Metric monads

J. Rosický

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Varieties of universal algebra (= categories algebras of Lawvere theories) correspond to finitary monads on $\mathbf{Set}$ (= monads preserving filtered colimits) [Linton 1965].

Lawvere theories deal with (X, Y)-ary operations interpreted as mappings $A^X \to A^Y$; they are Y-tuples of X-ary operations. [Linton 1969] showed that algebras for every monad on $\mathbf{Set}$ are described by equations between such operations. In particular, λ-ary monads (= monads preserving λ-filtered colimits) correspond to λ-ary equational theories (= X, Y are λ-presentable).

The reason is that (X, Y)-ary operations correspond to morphisms $FY \to FX$ between free algebras and free algebras are dense in algebras.

Motivated by Linton, I introduced the following concept in 1977.

Let $\mathcal{K}$ be a category. A *type* t over $\mathcal{K}$ is a class Ω equipped with a mapping $t : \Omega \to \text{ob}(\mathcal{K}) \times \text{ob}(\mathcal{K})$ such that $\Omega^{X,Y} = t^{-1}(X, Y)$ are sets for every $X, Y \in \text{ob}(\mathcal{K})$. Elements of $\Omega^{X,Y}$ are called (X, Y)-*ary operation symbols* of type t.

Terms of type t are inductively defined as follows:

1. Every $\omega \in \Omega$ is a $t(\omega)$-ary term,
2. Every morphism $f : Y \to X$ of $\mathcal{K}$ determines an (X, Y)-ary term x_f,
3. If p is an (X, Y)-ary term and q an (Y, Z)-ary term then qp is an (X, Z)-ary term,
4. $x_{gf} = x_g x_f$, and
5. $(pq)r = p(qr)$.

Equations $p = q$ of type t are pairs (p, q) of terms of type t. An *equational theory* of type t is a class E of equations of type t.

In what follows, we will denote the set $\mathcal{K}(X, A)$ as A^X. Similarly, $A^f = \mathcal{K}(f, A)$ and $h^X = \mathcal{K}(X, h)$.

An *algebra* of type t is an object A of $\mathcal{K}$ equipped with mappings $\omega_A : A^X \to A^Y$ for every (X, Y)-ary operation symbol ω of t. Terms are interpreted in A as follows:

1. $(x_f)_A = A^f$, and
2. $(qp)_A = q_A p_A$.

An algebra A *satisfies* an equation $p = q$ if $p_A = q_A$. It satisfies a theory E if it satisfies all equations of E.
A *homomorphism* $h : A \to B$ of t-algebras are morphisms $h : A \to B$ such that $h^Y \omega_A = \omega_B h^X$ for every $\omega \in \Omega^{X,Y}$.
$\boldsymbol{Alg}(E)$ will denote the category of E-algebras and $U_E : \boldsymbol{Alg}(E) \to \mathcal{K}$ will be the forgetful functor.

My original definition of terms did not contain condition 5. But it does not change the interpretation of terms on algebras.

My equational theories and algebras coincide with pretheories and algebras in the sense of [Bourke, Garner 2019].

Let $U : \mathcal{A} \to \mathcal{K}$ be a faithful functor, i.e., $\mathcal{A}$ is *concrete* over $\mathcal{K}$. For $X, Y \in \mathrm{ob}(\mathcal{K})$, let $\Omega^{X,Y}$ consist of natural transformations $U^X \to U^Y$ where $U^X = \mathcal{K}(X, U-)$. Since $\Omega(X, Y)$ do not need to be sets, the resulting "type" is not legitimate. Ignoring this, we get terms where $x_f = U^f$ and $\omega\omega'$ is the composition. This yields the "equational theory" E_U and the functor $H_U : \mathcal{A} \to \boldsymbol{Alg}(E_U)$ such that $H_U(A)$ is A equipped with components ω_A as operations.

Assume that U has a left adjoint F. Then natural transformations $\omega : U^X \to U^Y$ correspond to natural transformations $\mathcal{A}(FX, -) \to \mathcal{A}(FY, -)$ and thus to morphisms $FY \to FX$. Hence we get a type t_U and an equational theory E_U.

This is due to Linton and it goes back to [Lawvere 1963]. The fundamental result of Linton is that if U is monadic then the comparison functor $H_U : \mathcal{K}^T \to \boldsymbol{Alg}(E_U)$ from the category of T-algebras is an equivalence. Moreover, given a T-algebra, elements $a \in A^X$ correspond to morphisms $FX \to A$.

Let $\mathcal{K}$ be a locally λ-presentable category. We say that a type t over $\mathcal{K}$ is λ-*ary* if all arities X and Y of its operation symbols are λ-presentable. A theory is λ-ary if its type is λ-ary. In the terminology of [Bourke, Garner], our λ-ary theories coincide with $\mathcal{K}_\lambda$-pretheories where $\mathcal{K}_\lambda$ denotes the (representative) small full subcategory consisting of λ-presentable objects.

Following [Bourke, Garner], λ-ary monads on $\mathcal{K}$ correspond to λ-ary equational theories over $\mathcal{K}$.

Let E be a λ-ary equational theory over a locally λ-presentable category $\mathcal{K}$ and $\mathcal{A} \subseteq \mathcal{K}_\lambda$ be a dense subcategory consisting of (some) λ-presentable objects. Let E' be a subtheory of E consisting of (X, Y)-ary operations where $Y \in \mathcal{A}$. Then the reduct functor $\boldsymbol{Alg}(E) \to \boldsymbol{Alg}(E')$ is an equivalence.

Let $\lambda' < \lambda$. We say that a type is (λ', λ)-ary if all arities X are λ'-presentable and all arites of its operation symbols Y are λ-presentable.

Lemma 1. Monads given by (λ', λ)-ary equational theories on a locally λ-presentable category preserve λ'-filtered colimits.

The category ***Pos*** of posets and monotone mappings is locally finitely presentable. Hence finitary monads on ***Pos*** correspond to finitary equational theories. Since $\{1, 2\}$ is dense in ***Pos***, it suffices to take only $(X, 1)$ and $(X, 2)$-ary operations. The first are usual X-ary operations and the second express an inequation $f \leq g$ between X-ary operations. Hence finitary monads on ***Pos*** correspond to finitary inequational theories, which was shown in [Adámek, Ford, Milius, Schröder 2020].

The category of simplicial sets is locally finitely presentable. Hence finitary monads on correspond to finitary equational theories. Since simplices Δ_n are dense, it suffices to take only (X, Δ_n)-ary operations. (X, Δ_0)-ary operations are usual X-ary operation. An (X, Δ_1)-ary operation expresses a homotopy between two X-ary operations because $A^X \to A^{\Delta_1}$ correspond to $\Delta_1 \times A^X \to A$. For instance, we capture homotopy monoids where associativity is only up to homotopy.

The category ***Met*** of (generalized) metric spaces (distances ∞ are allowed) and non-expanding maps is locally $\aleph_1$-presentable. $\aleph_1$-presentable metric spaces are precisely those having cardinality $\leq \aleph_0$. Hence $\aleph_1$-ary monads on correspond to $\aleph_1$-ary equational theories. $1, 2_\varepsilon$ are dense where 2_ε has two points with distance ε. An $(X, 2_\varepsilon)$-ary operation expresses that two X-ary operations have distance $\leq \varepsilon$. Thus they give "metric equations" $f =_\varepsilon g$ used in [Weawer 1995].

However, ***Met*** is isometry-locally finitely generated in the sense of [Di Liberti, JR 2020]. A metric space is finitely generated w.r.t. isometries iff it is finite.

Let $\mathcal{K}$ be a $\mathcal{M}$-locally λ-generated category. We say that a type t over $\mathcal{K}$ is λ-*ary* if all arities X and Y of its operation symbols are λ-generated w.r.t. $\mathcal{M}$.

Every $\mathcal{M}$-locally λ-generated category is locally μ-presentable for some $\mu \geq \lambda$.

Theorem 1. Let $\mathcal{K}$ be $\mathcal{M}$-locally λ-generated and locally μ-presentable for $\lambda \leq \mu$. Let E be a λ-ary equational theory over $\mathcal{K}$. Then $\textbf{\textit{Alg}}(E)$ monadic and the induced monad T_E is μ-ary and sends λ-directed colimits of $\mathcal{M}$-morphisms to λ-directed colimits.

But T_E does not need to preserve $\mathcal{M}$-morphisms.

Theorem 2. Let $\mathcal{K}$ be $\mathcal{M}$-locally λ-generated and locally μ-presentable for $\lambda \leq \mu$. Let T be a μ-ary monad on $\mathcal{K}$ preserving $\mathcal{M}$-morphisms and λ-directed colimits of $\mathcal{M}$-morphisms. Then there is a λ-ary equational theory E such that $\mathcal{K}^T$ and $\textbf{\textit{Alg}}(E)$ are equivalent (as concrete categories over $\mathcal{K}$).

Finitary equational theories over ***Met*** are precisely those whose operations have finite metric spaces as arities. The corresponding monads preserve $\aleph_1$-directed colimits and send directed colimits of isometries to directed colimits. Conversely, if a monad preserves $\aleph_1$-directed colimits, isometries and sends directed colimits of isometries to directed colimits then it is given by a finitary equational theory. But monads given by finitary equational theories do not need to preserve isometries.

Every finite discrete metric space (= distinct points have distance ∞) is perfectly presentable, i.e., its hom-functor preserves sifted colimits (= filtered colimits and reflexive coequalizers). Hence monads for metric algebras of [Weawer] preserve sifted colimits.

This property has every monad induced by an operad on ***Met***. Recall that ***Met*** is symmetric monoidal closed category where $A \otimes B$ is $A \times B$ with the $+$-metric making the distance $d((x,y),(x',y'))$ equal to $d(x,x') + d(y,y')$. For instance, the monad for monoids where the monoid operation is $\cdot : M \otimes M \to M$.

Similarly, the monad for generalized normed spaces which are monoids in ***Met*** equipped with unary operations $c \cdot -$, $|c| \leq 1$. $\| x + y \| \leq \| x \| + \| y \|$ follows from $+$ being nonexpanding.

It does not seem that these monads are given by finitary equational theories.

The unit ball functor ***Ban*** $\to$ ***Met*** is monadic where ***Ban*** is the category of Banach spaces and linear maps of norm ≤ 1 (see [JR 2020]. The correspoding monad preserves $\aleph_1$-filtered colimits but it does not send directed colimits of isometries to directed colimits.

Everything can be made enriched. Given a symmetric monoidal closed category $\mathcal{V}$ and a $\mathcal{V}$-category $\mathcal{K}$, we denote the $\mathcal{V}$-object $\mathcal{K}(X, A)$ as A^X.

We can immediately define enriched equational theory and they correspond to enriched pretheories of [Bourke, Garner]. If $\mathcal{V}$ is locally λ-presentable as a closed category and $\mathcal{K}$ is locally λ-presentable $\mathcal{V}$-category then [Bourke, Garner] showed that enriched λ-ary equational theories over $\mathcal{K}$ correspond to enriched λ-ary monads on $\mathcal{K}$.

[Linton 1966] was made enriched in [Dubuc 1970]. [Power 1999] made enriched [Linton 1965] by showing that enriched finitary monads on a symmetric monoidal closed locally finitely presentable (as a closed category) $\mathcal{V}$ correspond to enriched Lawvere theories over $\mathcal{V}$ where finite products are replaced by finite cotensors. [Nishizawa, Power 2009] did the same for a locally finitely presentable $\mathcal{V}$-category $\mathcal{K}$ but their Lawvere theories were uncomfortable. Finally, [Bourke, Garner 2019] made the final step.

For an enriched equational theory over ***Pos*** operations $A^X \to A^Y$ are monotone. Hence enriched finitary theories over ***Pos*** correspond to inequational theories for coherent algebras from [Adámek, Ford, Milius, Schröder 2020]. Hence we get their result that they correspond to enriched finitary monads on ***Pos***.

For an enriched equational theory over ***Met*** operations $A^X \to A^Y$ are nonexpanding. The special case when arities X are finite discrete and arities Y are finite correspond to finitary unconditional equational theories for quantitative algebras from [Mardare, Panangaden, Plotkin 2017].

Met is isometry-locally finitely generated as a closed category. This means that it is an enriched isometry-locally finitely generated (see [Di Liberti, JR]) with the tensor unit 1 finitely generated w.r.t. isometries and with the tensor product of two objects finitely generated w.r.t. isometries being finitely generated w.r.t. isometries.

Theorems 1 and 2 can be made enriched and we get connections between finitary enriched monads on ***Met*** and finitary enriched equational theories over ***Met***.

Sifted colimits are those which commute in ***Set*** with finite products. Similarly, they are those which commute with finite products in ***Met***. This follows from the fact that discrete metric spaces form a coreflective full subcategory of ***Met*** closed under products. But they do not form an enriched coreflective subcategory. Hence ***Met*** might contain other weighted colimits commuting with finite products.

This happens in ***Pos*** where finite products commute with reflexive coinserters (see [Bourke 2010]). Every finite poset can be obtained from finite discrete posets by means of reflexive coinserters and hom-functors $\boldsymbol{Pos}(A, -) : \boldsymbol{Pos} \to \boldsymbol{Pos}$ of finite discrete posets A preserve reflexive coinserters. An argument analogous to Lemma 1 implies that enriched monads given by enriched finitary equational theories over ***Pos*** given by (X, Y)-ary operations with X discrete preserves not only filtered colimits but also reflexive coinserters. This is shown in [Adámek, Dostál, Velebil 2020] where, moreover, there is proved the converse. This can be also shown using (X, Y)-ary operations.

The category ***CMet*** of complete metric spaces is locally $\aleph_1$-presentable as a closed category and the unit ball functor ***Ban*** $\to$ ***CMet*** is monadic ([JR 2020]). The corresponding monad preserves directed colimits but ***CMet*** is not isometry-locally finitely generated.

The category ***Ban*** is symmetric monoidal closed w.r.t. the projective tensor product $\| x \cdot y \| = \| x \| \| y \|$. Also, ***Ban*** is isometry-locally finitely presentable where finite-dimensional Banach spaces are finitely generated w.r.t. isometries.

Unital Banach algebras are monadic over ***Ban*** and the corresponding monad preserves filtered colimits. Here, $\| x \cdot y \leq \| x \| \| y \|$ because $\cdot$ has norm ≤ 1. The same holds for unital involutive Banach algebras.

Unital C^*-algebras are involutive Banach algebras satisfying $\| x^* \cdot x \| = \| x \|^2$. The forgetful functor ***CAlg*** $\to$ ***Ban*** has a left adjoint but I do know know whether it is monadic.