] who explained the relationship between such adjunctions and codescent objects.

We focus heavily upon these adjoints and our treatment is based upon that of [34]. We consider also the notion of flexible algebra [8], in order to pave the way for the *pie algebras* of Chapter 9. We then consider Power's coherence result [44] which constituted the first use of the enhanced property of the (Bijective on objects/fully faithful)-factorisation system on Cat, and its relationship to the above adjoints as recognised by Lack [34]. These results will be used in the short and largely expository Chapter 7. Finally we recall the notion of strongly finitary 2-monad as introduced in [27], a variant of which will play an important role in Chapters 8,9 and 10.

6.1 2-monads and their algebras

Definition 6.1. A 2-monad (T, η, μ) is a monad in the 2-category 2-CAT. Thus it consists of a 2-category $\mathcal{A}$, a 2-functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ and 2-natural transformations $\mu : T^2 \Longrightarrow T$ and $\eta : 1_{\mathcal{A}} \Longrightarrow T$ such that the diagrams:

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta_T"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, equal, "1"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

commute.

Notation 6.2. A 2-monad consists of a triple (T, η, μ) but we sometimes abbreviate it to simply T; the 2-natural transformations η and μ taken for granted.

Definition 6.3. Let (T, η, μ) be a 2-monad on a 2-category $\mathcal{A}$.

1. A lax T-algebra is given by a quadruple (A, a, α, α_0) consisting of an object $A \in \mathcal{A}$, a 1-cell $a : TA \longrightarrow A$ and 2-cells $\alpha : a \circ Ta \Longrightarrow a \circ \mu_A$ and $\alpha_0 : 1 \Longrightarrow a \circ \eta_A$ satisfying the equation:

$$\begin{tikzcd}
T^3A \arrow[rr, "T^2a"] \arrow[dr, "T\mu_A"] \arrow[dd, "\mu_{TA}"'] & & T^2A \arrow[dr, "Ta"] \arrow[d, phantom, "T\alpha \Searrow"] & \\
& T^2A \arrow[rr, "Ta"'] \arrow[dd, "\mu_A"] & {} & TA \arrow[dd, "a"] \\
T^2A \arrow[dr, "\mu_A"'] & & \alpha \Downarrow & \\
& TA \arrow[rr, "a"] & & A
\end{tikzcd}
=
\begin{tikzcd}
T^3A \arrow[rr, "T^2a"] \arrow[dd, "\mu_{TA}"'] & & T^2A \arrow[dr, "Ta"] \arrow[dd, "\mu_A"'] & \\
& & & TA \arrow[dd, "a"] \\
T^2A \arrow[rr, "Ta"] \arrow[dr, "\mu_A"'] & & TA \arrow[dr, "a"] \arrow[r, phantom, "\alpha \Downarrow"] \arrow[d, phantom, "\alpha \Searrow"'] & {} \\
& TA \arrow[rr, "a"] & {} & A
\end{tikzcd}$$

and such that the composite 2-cells:

$$\begin{tikzcd}
TA \arrow[d, "T\eta_A"] \arrow[dr, bend left, "1"] \arrow[dd, bend right=60, "1"'] & \Downarrow T\alpha_0 \\
T^2A \arrow[r, "Ta"'] \arrow[d, "\mu_A"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"] \arrow[ur, phantom, "\alpha \Downarrow"] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TA \arrow[r, "a"] \arrow[d, "\eta_{TA}"] \arrow[dd, bend right=60, "1"'] & A \arrow[d, "\eta_A"'] \arrow[dd, bend left=60, "1"] \\
T^2A \arrow[r, "Ta"'] \arrow[d, "\mu_A"] & TA \arrow[d, "a"'] \arrow[r, phantom, "\overset{\Longleftarrow}{\alpha_0}"] & {} \\
TA \arrow[r, "a"] \arrow[ur, phantom, "\alpha \Downarrow"] & A
\end{tikzcd}$$

are both identities. If the coherence 2-cells α and α_0 are both isomorphisms then (A, a, α, α_0) is said to be a pseudo algebra. If they are identities then (A, a, α, α_0) is said to be a strict algebra.

2. A lax morphism $(f, \overline{f}) : (A, a, \alpha, \alpha_0) \longrightarrow (B, b, \beta, \beta_0)$ of lax algebras consists of a 1-cell $f : A \longrightarrow B$ and

2-cell $\overline{f} : b \circ Tf \Longrightarrow f \circ a$ satisfying the equations:

$$\begin{tikzcd}
T^2A \arrow[rr, "T^2f"] \arrow[dr, "Ta"] \arrow[dd, "\mu_A"'] & & T^2B \arrow[dr, "Tb"] \arrow[dl, phantom, "T\overline{f}\Searrow"] & \\
& TA \arrow[rr, "Tf"'] \arrow[dd, "a"] \arrow[dl, phantom, "\alpha\Downarrow"] & & TB \arrow[dd, "b"] \arrow[dll, phantom, "\overline{f}\Downarrow"] \\
TA \arrow[dr, "a"'] & & & \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[rr, "T^2f"] \arrow[dd, "\mu_A"'] & & T^2B \arrow[dr, "Tb"] \arrow[dd, "\mu_B"'] & \\
& & \arrow[r, phantom, "\beta\Downarrow"] & TB \arrow[dd, "b"] \\
TA \arrow[rr, "Tf"] \arrow[dr, "a"'] & & TB \arrow[dr, "b"] \arrow[dl, phantom, "\overline{f}\Searrow"] & \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \arrow[dd, bend left=60, "1"] \\
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{f}\Downarrow"] & TB \arrow[d, "b"] \arrow[r, phantom, "\overset{\Longleftarrow}{\beta_0}"] & {} \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dd, bend left=60, "1"] & B \arrow[dd, bend left=60, "1"] \\
TA \arrow[d, "a"'] \arrow[r, phantom, "\overset{\Longleftarrow}{\alpha_0}"] & {} \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

If the coherence 2-cell $\overline{f}$ is an isomorphism then $(f, \overline{f})$ is said to be a pseudo morphism. If it is an identity then $(f, \overline{f})$ is a strict morphism.

3. A transformation (also called an algebra 2-cell) between lax morphisms $\theta : (f, \overline{f}) \Longrightarrow (g, \overline{g})$ consists of a 2-cell $\theta : f \Longrightarrow g$ satisfying the equation:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{f}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B
\arrow[from=U, to=D, phantom, "\theta\Downarrow"]
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, bend left, "Tf"{name=U}] \arrow[r, bend right, "Tg"'{name=D}] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{g}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "g"'] & B
\arrow[from=U, to=D, phantom, "T\theta\Downarrow"]
\end{tikzcd}$$

Remark 6.4. Each 2-monad has several associated 2-categories of algebras. Of primary interest to us will be the 2-categories T-Alg$_\mathrm{s}$, T-Alg, T-Alg$_\mathrm{l}$ and Ps-T-Alg.

Definition 6.5.

- T-Alg$_\mathrm{s}$ is the 2-category of strict algebras, strict algebra morphisms and transformations.
- T-Alg is the 2-category of strict algebras, pseudomorphisms and transformations.
- T-Alg$_\mathrm{l}$ has strict algebras, lax morphisms and transformations.
- Ps-T-Alg has pseudoalgebras, pseudomorphisms and transformations.

Remark 6.6. Each 2-monad T on $\mathcal{A}$ induces a 2-adjunction:

$$\begin{tikzcd}
\text{T-Alg}_\text{s} \arrow[r, shift right=2, "U^T"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right=2, "F^T"']
\end{tikzcd}$$

Given $A \in \mathcal{A}$ we have $F^T A = (TA, \mu_A)$. The unit of the 2-adjunction is provided by the unit of the 2-monad, whilst the counit component at an algebra (A, a) is the algebra morphism $a : (TA, \mu_A) \longrightarrow (A, a)$. The 2-category T-Alg$_\text{s}$ is a sub 2-category of each of the other 2-categories of algebras. Postcomposing the left 2-adjoint $F^T : \mathcal{A} \longrightarrow$ T-Alg$_\text{s}$ with the respective inclusions $\iota :$ T-Alg$_\text{s} \longrightarrow$ T-Alg and $\iota :$ T-Alg$_\text{s} \longrightarrow$ Ps-T-Alg gives left biadjoints to the evident forgetful 2-functors. Thus we have:

$$\begin{tikzcd}
\text{T-Alg} \arrow[r, shift right=2, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right=2, "F"']
\end{tikzcd}$$

and:

$$\begin{tikzcd}
\text{Ps-T-Alg} \arrow[r, shift right=2, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right=2, "F"']
\end{tikzcd}$$

Given a pseudoalgebra (A, a, α, α_0) the counit component of this latter biadjunction is the pseudomorphism of algebras $(a, \alpha) : (TA, \mu_A) \longrightarrow (A, a, \alpha, \alpha_0)$. If the algebra in question is strict, as in the case of T-Alg, then this reduces to the strict algebra map $a : (TA, \mu_A) \longrightarrow (A, a)$. Given a pseudomorphism $(f, \overline{f}) : (A, a, \alpha, \alpha_0) \longrightarrow (B, b, \beta, \beta_0)$ the component of the pseudonatural counit is the algebra 2-cell:

$$\begin{tikzcd}
(TA, \mu_A) \arrow[r, "{(a,\alpha)}"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overline{f}^{-1}\Downarrow"] & (A, a, \alpha, \alpha_0) \arrow[d, "{(f,\overline{f})}"] \\
(TB, \mu_B) \arrow[r, "{(b,\beta)}"'] & (B, b, \beta, \beta_0)
\end{tikzcd}$$

6.2 Strict algebras and strict coherence data

In this section we describe the left 2-adjoints to the inclusions of T-Alg$_\text{s}$ into T-Alg, T-Alg$_\text{l}$ and Ps-T-Alg first constructed in [8], and in particular the relationship of these adjoints with codescent objects. The construction of the left 2-adjoints using codescent objects presented here follows [34].

Remark 6.7. For a 2-monad T the adjunction:

$$\begin{tikzcd}
\text{T-Alg}_\text{s} \arrow[r, shift right=2, "U^T"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right=2, "F^T"']
\end{tikzcd}$$

induces a 2-comonad $F^T U^T$ on T-Alg$_\text{s}$. Such a comonad is precisely a comonoid in the strict monoidal category 2-CAT(T-Alg$_\text{s}$, T-Alg$_\text{s}$) = [T-Alg$_\text{s}$, T-Alg$_\text{s}$], with the monoidal structure given by composition of 2-functors. Now $(\Delta_+, \oplus, [-1])$ is the free strict monoidal category containing a monoid. Therefore Δ_+^{op}, with the opposite monoidal structure, is the free strict monoidal category containing a comonoid. Consequently the 2-comonad corresponds exactly to a strict monoidal functor $\Delta_+^{op} \longrightarrow$ [T-Alg$_\text{s}$, T-Alg$_\text{s}$]. As 2-CAT is partially closed this transposes to a 2-functor T-Alg$_\text{s} \longrightarrow [\Delta_+^{op}$, T-Alg$_\text{s}$]. Restricting along the inclusion $\Delta \longrightarrow \Delta_+$ gives a 2-functor T-Alg$_\text{s} \longrightarrow [\Delta^{op}$, T-Alg$_\text{s}$]; thus each algebra has an associated simplicial object. Restricting again, now along the inclusion $\Delta_2 \longrightarrow \Delta$, induces a 2-functor $\text{T-Alg}_\text{s} \longrightarrow [\Delta_2^{op}, \text{T-Alg}_\text{s}]$. Therefore each algebra (A, a) has associated strict reflexive coherence data in T-Alg_s which we refer to as the resolution of (A, a):

$$\begin{tikzcd}
Res(A,a) \arrow[r, phantom, "="] & (T^3A, \mu_{T^2A}) \arrow[r, shift left=6, "\mu_{TA}"] \arrow[r, shift left=0, "T\mu_A" description] \arrow[r, shift right=6, "T^2a"'] & (T^2A, \mu_{TA}) \arrow[l, shift right=3, "T\eta_{TA}" description] \arrow[l, shift left=3, "T^2\eta_A" description] \arrow[r, shift left=4, "\mu_A"] \arrow[r, shift right=4, "Ta"'] & (TA, \mu_A) \arrow[l, "T\eta_A" description]
\end{tikzcd}$$

A codescent cocone to this strict (reflexive) coherence data is a triple $((B, b), f, \theta)$ consisting of:

1. An algebra (B, b).
2. An algebra morphism $f : (TA, \mu_A) \longrightarrow (B, b)$.
3. An algebra transformation:

$$\begin{tikzcd}
 & (TA, \mu_A) \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow \theta"] & \\
(T^2A, \mu_{TA}) \arrow[ur, "Ta"] \arrow[dr, "\mu_A"'] & & (B, b) \\
 & (TA, \mu_A) \arrow[ur, "f"'] &
\end{tikzcd}$$

Transposing across the adjunction $\text{T-Alg}_\text{s}((TA, \mu_A), (B, b)) \cong \mathcal{A}(A, B)$ we see that the algebra morphism $F^TA = (TA, \mu_A) \longrightarrow (B, b)$ corresponds uniquely to a morphism $\overline{f} : A \longrightarrow B$ in $\mathcal{A}$. Transposing across the adjunction $\text{T-Alg}_\text{s}((T^2A, \mu_A), (B, b)) \cong \mathcal{A}(TA, B)$ the algebra transformation θ corresponds exactly to a 2-cell:

$$\begin{tikzcd}
TA \arrow[r, "T\overline{f}"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{\theta}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "\overline{f}"'] & B
\end{tikzcd}$$

Thus to give the "data" $((B, b), f, \theta))$ for a codescent cocone is equally to give the "data" $(\overline{f}, \overline{\theta})$ for a lax algebra morphism from (A, a) to (B, b). Furthermore the equations for a codescent cocone of Section 2.2 correspond exactly to the equations for a lax algebra morphism. Denoting by $W : \Delta_2 \longrightarrow \text{Cat}$ the weight for codescent objects of strict reflexive coherence data what we have observed is a bijection of sets:

$$[\Delta_2^{op}, \text{T-Alg}_\text{s}](W, \text{T-Alg}_\text{s}(Res(A, a), (B, b)) \cong \text{T-Alg}_\text{l}((A, a), (B, b))$$

between codescent cocones and lax algebra morphisms. This extends to an isomorphism of categories 2-natural in (B, b).

Furthermore the 2-cell θ of the codescent cocone is invertible if and only if its transpose $\overline{\theta}$ is so. Letting $W_i : \Delta_2 \longrightarrow \text{Cat}$ denote the weight for isocodescent objects we then have an isomorphism of categories:

$$[\Delta_2^{op}, \text{T-Alg}_\text{s}](W_i, \text{T-Alg}_\text{s}(Res(A, a), (B, b)) \cong \text{T-Alg}((A, a), (B, b))$$

2-natural in (B, b).

Proposition 6.8. 1. The inclusion $\text{T-Alg}_\text{s} \longrightarrow \text{T-Alg}_\text{l}$ has a left 2-adjoint if and only if for each algebra (A, a) the codescent object of its resolution $Res(A, a)$ exists in T-Alg_s. This is the case whenever T-Alg_s has codescent objects of strict reflexive coherence data.

2. The inclusion $\text{T-Alg}_\text{s} \longrightarrow \text{T-Alg}$ has a left 2-adjoint if and only if for each algebra (A, a) the isocodescent object of its resolution $Res(A, a)$ exists in T-Alg_s. This is the case whenever T-Alg_s has isocodescent objects of strict reflexive coherence data.

Proof.

1. The inclusion $\iota : \text{T-Alg}_{\mathrm{s}} \longrightarrow \text{T-Alg}_{\mathrm{l}}$ is the identity on objects. Therefore to give a left 2-adjoint to it is to give, for each algebra (A, a), an algebra $(A, a)'$ and an isomorphism of categories: $\text{T-Alg}_{\mathrm{s}}((A, a)', (B, b)) \cong \text{T-Alg}_{\mathrm{l}}((A, a), (B, b))$ 2-natural in (B, b). We have, by the above discussion, an isomorphism: $[\Delta_2^{op}, \text{T-Alg}_{\mathrm{s}}](W, \text{T-Alg}_{\mathrm{s}}(Res(A, a), (B, b))) \cong \text{T-Alg}_{\mathrm{l}}((A, a), (B, b))$ 2-natural in (B, b). Thus to give a left 2-adjoint is equally to give an algebra $(A, a)'$ and an isomorphism of categories: $\text{T-Alg}_{\mathrm{s}}((A, a)', (B, b)) \cong [\Delta_2^{op}, \text{T-Alg}_{\mathrm{s}}](W, \text{T-Alg}_{\mathrm{s}}(Res(A, a), (B, b)))$ 2-natural in (B, b). This is precisely the assertion that the codescent object of $Res(A, a)$ exists and equals $(A, a)'$. Since $Res(A, a)$ constitutes strict reflexive coherence data it follows that if $\text{T-Alg}_{\mathrm{s}}$ has such codescent objects the left 2-adjoint exists.

2. Given the isomorphism of categories: $[\Delta_2^{op}, \text{T-Alg}_{\mathrm{s}}](W_i, \text{T-Alg}_{\mathrm{s}}(Res(A, a), (B, b)) \cong \text{T-Alg}((A, a), (B, b))$ the argument is just the same as for the first part of the proposition.

□

Remark 6.9. In Chapter 1, the overview, we remarked that the first appearance of descent objects [49], then unnamed, was in the context of two dimensional universal algebra. Street's observation was very closely connected to the constructions of the left adjoints $\text{T-Alg}, \text{T-Alg}_{\mathrm{l}} \longrightarrow \text{T-Alg}_{\mathrm{s}}$ of Proposition 6.8 and we describe this relationship now. Let (A, a) and (B, b) be strict algebras for a 2-monad T on $\mathcal{A}$. In [49] Street observes that $\text{T-Alg}_{\mathrm{l}}((A, a), (B, b))$ is the descent object in Cat of the Δ_2^- indexed diagram:

$$\begin{tikzcd}
\text{T-Alg}_{\mathrm{l}}((A,a),(B,b)) \arrow[r, "U"] & \mathcal{A}(A,B) \arrow[r, shift left=2, "\mathcal{A}(1,b)\circ T_{A,B}"] \arrow[r, shift right=2, "\mathcal{A}(a,1)"'] & \mathcal{A}(TA,B) \arrow[l, "\mathcal{A}(\eta_A,1)" description] \arrow[r, shift left=2, "\mathcal{A}(1,b)\circ T_{TA,TB}"] \arrow[r, "\mathcal{A}(\mu_A,1)" description] \arrow[r, shift right=2, "\mathcal{A}(Ta,1)"'] & \mathcal{A}(T^2A,B)
\end{tikzcd}$$

Given a lax algebra morphism $(f, \overline{f}) : (A, a) \longrightarrow (B, b)$ we have $\mathcal{A}(1, b) \circ T_{A,B} \circ U(f, \overline{f}) = \mathcal{A}(1, b) \circ Tf = b \circ Tf$ whilst $\mathcal{A}(a, 1) \circ U(f, \overline{f}) = \mathcal{A}(a, 1) \circ f = f \circ a$ so that the natural transformation exhibiting $\text{T-Alg}_{\mathrm{l}}((A, a), (B, b))$ as the descent object has value at $(f, \overline{f})$ its structure 2-cell $\overline{f} : b \circ Tf \Longrightarrow f \circ a$; the lax algebra morphism equations under this correspondence becoming those for a descent cone.
Since the object B underlies a strict algebra (B, b) the diagram above may be transposed through the adjunction:

$$\begin{tikzcd}
\text{T-Alg}_{\mathrm{s}} \arrow[r, shift right=2, "U^T"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right=2, "F^T"']
\end{tikzcd}$$

to give a diagram in $\text{T-Alg}_{\mathrm{s}}$ as on the top row below where we have abbreviated (B, b) to B for clarity and similarly abbreviated the free algebras which appear. The calculations of Remark 6.7 correspond to the fact that this transposed diagram is precisely the image of the resolution of the algebra (A, a) under the contravariant hom functor $\text{T-Alg}_{\mathrm{s}}(-, (B, b)) : \text{T-Alg}_{\mathrm{s}}^{op} \longrightarrow \text{Cat}$. We thus obtain a natural isomorphism of Δ_2^- objects in Cat as indicated by the rightmost three vertical isomorphisms below.

$$\begin{tikzcd}
\text{T-Alg}_{\mathrm{s}}(A',B) \arrow[r] \arrow[d, "\cong"'] & \text{T-Alg}_{\mathrm{s}}(TA,B) \arrow[r, shift left=2, "\text{T-Alg}_{\mathrm{s}}(\mu_A{,}1)"] \arrow[r, shift right=2, "\text{T-Alg}_{\mathrm{s}}(Ta{,}1)"'] \arrow[d, "\cong"'] & \text{T-Alg}_{\mathrm{s}}(T^2A,B) \arrow[l, "\text{T-Alg}_{\mathrm{s}}(T\eta_A{,}1)" description] \arrow[r, shift left=2, "\text{T-Alg}_{\mathrm{s}}(\mu_{TA}{,}1)"] \arrow[r, "\text{T-Alg}_{\mathrm{s}}(T\mu_A{,}1)" description] \arrow[r, shift right=2, "\text{T-Alg}_{\mathrm{s}}(T^2a{,}1)"'] \arrow[d, "\cong"'] & \text{T-Alg}_{\mathrm{s}}(T^3A,B) \arrow[d, "\cong"'] \\
\text{T-Alg}_{\mathrm{l}}((A,a),(B,b)) \arrow[r] & \mathcal{A}(A,B) \arrow[r, shift left=2, "\mathcal{A}(1{,}b)\circ T_{A,B}"] \arrow[r, shift right=2, "\mathcal{A}(a{,}1)"'] & \mathcal{A}(TA,B) \arrow[l, "\mathcal{A}(\eta_A{,}1)" description] \arrow[r, shift left=2, "\mathcal{A}(1{,}b)\circ T_{TA,TB}"] \arrow[r, "\mathcal{A}(\mu_A{,}1)" description] \arrow[r, shift right=2, "\mathcal{A}(Ta{,}1)"'] & \mathcal{A}(T^2A,B)
\end{tikzcd}$$

Now an algebra $(A, a)'$, above abbreviated to A', is the codescent object in $\text{T-Alg}_{\mathrm{s}}$ of the resolution of (A, a) if and only if its image under $\text{T-Alg}_{\mathrm{s}}(-, (B, b)) : \text{T-Alg}_{\mathrm{s}}^{op} \longrightarrow \text{Cat}$ is the descent object of the corresponding Δ_2^- object in Cat, the top row above. Therefore combining Street's observation with the fact that descent objects are invariant up to an isomorphism of Δ_2^- diagrams we see that $(A, a)'$ is the isocodescent object of the resolution of (A, a) if and only if we have an isomorphism $\text{T-Alg}_{\mathrm{s}}((A, a)', (B, b)) \cong \text{T-Alg}_{\mathrm{l}}((A, a), (B, b))$.

Similarly T-Alg$((A,a),(B,b))$ is an isodescent object in Cat; this fact similarly giving a description of the left adjoint to the inclusion $\text{T-Alg}_s \longrightarrow \text{T-Alg}$ in terms of isocodescent objects.

Remark 6.10. Let (A,a) be an algebra for T. The algebra $(A,a)'$ defined by the natural isomorphism $\text{T-Alg}_s((A,a)',(B,b)) \cong \text{T-Alg}((A,a),(B,b))$ is often referred to as the "pseudomorphism classifier" of (A,a). The unit of the representation is a pseudomorphism $\lambda_A : (A,a) \longrightarrow (A,a)'$ with the universal property that any pseudomorphism $(A,a) \longrightarrow (B,b)$ factors uniquely through it via a strict morphism $(A,a)' \longrightarrow (B,b)$, and with the evident 2-dimensional universal property. In the case of the representing algebra $(A,a)'$ of $\text{T-Alg}_s((A,a)',(B,b)) \cong \text{T-Alg}_l((A,a),(B,b))$ we call $(A,a)'$ the "lax morphism classifier" of (A,a) and it has a similar universal property with respect to lax algebra morphisms.

Remark 6.11. Consider the 2-adjunction:

$$\begin{tikzcd}
\text{T-Alg}_s \arrow[r, bend right, "\iota"'] \arrow[r, phantom, "\perp"] & \text{T-Alg} \arrow[l, bend right, "(-)'"']
\end{tikzcd}$$

The unit of the adjunction at an algebra (A,a) is a pseudomorphism $\lambda_A : (A,a) \longrightarrow (A,a)'$. The counit is a strict algebra morphism $p_A : (A,a)' \longrightarrow (A,a)$. The triangle equations for the adjunction are:

$$\begin{tikzcd}
(A,a) \arrow[r, "\lambda_A"] \arrow[dr, "1"'] & (A,a)' \arrow[d, "p_A"] \\
& (A,a)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(A,a)' \arrow[r, "(\lambda_A)'"] \arrow[dr, "1"'] & (A,a)'' \arrow[d, "p_{A'}"] \\
& (A,a)'
\end{tikzcd}$$

where the left triangle lies in T-Alg and the right triangle in T-Alg_s. The left triangle asserts that λ_A is a section of p_A in T-Alg. In fact p_A is, at least if $\mathcal{A}$ is sufficiently complete, a surjective equivalence.

Proposition 6.12 (Blackwell-Kelly-Power)**.** Let $\mathcal{A}$ be a 2-category which admits pseudolimits of arrows and suppose that the left 2-adjoint to the inclusion $\iota : \text{T-Alg}_s \longrightarrow \text{T-Alg}$ exists. Then for each algebra (A,a) the strict algebra morphism $p_A : (A,a)' \longrightarrow (A,a)$ is a surjective equivalence in T-Alg, with $\lambda_A : (A,a) \longrightarrow (A,a)'$ its equivalence inverse.

Proof. See [8]. Note that in [8] all results are proven in the context of a complete and cocomplete 2-category $\mathcal{A}$ and a 2-monad with rank. However the above assumptions are all that is required for this particular result. ☐

6.3 Flexible algebras

In this section we assume that the 2-monad T satisfies the assumptions of Proposition 6.12.

Remark 6.13. One of the triangle equations for the adjunction:

$$\begin{tikzcd}
\text{T-Alg}_s \arrow[r, bend right, "\iota"'] \arrow[r, phantom, "\perp"] & \text{T-Alg} \arrow[l, bend right, "(-)'"']
\end{tikzcd}$$

asserts that for each algebra (A,a) the strict algebra morphism p_A has section λ_A in T-Alg. This motivates the definition of flexible algebra below.

Definition 6.14. An algebra is said to be flexible [8] if the strict algebra map $p_A : (A,a)' \longrightarrow (A,a)$ has a section in T-Alg_s.

Proposition 6.15. The following are equivalent:

1. (A, a) is flexible.
2. $p_A : (A, a)' \longrightarrow (A, a)$ is a surjective equivalence in T-Alg$_\mathrm{s}$.
3. (A, a) is a retract in T-Alg$_\mathrm{s}$ of $(B, b)'$ for some algebra (B, b).

Proof. If (A, a) is flexible then by definition p_A has a section $r : (A, a) \longrightarrow (A, a)'$ in T-Alg$_\mathrm{s}$. We claim that r is the equivalence inverse of p_A in T-Alg$_\mathrm{s}$. We must show that $rp_A \cong 1$. We have $p_A r p_A = p_A$ as r is a section. Now p_A is an equivalence in T-Alg by Proposition 6.12; therefore it is fully faithful in T-Alg. Consequently there exists a unique isomorphism $rp_A \cong 1 \in$ T-Alg which postcomposed with p_A yields the identity. As the inclusion T-Alg$_\mathrm{s} \longrightarrow$ T-Alg is locally fully faithful this 2-cell isomorphism lies in T-Alg$_\mathrm{s}$. Thus $(1 \implies 2)$.
If $p_A : (A, a)' \longrightarrow (A, a)$ is a surjective equivalence in T-Alg$_\mathrm{s}$ then certainly p_A has a section so that (A, a) is a retract of $(A, a)'$. Thus $(2 \implies 3)$.
Suppose that (A, a) is a retract in T-Alg$_\mathrm{s}$ of $(B, b)'$ for some algebra (B, b): there exist strict algebra morphisms $r : (A, a) \longrightarrow (B, b)'$ and $s : (B, b)' \longrightarrow (A, a)$ such that $sr = 1$. Then the strict map:

$$\begin{tikzcd}(A, a) \arrow[r, "r"] & (B, b)' \arrow[r, "(\lambda_B)'"] & (B, b)'' \arrow[r, "s'"] & (A, a)'\end{tikzcd}$$

provides the section of p_A. For we have $p_A \circ s' \circ (\lambda_B)' \circ r = s \circ p_{B'} \circ (\lambda_B)' \circ r = s \circ r = 1$ first using naturality of p, then the triangle equation $p_{B'} \circ (\lambda_B)' = 1$ and finally the section $sr = 1$. Thus $(3 \implies 1)$. ☐

A particularly useful property of flexible algebras is the following.

Proposition 6.16. Let (A, a) be a flexible algebra. Given a pseudomorphism $(f, \overline{f}) : (A, a) \longrightarrow (B, b)$ there exists a strict morphism $g : (A, a) \longrightarrow (B, b)$ such that $g \cong (f, \overline{f})$.

Proof. Given a pseudomorphism $(f, \overline{f}) : (A, a) \longrightarrow (B, b)$ the square on the left below:

$$\begin{tikzcd}(A, a)' \arrow[r, "(f,\overline{f})'"] \arrow[d, "p_A"'] & (B, b)' \arrow[d, "p_B"] \\ (A, a) \arrow[r, "(f,\overline{f})"'] & (B, b)\end{tikzcd} \qquad \begin{tikzcd}(A, a)' \arrow[r, "(f,\overline{f})'"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\cong"] & (B, b)' \arrow[d, "p_B"] \\ (A, a) \arrow[r, "(f,\overline{f})"'] & (B, b)\end{tikzcd}$$

need not commute since p is 2-natural only in strict algebra morphisms. However both paths of the square do agree once precomposed with $\lambda_A : (A, a) \longrightarrow (A, a)'$ since we have $p_B \circ (f, \overline{f})' \circ \lambda_A = p_B \circ \lambda_B \circ (f, \overline{f}) = (f, \overline{f}) = (f, \overline{f}) \circ p_A \circ \lambda_A$ using naturality of λ and that $p \circ \lambda = 1$. Now $\lambda_A : (A, a) \longrightarrow (A, a)'$ is an equivalence by Proposition 6.12 and is therefore co-fully faithful, so that there exists a unique 2-cell, as on the right above, which becomes an identity upon precomposition with λ_A. Any co-fully faithful arrow is liberal, thus the 2-cell is an isomorphism. By assumption (A, a) is flexible. Therefore there exists a section r of p_A in T-Alg$_\mathrm{s}$. We now have the composite 2-cell isomorphism:

$$\begin{tikzcd}(A, a) \arrow[r, "r"] \arrow[dr, "1"'] & (A, a)' \arrow[r, "(f,\overline{f})'"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\cong"] & (B, b)' \arrow[d, "p_B"] \\ & (A, a) \arrow[r, "(f,\overline{f})"'] & (B, b)\end{tikzcd}$$

Since each of p_B, $(f, \overline{f})'$ and r are strict algebra maps the 2-cell isomorphism $p_B \circ (f, \overline{f})' \circ r \cong (f, \overline{f})$ proves the claim. ☐

Example 6.17. Each algebra $(A, a)'$ is flexible by Proposition 6.15(3).

Proposition 6.18. Each free algebra is flexible.

Proof. Consider a free algebra (TA, μ_A). We will construct a section of $p_{TA} : (TA, \mu_A)' \longrightarrow (TA, \mu_A)$. The pseudomorphism $\lambda_A : (TA, \mu_A) \longrightarrow (TA, \mu_A)'$ has underlying map: $\lambda_A : TA \longrightarrow TA' = U^T((TA, \mu_A)')$ whose transpose across the adjunction:

$$\begin{tikzcd} \text{T-Alg}_s \arrow[r, phantom, "\perp"] \arrow[r, bend right, "U^T"'] & \mathcal{A} \arrow[l, bend right, "F^T"'] \end{tikzcd}$$

we denote by:

$$(T^2A, \mu_{TA}) \xrightarrow{(\lambda_A)^t} (TA', \mu_{TA'})$$

We claim that the composite:

$$(TA, \mu_A) \xrightarrow{T\eta_A} (T^2A, \mu_{TA}) \xrightarrow{(\lambda_A)^t} (TA', \mu_{TA'}) \xrightarrow{p_{TA}} (TA, \mu_A)$$

is the identity, for which it suffices to show that its transpose under the same adjunction equals the transpose of the identity on (TA, μ_A). The transpose of the above composite equals:

$$A \xrightarrow{\eta_A} TA \xrightarrow{T\eta_A} T^2A \xrightarrow{(\lambda_A)^t} TA' \xrightarrow{p_{TA}} TA$$

By naturality of η_A the first two components equal $\eta_{TA} \circ \eta_A$. We have $(\lambda_A)^t \circ \eta_{TA} = \lambda_A$ undoing the original transposition. Therefore the composite equals $p_{TA} \circ \lambda_{TA} \circ \eta_A = \eta_A$ the final equality holding as $p \circ \lambda = 1$. Since η_A is the transpose of the identity on (TA, μ_A) the claimed composite is indeed the identity in T-Alg$_s$. Therefore $(\lambda_A)^t \circ T\eta_A$ is a section of p_{TA}. ☐

Remark 6.19. In the remainder of this section we suppose that T is a 2-monad with rank on a complete and cocomplete category. In that case T-Alg$_s$ is both complete and cocomplete by Proposition 3.8 of [8].

Definition 6.20. Flexible limits are those limits which may be constructed from pie limits: products, inserter and equifiers, together with splittings of idempotents. Flexible colimits are those colimits which may be constructed from pie colimits: coproducts, coinserters and coequifiers, together with splittings of idempotents.

Proposition 6.21. Each algebra $(A, a)'$ is contained in the closure of the free algebras in T-Alg$_s$ under pie colimits. Each flexible algebra is contained in the closure of the free algebras in T-Alg$_s$ under flexible colimits.

Proof. By Proposition 6.8(2) the algebra $(A, a)'$ is the isocodescent object of the strict reflexive coherence data in T-Alg$_s$:

$$\begin{tikzcd} (T^3A, \mu_{T^2A}) \arrow[r, shift left=2, "\mu_{TA}"] \arrow[r, "T\mu_A" description] \arrow[r, shift right=2, "T^2a"'] & (T^2A, \mu_{TA}) \arrow[r, shift left=2, "\mu_A"] \arrow[r, shift right=2, "Ta"'] & (TA, \mu_A) \arrow[l, "T\eta_A" description] \end{tikzcd}$$

each component of which is a free algebra. By Remark 2.18 isocodescent objects may be constructed from coinserters and coequifiers, thus $(A, a)'$ lies in the closure of the frees in T-Alg$_s$ under coinserters and coequifiers, and therefore under pie-colimits. By Proposition 6.15(3) each flexible algebra (B, b) is a retract of an algebra of the form $(A, a)'$. Given then $r : (B, b) \longrightarrow (A, a)'$ and $s : (A, a)' \longrightarrow (B, b)$ such that $sr = 1$ the flexible algebra (B, b) is the splitting of the idempotent rs on $(A, a)'$. Therefore each flexible algebra is contained in the closure of the frees under coinserters, coequifiers and splittings of idempotents; thus flexible colimits. ☐

Proposition 6.22. The flexible algebras are closed in T-Alg$_s$ under flexible colimits. Therefore the flexible algebras are precisely those algebras contained in the closure of the free algebras under flexible colimits in T-Alg$_s$.

Proof. It is straightforward to verify directly, using the relevant universal property, that any coproduct, coinserter, coequifier or idempotent splitting of flexible algebras is again flexible. Therefore the flexible algebras are closed under flexible colimits. By Proposition 6.21 each flexible algebra is a flexible colimit of free algebras and the result follows. ☐

6.4 Pseudoalgebras and general coherence data

We have so far considered the cases of codescent objects of strict coherence data, and strict reflexive coherence data and these were sufficient to describe the left adjoints to the inclusions of T-Alg$_\mathrm{s}$ into T-Alg and T-Alg$_\mathrm{l}$. If we are interested in pseudoalgebras we need to consider a more general notion. Recall that the resolution of a strict algebra constitutes strict reflexive coherence data. A pseudoalgebra again has a "resolution" but the corresponding coherence data is no longer "strict". The weights for strict (reflexive) coherence data have indexing categories respectively Δ_2^- and Δ_2. We now describe the indexing 2-categories ${\Delta_2^-}'$ and Δ_2' which are the indexing categories for the weaker notion of coherence data and its reflexive counterpart. Both are sub 2-categories of Δ', which may be constructed using a factorisation system on 2-CAT which we now describe.

Remark 6.23. 2-CAT has an orthogonal factorisation system (E, M) in which:

- $E = \{$2-functors which are bijective on objects and arrows$\}$
- $M = \{$Locally fully faithful 2-functors$\}$

A 2-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ may be factored through another 2-category $\mathcal{C}$ whose underlying category $\mathcal{UC}$ equals $\mathcal{UA}$. Given a parallel pair of arrows $f, g : A \rightrightarrows B$ of $\mathcal{C}$ a 2-cell $f \Longrightarrow g$ of $\mathcal{C}$ is a triple $\langle f, \alpha : Ff \Longrightarrow Fg, g\rangle$. Composition of 2-cells in $\mathcal{C}$ is inherited from that of $\mathcal{B}$, giving $\mathcal{C}$ the structure of a 2-category. We therefore obtain a factorisation:

$$\begin{tikzcd}\mathcal{A} \arrow[r, "F_1"] & \mathcal{C} \arrow[r, "F_2"] & \mathcal{B}\end{tikzcd}$$

In this factorisation F_1 is identity on objects and arrows, whilst the action of F_2 on objects and arrows of $\mathcal{A}$ agrees with the action of F. Given a 2-cell $\alpha : f \Longrightarrow g$, F_1 and F_2 act as:

$$f \overset{\theta}{\Longrightarrow} g \quad \overset{F_1}{\longmapsto} \quad \langle f, Ff \overset{F\theta}{\Longrightarrow} Fg, g\rangle \quad \overset{F_2}{\longmapsto} \quad Ff \overset{F\theta}{\Longrightarrow} Fg \quad .$$

Remark 6.24. We now describe a specific case of this factorisation which we will employ to obtain the 2-categories ${\Delta_2^-}'$ and Δ_2'. As described in Proposition 2.13 the simplicial category Δ is freely generated by the face and degeneracy operaters subject to the simplicial identities. To be precise it is a quotient of a free category on a directed graph exhibited by a functor:

$$q : FG \longrightarrow \Delta$$

The graph G has the same objects as Δ and directed edges the face and degeneracy operators $\sigma_j^n : [n-1] \longrightarrow [n]$ and $\rho_j^n : [n+1] \longrightarrow [n]$. The free category FG on G has the same objects as G. Its morphisms are composable strings of the generating words $[\sigma_j^n] : [n-1] \longrightarrow [n]$ and $[\rho_j^n] : [n+1] \longrightarrow [n]$, with empty strings playing the role of identity morphisms. The functor $q : FG \longrightarrow \Delta$ is the identity on objects and is uniquely determined by its action on the generating arrows, whereupon we have $q[\sigma_j^n] = \sigma_j^n$ and $q[\rho_j^n] = \rho_j^n$.
We may view FG and Δ as locally discrete 2-categories upon which $q : FG \longrightarrow \Delta$ becomes a 2-functor. Factorising it as bijective on objects and arrows followed by locally fully faithful:

$$\begin{tikzcd}FG \arrow[r] & \Delta' \arrow[r, "p"] & \Delta\end{tikzcd}$$

gives the 2-category Δ'. The underlying category of Δ' agrees with that of FG. For parallel arrows:

$$\begin{tikzcd}{[i]} \arrow[r, shift left, "{[\alpha_1 \ldots \alpha_n]}"] \arrow[r, shift right, "{[\beta_1 \ldots \beta_m]}"'] & {[j]}\end{tikzcd}$$

of FG there exists a unique 2-cell

$$\begin{tikzcd}
{[i]} \arrow[r, bend left, "{[\alpha_1 \ldots \alpha_n]}"] \arrow[r, bend right, "{[\beta_1 \ldots \beta_m]}"'] \arrow[r, phantom, "\cong"] & {[j]}
\end{tikzcd}$$

whenever the equation $\alpha_n \circ \ldots \alpha_1 = \beta_m \circ \ldots \beta_1$ holds in Δ.
As equality is an equivalence relation it follows that Δ' is locally an equivalence relation. In particular each such 2-cell is not only unique, but by reflexivity an isomorphism. Consequently it is not necessary to either orient or label the 2-cells of Δ', each of these being determined precisely by its domain and codomain 1-cells. As Δ' is locally a preorder composition of 2-cells is automatic. As each equation between 1-cells of Δ is generated by one of the simplicial identities of Proposition 2.13, each 2-cell of Δ' is generated by one of the generating isomorphisms:

- For $j < i \in [n+1]$: $\quad [\delta_i^{n+1}\delta_j^n] \cong [\delta_j^{n+1}\delta_{i-1}^n]$.
- For $j \leq i \in [n-1]$: $\quad [\sigma_i^{n-1}\sigma_j^n] \cong [\sigma_j^{n-1}\sigma_{i+1}^n]$.
- For all $j \in [n]$ and $i \in [n-1]$:

$$[\sigma_i^{n-1}\delta_j^n] \cong \begin{cases} [\delta_j^{n-1}\rho_{i-1}^{n-2}] & \text{if } j < i \\ [\ \ ]_{[n]} & \text{if } j = i \text{ or } j = i+1 \\ [\delta_{j-1}^{n-1}\sigma_i^{n-2}] & \text{if } j > i+1 \end{cases}$$

Now $q : FG \longrightarrow \Delta$ is the identity on objects and full on 1-cells. Therefore the 2-functor $p : \Delta' \longrightarrow \Delta$ is both the identity on objects and full on 1-cells, in addition to being locally fully faithful. Consequently p is the identity on objects and locally a surjective equivalence.

Definition 6.25. The 2-category ${\Delta_2^-}'$ is the locally full sub 2-category of Δ' with generating morphisms the maps:

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=2, "{[\delta_1]}"] \arrow[r, shift right=2, "{[\delta_0]}"'] & {[1]} \arrow[l, "{[\sigma_0]}" description] \arrow[r, shift left=2, "{[\delta_2]}"] \arrow[r, "{[\delta_1]}" description] \arrow[r, shift right=2, "{[\delta_0]}"'] & {[2]}
\end{tikzcd}$$

whilst Δ_2' is the full sub 2-category of Δ' with objects [0], [1] and [2].

Remark 6.26. Restricting the 2-functor $p : \Delta' \longrightarrow \Delta$ to Δ_2' of Remark 6.24 gives a 2-functor $p : \Delta_2' \longrightarrow \Delta_2$. As $p : \Delta' \longrightarrow \Delta$ is identity on objects and locally a surjective equivalence the same properties hold for $p : \Delta_2' \longrightarrow \Delta_2$. This will be of importance in Chapter 8.

Definition 6.27. Let $\mathcal{A}$ be a 2-category.

1. A 2-functor $A : ({\Delta_2^-}')^{op} \longrightarrow \mathcal{A}$ is called coherence data. Such a 2-functor consists of a diagram of objects and 1-cells:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

   together with isomorphisms $dp \cong dm$, $cq \cong cm$, $cp \cong dq$ and $di \cong ci \cong 1$. No equations are required to hold between the 1-cells but "all diagrams of 2-cells commute". This corresponds to the fact that $({\Delta_2^-}')^{op}$ is locally a preorder. Furthermore we need not label or orient any of these 2-cells; for they are uniquely determined by their domains and codomains and are invertible.

2. A 2-functor $A : (\Delta_2')^{op} \longrightarrow \mathcal{A}$ is reflexive coherence data. This consists of a diagram of objects and 1-cells:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=4, "p"] \arrow[r, leftarrow, shift left=2, "l" description] \arrow[r, "m" description] \arrow[r, leftarrow, shift right=2, "r" description] \arrow[r, shift right=4, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "c"'] & A_0
\end{tikzcd}$$

together with the isomorphisms required for coherence data. Furthermore there are additional isomorphisms $li \cong ri$, $pl \cong id$, $ml \cong ql \cong 1$, $qr \cong ic$ and $mr \cong pr \cong 1$. Again there are no equations between 1-cells but all diagrams of 2-cells must commute.

Example 6.28. Consider a pseudoalgebra (A, a, α, α_0). Its putative resolution:

$$\begin{tikzcd}
Res(A,a,\alpha,\alpha_0) & = & (T^3A, \mu_{T^2A}) \arrow[r, shift left=4, "\mu_{TA}"] \arrow[r, leftarrow, shift left=2, "T\eta_{TA}" description] \arrow[r, "T\mu_A" description] \arrow[r, leftarrow, shift right=2, "T^2\eta_A" description] \arrow[r, shift right=4, "T^2a"'] & (T^2A, \mu_{TA}) \arrow[r, shift left=2, "\mu_A"] \arrow[r, leftarrow, "T\eta_A" description] \arrow[r, shift right=2, "Ta"'] & (TA, \mu_A)
\end{tikzcd}$$

is no longer strict coherence data; but equipped with the 2-cell isomorphisms $T\alpha_0 : 1 \cong Ta \circ T\eta_A$, $T\alpha : Ta \cong T^2\alpha \cong Ta \circ T\mu_a$ and $T^2\alpha_0 : 1 \cong T^2a \circ T^2\eta_A$, the other necessary isomorphisms being identities, becomes reflexive coherence data.

Remark 6.29. In the case of strict algebras we gave a conceptual explanation as to why each strict algebra gives rise to strict coherence data. Namely $(\Delta_+^{op}, \oplus, [-1])$ is the free strict monoidal category containing a comonoid. The 2-monad T induces a 2-comonad on T-Alg_s, a comonoid in $[\text{T-Alg}_s, \text{T-Alg}_s]$ and thus a corresponding functor $\Delta_+^{op} \longrightarrow [\text{T-Alg}_s, \text{T-Alg}_s]$ whose transpose:

$$\text{T-Alg}_s \longrightarrow [\Delta_+^{op}, \text{T-Alg}_s]$$

may be restricted to assign strict reflexive coherence data to each strict algebra.
A conceptual explanation for the coherence data associated to a pseudoalgebra follows from work of Lack [33] which we very briefly outline here. Each 2-monad induces a pseudo-comonad on Ps-T-Alg, a pseudo-comonoid in the Gray monoid Gray(Ps-T-Alg, Ps-T-Alg). We have only considered Δ' but of course this may be extended to Δ'_+. This 2-category admits the structure of a Gray monoid, the universal Gray monoid containing a pseudo-monoid and thus $(\Delta'_+)^{op}$ is the universal Gray-monoid containing a pseudo-comonoid. Therefore we have a 2-functor $(\Delta'_+)^{op} \longrightarrow \text{Gray(Ps-T-Alg, Ps-T-Alg)}$ corresponding to the pseudocomonad. Its transpose is a 2-functor:

$$\text{Ps-T-Alg} \longrightarrow \text{Gray}((\Delta'_+)^{op}, \text{Ps-T-Alg})$$

whose restriction assigns the reflexive coherence data to a pseudoalgebra described in Example 6.28.

Remark 6.30. As in the strict case the codescent object of reflexive coherence data will be simply the codescent object of the underlying coherence data. Since our interest in both types of coherence data is primarily with regards their codescent objects we view "reflexivity" as a property of coherence data. Thus we say that coherence data $(\Delta_2^{-\prime})^{op} \longrightarrow \mathcal{A}$ is reflexive if it underlies reflexive coherence data $(\Delta_2')^{op} \longrightarrow \mathcal{A}$.

Definition 6.31. Given coherence data:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "c"'] & A_0
\end{tikzcd}$$

reflexive or not, its codescent object and universal cocone consists of a triple (A, f, α): an object, 1-cell and 2-cell as depicted:

$$\begin{tikzcd}
 & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, "\alpha"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & A \\
 & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

satisfying the two identities:

1.

$$\begin{tikzcd}
 & A_1 \arrow[r, "d"] \arrow[d, phantom, "\cong"] & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, shorten=6mm, "\alpha"] & \\
A_2 \arrow[ur, "p"] \arrow[r, "m"] \arrow[dr, "q"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[d, phantom, "\cong"] & & A \\
 & A_1 \arrow[r, "c"'] & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & A_1 \arrow[rr, "d"] \arrow[dr, "c"] & & A_0 \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
A_2 \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\cong"] & {} & A_0 \arrow[rr, "f"] & {} \arrow[d, phantom, "\Downarrow\alpha"] & A \\
 & A_1 \arrow[ur, "d"] \arrow[rr, "c"'] & & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

2.

$$\begin{tikzcd}
 & & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, shorten=4mm, "\alpha"] & \\
A_0 \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[r, "i"] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & A \\
 & & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
= A_0 \xrightarrow{f} a$$

The "cocone" is the universal one with these properties: the universal property of the codescent object is the same as the universal property of the codescent object of strict coherence data described in Section 2.2. The isocodescent object, as in the case of strict coherence data, comes equipped with cocone a triple (A, f, α) as above but the 2-cell α is invertible, and furthermore this cocone has its universal property only with respect to other such cocones.

Remark 6.32. We have not shown that the notion of codescent object of coherence data describes a weighted colimit; we have not given the weight. The weight is not so simple as for the case of strict coherence data. In order to describe the weight we firstly observe that the codescent object of coherence data may be formed using coinserters and coequifiers if $\mathcal{A}$ has these colimits. Given coherence data as in Definition 6.31 we firstly form the coinserter:

$$\begin{tikzcd}
 & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, shorten=4mm, "\alpha"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & A \\
 & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

of its underlying graph. Next form the coequifier $(B, g : A \longrightarrow B)$ of the parallel pair of 2-cells:

$$\begin{tikzcd}
 & A_1 \arrow[r, "d"] \arrow[d, phantom, "\cong"] & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, shorten=6mm, "\alpha"] & \\
A_2 \arrow[ur, "p"] \arrow[r, "m"] \arrow[dr, "q"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[d, phantom, "\cong"] & & A \\
 & A_1 \arrow[r, "c"'] & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & A_1 \arrow[rr, "d"] \arrow[dr, "c"] & & A_0 \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
A_2 \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\cong"] & {} & A_0 \arrow[rr, "f"] & {} \arrow[d, phantom, "\Downarrow\alpha"] & A \\
 & A_1 \arrow[ur, "d"] \arrow[rr, "c"'] & & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

Finally form the coequifier $(C, h : B \longrightarrow C)$ of the parallel pair of 2-cells:

$$\begin{tikzcd}
 & & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, shorten=4mm, "\alpha"] & & \\
A_0 \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[r, "i"] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & & A \arrow[r, "g"] & B \\
 & & A_0 \arrow[ur, "f"'] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A_0 \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1"] & A \arrow[r, "g"] & B
\end{tikzcd}$$

It is straightforward to check now that the cocone $(C, hgf, hg\alpha)$ has the universal property required to exhibit C as the codescent object of the coherence data. The Yoneda embedding $Y : ({\Delta_2^-}')^{op} \longrightarrow [{\Delta_2^-}', \mathrm{Cat}]$

constitutes coherence data in $[\Delta_2^{-\prime}, \mathrm{Cat}]$. As the presheaf 2-category $[\Delta_2^{-\prime}, \mathrm{Cat}]$ is cocomplete we may calculate its codescent object in $[\Delta_2^{-\prime}, \mathrm{Cat}]$ using coinserters and coequifiers as described to obtain a 2-functor $W : \Delta_2^{-\prime} \longrightarrow \mathrm{Cat}$ and this is the weight for codescent objects of coherence data. It is easy to see this directly and we briefly justify the claim now. Given coherence data $X : \Delta_2^{-op} \longrightarrow \mathcal{A}$ let $\mathrm{Cocone}(X, A)$ denote the evident category of codescent cones from X to an object A of $\mathcal{A}$. The codescent object of X is the universal cocone: we have an isomorphism $\mathrm{Cocone}(X, A) \cong \mathcal{A}(QX, A)$ where we denote by QX the codescent object of X. To give the codescent object of X is therefore to give a representation of $\mathrm{Cocone}(X, 1) : \mathcal{A} \longrightarrow \mathrm{Cat}$. As W is the codescent object of the Yoneda embedding we have $[\Delta_2^{-}, \mathrm{Cat}](W, A(X-, A)) \cong \mathrm{Cocone}(Y, A(X-, A))$. Using Yoneda's lemma we see that $\mathrm{Cocone}(Y, A(X-, A)) \cong \mathrm{Cocone}(X, A)$ naturally in A as required. Thus $W : \Delta_2^{-} \longrightarrow \mathrm{Cat}$ is the required weight.

One may form the weight for isocodescent objects of coherence data from the weight for codescent objects by additionally forming a coinverter, just as described in Remark 2.18. A similar approach again will yield the weights for codescent and isocodescent objects of reflexive coherence data.

Remark 6.33. Strict coherence data is a special case of general coherence data; all of the defining 2-cell isomorphisms are required to be identities. The codescent object of strict coherence data, viewed as general coherence data, as described above in Definition 6.31, is exactly the codescent object of the strict coherence data as described in Section 2.2. The cocone equations of Definition 6.31 reduce to the equations for a codescent cocone of Section 2.2 when the 2-cell isomorphisms of the coherence data are identities. Thus the notion of codescent object introduced in this section is a generalisation of the strict case, specialising to the preceding notion when the coherence data is actually strict. These remarks also hold for the case of the isocodescent object.

Proposition 6.34. The inclusion of $\text{T-Alg}_{\mathrm{s}} \longrightarrow \text{Ps-T-Alg}$ has a left 2-adjoint if and only if for each pseudoalgebra (A, a, α, α_0) the isocodescent object of its resolution exists in $\text{T-Alg}_{\mathrm{s}}$. This is the case whenever $\text{T-Alg}_{\mathrm{s}}$ has isocodescent objects of reflexive coherence data.

Proof. Denoting by $W : \Delta_2' \longrightarrow \mathrm{Cat}$ the weight for isocodescent objects of reflexive coherence data we have an isomorphism of categories:

$$[(\Delta_2')^{op}, \text{T-Alg}_{\mathrm{s}}](W, \text{T-Alg}_{\mathrm{s}}(Res(A, a, \alpha, \alpha_0), (B, b)) \cong \text{Ps-T-Alg}((A, a, \alpha, \alpha_0), (B, b))$$

2-natural in (B, b). The argument now proceeds exactly as described in Proposition 6.8. ☐

6.5 Enhanced factorisation systems, bijections on objects and calculations of $(-)'$

In this section we consider some cases in which the left adjoints to the inclusions of $\text{T-Alg}_{\mathrm{s}}$ into T-Alg and Ps-T-Alg are easily calculated. The calculations of $(-)'$ we here describe constitute the first use of enhanced factorisation systems. The basic result concerning factorisation systems and the theory of 1-dimensional monads, which is well known, is the following. We state that proposition in terms of 2-monads, the generalisation to the 2-dimensional case being trivial.

Proposition 6.35. Let $\mathcal{A}$ be a 2-category with an orthogonal factorisation system (E, M). Let T be a 2-monad on $\mathcal{A}$ and suppose that $T(E) \subseteq E$. Then the factorisation system lifts to an orthogonal factorisation system $(\overline{E}, \overline{M})$ on $\text{T-Alg}_{\mathrm{s}}$ where $\overline{E} = \{f \in \text{T-Alg}_{\mathrm{s}} : U^T f \in E\}$ and $\overline{M} = \{f \in \text{T-Alg}_{\mathrm{s}} : U^T f \in M\}$.

Proof. Given an algebra morphism $f : (A, a) \longrightarrow (B, b)$ as on the left below:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "Te"] \arrow[d, "a"'] & TC \arrow[r, "Tm"] & TB \arrow[d, "b"] \\ TA \arrow[r, "e"'] & C \arrow[r, "m"'] & B \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "Te"] \arrow[d, "a"'] & TC \arrow[r, "Tm"] \arrow[d, "c"] & TB \arrow[d, "b"] \\ TA \arrow[r, "e"'] & C \arrow[r, "m"'] & B \end{tikzcd}$$

we may factor the underlying morphism $f : A \longrightarrow B$ as $f = me$ with $m \in M$ and $e \in E$. Upon doing so the diagram on the left above may be rewritten as the middle diagram. Now $Te \in E$ by assumption and $m \in M$ so that there exists a unique arrow $c : TC \longrightarrow C$ as on the right above, rendering both squares of that diagram commutative. By uniqueness of c and naturality of η and μ it is straightforward to see that the pair (C, c) is an algebra and that we have a pair of algebra morphisms $e : (A, a) \longrightarrow (C, c)$ and $m : (C, c) \longrightarrow (B, b)$ whose composite equals $f : (A, a) \longrightarrow (B, b)$. That the classes $\overline{E}$ and $\overline{M}$ are closed under composition and contain the isomorphisms follows from the corresponding facts for E and M, as does orthogonality of the classes $\overline{E}$ and $\overline{M}$. A full proof of the 1-dimensional case may be found in [2]. $\square$

Remark 6.36. The construction of a strict algebra from a pseudoalgebra using enhanced factorisations systems described in the following Remark is due to Power [44]. We separate this construction from Power's result so as to give the construction in a suitable generality which will allow us to draw the connection with the later work of Lack. Power proved that in the case of a 2-monad on Cat which preserves bijections on objects each pseudoalgebra is equivalent to a strict algebra. This was achieved by associating to each pseudoalgebra (A, a, α, α_0) a strict one equivalent to it. In [34] Lack showed that the strict algebra obtained by Power's method is precisely $(A, a, \alpha, \alpha_0)'$, by exhibiting its universal property.

Remark 6.37. Let (E, M) be an enhanced factorisation system on a 2-category $\mathcal{A}$ and T a 2-monad on $\mathcal{A}$ such that $TE \subseteq E$. At a pseudoalgebra (A, a, α, α_0), the counit of the biadjunction :

$$\begin{tikzcd}
\text{Ps-T-Alg} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right, "F"']
\end{tikzcd}$$

is the pseudomorphism of algebras: $(a, \alpha) : (TA, \mu_A) \longrightarrow (A, a, \alpha, \alpha_0)$ on the left below:

$$\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\alpha\Downarrow"] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "Te"] \arrow[d, "\mu_A"'] & TB \arrow[r, "Tm"] \arrow[d, phantom, "\alpha\Downarrow"] & TA \arrow[d, "a"] \\
TA \arrow[r, "e"'] & B \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "Te"] \arrow[d, "\mu_A"'] & TB \arrow[r, "Tm"] \arrow[d, "b"] \arrow[dr, phantom, "\beta\Downarrow"] & TA \arrow[d, "a"] \\
TA \arrow[r, "e"'] & B \arrow[r, "m"'] & A
\end{tikzcd}$$

We may factor the underlying morphism $a : TA \longrightarrow A$ as $a = me$ with $m \in M$ and $e \in E$ so that the diagram on the left above may be rewritten as the middle diagram above. Now $Te \in E$ by assumption and $m \in M$. Therefore as (E, M) is enhanced there exists a unique 1-cell $b : TB \longrightarrow B$ and isomorphism $\beta : a \circ Tm \cong m \circ b$ such that the left square of the right diagram above is commutative and such that $\beta \circ Tm = \alpha$. It follows from the uniqueness of the construction that (B, b) is a strict algebra and that $e : (A, a) \longrightarrow (B, b)$ is a strict algebra map and $(m, \beta) : (B, b) \longrightarrow (A, a, \alpha, \alpha_0)$ a pseudomorphism of algebras. Furthermore we have a factorisation of the algebra morphism (a, α) as $(m, \beta) \circ e$.

Theorem 6.38 (Power)**.** Let T be a 2-monad on Cat which preserves bijections on objects. Then each pseudoalgebra is equivalent to a strict algebra.

Proof. The (Bijective on objects/fully faithful) factorisation system on Cat is enhanced. Therefore given a 2-monad T on Cat preserving bijections on objects one may apply the construction of Remark 6.37 to obtain, for each pseudoalgebra (A, a, α, α_0) a strict algebra (B, b) a pseudomorphism $(m, \beta) : (B, b) \longrightarrow (A, a, \alpha, \alpha_0)$. Power showed this map to be an equivalence by describing its equivalence inverse. $\square$

Remark 6.39. In that paper Power generalised the result to a 2-monad based on Cat^X for a set X, and furthermore to Cat_g the 2-category with the same underlying category as Cat but with 2-cells the natural isomorphisms, the interest in this latter 2-category being that 2-monads on Cat_g can describe contravariant structure. In [34] Lack isolated those aspects of the (Bijective on objects/fully faithful) factorisation system on Cat which enabled Power's construction to work. The following result is, stated in a less compact form, Theorem 4.10 of [34].

Proposition 6.40 (Lack)**.** Let (E, M) be an enhanced factorisation system on a 2-category $\mathcal{A}$ with the following property:

- Let $m : A \longrightarrow B \in M$ and $f : B \longrightarrow A$. If $mf \cong 1$ then $1 \cong fm$.

Suppose T is a 2-monad on $\mathcal{A}$ such that $T(E) \subseteq E$.

1. The inclusion $\iota : \text{T-Alg}_{\text{s}} \longrightarrow \text{Ps-T-Alg}$ has a left 2-adjoint.

   $$\begin{tikzcd} \text{T-Alg}_{\text{s}} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\iota"'] & \text{Ps-T-Alg} \arrow[l, bend right, "(-)'"'] \end{tikzcd}$$

   At a pseudoalgebra (A, a, α, α_0) we have $(A, a, \alpha, \alpha_0)' = (B, b)$ the strict algebra as constructed in Remark 6.37 and the counit component at this pseudoalgebra is the pseudomorphism

   $$(m, \beta) : (B, b) \longrightarrow (A, a, \alpha, \alpha_0)$$

2. Each counit component is an equivalence. Therefore each pseudoalgebra is equivalent to a strict algebra.

6.6 Strongly finitary 2-monads

A class of 2-functor known as strongly finitary will be important in Chapters 8,9 and 10. In Chapter 8 we will be led in a natural manner to a certain definition of "strongly finitary", which will differ somewhat from the notion which appears in the literature: [27],[28],[34] but will agree in the most important case of a 2-functor based on Cat. We briefly recall the notion of strongly finitary V-functor, as originally defined, and then pass to the case of $V = \text{Cat}$ where we review the relevant results.

Remark 6.41. In order to describe the notion of strongly finitary V-functor we must recall a couple of facts about enriched categories. If V is a symmetric monoidal closed category then the closed structure enriches V to a V-category $\mathbf{V}$.
We have a forgetful 2-functor $(-)_0 : \text{V-CAT} \longrightarrow \text{CAT}$. Given a V-category A the category A_0 has the same objects as A and given $X, Y \in A$ the hom-set $A_0(X, Y)$ is given by $V(i, A(X, Y))$ where i denotes the monoidal unit for V. The category V may be recovered from $\mathbf{V}$, up to isomorphism of categories, by taking its image under the forgetful 2-functor $(-)_0 : \text{V-CAT} \longrightarrow \text{CAT}$.
If the category V admits copowers of the unit i by all sets then the functor $V(i, -) : V \longrightarrow \text{Set}$ has a left adjoint, which assigns to a set X the copower $X.i \in V$. This lifts to a left 2-adjoint $F : \text{Set-CAT} = \text{CAT} \longrightarrow \text{V-CAT}$ to $(-)_0$. It assigns to a category the free V-category upon it. At a category A, $F(A)$ is the V-category with the same objects as A and with $FA(X, Y) = A(X, Y).i$. Let Set_f denote the skeletal category of finite sets. Then as V is cocomplete we have the functor $\text{Set}_f \longrightarrow V \cong \mathbf{V}_0$ assigning to a finite set $n \in \text{Set}_f$ the copower $n.i \in V$. Its transpose across the adjunction corresponds to a V-functor $\iota : F(\text{Set}_f) \longrightarrow \mathbf{V}$.

Definition 6.42 ([27])**.** Let V be a complete and cocomplete symmetric monoidal closed category, in which case we have the V-functor $\iota : F(\text{Set}_f) \longrightarrow \mathbf{V}$ described in the preceding Remark. An endo-V-functor $T : \mathbf{V} \longrightarrow \mathbf{V}$ is said to be strongly finitary if it is the left Kan extension in V-CAT of its restriction along ι.

Remark 6.43. The case of primary interest in [27] was the case $V = \mathcal{U}\text{Cat}$ the category of small categories with its cartesian closed structure. In that case $\mathbf{V} = \text{Cat}$. As the monoidal unit for $\mathcal{U}\text{Cat}$ is the terminal object 1 it is straightforward to see that $F(\text{Set}_f) = \text{Set}_f$ viewed as a locally discrete 2-category, and that the 2-functor $\iota : \text{Set}_f \longrightarrow \text{Cat}$ is the 2-functor which views each finite set as a discrete category. Therefore an endo 2-functor of Cat is said to be strongly finitary if it is the left Kan extension in 2-CAT of its restriction along $\iota : \text{Set}_f \longrightarrow \text{Cat}$.

Definition 6.44. A 2-monad (T, η, μ) on Cat is said to be strongly finitary [27] if its endo 2-functor part is so.

Remark 6.45. In [27] it was shown that strongly finitary 2-monads on Cat describe precisely those kinds of equational structure borne by categories, the arity of whose operations are both finite and discrete, objects of Set_f. For instance there are strongly finitary 2-monads whose strict algebras are: monoidal categories, strict monoidal categories, distributive categories, categories with finite products, categories with finite coproducts and so on, but are limited to such cases. However one cannot describe a strongly finitary 2-monad whose algebras are categories with chosen equalisers. In order to describe categories with equalisers one requires an operation $A^C \longrightarrow A$ where C is the category consisting of a parallel pair: $\{\mathbf{0} \rightrightarrows \mathbf{1}\}$. Whilst C is a finitely presentable category it is not an element of Set_f, both finitely presentable and discrete.

Remark 6.46. The relevance of strongly finitary 2-monads on Cat to the concerns of this thesis are indicated by the following result of Lack [34].

Proposition 6.47 (Lack)**.** *Any strongly finitary 2-functor on Cat preserves codescent objects of reflexive coherence data.*

Proof. This appears in Example 4.5 of [34], the main work of that argument being due to Proposition 4.3 of the same paper. □

Remark 6.48. Lack only briefly sketched a proof of the above result in [34]. We give a full proof in Chapter 8.

Chapter 7

2-monads of the form $Cat(T)$

This is a short and largely expository chapter which we include as it connects those 2-functors of the form $Cat(T)$ for $T \in \mathrm{Cat}_{\mathrm{pb}}$ much studied in Chapters 3 and 4 with two dimensional monad theory and furthermore the main results follow easily from the theory of those chapters. The main results are based on those of Batanin [5] with minor improvements, as described within. We now summarise the contents of the chapter. If $T = (T, \eta, \mu)$ is a monad in the 2-category $\mathrm{Cat}_{\mathrm{pb}}$ then its image under the 2-functor $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$ is a 2-monad $Cat(T)$. We show that in this case pseudo-morphism classifiers admit a simple description. Furthermore we show that if T is a cartesian monad then in special cases the lax morphism classifier of a strict algebra also exists and admits a simple description. We use this to explain an observation of Bénabou [6] which concerns the connection between the strict monoidal category $(\Delta_+, \oplus, [-1])$ and the resolution of the terminal strict monoidal category.

7.1 $Cat(T)$-algebras

Remark 7.1. If (T, η, μ) is a monad on $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$ then its image under $Cat(-) : \mathrm{Cat}_{\mathrm{pb}} \longrightarrow \mathrm{Rep}$ is a 2-monad $(Cat(T), Cat(\eta), Cat(\mu))$ on $Cat(\mathcal{E}) \in \mathrm{Rep}$. A strict $Cat(T)$-algebra (A, a) is an internal category A and an internal functor $a : Cat(T)A \longrightarrow A$ as on the left below:

$$\begin{tikzcd}
TA_2 \arrow[r, "a_2"] \arrow[d, shift right=2, "Tp_a"'] \arrow[d] \arrow[d, shift left=2, "Tq_a"] & A_2 \arrow[d, shift right=2, "p_a"'] \arrow[d, "m_a"] \arrow[d, shift left=2, "q_a"] \\
TA_1 \arrow[r, "a_1"] \arrow[d, shift right=2, "Td_a"'] \arrow[d, shift left=2, "Tc_a"] & A_1 \arrow[d, shift right=2, "d_a"'] \arrow[d, shift left=2, "c_a"] \\
TA_0 \arrow[u] \arrow[r, "a_0"] & A_0 \arrow[u, "i_a"]
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A_i \arrow[r, "Ta_i"] \arrow[d, "\mu_{A_i}"'] & TA_i \arrow[d, "a_i"] \\
TA_i \arrow[r, "a_i"'] & A_i
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TA_i \arrow[d, "a_i"] \\
TA_i \arrow[ur, "\eta_{A_i}"] \arrow[r, "1"'] & A_i
\end{tikzcd}$$

such that for $i = 0, 1, 2$ both diagrams on the right above are commutative. The diagrams on the right express the fact that (A, a) is an algebra: we have the equalities of internal functors $a \circ Cat(\mu)_A = a \circ Cat(T)a$ and $a \circ Cat(\eta)_A = 1$. Equivalently they exhibit each pair (A_i, a_i) as an algebra for the monad T. Under this second viewpoint each of the arrows defining the internal category structure of A is precisely a morphism of T-algebras. Thus the above $Cat(T)$-algebra is equally an internal category:

$$\begin{tikzcd}
(A_2, a_2) \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] & (A_1, a_1) \arrow[r, shift left=2, "d_a"] \arrow[r, shift right=2, "c_a"'] & (A_0, a_0) \arrow[l, "i_a" description]
\end{tikzcd}$$

in $\mathrm{T\text{-}Alg_s}$, the ordinary Eilenberg-Moore category for the monad T. This correspondence between objects of $\mathrm{Cat(T)\text{-}Alg_s}$ and $Cat(\mathrm{T\text{-}Alg_s})$ equally holds for 1-cells and 2-cells so that we have an isomorphism of 2-categories $\mathrm{Cat(T)\text{-}Alg_s} \cong Cat(\mathrm{T\text{-}Alg_s})$.

7.2 Pseudoalgebras and pseudomorphisms

Remark 7.2. We now move towards describing the left 2-adjoints to the inclusions $\iota : \mathrm{Cat(T)\text{-}Alg_s} \longrightarrow \mathrm{Cat(T)\text{-}Alg}$ and $\iota : \mathrm{Cat(T)\text{-}Alg_s} \longrightarrow \mathrm{Ps\text{-}Cat(T)\text{-}Alg}$. Giving a left adjoint to the second inclusion will automatically give the left adjoint to the first inclusion, since Cat(T)-Alg is a full subcategory of Ps-Cat(T)-Alg containing $\mathrm{Cat(T)\text{-}Alg_s}$. Therefore we focus upon constructing the adjunction:

$$\begin{tikzcd}
\mathrm{Cat(T)\text{-}Alg_s} \arrow[r, shift right=2, "\iota"'] \arrow[r, phantom, "\perp"] & \mathrm{Ps\text{-}Cat(T)\text{-}Alg} \arrow[l, shift right=2, "(-)'"']
\end{tikzcd}$$

This is a straightforward application of Proposition 6.40 and observed by Batanin in Theorem 7.1 of [5].

Proposition 7.3. Let (T, η, μ) be a monad in $\mathrm{Cat}_{\mathrm{pb}}$ on a category $\mathcal{E}$ and consider the induced 2-monad $(Cat(T), Cat(\eta), Cat(\mu))$ on $Cat(\mathcal{E})$. The (Bijective on objects / fully faithful) factorisation system on $Cat(\mathcal{E})$ satisfies the assumptions of Proposition 6.40 and any 2-monad of the form $Cat(T)$ preserves bijections on objects. Therefore we have a 2-adjunction:

$$\begin{tikzcd}
\mathrm{Cat(T)\text{-}Alg_s} \arrow[r, shift right=2, "\iota"'] \arrow[r, phantom, "\perp"] & \mathrm{Ps\text{-}Cat(T)\text{-}Alg} \arrow[l, shift right=2, "(-)'"']
\end{tikzcd}$$

and given a $Cat(T)$ pseudoalgebra (A, a, α, α_0) we may construct $(A, a, \alpha, \alpha_0)'$ by factoring the internal functor underlying the counit $(a, \alpha) : (Cat(T)A, Cat(\mu)_A) \longrightarrow (A, a, \alpha, \alpha_0)$ as bijective on objects followed by fully faithful.

Proof. It suffices to show that the (Bijective on objects/fully faithful) factorisation system on $Cat(\mathcal{E})$ and the 2-monad $Cat(T)$ satisfy the assumptions of Proposition 6.40. By Corollary 3.69 this is an enhanced factorisation system. Consider a fully faithful internal functor $m : X \longrightarrow Y$ and an internal functor $f : Y \longrightarrow X$ together with an isomorphism $\alpha : mf \cong 1$. Then we have $\alpha m : mfm \cong m$. As m is fully faithful there exists a unique 2-cell $\beta : fm \Longrightarrow 1$ and furthermore β is an isomorphism as the fully faithful arrow m reflects isomorphisms.
Consider a bijective on objects internal functor, an internal functor $f : X \longrightarrow Y$ such that $f_0 : X_0 \longrightarrow Y_0$ is an isomorphism. The internal functor $Cat(T)f : Cat(T)X \longrightarrow Cat(T)Y$ has objects components $Tf_0 : TX_0 \longrightarrow TY_0$ which is again an isomorphism. Thus Tf is bijective on objects. Therefore we may apply Proposition 6.40 to deduce the result. ☐

7.3 Lax morphisms

In the preceding section we considered the case of a monad (T, η, μ) in $\mathrm{Cat}_{\mathrm{pb}}$ and the induced 2-monad $Cat(T)$ on $Cat(\mathcal{E})$. We now suppose that T is a cartesian monad: not only is $T : \mathcal{E} \longrightarrow \mathcal{E}$ required to preserve pullbacks but the natural transformations η and μ must be cartesian natural transformations.

Lemma 7.4. If T is a cartesian monad then $Cat(T)$ is a cartesian 2-monad: both $Cat(\eta)$ and $Cat(\mu)$ are cartesian (by which we mean their underlying natural transformations are cartesian).

Proof. This is easy to see. It suffices to show that given a cartesian natural transformation $\theta : F \Longrightarrow G \in \mathrm{Cat}_{\mathrm{pb}}(\mathcal{A}, \mathcal{B})$ the induced 2-natural transformation $Cat(\theta) : Cat(F) \Longrightarrow Cat(G)$ is cartesian. Given an internal functor $f : X \longrightarrow Y \in Cat(\mathcal{A})$ we must show that the square on the left below is a pullback:

$$\begin{tikzcd}
Cat(F)X \arrow[r, "Cat(\theta)X"] \arrow[d, "Cat(F)f"'] & Cat(G)X \arrow[d, "Cat(G)f"] \\
Cat(F)Y \arrow[r, "Cat(\theta)Y"'] & Cat(G)Y
\end{tikzcd}
\qquad
\begin{tikzcd}
FX_i \arrow[r, "\theta_{X_i}"] \arrow[d, "Ff_i"'] & GX_i \arrow[d, "Gf_i"] \\
FY_i \arrow[r, "\theta_{Y_i}"'] & GY_i
\end{tikzcd}$$

For $i = 0, 1, 2$ the components of that square are the arrows in $\mathcal{B}$ on the right. The square on the right is a pullback as θ is cartesian. As pullbacks are pointwise in $Cat(\mathcal{B})$ it follows that the square on the left is a pullback in $Cat(\mathcal{B})$. Therefore $Cat(\theta)$ is cartesian. ☐

Remark 7.5. We are interested in lax morphism classifiers for a 2-monad $Cat(T)$ induced by cartesian T. Given a strict $Cat(T)$-algebra (A, a) its lax morphism classifier $(A, a)'$, if it exists, is defined by an isomorphism: $\text{Cat(T)-Alg}_{\text{s}}((A, a)', (B, b)) \cong \text{Cat(T)-Alg}_{\text{l}}((A, a), (B, b))$ 2-natural in (B, b), and, if it exists, is the codescent object in $\text{Cat(T)-Alg}_{\text{s}}$ of the strict reflexive coherence data constituting the resolution of (A, a):

$$\begin{tikzcd}[column sep=huge]
(\boldsymbol{T}^4A, \boldsymbol{\mu}_{\boldsymbol{T}^3A}) \arrow[r, shift left=9, "\boldsymbol{\mu}_{\boldsymbol{T}^2A}"] \arrow[r, leftarrow, shift left=6, "\boldsymbol{T\eta}_{\boldsymbol{T}^2A}" description] \arrow[r, shift left=3, "\boldsymbol{T\mu}_{\boldsymbol{T}A}" description] \arrow[r, leftarrow, "\boldsymbol{T}^2\boldsymbol{\eta}_{\boldsymbol{T}A}" description] \arrow[r, shift right=3, "\boldsymbol{T}^2\boldsymbol{\mu}_A" description] \arrow[r, leftarrow, shift right=6, "\boldsymbol{T}^3\boldsymbol{\eta}_A" description] \arrow[r, shift right=9, "\boldsymbol{T}^3a"'] & (\boldsymbol{T}^3A, \boldsymbol{\mu}_{\boldsymbol{T}^2A}) \arrow[r, shift left=6, "\boldsymbol{\mu}_{\boldsymbol{T}A}"] \arrow[r, leftarrow, shift left=3, "\boldsymbol{T\eta}_{\boldsymbol{T}A}" description] \arrow[r, "\boldsymbol{T\mu}_A" description] \arrow[r, leftarrow, shift right=3, "\boldsymbol{T}^2\boldsymbol{\eta}_A" description] \arrow[r, shift right=6, "\boldsymbol{T}^2a"'] & (\boldsymbol{T}^2A, \boldsymbol{\mu}_{\boldsymbol{T}A}) \arrow[r, shift left=3, "\boldsymbol{\mu}_A"] \arrow[r, leftarrow, "\boldsymbol{T\eta}_A" description] \arrow[r, shift right=3, "\boldsymbol{T}a"'] & (\boldsymbol{T}A, \boldsymbol{\mu}_A)
\end{tikzcd}$$

Here we have drawn the 3-truncation of the simplicial object associated to the algebra (A, a) whose 2-truncation is the resolution of (A, a). We have abbreviated $(Cat(T), Cat(\eta), Cat(\mu))$ as $(\boldsymbol{T}, \boldsymbol{\eta}, \boldsymbol{\mu})$ for clarity.

By Lemma 7.4 $Cat(\mu) = \boldsymbol{\mu}$ is cartesian and so the square on the left below is a pullback in $Cat(\mathcal{E})$:

$$\begin{tikzcd}
\boldsymbol{T}^3A \arrow[r, "\boldsymbol{\mu}_{\boldsymbol{T}A}"] \arrow[d, "\boldsymbol{T}^2a"'] & \boldsymbol{T}^2A \arrow[d, "\boldsymbol{T}a"] \\
\boldsymbol{T}^2A \arrow[r, "\boldsymbol{\mu}_A"'] & \boldsymbol{T}A
\end{tikzcd}
\qquad
\begin{tikzcd}
(\boldsymbol{T}^3A, \boldsymbol{\mu}_{\boldsymbol{T}^2A}) \arrow[r, "\boldsymbol{\mu}_{\boldsymbol{T}A}"] \arrow[d, "\boldsymbol{T}^2a"'] & (\boldsymbol{T}^2A, \boldsymbol{\mu}_{\boldsymbol{T}A}) \arrow[d, "\boldsymbol{T}a"] \\
(\boldsymbol{T}^2A, \boldsymbol{\mu}_{\boldsymbol{T}A}) \arrow[r, "\boldsymbol{\mu}_A"'] & (\boldsymbol{T}A, \boldsymbol{\mu}_A)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\boldsymbol{T}^4A, \boldsymbol{\mu}_{\boldsymbol{T}^3A}) \arrow[r, "\boldsymbol{\mu}_{\boldsymbol{T}^2A}"] \arrow[d, "\boldsymbol{T}^3a"'] & (\boldsymbol{T}^3A, \boldsymbol{\mu}_{\boldsymbol{T}^2A}) \arrow[d, "\boldsymbol{T}^2a"] \\
(\boldsymbol{T}^3A, \boldsymbol{\mu}_{\boldsymbol{T}^2A}) \arrow[r, "\boldsymbol{\mu}_{\boldsymbol{T}A}"'] & (\boldsymbol{T}^2A, \boldsymbol{\mu}_{\boldsymbol{T}A})
\end{tikzcd}$$

As the forgetful functor from $U^{Cat(T)}$: Cat(T)-Alg$_{\mathrm{s}}$ $\longrightarrow Cat(\mathcal{E})$ creates pullbacks the middle square above is a pullback in Cat(T)-Alg$_{\mathrm{s}}$. The square on the right is now easily seen to be a pullback square. Pasting it with the middle square gives a pullback square using naturality and the cartesian property of $\boldsymbol{\mu}$, and thus it is a pullback itself. Therefore $Res(A, a)$ is an internal category in Cat(T)-Alg$_{\mathrm{s}}$.

Remark 7.6. The following proposition is essentially Theorem 7.2 of [5] with the exception that Batanin considers only the terminal algebra, whereas in the following result we allow any discrete algebra. The proof as presented here is somewhat different to Batanin's. We use the theory developed in the preceding chapters.

Proposition 7.7. *Let T be a cartesian monad on $\mathcal{E}$ and consider the induced cartesian 2-monad $Cat(T)$ on $Cat(\mathcal{E})$. Consider a strict $Cat(T)$-algebra (A, a) such that A is a discrete internal category. Then the lax morphism classifier $(A, a)' = (A', a') \in$ Cat(T)-Alg$_{\mathrm{s}}$ exists and has underlying internal category A':*

$$\begin{tikzcd}[column sep=huge]
T^3A_0 \arrow[r, shift left=3, "\mu_{TA_0}"] \arrow[r, "T\mu_{A_0}" description] \arrow[r, shift right=3, "T^2a_0"'] & T^2A_0 \arrow[r, shift left=3, "\mu_{A_0}"] \arrow[r, shift right=3, "Ta_0"'] & TA_0 \arrow[l, "T\eta_{A_0}" description]
\end{tikzcd}$$

and structure map $a' : Cat(T)A' \longrightarrow A'$ the internal functor with $a'_i = \mu_{A'_i}$ for $i = 0, 1, 2$. Under the isomorphism of 2-categories Cat(T)-Alg$_{\mathrm{s}}$ $\cong$ Cat(T-Alg$_{\mathrm{s}}$) the Cat(T)-algebra $(A, a)'$ is equally the internal category in T-Alg$_{\mathrm{s}}$:

$$\begin{tikzcd}[column sep=huge]
(T^3A_0, \mu_{T^2A_0}) \arrow[r, shift left=3, "\mu_{TA_0}"] \arrow[r, "T\mu_{A_0}" description] \arrow[r, shift right=3, "T^2a"'] & (T^2A_0, \mu_{TA_0}) \arrow[r, shift left=3, "\mu_{A_0}"] \arrow[r, shift right=3, "Ta"'] & (TA_0, \mu_{A_0}) \arrow[l, "T\eta_{A_0}" description]
\end{tikzcd}$$

Proof. The forgetful 2-functor $U^{Cat(T)}$: Cat(T)-Alg$_{\mathrm{s}}$ $\longrightarrow Cat(\mathcal{E})$ creates all limits that $Cat(\mathcal{E})$ has. $Cat(\mathcal{E})$ has cotensors with **2** and so $U^{Cat(T)}$ preserves them. It additionally reflects isomorphisms and so by Corollary 4.3(2) reflects discreteness. Therefore the $Cat(T)$-algebra (A, a) is discrete precisely if its underlying internal category A is discrete. Consider the resolution of (A, a):

$$\begin{tikzcd}[column sep=huge]
(\boldsymbol{T}^3A, \boldsymbol{\mu}_{\boldsymbol{T}^2A}) \arrow[r, shift left=6, "\boldsymbol{\mu}_{\boldsymbol{T}A}"] \arrow[r, "\boldsymbol{T}\boldsymbol{\mu}_A" description] \arrow[r, shift right=6, "\boldsymbol{T}^2a"'] & (\boldsymbol{T}^2A, \boldsymbol{\mu}_{\boldsymbol{T}A}) \arrow[l, shift right=3, "\boldsymbol{T}\boldsymbol{\eta}_{\boldsymbol{T}A}" description] \arrow[l, shift left=3, "\boldsymbol{T}^2\boldsymbol{\eta}_A" description] \arrow[r, shift left=3, "\boldsymbol{\mu}_A"] \arrow[r, shift right=3, "\boldsymbol{T}a"'] & (\boldsymbol{T}A, \boldsymbol{\mu}_A) \arrow[l, "\boldsymbol{T}\boldsymbol{\eta}_A" description]
\end{tikzcd}$$

As described in Remark 7.5 this is an internal category in Cat(T)-Alg$_{\mathrm{s}}$. $Cat(T)$ is a morphism of Rep and so it preserves cotensors with **2** and therefore discreteness. In particular each internal category $Cat(T)^nA = \boldsymbol{T}^nA$ is a discrete internal category. As $U^{Cat(T)}$: Cat(T)-Alg$_{\mathrm{s}}$ $\longrightarrow Cat(\mathcal{E})$ reflects discreteness we see that the resolution of (A, a) is a pointwise discrete category in Cat(T)-Alg$_{\mathrm{s}}$, in particular a catead. By Remark 7.1 Cat(T)-Alg$_{\mathrm{s}}$ is isomorphic to Cat(T-Alg$_{\mathrm{s}}$). As T-Alg$_{\mathrm{s}}$ has pullbacks it follows by Theorem 3.65 that Cat(T-Alg$_{\mathrm{s}}$) has codescent objects of cateads, and by Theorem 3.64 they are computed by the 2-functor:

$$Kat(Cat(\text{T-Alg}_{\mathrm{s}})) \xrightarrow{j} Cat(\mathcal{U}Cat(\text{T-Alg}_{\mathrm{s}})) \xrightarrow{Cat(ob)} Cat(\text{T-Alg}_{\mathrm{s}})$$

We may therefore compute them in Cat(T)-Alg$_{\mathrm{s}}$, via Cat(T-Alg$_{\mathrm{s}}$), using the given isomorphism of 2-categories:

$$Kat(\mathrm{Cat(T)\text{-}Alg_s}) \cong Kat(Cat(\mathrm{T\text{-}Alg_s})) \xrightarrow{j} Cat(\mathcal{U}Cat(\mathrm{T\text{-}Alg_s})) \xrightarrow{Cat(ob)} Cat(\mathrm{T\text{-}Alg_s}) \cong \mathrm{Cat(T)\text{-}Alg_s}$$

Applying the first three components of this 2-functor to the resolution of (A, a) gives the internal category in T-Alg$_{\mathrm{s}}$ obtained by taking the objects of each component of the resolution of (A, a), each viewed as an object of Cat(T-Alg$_{\mathrm{s}}$). We obtain the internal category in T-Alg$_{\mathrm{s}}$:

$$\begin{tikzcd}
(T^3A_0, \mu_{T^2A_0}) \arrow[r, shift left=3, "\mu_{TA_0}"] \arrow[r, "T\mu_{A_0}" description] \arrow[r, shift right=3, "T^2a"'] & (T^2A_0, \mu_{TA_0}) \arrow[r, shift left=3, "\mu_{A_0}"] \arrow[r, shift right=3, "Ta"'] & (TA_0, \mu_{A_0}) \arrow[l, "T\eta_{A_0}" description]
\end{tikzcd}$$

as claimed. The image of this internal category in T-Alg$_{\mathrm{s}}$ under the isomorphism Cat(T-Alg$_{\mathrm{s}}$) ≅ Cat(T)-Alg$_{\mathrm{s}}$ is the claimed $Cat(T)$-algebra. ☐

Example 7.8. The free monoid monad (T, η, μ) on Set is the prototypical example of a cartesian monad. The objects of T-Alg$_{\mathrm{s}}$ are of course precisely monoids. A $Cat(T)$-algebra is a small strict monoidal category: the isomorphism Cat(T)-Alg$_{\mathrm{s}}$ ≅ Cat(T-Alg$_{\mathrm{s}}$) now the assertion that a small strict monoidal category is equally a monoid in Cat =Cat(Set) or an internal category in the category of monoids T-Alg$_{\mathrm{s}}$. 1-cells and 2-cells of Cat(T)-Alg$_{\mathrm{s}}$ are furthermore strict monoidal functors and monoidal transformations between them. Cat(T)-Alg$_{\mathrm{l}}$ is the 2-category of small strict monoidal categories, lax monoidal functors and monoidal transformations.

A lax monoidal functor $1 \longrightarrow (A, \otimes, i)$ from the terminal strict monoidal category to another strict monoidal category $(A, \otimes, i)$ is precisely a monoid in $(A, \otimes, i)$. The simplicial category $(\Delta_+, \oplus, [-1])$ with its strict monoidal structure is the free strict monoidal category containing a monoid: thus interpreting these strict monoidal categories as $Cat(T)$-algebras we have Cat(T)-Alg$_{\mathrm{s}}((\Delta_+, \oplus, [-1]), (A, \otimes, i)) \cong$ Cat(T)-Alg$_{\mathrm{l}}(1, (A, \otimes, i))$ 2-naturally in $(A, \otimes, i)$. Therefore, by its universal property, it must be the case that $(\Delta_+, \oplus, [-1]) = 1'$ the lax morphism classifier for the terminal strict monoidal category. Since the monad T is cartesian and the terminal strict monoidal category discrete, Proposition 7.7 asserts that, interpreted as a category internal to the category of monoids T-Alg$_{\mathrm{s}}$, we have:

$$\begin{tikzcd}
(\Delta_+, \oplus, [-1]) \arrow[r, phantom, "="] & (T^31, \mu_{T^21}) \arrow[r, shift left=3, "\mu_{T1}"] \arrow[r, "T\mu_1" description] \arrow[r, shift right=3, "T^2!"'] & (T^21, \mu_{T1}) \arrow[r, shift left=3, "\mu_1"] \arrow[r, shift right=3, "T!"'] & (T1, \mu_1) \arrow[l, "T\eta_1" description]
\end{tikzcd}$$

where $! : T1 \longrightarrow 1$ is unique arrow defining the algebra structure for the terminal monoid 1. This was observed directly by Bénabou [6]. The free monoid on 1 $(T1, \mu_1) = (\mathbf{N}, +, 0)$ is the set of natural numbers with addition. The free monoid (T^21, μ_{T1}) has elements: finite sequences $[a_1a_2 \dots a_n]$ of natural numbers. The monoid morphisms $\mu_{T1}, T! : (T^21, \mu_{T1}) \rightrightarrows (T1, \mu_1)$ respectively add the elements of, and count the number of elements in, a sequence of natural numbers. Interpreting these morphisms now as the domain and codomain morphisms for the above internal category the element $[a_1a_2 \dots a_n]$ of the monoid (T^21, μ_{T1}) becomes an arrow:

$$\sum_{i=1\dots n} a_i \xrightarrow{[a_1a_2\dots a_n]} n$$

Identifying each natural number $n \in \mathbf{N}$ with the n-element ordinal, [n-1], the "morphism":

$$[(\sum_{i=1\dots n} a_i) - 1] \xrightarrow{[a_1a_2\dots a_n]} [n-1]$$

may be thought of as an ordered partition of the ordinal which constitutes its domain into ordered sets of $a_1, \dots a_n$ elements; now the morphism may be interpreted as sending the least a_1 elements to $0 \in [n-1]$,

the next least a_2 elements to $1 \in [n-1]$ and so on. Thus morphisms of Δ_+ correspond precisely to such sequences of natural numbers. One may carry on in this manner to see that the category Δ_+ and its strict monoidal structure is encoded precisely by this internal category in T-Alg_{s} =Mon.
Of course any monoid M may be interpreted as a discrete strict monoidal category $[M]$, a discrete $Cat(T)$-algebra. Thus there corresponds a strict monoidal category M': its explicit description as a strict monoidal category may be calculated from the formula of Proposition 7.7.

Chapter 8

Cat as a free completion and strongly finitary 2-functors

In Chapter 4 we characterised those 2-categories which are of the form $Cat(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks, up to 2-equivalence. Furthermore we characterised those 2-functors of the form $Cat(F) : Cat(A) \longrightarrow Cat(B)$ arising from pullback preserving functors $F : \mathcal{A} \longrightarrow \mathcal{B}$, up to 2-natural isomorphism. By Corollary 4.29, a 2-functor $H : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ is of this form if and only if:

- H preserves cotensors with $\mathbf{2}$, pullbacks and bijections on objects.

or equivalently by Corollary 4.21(2):

- H preserves cotensors with $\mathbf{2}$, pullbacks and codescent objects of cateads.

In this chapter we begin by studying a broader class of 2-functor, those 2-functors based on $Cat(\mathcal{E})$ which are the left Kan extension of their restriction along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$, and the relationship of such 2-functors with codescent objects. We restrict to the special case where is $\mathcal{E}$ is locally finitely presentable and characterise those 2-functors which are the left Kan extension of their restriction along the composite inclusion:

$$\mathcal{E}_f \longrightarrow \mathcal{E} \longrightarrow Cat(\mathcal{E})$$

the "strongly finitary 2-functors". We prove these to be precisely the 2-functors which preserve codescent objects of cateads and filtered colimits.

We use this to characterise Cat as the free completion of Set_f under codescent objects of reflexive coherence data and filtered colimits. We equally show it to be the free completion of Set_f under 2-dimensional sifted colimits, and study examples of sifted colimits.

8.1 Left Kan extensions along the discrete embedding

Consider the 2-functor $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ assigning to an object of $\mathcal{E}$ the canonical discrete internal category upon it. In the following proposition we characterise those 2-functors based on $Cat(\mathcal{E})$ which are the left Kan extension of their restriction along $[-]$. This is the foundational result of the chapter.

Theorem 8.1. Let $\mathcal{E}$ be a category with pullbacks.

1. Consider a 2-functor $T : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$ where $\mathcal{A}$ is an arbitrary 2-category and its restriction along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[dr, "T"] & \\
\mathcal{E} \arrow[u, "{[-]}"] \arrow[r, "T\circ[-]"'] & \mathcal{A}
\end{tikzcd}$$

   Suppose that T preserves codescent objects of pointwise discrete categories in $Cat(\mathcal{E})$. Then the identity 2-cell on $T \circ [-]$ exhibits T as the left Kan extension (in 2-CAT) of its restriction along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$.

2. Suppose that $\mathcal{A}$ has codescent objects of strict reflexive coherence data and consider $T : \mathcal{E} \longrightarrow \mathcal{A}$ any 2-functor. Its left Kan extension along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ in 2-CAT exists and preserves codescent objects of pointwise discrete categories in $Cat(\mathcal{E})$.

Proof. 1. The key to proving the first part of this theorem is the canonical presentation of an internal category as a codescent object of discrete internal categories described in Example 4.19. We recall that presentation now. To each internal category X in $\mathcal{E}$ there is an associated pointwise discrete category in $Cat(\mathcal{E})$:

$$\begin{tikzcd}
{[X_2]} \arrow[r, shift left=2, "{[p_x]}"] \arrow[r, "{[m_x]}" description] \arrow[r, shift right=2, "{[q_x]}"'] & {[X_1]} \arrow[r, shift left=2, "{[d_x]}"] \arrow[r, shift right=2, "{[c_x]}"'] & {[X_0]} \arrow[l, "{[i_x]}" description]
\end{tikzcd}$$

whose codescent object in $Cat(\mathcal{E})$ is X. Its exhibiting cocone is:

$$\begin{tikzcd}
& {[X_0]} \arrow[dr, "\epsilon_x"] \arrow[dd, phantom, "\Downarrow\theta_x"] & \\
{[X_1]} \arrow[ur, "{[d_x]}"] \arrow[dr, "{[c_x]}"'] & & X \\
& {[X_0]} \arrow[ur, "\epsilon_x"'] &
\end{tikzcd}$$

Now suppose that we are given a 2-functor $F : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$ and 2-natural transformation:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[dr, "F"] & \\
\mathcal{E} \arrow[u, "{[-]}"] \arrow[r, "T\circ[-]"'] \arrow[ur, phantom, "\eta\Uparrow"] & \mathcal{A}
\end{tikzcd}$$

We must show that there exists a unique 2-natural transformation $\hat{\eta} : T \Longrightarrow F$ such that for each $A \in \mathcal{E}$ we have $\hat{\eta}_{[A]} = \eta_A : T[A] \longrightarrow F[A]$. For $\hat{\eta}$ to satisfy these requirements the right hand square of the following diagram must commute:

$$\begin{tikzcd}
T[X_2] \arrow[r, shift left=2, "{T[p_x]}"] \arrow[r, "{T[m_x]}" description] \arrow[r, shift right=2, "{T[q_x]}"'] \arrow[d, "\eta_{X_2}"'] & T[X_1] \arrow[r, shift left=2, "{T[d_x]}"] \arrow[r, shift right=2, "{T[c_x]}"'] \arrow[d, "\eta_{X_1}"] & T[X_0] \arrow[l, "{T[i_x]}" description] \arrow[r, "T\epsilon_x"] \arrow[d, "\eta_{X_0}"] & TX \arrow[d, "\hat{\eta}_X"] \\
F[X_2] \arrow[r, shift left=2, "{F[p_x]}"] \arrow[r, "{F[m_x]}" description] \arrow[r, shift right=2, "{F[q_x]}"'] & F[X_1] \arrow[r, shift left=2, "{F[d_x]}"] \arrow[r, shift right=2, "{F[c_x]}"'] & F[X_0] \arrow[l, "{F[i_x]}" description] \arrow[r, "F\epsilon_x"'] & FX
\end{tikzcd}$$

(the remainder of the diagram is automatically commutative by naturality of η).
Furthermore if $\hat{\eta}$ is to be 2-natural we must additionally have:

$$\hat{\eta}_X \circ T\theta_x = F\theta_x \circ \eta_{X_1}$$

Now the triple $(FX, F\epsilon_x \circ \eta_{X_0}, F\theta_x \circ \eta_{X_1})$ constitutes a codescent cocone to the top row. By assumption T preserves codescent objects of pointwise discrete categories. Therefore the top row has codescent object TX with exhibiting cocone $(TX, T\epsilon_x, T\theta_x)$. By the universal property of TX there is a unique 1-cell $\hat{\eta}_X : TX \longrightarrow FX$ out of the codescent object such that $\hat{\eta}_X \circ T\epsilon_x = F\epsilon_x \circ \eta_{X_0}$ and $\hat{\eta}_X \circ T\theta_x = F\theta_x \circ \eta_{X_1}$. It remains therefore to show that $\hat{\eta} : T \Longrightarrow F$ is indeed 2-natural and that $\hat{\eta} \circ [-] = \eta$. 2-naturality is straightforward. It remains then to verify that $\hat{\eta}_{[A]} = \eta_A$ for each $A \in \mathcal{E}$. The internal category $[A]$ is presented canonically as the codescent object of the pointwise discrete category:

$$\begin{tikzcd}
{[A]} \arrow[r, shift left=2, "{[1]}"] \arrow[r, "{[1]}" description] \arrow[r, shift right=2, "{[1]}"'] & {[A]} \arrow[r, shift left=2, "{[1]}"] \arrow[r, shift right=2, "{[1]}"'] & {[A]} \arrow[l, "{[1]}" description]
\end{tikzcd}$$

with exhibiting 1 and 2-cells $(\epsilon_{[A]}, \theta_{[A]})$ both identities. Thus the equation $\hat{\eta}_{[A]} \circ T\epsilon_{[A]} = F\epsilon_{[A]} \circ \eta_A$ becomes $\hat{\eta}_{[A]} = \eta_A$ as required.

2. We are given a functor $T : \mathcal{E} \longrightarrow \mathcal{A}$ and must extend it along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$. We will describe its left Kan extension $\hat{T}$ and show that we may extend it precisely; so that $\hat{T} \circ [-] = T$. Each internal category:

$$\begin{tikzcd}
X = X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "i_x" description]
\end{tikzcd}$$

constitutes strict reflexive coherence data in $\mathcal{E}$. Therefore its image under T:

$$\begin{tikzcd}
TX_2 \arrow[r, shift left=2, "Tp_x"] \arrow[r, "Tm_x" description] \arrow[r, shift right=2, "Tq_x"'] & TX_1 \arrow[r, shift left=2, "Td_x"] \arrow[r, shift right=2, "Tc_x"'] & TX_0 \arrow[l, "Ti_x" description]
\end{tikzcd}$$

constitutes strict reflexive coherence data in $\mathcal{A}$. We define $\hat{T}X$ to be the codescent object of this coherence data in $\mathcal{A}$, which exists by assumption, and denote the exhibiting cocone by $(\hat{T}X, \alpha_x, \phi_x)$. Each internal functor $f : X \longrightarrow Y$ induces a morphism of reflexive coherence data in $\mathcal{A}$:

$$\begin{tikzcd}
TX_2 \arrow[r, shift left=2, "Tp_x"] \arrow[r, "Tm_x" description] \arrow[r, shift right=2, "Tq_x"'] \arrow[d, "Tf_2"'] & TX_1 \arrow[r, shift left=2, "Td_x"] \arrow[r, shift right=2, "Tc_x"'] \arrow[d, "Tf_1"] & TX_0 \arrow[l, "Ti_x" description] \arrow[d, "Tf_0"] \\
TY_2 \arrow[r, shift left=2, "Tp_y"] \arrow[r, "Tm_y" description] \arrow[r, shift right=2, "Tq_y"'] & TY_1 \arrow[r, shift left=2, "Td_y"] \arrow[r, shift right=2, "Tc_y"'] & TY_0 \arrow[l, "Ti_y" description]
\end{tikzcd}$$

and thus a unique morphism between the codescent objects $\hat{T}f : \hat{T}X \longrightarrow \hat{T}Y$ such that $\hat{T}f \circ \alpha_x = \alpha_y \circ Tf_0$ and $\hat{T}f \circ \phi_x = \phi_y \circ Tf_1$.
Letting $Q : [\Delta_2^{op}, \mathcal{A}] \longrightarrow \mathcal{A}$ denote the codescent 2-functor we see that we have defined $\hat{T}$ on objects and 1-cells as the composite:

$$\begin{tikzcd}
\mathcal{U}Cat(\mathcal{E}) \arrow[r, "\iota"] & {[\Delta_2^{op}, \mathcal{E}]} \arrow[r, "{[\Delta_2^{op}, \mathcal{U}T]}"] & {[\Delta_2^{op}, \mathcal{U}\mathcal{A}]} \arrow[r, "\mathcal{U}Q"] & \mathcal{U}\mathcal{A}
\end{tikzcd}$$

and this is functorial.
With regards to the construction of $\hat{T}$ it remains to define it on 2-cells. Each internal natural transformation $\eta : f \Longrightarrow g$ gives rise to a diagram in $\mathcal{A}$:

$$\begin{tikzcd}
TX_2 \arrow[r, shift left=2, "Tp_x"] \arrow[r, "Tm_x" description] \arrow[r, shift right=2, "Tq_x"'] \arrow[d, shift right, "Tf_2"'] \arrow[d, shift left, "Tg_2"] & TX_1 \arrow[r, shift left=2, "Td_x"] \arrow[r, shift right=2, "Tc_x"'] \arrow[d, shift right, "Tf_1"'] \arrow[d, shift left, "Tg_1"] & TX_0 \arrow[l, "Ti_x" description] \arrow[d, shift right, "Tf_0"'] \arrow[d, shift left, "Tg_0"] \arrow[ld, "T\overline{\eta}"'] \\
TY_2 \arrow[r, shift left=2, "Tp_y"] \arrow[r, "Tm_y" description] \arrow[r, shift right=2, "Tq_y"'] & TY_1 \arrow[r, shift left=2, "Td_y"] \arrow[r, shift right=2, "Tc_y"'] & TY_0 \arrow[l, "Ti_y" description]
\end{tikzcd}$$

We claim that there exists a unique 2-cell $\hat{T}\eta : \hat{T}f \Longrightarrow \hat{T}g$ out of the codescent object $\hat{T}X$ such that:

$$\begin{tikzcd}
& TY_0 \arrow[rrd, bend left, "\alpha_y"] & & \\
TX_0 \arrow[ru, "Tf_0"] \arrow[r, "\alpha_x"] \arrow[rd, "Tg_0"'] & \hat{T}X \arrow[rr, bend left, "\hat{T}f"] \arrow[rr, bend right, "\hat{T}g"'] & \Downarrow \hat{T}\eta & \hat{T}Y \\
& TY_0 \arrow[rru, bend right, "\alpha_y"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & TY_0 \arrow[rd, "\alpha_y"] & \\
TX_0 \arrow[rru, bend left, "Tf_0"] \arrow[r, "T\overline{\eta}"] \arrow[rrd, bend right, "Tg_0"'] & TY_1 \arrow[ru, "Td_y"] \arrow[rd, "Tc_y"'] & \Downarrow \phi_y & \hat{T}Y \\
& & TY_0 \arrow[ru, "\alpha_y"'] &
\end{tikzcd}$$

By the 2-dimensional universal property of the codescent object $\hat{T}X$ to give a 2-cell $\hat{T}f \Longrightarrow \hat{T}g$ is equally to give a 2-cell $\rho : \hat{T}f \circ \alpha_x = \alpha_y \circ Tf_0 \Longrightarrow \alpha_y \circ Tg_0 = \hat{T}g \circ \alpha_x$ such that the square on the left below commutes:

$$\begin{tikzcd}
\hat{T}f \circ \alpha_x \circ Td_x \arrow[r, Rightarrow, "\hat{T}f \circ \phi_x"] \arrow[d, Rightarrow, "\rho \circ Td_x"'] & \hat{T}f \circ \alpha_x \circ Tc_x \arrow[d, Rightarrow, "\rho \circ Tc_x"] \\
\hat{T}g \circ \alpha_x \circ Td_x \arrow[r, Rightarrow, "\hat{T}g \circ \phi_x"'] & \hat{T}g \circ \alpha_x \circ Tc_x
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha_y \circ Tf_0 \circ Td_x \arrow[r, Rightarrow, "\phi_y \circ Tf_1"] \arrow[d, Rightarrow, "\rho \circ Td_x"'] & \alpha_y \circ Tf_0 \circ Tc_x \arrow[d, Rightarrow, "\rho \circ Tc_x"] \\
\alpha_y \circ Tg_0 \circ Td_x \arrow[r, Rightarrow, "\phi_y \circ Tg_1"'] & \alpha_y \circ Tg_0 \circ Tc_x
\end{tikzcd}$$

commutes. By definition of $\hat{T}$ on 1-cells the square on the left equals the square on the right above. Let ρ be the 2-cell:

$$\begin{tikzcd}
& & TY_0 \arrow[rd, "\alpha_y"] & \\
TX_0 \arrow[rru, bend left, "Tf_0"] \arrow[r, "T\overline{\eta}"'] \arrow[rrd, bend right, "Tg_0"'] & TY_1 \arrow[ru, "Td_y"] \arrow[rd, "Tc_y"'] & \Downarrow \phi_y & \hat{T}Y \\
& & TY_0 \arrow[ru, "\alpha_y"'] &
\end{tikzcd}$$

Substituting this 2-cell into the above square the two paths around the square become the two composite 2-cells below:

(1)

$$\begin{tikzcd}
& & TY_1 \arrow[rr, "Td_y"] \arrow[rdd, "Tc_y"'] & & TY_0 \arrow[rdd, "\alpha_y"] & \\
& & & & \Downarrow \phi_y & \\
TX_1 \arrow[rruu, "Tf_1"] \arrow[rdd, "Tc_x"'] & & & TY_0 \arrow[rr, "\alpha_y"] & & \hat{T}Y \\
& & & & \Downarrow \phi_y & \\
& TX_0 \arrow[r, "T\overline{\eta}"'] & TY_1 \arrow[ruu, "Td_y"] \arrow[rr, "Tc_y"'] & & TY_0 \arrow[ruu, "\alpha_y"'] &
\end{tikzcd}$$

and

(2)

$$\begin{tikzcd}
& TX_0 \arrow[r, "T\overline{\eta}"] & TY_1 \arrow[rr, "Td_y"] \arrow[rdd, "Tc_y"'] & & TY_0 \arrow[rdd, "\alpha_y"] & \\
& & & & \Downarrow \phi_y & \\
TX_1 \arrow[ruu, "Td_x"] \arrow[rrdd, "Tg_1"'] & & & TY_0 \arrow[rr, "\alpha_y"] & & \hat{T}Y \\
& & & & \Downarrow \phi_y & \\
& & TY_1 \arrow[ruu, "Td_y"] \arrow[rr, "Tc_y"'] & & TY_0 \arrow[ruu, "\alpha_y"'] &
\end{tikzcd}$$

Consequently we must prove the equality of these composite 2-cells. Consider the map $(f_1, \overline{\eta} \circ c_x) : X_1 \longrightarrow Y_2$ into the pullback Y_2, which postcomposed with p_y and q_y yields f_1 and $\overline{\eta} \circ c_x$ respectively. By functoriality of T (1) equals:

$$\begin{tikzcd}
& & TY_1 \arrow[rr, "Td_y"] \arrow[dr, "Tc_y"'] & {} \arrow[d, phantom, "\Downarrow\phi_y"] & TY_0 \arrow[dr, "\alpha_y"] & \\
TX_1 \arrow[r, "T(f_1{,}\overline{\eta}\circ c_x)"] & TY_2 \arrow[ur, "Tp_y"] \arrow[dr, "Tq_y"'] & & TY_0 \arrow[rr, "\alpha_y"] \arrow[d, phantom, "\Downarrow\phi_y"] & & \hat{T}Y \\
& & TY_1 \arrow[rr, "Tc_y"'] \arrow[ur, "Td_y"] & {} & TY_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}
=
\begin{tikzcd}
& & & TY_0 \arrow[dr, "\alpha_y"] & \\
TX_1 \arrow[r, "T(f_1{,}\overline{\eta}\circ c_x)"] & TY_2 \arrow[r, "Tm_x"] & TY_1 \arrow[ur, "Td_y"] \arrow[dr, "Tc_y"'] & \Downarrow\phi_y & \hat{T}Y \\
& & & TY_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}$$

where the second equation holds as $(\hat{T}Y, \alpha_y, \phi_y)$ is a codescent cocone. Similarly the right hand side (2) equals:

$$\begin{tikzcd}
& & TY_1 \arrow[rr, "Td_y"] \arrow[dr, "Tc_y"'] & {} \arrow[d, phantom, "\Downarrow\phi_y"] & TY_0 \arrow[dr, "\alpha_y"] & \\
TX_1 \arrow[r, "T(\overline{\eta}\circ d_x{,}g_1)"] & TY_2 \arrow[ur, "Tp_y"] \arrow[dr, "Tq_y"'] & & TY_0 \arrow[rr, "\alpha_y"] \arrow[d, phantom, "\Downarrow\phi_y"] & & \hat{T}Y \\
& & TY_1 \arrow[rr, "Tc_y"'] \arrow[ur, "Td_y"] & {} & TY_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}
=
\begin{tikzcd}
& & & TY_0 \arrow[dr, "\alpha_y"] & \\
TX_1 \arrow[r, "T(\overline{\eta}\circ d_x{,}g_1)"] & TY_2 \arrow[r, "Tm_y"] & TY_1 \arrow[ur, "Td_y"] \arrow[dr, "Tc_y"'] & \Downarrow\phi_y & \hat{T}Y \\
& & & TY_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}$$

That (1) and (2) agree now follows from the commutativity of the naturality square:

$$\begin{tikzcd}
X_1 \arrow[r, "(\overline{\eta}\circ d_x{,}g_1)"] \arrow[d, "(f_1{,}\overline{\eta}\circ c_x)"'] & Y_2 \arrow[d, "m_y"] \\
Y_2 \arrow[r, "m_y"'] & X_1
\end{tikzcd}$$

of the internal natural transformation η and functoriality of T. Therefore by the 2-dimensional universal property of the codescent object $\hat{T}X$ there exists a unique 2-cell:

$$\begin{tikzcd}
\hat{T}X \arrow[r, bend left, "\hat{T}f"] \arrow[r, bend right, "\hat{T}g"'] \arrow[r, phantom, "\Downarrow\hat{T}\eta"] & \hat{T}Y
\end{tikzcd}$$

such that precomposition with the codescent morphism $\alpha_x : TX_0 \longrightarrow \hat{T}X$ gives:

$$\begin{tikzcd}
TX_0 \arrow[r, "\alpha_x"] & \hat{T}X \arrow[r, bend left, "\hat{T}\overline{f}"] \arrow[r, bend right, "\hat{T}\overline{g}"'] \arrow[r, phantom, "\Downarrow\hat{T}\eta"] & \hat{T}Y
\end{tikzcd}
=
\begin{tikzcd}
& & TY_0 \arrow[dr, "\alpha_y"] & \\
TX_0 \arrow[urr, bend left, "Tf_0"] \arrow[drr, bend right, "Tg_0"'] \arrow[r, "T\overline{\eta}"'] & TY_1 \arrow[ur, "Td_y"] \arrow[dr, "Tc_y"'] & \Downarrow\phi_y & \hat{T}Y \\
& & TY_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}$$

We need to prove that $\hat{T}$ so defined is a 2-functor. We have seen that it has an underlying functor, so that it suffices to prove that it preserves vertical and horizontal composition of 2-cells. We consider vertical composition first.

Given a vertically composable pair of 2-cells in $Cat(\mathcal{E})$:

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\rho"] \arrow[r, phantom, shift right=3, "\Downarrow\lambda"] & Y
\end{tikzcd}$$

we must show that $\hat{T}(\lambda \circ \rho) = \hat{T}\lambda \circ \hat{T}\rho$. The composite internal natural transformation $\lambda \circ \rho$ has arrow component:

$$\begin{tikzcd}
X_0 \arrow[r, "(\overline{\rho},\overline{\lambda})"] & Y_2 \arrow[r, "m_y"] & Y_1
\end{tikzcd}$$

Therefore, by the definition of $\hat{T}$ on 2-cells, $\hat{T}(\lambda \circ \rho)$ is the unique 2-cell which precomposed with α_x equals the left hand composite 2-cell below:

$$\begin{tikzcd}
 & & & TY_0 \arrow[dr, "\alpha_y"] & & & & & TY_1 \arrow[rr, "Td_y"] \arrow[dr, "Tc_y"'] & & TY_0 \arrow[dr, "\alpha_y"] \arrow[dl, phantom, "\Downarrow\phi_y"] & \\
TX_0 \arrow[r, "T(\overline{\rho},\overline{\lambda})"] & TY_2 \arrow[r, "Tm_y"] & TY_1 \arrow[ur, "Td_y"] \arrow[dr, "Tc_y"'] & \Downarrow\phi_y & \hat{T}Y & = & TX_0 \arrow[r, "T(\overline{\rho},\overline{\lambda})"] & TY_2 \arrow[ur, "Tp_y"] \arrow[dr, "Tq_y"'] & & TY_0 \arrow[rr, "\alpha_y"] \arrow[dr, phantom, "\Downarrow\phi_y"] & & \hat{T}Y \\
 & & & TY_0 \arrow[ur, "\alpha_y"'] & & & & & TY_1 \arrow[ur, "Td_y"] \arrow[rr, "Tc_y"'] & & TY_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & TY_1 \arrow[rr, "Td_y"] \arrow[dr, "Tc_y"'] & & TY_0 \arrow[dr, "\alpha_y"] \arrow[dl, phantom, "\Downarrow\phi_y"] & & & & & \\
= & TX_0 \arrow[ur, "T\overline{\rho}"] \arrow[dr, "T\overline{\lambda}"'] & & TY_0 \arrow[rr, "\alpha_y"] \arrow[dr, phantom, "\Downarrow\phi_y"] & & \hat{T}Y & = & TX_0 \arrow[r, "\alpha_x"] & \hat{T}X \arrow[r, bend left=60, "\hat{T}f"] \arrow[r, "\hat{T}g" description] \arrow[r, bend right=60, "\hat{T}h"'] \arrow[r, phantom, shift left=3, "\Downarrow\hat{T}\rho"] \arrow[r, phantom, shift right=3, "\Downarrow\hat{T}\lambda"] & \hat{T}Y \\
 & & TY_1 \arrow[ur, "Td_y"] \arrow[rr, "Tc_y"'] & & TY_0 \arrow[ur, "\alpha_y"'] & & & & &
\end{tikzcd}$$

The first equation holds as the triple $(\hat{T}Y, \alpha_y, \phi_y)$ is a codescent cocone. The second equation holds as the functor T preserves the equations $p_y \circ (\overline{\rho}, \overline{\lambda}) = \overline{\rho}$ and $q_y \circ (\overline{\rho}, \overline{\lambda}) = \overline{\lambda}$. The final equation holds by the definition of the 2-cells $\hat{T}\rho$ and $\hat{T}\lambda$. Therefore $\hat{T}$ preserves vertical composition of 2-cells.
We need to show that $\hat{T}$ preserves horizontal composition of 2-cells. Horizontal composition may be defined in terms of vertical composition and left and right whiskering. Having shown that $\hat{T}$ preserves vertical composition of 2-cells it therefore suffices to show that it preserves left and right whiskering. We will prove that $\hat{T}$ preserves left whiskering firstly.
Consider a 2-cell $\eta \in Cat(\mathcal{E})$ with a 1-cell to its left:

$$\begin{tikzcd}
W \arrow[r, "r"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & Y
\end{tikzcd}$$

We must prove that $\hat{T}\eta \circ \hat{T}r = \hat{T}(\eta \circ r)$. The internal natural transformation $\eta \circ r$ has arrow component the composite:

$$\begin{tikzcd}
W_0 \arrow[r, "r_0"] & X_0 \arrow[r, "\overline{\eta}"] & Y_1
\end{tikzcd}$$

By definition $\hat{T}(\eta \circ r)$ is the unique 2-cell which precomposes with $\alpha_w : TW_0 \longrightarrow \hat{T}W$ to give the composite: $\phi_y \circ T(\overline{\eta}) \circ T(r_0)$. It suffices therefore to verify the equality:

$$\hat{T}(\eta) \circ \hat{T}(r) \circ \alpha_w = \phi_y \circ T(\overline{\eta}) \circ T(r_0)$$

We have:

$$\hat{T}(\eta) \circ \hat{T}(r) \circ \alpha_w = \hat{T}(\eta) \circ \alpha_x \circ Tr_0 = \phi_y \circ T(\overline{\eta}) \circ T(r_0)$$

the first equation holding by the definition of $\hat{T}$ on 1-cells and the second equation by its definition on 2-cells.
With regards to right whiskering, consider again the 2-cell $\eta : f \Longrightarrow g$ as above; now with a 1-cell $s : Y \longrightarrow Z$ to its right. We must show that $\hat{T}s \circ \hat{T}\eta = \hat{T}(s \circ \eta)$. The internal natural transformation $s \circ \eta$ has arrow component the composite:

$$\begin{tikzcd} X_0 \arrow[r, "\overline{\eta}"] & Y_1 \arrow[r, "s_1"] & Z_1 \end{tikzcd}$$

Therefore $\hat{T}(s \circ \eta)$ is the unique 2-cell which precomposes with $\alpha_x : TX_0 \longrightarrow \hat{T}X$ to give $\phi_z \circ Ts_1 \circ T\overline{\eta}$. Consequently it suffices to show that:

$$\hat{T}s \circ \hat{T}\eta \circ \alpha_x = \phi_z \circ Ts_1 \circ T\overline{\eta}$$

We have:

$$\hat{T}s \circ \hat{T}\eta \circ \alpha_x = \hat{T}s \circ \phi_y \circ T\overline{\eta} = \phi_z \circ Ts_1 \circ T\overline{\eta}$$

first using the definition of $\hat{T}$ on 2-cells and then on 1-cells. Therefore $\hat{T}$ preserves whiskering on both sides and so is 2-functorial. In order to prove that the 2-functor $\hat{T}$ is the left Kan extension of T along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ it suffices, by Part 1 of the present theorem, to prove that $\hat{T}$ preserves codescent objects of pointwise discrete categories and restricts along $[-]$ to precisely T.
We show firstly that we may choose $\hat{T}$ so that it restricts to T. For $A \in \mathcal{E}$ the internal category $[A]$ is the diagram in $\mathcal{E}$:

$$\begin{tikzcd} A \arrow[r, shift left=2, "1"] \arrow[r, "1" description] \arrow[r, shift right=2, "1"'] & A \arrow[r, shift left=2, "1"] & A \arrow[l, "1" description] \arrow[l, shift left=2, leftarrow, "1"'] \end{tikzcd}$$

with image under T the diagram in $\mathcal{A}$:

$$\begin{tikzcd} TA \arrow[r, shift left=2, "1"] \arrow[r, "1" description] \arrow[r, shift right=2, "1"'] & TA \arrow[r, shift left=2, "1"] & TA \arrow[l, "1" description] \arrow[l, shift left=2, leftarrow, "1"'] \end{tikzcd}$$

The identity cocone $(TA, 1, 1)$ exhibits TA as the codescent object in $\mathcal{A}$ and so we may set $(\hat{T}[A], \alpha_{[A]}, \phi_{[A]}) = (TA, 1, 1)$. This choice sets $\hat{T}[-] = T$ on objects, and it easily follows from the definition of $\hat{T}$, combined with the above choices of codescent cocones, that $\hat{T} \circ [-] = T$ on 1 and 2-cells too.
It remains then to show that $\hat{T}$ preserves codescent objects of pointwise discrete categories in $Cat(\mathcal{E})$. Pointwise discrete categories in $Cat(\mathcal{E})$ are the objects of $Cat(Disc(Cat(\mathcal{E})))$. As described in Example 4.19 each of these is isomorphic to one of the canonical pointwise discrete categories in $Cat(\mathcal{E})$; those of the form:

$$\begin{tikzcd} {[X_2]} \arrow[r, shift left=2, "{[p_x]}"] \arrow[r, "{[m_x]}" description] \arrow[r, shift right=2, "{[q_x]}"'] & {[X_1]} \arrow[r, shift left=2, "{[d_x]}"] \arrow[r, shift right=2, "{[c_x]}"'] & {[X_0]} \arrow[l, "{[i_x]}" description] \end{tikzcd}$$

for some internal category X. Consequently it suffices to show that $\hat{T}$ preserves codescent objects of these. As described in Part 1 of the present theorem the codescent object of the above diagram in $Cat(\mathcal{E})$ is the internal category X itself, with exhibiting cocone $(X, \epsilon_x, \theta_x)$.
Since $\hat{T} \circ [-] = T$ the image of the above pointwise discrete category under $\hat{T}$ is simply:

$$\begin{tikzcd} TX_2 \arrow[r, shift left=2, "Tp_x"] \arrow[r, "Tm_x" description] \arrow[r, shift right=2, "Tq_x"'] & TX_1 \arrow[r, shift left=2, "Td_x"] \arrow[r, shift right=2, "Tc_x"'] & TX_0 \arrow[l, "Ti_x" description] \end{tikzcd}$$

whose codescent object is, by definition, $\hat{T}X$. Thus $\hat{T}$ does preserve the above codescent "object". We will show that $\hat{T}\epsilon_x = \alpha_x$ and $\hat{T}\theta_x = \phi_x$ so that $\hat{T}$ takes the universal cocone in $Cat(\mathcal{E})$ to the corresponding universal cocone in $\mathcal{A}$.
Consider $\hat{T}\epsilon_x : \hat{T}[X_0] = TX_0 \longrightarrow \hat{T}X$. By the definition of $\hat{T}$ on 1-cells we have:

$$\hat{T}\epsilon_x \circ \alpha_{[X_0]} = \alpha_x \circ T((\epsilon_x)_0)$$

Now by definition $\alpha_{[X_0]}$ is the identity 1-cell on TX_0. As described in Example 4.19 $(\epsilon_x)_0$ is also an identity. Thus the equation reduces to $\hat{T}\epsilon_x = \alpha_x$.
The 2-cell:

$$\begin{tikzcd}
\hat{T}[X_1] \arrow[r, bend left, "\hat{T}(\epsilon_x \circ [d_x])"] \arrow[r, bend right, "\hat{T}(\epsilon_x \circ [c_x])"'] \arrow[r, phantom, "\Downarrow \hat{T}(\theta_x)"] & \hat{T}X
\end{tikzcd}$$

is, by definition, the unique one such that $\hat{T}(\theta_x) \circ \alpha_{[X_0]} = \phi_x \circ T\overline{\theta_x}$. Now $\alpha_{[X_0]}$ is an identity 1-cell. As described in Example 4.19 $\overline{\theta_x} : [X_1]_0 = X_1 \longrightarrow X_1$ is the identity 1-cell on X_1. Thus the equation reduces to $\hat{T}\theta_x = \phi_x$ as required.

□

Example 8.2. Consider a functor $F : \mathcal{A} \longrightarrow \mathcal{B} \in \text{Cat}_{\text{pb}}$. Then $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves codescent objects of cateads by Theorem 3.66, in particular codescent objects of pointwise discrete categories. By Theorem 8.1(1) each such 2-functor is consequently the left Kan extension of its restriction along $[-] : \mathcal{A} \longrightarrow Cat(\mathcal{A})$.

8.2 Pointwise Kan extensions

In this section we recall the notion of pointwise Kan extension [42], observe that the Kan extensions of the preceding section are pointwise, and study those properties of pointwise Kan extensions which will be of importance in the remainder of the chapter. We begin by discussing some generalities concerning Kan extensions, which are well known for the 1-dimensional case, and carry over immediately to the 2-dimensional case which concerns us.
Consider a left Kan extension in 2-CAT:

$$\begin{tikzcd}
\mathcal{B} \arrow[dr, "L"] & \\
\mathcal{A} \arrow[u, "J"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{C}
\end{tikzcd}$$

If the 2-category $\mathcal{A}$ is small and $\mathcal{C}$ cocomplete then the left Kan extension of any such 2-functor $F : \mathcal{A} \longrightarrow \mathcal{C}$ exists for any J. Furthermore the left Kan extension admits a simple pointwise description:

$$LX = \int^{Y \in \mathcal{A}} \mathcal{B}(J(Y), X).FY$$

in terms of colimits: coends and copowers. Globally the left Kan extension is the composite 2-functor:

$$L = \int^{Y \in \mathcal{A}} \mathcal{B}(J(Y), -).FY$$

Colimit preservation properties of such left Kan extensions are easily seen to follow from this explicit formula, for instance Proposition 8.6 below. If $\mathcal{C}$ is insufficiently cocomplete then this description of the Kan extension in terms of colimits cannot be applied; however we will see in the same proposition that if the Kan extension in question is "pointwise" its colimit preservation properties may be understood just as well as if $\mathcal{C}$ were cocomplete.

Definition 8.3. A left Kan extension $L : \mathcal{B} \longrightarrow \mathcal{C}$ is said to be pointwise if it is preserved by the representable $\mathcal{B}(-, X) : \mathcal{B} \longrightarrow \mathrm{Cat}^{op}$ for each $X \in \mathcal{B}$.

Remark 8.4. It is immediate that if $\mathcal{A}$ is small and $\mathcal{C}$ cocomplete, so that the above colimit formula applies, then each left Kan extension is pointwise. For the representables $\mathcal{B}(-, X) : \mathcal{B} \longrightarrow \mathrm{Cat}^{op}$ preserve all colimits and therefore those colimits defining such left Kan extensions.

Example 8.5. Consider a 2-functor $T : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$ which preserves codescent objects of pointwise discrete categories in $Cat(\mathcal{E})$. By Theorem 8.1(1) T is the left Kan extension of its restriction along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$. Each representable $\mathcal{A}(-, X) : \mathcal{A} \longrightarrow \mathrm{Cat}^{op}$ preserves all colimits; thus the composite $\mathcal{A}(-, X) \circ T : Cat(\mathcal{E}) \longrightarrow \mathrm{Cat}^{op}$ also preserves codescent objects of pointwise discrete categories. By Theorem 8.1(1) again this composite is the left Kan extension of its restriction, $\mathcal{A}(-, X) \circ T \circ [-]$, along $[-]$. Therefore each representable $\mathcal{A}(-, X) : \mathcal{A} \longrightarrow \mathrm{Cat}^{op}$ preserves the Kan extension and it is pointwise.

Proposition 8.6. Consider a pointwise left Kan extension in 2-CAT:

$$\begin{tikzcd}
\mathcal{B} \arrow[dr, "L"] & \\
\mathcal{A} \arrow[u, "J"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{C}
\end{tikzcd}$$

with $\mathcal{A}$ small. Then L preserves any colimit existing in $\mathcal{B}$ that is preserved by the representable $\mathcal{B}(J(X), -) : \mathcal{B} \longrightarrow \mathrm{Cat}$ for each $X \in \mathcal{A}$.

Proof. We begin by supposing that $\mathcal{C}$ is cocomplete and then deduce the general case.

1. If $\mathcal{C}$ is cocomplete then we have:

   $$L = \int^{Y \in \mathcal{A}} \mathcal{B}(J(Y), -).FY$$

   Each 2-functor $\mathcal{B}(J(Y), -).FY : \mathcal{B} \longrightarrow \mathcal{C}$ is the composite:

   $$\mathcal{B} \xrightarrow{\mathcal{B}(J(Y),-)} \mathrm{Cat} \xrightarrow{-.FY} \mathcal{C}$$

   The 2-functor $-.FY : \mathrm{Cat} \longrightarrow \mathcal{C}$ preserves all colimits as it is the left 2-adjoint of $\mathcal{C}(FY, -) : \mathcal{C} \longrightarrow \mathrm{Cat}$. Therefore each 2-functor $\mathcal{B}(J(Y), -).FY$ preserves any colimit preserved by $\mathcal{B}(J(Y), -) : \mathcal{B} \longrightarrow \mathrm{Cat}$ for each $Y \in \mathcal{A}$. As L is a colimit of the 2-functors $\mathcal{B}(J(Y), -).FY$ it preserves any colimit preserved by each of these.

2. If $\mathcal{C}$ is not necessarily cocomplete but L is pointwise then each representable $\mathcal{C}(-, X) : \mathcal{C} \longrightarrow \mathrm{Cat}^{op}$ preserves the Kan extension so that we have a left Kan extension along J:

   $$\begin{tikzcd}
   \mathcal{B} \arrow[dr, "L"] & & \\
   \mathcal{A} \arrow[u, "J"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{C} \arrow[r, "\mathcal{C}(-{,}X)"] & \mathrm{Cat}^{op}
   \end{tikzcd}$$

   for each $X \in \mathcal{C}$. Cat is complete and so Cat^{op} cocomplete. Therefore we may apply the first part of the proof to deduce that for each $X \in \mathcal{C}$ the composite $\mathcal{C}(-, X) \circ L$ preserves any colimit preserved by each representable $\mathcal{B}(J(Y), -) : \mathcal{B} \longrightarrow \mathrm{Cat}$. Of course L itself preserves any colimit preserved by $\mathcal{C}(-, X) \circ L$ for each representable $\mathcal{C}(-, X)$ as colimits are defined representably.

□

8.3 Locally finitely presentable categories and strongly finitary 2-functors

In this section we concentrate on 2-categories of the form $Cat(\mathcal{E})$ where $\mathcal{E}$ is a locally finitely presentable category. We recall that concept now.

Definition 8.7. An object X of a category $\mathcal{E}$ is finitely presentable if $\mathcal{E}(X,-) : \mathcal{E} \longrightarrow \text{Set}$ preserves filtered colimits.

Notation 8.8. $\mathcal{E}_f$ denotes a skeletal full subcategory of $\mathcal{E}$ containing, up to isomorphism, precisely the finitely presentable objects of $\mathcal{E}$ and $\iota : \mathcal{E}_f \longrightarrow \mathcal{E}$ the evident inclusion.

Definition 8.9. A category $\mathcal{E}$ is said to be locally finitely presentable if $\mathcal{E}_f$ is small and finitely cocomplete, and the inclusion $\iota : \mathcal{E}_f \longrightarrow \mathcal{E}$ exhibits $\mathcal{E}$ as the free completion of $\mathcal{E}_f$ under filtered colimits.

Remark 8.10. The notion of locally finitely presentable category was introduced in [20] and describes those categories of models for a (many sorted) finite limit theory. Recall that the category of algebras for a finitary monad T on the category of sets may be identified with $FPP(\mathbf{L}_T, \text{Set})$ the category of finite product preserving functors from the opposite of the Lawvere theory corresponding to T. The Lawvere theory T has a single generating object and finite products. Thus finitary monads on Set describe "single sorted finite product theories". Finite limit theories are capable of expressing many sorted structures with arities expressible using finite limits. An example is the category of $\mathcal{U}\text{Cat}$ of small categories. The notion of small category is typically presented with two sorts[1] and requires pullbacks to express the partial operation of composition. We have $\mathcal{U}\text{Cat} \simeq Lex(\mathbf{T}, \text{Set})$ the category of finite limit preserving functors from the theory of categories $\mathbf{T}$, a small category with finite limits. The following proposition collects those basic facts about locally finitely presentable categories that we will require.

Proposition 8.11.

1. Let $\mathcal{E}$ be a locally finitely presentable category. Its corresponding theory is the category $\mathcal{E}_f$. The functor $\mathcal{E}(\iota-, 1) : \mathcal{E} \longrightarrow [(\mathcal{E}_f)^{op}, \text{Set}]$ induced by the inclusion $\iota : \mathcal{E}_f \longrightarrow \mathcal{E}$ is fully faithful and preserves all limits and filtered colimits. Furthermore it has a left adjoint, thus exhibiting $\mathcal{E}$ as a reflective subcategory of $[(\mathcal{E}_f)^{op}, \text{Set}]$.
2. Any locally finitely presentable category is both small complete and cocomplete.
3. If $\mathcal{E}$ is locally finitely presentable and $\mathcal{A}$ a small category then the functor category $[\mathcal{A}, \mathcal{E}]$ is also locally finitely presentable.
4. If $\mathcal{E}$ is locally finitely presentable then finite limits commute with filtered colimits in $\mathcal{E}$.
5. Any functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between locally finitely presentable categories which preserves all limits and filtered colimits has a left adjoint.
6. If $\mathcal{E}$ is locally finitely presentable and $\mathbf{T}$ is a small finitely complete category then $Lex(\mathbf{T}, \mathcal{E})$ is locally finitely presentable. In particular if $\mathcal{E}$ is locally finitely presentable then so is $\mathcal{U}Cat(\mathcal{E})$.

Proof. The proofs of these claims may be found in [1]. □

Remark 8.12. The main aim of the following proposition is to prove that if $\mathcal{E}$ is locally finitely presentable then the 2-category $Cat(\mathcal{E})$ is both small complete and cocomplete.

Proposition 8.13.

1. Let $\mathcal{E}$ be a locally finitely presentable category. Then $Cat(\mathcal{E})$ is small complete as a 2-category.

---

[1] It is however possible to give a single sorted presentation of the notion of small category by identifying objects with the identity arrows upon them.

2. If a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between locally finitely presentable categories preserves limits and filtered colimits then the induced 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves limits and filtered colimits.

3. If $\mathcal{E}$ is locally finitely presentable then the 2-category $Cat(\mathcal{E})$ is small cocomplete as a 2-category.

Proof. 1. It is straightforward to see that $Cat(\mathcal{E})$ is complete as a 2-category if $\mathcal{E}$ is complete, which is the case as any locally finitely presentable category is complete. For a small category $\mathcal{J}$ if $\mathcal{E}$ has $\mathcal{J}$-limits then so does $Cat(\mathcal{E})$, pointwise, and furthermore it may be verified directly that these are conical $\mathcal{J}$-limits in the 2-dimensional sense. As $\mathcal{E}$ has pullbacks $Cat(\mathcal{E})$ has cotensors with **2** by Proposition 3.19(1). Consequently $Cat(\mathcal{E})$ has conical $\mathcal{J}$-limits for all small categories $\mathcal{J}$ and cotensors with **2**; this is sufficient to show that it admits all small weighted limits.

2. As F preserves conical $\mathcal{J}$-limits for any small category $\mathcal{J}$ it follows that $Cat(F)$ also preserves conical $\mathcal{J}$-limits, as those limits are pointwise in both $Cat(\mathcal{A})$ and $Cat(\mathcal{B})$. Since F preserves pullbacks $Cat(F)$ preserves cotensors with **2** by Proposition 3.19(3). Therefore $Cat(F)$ preserves all limits.
As each of $Cat(\mathcal{A})$ and $Cat(\mathcal{B})$ admits cotensors with **2** it will suffice, using Proposition 2.5, to understand filtered colimits in the respective underlying categories and to show that $\mathcal{U}Cat(F) : \mathcal{U}Cat(\mathcal{A}) \longrightarrow \mathcal{U}Cat(\mathcal{B})$ preserves filtered colimits. Finite limits commute with filtered colimits in both $\mathcal{A}$ and $\mathcal{B}$ by Proposition 8.11(4), in particular pullbacks. It follows that $\mathcal{U}Cat(\mathcal{A})$ is closed in $[\Delta_2^{op}, \mathcal{A}]$ under filtered colimits and similarly for $\mathcal{B}$. In order to show that $\mathcal{U}Cat(F)$ preserves filtered colimits it consequently suffices to show that $[\Delta_2^{op}, F] : [\Delta_2^{op}, \mathcal{A}] \longrightarrow [\Delta_2^{op}, \mathcal{B}]$ does so, which is the case as F preserves filtered colimits.

3. In order to show that $Cat(\mathcal{E})$ admits all small colimits it will suffice to exhibit it as a full reflective sub 2-category of a cocomplete 2-category $\mathcal{A}$: we must describe a 2-adjunction:

$$\begin{tikzcd} Cat(\mathcal{E}) \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right, "F"'] \end{tikzcd}$$

with counit an isomorphism. For if we can do so then to compute a colimit in $Cat(\mathcal{E})$ we may form the colimit of the corresponding diagram in $\mathcal{A}$ and take its reflection under F to obtain the colimit in $\mathcal{A}$. As $\mathcal{E}$ is locally finitely presentable the functor $\mathcal{E}(\iota -, 1) : \mathcal{E} \longrightarrow [(\mathcal{E}_f)^{op}, \text{Set}]$ is fully faithful and preserves all limits and filtered colimits by Proposition 8.11(1). As $\mathcal{E}_f$ is small the category $[\mathcal{E}_f^{op}, \text{Set}]$ is locally finitely presentable by Proposition 8.11(3). Taking the image of this functor under $Cat(-) : \text{Cat}_{\text{pb}} \longrightarrow \text{Rep}$ gives a 2-functor $Cat(\mathcal{E}(\iota -, 1)) : Cat(\mathcal{E}) \longrightarrow Cat([(\mathcal{E}_f)^{op}, \text{Set}])$ which preserves all limits and filtered colimits by the second part of the present proposition. Therefore its underlying functor $\mathcal{U}Cat(\mathcal{E}(\iota -, 1)) : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{U}Cat([(\mathcal{E}_f)^{op}, \text{Set}])$ preserves all ordinary limits and filtered colimits. By Proposition 8.11 Parts 3 and 6 the domain and codomain of this functor are locally finitely presentable. By Proposition 8.11(5) it consequently has a left adjoint. Furthermore it is straightforward to see that $\mathcal{U}Cat(\mathcal{E}(\iota -, 1))$ is fully faithful since $\mathcal{E}(\iota -, 1)$ is so. Therefore the counit of the adjunction is an isomorphism. As $Cat(\mathcal{E}(\iota -, 1))$ preserves cotensors with **2** this lifts, by Proposition 3.1 of [8], to a 2-adjunction:

$$\begin{tikzcd} Cat(\mathcal{E}) \arrow[r, bend right, "Cat(\mathcal{E}(\iota -, 1))"'] \arrow[r, phantom, "\perp"] & Cat([(\mathcal{E}_f)^{op}, \text{Set}]) \arrow[l, bend right] \end{tikzcd}$$

with the same counit components, thus again is a reflection. Now $Cat([(\mathcal{E}_f)^{op}, \text{Set}]) \cong [(\mathcal{E}_f)^{op}, \text{Cat}]$. As this presheaf 2-category is cocomplete it follows that $Cat([(\mathcal{E}_f)^{op}, \text{Set}])$ is too. Therefore the reflection exhibits $Cat(\mathcal{E})$ as cocomplete.

□

Definition 8.14. Let $\mathcal{E}$ be locally finitely presentable and view both $\mathcal{E}$ and $\mathcal{E}_f$ as locally discrete 2-categories.

- A 2-functor $T : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$ is said to be strongly finitary if it is the left Kan extension in 2-CAT of its restriction along the inclusion:

$$\mathcal{E}_f \xrightarrow{\iota} \mathcal{E} \xrightarrow{[-]} Cat(\mathcal{E})$$

.

- A 2-monad on $Cat(\mathcal{E})$ is said to be strongly finitary if its endo 2-functor part is so.

Remark 8.15. Consider the case $\mathcal{E}$ = Set. Set$_f$ is the skeletal category of finite sets and the composite embedding Set$_f \longrightarrow$Set$\longrightarrow Cat$(Set) = Cat is just the embedding $\iota :$ Set$_f \longrightarrow$Cat described in Remark 6.43. Therefore if we restrict our attention to endo 2-functors of Cat the notion of a "strongly finitary" 2-functor introduced in Definition 8.14 agrees with the notion of "strongly finitary" 2-functor of [27] described in Remark 6.43.

Theorem 8.16. Let $\mathcal{E}$ be locally finitely presentable and $\mathcal{A}$ a 2-category with filtered colimits and codescent objects of reflexive coherence data.

1. Any 2-functor $T : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$ which preserves codescent objects of pointwise discrete categories and filtered colimits is strongly finitary.

2. Given a 2-functor $T : \mathcal{E}_f \longrightarrow \mathcal{A}$ its left Kan extension along $\mathcal{E}_f \xrightarrow{\iota} \mathcal{E} \xrightarrow{[-]} Cat(\mathcal{E})$ exists, is pointwise, and preserves codescent objects of cateads and filtered colimits.

Proof. 1. Consider the diagram:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[ddr, "T"] & \\
\mathcal{E} \arrow[u, "{[\ ]}"] \arrow[dr, "T\circ[\ ]"] & \\
\mathcal{E}_f \arrow[u, "\iota"] \arrow[r, "T\circ[\ ]\circ\iota"'] & \mathcal{A}
\end{tikzcd}$$

with both triangles commuting.
Suppose that T preserves codescent objects of pointwise discrete categories and filtered colimits. By Proposition 8.1(1) the top triangle exhibits T as the left Kan extension of its restriction along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$. The underlying functor of the inclusion $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ has right adjoint $ob : \mathcal{U}Cat(\mathcal{E}) \longrightarrow \mathcal{E}$ and thus preserves all 1-dimensional colimits. As $\mathcal{E}$ has pullbacks $Cat(\mathcal{E})$ has cotensors with **2** and so by Proposition 2.5 any filtered colimit in $\mathcal{U}Cat(\mathcal{E})$ is immediately a filtered colimit in $Cat(\mathcal{E})$. It follows then that the 2-functor $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ preserves filtered colimits, since its underlying functor does. As both T and $[-]$ preserve filtered colimits so does the composite $T \circ [-]$. Now the inclusion $\mathcal{E}_f \longrightarrow \mathcal{E}$ exhibits $\mathcal{E}$ as the free completion of $\mathcal{E}_f$ under filtered colimits, and the 2-category $\mathcal{A}$ has them. Therefore, by the universal property of the free completion, the lower triangle exhibits $T \circ [-]$ as the left Kan extension of its restriction along $\iota : \mathcal{E}_f \longrightarrow \mathcal{E}$. As each triangle exhibits its diagonal as a left Kan extension it follows that the outer triangle exhibits the diagonal T as the left Kan extension of its restriction along the composite vertical arrow. Therefore T is strongly finitary.

2. Given $T : \mathcal{E}_f \longrightarrow \mathcal{A}$ we form its left Kan extension along $\mathcal{E}_f \xrightarrow{\iota} \mathcal{E} \xrightarrow{[-]} Cat(\mathcal{E})$ in two stages. Since $\mathcal{A}$ has filtered colimits, and $\mathcal{E}$ is the free completion of $\mathcal{E}_f$ under filtered colimits, the left Kan extension $L_1 : \mathcal{E} \longrightarrow \mathcal{A}$ of T along $\iota : \mathcal{E}_f \longrightarrow \mathcal{E}$ exists and preserves filtered colimits. Now $\mathcal{A}$ has codescent objects of reflexive coherence data. Therefore by Theorem 8.1(2) the left Kan extension of $L_2 : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$

of L_1 along $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ exists so that we have a diagram:

$$\begin{tikzcd}
Cat(\mathcal{E}) \arrow[rrdd, "L_2"] & & \\
\mathcal{E} \arrow[u, "{[-]}"] \arrow[rrd, "L_1"] \arrow[ur, phantom, "\Nearrow"] & & \\
\mathcal{E}_f \arrow[u, "\iota"] \arrow[rr, "T"'] \arrow[ur, phantom, "\Nearrow"] & & \mathcal{A}
\end{tikzcd}$$

Furthermore L_2 preserves codescent objects of pointwise discrete categories by the same result. Both triangles are left Kan extensions so that the outer triangle exhibits L_2 as the left Kan extension of T along $\mathcal{E}_f \xrightarrow{\iota} \mathcal{E} \xrightarrow{[-]} Cat(\mathcal{E})$. In order to show that L_2 is a pointwise Kan extension it will suffice to show that both 2-cells individually exhibit L_1 and L_2 as pointwise Kan extensions. As $\iota : \mathcal{E}_f \longrightarrow \mathcal{E}$ and $[-] : \mathcal{E} \longrightarrow Cat(\mathcal{E})$ are fully faithful each 2-cell is an isomorphism. Therefore each of L_1 and L_2 is equally the Kan extension of its restriction along ι and $[-]$ respectively. Consequently it will suffice to show that that the identity 2-cells:

$$\begin{tikzcd}
\mathcal{E} \arrow[rd, "L_1"] & \\
\mathcal{E}_f \arrow[u, "\iota"] \arrow[r, "L_1 \circ \iota"'] & A
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Cat(\mathcal{E}) \arrow[rd, "L_2"] & \\
\mathcal{E} \arrow[u, "{[-]}"] \arrow[r, "L_2 \circ [-]"'] & A
\end{tikzcd}$$

exhibit L_1 and L_2 as pointwise Kan extensions. Since L_2 preserves codescent objects of pointwise discrete categories the Kan extension on the right is pointwise, by Example 8.5.

Consider the composite:

$$\begin{tikzcd}
\mathcal{E} \arrow[rd, "L_1"] & & \\
\mathcal{E}_f \arrow[u, "\iota"] \arrow[r, "L_1 \circ \iota"'] & \mathcal{A} \arrow[r, "\mathcal{A}(-{,}X)"'] & \mathrm{Cat}^{op}
\end{tikzcd}$$

As $\mathcal{A}(-,X)$ preserves all colimits and L_1 preserves filtered colimits; the composite $\mathcal{A}(-,X) \circ L_1$ preserves filtered colimits. Furthermore Cat^{op} has filtered colimits so that, by the universal property of the free completion $\mathcal{E}$, $\mathcal{A}(-,X) \circ L_1$ is the left Kan extension of its restriction along ι. Thus $\mathcal{A}(-,X)$ preserves the Kan extension; it is pointwise. Consequently L_2 is the pointwise left Kan extension of T along $[-] \circ \iota$.

The category $\mathcal{E}_f$ is small and L_2 pointwise so, by Proposition 8.6, L_2 preserves any colimit preserved by $Cat(\mathcal{E})([A],-) : Cat(\mathcal{E}) \longrightarrow Cat$ for each $A \in \mathcal{E}_f$. As described in Example 4.31 we have $Cat(\mathcal{E})([A],-) \cong Cat(\mathcal{E}(A,-))$. Therefore this 2-functor preserves codescent objects of cateads by Theorem 3.66 and so L_2 preserves codescent objects of cateads. As $A \in \mathcal{E}_f$ the functor $\mathcal{E}([A],-)$ preserves filtered colimits. It also preserves all limits and so by Proposition 8.13(2) $Cat(\mathcal{E}(A,-)) \cong Cat(\mathcal{E})([A],-)$ preserves filtered colimits. Therefore L_2 preserves filtered colimits.

□

Corollary 8.17. Let $\mathcal{E}$ and $\mathcal{A}$ be as in Theorem 8.16. Let $T : Cat(\mathcal{E}) \longrightarrow \mathcal{A}$ preserve filtered colimits. Then T preserves codescent objects of cateads if and only if T preserves codescent objects of pointwise discrete categories.

Proof. It suffices to show that if T preserves filtered colimits and codescent objects of pointwise discrete categories then it preserves codescent objects of cateads. If T preserves these colimits it is strongly finitary by Theorem 8.16(1) and therefore by Theorem 8.16(2) preserves codescent objects of cateads. □

Corollary 8.18. Consider an endo 2-functor T on $\mathrm{Cat}(\mathcal{E})$ for a locally finitely presentable category $\mathcal{E}$. The following are equivalent:

1. T is strongly finitary.
2. T preserves filtered colimits and codescent objects of cateads.
3. T preserves filtered colimits and codescent objects of pointwise discrete categories.

Proof. If T is strongly finitary then it is the left Kan extension of its restriction along the inclusion $\mathcal{E}_f \longrightarrow \mathcal{E} \longrightarrow Cat(\mathcal{E})$. Now $Cat(\mathcal{E})$ is cocomplete by Proposition 8.13(3). Therefore by Theorem 8.16(2) T preserves filtered colimits and codescent objects of cateads. Thus ($1 \implies 2$). Clearly ($2 \implies 3$). By Theorem 8.16(1) we see that ($3 \implies 1$). $\square$

8.4 Cat as a free completion

We now specialise to the case of $\mathcal{E} = \mathrm{Set}$, so that $Cat(\mathcal{E}) = \mathrm{Cat}$. The category of sets is finitely presentable; the finitely presentable objects are the finite sets. Therefore Set_f is the skeletal category of finite sets. It has objects: $n \in N$ where n denotes a finite set with n elements. The arrows of Set_f are just functions between those sets.

Remark 8.19. The composite inclusion $\mathrm{Set}_f \xrightarrow{\iota} \mathrm{Set} \xrightarrow{[-]} \mathrm{Cat}$, which we abbreviate to $\iota : \mathrm{Set}_f \longrightarrow \mathrm{Cat}$, sends the set n to the discrete category with n objects; which we again denote by n. By Proposition 8.6 any pointwise Kan extension along $\iota : \mathrm{Set}_f \longrightarrow \mathrm{Cat}$ preserves any colimit preserved by each representable 2-functor $\mathrm{Cat}(n, -) : \mathrm{Cat} \longrightarrow \mathrm{Cat}$ for $n \in N$. Given a category C we have $\mathrm{Cat}(n, C) = C^n$, the n-fold product of C with itself. Consider the diagonal 2-functor $\triangle : \mathrm{Cat} \longrightarrow [n, \mathrm{Cat}]$ which assigns to a category C the constant 2-functor at C. $\mathrm{Cat}(n, -)$ may be decomposed as the composite:

$$\mathrm{Cat} \xrightarrow{\triangle} [n, \mathrm{Cat}] \xrightarrow{\Pi_n} \mathrm{Cat}$$

where Π_n is the 2-functor which takes the product of an n-tuple of categories. Now $\triangle$ is left 2-adjoint to Π_n and thus preserves all colimits. Therefore $\mathrm{Cat}(n, -)$ preserves any colimit preserved by $\Pi_n : [n, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ and so a pointwise Kan extension along $\iota : \mathrm{Set}_f \longrightarrow \mathrm{Cat}$ preserves any colimit preserved by $\Pi_n : \mathrm{Cat} \longrightarrow \mathrm{Cat}$ for each $n \in \mathbf{N}$. In other words those colimits which commute with finite products in Cat. Such colimits are called *sifted colimits*, which in the 1-dimensional setting were introduced in [20] and have been considered in the enriched setting in [38].

Definition 8.20. A weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ is sifted if $\Pi_n : [n, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ preserves W-colimits for each $n \in N$.

Corollary 8.21. Any pointwise Kan extension along $\iota : \mathrm{Set}_f \longrightarrow \mathrm{Cat}$ preserves sifted colimits. Thus strongly finitary 2-functors on Cat preserve sifted colimits.

Proof. This is immediate by Remark 8.19. $\square$

Example 8.22. Filtered colimits are sifted. In the language of the preceding definition each small filtered category $\mathcal{J}$ corresponds to a weight $\triangle(1) : \mathcal{J} \longrightarrow \mathrm{Cat}$ and Π_n preserves $\triangle(1)$-colimits for each $\mathcal{J}$. This is true as the underlying category of Cat is locally finitely presentable, and finite limits commute with filtered colimits in any locally finitely presentable category. That this fact lifts to the 2-category Cat follows immediately from Proposition 2.5 as Cat has cotensors with **2**.

Remark 8.23. Our main aim is to firstly show that codescent objects of strict reflexive coherence data are sifted colimits and to deduce that Cat is the free completion of Set_f under codescent objects of strict reflexive coherence data and filtered colimits. We firstly recall the precise notion of free completion. Our terminology follows that of Kelly-Schmidt [31]. They consider the general case of categories enriched over a symmetrical monoidal category V. This agrees with the present situation upon taking V to be Cat; in which case V-CAT = 2-CAT.

Definition 8.24. Let Φ be a class of weights. We consider the locally full sub 2-category Φ-Cocts of 2-CAT. The objects of Φ-Cocts are those 2-categories with all Φ colimits; that is W-colimits for each weight $W \in \Phi$. The morphisms of Φ-Cocts are those 2-functors which preserve W-colimits for each $W \in \Phi$. We let $U : \Phi\text{-Cocts} \longrightarrow \text{2-CAT}$ denote the forgetful 2-functor.

Definition 8.25. Consider a fully faithful 2-functor $\iota : \mathcal{A} \longrightarrow \mathcal{B}$ with $\mathcal{B} \in \Phi$-Cocts. Restriction along ι induces a functor:

$$\Phi\text{-Cocts}(\mathcal{B}, \mathcal{C}) \longrightarrow \text{2-CAT}(\mathcal{A}, U\mathcal{C})$$

for each $\mathcal{C} \in \Phi$-Cocts. If this has an equivalence inverse for each $\mathcal{C}$, and this is 2-natural in $\mathcal{C}$, then we say that $\iota : \mathcal{A} \longrightarrow \mathcal{B}$ exhibits $\mathcal{B}$ as the free completion of $\mathcal{A}$ under Φ-colimits.

Remark 8.26. Suppose that $\mathcal{C} \in \Phi$-Cocts, and that left Kan extensions along $\iota : \mathcal{A} \longrightarrow \mathcal{B}$ into $\mathcal{C}$ exist. Then we have an adjunction:

$$\begin{tikzcd} \text{2-CAT}(\mathcal{B}, \mathcal{C}) \arrow[r, "Res_\iota"', shift right] \arrow[r, phantom, "\perp"] & \text{2-CAT}(\mathcal{A}, \mathcal{C}) \arrow[l, "Lan_\iota"', shift right] \end{tikzcd}$$

which is 2-natural in $\mathcal{C}$. Furthermore the unit is an isomorphism as $\iota : \mathcal{A} \longrightarrow \mathcal{B}$ is fully faithful.
If each such left Kan extension along ι preserves Φ-colimits then this adjunction restricts to another:

$$\begin{tikzcd} \Phi\text{-Cocts}(\mathcal{B}, \mathcal{C}) \arrow[r, "Res_\iota"', shift right] \arrow[r, phantom, "\perp"] & \text{2-CAT}(\mathcal{A}, U\mathcal{C}) \arrow[l, "Lan_\iota"', shift right] \end{tikzcd}$$

which is again 2-natural in $\mathcal{C}$. If we suppose furthermore that each $T : \mathcal{B} \longrightarrow \mathcal{C} \in \Phi$-Cocts is the left Kan extension of its restriction along ι then the counit is also an isomorphism. Therefore under these circumstances $\iota : \mathcal{A} \longrightarrow \mathcal{B}$ exhibits $\mathcal{B}$ as the free completion of $\mathcal{A}$ under Φ-colimits.

Remark 8.27. We wish to show that Cat is the free completion of Set_f under filtered colimits and codescent objects of reflexive coherence data. Using the above terminology this is to show that Cat is the free completion of Set_f under Φ-colimits where:

$$\Phi = \begin{cases} \triangle(1) : \mathcal{J}^{op} \longrightarrow \text{Cat} & \text{for each small filtered category } \mathcal{J} \\ \iota : \Delta_2 \longrightarrow \text{Cat} & \text{the weight for reflexive codescent objects} \end{cases}$$

Lemma 8.28. Consider a weight $W : \mathcal{J} \longrightarrow \text{Cat}$. Suppose that, in Cat, finite products commute with W-colimits of the representables $\mathcal{J}(-, j) : \mathcal{J}^{op} \longrightarrow \text{Cat}$ for each $j \in \mathcal{J}$. Then finite products commute with all W-colimits in Cat.

Proof. A proof is given in Lemma 4.1 of [34]. $\square$

Remark 8.29. In Proposition 4.3 of [34] Lack states that finite products commute with codescent objects of reflexive coherence data in Cat, meaning both strict and general reflexive coherence data, therefore covering our Propositions 8.30 and 8.41 below. A full proof is not given but two approaches are sketched. The first approach suggested, and upon which our argument will be based, is to calculate codescent objects of representables and then apply Lemma 8.28 above. With regards codescent objects of strict reflexive coherence data this is straightforward; the representables of Δ_2 are simple as we show in the following proposition.
On the other hand the 2-category Δ_2' is itself complicated, its underlying category being freely generated. Consequently the Δ_2'-representables required for the consideration of general reflexive coherence data are not easily described and so computing their codescent objects directly is unlikely to be straightforward.
Our approach will be to first prove that finite products commute with codescent objects of strict reflexive coherence data. We will then replace the Δ_2'-representables by strict reflexive coherence data with the same codescent objects, enabling us to deduce the general case in Proposition 8.41.

Proposition 8.30 (See Remark 8.29)**.** Finite products commute with codescent objects of strict reflexive coherence data in Cat.

Proof. By Lemma 8.28 it suffices to show that finite products commute with codescent objects of the representables $\Delta_2(-,i) : \Delta_2^{op} \longrightarrow \mathrm{Cat}$. Each representable $\Delta(-,i) : \Delta^{op} \longrightarrow \mathrm{Cat}$ defines an internal category in Cat. As Δ_2 is a full subcategory of Δ the representable $\Delta_2(-,i)$ is just the restriction:

$$\Delta_2^{op} \longrightarrow \Delta^{op} \xrightarrow{\Delta(-,i)} \mathrm{Cat}$$

and so also an internal category in Cat. Furthermore Δ_2 is a locally discrete 2-category so that each category $\Delta_2(j,i)$ is discrete. Therefore each representable $\Delta_2(-,i) : \Delta_2^{op} \longrightarrow \mathrm{Cat}$ is a pointwise discrete category in Cat, and so a catead. By Corollary 4.33 $\Pi_n : [n, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ preserves codescent objects of cateads. Each n-tuple of representables Δ_2 representables constitutes a single catead in $[n, \mathrm{Cat}]$. That Π_n preserves its codescent object is then just the assertion that Π_n preserves codescent objects of the Δ_2 representables. □

Theorem 8.31. Let Φ be a class of sifted weights containing those weights for filtered colimits and codescent objects of strict reflexive coherence data. Then Cat is the free completion of Set_f under Φ-colimits. In particular:

1. Cat is the free completion of Set_f under filtered colimits and codescent objects of strict reflexive coherence data.
2. Cat is the free completion of Set_f under sifted colimits.

Proof. By Remark 8.26 we need to show three things. For a Φ-cocomplete 2-category $\mathcal{A}$:

1. Left Kan extensions into $\mathcal{A}$ along $\iota : \mathrm{Set}_f \longrightarrow \mathrm{Cat}$ exist.
2. Such left Kan extensions preserve Φ-colimits.
3. Any Φ-cocontinuous 2-functor $T : \mathrm{Cat} \longrightarrow \mathcal{A}$ is the left Kan extension of its restriction along ι.

The class Φ contains the weights for filtered colimits and strict reflexive codescent objects. Thus any Φ-cocomplete 2-category has filtered colimits and reflexive codescent objects. Theorem 8.16(2) shows, taking $\mathcal{E}$ =Set, that such left Kan extensions exist and are pointwise. Thus (1) is verified. By Corollary 8.21 any pointwise Kan extension along ι preserves all sifted colimits, thus all Φ-colimits. Thus (2) is verified. Theorem 8.16(1) asserts, taking $\mathcal{E}$ = Set, that any 2-functor which preserves codescent objects of pointwise discrete categories and filtered colimits is the left Kan extension of its restriction along $\iota : \mathrm{Set}_f \longrightarrow \mathrm{Cat}$. As any Φ-cocontinuous 2-functor preserves these colimits we have verified (3). □

Remark 8.32. In Part 2 of Theorem 8.31 we proved that Cat is the free completion of Set_f under sifted colimits. This is closely related to work of Lack and Rosický on enriched sifted colimits and strongly finitary functors [38].

8.5 Sifted colimits

In this section we consider examples of sifted colimits in Cat. Our primary interest is to show that (iso)codescent objects of reflexive coherence data are sifted colimits, not only in the strict case. Our approach to understanding codescent objects of general reflexive coherence data will be to relate them to codescent objects of strict reflexive coherence data, whose codescent objects we now know to be sifted. In order to deal with the cases of codescent objects and isocodescent objects concurrently we should firstly show that finite products commute with isocodescent objects of strict reflexive coherence data. In order to do so we need to consider the case of reflexive coinverters, another sifted colimit.

Definition 8.33. The weight for reflexive coinverters is:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "l"{name=U}] \arrow[r, bend right, "m"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\rho"'] & \cdot \arrow[l, dotted, "n"] \arrow[r, mapsto, "W"] & \mathbf{1} \arrow[r, bend left, "0"{name=U2}] \arrow[r, bend right, "1"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2] & \mathrm{I}(\mathbf{2}) \arrow[l, dotted] \end{tikzcd}$$

The 2-category on the left, which we denote by $\mathcal{J}$, has a pair of parallel 1-cells l and m with a single 2-cell ρ between them, with a 1-cell n in the opposite direction such that $nl = nm = 1$ and $n\rho = 1$. It is the 2-category freely generated by these equations.
A 2-functor $\mathcal{J}^{op} \longrightarrow \mathcal{A}$ consists of a "reflexive 2-cell" in $\mathcal{A}$:

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\theta"'] & B \arrow[l, dotted, "k"] \end{tikzcd}$$

That is, a 2-cell θ and a morphism $k : B \longrightarrow A$ such that precomposing with k gives $fk = gk = 1_B$ and θk is an identity 2-cell: k splits the 2-cell.
The weighted colimit of this 2-functor is simply the coinverter of the underlying 2-cell.

Remark 8.34. The colimit described in Definition 8.33 above simply takes the coinverter of the underlying 2-cell of a reflexive 2-cell. Therefore the splitting k plays no role in the colimit itself. As we are primarily interested in reflexive 2-cells with regards to computing their coinverters we view "reflexivity" as a property of a 2-cell.

Example 8.35. Given an object A of a 2-category $\mathcal{A}$ consider its cotensor with $\mathbf{2}$. The universal 2-cell:

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, ""'{name=D}] \arrow[Rightarrow, from=U, to=D] & A \end{tikzcd}$$

is reflexive, with splitting $i : A \longrightarrow A^{\mathbf{2}}$ the unique 1-cell induced by the identity 2-cell on A.

Proposition 8.36 (Kelly-Lack-Walters)**.** Finite products commute with reflexive coinverters in Cat.

Proof. A proof is given in [28] using a 3×3 lemma analogous to the well known 3×3 lemma for reflexive coequalisers [24]. ☐

Example 8.37. Given a category A consider the universal 2-cell:

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, ""'{name=D}] \arrow[Rightarrow, from=U, to=D] & A \end{tikzcd}$$

defining its cotensor with $\mathbf{2}$. Its coinverter is a groupoid, the groupoid obtained by freely inverting all the arrows of A, and thus provides the left 2-adjoint to the inclusion $\iota : \mathrm{Gpd} \longrightarrow \mathrm{Cat}$. As described in Example 8.35 each such 2-cell is reflexive, thus coinverters of such 2-cells commute with finite products by Proposition 8.36. As finite products also commute with cotensors with $\mathbf{2}$ it follows that the left adjoint to the inclusion preserves finite products. Consider now the composite ordinary adjunction:

$$\begin{tikzcd} \mathcal{U}\mathrm{Gpd} \arrow[r, shift right=1.5ex, "\mathcal{U}\iota"'] \arrow[r, phantom, "\perp"] & \mathcal{U}\mathrm{Cat} \arrow[l, shift right=1.5ex] \arrow[r, shift right=1.5ex, "\mathrm{Cat}(j-,1)"'] \arrow[r, phantom, "\perp"] & {[\Delta^{op}, \mathrm{Set}]} \arrow[l, shift right=1.5ex] \end{tikzcd}$$

which assigns to a simplicial set its "fundamental groupoid". As described in Example 2.22 the left adjoint to the nerve functor $\mathrm{Cat}(j-,1)$ computes the codescent object of the underlying strict reflexive coherence data of a simplicial set. By Proposition 8.30 this preserves finite products. Consequently the composite left adjoint preserves finite products. Thus we recover the well known result of [19] that the fundamental groupoid functor preserves finite products.

Proposition 8.38. Finite products commute with isocodescent objects of strict reflexive coherence data.

Proof. By Lemma 8.28 it suffices to show that finite products commute with isocodescent objects of the representables of $[\Delta_2^{op}, \mathrm{Cat}]$. The terminal object clearly commutes with isocodescent objects, as it does with codescent objects in Proposition 8.30. Thus it suffices to consider binary products of representables. We also observed in Proposition 8.30 that each representable is a pointwise discrete category in Cat. Therefore it suffices to show that binary products commute with isocodescent objects of pointwise discrete categories in Cat. Consider a pointwise discrete category X in Cat:

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "p_x"] \arrow[r, "m_x" description] \arrow[r, shift right=2, "q_x"'] & X_1 \arrow[r, shift left=2, "d_x"] \arrow[r, shift right=2, "c_x"'] & X_0 \arrow[l, "i_x" description]
\end{tikzcd}$$

We may form its isocodescent object $IQ(X)$ in two steps. Firstly form its codescent object QX, exhibited by the cocone (QX, α_x, θ_x):

$$\begin{tikzcd}
& X_0 \arrow[dr, "\alpha_x"] \arrow[dd, Rightarrow, "\theta_x"] & \\
X_1 \arrow[ur, "d_x"] \arrow[dr, "c_x"'] & & QX \\
& X_0 \arrow[ur, "\alpha_x"'] &
\end{tikzcd}$$

The coinverter of θ_x, which we denote by $I(\theta_x)$ is then the isocodescent object $IQ(X)$ of the pointwise discrete category X. If the 2-cell θ_x were reflexive then we could deduce the result from the fact that finite products commute with both codescent objects of pointwise discrete categories and coinverters of reflexive 2-cells. The 2-cell is however not necessarily reflexive. Our approach will be to replace it with a reflexive 2-cell with the same coinverter.

The arrow category $(QX)^{\mathbf{2}}$ has universal 2-cell:

$$\begin{tikzcd}
(QX)^{\mathbf{2}} \arrow[r, bend left=40, "d", ""{name=U, below}] \arrow[r, bend right=40, "c"', ""{name=D, above}] & QX \arrow[from=U, to=D, Rightarrow, "\eta_x"]
\end{tikzcd}$$

By the universal property of the arrow category the 2-cell θ_x induces a unique functor $\iota_x : X_1 \longrightarrow (QX)^{\mathbf{2}}$ such that $\eta_x \circ \iota_x = \theta_x$.

Now the category QX is exactly that presented by the internal category X (as discussed in Example 2.21). Thus it has objects X_0 and arrows X_1 so that the arrow category $(QX)^{\mathbf{2}}$ has object set exactly X_1. Moreover the comparison $\iota_x : X_1 \longrightarrow (QX)^{\mathbf{2}}$ is bijective on objects.

Any bijective on objects functor is, in particular, liberal (as defined in Notation 2.28). Consider then a morphism $f : QX \longrightarrow C$ with $f\eta$ is invertible. Now $f\theta_x = f\eta_x\iota_x$. As ι_x is liberal $f\eta_x\iota_x$ is invertible if and only if $f\eta_x$ is so. Therefore a morphism $f : X \longrightarrow C$ coinverts η_x if and only if it coinverts θ_x, and so these 2-cells have the same coinverter: $I(\theta_x) = I(\eta_x)$. Consequently we have $IQ(X) = I(\theta_x) = I(\eta_x)$.

Consider another pointwise discrete category Y:

$$\begin{tikzcd}
Y_2 \arrow[r, shift left=2, "p_y"] \arrow[r, "m_y" description] \arrow[r, shift right=2, "q_y"'] & Y_1 \arrow[r, shift left=2, "d_y"] \arrow[r, shift right=2, "c_y"'] & Y_0 \arrow[l, "i_y" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y_0 \arrow[dr, "\alpha_y"] \arrow[dd, Rightarrow, "\theta_y"] & \\
Y_1 \arrow[ur, "d_y"] \arrow[dr, "c_y"'] & & QY \\
& Y_0 \arrow[ur, "\alpha_y"'] &
\end{tikzcd}$$

with codescent object QY and universal cocone as on the right above. By Proposition 8.30 finite products commute with codescent objects of strict reflexive coherence data we have $Q(X \times Y) \cong QX \times QY$ with exhibiting cocone $(QX \times QY, \alpha_x \times \alpha_y, \theta_x \times \theta_y)$. Therefore $IQ(X \times Y) \cong I(QX \times QY) = I(\theta_x \times \theta_y)$. Now finite products commute with cotensors with $\mathbf{2}$ so that $(Q(X \times Y))^{\mathbf{2}} = (QX)^{\mathbf{2}} \times (QY)^{\mathbf{2}}$ with universal 2-cell $\eta_x \times \eta_y$ and the unique map $\iota_{x \times y} : X_1 \times Y_1 \longrightarrow (QX \times QY)^{\mathbf{2}}$ is just $i_x \times i_y : X_1 \times Y_1 \longrightarrow (QX)^{\mathbf{2}} \times (QY)^{\mathbf{2}}$.

As finite products commute with bijections on objects $i_x \times i_y$ is bijective on objects and therefore liberal. Consequently we have $IQ(X \times Y) \cong I(\theta_x \times \theta_y) \cong I(\eta_x \times \eta_y)$. As both η_x and η_y are reflexive 2-cells we have $I(\eta_x \times \eta_y) \cong I(\eta_x) \times I(\eta_y)$ by Proposition 8.36. Since $IQ(X) = I(\eta_x)$ and $IQ(Y) = I(\eta_y)$ we deduce that $IQ(X \times Y) \cong IQX \times IQY$ as required. □

Remark 8.39. Lemma 8.40 below shows when two pieces of general coherence data have the same codescent object and will be employed in the next chapter, as well as this one. Firstly we motivate the lemma by analogy. Let C be a category and consider a morphism of graphs in C:

$$\begin{tikzcd}
a_1 \arrow[r, shift left, "d_a"] \arrow[r, shift right, "c_a"'] \arrow[d, "f_1"'] & a_0 \arrow[d, "1"] \\
b_1 \arrow[r, shift left, "d_b"] \arrow[r, shift right, "c_b"'] & a_0
\end{tikzcd}$$

in which the arrow f_1 is an epimorphism. Then both rows have the same coequaliser. That f_1 is required to be an epimorphism ensures that it detects equality of 1-cells out of its codomain. Therefore any arrow out of a_0 which coequalises the upper graph also coequalises the lower one. With regards codescent objects, no equality of 1-cells will be required, only equality of 2-cells. A 1-cell detects equality of parallel 2-cells out of its codomain if it is cofaithful and so naturally this will be an important notion regarding invariance of codescent objects. As we will need to "construct" a 2-cell co-fully faithfulness will also be important.

Lemma 8.40. Consider a pseudonatural transformation of coherence data $f : A \longrightarrow B$ as depicted below:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] \arrow[d, "f_2"'] \arrow[dr, phantom, "\cong"] & A_1 \arrow[d, "f_1"] \arrow[dr, phantom, "\cong"] & B_0 \arrow[l, "i_a" description] \arrow[l, shift right=2, "d_a"'] \arrow[l, shift left=2, "c_a"] \arrow[d, "f_0=1"] \\
B_2 \arrow[r, shift left=2, "p_b"] \arrow[r, "m_b" description] \arrow[r, shift right=2, "q_b"'] & B_1 & B_0 \arrow[l, "i_b" description] \arrow[l, shift right=2, "d_b"'] \arrow[l, shift left=2, "c_b"]
\end{tikzcd}$$

such that:

- f_0 is an identity,
- f_1 is co-fully faithful and
- f_2 is co-faithful.

1. Suppose that:

$$\begin{tikzcd}
& B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow \overline{\alpha}"] & \\
B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & QB \\
& B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

exhibits QB as the codescent object of the bottom row.
Then:

$$\begin{tikzcd}
& B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\alpha"] & \\
A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & \Downarrow \overline{\alpha} & QB \\
& B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

exhibits QB as the codescent object of the top row.

2. This equally applies to the case of the isocodescent object.

Proof. 1. The coherence data A and B come equipped with 2-cell isomorphisms as described in Definition 6.27. For instance we have an invertible 2-cell $d_a i_a \cong 1$. Since $\Delta_2^{-'}$ is locally a preorder each such coherence isomorphism is determined by its source and target 1-cells and so we label each coherence 2-cell isomorphism by τ_a for the coherence data A, and by τ_b for B.

We must firstly prove that the above composite 2-cell is indeed a codescent cocone to the top row, that is satisfies the cocone equations (1) and (2) of Definition 6.31. Firstly we show that equation (2) holds, which states that the 2-cell on the left below is the identity on $\alpha : B_0 \longrightarrow QB$. We have the following string of equalities:

$$\begin{tikzcd}
 & & B_0 \arrow[r, "1"] & B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow\overline{\alpha}"] & \\
B_0 \arrow[r, "i_a"] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "\Downarrow\tau_a"] \arrow[dr, phantom, "\Downarrow\tau_a"] & A_1 \arrow[r, "f_1"] \arrow[ur, "d_a"] \arrow[dr, "c_a"'] \arrow[ur, phantom, near end, "\Downarrow f_d"] \arrow[dr, phantom, near end, "\Downarrow f_c^{-1}"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & QB \\
 & & B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & & B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow\overline{\alpha}"] & \\
B_0 \arrow[r, bend left, "i_b"] \arrow[r, "f_1 i_a" description] \arrow[r, bend right, "i_b"'] \arrow[urr, bend left=40, "1"] \arrow[drr, bend right=40, "1"'] \arrow[r, phantom, shift left=3, "\Downarrow\tau_b"] \arrow[r, phantom, shift left=1.5, "\Downarrow f_i^{-1}"] \arrow[r, phantom, shift right=1.5, "\Downarrow f_i"] \arrow[r, phantom, shift right=3, "\Downarrow\tau_b"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & QB \\
 & & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

$$= \begin{tikzcd}
 & & B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow\overline{\alpha}"] & \\
B_0 \arrow[r, "i_b"] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "\Downarrow\tau_b"] \arrow[dr, phantom, "\Downarrow\tau_b^{-1}"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & QB \\
 & & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}
= \quad B_0 \xrightarrow{\alpha} QB$$

Only the first equality requires much justification; the second holds upon cancelling inverses whilst the third holds as $(QB, \alpha, \overline{\alpha})$ is a codescent cocone to the bottom row. The first equation requires us to use the pseudonaturality of f. By pseudonaturality we have the equation:

$$\begin{tikzcd}
B_0 \arrow[r, "i_a"] \arrow[rr, bend left=50, "1"] \arrow[d, "1"] \arrow[dr, phantom, "\Downarrow f_i"] & A_1 \arrow[r, "d_a"] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow f_d"] \arrow[u, phantom, "\Downarrow\tau_a"] & B_0 \arrow[d, "1"] \\
B_0 \arrow[r, "i_b"] & B_1 \arrow[r, "d_b"] & B_0
\end{tikzcd}
=
\begin{tikzcd}
B_0 \arrow[rr, "1"] \arrow[d, "1"] & & B_0 \arrow[d, "1"] \\
B_0 \arrow[rr, "1"] \arrow[dr, "i_b"'] & \Downarrow\tau_b & B_0 \\
 & B_1 \arrow[ur, "d_b"'] &
\end{tikzcd}$$

We may rewrite this equation by pasting the 2-cell $(f_i)^{-1}$ on the lower left side of each of the above diagrams to obtain:

$$(1) \quad \begin{tikzcd}
 & & B_0 \arrow[r, "1"] & B_0 \\
B_0 \arrow[r, "i_a"] \arrow[urr, bend left, "1"] \arrow[ur, phantom, "\Downarrow\tau_a"] & A_1 \arrow[r, "f_1"] \arrow[ur, "d_a"] \arrow[ur, phantom, near end, "\Downarrow f_d"] & B_1 \arrow[ur, "d_b"] &
\end{tikzcd}
=
\begin{tikzcd}
 & & B_0 \\
B_0 \arrow[r, bend left, "i_b"] \arrow[r, "f_1 i_a"'] \arrow[urr, bend left=40, "1"] \arrow[r, phantom, shift left=3, "\Downarrow\tau_b"] \arrow[r, phantom, shift left=1, "\Downarrow f_i^{-1}"] & B_1 \arrow[ur, "d_b"] &
\end{tikzcd}$$

We use this equation enables us to equate the top left sides of the composite 2-cells on either side of the first equation. We also need to equate the lower left hand sides. By pseudonaturality of $f : A \longrightarrow B$

we equally have:

$$\begin{tikzcd}
 & & B_0 \arrow[r, "1"] & B_0 \\
B_0 \arrow[r, "i_a"] \arrow[urr, bend left, "1"] & A_1 \arrow[ur, "c_a"] \arrow[r, "f_1"] \arrow[u, phantom, "\Downarrow\tau_a"] & B_1 \arrow[ur, "c_b"] \arrow[u, phantom, "\Downarrow f_c"] &
\end{tikzcd}
=
\begin{tikzcd}
 & & B_0 \\
B_0 \arrow[rr, "f_1 i_a"] \arrow[urr, bend left, "1"] \arrow[r, phantom, "\Downarrow\tau_b"] \arrow[rr, bend left=20, "i_b"] & \arrow[u, phantom, "\Downarrow f_i^{-1}"] & B_1 \arrow[u, "c_b"]
\end{tikzcd}$$

Inverting both composites gives:

$$(2) \qquad \begin{tikzcd}
B_0 \arrow[r, "i_a"] \arrow[drr, bend right, "1"'] & A_1 \arrow[r, "f_1"] \arrow[dr, "c_a"] \arrow[d, phantom, "\Downarrow\tau_a"] & B_1 \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & \\
 & & B_0 \arrow[r, "1"'] & B_0
\end{tikzcd}
=
\begin{tikzcd}
B_0 \arrow[rr, "f_1 i_a"] \arrow[rr, bend right=20, "i_b"'] \arrow[drrr, bend right, "1"'] & \arrow[d, phantom, "\Downarrow f_i"] & B_1 \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow\tau_b"] & \\
 & & & B_0
\end{tikzcd}$$

noting that the inverse of τ_a is just τ_a again, by our abuse of notation. Combining the 2-cells (1) and (2) and cancelling f_i gives the first equation.

In order to show that we have a codescent cocone it remains to establish the codescent cocone equation (1) of Definition 6.31 holds. This uses similar, though lengthier, techniques to those described for equation (2) above and we continue with the proof of this in Appendix 12.2(1). Thus we have a cocone to the top row. It remains to verify its universal property. Given another cocone to the top row $(C, \beta, \overline{\beta})$, we have the 2-cell:

$$\begin{tikzcd}
B_1 \arrow[r, "d_b"] & B_0 \arrow[dr, "\beta"] & \\
A_1 \arrow[u, "f_1"] \arrow[ur, "d_a"'] \arrow[d, "f_1"'] \arrow[dr, "c_a"] \arrow[u, phantom, shift left=4, "\Downarrow f_d^{-1}"] \arrow[d, phantom, shift left=4, "\Downarrow f_c"] & \Downarrow\overline{\beta} & C \\
B_1 \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

As f_1 is co-fully faithful, there exists a unique 2-cell:

$$\begin{tikzcd}
 & B_0 \arrow[dr, "\beta"] & \\
B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & \Downarrow\overline{\beta}' & C \\
 & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

which, precomposed with f_1, yields the composite 2-cell above it. We claim that the triple $(C, \beta, \overline{\beta}')$ is a cocone to the bottom row. Firstly we must show that the first and last composite 2-cells of the following string are equal:

$$\begin{tikzcd}
 & & B_0 \arrow[dr, "\beta"] & \\
B_0 \arrow[r, "i_b"] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[u, phantom, "\Downarrow\tau_b"] \arrow[d, phantom, "\Downarrow\tau_b"] & \Downarrow\overline{\beta}' & C \\
 & & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & & B_0 \arrow[r, "1"] & B_0 \arrow[dr, "\beta"] & \\
B_0 \arrow[r, "i_a"] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] \arrow[u, phantom, "\Downarrow\tau_a"] \arrow[d, phantom, "\Downarrow\tau_a"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[u, phantom, "\Downarrow f_d"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & \Downarrow\overline{\beta}' & C \\
 & & B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

$$= \begin{tikzcd}
 & & B_0 \arrow[dr, "\beta"] & \\
B_0 \arrow[r, "i_b"] \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] & A_1 \arrow[ur, "d_a"] \arrow[dr, "c_a"'] \arrow[u, phantom, "\Downarrow\tau_a"] \arrow[d, phantom, "\Downarrow\tau_a"] & \Downarrow\overline{\beta} & C \\
 & & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}
= \begin{tikzcd} B_0 \arrow[r, "\beta"] & C \end{tikzcd}$$

Combining the equations (1) and (2), and cancelling inverses, enables us to deduce the first equation. The second equation follows by the definition of $\overline{\beta}'$ precomposed with f_1 upon cancelling f_d and f_c^{-1} by their inverses. The last equation holds as the triple $(C, \beta, \overline{\beta})$ is a cocone to the top row. We must also show that the triple $(C, \beta, \overline{\beta}')$ satisfies the cocone condition (1) of Definition 6.31 and we continue this in Appendix 12.2(2). Therefore the triple $(C, \beta, \overline{\beta}')$ is a cocone to the bottom row. Consequently we obtain a unique arrow out of the codescent object $k : QB \longrightarrow C$ such that $k \circ \alpha = \beta$ and such that:

$$\begin{tikzcd}
 & B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow \overline{\alpha}"] & & \\
B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & QB \arrow[r, "k"] & C \\
 & B_0 \arrow[ur, "\alpha"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B_0 \arrow[dr, "\beta"] \arrow[dd, phantom, "\Downarrow \overline{\beta}'"] & \\
B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & C \\
 & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

The condition concerning the equality of 2-cells may be replaced by:

$$\begin{tikzcd}
 & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow \overline{\alpha}"] & & \\
A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & & QB \arrow[r, "k"] & C \\
 & B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\alpha"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\beta"] \arrow[dd, phantom, "\Downarrow \overline{\beta}'"] & \\
A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & & C \\
 & B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

since f_1 is co-fully faithful and the 2-cells f_d and f_c^{-1} both isomorphisms. We easily see upon cancelling isomorphisms that this second composite 2-cell is exactly $\overline{\beta}$. Therefore the cocone

$$\begin{tikzcd}
 & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow \overline{\alpha}"] & \\
A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & & QB \\
 & B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

indeed satisfies the one dimensional aspect of the universal property of the codescent object of the top row. One may at length, in a similar manner, verify the two dimensional aspect of the universal property of the codescent object. In any case the 2-categories of interest to us will have cotensors with **2**. For such 2-categories the two dimensional universal property follows from the one dimensional, using Proposition 2.5.

2. The case of the isocodescent object proceeds just as the case of the codescent object presented in the first part of the lemma.

☐

Proposition 8.41 (See Remark 8.29)**.** Finite products commute with both codescent objects and isocodescent objects of general reflexive coherence data in Cat.

Proof. The cases of codescent and isocodescent objects are identical. We consider the former; the latter can be proved by replacing each appearance of codescent object by isocodescent object, as all of the remarks we make apply equally to either of these cases.
Denote by $Q' : [\Delta_2'^{op}, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ and $Q : [\Delta_2^{op}, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ the 2-functors which send reflexive coherence data/ strict reflexive coherence data to their respective codescent objects. Consider the 2-functor $p : \Delta_2' \longrightarrow \Delta_2$ of Remark 6.26. Restriction along its opposite gives a 2-functor:

$$p^* = [p^{op}, \mathrm{Cat}] : [\Delta_2^{op}, \mathrm{Cat}] \longrightarrow [\Delta_2'^{op}, \mathrm{Cat}]$$

which views strict coherence data as general coherence data. By Remark 6.33 it is clear that we have a 2-natural isomorphism $Q' \circ p^* \cong Q$.

We need to show that $Q' : [{\Delta'_2}^{op}, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ commutes with finite products. It is clear that it preserves the terminal object, which is just the constant diagram at $\mathbf{1}$. In order to show that finite products are preserved then it suffices to consider the case of binary products. By Lemma 8.28 above we need only show that binary products commute with codescent objects of Δ'_2 representables.

The key step in our argument will be to associate to each representable $\Delta'_2(-,i)$, strict reflexive coherence data $X_i : \Delta_2^{op} \longrightarrow \mathrm{Cat}$ and a pseudonatural transformation $\theta_i : \Delta'_2(-,i) \longrightarrow p^* \circ X_i$ such that:

1. $\theta_i(0)$ is an identity.
2. $\theta_i(1)$ and $\theta_i(2)$ are equivalences.

Then by Lemma 8.40, we will have $Q'(\Delta'_2(-,i)) \cong Q'(p^* \circ X_i)$. Furthermore the properties of the maps θ_i are stable under products in $[\Delta'_2, \mathrm{Cat}]$ so that $\theta_i \times \theta_j$ again satisfies them. This gives the first of the following string of isomorphisms:

$$Q'(\Delta'_2(-,i) \times \Delta'_2(-,j)) \cong Q'(p^* \circ X_i \times p^* \circ X_j) \cong Q'(p^* \circ (X_i \times X_j)) \cong Q(X_i \times X_j) \cong$$

$$Q(X_i) \times Q(X_j) \cong Q'(p^* X_i) \times Q'(p^* X_j) \cong Q'(\Delta'_2(-,i)) \times Q'(\Delta'_2(-,j))$$

The second isomorphism holds as p^* preserves products. For since p^* is defined by restriction it has a left 2-adjoint given by left Kan extension. The third isomorphism holds as $Q'p^* \cong Q$. The fourth holds since finite products commute with codescent objects of strict reflexive coherence data in Cat (by Proposition 8.30 above). The fifth isomorphisms again uses the isomorphism $Q'p^* \cong Q$ whilst the final isomorphism comes from the isomorphisms $Q'(\Delta'_2(-,i)) \cong Q'(p^* \circ X_i)$ for each $i \in \Delta'_2$.

It remains then to construct, for each representable $\Delta'_2(-,i)$, a pseudonatural transformation $\theta_i : \Delta'_2(-,i) \longrightarrow p^* \circ X_i$ satisfying properties (1) and (2) which we now proceed to do.

The 2-functor $p : \Delta'_2 \longrightarrow \Delta_2$ yields a 2-natural transformation:

$$\begin{tikzcd}
\Delta'_2 \arrow[r, "p"] \arrow[d, "Y"'] & \Delta_2 \arrow[d, "Y"] \\
{[{\Delta'_2}^{op}, Cat]} & {[{\Delta_2}^{op}, Cat]} \arrow[l, "p^*"']
\arrow[from=1-1, to=2-2, phantom, "\Longrightarrow"]
\end{tikzcd}$$

whose components are themselves 2-natural transformations $p_{-,i} : \Delta'_2(-,i) \longrightarrow \Delta_2(p-, pi) = p^*(\Delta_2(-,pi))$. The 2-functor $p : \Delta'_2 \longrightarrow \Delta_2$ is the identity on objects by Remark 6.26 so that $p^*(\Delta_2(-,pi)) = p^*\Delta_2(-,i)$. The components of the 2-natural transformation $p_{-,i} : \Delta'_2(-,i) \longrightarrow p^*\Delta_2(-,i)$ are given by the action of the 2-functor p on the hom-categories: $p_{i,j} : \Delta'_2(i,j) \longrightarrow \Delta_2(i,j)$ so that each $p_{-,i}$ is a pointwise surjective equivalence again by Remark 6.26. Indeed this gives a 2-natural transformation from the representable $\Delta'_2(-,i)$ to strict reflexive coherence data, viewed as a presheaf on Δ'_2, and satisfies condition (2) but not (1) since $p_{-,i}(0) = p_{0,i}$ is not the identity; consequently we need to alter the "objects" of $\Delta_2(-,i)$.

The representable $\Delta_2(-,i)$ is an internal category in Cat. The functor $p_{0,i} : \Delta'_2(0,i) \longrightarrow \Delta_2(0,i) = ob(\Delta_2(-,i))$ may then be lifted along the fibration $ob : \mathcal{U}Cat(\mathcal{U}\mathrm{Cat}) \longrightarrow \mathcal{U}\mathrm{Cat}$ to give an internal category $X_i \in Cat(\mathcal{U}\mathrm{Cat})$ and its cartesian lift $\phi : X_i \longrightarrow \Delta_2(-,i)$. These satisfy $X_i(0) = \Delta'_2(0,i)$ and $\phi_0 = p_{0,i}$. Explicitly $X_i(1)$ and $X_i(2)$ are the pullbacks[2]:

$$\begin{tikzcd}
X_i(1) \arrow[r, "\phi_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta_2(1,i) \arrow[d, "{(d,c)}"] \\
\Delta'_2(0,i)^2 \arrow[r, "{p_{0,i}}^2"] & \Delta_2(0,i)^2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_i(2) \arrow[r, "\phi_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta_2(2,i) \arrow[d, "{(dp,dq=cp,cq)}"] \\
\Delta'_2(0,i)^3 \arrow[r, "{p_{0,i}}^3"] & \Delta_2(0,i)^3
\end{tikzcd}$$

---

[2] The description of the cartesian lift of the fibration $ob : \mathcal{U}Cat(\mathcal{U}\mathrm{Cat}) \longrightarrow \mathcal{U}\mathrm{Cat}$ given here is a simplification of the description given in Proposition 2.61. This simplification is possible because Cat has products and is indeed the better known construction. [11]

with ϕ_2 and ϕ_1 constructed by pulling back the surjective equivalence $\theta_0 = p_{0,i}$. Now surjective equivalences are preserved by products and stable under pullback in Cat; thus both ϕ_1 and ϕ_2 are surjective equivalences. Therefore we have a pointwise surjective equivalence of internal categories $\phi : X_i \longrightarrow \Delta_2(-,i)$, both internal categories in particular constituting strict reflexive coherence data in Cat. We now factor $p_{-,i} : \Delta_2'(-,i) \longrightarrow p^*\Delta_2(-,i)$ through $p^*\phi : p^*X_i \longrightarrow p^*\Delta_2(-,i)$ to obtain the required pseudonatural transformation $\theta : \Delta_2'(-,i) \longrightarrow p^*X_i$.

We have $X_i(0) = \Delta_2'(0,i)$ and correspondingly define θ_0 to be the identity. The surjective equivalences $\phi_1 : X_i(1) \longrightarrow \Delta_2(1,i)$ and $\phi_2 : X_i(2) \longrightarrow \Delta_2(2,i)$ each have equivalence inverses; sections $r_1 : \Delta_2(1,i) \longrightarrow X_i(1)$ and $r_2 : \Delta_2(2,i) \longrightarrow X_i(2)$. We define θ_1 and θ_2 respectively to be the composites:

$$\theta_1 = \Delta_2'(1,i) \xrightarrow{p_{1,i}} \Delta_2(1,i) \xrightarrow{r_1} X_i(1) \qquad \text{and} \qquad \theta_2 = \Delta_2'(2,i) \xrightarrow{p_{2,i}} \Delta_2(2,i) \xrightarrow{r_2} X_i(2).$$

As composites of equivalences both of θ_1 and θ_2 are equivalences whilst θ_0 is the identity. Therefore upon extending to a pseudonatural transformation θ our argument will be complete, the arrow components of θ satisfying properties (1) and (2).

Observe that for each $j \in \Delta_2'$ we have $\phi_j \circ \theta_j = p_{j,i}$ as r_1 and r_2 are sections. Now given an arrow $r : j_1 \longrightarrow j_2$ of Δ_2' consider the square:

$$\begin{tikzcd}
\Delta_2'(j_2,i) \arrow[r, "\theta_{j_2}"] \arrow[d, "\Delta_2'(r,i)"'] & X_i(j_2) \arrow[d, "X_i(pr)"] \\
\Delta_2'(j_1,i) \arrow[r, "\theta_{j_1}"'] & X_i(j_1)
\end{tikzcd}$$

Postcomposing both paths of the square with $\phi_{j_1} : X_i(j_1) \longrightarrow \Delta_2(j_1,i)$ gives the commuting square:

$$\begin{tikzcd}
\Delta_2'(j_2,i) \arrow[r, "p_{(j_2,i)}"] \arrow[d, "\Delta_2'(r,i)"'] & \Delta_2(j_2,i) \arrow[d, "\Delta_2(pr,i)"] \\
\Delta_2'(j_1,i) \arrow[r, "p_{(j_1,i)}"'] & \Delta_2(j_1,i)
\end{tikzcd}$$

The equivalence $\phi_{j_1} : X_i(j_1) \longrightarrow \Delta_2(j_1,i)$ is in particular fully faithful. Therefore there exists a unique 2-cell:

$$\begin{tikzcd}
\Delta_2'(j_2,i) \arrow[r, "\theta_{j_2}"] \arrow[d, "\Delta_2'(r,i)"'] \arrow[dr, phantom, "\Downarrow \theta_r"] & X_i(j_2) \arrow[d, "X_i(pr)"] \\
\Delta_2'(j_1,i) \arrow[r, "\theta_{j_1}"'] & X_i(j_1)
\end{tikzcd}$$

which upon postcomposition with ϕ_{j_1} yields the identity 2-cell square of the square above it. The 2-cell θ_r is an isomorphism since any fully faithful functor reflects isomorphisms.

To see that this description of θ makes it into a pseudonatural transformation requires us only to verify certain equations between 2-cells. As $p_{-,i} = \phi \circ \theta$ we may deduce that these equations hold using the faithfulness of the components of ϕ and the 2-naturality of $p_{-,i}$. Thus we obtain a pseudonatural transformation:

$$\theta_i = \theta : \Delta_2'(-,i) \longrightarrow p^*X_i$$

satisfying (1) and (2). □

Remark 8.42. We conclude our examples of sifted colimits by considering the case of Kleisli objects which are not only sifted but commute with all conical limits whose indexing 2-category is locally discrete.

Proposition 8.43. Let $\mathcal{J}$ be a small category and consider the limit 2-functor $lim : [\mathcal{J}, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ which assigns to a diagram its conical $\mathcal{J}$-limit. Then lim preserves Kleisli objects. In particular Kleisli objects commute with finite products.

Proof. Kleisli objects in Cat may be constructed from Eilenberg-Moore objects as follows. Let T be a monad on a category C. We have the canonical left adjoint arrow $f^T : C \longrightarrow C^T$ to the Eilenberg-Moore object. The Kleisli object may be constructed by factoring this as bijective on objects followed by fully faithful.

$$\begin{tikzcd}
C \arrow[rr, "f^T"] \arrow[dr, "f_T"'] & & C^T \\
& C_T \arrow[ur] &
\end{tikzcd}$$

Kleisli objects in $[\mathcal{J}, \mathrm{Cat}]$ are pointwise in Cat. Namely they are obtained by factoring the map to the Eilenberg-Moore object as pointwise bijective on objects followed by pointwise fully faithful. Now $lim : [\mathcal{J}, \mathrm{Cat}] \longrightarrow \mathrm{Cat}$ preserves all limits, thus Eilenberg-Moore objects, and fully faithfulness. Furthermore if $\mathcal{J}$ is locally discrete it takes pointwise bijections on objects to bijections on objects (as described in the proof of Corollary 4.33). In summary lim preserves Eilenberg-Moore objects and takes the (pointwise bijective on objects/pointwise fully faithful) factorisation on $[\mathcal{J}, \mathrm{Cat}]$ to the (bijective on objects/fully faithful)-factorisation on Cat. Consequently it preserves the construction of Kleisli objects.
The above certainly applies if $\mathcal{J}$ is a finite discrete category; thus we deduce that Kleisli objects commute with finite products in Cat. ☐

Remark 8.44. In the following we describe various characterising descriptions of the strongly finitary 2-functors on Cat, both minimal and maximal.

Corollary 8.45. Let T be a 2-functor on Cat. The following are equivalent.

1. T is strongly finitary.
2. T preserves filtered colimits and codescent objects of pointwise discrete categories.
3. T preserves filtered colimits and codescent objects of cateads.
4. T preserves filtered colimits and codescent objects of strict reflexive coherence data.
5. T preserves sifted colimits. These include: filtered colimits, (iso)codescent objects of reflexive coherence data, both general and strict, reflexive coinverters and Kleisli objects.

Proof. The equivalence of (1) and (5) is immediate as Cat is the free completion of Set_f under sifted colimits. Certainly (5) implies (4), (4) implies (3) and (3) implies (2) whilst (2) implies (1) by Theorem 8.16(1). applied in the case of $\mathcal{E} = \mathrm{Set}$. ☐

Example 8.46. Every strongly finitary 2-monad on Cat preserves codescent objects of cateads and filtered colimits. As each such 2-monad preserves codescent objects of cateads it preserves bijections on objects by Corollary 4.21(1). It is worth remarking that the converse is not true: there exist 2-monads which preserve filtered colimits and bijections on objects but are not strongly finitary. Consider the reflection:

$$\begin{tikzcd}
\mathrm{Gpd} \arrow[r, bend right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend right, "R"']
\end{tikzcd}$$

which induces a 2-monad $T = \iota R$ on Cat. The underlying functor $\mathcal{U}\iota : \mathcal{U}\mathrm{Gpd} \longrightarrow \mathcal{U}\mathrm{Cat}$ has a right adjoint which assigns to a category C the groupoid with the same objects as C and morphisms the isomorphisms in C. Consequently $\mathcal{U}\iota$ preserves filtered colimits. Both Gpd and Cat have cotensors with $\mathbf{2}$ and so these filtered colimits in $\mathcal{U}\mathrm{Gpd}$ and $\mathcal{U}\mathrm{Cat}$ are immediately filtered colimits in Gpd and Cat by Proposition 2.5.

Consequently ι preserves filtered colimits and it follows that $T = \iota R$ also preserves filtered colimits.
Given a category A, TA is the the coinverter of the 2-cell:

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

The unit $\eta_A : A \longrightarrow TA$ is the coinverter morphism which is clearly bijective on objects. We may see this abstractly upon applying Corollary 2.44. Given a bijective on objects functor $f : A \longrightarrow B$ we then have $Tf \circ \eta_A = \eta_B \circ f$. Since $\eta_B \circ f$ is a composite of bijections on objects it is itself bijective on objects. As η_A is bijective on objects it then follows that Tf is too. Therefore T preserves bijections on objects.
On the other hand T is not strongly finitary. Given a discrete category X we have $TX = X$ which implies that the restriction of T along the inclusion : $\mathrm{Set}_f \longrightarrow \mathrm{Cat}$ is simply the inclusion itself. But the identity 2-functor on Cat is the left Kan extension of the inclusion. Alternatively we can see that T is not strongly finitary by observing it does not preserve codescent objects of pointwise discrete categories. Given a category $A \in \mathrm{Cat}$ we have its presentation as a pointwise discrete category:

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] & A_1 \arrow[r, shift left=2, "d_a"] \arrow[r, shift right=2, "c_a"'] & A_0 \arrow[l, "i_a" description] \end{tikzcd}$$

The codescent object of this pointwise discrete category in Cat is precisely A. Unless A is itself a groupoid it will not be the case that $TA \cong A$. On the other hand the above pointwise discrete category in Cat is unaltered by the action of T as each of its components are discrete categories. Therefore taking its image under T and then the codescent object in Cat we do recover A. Consequently T does not preserve codescent objects of pointwise discrete categories and so cannot be strongly finitary.

Chapter 9

Pie Algebras for strongly finitary 2-monads

In this chapter we focus upon strongly finitary 2-monads on $Cat(\mathcal{E})$. In Chapter 4 we introduced a 2-categorical notion of projectivity. We begin this chapter by investigating projectives in T-Alg_{s} and show that the free algebras on discrete internal categories are projective. We introduce the notion of *pie algebra*, a natural type of flexible algebra, and prove that the pie algebras are precisely those covered by the free algebras on discrete internal categories. We consider the 2-category $\text{T-Alg}_{\text{pie}}$, which we show to be biequivalent to T-Alg, and investigate limits therein, introducing the notion of *cone bilimit*. We consider the pie weights of [45] and show that they are precisely the pie algebras for a certain 2-monad. Our characterisation of the pie algebras then gives a new characterisation of the pie weights, which we show to be equivalent to that of [45].

9.1 Codescent morphisms and projectives in T-Alg$_\mathrm{s}$

In Chapter 4 we characterised the projectives in $Cat(\mathcal{E})$ for a category $\mathcal{E}$ with pullbacks as the discrete internal categories. In this section we consider projectives in T-Alg$_\mathrm{s}$ for a strongly finitary 2-monad T on $Cat(\mathcal{E})$. Projectives are defined relative to codescent morphisms and so understanding codescent morphisms in T-Alg$_\mathrm{s}$ is our first goal. We achieve such an understanding by studying the factorisation system on T-Alg$_\mathrm{s}$ obtained by lifting the (Bijective on objects/fully faithful) factorisation system on $Cat(\mathcal{E})$.

Remark 9.1. In this section we will introduce assumptions only as they are required. However we begin by compiling all of the properties of strongly finitary 2-monads which we will have need to call upon.

Proposition 9.2. Let $\mathcal{E}$ be locally finitely presentable and T a strongly finitary 2-monad on $Cat(\mathcal{E})$.

1. T preserves codescent objects of cateads and filtered colimits.
2. T preserves bijections on objects.
3. $Cat(\mathcal{E})$ is both complete and cocomplete.
4. T-Alg$_\mathrm{s}$ is both complete and cocomplete.

Proof. 1. This was proven in Corollary 8.18, in which moreover we showed that every 2-functor which preserves such colimits is strongly finitary.

2. By Corollary 4.21(1) any 2-functor on $Cat(\mathcal{E})$ which preserves codescent objects of cateads preserves bijections on objects. Therefore, by the first part of the present proposition, any strongly finitary 2-monad preserves bijections on objects.

3. By Proposition 8.13 parts 1 and 3.

4. The forgetful 2-functor $U^T : \text{T-Alg}_\mathrm{s} \longrightarrow Cat(\mathcal{E})$ creates all limits that $Cat(\mathcal{E})$ has. Since $Cat(\mathcal{E})$ is complete it follows that T-Alg$_\mathrm{s}$ is also complete, and that $U^T : \text{T-Alg}_\mathrm{s} \longrightarrow Cat(\mathcal{E})$ preserves all limits. Any strongly finitary 2-functor preserves filtered colimits by Corollary 8.18. Since $Cat(\mathcal{E})$ is both complete and cocomplete it follows from Proposition 3.8 of [8] that T-Alg$_\mathrm{s}$ is also cocomplete.

□

Proposition 9.3. Let $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$.

1. Let T be a 2-monad on $Cat(\mathcal{E})$ which preserves bijections on objects. Then we have an orthogonal factorisation system (E, M) on T-Alg$_\mathrm{s}$ in which:
$E = \{f \in \text{T-Alg}_\mathrm{s} : U^T f \text{ is bijective on objects}\}$ and $M = \{f \in \text{T-Alg}_\mathrm{s} : f \text{ is fully faithful}\}$[1].
Furthermore if f is a codescent morphism in T-Alg$_\mathrm{s}$ then $U^T f$ is bijective on objects.
2. Suppose that T preserves codescent objects of cateads. Then T-Alg$_\mathrm{s}$ admits the orthogonal factorisation system of the first part of the present proposition. Furthermore an algebra morphism f is a codescent morphism if and only if $U^T f$ is bijective on objects. Therefore the factorisation system (E, M) on T-Alg$_\mathrm{s}$ becomes:
$E = \{f \in \text{T-Alg}_\mathrm{s} : f \text{ is a codescent morphism}\}$ and $M = \{f \in \text{T-Alg}_\mathrm{s} : f \text{ is fully faithful}\}$.
Furthermore codescent morphisms are effective in T-Alg$_\mathrm{s}$.

Proof. 1. Bijections on objects and fully faithful internal functors form a factorisation system on $Cat(\mathcal{E})$ by Corollary 2.62. Since T preserves bijections on objects we may apply Proposition 6.35 to obtain an (E, M) factorisation system on T-Alg$_\mathrm{s}$ such that $E = \{f \in \text{T-Alg}_\mathrm{s} : U^T f \text{ is bijective on objects}\}$ and $M = \{f \in \text{T-Alg}_\mathrm{s} : U^T f \text{ is fully faithful}\}$. It suffices then to show that a morphism $f : (A, a) \longrightarrow (B, b)$ of T-Alg$_\mathrm{s}$ is fully faithful if and only if $U^T f : A \longrightarrow B$ is so. As $\mathcal{E}$ has pullbacks $Cat(\mathcal{E})$ is an object

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[1] Here fully faithful is used in the representable sense: see Notation 2.28.

of Rep. Since $U^T : \text{T-Alg}_\text{s} \longrightarrow Cat(\mathcal{E})$ creates all limits it follows that $\text{T-Alg}_\text{s} \in \text{Rep}$ and that U^T is a morphism of Rep. By Corollary 3.53(1) U^T preserves fully faithfulness. As U^T also reflects isomorphisms it reflects fully faithfulness by Corollary 3.53(2). Therefore f is fully faithful if and only if $U^T f$ is so.
Let $f : (A,a) \longrightarrow (B,b)$ be a codescent morphism. By Proposition 2.34 codescent morphisms are orthogonal to fully faithful arrows. Since the morphisms of M are precisely the fully faithful algebra morphisms it now follows that $f \in E$ by Proposition 2.37. Thus $U^T f$ is bijective on objects.

2. By Corollary 4.21(1) T preserves bijections on objects. Therefore the results of the first part of the proposition hold so that T-Alg_s admits the factorisation system (E, M) described therein and any codescent morphism f has $U^T f$ bijective on objects.
Conversely consider an algebra morphism $f : (A,a) \longrightarrow (B,b)$ whose underlying map is bijective on objects. We will show that f is the codescent morphism exhibiting its codomain of its higher kernel. As all codescent morphisms are bijective on objects this will also show that codescent morphisms are effective in T-Alg_s.
Now $\mathcal{E}$ has pullbacks and so $Cat(\mathcal{E})$ is a representable 2-category. Therefore T-Alg_s is representable too. In particular it admits comma objects and pullbacks and therefore the construction of higher kernels. Consider the higher kernel of $f : (A,a) \longrightarrow (B,b)$:

$$\begin{tikzcd}
(f|f|f, x_{f|f|f}) \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & (f|f, x_{f|f}) \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & (A,a) \arrow[l, "i" description]
\end{tikzcd}$$

$$\begin{tikzcd}
& (A,a) \arrow[dr, "f"] & \\
(f|f, x_{f|f}) \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & (B,b) \\
& (A,a) \arrow[ur, "f"'] &
\end{tikzcd}$$

with (d, η, c) the triple exhibiting $(f|f, x_{f|f})$ as the comma object of $f : (A,a) \longrightarrow (B,b)$ and $x_{f|f|f} : T(f|f|f) \longrightarrow f|f|f$ and $x_{f|f} : T(f|f) \longrightarrow f|f$ the structure maps for the algebras of the higher kernel.
We must show that the codescent cocone $((B,b), f, \eta)$ exhibits (B,b) as the codescent object of the higher kernel of $f : (A,a) \longrightarrow (B,b)$. As T-Alg_s has cotensors with $\mathbf{2}$ it will suffice, by Proposition 2.5, to verify the 1-dimensional universal property of the codescent cocone. Consider then another codescent cocone $((X,x), g, \theta)$ to the higher kernel of f as below:

$$\begin{tikzcd}
& (A,a) \arrow[dr, "g"] & \\
(f|f, x_{f|f}) \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\theta & (X,x) \\
& (A,a) \arrow[ur, "g"'] &
\end{tikzcd}$$

Now U^T creates those limits that $Cat(\mathcal{E})$ has, in particular pullbacks and comma objects. Consequently U^T preserves higher kernels so that:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description]
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & B \\
& A \arrow[ur, "f"'] &
\end{tikzcd}$$

is the higher kernel of $f : A \longrightarrow B$ in $Cat(\mathcal{E})$. Now f is bijective on objects. Therefore, by Theorem 3.68, the codescent cocone (B, f, η) exhibits B as the codescent object of the higher kernel of $f : A \longrightarrow B$. Certainly the triple (X, g, θ) constitutes a codescent cocone in $Cat(\mathcal{E})$ and so we obtain, by the universal property of the codescent object B, a unique 1-cell $k : B \longrightarrow X$ such that $kf = g$ and $k\eta = \theta$. We claim that k underlies an algebra map $k : (B, b) \longrightarrow (X, x)$: we must verify that the square:

$$(2) \qquad \begin{tikzcd} TB \arrow[r, "Tk"] \arrow[d, "b"'] & TX \arrow[d, "x"] \\ B \arrow[r, "k"'] & X \end{tikzcd}$$

is commutative.

Now the higher kernel of $f : A \longrightarrow B$ is a catead. As T preserves codescent objects of cateads it preserves the codescent object of the higher kernel (1) so that the triple $(TB, Tf, T\eta)$ exhibits TB as the codescent object of the image of (1) under T. Consequently to show that the square (2) is commutative it suffices to show that both paths agree upon precomposition with the 1 and 2-cell Tf and $T\eta$ of the exhibiting cocone.

We have:

$$x \circ Tk \circ Tf = x \circ Tg = g \circ a$$

firstly using that $kf = g$ and secondly that $g : (A, a) \longrightarrow (X, x)$ is an algebra morphism. We have:

$$k \circ b \circ Tf = k \circ f \circ a = g \circ a$$

firstly using that $f : (A, a) \longrightarrow (B, b)$ is an algebra morphism and secondly that $kf = g$. Therefore both paths agree upon precomposition with Tf.

We have:

$$x \circ Tk \circ T\eta = x \circ T\theta = \theta \circ x_{f|f}$$

firstly using that $k\eta = \theta$ and secondly using that $\theta : gd \Longrightarrow gc$ is an algebra 2-cell. We have:

$$k \circ b \circ T\eta = k \circ \eta \circ x_{f|f} = \theta \circ x_{f|f}$$

firstly using that $\eta : fd \Longrightarrow fc$ is an algebra 2-cell and secondly using that $k\eta = \theta$.

Therefore the square (2) commutes and $k : (B, b) \longrightarrow (X, x)$ is an algebra morphism. This gives the required factorisation of the codescent cocone $((X, x), g, \theta)$ through $((B, b), f, \eta)$. That it is the unique such follows from the fact that U^T is faithful on 1 and 2-cells.

Therefore (B, b) is the codescent object of the higher kernel of $f : (A, a) \longrightarrow (B, b)$ with f the exhibiting codescent morphism.

□

Proposition 9.4. Let $\mathcal{E} \in \mathrm{Cat}_{\mathrm{pb}}$ and T a 2-monad on $Cat(\mathcal{E})$ which preserves bijections on objects. If $X \in Cat(\mathcal{E})$ is projective then so is $F^T X \in$ T-$\mathrm{Alg}_{\mathrm{s}}$. Consequently each free algebra on a discrete internal category is projective.

Proof. Suppose that X is projective in $Cat(\mathcal{E})$ and consider a codescent morphism $f : (A, a) \longrightarrow (B, b)$ in T-$\mathrm{Alg}_{\mathrm{s}}$. We must show that given any algebra morphism $g : F^T X \longrightarrow (B, b)$ there exists a unique factorisation:

$$\begin{tikzcd} F^T X \arrow[d, "\exists!"'] \arrow[dr, "g"] & \\ (A, a) \arrow[r, "f"'] & (B, b) \end{tikzcd}$$

rendering the triangle commutative. Transposing across the adjunction:

$$\begin{tikzcd} \text{T-Alg}_{\mathrm{s}} \arrow[r, shift right, "U^T"'] \arrow[r, phantom, "\perp"] & Cat(\mathcal{E}) \arrow[l, shift right, "F^T"'] \end{tikzcd}$$

we see that this is equally to give a factorisation in Cat($\mathcal{E}$):

$$\begin{tikzcd}
X \arrow[d, "\exists!"'] \arrow[dr, "g"] & \\
A \arrow[r, "f"', two heads] & B
\end{tikzcd}$$

Now $f : A \twoheadrightarrow B$ is bijective on objects by Proposition 9.3(1) and so a codescent morphism in $Cat(\mathcal{E})$ by Theorem 3.68. As X is projective a unique such factorisation exists. Transposing back across the adjunction yields the required factorisation.
In particular the projectives in $Cat(\mathcal{E})$ are precisely the discrete internal categories by Proposition 4.17. Therefore if X is a discrete internal category it follows that $F^T X$ is projective in T-Alg$_s$. ☐

Corollary 9.5. Let $\mathcal{E}$ be a locally finitely presentable category and T a strongly finitary 2-monad on $Cat(\mathcal{E})$.

1. The (Bijective on objects/fully faithful) factorisation system on $Cat(\mathcal{E})$ lifts to an orthogonal factorisation system (E, M) on T-Alg$_s$ in which the E's are the codescent morphisms, equally those algebra morphisms whose underlying internal functor is bijective on objects, and the M's the fully faithful algebra morphisms.
2. Codescent morphisms are effective in T-Alg$_s$.
3. Each free algebra on a discrete internal category is projective.

Proof. As $\mathcal{E}$ is locally finitely presentable it is complete and thus has pullbacks. Since T is strongly finitary it preserves both codescent objects of cateads and codescent morphisms by Proposition 9.2. The result follows upon applying Propositions 9.3 and 9.4. ☐

9.2 Pie algebras and a projective cover

Throughout this section we suppose $\mathcal{E}$ to be a locally finitely presentable category and T to be a strongly finitary 2-monad on $Cat(\mathcal{E})$, though Definition 9.7 makes sense so long as T-Alg$_s$ is cocomplete. In Corollary 9.5 we saw that for such T each free algebra on a discrete internal category is projective. In this section we examine the extent to which such algebras form a projective cover of T-Alg$_s$. In other words we ask:

- For which algebras (A, a) does there exist a discrete internal category X and a codescent morphism $F^T X \twoheadrightarrow (A, a)$?

We prove that the algebras covered by the "frees on discretes" are precisely the pie algebras, which we now introduce.

Remark 9.6. Recall the notion of flexible algebra from Chapter 6. Flexible algebras are defined relative to the adjunction:

$$\begin{tikzcd}
\text{T-Alg}_s \arrow[r, bend right, "\iota"'] \arrow[r, phantom, "\perp"] & \text{T-Alg} \arrow[l, bend right, "(-)'"']
\end{tikzcd}$$

which exists, by Proposition 6.8(2), if T-Alg$_s$ is sufficiently cocomplete. This is the case for strongly finitary T, by Proposition 9.2(4). Recall that an algebra (A, a) is said to be flexible if the counit component of the adjunction at (A, a), the strict algebra morphism $p_A : (A, a)' \twoheadrightarrow (A, a)$ has a section in T-Alg$_s$. In Proposition 6.22 we saw that an algebra is flexible precisely if it lies in the closure of the free algebras in T-Alg$_s$ under flexible colimits: pie colimits together with splittings of idempotents. This motivates the following definition.

Definition 9.7. An algebra $(A, a) \in \text{T-Alg}_{\text{s}}$ is said to be a pie algebra if it is contained in the closure of the free algebras under pie colimits (coproducts, coinserters and coequifiers) in T-Alg_{s}. $\text{T-Alg}_{\text{pie}}$ is the full sub 2-category of T-Alg_{s} with objects the pie algebras (noting that the morphisms are the strict ones). Furthermore we denote by $\text{T-Alg}_{\text{flex}}$ the full subcategory of T-Alg_{s} with objects the flexible algebras.

Example 9.8. Each free algebra is a pie algebra, by definition.

Example 9.9. Each algebra of the form $(A, a)'$ is a pie algebra by Proposition 6.21.

Remark 9.10. The flexible algebras are the closure of the free algebras under flexible colimits. As pie colimits are a subclass of flexible colimits it follows that each pie algebra is flexible and we have an inclusion $\text{T-Alg}_{\text{pie}} \longrightarrow \text{T-Alg}_{\text{flex}}$. The following proposition shows that, up to biequivalence, the 2-categories $\text{T-Alg}_{\text{pie}}$ and $\text{T-Alg}_{\text{flex}}$ are indistinguishable from T-Alg.

Proposition 9.11. Let $\mathcal{A}$ be a full sub 2-category of T-Alg_{s} containing each algebra of the form $(A, a)'$ and suppose further that each object of $\mathcal{A}$ is a flexible algebra. Then the inclusion $\mathcal{A} \longrightarrow \text{T-Alg}$ is a biequivalence. In particular both inclusions $\iota : \text{T-Alg}_{\text{flex}}, \text{T-Alg}_{\text{pie}} \longrightarrow \text{T-Alg}$ are biequivalences.

Proof. The inclusion $\mathcal{A} \longrightarrow \text{T-Alg}$ is the composite inclusion $\mathcal{A} \longrightarrow \text{T-Alg}_{\text{s}} \longrightarrow \text{T-Alg}$. We must show that this composite is biessentially surjective on objects (surjective up to equivalence) and locally an equivalence. For each algebra (A, a) the counit $p_a : (A, a)' \longrightarrow (A, a)$ is a surjective equivalence in T-Alg by Proposition 6.12. Since $(A, a)' \in \mathcal{A}$ the inclusion is therefore biessentially surjective.
Now given a pair of algebras (A, a) and (B, b) of $\mathcal{A}$ consider the inclusion $\mathcal{A}((A, a), (B, b)) \longrightarrow \text{T-Alg}(A, a), (B, b))$. This is certainly fully faithful, since the inclusions $\mathcal{A} \longrightarrow \text{T-Alg}_{\text{s}}$ and $\text{T-Alg}_{\text{s}} \longrightarrow \text{T-Alg}$ are both locally fully faithful. Thus we need only show it is essentially surjective. In other words we must show that given a pseudomorphism $(f, \overline{f}) : (A, a) \longrightarrow (B, b)$ there exists a strict morphism $g : (A, a) \longrightarrow (B, b)$ isomorphic to it, in $\text{T-Alg}((A, a), (B, b))$. As (A, a) is flexible this follows from Proposition 6.16.
As each object of $\text{T-Alg}_{\text{flex}}$ and $\text{T-Alg}_{\text{pie}}$ is flexible, and both 2-categories contain each algebra of the form $(A, a)'$ it now follows that both inclusions $\text{T-Alg}_{\text{flex}}, \text{T-Alg}_{\text{pie}} \longrightarrow \text{T-Alg}$ are biequivalences. ☐

Remark 9.12. Our aim is now to show that an algebra $(A, a) \in \text{T-Alg}_{\text{pie}}$ if and only if there exists a discrete internal category X and a codescent morphism $F^T X \longrightarrow (A, a)$.

Proposition 9.13. $\text{T-Alg}_{\text{pie}}$ is equally the closure of the free algebras on discrete internal categories under pie colimits.

Proof. It suffices to show that each free algebra is a pie-colimit of free algebras on discrete internal categories. By Example 4.19 each object $A \in Cat(\mathcal{E})$ is the codescent object of its canonical presentation via discrete internal categories:

$$\begin{tikzcd}
{[A_2]} \arrow[r, shift left=2, "{[p_a]}"] \arrow[r, "{[m_a]}" description] \arrow[r, shift right=2, "{[q_a]}"'] & {[A_1]} \arrow[r, shift left=2, "{[d_a]}"] \arrow[r, shift right=2, "{[c_a]}"'] & {[A_0]} \arrow[l, "{[i_a]}" description] \arrow[r, "\epsilon_a"] & A
\end{tikzcd}$$

with ϵ_a the exhibiting codescent morphism. As $F^T : Cat(\mathcal{E}) \longrightarrow \text{T-Alg}_{\text{s}}$ is a left 2-adjoint it preserves all colimits. Thus:

$$\begin{tikzcd}
F^T[A_2] \arrow[r, shift left=2, "{F^T[p_a]}"] \arrow[r, "{F^T[m_a]}" description] \arrow[r, shift right=2, "{F^T[q_a]}"'] & F^T[A_1] \arrow[r, shift left=2, "{F^T[d_a]}"] \arrow[r, shift right=2, "{F^T[c_a]}"'] & F^T[A_0] \arrow[l, "{F^T[i_a]}" description] \arrow[r, "F^T\epsilon_a"] & F^T A
\end{tikzcd}$$

exhibits $F^T A$ as a codescent object in T-Alg_{s} of strict coherence data each component of which is a free algebra on a discrete internal category. Codescent objects may be formed by coinserters and coequifiers, as described in Remark 2.18, and so the result follows. ☐

Proposition 9.14. Given a pie algebra (A, a) there exists a discrete internal category X and a codescent morphism $F^T X \longrightarrow (A, a) \in \text{T-Alg}_{\text{s}}$.

Proof. Let Cov denote the full sub 2-category of T-Alg_{s} with objects those algebras (A, a) for which there exists a discrete internal category X and codescent morphism $F^T X \longrightarrow (A, a)$. We must show that each pie algebra belongs to Cov. As the 2-category $\text{T-Alg}_{\text{pie}}$ is by definition the closure of the free algebras under pie-colimits it will suffice to verify that each free algebra belongs to Cov and that Cov is closed in T-Alg_{s} under pie-colimits: coproducts, coinserters and coequifiers.

For each free algebra $F^T A$ we have the codescent morphism $F^T(\epsilon_a) : F^T[A_0] \longrightarrow F^T A$ of Proposition 9.13. As $[A_0]$ is discrete therefore $F^T A \in \text{Cov}$.

Consider a pair of algebras (A, a) and (B, b) of Cov and a parallel pair of 2-cells:

$$\begin{tikzcd}
(A,a) \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & (B,b)
\arrow[from=U, to=D, phantom, "\theta\Downarrow\ \Downarrow\phi"]
\end{tikzcd}$$

with coequifier $h : (B, b) \longrightarrow (C, c)$ in T-Alg_{s}. Coequifier morphisms are orthogonal to fully faithful morphisms by Proposition 2.34. Codescent morphisms and fully faithful morphisms form a factorisation system (E, M) on T-Alg_{s} by Corollary 9.5. Therefore by Proposition 2.37 the coequifier morphism is a codescent morphism. By assumption there exists a codescent morphism $\alpha_b : F^T X \longrightarrow (B, b)$ with X a discrete internal category. Composing these codescent morphisms:

$$\begin{tikzcd}
F^T X \arrow[r, "\alpha_b"] & (B,b) \arrow[r, "h"] & (C,c)
\end{tikzcd}$$

yields another, since in any orthogonal factorisation system the class E is closed under composition. Therefore $(C, c) \in \text{Cov}$.

The case of coinserters is similar. For suppose we are given a parallel pair of algebra morphisms:

$$\begin{tikzcd}
(A,a) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (B,b)
\end{tikzcd}$$

with both (A, a) and (B, b) objects of Cov. Now form the coinserter (C, c) with exhibiting cocone:

$$\begin{tikzcd}
 & (B,b) \arrow[dr, "j"] & \\
(A,a) \arrow[ur, "f"] \arrow[dr, "g"'] & \eta\Downarrow & (C,c) \\
 & (B,b) \arrow[ur, "j"'] &
\end{tikzcd}$$

By Proposition 2.34 coinserter morphisms are orthogonal to fully faithful ones. Consequently it follows as before that $j : (B, b) \longrightarrow (C, c)$ is a codescent morphism. As $(B, b) \in \text{Cov}$ we have a codescent morphism $\alpha_b : F^T X \longrightarrow (B, b)$ with X a discrete internal category and the composite codescent morphism:

$$\begin{tikzcd}
F^T X \arrow[r, "\alpha_b"] & (B,b) \arrow[r, "j"] & (C,c)
\end{tikzcd}$$

proves that $(C, c) \in \text{Cov}$.

With regards the case of coproducts suppose we are given a family of algebras $(C_i, c_i) \in \text{Cov}$ indexed by some set I. Each is equipped with a codescent morphism:

$$\begin{tikzcd}
F^T X_i \arrow[r, "\alpha_i"] & (C_i, c_i)
\end{tikzcd}$$

with X_i a discrete internal category. Taking the I-indexed coproduct in T-Alg_{s} we obtain a map:

$$\begin{tikzcd}
\sum\limits_{i\in I} F^T X_i \arrow[r, "\sum_{i\in I}\alpha_i"] & \sum\limits_{i\in I} (C_i, c_i)
\end{tikzcd}$$

As each α_i is a codescent morphism it follows that their sum is one too; for coproducts commute with all colimits, in particular codescent objects. Now $\sum_{i \in I} F^T X_i \cong F^T(\sum_{i \in I} X_i)$ as F^T preserves coproducts. Thus we have the composite:

$$F^T(\sum_{i \in I} X_i) \cong \sum_{i \in I} F^T X_i \xrightarrow{\sum_{i \in I} \alpha_i} \sum_{i \in I} (C_i, c_i)$$

which is again a codescent morphism since the class E is closed under isomorphisms and composition. It remains to show then that the internal category $\sum_{i \in I} X_i$ is itself discrete. By assumption we have $X_i \in Disc(Cat(\mathcal{E}))$ for each $i \in I$ and so it suffices to show that the full sub 2-category $Disc(Cat(\mathcal{E}))$ is closed in $Cat(\mathcal{E})$ under coproducts. Combining Propositions 4.16(3) and 4.17 we see that the inclusion $\iota : Disc(Cat(\mathcal{E})) \longrightarrow UCat(\mathcal{E})$ has a right adjoint and therefore preserves all 1-dimensional colimits, in particular coproducts. The locally discrete 2-category $Disc(Cat(\mathcal{E}))$ trivially has cotensors with **2**, thus the coproducts are 2-dimensional coproducts by Proposition 2.5. As $Cat(\mathcal{E})$ itself has cotensors with **2** any 1-dimensional coproducts in $Cat(\mathcal{E})$ are 2-dimensional coproducts by the same proposition, and it follows that $\iota : Disc(Cat(\mathcal{E})) \longrightarrow Cat(\mathcal{E})$ preserves coproducts. Therefore $Disc(Cat(\mathcal{E}))$ is closed in $Cat(\mathcal{E})$ under coproducts as required. Therefore $\sum_{i \in I}(C_i, c_i) \in$ Cov. ☐

Remark 9.15. In order to prove the converse to Proposition 9.14 we will require the following lemma.

Lemma 9.16. Consider the adjunction:

$$\begin{tikzcd} \text{T-Alg}_\text{s} \arrow[r, bend right, "\iota"'] \arrow[r, phantom, "\perp"] & \text{T-Alg} \arrow[l, bend right, "(-)'"'] \end{tikzcd}$$

The counit at an algebra (A, a), $p_A : (A, a)' \longrightarrow (A, a)$ is both fully faithful and co-fully faithful in T-Alg$_\text{s}$.

Proof. As T-Alg$_\text{s}$ is complete and cocomplete it follows from Proposition 6.12 that p_A is a surjective equivalence in T-Alg. Therefore it is both fully faithful and co-fully faithful in T-Alg. We must show that it is both fully faithful and co-fully faithful in T-Alg$_\text{s}$. For suppose that we are given a pair of strict algebra maps $f_1, f_2 : (A, a) \longrightarrow (B, b)$ and a 2-cell:

$$\begin{tikzcd} (A,a)' \arrow[r, bend left, "f_1 p_A"] \arrow[r, bend right, "f_2 p_A"'] \arrow[r, phantom, "\Downarrow\phi"] & (B,b) \end{tikzcd}$$

As p_A is co-fully faithful in T-Alg there exists a unique 2-cell in T-Alg:

$$\begin{tikzcd} (A,a) \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\theta"] & (B,b) \end{tikzcd}$$

such that precomposing this 2-cell with p_A yields ϕ. But the inclusion $\iota :$ T-Alg$_\text{s} \longrightarrow$ T-Alg is locally fully faithful, thus θ is a 2-cell in T-Alg$_\text{s}$ and indeed the unique such. Consequently p_A is co-fully faithful in T-Alg$_\text{s}$. Similarly p_A is fully faithful in T-Alg$_\text{s}$ since it is fully faithful in T-Alg. ☐

Proposition 9.17. Consider a pie algebra (A, a) and suppose that $f : (A, a) \longrightarrow (B, b)$ is a codescent morphism in T-Alg$_\text{s}$. Then (B, b) is a pie algebra and $f : (A, a) \longrightarrow (B, b)$ is a codescent morphism in T-Alg$_\text{pie}$.

Proof. T-$\mathrm{Alg}_{\mathrm{s}}$ is complete so we may consider the higher kernel of $f : (A,a) \longrightarrow (B,b)$:

$$\begin{tikzcd}
(f|f|f) \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & (f|f) \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & (A,a) \arrow[l, "i" description] \arrow[r, "f"] & (B,b)
\end{tikzcd}$$

omitting to label the structure maps for the algebras $f|f|f$ and $f|f$. The codescent morphism $f : (A,a) \longrightarrow (B,b)$ necessarily exhibits (B,b) as the codescent object of its higher kernel (by Corollary 9.5(2)).
We consider the image of this coherence data under $(-)' :$ T-$\mathrm{Alg}_{\mathrm{s}} \longrightarrow$ T-$\mathrm{Alg}_{\mathrm{s}}$. The counit p is 2-natural so that we obtain a natural transformation of coherence data:

$$\begin{tikzcd}
(f|f|f) \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & (f|f) \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & (A,a) \arrow[l, "i" description] \\
(f|f|f)' \arrow[u, "p_{f|f|f}"] \arrow[r, shift left=2, "p'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "q'"'] & (f|f)' \arrow[u, "p_{f|f}"] \arrow[r, shift left=2, "d'"] \arrow[r, shift right=2, "c'"'] & (A,a)' \arrow[u, "p_A"'] \arrow[l, "i'" description]
\end{tikzcd} \tag{1}$$

where the codescent object of the top row is (B,b). Each pie algebra is flexible by Remark 9.10 and so the counit component $p_A : (A,a)' \longrightarrow (A,a)$ has a section $r : (A,a) \longrightarrow (A,a)' \in$ T-$\mathrm{Alg}_{\mathrm{s}}$. We "augment" the bottom row by:

$$\begin{tikzcd}
(A,a)' \arrow[r, shift left=1, "p_A"] & (A,a) \arrow[l, shift left=1, "r"]
\end{tikzcd}$$

to obtain new strict coherence data:

$$\begin{tikzcd}
(f|f|f)' \arrow[r, shift left=2, "p'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "q'"'] & (f|f)' \arrow[r, shift left=2, "p_A \circ d'"] \arrow[r, shift right=2, "p_A \circ c'"'] & (A,a) \arrow[l, "i' \circ r" description]
\end{tikzcd}$$

To see this constitutes strict coherence data we must verify the equations:

- $p_A \circ d' \circ p' = p_A \circ d' \circ m'$
- $p_A \circ c' \circ q' = p_A \circ c' \circ m'$
- $p_A \circ c' \circ p' = p_A \circ d' \circ q'$
- $p_A \circ d' \circ i' \circ r = 1$
- $p_A \circ c' \circ i' \circ r = 1$

The first three equations following immediately from the corresponding equations for the lower row of (1). Regarding the fourth equation we have

$$p_A \circ d' \circ i' \circ r = d \circ p_{f|f} \circ i' \circ r = d \circ i \circ p_A \circ r = 1 \circ p_A \circ r = 1$$

The first two equalities here use the naturality of $p : (-)' \Longrightarrow 1$. The third equality uses that $d \circ i = 1$ and the fourth that r is a section of p_A.
Verification of the fifth equation is similar to that of the fourth; now using that $c \circ i = 1$ as opposed to $d \circ i = 1$. The diagram:

$$\begin{tikzcd}
(f|f|f) \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & (f|f) \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & (A,a) \arrow[l, "i" description] \\
(f|f|f)' \arrow[u, "p_{f|f|f}"] \arrow[r, shift left=2, "p'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "q'"'] & (f|f)' \arrow[u, "p_{f|f}"] \arrow[r, shift left=2, "p_A \circ d'"] \arrow[r, shift right=2, "p_A \circ c'"'] & (A,a) \arrow[u, "1"'] \arrow[l, "i' \circ r" description]
\end{tikzcd}$$

now constitutes a natural transformation of strict coherence data. This clearly follows from the natural transformation of coherence data (1) with the exception that we must verify the equation $p_{f|f} \circ i' \circ r = i \circ 1$. We have $p_{f|f} \circ i' \circ r = i \circ p_A \circ r = i$ first using naturality of $p : (-)' \Longrightarrow 1$ and then that r is a section.
Both of $p_{f|f|f}$ and $p_{f|f}$ are co-fully faithful by Lemma 9.16 whilst we have the identity on (A, a) on the right hand side. Thus by Lemma 8.40 both rows have the same codescent object (B, b) with exhibiting codescent morphism $f : (A, a) \twoheadrightarrow (B, b)$. Each of $(f|f|f)'$ and $(f|f)'$ are pie algebras by Example 9.9 so that the lower row lies entirely in T-Alg$_{\text{pie}}$. Now T-Alg$_{\text{pie}}$ is closed under codescent objects as they may be formed using coinserters and coequifiers. Thus (B, b) also a pie algebra. □

Theorem 9.18. An algebra $(A, a) \in$ T-Alg$_{\text{s}}$ is a pie-algebra if and only if there exists a discrete internal category X and a codescent morphism $F^T X \twoheadrightarrow (A, a)$ in T-Alg$_{\text{s}}$.

Proof. We proved the "only if" part of this statement in Proposition 9.14.
Conversely suppose that there exists a codescent morphism $F^T X \twoheadrightarrow (A, a) \in$ T-Alg$_{\text{s}}$. Each free algebra is a pie algebra by definition so $F^T X \in$ T-Alg$_{\text{pie}}$. Applying Proposition 9.17 to this codescent morphism shows that (A, a) is also a pie algebra. □

Remark 9.19. For strongly finitary T we have seen that the codescent morphisms in T-Alg$_{\text{s}}$ are those whose underlying internal functor is bijective on objects. The characterisation of Theorem 9.18 then asserts that an algebra (A, a) is a pie algebra precisely if there exists a discrete internal category X and a bijective on objects algebra homomorphism $F^T X \twoheadrightarrow (A, a)$. Thus the pie algebras are those which are free at the level of objects.

Corollary 9.20. The free algebras on discrete internal categories form a projective cover of T-Alg$_{\text{pie}}$.

Proof. We have seen that the free algebras on discretes are projective in T-Alg$_{\text{s}}$ but should be careful and verify that they are projective in T-Alg$_{\text{pie}}$. The inclusion of T-Alg$_{\text{pie}}$ into T-Alg$_{\text{s}}$ preserve pie-colimits and so preserves codescent objects. Therefore each codescent morphism in T-Alg$_{\text{pie}}$ is a codescent morphism in T-Alg$_{\text{s}}$. As T-Alg$_{\text{pie}}$ is a full sub 2-category of T-Alg$_{\text{s}}$ the projectivity, in T-Alg$_{\text{pie}}$, of the free algebras on discretes may be deduced from their projectivity in T-Alg$_{\text{s}}$.
Now for each pie algebra (A, a) we have a codescent morphism $F^T X \twoheadrightarrow (A, a)$ in T-Alg$_{\text{s}}$ with X a discrete internal category. Since $F^T X$ is a pie algebra we may apply Proposition 9.17 to show that this a codescent morphism in T-Alg$_{\text{pie}}$. Thus the frees on discretes form a projective cover of T-Alg$_{\text{pie}}$. □

9.3 Limits in T-Alg$_{\text{pie}}$ and T-Alg$_{\text{flex}}$

In this section we consider limits in T-Alg$_{\text{pie}}$ and T-Alg$_{\text{flex}}$. These results do not require T to be strongly finitary but only that T-Alg$_{\text{s}}$ be complete and cocomplete, which, by Proposition 3.8 of [8], is the case so long as T is a 2-monad with rank on a complete and cocomplete 2-category. Of course these assumptions hold when T is strongly finitary.

Remark 9.21. By definition T-Alg$_{\text{pie}}$ is closed under pie colimits in T-Alg$_{\text{s}}$. As described in [8] T-Alg admits pie limits and therefore by Proposition 2.7 all bilimits. By Proposition 9.11 T-Alg$_{\text{pie}}$ is biequivalent to T-Alg and therefore admits all bilimits too, but in fact admits more than this.

Definition 9.22. Let $\mathcal{A}$ be a 2-category, $W : \mathcal{J} \twoheadrightarrow \text{Cat}$ a weight and $F : \mathcal{J} \twoheadrightarrow \mathcal{A}$ a diagram. Let $A \in \mathcal{A}$ and consider a cone $\eta : W \twoheadrightarrow \mathcal{A}(A, F-)$ which in turn induces a 2-natural transformation:

$$\mathcal{A}(-, A) \twoheadrightarrow [\mathcal{J}, \text{Cat}](W, \mathcal{A}(-, F-))$$

We say that η exhibits A as the cone bilimit of W weighted by F if and only for each $B \in \mathcal{A}$ the component:

$$\mathcal{A}(B, A) \twoheadrightarrow [\mathcal{J}, \text{Cat}](W, \mathcal{A}(B, F-))$$

is a surjective equivalence of categories.
Given $F : \mathcal{J}^{op} \longrightarrow \mathrm{Cat}$ and a cocone $\eta : W \longrightarrow \mathcal{A}(F-, A)$ we say that this cocone exhibits A as the cone bicolimit if the induced 2-natural transformation $\mathcal{A}(A, -) \longrightarrow [\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(F-, -))$ has each component a surjective equivalence.

Remark 9.23. The bilimit of F weighted by W is defined by a pseudonatural equivalence $\mathcal{A}(-, A) \simeq Ps(\mathcal{J}, \mathrm{Cat})(W, \mathcal{A}(-, F-))$. Thus, in particular, the "unit" $W \longrightarrow \mathcal{A}(A, F-)$ is only a pseudonatural transformation, a pseudo cone. Furthermore the universal property of the unit is that any other pseudo cone $W \longrightarrow \mathcal{A}(B, F-)$ factors through it via some arrow $B \longrightarrow A$, but only up to isomorphism. This fact corresponds to the equivalence of categories $\mathcal{A}(B, A) \simeq Ps(\mathcal{J}, \mathrm{Cat})(W, \mathcal{A}(B, F-))$. In the case of the cone bilimit we have an actual cone $W \longrightarrow \mathcal{A}(A, F-)$ and, corresponding to the surjective equivalence $\mathcal{A}(B, A) \longrightarrow [\mathcal{J}, \mathrm{Cat}](W, \mathcal{A}(B, F-))$, any other cone $W \longrightarrow \mathcal{A}(B, F-)$ factors through it exactly via some arrow $B \longrightarrow A$, not merely up to isomorphism. The main distinction between the cone bilimit, and the actual weighted limit, is that in the former case the factorisation need not be unique, but only unique up to isomorphism in $\mathcal{A}(B, A)$.

Proposition 9.24. Both $\text{T-Alg}_{\text{flex}}$ and $\text{T-Alg}_{\text{pie}}$ admit all cone bilimits.

Proof. In the following proof we denote each algebra (A, a) simply as A, the structure map $a : TA \longrightarrow A$ not being required. We will prove firstly that $\text{T-Alg}_{\text{flex}}$ has all cone bilimits. Given a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a diagram $F : \mathcal{J} \longrightarrow \text{T-Alg}_{\text{flex}}$ we must show the cone bilimit exists in $\text{T-Alg}_{\text{flex}}$. Firstly form the weighted limit A in T-Alg_{s} which comes equipped with a universal cone $\eta : W \longrightarrow \text{T-Alg}_{\text{s}}(A, F-)$. The algebra A' is of course a pie algebra by Example 9.9, and is in particular flexible. The counit map $p_A : A' \longrightarrow A$ induces a cone in $\text{T-Alg}_{\text{flex}}$:

$$W \xrightarrow{\eta} \text{T-Alg}_{\text{s}}(A, F-) \xrightarrow{p_A^*} \text{T-Alg}_{\text{flex}}(A', F-)$$

We claim that this cone exhibits A' as the cone bilimit in $\text{T-Alg}_{\text{flex}}$. We must show that the induced map:

$$\text{T-Alg}_{\text{flex}}(B, A') \xrightarrow{p_A^*} \text{T-Alg}_{\text{s}}(B, A) \xrightarrow{\eta^*} [\mathcal{J}, \mathrm{Cat}](W, \text{T-Alg}_{\text{flex}}(B, F-))$$

is 2-natural in B and a surjective equivalence for each flexible algebra B. Naturality is clear.
The map $\eta^* : \text{T-Alg}_{\text{s}}(B, A) \longrightarrow [\mathcal{J}, Cat](W, \text{T-Alg}_{\text{flex}}(B, F-))$ is an isomorphism as A is the limit in T-Alg_{s}. By Lemma 9.16 $p_A : A' \longrightarrow A$ is fully faithful and so the induced map $p_A^* : \text{T-Alg}_{\text{flex}}(B, A') \longrightarrow \text{T-Alg}_{\text{s}}(B, A)$ is also fully faithful. Thus the composite is fully faithful, being the composite of a fully faithful functor and an isomorphism of categories.
In order to show the map is a surjective equivalence it remains to show it is surjective on objects. Suppose then that we are given a cone $\theta : W \longrightarrow \text{T-Alg}_{\text{flex}}(B, F-) = \text{T-Alg}_{\text{s}}(B, F-)$. Then we have a unique 1-cell $g : B \longrightarrow A$ to the limit in T-Alg_{s} such that the triangle on the left below commutes:

$$\begin{tikzcd}
W \arrow[r, "\eta"] \arrow[dr, "\theta"'] & \text{T-Alg}_{\text{s}}(A, F-) \arrow[r, "p_A^*"] \arrow[d, "g^*"] & \text{T-Alg}_{\text{flex}}(A', F-) \arrow[dl, "h^*"] \\
& \text{T-Alg}_{\text{flex}}(B, F-) &
\end{tikzcd}$$

It remains to find a 1-cell $h : B \longrightarrow A'$ such that the triangle on the right commutes. Now B is flexible so that we have a section $r : B \longrightarrow B'$ of $p_B : B' \longrightarrow B$. We claim that the composite:

$$B \xrightarrow{r} B' \xrightarrow{g'} A'$$

is the required map.
It suffices to check that:

$$B \xrightarrow{r} B' \xrightarrow{g'} A' \xrightarrow{p_A} A \quad = \quad B \xrightarrow{g} A$$

We have $p_A \circ g' \circ r = g \circ p_B \circ r = g$ first using naturality of p, and then using that r is a section of p_B. Consequently T-Alg$_{\text{flex}}$ has all cone bilimits.
It is clear that T-Alg$_{\text{pie}}$ is closed in T-Alg$_{\text{flex}}$ under cone bilimits as T-Alg$_{\text{pie}}$ contains each algebra of the form A' and each pie algebra is flexible. Thus T-Alg$_{\text{pie}}$ has all cone bilimits and of course these are preserved by the inclusion into T-Alg$_{\text{flex}}$. ☐

Remark 9.25. In fact T-Alg admits all cone bicolimits. This is the essential content of Remarks 5.10 and 5.11 of [8].

9.4 Pie algebras and pie weights

In this section we begin by recalling the "closure of a class of weights" of Albert and Kelly [3] and the corresponding notion of a pie weight. We describe the motivation behind Kelly and coauthors' question in [7] as to whether an explicit characterisation could be found for those pie weights, and discuss the resulting paper of Power and Robinson [45] which answered that question affirmatively. We show that the pie weights are the pie algebras for a strongly finitary 2-monad and apply our characterisation of pie algebras to give an alternative characterisation of the pie weights. We describe how this is equivalent to the characterisation of Power and Robinson thereby giving another proof of their characterisation.

Definition 9.26. Given a class Φ of weights its closure Φ^* is a class of weights indexed by the small 2-categories. Given a small 2-category $\mathcal{J}$, the 2-category $\Phi^*[\mathcal{J}]$ is the closure of the representables in $[\mathcal{J}, \mathrm{Cat}]$ under Φ-colimits. For a $\mathcal{J}$-indexed weight W we say $W \in \Phi^*$ if $W \in \Phi^*[\mathcal{J}]$.

Proposition 9.27 (Albert and Kelly)**.** Let Φ be a class of weights and $\mathcal{A}$ a 2-category with Φ-limits. Then $\mathcal{A}$ has Φ^*-limits. If $U : \mathcal{A} \longrightarrow \mathcal{B}$ is a 2-functor preserving Φ-limits then U preserves all Φ^* limits.

Remark 9.28. Letting Φ consist of the weights for products, inserters and equifiers Φ^* is the class of *pie weights*. As described in [8] given a 2-monad T on a complete 2-category $\mathcal{A}$ the 2-category T-Alg has pie limits and the forgetful 2-functor U : T-Alg $\longrightarrow \mathcal{A}$ preserves them. Motivated by this fact and Proposition 9.27 above, an explicit description of the pie-weights was asked for in [7]. Such an explicit characterisation would enable one to determine from an arbitrary weight W whether T-Alg has W-limits preserved by the forgetful 2-functor. The paper [45] of Power and Robinson was devoted to answering that question. Their result is the following.

Theorem 9.29 (Power and Robinson)**.** Let $\mathcal{J}$ be a small 2-category and consider a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$. Now consider the underlying functor $W : \mathcal{U}\mathcal{J} \longrightarrow \mathcal{U}\mathrm{Cat}$ and the composite:

$$\mathcal{U}\mathcal{J} \xrightarrow{\mathcal{U}W} \mathcal{U}\mathrm{Cat} \xrightarrow{ob} \mathrm{Set}$$

W is a pie weight if and only if each connected component of the category of elements $el(ob \circ \mathcal{U}W)$ has an initial object.

Remark 9.30. To describe the connection with pie algebras we will begin by describing $[\mathcal{J}, \mathrm{Cat}]$ as the 2-category of algebras for a strongly finitary 2-monad. Let ob$\mathcal{J}$ denote the set of objects of the 2-category $\mathcal{J}$ and $\iota : ob\mathcal{J} \longrightarrow \mathcal{J}$ the evident inclusion. Restriction along ι induces a 2-functor $R : [\mathcal{J}, \mathrm{Cat}] \longrightarrow [ob\mathcal{J}, \mathrm{Cat}]$ which has both adjoints given by left and right Kan extension:

$$\begin{tikzcd}
{[\mathcal{J}, \mathrm{Cat}]} \arrow[r, "R" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & {[ob\mathcal{J}, \mathrm{Cat}]} \arrow[l, shift right=4, "L"'] \arrow[l, shift left=4] \arrow[r, shift right=1.5, "I^{-1}"'] \arrow[r, phantom, "\perp"] & Cat([ob\mathcal{J}, \mathrm{Set}]) \arrow[l, shift right=1.5, "I"']
\end{tikzcd}$$

Consequently the restriction 2-functor R preserves both limits and colimits. In fact since limits and colimits are pointwise in $[\mathcal{J}, \mathrm{Cat}]$ R actually creates limits and colimits. We have the isomorphism of 2-categories $I : Cat(ob\mathcal{J}, \mathrm{Set}) \cong [ob\mathcal{J}, \mathrm{Cat}]$ and furthermore the category $[ob\mathcal{J}, \mathrm{Set}]$ is locally finitely presentable as $ob\mathcal{J}$ is small. Therefore the 2-monad $T = I^{-1}RLI$ on $Cat(ob\mathcal{J}, \mathrm{Set})$ creates all colimits. In particular it preserves codescent objects of cateads and filtered colimits and as such is strongly finitary by Theorem 8.16(1).
Furthermore as $\iota : ob\mathcal{J} \longrightarrow \mathcal{J}$ is bijective on objects R reflects isomorphisms. Therefore by Beck's theorem $I^{-1}R$ is strictly monadic and $[\mathcal{J}, \mathrm{Cat}] \cong \text{T-Alg}_{\text{s}}$.
The left Kan extension 2-functor L admits a simple explicit description. Given a $\mathcal{J}$-indexed family $\Omega : ob\mathcal{J} \longrightarrow \mathrm{Cat}$ its left Kan extension along ι, $L(\Omega)$, acts on an object a of $\mathcal{J}$ by $L(\Omega)(a) = \sum_{x \in \mathcal{J}} \mathcal{J}(x, a) \times \Omega(x)$.

Proposition 9.31. Consider the strongly finitary 2-monad T on $Cat([ob\mathcal{J}, \mathrm{Set}])$ described in Remark 9.30 with $[\mathcal{J}, \mathrm{Cat}] \cong \text{T-Alg}_{\text{s}}$. Each representable in $[J, \mathrm{Cat}]$ is a free algebra, and each free algebra on a discrete internal category is a coproduct of representables.

Proof. It suffices to show that each representable is in the image of the left 2-adjoint LI and that for each discrete internal category X of $Cat([ob\mathcal{J}, \mathrm{Set}])$ the 2-functor $LI(X)$ is a coproduct of representables.
Given $c \in \mathcal{J}$ we must show that the representable $\mathcal{J}(c, -)$ is free. Consider the characteristic family $\chi_c : ob\mathcal{J} \longrightarrow \mathrm{Cat}$ defined by $\{\chi_c(d) = \emptyset \text{ if } c \neq d \text{ and } \chi_c(c) = 1\}$. Then $L(\chi_c)(a) = \sum_x \mathcal{J}(x, a) \times \chi_c(a) = \mathcal{J}(c, a)$ and thus we see $L(\chi_c) = \mathcal{J}(c, -)$.[2] Therefore each representable is free.
A discrete object of $Cat[ob\mathcal{J}, \mathrm{Set}]$ becomes identified with the corresponding $\mathcal{J}$-indexed family of discrete categories under the isomorphism I: a family $\Omega : ob\mathcal{J} \longrightarrow \mathrm{Cat}$ such that each $\Omega(c)$ a discrete category. It suffices to show then that for each such Ω the 2-functor $L(\Omega)$ is a coproduct of representables. Each such Ω may be described as a coproduct of characteristic families: we have $\Omega(c) = \Omega(c).\chi_c(c)$ for each $c \in \mathcal{J}$, so that $\Omega = \sum_c \Omega(c).\chi_c$. Using the fact that L preserves colimits we then have $L(\Omega) = L(\sum_c \Omega(c).\chi_c) = \sum_c L(\Omega(c).\chi_c) = \sum_c \Omega(c).L(\chi_c) = \sum_c \Omega(c).J(c, -)$, a coproduct of representables. ☐

Corollary 9.32. The pie weights of $[\mathcal{J}, \mathrm{Cat}]$ are equally the pie algebras for T.

Proof. As each representable is free the closure of the representables under coproducts, coinserters and coequifiers is a full sub 2-category of $\text{T-Alg}_{\text{pie}}$. By Proposition 9.13 $\text{T-Alg}_{\text{pie}}$ is equally the closure of the free algebras on discretes under coproducts, coinserters and coequifiers. By Proposition 9.31, each free algebra on a discrete category is a coproduct of representables. Therefore we see that the closure of the representables under coproducts, coinserters and coequifiers equally contains $\text{T-Alg}_{\text{pie}}$. Consequently a weight is pie if and only if it is a pie algebra. ☐

Corollary 9.33. $W : J \longrightarrow \mathrm{Cat}$ is a pie weight if and only if there exists a family of discrete categories $X : ob\mathcal{J} \longrightarrow \mathrm{Cat}$ and a 2-natural transformation $L(X) \Longrightarrow W$ which is pointwise bijective on objects.

Proof. Theorem 9.18 asserts that an algebra $(A, a) \in \text{T-Alg}_{\text{s}}$ is a pie algebra if and only if there exists a discrete internal category $X \in Cat([ob\mathcal{J}, \mathrm{Set}])$ and a codescent morphism $F^T X \longrightarrow (A, a)$. Interpreting this fact across the isomorphism of 2-categories $[\mathcal{J}, \mathrm{Cat}] \cong \text{T-Alg}_{\text{s}}$ gives the result, as the codescent morphisms in $[\mathcal{J}, \mathrm{Cat}]$ are precisely the pointwise bijections on objects. ☐

Remark 9.34. A common intuition about pie limits and colimits is that they "force no equations between objects". The above statement makes precise this intuition: a weight is pie if and only if it is free at the level of objects. Bearing in mind that we have characterised the pie weights it must be the case that our characterisation is equivalent to that of Power and Robinson. We give a direct proof of that equivalence below, thereby reaffirming their result.

Proposition 9.35. Consider a weight $W : \mathcal{J} \longrightarrow \mathrm{Cat}$. The following are equivalent:

1. Each connected component of $el(ob \circ \mathcal{U}W)$ has an initial object.

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[2] Indeed the Yoneda lemma is essentially equivalent to the statement that the representables are free on characteristic families. For since L is left adjoint to R we have $[\mathcal{J}, \mathrm{Cat}](\mathcal{J}(c, -), G) = [\mathcal{J}, \mathrm{Cat}](L\chi_c, G) \cong [ob\mathcal{J}, \mathrm{Cat}](\chi_c, RG) \cong Gc$.

2. There exists a family of sets $X : ob\mathcal{J} \longrightarrow \mathrm{Cat}$ and a 2-natural transformation $\alpha : L(X) \Longrightarrow W$ which is pointwise bijective on objects.

Proof. Consider what it means to give a family of sets $X : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a 2-natural transformation $L(X) \Longrightarrow W$ which is pointwise bijective on objects. Any 2-natural transformation $L(X) \Longrightarrow W$ is of the form:

$$L(X) \xRightarrow{L\theta} LRW \xRightarrow{\epsilon_W} W$$

for some map $\theta : X \Longrightarrow RW$ in $[ob\mathcal{J}, \mathrm{Cat}]$ where $\epsilon_W : LRW \Longrightarrow W$ is the counit component at W. Such a θ consists of a family of functors $\theta_c : X(c) \longrightarrow W(c)$ for each $c \in \mathcal{J}$, which is really just a family of functions as each $X(c)$ is discrete. The component of the 2-natural transformation at $c \in \mathcal{J}$ is:

$$\sum_{d \in \mathcal{J}} \mathcal{J}(d,c) \times X(d) \xrightarrow{\sum_{d \in \mathcal{J}} \mathcal{J}(d,c) \times \theta_d} \sum_{d \in \mathcal{J}} \mathcal{J}(d,c) \times W(d) \xrightarrow{\epsilon_W(c)} W(c)$$

It takes a pair $(\alpha : d \longrightarrow c, x \in X(d))$ to the element $W\alpha(\theta_d(x)) \in W(c)$. For this map to be bijective on objects the first component must certainly be injective on objects. This implies that $\theta_c : X(c) \longrightarrow W(c)$ is injective. For if $\theta_c(x) = \theta_c(y)$ then:

$$\sum_{d \in \mathcal{J}} \mathcal{J}(d,c) \times \theta_d : \sum_{d \in \mathcal{J}} \mathcal{J}(d,c) \times X(d) \longrightarrow \sum_{d \in \mathcal{J}} \mathcal{J}(d,c) \times W(d)$$

would identify $(1_c : c \longrightarrow c, x \in X(c))$ and $(1_c : c \longrightarrow c, y \in X(c))$.
Consequently if this 2-natural transformation is pointwise bijective on objects it follows that $\theta_c : X(c) \longrightarrow W(c)$ is injective for all $c \in \mathcal{J}$ so that $X(c)$ may be taken to be a subset of $W(c)$ for each c, and θ_c the subset inclusion. To say that there exists a family of sets $X : \mathcal{J} \longrightarrow \mathrm{Cat}$ and a pointwise bijective on objects 2-natural transformation $L(X) \Longrightarrow W$ is equally then to say that:

1. For each $c \in \mathcal{J}$ there is a set of objects $X(c) \subset W(c)$ such that given $y \in W(c)$ there exists a unique $x_y \in X(d_y)$ for some $d_y \in \mathcal{J}$ and arrow $\alpha_y : d_y \longrightarrow c$ such that $W\alpha_y(x_y) = y$. Moreover $\alpha : d_y \longrightarrow c$ is the unique 1-cell with this property.

Recall the category $el(ob \circ \mathcal{U}W)$. Objects are pairs (c, x) with $x \in Wc$. A morphism $\alpha : (c, x) \longrightarrow (d, y)$ is an arrow $\alpha : c \longrightarrow d \in \mathcal{J}$ such that $W\alpha(x) = y$. The above statement is clearly a statement about the category of elements. In that language it reads:

2. The category of elements $el(ob \circ \mathcal{U}W)$ has a distinguished set of objects X such that given any object $y \in el(ob \circ \mathcal{U}W)$ there exists a unique $x \in X$ such that there exists an arrow $\alpha : x \longrightarrow y$, and moreover α is the only such arrow.

In this reformulation the set $X = \{(x, c) : x \in X(c), c \in \mathcal{J}\}$.
Now suppose that each connected component of the category of elements has an initial object. Taking X to be the set of initial objects of the connected components (*but just one initial object from each connected component*) clearly gives such a set. For every object of a category lies in a single connected component, and then there exists a unique arrow from the chosen initial object of that connected component to it. Therefore the set X satisfies (2).
Conversely suppose that a set X satisfying (2) exists. We show that this implies that each connected component has an initial object and that the set X contains precisely these initial objects (one for each connected component). Let y be any object and suppose that $x \in X$ is the unique object of X such that there exists a map $x \longrightarrow y$. Suppose z is connected to y. We must show that there exists a unique morphism from x to z. To say that y is connected to z is to say that there exists a finite string of morphisms of alternating direction connecting x and y of the form:

$$y \longrightarrow y_1 \longleftarrow y_2 \longrightarrow y_3 \ldots y_{n-1} \longrightarrow y_n \longleftarrow z$$

(noting that this shape covers strings of the shape $y \longleftarrow y_1 \longrightarrow y_2 \longleftarrow y_3 \ldots y_{n-1} \longleftarrow y_n \longrightarrow z$ and all others since we can put identities on either end of the string).
We can compose our arrow $x \longrightarrow y$ with the first of the string to obtain an arrow $x \longrightarrow y \longrightarrow y_1$. Thus x is the unique element of X corresponding to y_1. Suppose that there was some other element of $x' \in X$ and an arrow $x' \longrightarrow y_2$. Then we would have a pair of arrows from elements of X to y_1: the map $x \longrightarrow y \longrightarrow y_1$ and the map $x' \longrightarrow y_2 \longrightarrow y_1$, contradicting our assumption (2). Therefore there must exist an arrow $x \longrightarrow y_2$. Continuing in this manner inductively we see that there must exist an arrow $x \longrightarrow z$. It is, by (2), the unique such arrow. Therefore x is indeed an initial object in the connected component of y. Since each object has an element of X associated to it, it follows that each connected component has an initial object and that the set X is the set of initial objects of the connected components, one from each.

□

Chapter 10

Cateads effective in T-Alg

In this chapter we begin by examining the extent to which the (bijective on objects/fully faithful)-factorisation system on $Cat(\mathcal{E})$ lifts to T-Alg and admits its description in terms of higher kernels and codescent objects therein; establishing, in particular, that this is the case for a strongly finitary 2-monad on $Cat(\mathcal{E})$. We then establish conditions upon a 2-monad T on $Cat(\mathcal{E})$ under which T-Alg has codescent objects of cateads and cateads are effective. The consideration of cateads in T-Alg requires some care; for instance a catead is by definition a type of internal category, yet T-Alg does not necessarily have pullbacks. We begin the second section by clarifying such issues. Our main result regarding the effectiveness of cateads may be summarised as follows:

- Let $\mathcal{A}$ be a representable 2-category with codescent objects of cateads and suppose that cateads are effective in $\mathcal{A}$. Let T be a 2-monad upon $\mathcal{A}$ and suppose that both T and T^2 preserve codescent objects of cateads. Then T-Alg has codescent objects of cateads, admits the construction of higher kernels and furthermore cateads are effective in T-Alg. In particular this is the case for any strongly finitary 2-monad on Cat or any 2-monad of the form $Cat(T)$ for a monad $T \in \mathrm{Cat}_{\mathrm{pb}}$.

10.1 Lifting the factorisation system to T-Alg

In this section we consider sufficient conditions upon a 2-monad T on $Cat(\mathcal{E})$ under which the (bijective on objects/fully faithful)-factorisation system lifts to T-Alg and has its universal property therein: namely agreeing with the factorisation of an algebra morphism through the codescent object of its higher kernel. We begin Section 10.2 by considering higher kernels in T-Alg for a 2-monad on a representable 2-category. We should remark that all of the results in this section, and indeed the chapter, may be more easily seen to hold for T-Alg_s.

Proposition 10.1. Let T a 2-monad on a representable 2-category $\mathcal{A}$.

1. T-Alg has comma objects and U preserves them.
2. T-Alg admits the construction of higher kernels and U preserves them. The higher kernel of an algebra morphism may be constructed so as to live entirely in T-Alg_s.

Proof. 1. That this is the case is well known though has, not quite, appeared in published form. The relevant article [8] starts with the assumption that the base 2-category $\mathcal{A}$ is complete. In particular comma objects are constructed using products and inserters. In that paper the authors remark that lesser assumptions are required to construct pie limits such as comma objects in T-Alg, but because the main examples are based upon a complete 2-category they work within this framework. We therefore briefly describe how the construction of comma objects may be accomplished if $\mathcal{A}$ is only a representable 2-category and is not presumed to have products.

Given pseudomorphisms $(f, \overline{f}) : (A, a) \longrightarrow (X, x)$ and $(g, \overline{g}) : (B, b) \longrightarrow (X, x)$ we must construct the corresponding comma object. As $\mathcal{A}$ is a representable 2-category it has comma objects; thus we form the comma object of $f|g$ in $\mathcal{A}$ as on the left below:

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
f|g \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \eta & X \\
& B \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& TA \arrow[rr, "a"] \arrow[dr, "Tf"] & \Downarrow \overline{f}^{-1} & A \arrow[dr, "f"] & \\
T(f|g) \arrow[ur, "Td"] \arrow[dr, "Tc"'] & \Downarrow T\eta & TX \arrow[rr, "x"] & \Downarrow \overline{g} & X \\
& TB \arrow[rr, "b"'] \arrow[ur, "Tg"'] & & B \arrow[ur, "g"'] &
\end{tikzcd}$$

By the universal property of the comma object the composite 2-cell in $\mathcal{A}$ on the right above induces a unique arrow $x_{f|g} : T(f|g) \longrightarrow f|g$ such that $d \circ x_{f|g} = a \circ Td$, $c \circ x_{f|g} = a \circ Tc$ and such that $\eta \circ x_{f|g}$ equals the composite 2-cell on the right above. Using the universal property of the comma object $f|g$ it is not difficult to see that $x_{f|g} : T(f|g) \longrightarrow f|g$ is the structure map of an algebra $(f|g, x_{f|g})$. The equations $d \circ x_{f|g} = a \circ Td$ and $c \circ x_{f|g} = a \circ Tc$ then assert that we have strict algebra morphisms $d, c : (f|g, x_{f|g}) \rightrightarrows (A, a)$ and the equation concerning 2-cells is easily seen to assert precisely that we have an algebra 2-cell:

$$\begin{tikzcd}
& (A,a) \arrow[dr, "{(f,\overline{f})}"] & \\
(f|g, x_{f|g}) \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \eta & (B,b) \\
& (X,x) \arrow[ur, "{(g,\overline{g})}"'] &
\end{tikzcd}$$

One then verifies directly that the comma cone $((f|g, x_{f|g}), d, c, \eta)$ in T-Alg exhibits $(f|g, x_{f|g})$ as the comma object. It is immediate, by construction, that this comma object is preserved by $U : \text{T-Alg} \longrightarrow \mathcal{A}$. Furthermore an algebra morphism $(h, \overline{h}) : (D, d) \longrightarrow (f|g, x_{f|g})$ is a strict morphism precisely if its composite with each of the comma projections is strict. To see this observe that since the comma projections d and c are themselves strict the 2-cell components of the algebra morphisms $(h, \overline{h}) \circ d$ and $(h, \overline{h}) \circ c$ are $d \circ \overline{h}$ and $c \circ \overline{h}$ respectively. The 2-dimensional universal property of the comma object $f|g$ ensures that $\overline{h}$ is an identity if and only if both $d \circ \overline{h}$ and $c \circ \overline{h}$ are identites.

2. In order to form the higher kernel of an algebra morphism $(f,\overline{f}):(A,a)\longrightarrow(B,b)$ we firstly form its comma object with comma cone $((f|f,x_{f|f}),d,c,\eta)$ which exists by the first part of the proposition and furthermore the projections d and c may be taken to be strict. To form the higher kernel of $(f,\overline{f})$ we must form the pullback of d and c. Since these are strict morphisms the pullback $(f|f|f,x_{f|f|f})$ may be constructed in T-Alg$_\mathrm{s}$. T-Alg$_\mathrm{s}$ of course admits pullbacks as $\mathcal{A}$ is representable and U^T : T-Alg$_\mathrm{s}\longrightarrow\mathcal{A}$ creates limits. The inclusion of T-Alg$_\mathrm{s}$ into T-Alg preserves all limits so that this remain a pullback in T-Alg, and furthermore that pullback is preserved by U : T-Alg $\longrightarrow\mathcal{A}$. The unit and composition maps for the higher kernel exist immediately, and may themselves be taken to be strict morphisms as, by definition, their composites with the comma projections are respectively the strict pairs of algebra morphisms (dp,cq) and $(1,1)$. To witness the associativity of the higher kernel we must further be able to form the pullback of p and q in T-Alg. Again this exists as p and q are themselves strict morphisms. ☐

Remark 10.2. We here observe that U : T-Alg$\longrightarrow\mathcal{A}$ reflects isomorphisms. This is a special case of the fact that U reflects adjoint equivalences in such a manner that the unit and counit of the equivalence lift directly to algebra 2-cells [26]. Given an algebra morphism $(f,\overline{f}):(A,a)\longrightarrow(B,b)\in$ T-Alg such that $f:A\longrightarrow B$ is invertible the inverse algebra morphism has underlying component $f^{-1}:B\longrightarrow A$ and 2-cell component the isomorphism:

$$\begin{tikzcd}
TB \arrow[r, "Tf^{-1}"] \arrow[d, "1"'] & TA \arrow[dl, "Tf"] \arrow[dd, "a"] \\
TB \arrow[dd, "b"'] & \overline{f}^{-1}\Downarrow \\
 & A \arrow[dl, "f"'] \arrow[d, "1"] \\
B \arrow[r, "f^{-1}"'] & A
\end{tikzcd}$$

Corollary 10.3. Let $\mathcal{A}$ be a representable 2-category and T a 2-monad on $\mathcal{A}$. An algebra morphism $(f,\overline{f}):(A,a)\longrightarrow(B,b)$ is fully faithful if and only if $f:A\longrightarrow B$ is.

Proof. By Proposition 10.1(1) T-Alg has comma objects and U : T-Alg $\longrightarrow\mathcal{A}$ preserves them. Therefore U preserves fully faithfulness by Corollary 3.53(1). By Remark 10.2 U reflects isomorphisms and so by the second part of the same corollary reflects fully faithfulness. ☐

Remark 10.4. We now turn our attention to lifting the (bijective on objects/fully faithful)-factorisation system on $Cat(\mathcal{E})$ to T-Alg. Firstly we observe the following.

Remark 10.5. Consider a category with pullbacks $\mathcal{E}$. If T is a 2-monad on $Cat(\mathcal{E})$ which preserves bijections on objects then any algebra morphism $(f,\overline{f}):(A,a)\longrightarrow(B,b)$ may be factored as a strict algebra morphism $e:(A,a)\longrightarrow(X,x)$ followed by a pseudomorphism $(m,\overline{m}):(X,x)\longrightarrow(B,b)$ where $e:A\longrightarrow X$ is bijective on objects and $m:X\longrightarrow B$ fully faithful. The method is just as described in Remark 6.37. Namely the pseudomorphism $(f,\overline{f})$ on the left below:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{f}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "Te"] \arrow[d, "a"'] & TX \arrow[r, "Tm"] \arrow[d, phantom, "\overline{f}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "e"'] & X \arrow[r, "m"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
TA \arrow[r, "Te"] \arrow[d, "a"'] & TX \arrow[r, "Tm"] \arrow[d, "x"] \arrow[dr, phantom, "\overline{m}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[r, "e"'] & X \arrow[r, "m"'] & B
\end{tikzcd}$$

has its underlying map f factored as bijective on objects followed by fully faithful so that the first two diagrams are equal. As T preserves bijections on objects Te is bijective on objects whilst m is fully faithful. As the (bijective on objects/fully faithful) factorisation system on $Cat(\mathcal{E})$ is enhanced there consequently

exists a unique arrow $x : TX \longrightarrow X$ and 2-cell isomorphism $\overline{m}$ as depicted in the right diagram, such that the left square of that diagram commutes and such that $\overline{m} \circ Te = \overline{f}$. The pair (X, x) then becomes a strict algebra, e a strict algebra morphism, and $(m, \overline{m})$ a pseudomorphism.
Observe further that since $Cat(\mathcal{E})$ is representable the morphism $(m, \overline{m})$ is fully faithful in T-Alg by Corollary 10.3.

Remark 10.6. In the following proposition we consider sufficient conditions upon a 2-monad T under which the factorisation of Remark 10.5 agrees with the factorisation of an algebra morphism through the codescent objects of its higher kernel.

Proposition 10.7. *Let $\mathcal{E}$ be a category with pullbacks and T a 2-monad on $Cat(\mathcal{E})$ which preserves codescent objects of cateads. Consider an algebra morphism $(f, \overline{f}) : (A, a) \longrightarrow (B, b) \in$ T-Alg.*

1. *The codescent object of the higher kernel of $(f, \overline{f})$ exists and is preserved by U. Furthermore the factorisation of $(f, \overline{f})$ through the codescent object of its higher kernel agrees with its factorisation as a bijective on objects algebra morphism followed by a fully faithful one described in Remark 10.5.*

2. *If $f : A \longrightarrow B$ is bijective on objects then $(f, \overline{f})$ is the codescent morphism of its higher kernel.*

In particular these results holds when T is a strongly finitary 2-monad on $Cat(\mathcal{E})$.

Proof. 1. Since $\mathcal{E}$ has pullbacks $Cat(\mathcal{E})$ is a representable 2-category. Therefore by Proposition 10.1(2) T-Alg admits the construction of higher kernels and U preserves them. Consider the higher kernel of $(f, \overline{f})$:

$$\begin{tikzcd}
(f|f|f, x_{f|f|f}) \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & (f|f, x_{f|f}) \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & (A, a) \arrow[l, "i" description]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (A, a) \arrow[dr, "{(f,\overline{f})}"] & \\
(f|f, x_{f|f}) \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & (B, b) \\
 & (A, a) \arrow[ur, "{(f,\overline{f})}"'] &
\end{tikzcd}$$

with comma cone on the right above. The higher kernel may be taken to live entirely in T-Alg$_{\text{s}}$ by Proposition 10.1(2) and we label the algebra morphisms involved accordingly. T preserves codescent objects of cateads by assumption, and thus bijections on objects by Corollary 4.21(1). Therefore we may factor $(f, \overline{f})$ as $e : (A, a) \longrightarrow (X, x)$ followed by $(m, \overline{m})$ as described in Remark 10.5. In particular e is bijective on objects and m fully faithful. By Corollary 10.3 $(m, \overline{m})$ is fully faithful in T-Alg. Consequently the algebra 2-cell η factors uniquely through $(m, \overline{m})$ to give an algebra 2-cell as on the left below:

$$\begin{tikzcd}
 & (A, a) \arrow[dr, "e"] & \\
(f|f, x_{f|f}) \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & (X, x) \\
 & (A, a) \arrow[ur, "e"'] &
\end{tikzcd}$$

Now the triple $((B, b), (f, \overline{f}), \eta)$ constitutes a codescent cocone to the higher kernel of $(f, \overline{f})$. As $(m, \overline{m})$ is fully faithful it reflects equations between 2-cells so that the triple $((X, x), e, \alpha)$ is also a codescent cocone to the higher kernel of $(f, \overline{f})$. We claim it is the universal codescent cocone. Observe that since T-Alg has cotensors with **2** it will suffice, by Proposition 2.5, to verify its one dimensional universal property.
We begin by considering the codescent cocone to the underlying strict coherence data in $Cat(\mathcal{E})$ as

drawn below:

$$\begin{tikzcd} (1) & f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "e"] & \\ f|f \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\alpha & X \\ & A \arrow[ur, "e"'] & \end{tikzcd}$$

By Proposition 10.1(2) U preserves the construction of higher kernels so that the strict coherence data is the higher kernel of the internal functor f whilst its factorisation $f = me$ is its factorisation as bijective on objects followed by fully faithful. Furthermore the codescent cocone (X, e, α) is that induced by the fully faithfulness of $m \in Cat(\mathcal{E})$. Therefore, by Theorem 3.67, the above codescent cocone in $Cat(\mathcal{E})$ exhibits X as the codescent object in $Cat(\mathcal{E})$. Given then another codescent cocone in T-Alg:

$$\begin{tikzcd} & (A,a) \arrow[dr, "(g{,}\overline{g})"] & \\ (f|f, x_{f|f}) \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\theta & (Y,y) \\ & (A,a) \arrow[ur, "(g{,}\overline{g})"'] & \end{tikzcd}$$

the triple (Y, g, θ) underlying it constitutes a codescent cocone to the higher kernel of f in $Cat(\mathcal{E})$. By the universal property of the codescent object X we now obtain a unique 1-cell $k : X \longrightarrow Y$ such that $k \circ e = g$ and $k \circ \alpha = \theta$. In order to show that the cocone $((X, x), e, \alpha)$ is the universal such it will suffice to verify that k admits a unique extension to an algebra morphism $(k, \overline{k}) : (X, x) \longrightarrow (Y, y)$ such that $(k, \overline{k}) \circ e = (g, \overline{g})$ and $(k, \overline{k}) \circ \alpha = \theta$. Upon verifying the first of these two conditions the second will hold immediately as U : T-Alg $\longrightarrow Cat(\mathcal{E})$ is locally faithful. Thus it suffices to show that there exists a unique 2-cell isomorphism:

$$\begin{tikzcd} TX \arrow[r, "Tk"] \arrow[d, "x"'] \arrow[dr, phantom, "\overline{k}\Downarrow"] & TY \arrow[d, "y"] \\ X \arrow[r, "k"'] & Y \end{tikzcd}$$

such that $(k, \overline{k})$ is an algebra morphism and such that $\overline{k} \circ Te = \overline{g}$. Now T preserves codescent objects of cateads by assumption and so preserves the codescent object in $Cat(\mathcal{E})$ of diagram (1) so that its image under T has codescent object TX with universal cocone $(TX, Te, T\alpha)$. Using that $e : (A, a) \longrightarrow (X, x)$ is a strict algebra morphism and that $k \circ e = g$ we see that $y \circ Tk \circ Te = y \circ Tg$ and that $k \circ x \circ Te = k \circ e \circ a = g \circ a$. Therefore we have the 2-cell isomorphism $\overline{g} : y \circ Tk \circ Te \Longrightarrow g \circ a = k \circ x \circ Te$. By the 2-dimensional universal property of the codescent object TX there exists a unique 2-cell $\overline{k} : y \circ Tk \Longrightarrow k \circ x$ such that $\overline{k} \circ Te = \overline{g}$ if and only if the square on the left below commutes:

$$\begin{tikzcd} y \circ Tk \circ Te \circ Td \arrow[r, Rightarrow, "\overline{g} \circ Td"] \arrow[d, Rightarrow, "y \circ Tk \circ T\alpha"'] & k \circ x \circ Te \circ Td \arrow[d, Rightarrow, "k \circ x \circ T\alpha"] \\ y \circ Tk \circ Te \circ Tc \arrow[r, Rightarrow, "\overline{g} \circ Tc"'] & k \circ x \circ Te \circ Tc \end{tikzcd} \qquad \begin{tikzcd} y \circ Tg \circ Td \arrow[r, Rightarrow, "\overline{g} \circ Td"] \arrow[d, Rightarrow, "y \circ T\theta"'] & g \circ a \circ Td \arrow[d, Rightarrow, "\theta \circ x_{f|f}"] \\ g \circ a \circ Tc \arrow[r, Rightarrow, "\overline{g} \circ Tc"'] & g \circ a \circ Tc \end{tikzcd}$$

We have $y \circ Tk \circ T\alpha = y \circ T\theta$ using that $k \circ \alpha = \theta$. Similarly we have $k \circ x \circ T\alpha = k \circ \alpha \circ x_{f|f} = \theta \circ x_{f|f}$ using firstly that $\alpha : e \circ d \Longrightarrow e \circ c$ is an algebra 2-cell between strict algebra morphisms, and then using the equation $k \circ \alpha = \theta$. Consequently the above square reduces to that on the right. That square simply asserts that $\theta : (g, \overline{g}) \circ d \Longrightarrow (g, \overline{g}) \circ c$ is an algebra 2-cell and so indeed commutes. Therefore we obtain a unique 2-cell $\overline{k} : y \circ Tk \Longrightarrow k \circ x$ such that $\overline{k} \circ Te = \overline{g}$ as required. One may construct the inverse of $\overline{k}$ using the inverse of $\overline{g}$ in the same manner we constructed $\overline{k}$ itself; thus $\overline{k}$ is invertible. It remains to verify that $(k, \overline{k}) : (X, x) \longrightarrow (Y, y)$ is an algebra morphism. Consider the following string of equalities:

$$\begin{tikzcd}
T^2A \arrow[r, "T^2e"] \arrow[d, "Ta"'] & T^2X \arrow[r, "T^2k"] \arrow[d, "Tx"'] \arrow[dr, phantom, "T\overline{k}\Downarrow"] & T^2Y \arrow[d, "Ty"] & & T^2A \arrow[r, "T^2g"] \arrow[d, "Ta"'] \arrow[dr, phantom, "T\overline{g}\Downarrow"] & T^2Y \arrow[d, "\mu_Y"] & & T^2A \arrow[r, "T^2g"] \arrow[d, "\mu_A"'] & T^2Y \arrow[d, "\mu_Y"] & & T^2A \arrow[r, "T^2e"] \arrow[d, "\mu_A"'] & T^2X \arrow[r, "T^2k"] \arrow[d, "\mu_X"'] & T^2Y \arrow[d, "\mu_Y"] \\
TA \arrow[r, "Te"] \arrow[d, "a"'] & TX \arrow[r, "Tk"] \arrow[d, "x"'] \arrow[dr, phantom, "\overline{k}\Downarrow"] & TY \arrow[d, "y"] & = & TA \arrow[r, "Tg"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{g}\Downarrow"] & TY \arrow[d, "y"] & = & TA \arrow[r, "Tg"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{g}\Downarrow"] & TY \arrow[d, "y"] & = & TA \arrow[r, "Te"] \arrow[d, "a"'] & TX \arrow[r, "Tk"] \arrow[d, "x"'] \arrow[dr, phantom, "\overline{k}\Downarrow"] & TY \arrow[d, "y"] \\
A \arrow[r, "e"'] & X \arrow[r, "k"'] & Y & & A \arrow[r, "g"'] & Y & & A \arrow[r, "g"'] & Y & & A \arrow[r, "e"'] & X \arrow[r, "k"'] & Y
\end{tikzcd}$$

Consider the first and last of these four composites. Both composites consist of four squares. The multiplicative axiom for an algebra morphism asserts the two rightmost vertically composable squares of each of these composites agree. Now T preserves codescent objects of cateads by assumption and thus codescent morphisms by Proposition 4.20(1). Consequently T^2e is a codescent morphism and so cofaithful (by Lemma 2.29). Therefore it suffices to verify the equality of the first and last of the four diagrams. Using that $k \circ e = g$ and $\overline{k} \circ Te = \overline{g}$ we see that the first and second composites agree. The second and third agree as $(g, \overline{g})$ is an algebra morphism. The third diagram equals the fourth using that $k \circ e = g$ and $\overline{k} \circ Te = \overline{g}$ again.
It is relatively straightforward to verify the unital axiom for an algebra morphism, now using the cofaithfulness of the codescent morphism e. Consequently $(k, \overline{k})$ is an algebra morphism as required, completing the proof.

2. Suppose that the internal functor $f : A \longrightarrow B$ underlying $(f, \overline{f})$ is bijective on objects. Factorising $(f, \overline{f}) = (m, \overline{m}) \circ e$ as in the first part of the proposition gives $e : A \longrightarrow B$ bijective on objects and m fully faithful. Since f is itself bijective on objects it follows that m is an isomorphism. By Remark 10.2 U : T-Alg $\longrightarrow Cat(\mathcal{E})$ reflects isomorphisms so that $(m, \overline{m})$ is an isomorphism in T-Alg. By the first part of the proposition $e : (A, a) \longrightarrow (X, x)$ is the codescent morphism exhibiting (X, x) as the codescent object of the higher kernel of $(f, \overline{f})$. Now we have $(f, \overline{f}) = (m, \overline{m}) \circ e$ with $(m, \overline{m})$ an isomorphism and so it follows that $(f, \overline{f})$ is the codescent morphism of its higher kernel.

If T is a strongly finitary 2-monad on $Cat(\mathcal{E})$ for a locally finitely presentable category $\mathcal{E}$ then certainly $\mathcal{E}$ has pullbacks and, by Corollary 8.18, T preserves codescent objects of cateads. Consequently the results of the proposition hold. ☐

Proposition 10.8. Let $\mathcal{E}$ be a category with pullbacks and T a 2-monad on $Cat(\mathcal{E})$ which preserves codescent objects of cateads.

1. The codescent morphisms in T-Alg are precisely those morphisms orthogonal to fully faithful ones.
2. An algebra morphism $(f, \overline{f})$ is a codescent morphism if and only if $f : A \longrightarrow B$ is bijective on objects.
3. We have an enhanced factorisation system (E, M) on T-Alg where E is the class of codescent morphisms and M the class of fully faithful morphisms.
4. Codescent morphisms are effective in T-Alg.

Proof. 1. Let E denote the class of morphisms orthogonal to the fully faithful morphisms in T-Alg. By Proposition 2.34 codescent morphisms are orthogonal to fully faithful ones. Therefore to prove the claim it suffices to show that each morphism $(f, \overline{f}) : (A, a) \longrightarrow (B, b) \in E$ is a codescent morphism.

We may factor $(f,\overline{f}) = (m,\overline{m}) \circ e$ as a bijective on objects algebra morphism followed by a fully faithful one (as described in Remark 10.5). By Proposition 10.7(1) e is a codescent morphism so that $e \in E$. As $(f,\overline{f})$ is orthogonal to each fully faithful morphism there exists a unique algebra morphism $(k,\overline{k}) : (B,b) \longrightarrow (X,x)$ rendering commutative both triangles of the left square below:

$$\begin{tikzcd}
(A,a) \arrow[r, "(f{,}\overline{f})"] \arrow[d, "e"'] & (B,b) \arrow[d, "1"] \arrow[dl, "(k{,}\overline{k})"'] \\
(X,x) \arrow[r, "(m{,}\overline{m})"'] & (B,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,a) \arrow[r, "e"] \arrow[d, "e"'] & (X,x) \arrow[d, "(m{,}\overline{m})"] \arrow[dl, "(k{,}\overline{k})\circ(m{,}\overline{m})"'] \\
(X,x) \arrow[r, "(m{,}\overline{m})"'] & (B,b)
\end{tikzcd}$$

To see that $(k,\overline{k})$ is the inverse of $(m,\overline{m})$ it therefore suffices to show that $(k,\overline{k}) \circ (m,\overline{m})$ is the identity on (X,x). The commuting triangles of the left square ensure that both triangles of the right square above commute. The identity on (X,x) is equally a diagonal filler for that square. Since $e \in E$ and $(m,\overline{m})$ is fully faithful it follows that $(k,\overline{k}) \circ (m,\overline{m})$ is the identity as required. Therefore $(m,\overline{m})$ is invertible. As codescent morphisms are closed under right composition with isomorphisms it follows that $f = e \circ (m,\overline{m})$ is a codescent morphism.

2. We saw in Proposition 10.7(2) that each algebra morphism $(f,\overline{f})$ with f bijective on objects is a codescent morphism. Conversely if f is a codescent morphism we have $(f,\overline{f}) \in E$ by Proposition 2.34. As described in the proof of the first part of the proposition factoring it as $(m,\overline{m}) \circ e$ gives $(m,\overline{m})$ an isomorphism and $e : A \longrightarrow X$ bijective on objects. Therefore m is bijective on objects and so $f = me$ is bijective on objects.

3. By the first part of the proposition the codescent morphisms are precisely the class E of morphisms orthogonal to the fully faithful ones, which we denote by M. The class of morphisms orthogonal to any fixed class is clearly closed under composition and isomorphisms, and consequently the class E satisfies these conditions. Certainly the class M of fully faithful morphisms is closed under composition and isomorphisms. Codescent morphisms in any 2-category are strongly orthogonal to fully faithful ones by Proposition 2.34. Thus (E,M) is an enhanced factorisation system on T-Alg as required.

4. Each codescent morphism $(f,\overline{f})$ has f bijective on objects by the second part of the present proposition. Applying Proposition 10.7(2) we deduce that codescent morphisms are effective.

□

10.2 Cateads in T-Alg

In the previous section we considered certain cateads, higher kernels, in T-Alg which we saw could be formed in the representable 2-category T-Alg$_s$. In this section we consider general cateads in T-Alg and are faced with the problem that T-Alg need not have all pullbacks even if the base 2-category $\mathcal{A}$ does. Therefore the notions of internal category and consequently catead in T-Alg require some care.

Remark 10.9. Recall that in a 2-category $\mathcal{A}$ with pullbacks an internal category X may be defined as strict coherence data:

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "i" description]
\end{tikzcd}$$

satisfying the following *properties* (1), (2) and (3):

1. The square:

$$\begin{tikzcd} X_2 \arrow[r, "p"] \arrow[d, "q"'] & X_1 \arrow[d, "c"] \\ X_1 \arrow[r, "d"'] & X_0 \end{tikzcd}$$

   is a pullback.

2. The induced arrows $(i,1),(1,i) : X_1 \rightrightarrows X_2$ satisfy $m \circ (i,1) = 1 = m \circ (1,i)$.

3. Consider the object of composable triples, the pullback:

$$\begin{tikzcd} X_3 \arrow[r] \arrow[d] & X_2 \arrow[d, "p"] \\ X_2 \arrow[r, "q"'] & X_1 \end{tikzcd}$$

   and the induced arrows $(m,1),(1,m) : X_3 \rightrightarrows X_2$. These satisfy $m \circ (m,1) = m \circ (1,m)$.

The internal category is a catead if:

4 The span $(d,c) : X_0 \longrightarrow X_0$ forms a two sided discrete fibration.

Remark 10.10. On the other hand if we cannot be sure that $\mathcal{A}$ has all of the pullbacks above then the above definition of internal category is insufficient; if we don't know whether the pullback X_3 exists we cannot view associativity of $m : X_2 \longrightarrow X_1$ as merely a property of the diagram. A solution to this problem would be to introduce the object of composable triples as part of the diagram defining a catead, or to define the notion of internal category representably. However we will see that in the case of T-Alg this is unnecessary. The following Remark indicates further difficulties to be overcome.

Remark 10.11. Consider the case of a 2-monad T on a representable 2-category $\mathcal{A}$. In that case T-Alg_s also has pullbacks and cotensors with $\mathbf{2}$, and the forgetful 2-functor $U^T : \text{T-Alg}_\text{s} \longrightarrow \mathcal{A}$ preserves them. In particular, since T-Alg_s has pullbacks the definition of internal category and catead of Remark 10.9 is sufficient. As U^T is a morphism of Rep it preserves preserves two sided discrete fibrations by Corollary 3.41(1). As U^T preserves both pullbacks and two sided discrete fibrations it preserves cateads.
We will indeed prove that each of these statements is true for T-Alg and the forgetful 2-functor U : T-Alg $\longrightarrow \mathcal{A}$. However the situation is not so straightforward as before. T-Alg does not have pullbacks in general and U : T-Alg $\longrightarrow \mathcal{A}$ is only known to preserve pie limits that $\mathcal{A}$ has. Furthermore, if a pullback happens to exist in T-Alg it is unknown whether it need be preserved. In Corollary 3.41(1) we characterised two sided discrete fibrations using pullbacks and comma objects. However pullbacks are not pie limits and so we cannot use this characterisation to deduce that U : T-Alg $\longrightarrow \mathcal{A}$ preserves two sided discrete fibrations. Therefore we cannot so easily deduce that U : T-Alg $\longrightarrow \mathcal{A}$ preserves cateads.

Remark 10.12. The content of the results of this section is to clarify these issues and show that the situation of cateads in T-Alg is as simple as can be hoped for. These results are collected in Proposition 10.14 and Corollary 10.16 below. The key to proving those results is the following proposition, which enables us to replace two sided discrete fibrations in T-Alg by isomorphic strict ones.

Proposition 10.13. Let $\mathcal{A}$ be an arbitrary 2-category and T any 2-monad upon it. Suppose the span:

$$\begin{tikzcd} & (A,a) \arrow[dl, "(p\text{,}\overline{p})"'] \arrow[dr, "(q\text{,}\overline{q})"] & \\ (B,b) & & (C,c) \end{tikzcd}$$

is a two-sided discrete fibration in T-Alg. Then there exists a strict span:

$$\begin{tikzcd}
 & (A,a'') \arrow[dl, "p"'] \arrow[dr, "q"] & \\
(B,b) & & (C,c)
\end{tikzcd}$$

and a span isomorphism between them.

Proof. The pseudonaturality component of the counit of the biadjunction:

$$\begin{tikzcd}
\text{T-Alg} \arrow[r, bend right=20, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right=20, "F"']
\end{tikzcd}$$

at the algebra morphism $(p,\overline{p}) : (A,a) \longrightarrow (B,b)$ is a 2-cell isomorphism in T-Alg, whose inverse is the 2-cell:

$$\begin{tikzcd}
(TA,\mu_A) \arrow[r, "Tp"] \arrow[d, "a"] & (TB,\mu_B) \arrow[d, "b"] \arrow[dl, Rightarrow, shorten=4mm, "\overline{p}"'] \\
(A,a) \arrow[r, "{(p,\overline{p})}"'] & (B,b)
\end{tikzcd}$$

Using the lifting property of the fibration $(p,\overline{p})$ we lift this 2-cell to obtain a pseudomorphism $(a',\overline{a'})$, its $(p,\overline{p})$-lift, and algebra 2-cell $\theta : (a',\overline{a'}) \Longrightarrow a$. The pair $((a',\overline{a'}),\theta)$ is unique in satisfying:

- $(p,\overline{p}) \circ (a',\overline{a'}) = b \circ Tp$.
-
$$\begin{tikzcd}
(TA,\mu_A) \arrow[r, "Tp"] \arrow[d, "a"] \arrow[d, bend left=60, "{(a',\overline{a'})}"] \arrow[d, phantom, shift left=3, "\overset{\theta}{\Leftarrow}"] & (TB,\mu_B) \arrow[d, "b"] \\
(A,a) \arrow[r, "{(p,\overline{p})}"'] & (B,b)
\end{tikzcd}
=
\begin{tikzcd}
(TA,\mu_A) \arrow[r, "Tp"] \arrow[d, "a"] & (TB,\mu_B) \arrow[d, "b"] \arrow[dl, Rightarrow, shorten=4mm, "\overline{p}"'] \\
(A,a) \arrow[r, "{(p,\overline{p})}"'] & (B,b)
\end{tikzcd}$$
- $(q,\overline{q}) \circ \theta$ is an identity 2-cell.

It is straightforward to see that θ is an isomorphism, as it is the $(p,\overline{p})$-lift of an isomorphism $\overline{p}$. Now consider the composite algebra 2-cell:

$$\begin{tikzcd}
(TA,\mu_A) \arrow[r, "Tq"] \arrow[d, "a"] \arrow[d, bend right=60, "{(a',\overline{a'})}"'] \arrow[d, phantom, shift right=3, "\overset{\theta}{\Rightarrow}"] & (TC,\mu_c) \arrow[d, "c"] \\
(A,a) \arrow[r, "{(q,\overline{q})}"'] \arrow[ur, Rightarrow, shorten=4mm, "\overline{q}^{-1}"'] & (C,c)
\end{tikzcd}$$

where $\overline{q}^{-1}$ is the component of the counit at $(q,\overline{q})$. We take its $(q,\overline{q})$-lift to obtain a morphism $(a'',\overline{a''})$ and algebra 2-cell: $\phi : (a',\overline{a'}) \Longrightarrow (a'',\overline{a''})$. The pair $((a'',\overline{a''}),\phi)$ is unique in satisfying:

- $(q,\overline{q}) \circ (a'',\overline{a''}) = c \circ Tq$.
-
$$\begin{tikzcd}
(TA,\mu_A) \arrow[r, "Tq"] \arrow[d, "{(a'',\overline{a''})}"] \arrow[d, bend right=60, "{(a',\overline{a'})}"'] \arrow[d, phantom, shift right=3, "\overset{\phi}{\Rightarrow}"] & (TC,\mu_c) \arrow[d, "c"] \\
(A,a) \arrow[r, "{(q,\overline{q})}"'] & (C,c)
\end{tikzcd}
=
\begin{tikzcd}
(TA,\mu_A) \arrow[r, "Tq"] \arrow[d, "a"] \arrow[d, bend right=60, "{(a',\overline{a'})}"'] \arrow[d, phantom, shift right=3, "\overset{\theta}{\Rightarrow}"] & (TC,\mu_c) \arrow[d, "c"] \\
(A,a) \arrow[r, "{(q,\overline{q})}"'] \arrow[ur, Rightarrow, shorten=4mm, "\overline{q}^{-1}"'] & (C,c)
\end{tikzcd}$$

- $(p, \overline{p}) \circ \phi$ is an identity 2-cell.

Again ϕ is an isomorphism, being the lifting of an isomorphism.
We claim that (A, a'') is a strict algebra, and that we have an algebra isomorphism $(1_A, \theta\phi^{-1}) : (A, a) \longrightarrow (A, a'')$ whose underlying arrow is an identity, as below:

$$\begin{tikzcd}
TA \arrow[d, bend right=70, "a"'] \arrow[d, "a'" description] \arrow[d, bend left=70, "a''"] \arrow[d, phantom, shift right=3ex, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3ex, "\overset{\Leftarrow}{\phi^{-1}}"] \\
A
\end{tikzcd}$$

Consider the composite 2-cells in T-Alg:

$$\begin{tikzcd}[column sep=large, row sep=huge]
(T^2A, \mu_{TA}) \arrow[r, "Ta'" description] \arrow[r, bend left=50, "Ta''"] \arrow[r, bend right=50, "Ta"'] \arrow[r, phantom, shift left=2ex, "\Downarrow T\phi^{-1}"] \arrow[r, phantom, shift right=2ex, "\Downarrow T\theta"] \arrow[d, "\mu_A"'] & (TA, \mu_A) \arrow[d, "{(a', \overline{a'})}" description] \arrow[d, bend right=50, "a"'] \arrow[d, bend left=50, "{(a'', \overline{a''})}"] \arrow[d, phantom, shift right=3ex, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3ex, "\overset{\Leftarrow}{\phi^{-1}}"] \\
(TA, \mu_A) \arrow[r, "{(a', \overline{a'})}" description] \arrow[r, bend left=50, "a"] \arrow[r, bend right=50, "{(a'', \overline{a''})}"'] \arrow[r, phantom, shift left=2ex, "\Downarrow \theta^{-1}"] \arrow[r, phantom, shift right=2ex, "\Downarrow \phi"] & (A, a)
\end{tikzcd} \quad \text{(1)}$$

and

$$\begin{tikzcd}[column sep=large, row sep=huge]
 & (TA, \mu_A) \arrow[d, "{(a', \overline{a'})}" description] \arrow[d, bend right=50, "a"'] \arrow[d, bend left=50, "{(a'', \overline{a''})}"] \arrow[d, phantom, shift right=3ex, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3ex, "\overset{\Leftarrow}{\phi^{-1}}"] \\
(A, a) \arrow[ur, "\eta_A"] \arrow[r, "1_A"'] & (A, a)
\end{tikzcd} \quad \text{(2)}$$

and their underlying 2-cells in $\mathcal{A}$:

$$\begin{tikzcd}[column sep=large, row sep=huge]
T^2A \arrow[r, "Ta'" description] \arrow[r, bend left=50, "Ta''"] \arrow[r, bend right=50, "Ta"'] \arrow[r, phantom, shift left=2ex, "\Downarrow T\phi^{-1}"] \arrow[r, phantom, shift right=2ex, "\Downarrow T\theta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a'" description] \arrow[d, bend right=50, "a"'] \arrow[d, bend left=50, "a''"] \arrow[d, phantom, shift right=3ex, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3ex, "\overset{\Leftarrow}{\phi^{-1}}"] \\
TA \arrow[r, "a'" description] \arrow[r, bend left=50, "a"] \arrow[r, bend right=50, "a''"'] \arrow[r, phantom, shift left=2ex, "\Downarrow \theta^{-1}"] \arrow[r, phantom, shift right=2ex, "\Downarrow \phi"] & A
\end{tikzcd} \quad \text{(3)}$$

and

$$\begin{tikzcd}[column sep=large, row sep=huge]
 & TA \arrow[d, "a'" description] \arrow[d, bend right=50, "a"'] \arrow[d, bend left=50, "a''"] \arrow[d, phantom, shift right=3ex, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3ex, "\overset{\Leftarrow}{\phi^{-1}}"] \\
A \arrow[ur, "\eta_A"] \arrow[r, "1_A"'] & A
\end{tikzcd} \quad \text{(4)}$$

To say that the morphism $a'' : TA \longrightarrow A$ equips A with an algebra structure is to say precisely that the domain equals the codomain 1-cell for each of the 2-cells (3) and (4). In particular if both 2-cells are identities then (A, a'') is an algebra. In fact to say that both these 2-cells are identities is precisely to say that (A, a'') is an algebra and that $(1_A, \theta\phi^{-1}) : (A, a) \longrightarrow (A, a'')$ an algebra isomorphism. To see this involves a straightforward rewriting of the defining equations for a pseudomorphism of algebras, bearing in the mind that the 2-cells θ and ϕ are isomorphisms.
To show that this second pair of 2-cells are identities in $\mathcal{A}$, it will of course suffice to show that the first pair, (1) and (2), are identities in T-Alg. In order to see that (1) and (2) are identities in T-Alg it will suffice, by Proposition 2.76, to show that they become identities upon postcomposition with each of $(p, \overline{p})$ and $(q, \overline{q})$. The 2-functor U : T-Alg$\longrightarrow \mathcal{A}$ clearly reflects the property of a 2-cell being an identity, so that it will suffice to show that the underlying 2-cells (3) and (4) become identities when we compose each with p and q.

Post-composing (3) with p we obtain:

$$\begin{tikzcd}
T^2A \arrow[r, "Ta'" description] \arrow[r, bend left=50, "Ta''"] \arrow[r, bend right=50, "Ta"'] \arrow[r, phantom, shift left=3, "\Downarrow T\phi^{-1}"] \arrow[r, phantom, shift right=3, "\Downarrow T\theta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a'" description] \arrow[d, bend right=60, "a"'] \arrow[d, bend left=60, "a''"] \arrow[d, phantom, shift right=3, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3, "\overset{\Leftarrow}{\phi^{-1}}"] \\
TA \arrow[r, "a'" description] \arrow[r, bend left=50, "a"] \arrow[r, bend right=50, "a''"'] \arrow[r, phantom, shift left=3, "\Downarrow \theta^{-1}"] \arrow[r, phantom, shift right=3, "\Downarrow \phi"] & A \arrow[dr, "p"] \\
& & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[r, "Ta'" description] \arrow[r, bend left=50, "Ta''"] \arrow[r, bend right=50, "Ta"'] \arrow[r, phantom, shift left=3, "\Downarrow T\phi^{-1}"] \arrow[r, phantom, shift right=3, "\Downarrow T\theta"] \arrow[dd, "\mu_A"'] & TA \arrow[dr, "Tp"] \arrow[dd, "a"] & \\
& & TB \arrow[dd, "b"] \arrow[dl, phantom, "\Swarrow \overline{p}"] \\
TA \arrow[r, "a"] \arrow[dr, "Tp"'] & A \arrow[dr, "p"] \arrow[d, phantom, "\Swarrow \overline{p}^{-1}"] & \\
& TB \arrow[r, "b"] & B
\end{tikzcd}
=
\begin{tikzcd}
& T^2B \arrow[rr, "Tb"] \arrow[dr, phantom, "\Swarrow T\overline{p}"] & & TB \arrow[ddd, "b"] \\
T^2A \arrow[ur, "T^2p"] \arrow[rr, "Ta"] \arrow[d, "\mu_A"'] & & TA \arrow[ur, "Tp"] \arrow[d, "a"] \arrow[r, phantom, shift right=6, "\Swarrow \overline{p}"] & \\
TA \arrow[rr, "a"] \arrow[dr, "Tp"'] & & A \arrow[dr, "p"] \arrow[dl, phantom, "\Swarrow \overline{p}^{-1}"] & \\
& TB \arrow[rr, "b"] & & B
\end{tikzcd}$$

$$=
\begin{tikzcd}
& T^2B \arrow[rr, "\mu_B"] & & TB \arrow[ddd, "b"] \\
T^2A \arrow[ur, "T^2p"] \arrow[rr, "\mu_A"] \arrow[d, "\mu_A"'] & & TA \arrow[ur, "Tp"] \arrow[d, "a"] \arrow[r, phantom, shift right=6, "\Swarrow \overline{p}"] & \\
TA \arrow[rr, "a"] \arrow[dr, "Tp"'] & & A \arrow[dr, "p"] \arrow[dl, phantom, "\Swarrow \overline{p}^{-1}"] & \\
& TB \arrow[rr, "b"] & & B
\end{tikzcd}
=
\begin{tikzcd}
& & TB \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow \overline{p}"] & \\
T^2A \arrow[r, "\mu_A"] & TA \arrow[ur, "Tp"] \arrow[dr, "Tp"'] \arrow[r, "a"] & A \arrow[r, "p"] \arrow[d, phantom, "\Downarrow \overline{p}^{-1}"] & B \\
& & TB \arrow[ur, "b"'] &
\end{tikzcd}
= 1_{b \circ Tp \circ \mu_A}$$

To deduce the first equality we apply the equations $p \circ \theta = \overline{p}$ and $p \circ \phi^{-1} = 1_{pa'}$ (together with their consequences $p \circ \theta^{-1} = \overline{p}^{-1}$ and $p \circ \phi = 1_{pa'}$) which come from the defining equations for θ and ϕ as liftings. Applying T to these equations enables us to deduce the second equality. The third equality applies one of the two equations for $(p, \overline{p})$ to be a pseudomorphism of algebras. The fourth equality is just a simplification of the preceding 2-cell, and the fifth equality follows upon cancelling $\overline{p}$ and $\overline{p}^{-1}$.
Similar reasoning shows that the 2-cell (3) post-composed with q is an identity. This time we use the fact that we know the images of θ and ϕ under q, and one of the equations for $(q, \overline{q})$ to be a pseudomorphism of algebras. Thus (3) is an identity 2-cell.
Consider the 2-cell (4) post-composed with p:

$$\begin{tikzcd}
& TA \arrow[d, "a'" description] \arrow[d, bend right=60, "a"'] \arrow[d, bend left=60, "a''"] \arrow[d, phantom, shift right=3, "\overset{\Leftarrow}{\theta}"] \arrow[d, phantom, shift left=3, "\overset{\Leftarrow}{\phi^{-1}}"] & \\
A \arrow[ur, "\eta_A"] \arrow[r, "1_A"'] & A \arrow[r, "p"'] & B
\end{tikzcd}
=
\begin{tikzcd}
& TA \arrow[r, "Tp"] \arrow[d, "a"] \arrow[dr, phantom, "\Swarrow \overline{p}"] & TB \arrow[d, "b"] \\
A \arrow[ur, "\eta_A"] \arrow[r, "1_A"'] & A \arrow[r, "p"'] & B
\end{tikzcd}
= 1_p$$

To deduce the first equality we apply the equations $p \circ \theta = \overline{p}$ and $p \circ \phi^{-1} = 1_{pa'}$. The second equality holds as it is one of the two defining equations for $(p, \overline{p})$ to be a pseudomorphism of algebras.
Similar reasoning shows that (4) post-composed with q equals 1_q. Consequently (4) is an identity 2-cell and we have established that (A, a'') is an algebra and $(1_A, \theta\phi^{-1}) : (A, a) \longrightarrow (A, a'')$ an isomorphism of algebras.

Consider the inverse of this algebra isomorphism: $(1_A, \phi\theta^{-1})$, and the composite span:

$$\begin{tikzcd}
 & & (B,b) \\
(A,a'') \arrow[r, "{(1_A,\phi\theta^{-1})}"] & (A,a) \arrow[ur, "{(p,\overline{p})}"'] \arrow[dr, "{(q,\overline{q})}"] & \\
 & & (C,c)
\end{tikzcd}$$

The 1-cells underlying the top and bottom composite pseudomorphisms are just p and q respectively. We now show that these composites are actually strict morphisms.
Consider the 2-cell defining the top pseudomorphism:

$$\begin{tikzcd}
TA \arrow[r, "Tp"] \arrow[dd, bend right=60, "a''"'] \arrow[dd, "a'" description] \arrow[dd, bend left=60, "a"] & TB \arrow[dd, "b"] & & TA \arrow[r, "Tp"] \arrow[dd, "a'"'] \arrow[dd, bend left=60, "a"] & TB \arrow[dd, "b"] & & TA \arrow[rr, "Tp"] \arrow[dd, "Tp"'] \arrow[dr, "a"] & & TB \arrow[dd, "b"] & \\
\overset{\Leftarrow}{\phi} \quad \overset{\Leftarrow}{\theta^{-1}} & \Swarrow \overline{p} & = & \overset{\Leftarrow}{\theta^{-1}} & \Swarrow \overline{p} & = & & A \arrow[dr, "p"] \arrow[dl, phantom, "\Swarrow \overline{p}^{-1}"] & \Swarrow \overline{p} & = \quad 1_{b\circ Tp} \\
A \arrow[r, "p"'] & B & & A \arrow[r, "p"'] & B & & TB \arrow[rr, "b"'] & & B &
\end{tikzcd}$$

The first equality holds as $p \circ \phi = 1_{pa'}$. The second because $p \circ \theta^{-1} = \overline{p}^{-1}$. The final equality holds upon cancelling inverses.
Similarly the bottom pseudomorphism is strict, so that we have a commutative diagram as on the left below:

$$\begin{tikzcd}
 & & (B,b) & & & & & (B,b) & \\
(A,a'') \arrow[urr, bend left, "p"] \arrow[drr, bend right, "q"'] \arrow[r, "{(1_A,\phi\theta^{-1})}"] & (A,a) \arrow[ur, "{(p,\overline{p})}"'] \arrow[dr, "{(q,\overline{q})}"] & & \text{and so} & (A,a) \arrow[urr, bend left, "{(p,\overline{p})}"] \arrow[drr, bend right, "{(q,\overline{q})}"'] \arrow[r, "{(1_A,\theta\phi^{-1})}"] & (A,a'') \arrow[ur, "p"'] \arrow[dr, "q"] & & & \text{commutes.} \\
 & & (C,c) & & & & & (C,c) &
\end{tikzcd}$$

□

Proposition 10.14. Let $\mathcal{A}$ be a representable 2-category and T a 2-monad on $\mathcal{A}$.

1. Given a two sided discrete fibration in T-Alg:

$$\begin{tikzcd}
(A_1,a_1) \arrow[r, shift left, "{(d,\overline{d})}"] \arrow[r, shift right, "{(c,\overline{c})}"'] & (A_0,a_0)
\end{tikzcd}$$

   the pullback of its two legs exists and is preserved by $U : \text{T-Alg} \longrightarrow \mathcal{A}$.

2. Consider strict coherence data in T-Alg:

$$\begin{tikzcd}
(A_2,a_2) \arrow[r, shift left=2, "{(p,\overline{p})}"] \arrow[r, "{(m,\overline{m})}" description] \arrow[r, shift right=2, "{(q,\overline{q})}"'] & (A_1,a_1) \arrow[r, shift left=2, "{(d,\overline{d})}"] \arrow[r, shift right=2, "{(c,\overline{c})}"'] & (A_0,a_0) \arrow[l, "{(i,\overline{i})}" description]
\end{tikzcd}$$

   with A_2 the pullback, as in Remark 10.9. This pullback is preserved by Proposition 10.14(1). The further pullback:

$$\begin{tikzcd}
(A_3,a_3) \arrow[r] \arrow[d] & (A_2,a_2) \arrow[d, "{(p,\overline{p})}"] \\
(A_2,a_2) \arrow[r, "{(q,\overline{q})}"'] & (A_1,a_1)
\end{tikzcd}$$

   then exists and is preserved by $U : \text{T-Alg} \longrightarrow \mathcal{A}$.

3. $U : \text{T-Alg} \longrightarrow \mathcal{A}$ preserves two sided discrete fibrations.

Proof. 1. The two sided discrete fibration is span isomorphic in T-Alg to a strict span (f, g) by Proposition 10.13. Consider the following diagram:

$$\begin{tikzcd}
(C,c) \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & (B,b) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "{(k,\overline{k})}"'] & (A_0,a_0) \\
& (A_1,a_1) \arrow[ur, shift left, "{(d,\overline{d})}"] \arrow[ur, shift right, "{(c,\overline{c})}"'] &
\end{tikzcd}$$

in which $(k, \overline{k})$ is the span isomorphism. As $\mathcal{A}$ is a representable 2-category so is T-Alg_s, in particular it has pullbacks. Let the pair (r, s) exhibit (C, c) as the pullback of (g, f) in T-Alg_s; so that $r \circ g = s \circ f$. The inclusion $\iota : \text{T-Alg}_\text{s} \longrightarrow \text{T-Alg}$ preserves all limits that exist so that (C, c) is equally the pullback in T-Alg. We then have $(d, \overline{d}) \circ (k, \overline{k}) \circ s = f \circ s = g \circ r = (c, \overline{c}) \circ (k, \overline{k}) \circ r$. Since $(k, \overline{k}) : (B, b) \longrightarrow (A_1, a_1)$ is an isomorphism it is routine to verify directly that the projections $(k, \overline{k}) \circ r, (k, \overline{k}) \circ s : (C, c) \rightrightarrows (A_1, a_1)$ exhibit (C, c) as the pullback of the pair $(c, \overline{c}), (d, \overline{d}) : (A_1, a_1) \rightrightarrows (A_0, a_0)$ in T-Alg. Now $U^T : \text{T-Alg}_\text{s} \longrightarrow \mathcal{A}$ preserves the pullback diagram of the top row. Since $(k, \overline{k})$ is an isomorphism it is easy to see that $U : \text{T-Alg} \longrightarrow \mathcal{A}$ preserves the constructed pullback of the lower row.

2. Consider the following diagram:

$$\begin{tikzcd}
(D,d) \arrow[r, shift left, "t"] \arrow[r, shift right, "u"'] & (C,c) \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "{(l,\overline{l})}"'] & (B,b) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "{(k,\overline{k})}"'] & (A_0,a_0) \\
& (A_2,a_2) \arrow[r, shift left, "{(p,\overline{p})}"] \arrow[r, shift right, "{(q,\overline{q})}"'] & (A_1,a_1) \arrow[ur, shift left, "{(d,\overline{d})}"] \arrow[ur, shift right, "{(c,\overline{c})}"'] &
\end{tikzcd}$$

where the data on the top row, with exception of the pair (t, u), is the same as in Part 1; the vertical morphism $(k, \overline{k})$ again the span isomorphism. The vertical morphism $(l, \overline{l})$ is the unique arrow into the pullback (A_2, a_2) induced by the commutativity of $(d, \overline{d}) \circ (k, \overline{k}) \circ s = (c, \overline{c}) \circ (k, \overline{k}) \circ r$ from Part 1 of this proposition. From Part 1 we have that $(k, \overline{k}) \circ s, \circ (k, \overline{k}) \circ r : (C, c) \rightrightarrows (A_1, a_1)$ are themselves the projections of the same pullback diagram so that $(l, \overline{l})$ is an isomorphism. The arrows $t, u : (D, d) \rightrightarrows (C, c)$ exhibit (D, d) as the pullback, in T-Alg_s, of the strict morphisms $s, r : (C, c) \rightrightarrows (B, b)$. Again this is a pullback diagram in T-Alg since $\iota : \text{T-Alg}_\text{s} \longrightarrow \text{T-Alg}$ preserves all limits. Since $(l, \overline{l})$ is an isomorphism it is straightforward to see that the pair $(l, \overline{l}) \circ t, (l, \overline{l}) \circ u : (D, d) \longrightarrow (A_2, a_2)$ exhibit (D, d) as the pullback of the pair $(q, \overline{q}), (p, \overline{p}) : (A_2, a_2) \longrightarrow (A_1, a_1)$ as required. Furthermore this pullback is preserved, as the pullback of the top row is, and $(l, \overline{l})$ is an isomorphism.

3. Given a two sided discrete fibration $((p, \overline{p}), (q, \overline{q}))$ in T-Alg, consider the isomorphic strict span (f, g) in T-Alg_s, and span isomorphism in T-Alg:

$$\begin{tikzcd}
& (A,a) \arrow[ddl, "{(p,\overline{p})}"'] \arrow[ddr, "{(q,\overline{q})}"] \arrow[d, "{(k,\overline{k})}"] & \\
& (D,d) \arrow[dl, "f"] \arrow[dr, "g"'] & \\
(B,b) & & (C,c)
\end{tikzcd}$$

guaranteed by Proposition 10.13. The property of being a two sided discrete fibration is an isomorphism invariant of spans in any 2-category. Therefore (f, g) is a two sided discrete fibration in T-Alg. Suppose that we can show (f,g) to be a two sided discrete fibration in T-Alg_s and consider the underlying span isomorphism:

$$\begin{tikzcd}
& A \arrow[ddl, "p"'] \arrow[d, "k"] \arrow[ddr, "q"] & \\
& D \arrow[dl, "f"] \arrow[dr, "g"'] & \\
B & & C
\end{tikzcd}$$

in $\mathcal{A}$. As described in Remark 10.11 $U^T : \text{T-Alg}_\text{s} \longrightarrow \mathcal{A}$ preserves two sided discrete fibrations so that if (f,g) is a two sided discrete fibration in T-Alg_s then (f,g) is one in $\mathcal{A}$. Consequently it will suffice to show that (f,g) is a two sided discrete fibration in T-Alg_s. T-Alg_s is a representable 2-category and the inclusion $\iota : \text{T-Alg}_\text{s} \longrightarrow \text{T-Alg}$ preserves all limits, in particular comma objects and pullbacks. Furthermore the inclusion reflects isomorphisms, since the composite $U^T = U \circ \iota$ does so. Therefore, by Corollary 3.41(2), the inclusion $\iota : \text{T-Alg}_\text{s} \longrightarrow \text{T-Alg}$ reflects the property of being a two sided discrete fibration. Since (f,g) is a two sided discrete fibration in T-Alg it consequently follows that it is one in T-Alg_s.

□

Remark 10.15. By Proposition 10.14 we see that the definition of catead of Remark 10.9 is sufficient for the consideration of cateads in T-Alg, all of the necessary pullbacks being guaranteed to exist.

Corollary 10.16. Let T be a 2-monad on a representable 2-category $\mathcal{A}$. Then $U : \text{T-Alg} \longrightarrow \mathcal{A}$ preserves cateads.

Proof. Consider a catead in T-Alg:

$$\begin{tikzcd}
(A_2,a_2) \arrow[r, shift left=2, "(p{,}\overline{p})"] \arrow[r, "(m{,}\overline{m})" description] \arrow[r, shift right=2, "(q{,}\overline{q})"'] & (A_1,a_1) \arrow[r, shift left=2, "(d{,}\overline{d})"] \arrow[r, shift right=2, "(c{,}\overline{c})"'] & (A_0,a_0) \arrow[l, "(i{,}\overline{i})" description]
\end{tikzcd}$$

By Proposition 10.14(1) the pullback (A_2,a_2) is preserved by U. By Proposition 10.14(2) the pullback (A_3,a_3) exists and is preserved by $U : \text{T-Alg} \longrightarrow \mathcal{A}$. Therefore U preserves the internal category structure of the catead. By Proposition 10.14(3), U preserves two sided discrete fibrations. Therefore $U : \text{T-Alg} \longrightarrow \mathcal{A}$ preserves cateads. □

10.3 Cateads effective in T-Alg

The aim in this section is to establish sufficient conditions on a 2-monad T so that T-Alg has codescent objects of cateads and such that cateads are effective in T-Alg. We begin by assuming that T-Alg has codescent objects of cateads and that they are preserved by the forgetful 2-functor and deduce that cateads are effective. We then establish sufficient conditions on T so that T-Alg has codescent objects of cateads preserved by U.

Proposition 10.17. Let $\mathcal{A}$ be a representable 2-category with codescent objects of cateads and suppose that cateads are effective in $\mathcal{A}$. Consider a 2-monad T on $\mathcal{A}$ and suppose that T-Alg has codescent objects of cateads and the forgetful 2-functor $U : \text{T-Alg} \longrightarrow \mathcal{A}$ preserves them. Then cateads are effective in T-Alg.

Proof. Consider a catead in T-Alg:

$$\begin{tikzcd}
(A_2,a_2) \arrow[r, shift left=2, "(p{,}\overline{p})"] \arrow[r, "(m{,}\overline{m})" description] \arrow[r, shift right=2, "(q{,}\overline{q})"'] & (A_1,a_1) \arrow[r, shift left=2, "(d{,}\overline{d})"] \arrow[r, shift right=2, "(c{,}\overline{c})"'] & (A_0,a_0) \arrow[l, "(i{,}\overline{i})" description]
\end{tikzcd}$$

with codescent object (A, a) with universal cocone:

$$\begin{tikzcd}
 & (A_0,a_0) \arrow[dr, "{(f,\overline{f})}"] & \\
(A_1,a_1) \arrow[ur, "{(d,\overline{d})}"] \arrow[dr, "{(c,\overline{c})}"'] & \Downarrow \alpha & (A,a) \\
 & (A_0,a_0) \arrow[ur, "{(f,\overline{f})}"'] &
\end{tikzcd}$$

In order to show the catead is the higher kernel of its codescent morphism $(f, \overline{f}) : (A_0, a_0) \longrightarrow (A, a)$ it suffices to show that the domain and codomain maps of the catead $((d, \overline{d}), (c, \overline{c}))$, together with the 2-cell α exhibit (A_1, a_1) as the comma object of $(f, \overline{f}) : (A_0, a_0) \longrightarrow (A, a)$. That comma object exists, by Proposition 10.1, and so the comma cone $((A_1, a_1), (d, \overline{d}), (c, \overline{c}), \alpha)$ induces a unique morphism $(g, \overline{g}) : (A_1, a_1) \longrightarrow (f|f, x_{f|f})$ satisfying the evident constraints. In order to show that cateads are effective we must show that the unique comparison $(g, \overline{g})$ is an isomorphism in T-Alg. By Corollary 10.16 U preserves cateads and it preserves codescent objects of them by assumption. Therefore the underlying strict coherence data in $\mathcal{A}$:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description]
\end{tikzcd}$$

is a catead and the underlying cocone:

$$\begin{tikzcd}
 & A_0 \arrow[dr, "f"] & \\
A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \alpha & A \\
 & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

exhibits A as its codescent object. By Proposition 10.1 $U : \text{T-Alg} \longrightarrow \mathcal{A}$ preserves comma objects, so that the unique arrow into the comma object induced by the triple (d, α, c) is exactly $g : A_1 \longrightarrow f|f$. Since cateads are effective in $\mathcal{A}$ the morphism $g : A_1 \longrightarrow f|f$ is an isomorphism. As described in Remark 10.2 $U : \text{T-Alg} \longrightarrow \mathcal{A}$ reflects isomorphisms and so $(g, \overline{g}) : (A_1, a_1) \longrightarrow (f|f, x_{f|f})$ is an algebra isomorphism. Therefore cateads are effective in T-Alg. ☐

Remark 10.18. The following proposition concerns the extent to which codescent objects of strict reflexive coherence data lift to T-Alg. In particular it implies that if T preserves codescent objects of strict reflexive coherence data then T-Alg has such codescent objects and the forgetful 2-functor preserves them. It should be noted that it is not the case in general that if a class of colimits exists in the base 2-category and is preserved by T then it lifts to T-Alg; this being in contrast to the easily understood case of T-Alg_s. The substance of such a counterexample is contained in [7] and we give the details now. If $\mathcal{J}$ is a small 2-category then we may consider the inclusion $ob\mathcal{J} \longrightarrow \mathcal{J}$. Left Kan extension gives a left 2-adjoint to the 2-functor $[\mathcal{J}, \text{Cat}] \longrightarrow [ob\mathcal{J}, \text{Cat}]$ obtained by restriction along the inclusion. This restriction 2-functor is strictly monadic so that for the induced 2-monad T we have $\text{T-Alg}_\text{s} \cong [\mathcal{J}, \text{Cat}]$; and furthermore we have $\text{T-Alg} \cong Ps(\mathcal{J}, \text{Cat})$. Now $[ob\mathcal{J}, \text{Cat}]$ is cocomplete and so admits splittings of idempotents. As splittings of idempotents are an absolute colimit they are preserved by T. Yet in Example 6.2 of [7] an instance of a small 2-category $\mathcal{J}$ is given for which $Ps(\mathcal{J}, \text{Cat})$ does not admit splittings of idempotents.

Proposition 10.19. Let $\mathcal{A}$ be an arbitrary 2-category and T a 2-monad upon it. Consider strict coherence data:

$$\begin{tikzcd}
(A_2,a_2) \arrow[r, shift left=2, "{(p,\overline{p})}"] \arrow[r, "{(m,\overline{m})}" description] \arrow[r, shift right=2, "{(q,\overline{q})}"'] & (A_1,a_1) \arrow[r, shift left=2, "{(d,\overline{d})}"] \arrow[r, shift right=2, "{(c,\overline{c})}"'] & (A_0,a_0) \arrow[l, "{(i,\overline{i})}" description]
\end{tikzcd}$$

in T-Alg. Suppose that the codescent object of the underlying strict coherence data in $\mathcal{A}$ exists and is preserved by both T and T^2. Then the codescent object of the strict coherence data in T-Alg exists and is preserved by $U : \text{T-Alg} \longrightarrow \mathcal{A}$.

Proof. Consider the underlying coherence data in $\mathcal{A}$:

(1)
$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description] \end{tikzcd} \qquad \begin{tikzcd} & A_0 \arrow[dr, "f"] & \\ A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & A \\ & A_0 \arrow[ur, "f"'] & \end{tikzcd}$$

and its codescent object $A \in \mathcal{A}$ with universal cocone as on the right above. We will show that there exists an algebra structure $a : TA \longrightarrow A$ such that f becomes a strict algebra morphism and the triple $((A,a), f, \eta)$ a codescent cocone in T-Alg. Now T preserves the codescent object (1) in $\mathcal{A}$ by assumption; therefore to give a morphism with domain TA is to give a codescent cocone to the coherence data:

$$\begin{tikzcd} TA_2 \arrow[r, shift left=2, "Tp"] \arrow[r, "Tm" description] \arrow[r, shift right=2, "Tq"'] & TA_1 \arrow[r, shift left=2, "Td"] \arrow[r, shift right=2, "Tc"'] & TA_0 \arrow[l, "Ti" description] \end{tikzcd}$$

in $\mathcal{A}$. We claim that:

(2)
$$\begin{tikzcd} & TA_0 \arrow[r, "a_0"] & A_0 \arrow[dr, "f"] & \\ TA_1 \arrow[ur, "Td"] \arrow[r, "a_1"] \arrow[dr, "Tc"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & A \\ & TA_0 \arrow[r, "a_0"'] & A_0 \arrow[ur, "f"'] & \arrow[from=1-2, to=2-2, phantom, "\Downarrow\overline{d}"] \arrow[from=2-2, to=3-2, phantom, "\Downarrow\overline{c}^{-1}"] \end{tikzcd}$$

is such a codescent cocone. The unital axiom for the proposed codescent cocone now asserts that:

$$\begin{tikzcd} & & TA_0 \arrow[r, "a_0"] & A_0 \arrow[dr, "f"] & \\ TA_0 \arrow[r, "Ti"] & TA_1 \arrow[ur, "Td"] \arrow[r, "a_1"] \arrow[dr, "Tc"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & A \\ & & TA_0 \arrow[r, "a_0"'] & A_0 \arrow[ur, "f"'] & \arrow[from=1-3, to=2-3, phantom, "\Downarrow\overline{d}"] \arrow[from=2-3, to=3-3, phantom, "\Downarrow\overline{c}^{-1}"] \end{tikzcd} = \begin{tikzcd} TA_0 \arrow[r, "a_0"] & A_0 \arrow[r, "f"] & A \end{tikzcd}$$

Inserting inverse isomorphisms the left hand side above may be rewritten as the left composite below:

$$\begin{tikzcd} & & TA_0 \arrow[dr, "a_0"] & & \\ & TA_1 \arrow[ur, "Td"] \arrow[dr, "a_1"] & \Downarrow\overline{d} & A_0 \arrow[dr, "f"] & \\ TA_0 \arrow[ur, "Ti"] \arrow[r, "a_0"] \arrow[dr, "Ti"'] & A_0 \arrow[r, "i"] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & A \\ & TA_1 \arrow[ur, "a_1"'] \arrow[dr, "Tc"'] & \Downarrow\overline{c}^{-1} & A_0 \arrow[ur, "f"'] & \\ & & TA_0 \arrow[ur, "a_0"'] & & \arrow[from=2-2, to=3-2, phantom, "\Downarrow\overline{i}"] \arrow[from=3-2, to=4-2, phantom, "\Downarrow\overline{i}^{-1}"] \end{tikzcd} = \begin{tikzcd} & & & A_0 \arrow[dr, "f"] & \\ TA_0 \arrow[r, "a_0"] & A_0 \arrow[r, "i"] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\eta & A \\ & & & A_0 \arrow[ur, "f"'] & \end{tikzcd}$$

Using that $(d,\overline{d}) \circ (i,\overline{i}) = 1$ the top leftmost pair of 2-cells of the left composite above equal the identity. Similarly the lower leftmost pair equal the identity, now using that $(c,\overline{c}) \circ (i,\overline{i}) = 1$. Thus the left and right composites above agree. Now $\eta \circ i$ is the identity on f as the triple (A, f, η) is a codescent cocone. Thus the rightmost composite above is the identity on $f \circ a_0$ as required.

It remains to verify that (2) satisifies the multiplicative axiom to be a codescent cocone. This involves a lengthier, though straightforward, diagram chase and we omit the proof.

Upon observing that (2) constitutes a codescent cocone we obtain a unique arrow $a : TA \longrightarrow A$ out of the codescent object such the following two equations hold:

3. $a \circ Tf = f \circ a_0$.

4.

$$\begin{tikzcd}
 & TA_0 \arrow[rr, "a_0"] \arrow[dr, "Tf"] & & A_0 \arrow[dr, "f"] & \\
TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] & \arrow[d, phantom, "\Downarrow T\eta"] & TA \arrow[rr, "a"] & & A \\
 & TA_0 \arrow[rr, "a_0"'] \arrow[ur, "Tf"'] & & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & TA_0 \arrow[rr, "a_0"] \arrow[d, phantom, "\Downarrow \overline{d}"] & & A_0 \arrow[dr, "f"] & \\
TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] \arrow[rr, "a_1"] & & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \eta & A \\
 & TA_0 \arrow[rr, "a_0"'] \arrow[u, phantom, "\Downarrow \overline{c}^{-1}"] & & A_0 \arrow[ur, "f"'] &
\end{tikzcd}$$

Suppose that (A, a) is an algebra. Then equation (3) asserts precisely that $f : (A, a) \longrightarrow (B, b)$ is an algebra morphism. The above equality of 2-cells of (4) can be seen, upon pasting by $\overline{c}$ on the lower left of each, to assert that precisely that $\eta : f \circ (d, \overline{d}) \Longrightarrow f \circ (c, \overline{c})$ is an algebra 2-cell. Assuming then that (A, a) is an algebra we have a triple $((A, a), f, \eta)$ constituting an algebra, a strict algebra morphism and an algebra 2-cell in T-Alg. The triple constitutes a codescent cocone to the coherence data in T-Alg; for the underlying data (A, f, η) in $\mathcal{A}$ constitutes a codescent cocone to the coherence data in $\mathcal{A}$ and U : T-Alg $\longrightarrow \mathcal{A}$ is locally faithful. Therefore in order to show that the triple $((A, a), f, \eta)$ constitutes a codescent cocone in T-Alg it suffices to show that (A, a) is an algebra.

For this we must verify that the diagrams:

$$(5) \quad \begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\qquad \text{and} \qquad
(6) \quad \begin{tikzcd}
 & TA \arrow[d, "a"] \\
A \arrow[ur, "\eta_A"] \arrow[r, "1"'] & A
\end{tikzcd}$$

commute. We will prove that diagram (5) commutes, (6) being relatively straightforward. Now T^2 proves the codescent object of diagram (1) by assumption. Thus the image of that coherence data under T^2 has codescent object T^2A with universal cocone $(T^2A, T^2f, T^2\eta)$. In order to prove that the square (3) is commutative it suffices, by the universal property of the codescent object T^2A to prove that both paths of the square agree upon precomposition with both T^2f and $T^2\eta$.

We have the equation:

$$a \circ \mu_A \circ T^2f = a \circ Tf \circ \mu_{A_0} = f \circ a_0 \circ \mu_{A_0} = f \circ a_0 \circ Ta_0 = a \circ Tf \circ Ta_0 = a \circ Ta \circ T^2f$$

The first equality uses naturality of μ. The second equality uses equation (3). The third equality uses that (A_0, a_0) is an algebra. The fourth equality holds by equation (3); the fifth equality holds by equation (3) again, now under the image of T. In order to prove that the square (5) commutes it remains to show that both paths agree upon precomposition with $T^2\eta$. Upon precomposition with $T^2\eta$ the lower path of the square becomes the left composite below:

$$\begin{tikzcd}
 & T^2A_0 \arrow[dr, "T^2f"] & & & \\
T^2A_1 \arrow[ur, "T^2d"] \arrow[dr, "T^2c"'] & \Downarrow T^2\eta & T^2A \arrow[r, "Ta"] & TA \arrow[r, "a"] & A \\
 & T^2A_0 \arrow[ur, "T^2f"'] & & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & T^2A_0 \arrow[rr, "Ta_0"] \arrow[d, phantom, "\Downarrow T\overline{d}"] & & TA_0 \arrow[dr, "Tf"] & & \\
T^2A_1 \arrow[ur, "T^2d"] \arrow[dr, "T^2c"'] \arrow[rr, "Ta_1"] & & TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] & \Downarrow T\eta & TA \arrow[r, "a"'] & A \\
 & T^2A_0 \arrow[rr, "Ta_0"'] \arrow[u, phantom, "\Downarrow T\overline{c}^{-1}"] & & TA_0 \arrow[ur, "Tf"'] & &
\end{tikzcd}
\quad =$$

$$\begin{tikzcd}
 & T^2A_0 \arrow[r, "Ta_0"] \arrow[d, phantom, "\Downarrow T\overline{d}"] & TA_0 \arrow[r, "a_0"] \arrow[d, phantom, "\Downarrow\overline{d}"] & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, "\eta"] & \\
T^2A_1 \arrow[ur, "T^2d"] \arrow[r, "Ta_1"] \arrow[dr, "T^2c"'] & TA_1 \arrow[ur, "Td"] \arrow[r, "a_1"] \arrow[dr, "Tc"'] \arrow[d, phantom, "\Downarrow T\overline{c}^{-1}"] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[d, phantom, "\Downarrow\overline{c}^{-1}"] & & A \\
 & T^2A_0 \arrow[r, "Ta_0"'] & TA_0 \arrow[r, "a_0"'] & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & T^2A_0 \arrow[r, "\mu_{A_0}"] & TA_0 \arrow[r, "a_0"] \arrow[d, phantom, "\Downarrow\overline{d}"] & A_0 \arrow[dr, "f"] \arrow[dd, Rightarrow, "\eta"] & \\
T^2A_1 \arrow[ur, "T^2d"] \arrow[r, "\mu_{A_1}"] \arrow[dr, "T^2c"'] & TA_1 \arrow[ur, "Td"] \arrow[r, "a_1"] \arrow[dr, "Tc"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[d, phantom, "\Downarrow\overline{c}^{-1}"] & & A \\
 & T^2A_0 \arrow[r, "\mu_{A_0}"'] & TA_0 \arrow[r, "a_0"'] & A_0 \arrow[ur, "f"'] &
\end{tikzcd}
\quad =$$

$$\begin{tikzcd}
 & & TA_0 \arrow[dr, "Tf"] \arrow[dd, Rightarrow, "T\eta"] & & \\
T^2A_1 \arrow[r, "\mu_{A_1}"] & TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] & & TA \arrow[r, "a"'] & A \\
 & & TA_0 \arrow[ur, "Tf"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & T^2A_0 \arrow[dr, "T^2f"] \arrow[dd, Rightarrow, "T^2\eta"] & & & \\
T^2A_1 \arrow[ur, "T^2d"] \arrow[dr, "T^2c"'] & & T^2A \arrow[r, "\mu_A"] & TA \arrow[r, "a"] & A \\
 & T^2A_0 \arrow[ur, "T^2f"'] & & &
\end{tikzcd}$$

The first equality of the string holds by equation (4) under the image of T, whilst the second equality holds by equation (4). The third equality holds as both $(d, \overline{d})$ and $(c, \overline{c})$ are algebra morphisms. The fourth equality holds upon a further application of equation (4). The fifth equality holds by 2-naturality of μ. Therefore both paths of the square (5) agree upon precomposition with both T^2f and $T\eta$. Consequently the square (5) commutes.

Therefore (A, a) is indeed an algebra[1] and the triple $((A, a), f, \eta)$ a codescent cocone in T-Alg. We claim that this codescent cocone exhibits (A, a) as the codescent object of the coherence data in T-Alg.

Consider a second codescent cocone $((B, b), (g, \overline{g}), \theta)$ in T-Alg; consisting of an algebra (B, b), a pseudomorphism $(g, \overline{g}) : (A_0, a_0) \longrightarrow (B, b)$ and an algebra 2-cell $\theta : (g, \overline{g}) \circ (d, \overline{d}) \Longrightarrow (g, \overline{g}) \circ (c, \overline{c})$. To begin with we must show that the cocone $((A, a), f, \eta)$ satisfies the one dimensional universal property of the codescent object; that there exists a unique algebra morphism $(h, \overline{h})$ such that:

7. $(h, \overline{h}) \circ f = (g, \overline{g})$ and
8. $(h, \overline{h}) \circ \eta = \theta$.

The underlying triple (B, g, θ) in $\mathcal{A}$ constitutes a codescent cocone to the coherence data of (1). Therefore there exists a unique arrow out of the codescent object in $\mathcal{A}$, $h : A \longrightarrow B$, such that $h \circ f = g$ and $h \circ \eta = \theta$. It suffices to show that $h : A \longrightarrow B$ admits a unique extension to a pseudomorphism $(h, \overline{h}) : (A, a) \longrightarrow (B, b)$ such that (7) holds, (8) holding immediately as $h \circ \eta = \theta$. Given that $h \circ f = g$ and that f is a strict algebra morphism (7) amounts to the equality of 2-cells: $\overline{h} \circ Tf = \overline{g}$. By the 2-dimensional universal property of the codescent object T^2A a 2-cell $\overline{h} : b \circ Th \Longrightarrow h \circ a$ is uniquely determined by its precomposite with the codescent morphism $Tf : TA_0 \longrightarrow TA$, thus the required equation $\overline{h} \circ Tf = \overline{g}$ ensures that such a $\overline{h}$ is unique if it exists. To show that $\overline{h}$ does exist it suffices, by the 2-dimensional universal property of the codescent object TA, to exhibit the equality of 2-cells:

$$\begin{tikzcd}
b \circ Th \circ Tf \circ Td \arrow[r, Rightarrow, "b \circ Th \circ T\eta"] \arrow[d, Rightarrow, "\overline{g} \circ Td"'] & b \circ Th \circ Tf \circ Tc \arrow[d, Rightarrow, "\overline{g} \circ Tc"] \\
h \circ a \circ Tf \circ Td \arrow[r, Rightarrow, "h \circ a \circ T\eta"'] & h \circ a \circ Tf \circ Tc
\end{tikzcd}$$

---

[1] Observe that in order to prove that (A, a) is an algebra we required that both T and T^2 preserve the relevant codescent object, and not only T.

The lower left path of the square is the leftmost composite 2-cell below:

$$\begin{tikzcd}
 & TA_0 \arrow[dr, "Tf"] \arrow[drr, bend left, "b\circ Th\circ Tf"] & & \\
TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] & \Downarrow T\eta & TA \arrow[r, "h\circ a"'] \arrow[u, phantom, "\Downarrow \overline{g}"] & B \\
 & TA_0 \arrow[ur, "Tf"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & TA_0 \arrow[r, "a_0"] \arrow[rrr, bend left, "b\circ Th\circ Tf"] \arrow[d, phantom, "\Downarrow \overline{d}"] & A_0 \arrow[dr, "f"] & & \\
TA_1 \arrow[ur, "Td"] \arrow[r, "a_1"] \arrow[dr, "Tc"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \eta & A \arrow[r, "h"'] \arrow[u, phantom, "\Downarrow \overline{g}"] & B \\
 & TA_0 \arrow[r, "a_0"'] \arrow[u, phantom, "\Downarrow \overline{c}^{-1}"] & A_0 \arrow[ur, "f"'] & &
\end{tikzcd}
=$$

$$\begin{tikzcd}
 & TA_0 \arrow[r, "a_0"] \arrow[drr, bend left, "b\circ Tg"] \arrow[d, phantom, "\Downarrow \overline{d}"] & A_0 \arrow[dr, "g"] \arrow[r, phantom, "\Downarrow \overline{g}"] & \\
TA_1 \arrow[ur, "Td"] \arrow[r, "a_1"] \arrow[dr, "Tc"'] & A_1 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow \theta & B \\
 & TA_0 \arrow[r, "a_0"'] \arrow[u, phantom, "\Downarrow \overline{c}^{-1}"] & A_0 \arrow[ur, "g"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & TA_0 \arrow[dr, "Tg"] & & & \\
TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] & \Downarrow T\theta & TB \arrow[rr, "b"] & \Downarrow \overline{g} & B \\
 & TA_0 \arrow[ur, "Tg"] \arrow[rr, "a_0"'] & & A_0 \arrow[ur, "g"'] &
\end{tikzcd}
=$$

$$\begin{tikzcd}
 & TA_0 \arrow[dr, "Tf"] & & \\
TA_1 \arrow[ur, "Td"] \arrow[dr, "Tc"'] & \Downarrow T\eta & TA \arrow[r, "b\circ Th"] & B \\
 & TA_0 \arrow[ur, "Tf"'] \arrow[urr, bend right, "h\circ a\circ Tf"'] & \Downarrow \overline{g} &
\end{tikzcd}$$

The first equality holds by equation (4). The second equality holds as $h \circ f = g$ and $h \circ \eta = \theta$. To say that $\theta : (g, \overline{g}) \circ (d, \overline{d}) \Longrightarrow (g, \overline{g}) \circ (c, \overline{c})$ is an algebra 2-cell is to say that the equation:

$$\begin{tikzcd}
TA_1 \arrow[rr, "Td"] \arrow[dr, "Tc"] \arrow[dd, "a_1"'] & & TA_0 \arrow[dr, "Tg"] & \\
 & TA_0 \arrow[rr, "Tg"'] \arrow[dd, "a_0"] & T\theta \Rightarrow & TB \arrow[dd, "b"] \\
A_1 \arrow[dr, "c"'] & \overline{c} \Downarrow & \overline{g} \Downarrow & \\
 & A_0 \arrow[rr, "g"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
TA_1 \arrow[rr, "Td"] \arrow[dd, "a_1"'] & & TA_0 \arrow[dr, "Tg"] \arrow[dd, "a_0"] & \\
 & \overline{d} \Downarrow & \overline{g} \Downarrow & TB \arrow[dd, "b"] \\
A_1 \arrow[rr, "d"] \arrow[dr, "c"'] & & A_0 \arrow[dr, "g"] & \\
 & A_0 \arrow[rr, "g"'] & \theta \Rightarrow & B
\end{tikzcd}$$

holds, which upon cancelling $\overline{c}$ and its inverse gives the third equality. The fourth equality of the string holds as $T\theta = Th \circ T\eta$ and the final composite 2-cell is precisely the top right path of the square. Therefore we obtain a unique 2-cell $\overline{h} : b \circ Th \Longrightarrow h \circ a$ such that $\overline{h} \circ Tf = \overline{b}$. This equality ensures that once we verify that $(h, \overline{h})$ to be a pseudomorphism we will have verified the one dimensional universal property of the codescent cocone in T-Alg. In order to show that $(h, \overline{h})$ is a pseudomorphism we must first show that $\overline{h}$ is invertible. One may construct the inverse of $\overline{h}$ using the inverse of $\overline{g}$. More generally observe that codescent morphisms, in this case Tf, are liberal. It remains to verify the equations for an algebra morphism. One of the two equations asserts that the 2-cell:

$$\begin{tikzcd}
 & TA \arrow[r, "Th"] \arrow[d, "a"] \arrow[dr, phantom, "\overline{h}\Downarrow"] & TB \arrow[d, "b"] \\
A \arrow[ur, "\eta_A"] \arrow[r, "1"'] & A \arrow[r, "h"'] & B
\end{tikzcd}$$

is the identity on h. Now $b \circ Th \circ \eta_A = b \circ \eta_B \circ h = h$ so that the domain and codomain of the above 2-cell agree. Since codescent morphisms are co-faithful it consequently suffices to show that the above 2-cell becomes an identity upon precomposition with the codescent morphism f. We have the equation:

$$\overline{h} \circ \eta_A \circ f = \overline{h} \circ Tf \circ \eta_{A_0} = \overline{g} \circ \eta_{A_0} = 1$$

firstly using naturality of η, secondly using that $\overline{h} \circ Tf = \overline{g}$ and finally that $\overline{g}$ is an algebra morphism. Thus $\overline{h}\eta_A$ is indeed an identity. The other equation for an algebra morphism concerns the equality of a pair of composite 2-cells with domain object T^2A. To verify that equation in the case of $(h, \overline{h})$ one uses the co-faithfulness of the codescent morphism $T^2f : T^2A_0 \longrightarrow T^2A$, naturality of μ and that $(g, \overline{g})$ is an algebra morphism; this being just the same as our proof of the corresponding case in Proposition 10.7(1).
Therefore $(h, \overline{h})$ is an algebra morphism and so the codescent cocone $((A, a), f, \eta)$ satisfies the 1-dimensional aspect of the universal property of the codescent object. That it satisfies the two dimensional aspect may be verified in a similar manner. We omit the proof since T-Alg, in all the cases of interest, has cotensors with **2**, and in such situations the two dimensional universal property follows, by Proposition 2.5, from the one-dimensional universal property. ☐

Remark 10.20. Any strongly finitary 2-monad on Cat preserves all sifted colimits, by Corollary 8.21, and in particular preserves codescent objects of strict reflexive coherence data. Proposition 10.19 ensures that for such T, T-Alg admits codescent objects of strict reflexive coherence data and that U : T-Alg $\longrightarrow$ Cat preserves them. In [28] the authors' prove the analogous result for reflexive coinverters, another example of a sifted colimit that we have considered. One may also show, though we won't give proofs here, that for such T, T-Alg admits Kleisli objects and (iso)codescent objects objects of general reflexive coherence data and furthermore U preserves them. Again each such colimit is sifted as shown in Chapter 8. One may give proofs, in each case, using ad-hoc methods, much as those employed in the proof of Proposition 10.19. In the case of Kleisli objects this is not very difficult; in the case of the more general codescent objects such a proof would require a substantially lengthier version of the already long Proposition 10.19. What these colimits that lift to T-Alg have in common, in addition to being sifted, is that their defining weights are pie weights (see Remark 9.28). We will remark further upon this point in the Concluding Remarks of Chapter 11.

Corollary 10.21. *Let T be a 2-monad on a representable 2-category $\mathcal{A}$ with codescent objects of cateads. Suppose that both T and T^2 preserve codescent objects of cateads. Then T-Alg has codescent objects of cateads and U : T-Alg $\longrightarrow \mathcal{A}$ preserves them.*

Proof. Given a catead in T-Alg the strict coherence data underlying it is again a catead by Corollary 10.16. As both T and T^2 preserve its codescent object it follows from Proposition 10.19 that the codescent object of the catead in T-Alg exists and is preserved by U : T-Alg $\longrightarrow \mathcal{A}$. ☐

Theorem 10.22. *Let $T \in \mathrm{Cat}_{\mathrm{pb}}$ be a monad on a category with pullbacks $\mathcal{E}$ and consider the induced 2-monad $Cat(T)$ on $Cat(\mathcal{E})$. Then Cat(T)-Alg has codescent objects of cateads and they are effective.*

Proof. $Cat(\mathcal{E})$ is a representable 2-category as $\mathcal{E}$ has pullbacks and $Cat(T)$ is a morphism of Rep. By Corollary 10.16 U preserves cateads. If we can show that both $Cat(T)$ and $Cat(T)^2$ preserve codescent objects of cateads then it follows by Proposition 10.19 that T-Alg has codescent objects of cateads and U preserves them; then by Proposition 10.17 we see that cateads are effective in T-Alg. Both T and T^2 preserve pullbacks. Consequently by Theorem 3.66 both $Cat(T)$ and $Cat(T^2) = Cat(T)^2$ preserve codescent objects of cateads. ☐

Theorem 10.23. Let T be a strongly finitary 2-monad on Cat. Then T-Alg has codescent objects of strict reflexive coherence data, in particular cateads, and cateads are effective in T-Alg.

Proof. Certainly Cat is a representable 2-category and any strongly finitary 2-monad on Cat preserves codescent objects of strict reflexive coherence data, indeed all sifted colimits, by Corollary 8.45. Therefore both T and T^2 preserve codescent objects of strict reflexive coherence data and it follows from Proposition 10.19 that T-Alg has codescent objects of strict reflexive coherence data and these are preserved by U. Therefore by Proposition 10.17 cateads are effective in T-Alg. ☐

Example 10.24. In Example 8.46 we considered the 2-adjunction:

$$\begin{tikzcd}
\text{Gpd} \arrow[r, bend right, "\iota"'] \arrow[r, phantom, "\perp"] & \text{Cat} \arrow[l, bend right, "R"']
\end{tikzcd}$$

and remarked that the underlying functor of ι has a right adjoint. Consequently the functor underlying ι preserves coequalisers. Using Proposition 2.5 we see that ι preserve coequalisers as a 2-functor. It clearly reflects isomorphisms so that by Beck's monadicity theorem we have Gpd $\simeq$ T-Alg$_\text{s}$ for the 2-monad $T = \iota R$ on Cat. We observed that 2-monad is not strongly finitary. Indeed, as described in Example 8.46, the unit is bijective on objects; this is easily seen to imply that T-Alg$_\text{s}$ = T-Alg. We will show momentarily that cateads are not effective in Gpd, from which it follow by Proposition 10.23 that there does not exist *any* strongly finitary 2-monad for which Gpd $\simeq$ T-Alg or Gpd $\simeq$ T-Alg$_\text{s}$.

In order to see that cateads are not effective in Gpd consider a category A and the corresponding pointwise discrete category in Cat:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] & A_1 \arrow[r, shift left=2, "d_a"] \arrow[r, shift right=2, "c_a"'] & A_0 \arrow[l, "i_a" description]
\end{tikzcd}$$

This is equally a catead in Gpd as each discrete category is a groupoid. Its codescent object in Cat is just A. As Gpd is reflective sub 2-category of Cat we may form the codescent object of this catead in Cat and then take its image under the reflection R. Therefore the codescent object of this catead in Gpd is TA. Now TA has the same objects as A_0 and the codescent morphism is the bijective on objects inclusion $j : A_0 \longrightarrow TA$. The comma object $j|j$ in Gpd is simply the discrete category TA_1 with objects the arrows of TA. The elements of A_0 are of course the arrows of A itself. The induced functor (in fact function) into the comma object $A_1 \longrightarrow j|j = (TA)_1$ assigns to an arrow of A the corresponding invertible arrow of the groupoid TA. The comparison $A_1 \longrightarrow j|j = (TA)_1$ will not be an isomorphism unless the category A is a groupoid itself so that each object of A_1 is an invertible arrow. For a small explicit example consider $A = \mathbf{2}$. Then A_1 is the discrete category whose objects are the three arrows of A: two identity arrows and a single non-identity one. We have $TA = I(\mathbf{2})$ so that $j|j$ is the discrete category with four elements (those of A_1 together with an inverse to the only non-trivial arrow of A). The comparison $A_1 \longrightarrow j|j$ is evidently not an isomorphism. Consequently cateads are not effective in Gpd.

Chapter 11

Concluding Remarks

We conclude by remarking upon prospects for further development of the results of this thesis, beginning with the most concrete.

In Remark 10.20 we observed that for a strongly finitary 2-monad on Cat, T-Alg has codescent objects of strict reflexive coherence data and $U : \text{T-Alg} \longrightarrow \text{Cat}$ preserves them; the substance of that argument being Proposition 10.19. We mentioned in Remark 10.20 that various other colimits, each of which is both sifted and pie, lift to T-Alg for such T. We remarked further that similar ad-hoc methods to those of Proposition 10.19 may be employed to prove these claims, the length of such proofs being dependent upon the complexity of the colimit in question. Based upon as yet unpublished work of Lack and Power concerning the relationship between two dimensional Lawvere theories and strongly finitary 2-monads (the finitary case being touched upon in [35]) I have proven, for such T, that T-Alg admits all colimits whose weights are both sifted and pie; this including such cases as codescent objects of general reflexive coherence data, for which a proof along the lines of Proposition 10.19 would be very complicated. The characterisation of pie weights of [45] provides an easy method for determining whether a weight is pie. A method for determining whether a weight is sifted would therefore be useful; which combined with the characterisation of pie weights would enable one to determine whether T-Alg admits a certain colimit for strongly finitary T.

The property of codescent morphisms being effective played an important role in Chapter 4, being one of a list of properties characterising those 2-categories of the form $Cat(\mathcal{E})$ for $\mathcal{E}$ a category with pullbacks. It appears likely that this condition may be replaced by a more natural stability condition which we now discuss. Recall that in any 2-category with higher kernels and codescent objects of them a 1-cell $f : A \longrightarrow B$ factors through the codescent object of its higher kernel:

$$\begin{tikzcd}
f|f|f \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & f|f \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A \arrow[l, "i" description] \arrow[rr, "f"] \arrow[dr, "f_1"'] & & B \\
& & & C \arrow[ur, "f_2"'] &
\end{tikzcd}$$

where f_1 is a codescent morphism. If one can show that the induced arrow $f_2 : C \longrightarrow B$ is fully faithful (as is the case in $Cat(\mathcal{E})$) then it follows that codescent morphisms are effective. The one dimensional analogue of this situation is the factorisation of a morphism of a category through the coequaliser of its kernel pair; if regular epimorphisms are stable under pullback it follows that f_2 is a monomorphism. Returning to the two dimensional case one might initially anticipate that if codescent morphisms are stable under pullback then it should be the case that the induced arrow f_2 would be fully faithful. This does not appear to be the case. On the other hand the condition that regular epimorphisms are stable under pullback is equivalent to an apparently stronger condition: coequalisers of kernel pairs are stable under pullback. A 2-dimensional analogue of this condition, that codescent objects of higher kernels are stable under pullback, in a suitable sense, appears as though it may be sufficient to ensure that the induced arrow is fully faithful. This requires further investigation which would ideally result in the proposed stability axiom replacing the condition that "codescent morphisms are effective"; the stability condition would then additionally ensure that one obtains an enhanced factorisation system (E, M) with E the class of codescent morphisms, and M the class of fully faithful morphisms.

Categories of algebras for monads on Set may be characterised as those exact categories admitting a suitable projective cover [14]. It would be interesting to characterise T-Alg for strongly finitary 2-monads on Cat in such terms. On the other hand T-Alg does not admit a projective cover in the sense considered in this thesis. However T-Alg_{pie} does so and is biequivalent to T-Alg. It is likely that one could characterise T-Alg_{pie} up to 2-equivalence, and plausible that one could weaken the resulting characterising properties to obtain properties characterising T-Alg up to biequivalence. Another possibility is to replace the notion of codescent morphism by its bicategorical version, which amounts in Cat to replacing bijections on objects by essentially surjective on objects functors. T-Alg does admit a "projective cover" in this weaker sense, and it is plausible that one could characterise T-Alg up to biequivalence in these terms.

Chapter 12

Appendix

12.1 Completion of proof of Proposition 3.19

Proposition 12.1. Let $\mathcal{E}$ be a category with pullbacks.

1. Given $X \in Cat(\mathcal{E})$ its cotensor with $\mathbf{2}$, $X^{\mathbf{2}}$ exists. One may choose the cotensor and its universal 2-cell:

$$\begin{tikzcd} X^{\mathbf{2}} \arrow[r, bend left, "d_X"] \arrow[r, bend right, "c_X"'] \arrow[r, phantom, "\Downarrow\eta"] & X \end{tikzcd}$$

   to satisfy the following properties: $X^{\mathbf{2}}_0 = X_1$, $(d_X)_0 = d_x, (c_X)_0 = c_x$ and $\overline{\eta} : X^2_0 = X_1 \longrightarrow X_1$ is the identity on X_1.

2. A 2-cell in $Cat(\mathcal{E})$:

$$\begin{tikzcd} Y \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & X \end{tikzcd}$$

   exhibits Y as the cotensor of X with $\mathbf{2}$ if and only if:

   - $\overline{\theta} : Y_0 \longrightarrow X_1$ is an isomorphism.
   - The naturality square:

$$\begin{tikzcd} Y_1 \arrow[r, "(\overline{\theta}\circ d_y, g_1)"] \arrow[d, "(f_1, \overline{\theta}\circ c_y)"'] & X_2 \arrow[d, "m_x"] \\ X_2 \arrow[r, "m_x"'] & X_1 \end{tikzcd}$$

     is a pullback square.

3. Given $F : \mathcal{A} \longrightarrow \mathcal{B}$ of $\mathrm{Cat}_{\mathrm{pb}}$ the induced 2-functor $Cat(F) : Cat(\mathcal{A}) \longrightarrow Cat(\mathcal{B})$ preserves cotensors with $\mathbf{2}$.

Proof. We have verified Parts 1 and 2 of the Proposition in the case of Cat $= Cat(\mathrm{Set})$ and observed that Part 3 follows from Part 2. Therefore it suffices to prove Parts 1 and 2 for the general case. We will use the Yoneda embedding $y : \mathcal{E} \longrightarrow [\mathcal{E}^{op}, \mathrm{Set}]$ to deduce the general case. As $\mathcal{E}$ has pullbacks y preserves them and we obtain a 2-functor $Cat(y) : Cat(\mathcal{E}) \longrightarrow Cat[\mathcal{E}^{op}, \mathrm{Set}]$. This is easily seen to be 2-fully faithful as the Yoneda embedding is fully faithful. We shall show the claims of Parts 1 and 2 hold in $Cat[\mathcal{E}^{op}, \mathrm{Set}]$ and thereby deduce they hold in $\mathcal{E}$. Now an object of $Cat[\mathcal{E}^{op}, \mathrm{Set}]$ is an internal category:

$$\begin{tikzcd} F_2 \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & F_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & F_0 \arrow[l, "i" description] \end{tikzcd}$$

of presheaves F_i and natural transformations. Equally such an internal category may be seen as an object of $[\mathcal{E}^{op}, Cat(\mathrm{Set})]$; the above internal category corresponding to the 2-functor whose value at an object $e \in \mathcal{E}$ is the small category:

$$\begin{tikzcd} F_2 e \arrow[r, shift left=2, "p(e)"] \arrow[r, "m(e)" description] \arrow[r, shift right=2, "q(e)"'] & F_1 e \arrow[r, shift left=2, "d(e)"] \arrow[r, shift right=2, "c(e)"'] & F_0 e \arrow[l, "i(e)" description] \end{tikzcd}$$

This extends to an isomorphism of 2-categories $Cat[\mathcal{E}^{op}, \mathrm{Set}] \cong [\mathcal{E}^{op}, Cat(\mathrm{Set})]$. Since $Cat(\mathrm{Set}) =$ Cat is complete so therefore is $[\mathcal{E}^{op}, Cat(\mathrm{Set})]$ and consequently $Cat[\mathcal{E}^{op}, \mathrm{Set}]$. Now the evaluation 2-functors $ev_e : [\mathcal{E}^{op}, Cat(\mathrm{Set})] \longrightarrow Cat(\mathrm{Set})$ jointly create all limits. They correspond, across this isomorphism, to the pointwise evaluation 2-functors $Cat(ev_e) : Cat[\mathcal{E}^{op}, \text{Set}] \longrightarrow Cat(\text{Set})$, now for $ev_e : \mathcal{E}^{op} \longrightarrow \text{Set}$. Therefore the pointwise evaluation 2-functors jointly create cotensors with **2**. Now we have verified the claims of Parts 1 and 2 of the proposition in the case of Cat(Set). As the functors $ev_e : \mathcal{E}^{op} \longrightarrow \text{Set}$ jointly reflect isomorphisms and create pullbacks we deduce these conditions also hold in the case of $Cat[\mathcal{E}^{op}, \text{Set}]$.

Given then $X \in Cat(\mathcal{E})$ consider $Cat(y)X \in Cat[\mathcal{E}^{op}, \text{Set}]$ the internal category whose components $Cat(y)X_i$ are the representables $\mathcal{E}(-, X_i) = \hat{X}_i$. We choose the canonical value of Part 1 for the cotensor with **2**: $Z = (Cat(y)X)^{\mathbf{2}}$:

$$\begin{tikzcd} Z \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & {\Downarrow \eta} & Cat(y)X \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=huge, row sep=large] Z_2 \arrow[r, shift left, "f_2"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right=2, "p_Z"'] \arrow[d, "m_Z" description] \arrow[d, shift left=2, "q_Z"] & \hat{X}_2 \arrow[d, shift right=2, "\hat{p_X}"'] \arrow[d, "\hat{m_X}" description] \arrow[d, shift left=2, "\hat{q_X}"] \\ Z_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[d, shift right=2, "d_Z"'] \arrow[d, shift left=2, "c_Z"] & \hat{X}_1 \arrow[d, shift right=2, "\hat{d_X}"'] \arrow[d, shift left=2, "\hat{c_X}"] \\ \hat{X}_1 \arrow[u, "i_Z" description] \arrow[ur, "1" description] \arrow[r, shift left, "f_0=\hat{d_X}"] \arrow[r, shift right, "g_0=\hat{c_X}"'] & \hat{X}_0 \arrow[u, "\hat{i_X}" description] \end{tikzcd}$$

We wish to show that we take Z to be in the image of $Cat(y)$. By the second part of the proposition we know that Z_1 is a pullback as on the left below:

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & \hat{X}_2 \arrow[d, "\hat{m_x}"] \\ \hat{X}_2 \arrow[r, "\hat{m_x}"'] & \hat{X}_1 \end{tikzcd} \qquad\qquad \begin{tikzcd} P_1 \arrow[r] \arrow[d] & X_2 \arrow[d, "m_x"] \\ X_2 \arrow[r, "m_x"'] & X_1 \end{tikzcd}$$

Since y preserves pullbacks we may take the pullback in $\mathcal{E}$ and set $Z_1 = \hat{P}_1$. Then we have both Z_0 and Z_1 in the image of y. As y is fully faithful it follows also that d_Z and c_Z are in the image of y. As Z_2 is the pullback along these domain and codomain maps it follows that we may take $Z_2 = \hat{P}_2$ for the corresponding pullback in $\mathcal{E}$. Now we see that all of the objects of the internal categories Z and $Cat(y)X$ are in the image of y. By the fully faithfulness of y it follows that the structure maps of Z and the internal functor components of f and g are uniquely in the image of y. It follows that the internal natural transformation:

$$\begin{tikzcd} Z \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & {\Downarrow \eta} & Cat(y)X \end{tikzcd}$$

is the image under $Cat(y)$ of a 2-cell in $Cat(\mathcal{E})$:

$$\begin{tikzcd} X^{\mathbf{2}} \arrow[rr, bend left, "d_X"] \arrow[rr, bend right, "c_X"'] & {\Downarrow \eta_X} & X \end{tikzcd}$$

satisfying the conditions of Part 1 of the Proposition. As $Cat(y)$ is 2-fully faithful it follows that is indeed the cotensor in $Cat(\mathcal{E})$. Therefore we have verified Part 1 of the Proposition.

As $Cat(\mathcal{E})$ is closed in $Cat[\mathcal{E}^{op}, \text{Set}]$ it follows that $Cat(y)$ reflects cotensors with **2**. Since y reflects isomorphisms and pullbacks it follows that $Cat(y)$ reflects the characterising properties of Part 2, which we know to hold in $Cat[\mathcal{E}^{op}, \text{Set}]$. Thus the properties of Part 2 equally characterise cotensors with **2** in $Cat(\mathcal{E})$.

Part 3 follows immediately from Part 2. □

12.2 Completion of proof of Lemma 8.40

Lemma 12.2. Consider a pseudonatural transformation of coherence data $f : A \longrightarrow B$ as depicted below:

$$\begin{tikzcd}
A_2 \arrow[r, shift left=2, "p_a"] \arrow[r, "m_a" description] \arrow[r, shift right=2, "q_a"'] \arrow[d, "f_2"'] \arrow[dr, phantom, "\cong"] & A_1 \arrow[r, shift left=2, "d_a"] \arrow[r, leftarrow, "i_a" description] \arrow[r, shift right=2, "c_a"'] \arrow[d, "f_1"] \arrow[dr, phantom, "\cong"] & B_0 \arrow[d, "f_0=1"] \\
B_2 \arrow[r, shift left=2, "p_b"] \arrow[r, "m_b" description] \arrow[r, shift right=2, "q_b"'] & B_1 \arrow[r, shift left=2, "d_b"] \arrow[r, leftarrow, "i_b" description] \arrow[r, shift right=2, "c_b"'] & B_0
\end{tikzcd}$$

such that

- f_0 is an identity,
- f_1 is co-fully faithful and
- f_2 is co-faithful.

1. Suppose that

$$\begin{tikzcd}
& B_0 \arrow[dr, "\alpha"] \arrow[dd, phantom, "\Downarrow \overline{\alpha}"] & \\
B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] & & QB \\
& B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

   exhibits QB as the codescent object of the bottom row.
   Then

$$\begin{tikzcd}
& B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\alpha"] & \\
A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & \Downarrow \overline{\alpha} & QB \\
& B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

   exhibits QB as the codescent object of the top row.

2. This equally applies to the case of the isocodescent object.

Proof. In the main text we left the proof of two parts of this lemma to complete in the appendix. We prove these two details here, in order, continuing exactly where we left off in either case.

1. In order to show that we have a codescent cocone it remains to establish the codescent cocone equation (1) of Definition 6.31 holds. That is to show that the first and last composite 2-cells of the following string are equal:

$$\begin{tikzcd}
& A_1 \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\alpha"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"] \arrow[dr, "q_a"'] & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] \arrow[d, phantom, "\Downarrow \tau_a"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & \Downarrow \overline{\alpha} & QB \\
& A_1 \arrow[r, "c_a"'] & B_0 \arrow[r, "1"'] & B_0 \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
& & B_0 \arrow[dr, "1"] \arrow[d, phantom, "\Downarrow f_d"] & & \\
& A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & B_0 \arrow[dr, "\alpha"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[dr, "q_a"'] & B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] \arrow[dr, "q_b"] \arrow[d, phantom, "\Downarrow f_q^{-1}"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & \Downarrow \overline{\alpha} & QB \\
& A_1 \arrow[r, "f_1"'] \arrow[dr, "c_a"'] & B_1 \arrow[r, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & B_0 \arrow[ur, "\alpha"'] & \\
& & B_0 \arrow[ur, "1"'] & &
\end{tikzcd}$$

$$
=\quad
\begin{tikzcd}
 & & B_0 \arrow[d, phantom, "\Downarrow f_d"] \arrow[drr, "1"] & & \\
 & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[rr, "d_b"] \arrow[drr, "c_b"] \arrow[dd, phantom, "\Downarrow \tau_b"] & & B_0 \arrow[dr, "\alpha"] \arrow[d, phantom, "\Downarrow \overline{\alpha}"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[dr, "q_a"'] & B_2 \arrow[ur, "p_b"] \arrow[dr, "q_b"'] \arrow[d, phantom, "\Downarrow f_q^{-1}"] & & & B_0 \arrow[r, "\alpha"'] & QB \\
 & A_1 \arrow[r, "f_1"'] \arrow[dr, "c_a"'] & B_1 \arrow[urr, "d_b"] \arrow[rr, "c_b"'] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & & B_0 \arrow[ur, "\alpha"'] \arrow[u, phantom, "\Downarrow \overline{\alpha}"] & \\
 & & B_0 \arrow[urr, "1"'] & & &
\end{tikzcd}
\quad=\quad
\begin{tikzcd}
 & & & B_0 \arrow[d, phantom, "\Downarrow f_d"] \arrow[drr, "1"] & & & \\
 & A_1 \arrow[urr, "d_a"] \arrow[rr, "f_1"] \arrow[dr, "c_a"'] & & B_1 \arrow[rr, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & & B_0 \arrow[dr, "\alpha"] \arrow[d, phantom, "\Downarrow \overline{\alpha}"] & \\
A_2 \arrow[ur, "p_a"] \arrow[dr, "q_a"'] \arrow[rr, phantom, "\Downarrow \tau_a"] & & B_0 \arrow[rr, "1"] & & B_0 \arrow[rr, "\alpha"'] & & QB \\
 & A_1 \arrow[ur, "d_a"] \arrow[rr, "f_1"'] \arrow[drr, "c_a"'] & & B_1 \arrow[ur, "d_b"] \arrow[rr, "c_b"'] \arrow[d, phantom, "\Downarrow f_c^{-1}"] \arrow[u, phantom, "\Downarrow f_d"] & & B_0 \arrow[ur, "\alpha"'] \arrow[u, phantom, "\Downarrow \overline{\alpha}"] & \\
 & & & B_0 \arrow[urr, "1"'] & & &
\end{tikzcd}
$$

The first equation is again the only of these which require significant justification. The second equation holds since $(QB, \alpha, \overline{\alpha})$ is a codescent cocone to the bottom row. The third equation interchanges the two sides of a cube. This interchange is precisely the equation for pseudonaturality of f at $\tau_a : c_a p_a \cong d_a q_a$ upon pasting both sides of that equation on the top by f_c^{-1} and on the bottom by f_q^{-1}.
By pseudonaturality of f we have the equation:

$$
\begin{tikzcd}
A_2 \arrow[r, "m_a"] \arrow[d, "f_2"'] \arrow[rr, bend left=50, "d_a\circ p_a"] \arrow[rr, phantom, bend left=25, "\Downarrow \tau_a"] \arrow[dr, phantom, "\Downarrow f_m"] & A_1 \arrow[r, "d_a"] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow f_d"] & B_0 \arrow[d, "1"] \\
B_2 \arrow[r, "m_b"'] & B_1 \arrow[r, "d_b"'] & B_0
\end{tikzcd}
\quad=\quad
\begin{tikzcd}
A_2 \arrow[r, "m_a"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\Downarrow f_p"] & A_1 \arrow[r, "d_a"] \arrow[d, "f_1"] \arrow[dr, phantom, "\Downarrow f_d"] & B_0 \arrow[d, "1"] \\
B_2 \arrow[r, "p_b"] \arrow[rr, bend right=50, "d_b\circ m_b"'] \arrow[rr, phantom, bend right=25, "\Downarrow \tau_b"] & B_1 \arrow[r, "d_b"] & B_0
\end{tikzcd}
$$

Pasting f_m^{-1} on the lower left of both composites gives the equation:

$$
\begin{tikzcd}
 & A_1 \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"'] & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"'] & B_1 \arrow[ur, "d_b"] &
\end{tikzcd}
\quad=\quad
\begin{tikzcd}
 & & B_0 \arrow[d, phantom, "\Downarrow f_d"] \arrow[dr, "1"] & \\
 & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & B_0 \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[rr, bend right=40, "f_1\circ m_a"'] \arrow[rr, phantom, bend right=20, "\Downarrow f_m^{-1}"] & B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] & B_1 \arrow[ur, "d_b"'] &
\end{tikzcd}
$$

Similarly we have:

$$
\begin{tikzcd}
A_2 \arrow[r, "m_a"] \arrow[dr, "q_a"'] & A_1 \arrow[r, "f_1"] \arrow[dr, "c_a"'] \arrow[d, phantom, "\Downarrow \tau_a"] & B_1 \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & \\
 & A_1 \arrow[r, "c_a"'] & B_0 \arrow[r, "1"'] & B_0
\end{tikzcd}
\quad=\quad
\begin{tikzcd}
A_2 \arrow[r, "f_2"] \arrow[rr, bend left=40, "f_1\circ m_a"] \arrow[rr, phantom, bend left=20, "\Downarrow f_m"] \arrow[dr, "q_a"'] & B_2 \arrow[r, "m_b"] \arrow[dr, "q_b"'] \arrow[d, phantom, "\Downarrow f_q^{-1}"] & B_1 \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & \\
 & A_1 \arrow[r, "f_1"'] \arrow[dr, "c_a"'] & B_1 \arrow[r, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & B_0 \\
 & & B_0 \arrow[ur, "1"'] &
\end{tikzcd}
$$

These two equalities enable us to rewrite the first composite 2-cell, and, upon cancelling f_m and its inverse, to obtain the first equation. Thus we have a cocone to the top row.

2. We must show that the cocone condition (1) of Definition 6.31 is verified in the case of $(C, \beta, \overline{\beta}')$ which is to show that we have an equality of pasting diagrams:

$$\begin{tikzcd}
& B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow\tau_b"] & B_0 \arrow[dr, "\beta"] & \\
B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] \arrow[dr, "q_b"'] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow\tau_b"] & \Downarrow\overline{\beta}' & C \\
& B_1 \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}
=
\begin{tikzcd}
& B_1 \arrow[r, "d_b"] \arrow[dr, "c_b"] & B_0 \arrow[dr, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}'"] & \\
B_2 \arrow[ur, "p_b"] \arrow[dr, "q_b"'] & \Downarrow\tau_b & B_0 \arrow[r, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}'"] & C \\
& B_1 \arrow[ur, "d_b"] \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Since $f_2 : A_2 \longrightarrow B_2$ is cofaithful and the 2-cells f_p and f_q^{-1} both isomorphisms this is equally to show:

$$\begin{tikzcd}
& A_1 \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow\tau_b"] & B_0 \arrow[dr, "\beta"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[dr, "q_a"'] & B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] \arrow[dr, "q_b"'] \arrow[d, phantom, "\Downarrow f_q^{-1}"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow\tau_b"] & \Downarrow\overline{\beta}' & C \\
& A_1 \arrow[r, "f_1"'] & B_1 \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}
=
\begin{tikzcd}
& A_1 \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[dr, "c_b"] & B_0 \arrow[dr, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}'"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[dr, "q_a"'] & B_2 \arrow[ur, "p_b"] \arrow[dr, "q_b"'] \arrow[d, phantom, "\Downarrow f_q^{-1}"] & \Downarrow\tau_b & B_0 \arrow[r, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}'"] & C \\
& A_1 \arrow[r, "f_1"'] & B_1 \arrow[ur, "d_b"] \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}$$

It follows from the pseudonaturality of $f : A \longrightarrow B$ that the right composite 2-cell above equals the left hand side below:

$$\begin{tikzcd}
& A_1 \arrow[r, "f_1"] \arrow[dr, "c_a"] & B_1 \arrow[r, "d_b"] \arrow[dr, "c_b"] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & B_0 \arrow[dr, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}'"] & \\
A_2 \arrow[ur, "p_a"] \arrow[dr, "q_a"'] & \Downarrow\tau_a & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[r, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}'"] & C \\
& A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"'] & B_1 \arrow[ur, "d_b"] \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}
=
\begin{tikzcd}
& B_1 \arrow[dr, bend left, "d_b"] \arrow[d, phantom, "\Downarrow f_d^{-1}"] & & \\
& A_1 \arrow[u, "f_1"] \arrow[r, "d_a"] \arrow[dr, "c_a"] & B_0 \arrow[dr, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}"] & \\
A_2 \arrow[ur, "p_a"] \arrow[dr, "q_a"'] & \Downarrow\tau_a & B_0 \arrow[r, "\beta"] \arrow[d, phantom, "\Downarrow\overline{\beta}"] & C \\
& A_1 \arrow[ur, "d_a"] \arrow[r, "c_a"'] \arrow[d, "f_1"'] & B_0 \arrow[ur, "\beta"'] & \\
& B_1 \arrow[ur, bend right, "c_b"'] \arrow[u, phantom, "\Downarrow f_c"] & &
\end{tikzcd}$$

$$=\quad
\begin{tikzcd}
& B_1 \arrow[dr, bend left, "d_b"] \arrow[d, phantom, "\Downarrow f_d^{-1}"] & & \\
& A_1 \arrow[u, "f_1"] \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow\tau_a"] & B_0 \arrow[dr, "\beta"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"] \arrow[dr, "q_a"'] & A_1 \arrow[ur, "d_a"] \arrow[dr, "c_a"] \arrow[d, phantom, "\Downarrow\tau_a"] & \Downarrow\overline{\beta} & C \\
& A_1 \arrow[r, "c_a"'] \arrow[d, "f_1"'] & B_0 \arrow[ur, "\beta"'] & \\
& B_1 \arrow[ur, bend right, "c_b"'] \arrow[u, phantom, "\Downarrow f_c"] & &
\end{tikzcd}$$

The first equation above holds by the definition of $\overline{\beta}'$ precomposed with f_1 upon cancelling f_c^{-1} and f_d with their inverses. The second equation holds since $(C, \beta, \overline{\beta})$ is a cocone to the top row. The left hand composite above equals:

$$\begin{tikzcd}
& A_1 \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & B_0 \arrow[dr, "\beta"] \arrow[dd, phantom, "\Downarrow \overline{\beta}'"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[dr, "q_a"'] & B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] \arrow[dr, "q_b"'] \arrow[d, phantom, "\Downarrow f_q^{-1}"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[d, phantom, "\Downarrow \tau_b"] & & C \\
& A_1 \arrow[r, "f_1"'] & B_1 \arrow[r, "c_b"'] & B_0 \arrow[ur, "\beta"'] &
\end{tikzcd}
=
\begin{tikzcd}
& & B_1 \arrow[dr, "d_b"] \arrow[d, phantom, "\Downarrow f_d^{-1}"] & & \\
& A_1 \arrow[ur, "f_1"] \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \arrow[dr, "\beta"] \arrow[dd, phantom, "\Downarrow \overline{\beta}'"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"] \arrow[dr, "q_a"'] & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[dr, "c_a"'] \arrow[d, phantom, "\Downarrow \tau_a"] & B_1 \arrow[ur, "d_b"] \arrow[dr, "c_b"'] \arrow[d, phantom, "\Downarrow f_c^{-1}"] & & C \\
& A_1 \arrow[r, "c_a"'] \arrow[dr, "f_1"'] & B_0 \arrow[r, "1"'] \arrow[d, phantom, "\Downarrow f_c"] & B_0 \arrow[ur, "\beta"'] & \\
& & B_1 \arrow[ur, "c_a"'] & &
\end{tikzcd}$$

$$=
\begin{tikzcd}
& B_1 \arrow[dr, bend left, "d_b"] \arrow[d, phantom, "\Downarrow f_d^{-1}"] & & \\
& A_1 \arrow[u, "f_1"] \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & B_0 \arrow[dr, "\beta"] \arrow[dd, phantom, "\Downarrow \overline{\beta}"] & \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"] \arrow[dr, "q_a"'] & A_1 \arrow[ur, "d_a"'] \arrow[dr, "c_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & & C \\
& A_1 \arrow[r, "c_a"'] \arrow[d, "f_1"'] & B_0 \arrow[ur, "\beta"'] & \\
& B_1 \arrow[ur, bend right, "c_b"'] \arrow[u, phantom, "\Downarrow f_c"] & &
\end{tikzcd}$$

Only the first equation here requires some justification. The second equation holds by definition of $\overline{\beta}'$ precomposed with f_1 upon cancelling inverses. The main difference between the first two diagrams is that the former has $m_b f_2$ along its middle whilst the second has $f_1 m_a$. This change is obtained by an intermediate step, where one introduces the pair of inverse isomorphisms: $f_m^{-1} : m_b f_2 \cong f_1 m_a$ and $f_m : f_1 m_a \cong m_b f_2$ through the middle. We will equate the top left sections of each composite, and lower left sections separately: focusing only upon the top lefts. It will suffice to show that:

$$\begin{tikzcd}
& A_1 \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & B_0 \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] \arrow[dr, "m_a"'] & B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] \arrow[d, phantom, "\Downarrow f_m^{-1}"] & B_1 \arrow[ur, "d_b"] & \\
& A_1 \arrow[ur, "f_1"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& & B_1 \arrow[dr, "d_b"] \arrow[d, phantom, "\Downarrow f_d^{-1}"] & \\
& A_1 \arrow[ur, "f_1"] \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"'] & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"'] & B_1 \arrow[ur, "d_b"] &
\end{tikzcd}$$

Pasting the isomorphisms f_d and f_m top and bottom on either side this is equally to show:

$$\begin{tikzcd}
& & B_0 \arrow[dr, "1"] \arrow[d, phantom, "\Downarrow f_d"] & \\
& A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_p"] & B_1 \arrow[r, "d_b"] \arrow[d, phantom, "\Downarrow \tau_b"] & B_0 \\
A_2 \arrow[ur, "p_a"] \arrow[r, "f_2"] & B_2 \arrow[ur, "p_b"] \arrow[r, "m_b"] & B_1 \arrow[ur, "d_b"] &
\end{tikzcd}
=
\begin{tikzcd}
& A_1 \arrow[r, "d_a"] \arrow[d, phantom, "\Downarrow \tau_a"] & B_0 \arrow[r, "1"] \arrow[d, phantom, "\Downarrow f_d"] & B_0 \\
A_2 \arrow[ur, "p_a"] \arrow[r, "m_a"'] \arrow[dr, "f_2"'] & A_1 \arrow[ur, "d_a"] \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow f_m"] & B_1 \arrow[ur, "d_b"] & \\
& B_2 \arrow[ur, "m_b"'] & &
\end{tikzcd}$$

This latter equality holds by pseudonaturality of f. Thus we have justified the above string of equations and have therefore verified the codescent cocone condition (1) of Definition 6.31.

$\square$

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