Characterizations of lax orthogonal factorization systems

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Outline

2-monads and pseudomonads

A 2-monad (T, η, μ) on a 2-category $\mathscr{C}$ consists of a 2-functor $T\colon \mathscr{C} \to \mathscr{C}$ and a unit $\eta\colon 1 \to T$ and a multiplication $\mu\colon TT \to T$ rendering commutative

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & TT & \xleftarrow{T\eta} & T \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \mu} & \swarrow_{\mathrm{id}} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} TTT & \xrightarrow{\mu T} & TT \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T \end{array}$$

In the case of a pseudomonad, we only require T to be pseudofunctor and the squares to commute up to isomorphism (and these isomorphisms must satisfy some coherence conditions).

KZ 2-monads and pseudomonads

Usually one would think of 2-monads as being nicer than pseudomonads, as it involves less data and simpler coherence conditions.

However, in the Kock-Zöberlein (lax idempotent) case, pseudomonads are nicer!

A KZ pseudomonad can be defined in a number of ways, most commonly:

- as a pseudomonad T for which $T\eta \dashv \mu$ or equivalently $\mu \dashv \eta T$;
- as a pseudomonad T with a 2-cell $T\eta \Rightarrow \eta T$ satisfying some coherence axioms.

These structured definitions of KZ pseudomonads involve a lot of data. Thus we prefer to use a *property-like definition* - a definition that involves less data but more properties.

KZ pseudomonads

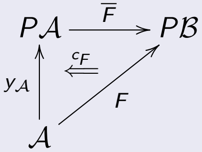
Definition (Marmolejo, Wood)

A *KZ pseudomonad* (P, y) on a bicategory $\mathscr{C}$ consists of
(i) An assignation on objects $P\colon \text{ob}\mathscr{C} \to \text{ob}\mathscr{C}$;
(ii) For every object $\mathcal{A} \in \mathscr{C}$, a 1-cell $y_{\mathcal{A}}\colon \mathcal{A} \to P\mathcal{A}$;
such that:

KZ pseudomonads

Definition (Marmolejo, Wood)

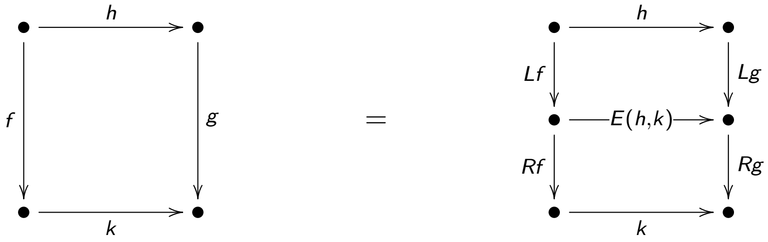
(a) For every pair of objects $\mathcal{A}$ and $\mathcal{B}$ and 1-cell $F\colon \mathcal{A} \to P\mathcal{B}$, there exists a left extension



with the exhibiting 2-cell c_F invertible;
(b) For every object $\mathcal{A} \in \mathscr{C}$, the left extension of $y_{\mathcal{A}}$ along itself is the identity
(c) For any 1-cell $G\colon \mathcal{B} \to P\mathcal{C}$, the corresponding left extension $\overline{G}\colon P\mathcal{B} \to P\mathcal{C}$ preserves the left extension $\overline{F}$.

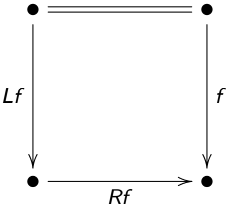
Functorial factorizations

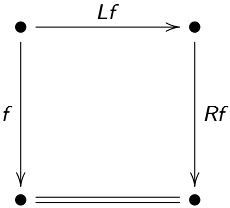
A functorial factorization is a functor $[\mathbf{2},\mathscr{C}] \to [\mathbf{3},\mathscr{C}]$ which is a section of the composition functor. This means any square as on the left below admits a decomposition:



Functorial factorizations

A functorial factorization yields a counit ε for L, and a unit η for R, which at each component f is given by the squares





respectively.

Algebraic weak factorization systems

Definition (Garner, Grandis, Tholen)

Given a category $\mathscr{C}$, an *algebraic weak factorization system* on $\mathscr{C}$ consists of:

- a functorial factorization (L, E, R) on $\mathscr{C}$
- an extension of (L, ε) to a comonad;
- an extension of (R, η) to a monad;

such that the canonical natural transformation $\lambda\colon LR \Rightarrow RL$ defines a mixed distributive law.

In an AWFS we have not just the property that there exists a filler, but an algebraic structure describing how they are constructed.

Pseudo algebraic weak factorization systems

Definition

Given a 2-category $\mathscr{C}$, a *pseudo algebraic weak factorization system* on $\mathscr{C}$ consists of:

- a pseudofunctorial factorization (L, E, R) on $\mathscr{C}$
- an extension of (L, ε) to a pseudocomonad;
- an extension of (R, η) to a pseudomonad;
- an extension of the canonical pseudo-natural transformation $\lambda\colon LR \Rightarrow RL$ to a mixed pseudo-distributive law.

Lax orthogonal AWFS

We will use a categorified version of Clementino & Lopez-Franco's definition, with pseudomonads in place of 2-monads.

Definition

Given a 2-category $\mathscr{C}$, a *lax orthogonal factorization system* on $\mathscr{C}$ is a pseudo algebraic weak factorization system (L, R) where both L and R are KZ.

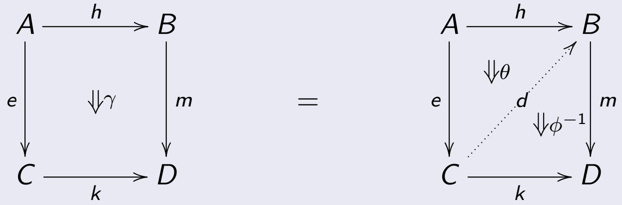
If L and R are both pseudo-idempotent, we call this a *pseudo orthogonal factorization system*.

In the case of a KZ pseudomonad, an algebraic structure is a property. So we again view the fillers as some property.

KZ-fillers

Definition (Clementino, Franco)

Two morphisms e and m are lax-orthogonal (written $e \perp_{\text{lax}} m$) if for any isomorphism γ as on the left below

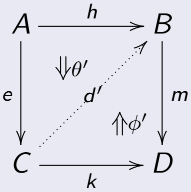


there exists a factorization containing a 1-cell d, and isomorphisms θ and ϕ as above.

KZ-fillers

Definition (Clementino, Franco)

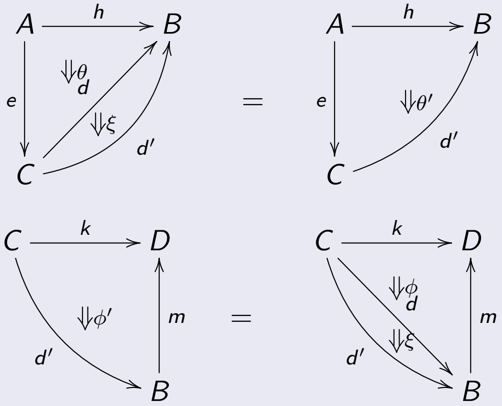
This is required to be universal in that, given any other "factorization" consisting of comparison 2-cells



such that $\theta' = \phi' e \cdot \gamma$, there exists a unique $\xi\colon d \Rightarrow d'$ such that

KZ-fillers

Definition (Clementino, Franco)



Lax orthogonal AWFS

Example

Every functor $F \colon \mathcal{A} \to \mathcal{B}$ factors as a left adjoint of a coreflection followed by an opfibration. The canonical factorization is given by

$$\mathcal{A} \to F \downarrow \mathcal{B} \to \mathcal{B}$$

where $F \downarrow \mathcal{B}$ has objects given by triples $(a, b, w \colon Fa \to b)$ and morphisms given by pairs $(h \colon a \to a', k \colon b \to b')$ yielding commuting squares

$$\begin{array}{ccc} Fa & \xrightarrow{w} & b \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle k} \\ Fa' & \xrightarrow[w']{} & b' \end{array}$$

Property like definition of Lax OFS?

	Usual definition	Property-like definition
KZ pseudomonads	$Ty \dashv \mu$	left extension definition
Lax OFS	KZ pseudo-AWFS	???

Goal

Definition

A *pseudo-orthogonal factorization system* $(\mathscr{E},\mathscr{M})$ on a 2-category $\mathscr{C}$ consists of two classes $\mathscr{E}$ and $\mathscr{M}$ of 1-cells of $\mathscr{C}$, such that:
(1) every $e \in \mathscr{E}$ is pseudo-orthogonal to every $m \in \mathscr{M}$;
(2) the classes $\mathscr{E}$ and $\mathscr{M}$ contain all equivalences and are closed under composition and invertible 2-cells;
(3) every 1-cell f in $\mathscr{C}$ admits a pseudo-factorization $f \cong m \cdot e$.

We want a definition like this for *lax* orthogonal factorization systems.

Differences with pseudo OFS

Just looking at the factorization of the identity tells us many of the differences. The canonical factorization of the identity on a category $\mathcal{A}$ is given by

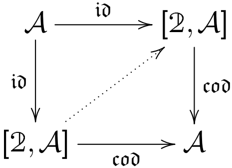
$$\mathcal{A} \xrightarrow{\mathfrak{id}} [\mathfrak{2}, \mathcal{A}] \xrightarrow{\mathfrak{cod}} \mathcal{A}$$

Thus we have already noticed:

1. Factorizations are not unique - two identities give another factorization

Differences with pseudo OFS

2. Even between two canonical factorizations

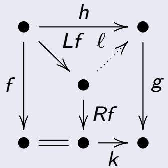


the filler is not unique - the identity and $\mathfrak{id} \cdot \mathfrak{cod}$ are two distinct fillers.

Universal property of factorizations in an AWFS

Fact (Bourke, Garner)

Given an AWFS (L, R), for any morphism f the factorization $f \cong Rf \cdot Lf$ is universal: given any R-algebra (g, x) and pair $(h, k) : f \to g$



there exists a unique filler ℓ such that $(\ell, k) : (Rf, \mu_f) \to (g, x)$ is an R-homomorphism (plus the dual property).

We can forget about the algebra structure maps as these are properties for KZ pseudomonads.

KZ algebras and homomorphisms

Definition

For a KZ pseudomonad (P, y) on a bicategory $\mathscr{C}$, a *P-pseudoalgebra* is an object $\mathcal{X}$ such that for all $F\colon \mathcal{A} \to \mathcal{X}$ there exists a left extension

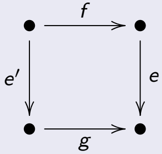
$$\begin{array}{ccc} P\mathcal{A} & \xrightarrow{\overline{F}} & \mathcal{X} \\ {\scriptstyle y_{\mathcal{A}}}\uparrow & \overset{c_F}{\Leftarrow} & \nearrow {\scriptstyle F} \\ \mathcal{A} & & \end{array}$$

with the exhibiting 2-cell c_F invertible. A *P-homomorphism* $\mathcal{X} \to \mathcal{Y}$ of P-pseudoalgebras is a morphism which preserves these left extensions.

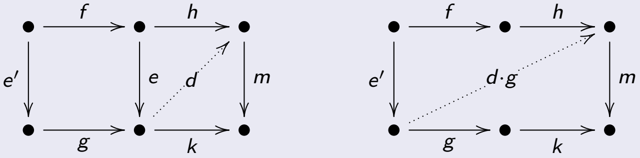
$\mathscr{E}$ and $\mathscr{M}$ homomorphisms

Definition

An $\mathscr{E}$-homomorphism is a square



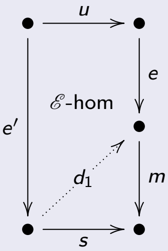
which preserves KZ-fillers of pseudocommuting squares

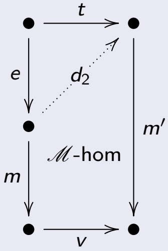


John's definition

Definition

A composable pair (e, m) is *KZ-universal* if



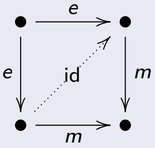


(1) for any square as on the left above, a filler d_1 is universal if and only if the indicated square is an $\mathscr{E}$-homomorphism;
(2) for any square as on the right above, a filler d_2 is universal if and only if the indicated square is an $\mathscr{M}$-homomorphism.

My definition

Definition

A composable pair (e, m) is *KZ-universal* if:
(1) the pair (e, m) respects the identity filler; meaning that the universal filler for the square

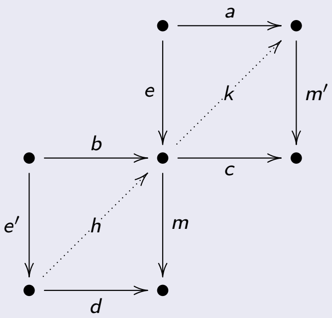


is the identity;

My definition

Definition

(2) the pair (e, m) respects binary composition of fillers; meaning that for any $e' \in \mathscr{E}$ and $m' \in \mathscr{M}$ and two universal fillers as below

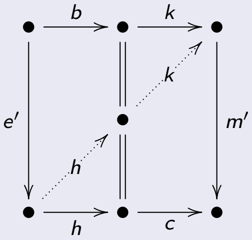


where $b \cong e \cdot u$ or $c \cong v \cdot m$,

My definition

Definition

the composite $k \cdot h$ is the universal filler of the square



given by composing the left top triangle with the right bottom triangle.

My definition

The motivation for my definition is fibrant replacement. Given any object X we can factor $X \to \mathbf{1}$ to get

$$X \xrightarrow{y_X} TX \xrightarrow{!} \mathbf{1}$$

and this T should be a KZ pseudomonad on $\mathscr{C}$.

It is almost immediate from my definition that T defines a KZ pseudomonad on $\mathscr{C}$.

KZ-orthogonal factorization systems

Theorem

These two definitions of a KZ-universal pair (e, m) are equivalent.

Definition

A *KZ-orthogonal factorization system* $(\mathscr{E}, \mathscr{M})$ consists of two classes of morphisms $\mathscr{E}$ and $\mathscr{M}$ such that:

1. we have $\mathscr{E} = {}^{\perp}\mathscr{M}$ and $\mathscr{E}^{\perp} = \mathscr{M}$;
2. every 1-cell f in $\mathscr{C}$ admits a pseudo-factorization $f \cong m \cdot e$ where (e, m) is KZ-universal.

KZ-orthogonal factorization systems

Definition

A *KZ-orthogonal factorization system* $(\mathscr{E},\mathscr{M})$ consists of two classes of morphisms $\mathscr{E}$ and $\mathscr{M}$ such that:

1. every $e \in \mathscr{E}$ is lax orthogonal to every $m \in \mathscr{M}$;
2. the classes $\mathscr{E}$ and $\mathscr{M}$ contain all equivalences and are closed under composition and invertible 2-cells;
3. every 1-cell f in $\mathscr{C}$ admits a pseudo-factorization $f \cong m \cdot e$ where (e, m) is KZ-universal.
4. for any KZ-universal pair (e, m),
    1. if m is the left adjoint of a reflection, and $\eta_m \cdot e$ is invertible, then $m \cdot e \in \mathscr{E}$;
    2. if e is the right adjoint of a reflection, and $m \cdot \eta_e$ is invertible, then $m \cdot e \in \mathscr{M}$.

Main theorem

Theorem

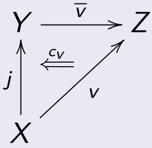
The following are in correspondence on a 2-category $\mathscr{C}$:

1. *lax-orthogonal factorization systems (L, R);*
2. *KZ-orthogonal factorization systems $(\mathscr{E}, \mathscr{M})$.*

KZ-orthogonality & Kan-injectives

Definition

In a 2-category $\mathscr{A}$, we say an object Z is kan-injective to a morphism $j\colon X \to Y$ if



for any $v\colon X \to Z$ there exists a left extension $\overline{v}$ as above exhibited by an invertible 2-cell.

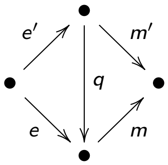
Lemma

The following are equivalent:

1. *f is lax orthogonal to g;*
2. *g is kan-injective to $(f, 1)\colon f \to 1$ in $[\mathbb{2}, \mathscr{C}]$.*

Similarities with pseudo OFS

In a pseudo OFS, between any two pseudofactorizations as below



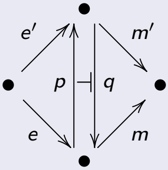
we have an induced equivalence q, and this equivalence is unique up to unique isomorphism. What happens in a lax OFS?

A general theme of lax OFSs vs pseudo OFSs is that equivalences are replaced by adjunctions.

Induced adjunctions

Theorem

Let (e, m) be a KZ-universal pair, and (e', m') be any pair, and suppose we have an isomorphism $\phi \colon m' \cdot e' \cong m \cdot e$. Then there exists an adjunction (and isomorphisms in the four triangles below)



coherent with ϕ and ϕ^{-1}, and such a coherent adjunction is unique up to unique isomorphism.

Semi-universal factorizations

Definition

We say a factorization is semi-KZ-universal if it is the whiskering of a KZ-universal pair (e, m) as below

$$\bullet \xrightarrow{e_w} \bullet \xrightarrow{e} \bullet \xrightarrow{m} \bullet \xrightarrow{m_w} \bullet$$

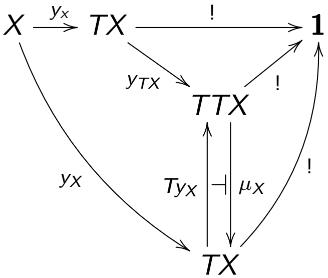
by an e_w and m_w in $\mathscr{E}$ and $\mathscr{M}$ respectively.

Fact

If (e', m') is semi-KZ-universal, then the induced adjunction $p \dashv q$ is a coreflection.

An example of semi-universal factorizations

Again consider fibrant replacement.



The induced adjunction $Ty_X \dashv \mu_X$ is a coreflection since the middle path is a semi-universal factorization.

Cases where we get a reflection

Denote the universal factorization of a morphism f as $Rf \cdot Lf$.

In a pseudo OFS, for any $e \in \mathscr{E}$ and $m \in \mathscr{M}$, we have that Re and Lm are equivalences. What happens in a lax OFS?

Lemma

For a morphism f:
(1) f lies in $\mathscr{E}$ if and only if Rf is a left adjoint part of a reflection and $\eta_{Rf} \cdot Lf$ is invertible;
(2) f lies in $\mathscr{M}$ if and only if Lf is a right adjoint part of a reflection and $Rf \cdot \eta_{Lf}$ is invertible.

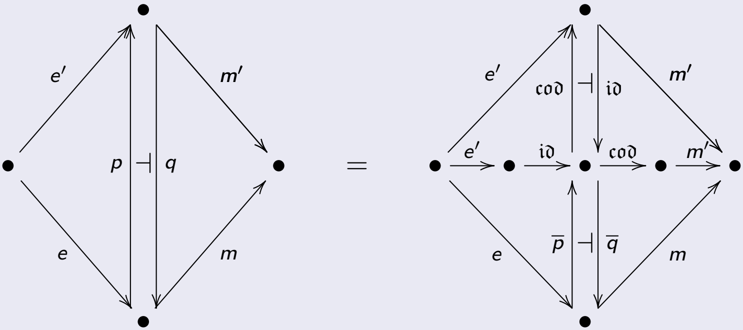
Example

Take f to be the identity, which has universal factorization $(\mathfrak{id}, \mathfrak{cod})$. Clearly $\mathfrak{cod}$ and $\mathfrak{id}$ comprise the reflection $\mathfrak{cod} \dashv \mathfrak{id}$.

Factoring induced adjunctions

Corollary

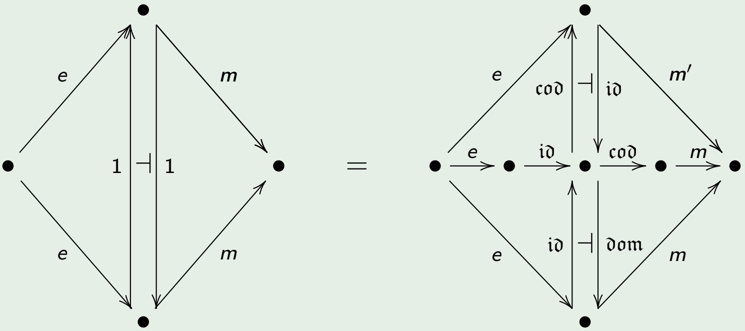
Any induced adjunction canonically factors as a reflection followed by a coreflection



Factoring induced adjunctions

Example

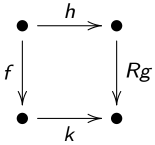
With adjoint triples $\mathfrak{cod} \dashv \mathfrak{id} \dashv \mathfrak{dom}$, the identity factors as

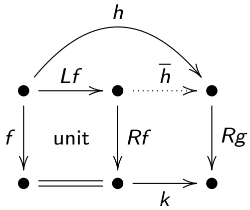


Still unsure if $\mathfrak{dom}$ exists for a general lax OFS.

Getting the KZ pseudomonad R

We have to define a KZ pseudomonad R on $[\mathbb{2}, \mathscr{C}]$. With $Rf \cdot Lf$ the KZ-universal factorization of f, the unit of R is as below

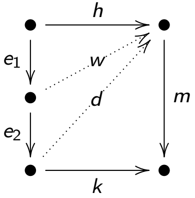




and given a square as on the left above, we factor through the unit as on the right above. The identity and composition axioms for the KZ pseudomonad follow from the corresponding axioms in my definition.

2-out-of-3 for homomorphisms

Given a square



(1) if we first construct d, then w as the uni filler for the top triangle d, then d is also the uni filler for the bottom square.
(2) if we first construct w, then d as the uni filler for the bottom square involving w, then d is also the uni filler for the original square.

2-out-of-3 for homomorphisms

Corollary

For a diagram

$$\begin{array}{ccc}
\bullet & \xrightarrow{a} & \bullet \\
{\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle e'_1} \\
\bullet & \xrightarrow{b} & \bullet \\
{\scriptstyle e_2}\downarrow & & \downarrow{\scriptstyle e'_2} \\
\bullet & \xrightarrow[c]{} & \bullet
\end{array}$$

the following are equivalent:

(1) the composite and the top square are $\mathscr{E}$-homomorphisms;

(2) the bottom and the top square are $\mathscr{E}$-homomorphisms.

Embeddings for a KZ pseudomonad

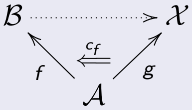
Definition

Given a KZ pseudomonad P on a 2-category $\mathscr{C}$, a 1-cell $f\colon \mathcal{A} \to \mathcal{B}$ is said to be *P-admissible* if Pf has a right adjoint res_f.

If the adjunction $Pf \dashv \mathrm{res}_f$ is a coreflection, we say f is a *P-embedding.*

Fact

A morphism f is a P-embedding if and only if for any P-pseudoalgebra $\mathcal{X}$ and 1-cell $g\colon \mathcal{A} \to \mathcal{X}$ there exists a left extension as below

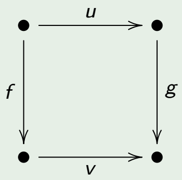


with the exhibiting 2-cell c_f invertible, and this left extension is preserved by P-homomorphisms.

Embeddings for a KZ pseudomonad

Example

Given a lax OFS (L, R), a morphism $(u, v) : f \to g$ as below



is Kan injective to $\mathscr{M}$ if and only if it's an R-embedding.

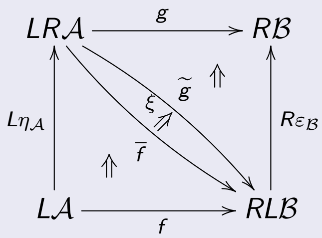
A mixed pseudo-distributive law of a pseudocomonad L and pseudomonad R consists of a pseudonatural transformation $\lambda \colon LR \Rightarrow RL$ and four invertible modifications subject to ten coherence axioms. In the KZ case, such a distributive law has a simpler description.

Mixed KZ pseudo-distributive laws

Theorem (80 percent complete)

Let (L, ε) be a KZ pseudocomonad and (R, η) be a KZ pseudomonad. Then a pseudo-distributive law $LR \Rightarrow RL$ consists of the assertions:

1. *for every $\mathcal{A} \in \mathscr{C}$, the morphism $L\eta_{\mathcal{A}} \colon L\mathcal{A} \to LR\mathcal{A}$ is a R-embedding;*
2. *for every $\mathcal{A} \in \mathscr{C}$, the morphism $R\varepsilon_{\mathcal{A}} \colon RL\mathcal{A} \to R\mathcal{A}$ is a L-embedding;*
3. *for any diagram as below which pastes to an isomorphism*



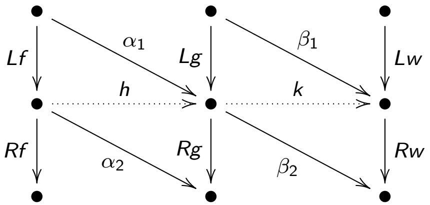
where $\overline{f}$ and $\widetilde{g}$ are the left extension and lifting, ξ is invertible.

The two sided Kleisli bicategory

Fact

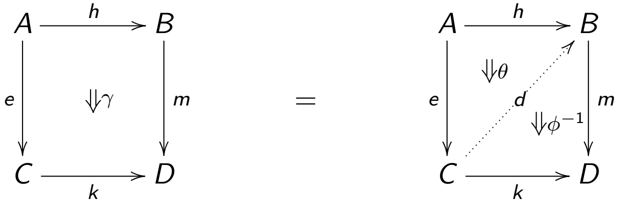
Given such a pseudo-distributive law, L preserves R-embeddings and R preserves L-embeddings. Thus, for example, if $(u, v) : f \to g$ *is kan injective to* $\mathscr{M}$, *then* $L(u, v)$ *is also kan injective to* $\mathscr{M}$.

The mixed pseudo-distributive law $\lambda \colon LR \Rightarrow RL$ yields the two-sided Kleisli bicategory $\mathbf{Kl}(\lambda)$ with objects those of $[\mathbb{2}, \mathscr{C}]$, and a morphism $f \nrightarrow g$ being a morphism $Lf \to Rg$ in $[\mathbb{2}, \mathscr{C}]$. Composition of a $f \nrightarrow g$ with a $g \nrightarrow w$ is given by constructing the diagram below



Fillers for lax commuting squares

We wish to extend the definition of a lax OFS to include fillers for squares as on the left below, where γ is only a comparison 2-cell



there exists a factorization containing a 1-cell d, 2-cell θ and isomorphism ϕ as above.

Fillers for lax commuting squares

Why extend the definition?

(1) A different example of two-dimensional factorizations (two dimensional generic factorizations) involves fillers for lax commuting squares.

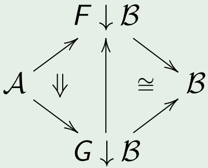
(2) An OFS can be defined as a suitable pair of classes $\mathscr{E}$ and $\mathscr{M}$, where each morphism f admits a factorization $m \cdot e$ which is unique up to unique isomorphism.

The obvious generalization fails for pseudo-OFS, since the definition of pseudo-OFS includes conditions concerning non-invertible 2-cells.

Example of such fillers

Example

Given functors $F, G\colon \mathcal{A} \to \mathcal{B}$ and a natural transformation $\alpha\colon F \Rightarrow G$ we have a factorization



where the middle functor sends $ga \to b$ to $fa \to ga \to b$.

Note the induced functor is in the opposite direction to the original natural transformation α.

F-bicategories

Definition

- an F-*bicategory* is a bicategory $\mathscr{A}$ equipped with an identity on objects, injective on 1-cells, locally fully faithful pseudofunctor $\mathscr{A}_{\mathsf{T}} \to \mathscr{A}$. The 1-cells of $\mathscr{A}_{\mathsf{T}}$ are called the *tight* 1-cells of $\mathscr{A}$ and are required to be closed under invertible 2-cells;
- an F-*pseudofunctor* $(\mathscr{A}, \mathscr{A}_T) \nrightarrow (\mathscr{B}, \mathscr{B}_T)$ is a pseudofunctor $F\colon \mathscr{A} \to \mathscr{B}$ which restricts to a pseudofunctor $F_T\colon \mathscr{A}_T \to \mathscr{B}_T$;
- a *lax* F-*natural transformation* $\alpha\colon F \Rightarrow G\colon (\mathscr{A}, \mathscr{A}_T) \to (\mathscr{B}, \mathscr{B}_T)$ is a lax natural transformation $\alpha\colon F \Rightarrow G$ such that both:
  1. for all $X \in \mathscr{A}$, $\alpha_X\colon FX \to GX$ is tight;
  2. for all $f\colon X \to Y$ tight, $\alpha_f\colon Gf \cdot \alpha_X \Rightarrow \alpha_Y \cdot Ff$ is invertible.

With modifications, this defines the tricategory F-**Bicat**.

F-bicategories

Example

Consider the bicategory $[\mathfrak{2},\mathscr{C}]_{\text{lax}}$ where squares are only required to commute up to comparison 2-cell; and take the tight maps as the pseudo-commuting squares. This is an F-bicategory.

Definition

A *KZ* F-*pseudomonad* on an F-bicategory $(\mathscr{C},\mathscr{C}_T)$ is a KZ pseudomonad in the tricategory F-**Bicat**, where the unit is fully pseudonatural.

Down-lax orthogonal AWFS

Definition

Given a 2-category $\mathscr{C}$, a *down-lax orthogonal factorization system* on $\mathscr{C}$ is a lax orthogonal factorization system on $\mathscr{C}$ such that

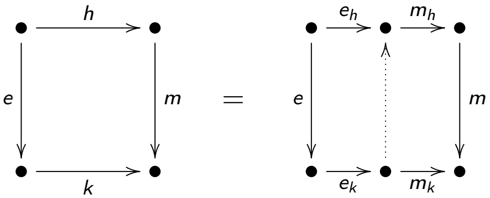
- L extends to a KZ F-pseudocomonad on $[\mathbb{2}, \mathscr{C}]_{\text{lax}}$;
- R extends to a KZ F-pseudomonad on $[\mathbb{2}, \mathscr{C}]_{\text{lax}}$;
- the mixed pseudo-distributive law $\lambda \colon LR \Rightarrow RL$ extends to a mixed F-pseudodistributive law.

If L and R are both pseudo-idempotent, we call this a *down-pseudo orthogonal factorization system*.

Down-KZ orthogonal factorization systems

In a weak factorization system it suffices to assume that we only have fillers for squares

to get fillers for general squares of the form



Down-KZ orthogonal factorization systems

This also works for one-dimensional orthogonal factorization systems. We want a result like this for 2-dimensional lax and pseudo orthogonal factorization systems.

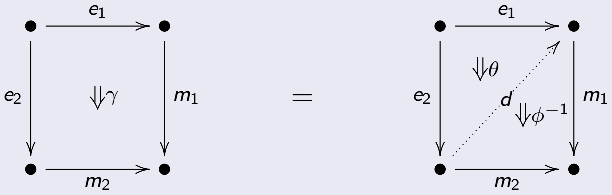
Definition

A *down-KZ orthogonal factorization system* $(\mathscr{E}, \mathscr{M})$ on a 2-category $\mathscr{C}$ consists of two classes $\mathscr{E}$ and $\mathscr{M}$ of 1-cells of $\mathscr{C}$, such that:
(1) the classes $\mathscr{E}$ and $\mathscr{M}$ contain all equivalences and are closed under composition and invertible 2-cells;
(2) every 1-cell f in $\mathscr{C}$ admits a pseudofactorizations $f \cong m \cdot e$ where (e, m) is KZ-universal;
(3) for any KZ-universal pair (e, m),
(a) if m is the left adjoint of a reflection, and $\eta_m \cdot e$ is invertible, then $m \cdot e \in \mathscr{E}$;
(b) if e is the right adjoint of a reflection, and $m \cdot \eta_e$ is invertible, then $m \cdot e \in \mathscr{M}$;

Down-KZ orthogonal factorization systems

Definition

(4) any square and 2-cell γ as on the left below

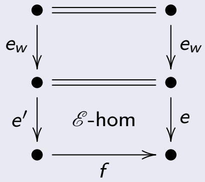


admits a universal filler (d, θ, ϕ) as on the right above with ϕ invertible;

Down-KZ orthogonal factorization systems

Definition

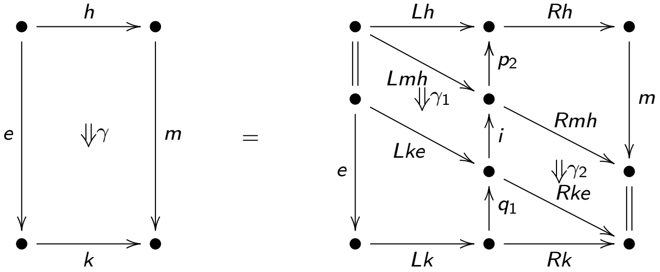
(5) for any $\mathscr{E}$-homomorphism as below



the composite above is also an $\mathscr{E}$-homomorphism.

Down-KZ orthogonal factorization systems

The proof relies on constructing a general filler as



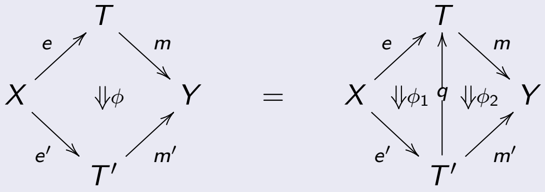
and constructing comparison 2-cells using units and counits of adjunctions.

Down-pseudo orthogonal factorization systems

Corollary

A down-pseudo orthogonal factorization system $(\mathscr{E},\mathscr{M})$ on a 2-category $\mathscr{C}$ consists of two classes $\mathscr{E}$ and $\mathscr{M}$ such that:

1. *the classes $\mathscr{E}$ and $\mathscr{M}$ contain all equivalences and are closed under composition and invertible 2-cells;*
2. *given any 2-cell $\phi\colon m\cdot e\Rightarrow m'\cdot e'$ there exists a factorization*



with ϕ_2 invertible, which is unique up to unique isomorphism.

Current work

Current work includes:

(1) finishing the proof of mixed pseudo-distributive law;

(2) find more examples and applications of lax orthogonal factorization systems;

(3) to better understand down-lax OFS.

The End

Thank you!