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**THEORY OF REGULARLY AND
RAPIDLY VARYING FUNCTIONS
ON TIME SCALES
AND ITS APPLICATION TO
DYNAMIC EQUATIONS**

Dissertation Thesis

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Introduction

The aims of this thesis are two. First, to establish the theory of regular and rapid variation on time scales, which would naturally supplement and extend the existing theory of regular and rapid variation from the continuous and discrete case. Second, to apply the obtained theory in a study of asymptotic behavior of solutions to linear and half-linear second order dynamic equations on time scales.

The theory of regular and rapid variation was studied at the first time for functions of real variable. It was initiated by Jovan Karamata in 1930, see [21], hence it is sometimes called Karamata theory. Nowadays, it is a very extensive theory, useful in many fields of mathematics; beside classic uses in asymptotic theory of functions, it is also used in, e.g., Tauberian theory, analytic number theory and theory of probability. A similar theory of regularly and rapidly varying sequences was initiated also by Karamata as a counterpart of the continuous case. However, in 1973, J. Galambos and E. Seneta, see [16], introduced an alternative theory for sequences based on a purely sequential conception. Their approach shows a new (alternative) way in the theory of regular and rapid variation, useful not only in this discrete case but convenient in our considerations. Finally, note that the theory of regular and rapid variation is a very good tool, which helps us to get precise information about asymptotic properties of solutions of differential and difference equations.

The theory of time scales was introduced by Stephan Hilger in his dissertation thesis in 1988, see [19], in order to unify the continuous and discrete calculus. Before, the theory of differential and difference equations was studied “separately”. The theory of dynamic equations allows us to prove certain results for these two cases simultaneously and moreover for any arbitrary general time scale.

This thesis is divided into six chapters. Each chapter is furthermore divided into a few sections with the exception of Chapter 1 and Chapter 6 - these two are devoted to introduction and conclusion of this thesis. In Chapter 2 we recall basic notations and state all basic statements that we will need later. The main part of the thesis are Chapter 3 and Chapter 4. In Chapter 3 we establish the theory of regular and rapid variation on general time scales with graininess $\mu(t) = o(t)$ (exceptionally, in some special cases the graininess $\mu(t) = O(t)$ “is allowed”). Note that for “bigger” graininess (as we show on examples later) it is impossible to

establish any reasonable theory for general time scales. As an application of our theory, we study the asymptotic properties of solutions to linear and half-linear second order dynamic equations on time scales. In Chapter 4 we establish corresponding theory and applications for the important time scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, which has the graininess $\mu(t) = (q - 1)t$, and hence it cannot be studied within previous (more general) case. Chapter 3 is based on the papers [P3], [P4], [P5] and Chapter 4 contains results of the papers [P2] and [P1]. In Chapter 5, oscillatory results from paper [P6] are attached. These results are not connected with the theory of regular (resp. rapid) variation. However, half-linear dynamic equation and studies of its oscillation is a strong point of connection with previous chapters.

Preliminaries

In this chapter we recall basic notations and state all basic statements that we will need later.

2.1 Continuous and discrete theory of regular and rapid variation

At first, recall the basic facts about regular variation. A measurable function f , $f : [a, \infty) \rightarrow (0, \infty)$, is said to be *regularly varying of index* ϑ , $\vartheta \in \mathbb{R}$, if it satisfies

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\vartheta \quad \text{for all } \lambda > 0; \quad (2.1)$$

we write $f \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *slowly varying*. Fundamental properties of regularly varying functions are that relation (2.1) holds uniformly on each compact λ -set in $(0, \infty)$ and $f \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$ if and only if it may be written in the form $f(x) = \varphi(x)x^\vartheta \exp\left\{\int_a^x \eta(s)/s \, ds\right\}$, where φ and η are measurable with $\varphi(x) \rightarrow C \in (0, \infty)$ and $\eta(x) \rightarrow 0$ as $x \rightarrow \infty$. For further reading on the continuous case we refer, e.g., [4, 17, 20, 21, 22, 23, 26, 27, 37].

In the basic theory of regularly varying sequences two main approaches are known. First, the approach by Karamata [21], based on a counterpart of the continuous definition: A positive sequence $\{f_k\}$, $k \in \{a, a+1, \dots\} \subset \mathbb{Z}$, is said to be *regularly varying of index* ϑ , $\vartheta \in \mathbb{R}$, if

$$\lim_{k \rightarrow \infty} \frac{f_{[\lambda k]}}{f_k} = \lambda^\vartheta \quad \text{for all } \lambda > 0, \quad (2.2)$$

where $[u]$ denotes the integer part of u . Second, the approach by Galambos and Seneta [16], based on a purely sequential definition: A positive sequence $\{f_k\}$ is said to be *regularly varying of index* ϑ if there exists a positive sequence $\{\omega_k\}$ satisfying

$$f_k \sim C\omega_k \quad \text{and} \quad \lim_{k \rightarrow \infty} k \left(1 - \frac{\omega_{k-1}}{\omega_k}\right) = \vartheta, \quad (2.3)$$

C being a positive constant. In [8], it was shown that these two definitions are equivalent. In [29], the second condition in (2.3) was suggested to replace

(equivalently) in the latter definition by $\lim_{k \rightarrow \infty} k \Delta \omega_k / \omega_k = \vartheta$. A regularly varying sequence can be represented as $f_k = \varphi_k k^\vartheta \prod_{j=a}^{k-1} (1 + \psi_j/j)$, see [29], or as $f_k = \varphi_k k^\vartheta \exp \left\{ \sum_{j=a}^{k-1} \psi_j/j \right\}$, where $\varphi_k \rightarrow C \in (0, \infty)$ and $\psi_k \rightarrow 0$ as $k \rightarrow \infty$, see [8, 16]. For further reading on the discrete case we refer, e.g., to [11].

Recall that the theory of regular variation can be viewed as the study of relations similar to (2.1) or (2.2), together with their wide applications, see, e.g., [4, 17, 20, 26, 27, 29, 30, 37]. There is a very practical way how regularly varying functions can be understood: Extension in a logical and useful manner of the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies “more slowly” than a power function. In [8, 16], see also [40], the so-called embedding theorem was established (and the converse result holds as well): If $\{y_k\}$ is a regularly varying sequence, then the function R (of a real variable), defined by $R(x) = y_{[x]}$, is regularly varying. Such a result makes it then possible to apply the continuous theory to the theory of regularly varying sequences. However, the development of a discrete theory, analogous to the continuous one, is not generally close, and sometimes far from a simple imitation of arguments for regularly varying functions, as noticed and demonstrated in [8]. Simply, the embedding theorem is just one of powerful tools, but sometimes it is not immediate that from a continuous results its discrete counterpart is easily obtained thanks to the embedding; sometimes it is even not possible to use this tool and the discrete theory requires a specific approach, different from the continuous one.

Now, recall the concept of rapid variation. A measurable function of real variable $f : [a, \infty) \rightarrow (0, \infty)$ is said to be *rapidly varying of index* ∞ if it satisfies

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \begin{cases} \infty & \text{for } \lambda > 1 \\ 0 & \text{for } 0 < \lambda < 1; \end{cases} \quad (2.4)$$

we write $f \in \mathcal{RPV}_{\mathbb{R}}(\infty)$. If

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \begin{cases} 0 & \text{for } \lambda > 1 \\ \infty & \text{for } 0 < \lambda < 1, \end{cases} \quad (2.5)$$

then f is said to be *rapidly varying of index* $-\infty$; we write $f \in \mathcal{RPV}_{\mathbb{R}}(-\infty)$. Note that it is easy to show that in relations (2.4) and (2.5) it is not necessary to include both cases $\lambda > 1$ and $0 < \lambda < 1$; more precisely,

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \infty \text{ (resp. } 0) \quad \forall \lambda > 1 \quad \Leftrightarrow \quad \lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = 0 \text{ (resp. } \infty) \quad \forall \lambda \in (0, 1).$$

For more information about rapid variation on \mathbb{R} , see for example [4, 21, 22, 27] and references therein.

In [28], the concept of rapidly varying sequences was introduced in the following way. Let $[u]$ denote the integer part of u . A positive sequence $\{f_k\}$, $k \in \{a, a+1, \dots\} \subset \mathbb{Z}$, is said to be *rapidly varying of index ∞* , if it satisfies

$$\lim_{k \rightarrow \infty} \frac{f_{[\lambda k]}}{f_k} = \begin{cases} \infty & \text{for } \lambda > 1 \\ 0 & \text{for } 0 < \lambda < 1; \end{cases} \quad (2.6)$$

we write $f \in \mathcal{RPV}_{\mathbb{Z}}(\infty)$. A positive sequence $\{f_k\}$, $k \in \{a, a+1, \dots\} \subset \mathbb{Z}$, is said to be *rapidly varying of index $-\infty$* , if it satisfies

$$\lim_{k \rightarrow \infty} \frac{f_{[\lambda k]}}{f_k} = \begin{cases} 0 & \text{for } \lambda > 1 \\ \infty & \text{for } 0 < \lambda < 1; \end{cases} \quad (2.7)$$

we write $f \in \mathcal{RPV}_{\mathbb{Z}}(-\infty)$. Note that the concept of rapidly varying sequence of index ∞ was introduced in [10] as $\lim_{k \rightarrow \infty} f_{[\lambda k]}/f_k = 0$ for $\lambda \in (0, 1)$. Similarly, as in previous case, in [10], it was shown that

$$\lim_{k \rightarrow \infty} \frac{f_{[\lambda k]}}{f_k} = \infty \text{ (resp. } 0) \quad \forall \lambda > 1 \quad \Leftrightarrow \quad \lim_{k \rightarrow \infty} \frac{f_{[\lambda k]}}{f_k} = 0 \text{ (resp. } \infty) \quad \forall \lambda \in (0, 1).$$

It is easy to see that the function $f(t) = a^t$ (resp. sequence $f_k = a^k$) with $a > 1$ is a typical representant of the class $\mathcal{RPV}_{\mathbb{R}}(\infty)$ (resp. $\mathcal{RPV}_{\mathbb{Z}}(\infty)$), while the function $f(t) = a^t$ (resp. sequence $f_k = a^k$) with $a \in (0, 1)$ is a typical representant of the class $\mathcal{RPV}_{\mathbb{R}}(-\infty)$ (resp. $\mathcal{RPV}_{\mathbb{Z}}(-\infty)$). Of course, these classes are much wider. In continuous case (for sequences, it is analogical), the extension is possible of previous examples to the class of functions where asymptotic behavior is that of an exponential function multiplied by a factor which varies “more slowly” than an exponential function, e.g., to the functions in the form of $f(t) = g(t)a^{h(t)}$, where $g \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$ or g is bounded both above and below by the positive constants, $h \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$ with $\vartheta > 0$ and $a \in (0, 1) \cup (1, \infty)$. The case $a \in (0, 1)$ stands for $f \in \mathcal{RPV}_{\mathbb{R}}(-\infty)$, while the case $a \in (1, \infty)$ stands for $f \in \mathcal{RPV}_{\mathbb{R}}(\infty)$.

As can we see, the forms of definitions of rapidly varying functions (2.4), (2.5) and rapidly varying sequences (2.6), (2.7), which include a parameter λ , correspond to the classic Karamata type definitions of regularly varying functions (2.1) and regularly varying sequences (2.2). In [27], it was shown that for any rapidly varying function f of index $-\infty$ for which $f'(t)$ increases one has

$$\lim_{t \rightarrow \infty} \frac{tf'(t)}{f(t)} = -\infty. \quad (2.8)$$

Conversely, if a continuously differentiable function f satisfies (2.8), then it is rapidly varying of index $-\infty$. In [28], it was similarly shown that if a positive sequence $\{f_k\}$ has the property Δf_k increases, then $f \in \mathcal{RPV}_{\mathbb{Z}}(-\infty)$ if and only if

$$\lim_{k \rightarrow \infty} \frac{k\Delta f_k}{f_k} = -\infty. \quad (2.9)$$

This results show us that under certain conditions there exists an alternative (in some cases – e.g. when studying asymptotic properties of differential or difference equations – more practical) possibility, how rapidly varying functions (resp. sequences) can be defined. For further reading of rapid variation in discrete case we refer to, e.g., [10, 11, 28, 30] and references therein.

2.2 Essentials on time scales

In this section we recall basic information concerning the calculus on time scales that are needed for further considerations.

Definition 2.1. Time scale \mathbb{T} is an arbitrary nonempty closed subset of \mathbb{R} .

We suppose that \mathbb{T} has inherited standard (Euclidean) topology on the real numbers \mathbb{R} . This is a (typical) example of time scale:

$$\mathbb{R}, \quad \mathbb{Z}, \quad \mathbb{N}, \quad \mathbb{N}_0, \quad [a, b], \quad [a, \infty),$$

$$h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}, \quad q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}, \quad q > 1.$$

Note that calculus on $h\mathbb{Z}$ is called *h-calculus*, while calculus on $q^{\mathbb{N}_0}$ is called *q-calculus*.

Definition 2.2. Let \mathbb{T} be a time scale. Define the *forward jump operator* σ for all $t \in \mathbb{T}$ such that $t < \sup \mathbb{T}$, by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the *backward jump operator* for all $t \in \mathbb{T}$ such that $t > \inf \mathbb{T}$, by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. If $\sigma(t) = t$, we say that t is *right-dense*, while if $\rho(t) = t$, we say that t is *left-dense*. If $\rho(t) < t < \sigma(t)$, we say that t is *isolated*, while if $\rho(t) = t = \sigma(t)$, we say that t is *dense*. If \mathbb{T} has a left-scattered maximum t_{\max} , then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{t_{\max}\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If \mathbb{T}^κ has a right-scattered minimum t_{\min} , then we define $\mathbb{T}^{\iota, \kappa} = \mathbb{T}^\kappa \setminus \{t_{\min}\}$, otherwise $\mathbb{T}^{\iota, \kappa} = \mathbb{T}^\kappa$. Finally, we also define $\mu(t) := \sigma(t) - t$ which is called the *graininess function*.

We will use the notation $f^\sigma(t) = f(\sigma(t))$, i.e., $f^\sigma = f \circ \sigma$. From the definition of the set \mathbb{T} is evident that $\sigma(t), \rho(t) \in \mathbb{T}$. If $\sup \mathbb{T} < \infty$, we define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$. Similarly, if $\inf \mathbb{T} > -\infty$, we define $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$. (In our case always $\sup \mathbb{T} = \infty$ and $\inf \mathbb{T} = a > -\infty$.) Finally, if every point $t \in \mathbb{T}$ is dense, we say that time scale \mathbb{T} is *continuous*, while if every point $t \in \mathbb{T}^{\iota, \kappa}$ is isolated, we say that time scale \mathbb{T} is *discrete*.

Remark 2.1. (i) In this thesis, by an interval $[a, b]$, where $a, b \in \mathbb{T}$, we will mean the set $\{t \in \mathbb{T} : a \leq t \leq b\}$. A symbol \mathcal{I}_a , where $a \in \mathbb{T}$, we will use for an infinite time scale interval $\{t \in \mathbb{T} : a \leq t < \infty\}$, i.e., for \mathbb{T} with $\sup \mathbb{T} = \infty$.

(ii) Let us remind that $\mu(t) = o(t)$ means that $\lim_{t \rightarrow \infty} (\mu(t)/t) = 0$, while $\mu(t) = O(t)$ means that there exists $c > 0$ such that $\mu(t)/t \leq c$ for each $t \in \mathbb{T}$.

Definition 2.3. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called Δ -differentiable at $t \in \mathbb{T}^\kappa$ with Δ -derivative $f^\Delta(t) \in \mathbb{R}$, if for any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We say that f is Δ -differentiable on \mathbb{T}^κ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

The following lemma shows some important properties of Δ -derivative.

Lemma 2.1 ([6, 19]). *Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be two functions, and let $t \in \mathbb{T}^\kappa$. Then we have*

(i) *If $f^\Delta(t)$ exists, then f is continuous at t .*

(ii) *If f is continuous at $t \in \mathbb{T}$ and t is right-scattered, then*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

(iii) *If $t \in \mathbb{T}$ is right-dense, then $f^\Delta(t)$ exists if and only if*

$$f^\Delta(t) = f'(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If $f^\Delta(t)$ exists, then*

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t). \tag{2.10}$$

(v) *If $f^\Delta(t)$ and $g^\Delta(t)$ exist, then fg is Δ -differentiable at t with*

$$(fg)^\Delta(t) = f^\sigma(t)g^\Delta(t) + f^\Delta(t)g(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

(vi) *Let f, g be such that $g(t)g^\sigma(t) \neq 0$ and $f^\Delta(t), g^\Delta(t)$ exist. Then f/g is Δ -differentiable at t with*

$$\frac{f}{g}^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

Remark 2.2.

$$f^\Delta(t) = \begin{cases} f'(t) & \text{for } \mathbb{T} = \mathbb{R}, \\ \Delta f(t) & \text{for } \mathbb{T} = \mathbb{Z}, \\ \frac{f^\sigma(t) - f(t)}{\mu(t)} & \text{for a discrete } \mathbb{T}. \end{cases}$$

Definition 2.4. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. We say that f is *rd-continuous* if it is continuous at each right-dense point in \mathbb{T} and $\lim_{s \rightarrow t^-} f(s)$ exists as a finite number for all left-dense points $t \in \mathbb{T}$. We write $f \in C_{\text{rd}}(\mathbb{T})$. If f is Δ -differentiable on a set \mathbb{T}^κ with $f^\Delta(t) \in C_{\text{rd}}(\mathbb{T}^\kappa)$, we write $f \in C_{\text{rd}}^1(\mathbb{T})$. If f is piecewise rd-continuously Δ -differentiable on \mathbb{T} , we write $f \in C_{\text{prd}}^1(\mathbb{T})$.

Definition 2.5. Let $f, F : \mathbb{T} \rightarrow \mathbb{R}$ be two functions and $f \in C_{\text{rd}}(\mathbb{T})$. If $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$, then F is said to be *antiderivative* of function f and we define the Δ -integral of f on $[a, b] \cap \mathbb{T}$ with $a, b \in \mathbb{T}$ by

$$\int_a^b f(t) \Delta t = F(b) - F(a)$$

and the Δ -integral of f on $[a, \infty) \cap \mathbb{T}$ by

$$\int_a^\infty f(s) \Delta s = \lim_{t \rightarrow \infty} \int_a^t f(s) \Delta s.$$

Note that the function F from Definition 2.5 always exists and is determined unambiguously. The following lemma shows some important properties of Δ -integral.

Lemma 2.2 ([6, 19]). *Let $f, g \in C_{\text{rd}}(\mathbb{T})$ and $a, b, c \in \mathbb{T}$ with $a \leq b \leq c$. Then*

- (i) $\int_t^{\sigma(t)} f(s) \Delta s = \mu(t)f(t)$ for $t \in \mathbb{T}^\kappa$.
- (ii) $\int_a^a f(t) \Delta t = 0$.
- (iii) $\int_a^c f(t) \Delta t = \int_a^b f(t) \Delta t + \int_b^c f(t) \Delta t$.
- (iv) $\int_a^b f(t)g^\Delta(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g^\sigma(t) \Delta t$ (*integration by parts*).

Remark 2.3.

$$\int_a^\infty f(s) \Delta s = \begin{cases} \int_a^\infty f(s) \, ds & \text{for } \mathbb{T} = \mathbb{R}, \\ \sum_{t=a}^\infty f(t) & \text{for } \mathbb{T} = \mathbb{Z}, \\ \sum_{t \in [a, \infty) \cap \mathbb{T}} \mu(t)f(t) & \text{for a discrete } \mathbb{T}. \end{cases}$$

Definition 2.6. We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is *regressive*, resp. *positive regressive* provided $1 + \mu(t)f(t) \neq 0$, resp. $1 + \mu(t)f(t) > 0$. If f is regressive and rd-continuous, we write $f \in \mathcal{R}(\mathbb{T})$, while if f is positive regressive and rd-continuous, we write $f \in \mathcal{R}^+(\mathbb{T})$.

Definition 2.7. We say that a function $e_f(t, s)$ is the *generalized exponential function* if $e_f(\cdot, t_0)$ is the unique solution of the initial value problem $y^\Delta = f(t)y$, $y(t_0) = 1$, where $f \in \mathcal{R}(\mathbb{T})$.

Here are some useful properties of generalized exponential function.

Lemma 2.3 ([6, 19]). *Let $f, g \in \mathcal{R}(\mathbb{T})$ and $e_f(t, s)$, $e_g(t, s)$ be two generalized exponential functions. Then we have*

- (i) $e_0(t, s) \equiv 1$ and $e_f(t, t) \equiv 1$ for all $s, t \in \mathbb{T}$.
- (ii) $e_f(t, s) = 1/e_f(s, t)$ for all $s, t \in \mathbb{T}$.
- (iii) $e_f(t, \tau)e_f(\tau, s) = e_f(t, s)$ for all $s, t, \tau \in \mathbb{T}$.
- (iv) $e_f(t, s)e_g(t, s) = e_{f+g+\mu(t)fg}(t, s)$ for all $s, t \in \mathbb{T}$.
- (v) If $f \in \mathcal{R}^+(\mathbb{T})$, then $e_f(t, s) > 0$ for all $s, t \in \mathbb{T}$.
- (vi) If $f \equiv c \in \mathbb{R}$, then $e_c(t, s) = \exp\{c(t - s)\}$ when $\mathbb{T} = \mathbb{R}$ and $s, t \in \mathbb{R}$.
- (vii) If $f \equiv c \in \mathbb{R}$, then $e_c(t, s) = (1 + c)^{(t-s)}$ when $\mathbb{T} = \mathbb{Z}$ and $s, t \in \mathbb{Z}$.

For further results on the calculus on time scales, see, for example [6, 19] and the references therein.

2.3 Dynamic equations on time scales

In this section, we recall some basic information about second order dynamic equations. We start with half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0 \quad (\text{HL}^\Delta\text{E})$$

on a time scale \mathbb{T} , where p and $1/r$ are real rd-continuous functions on \mathbb{T} with $r(t) \neq 0$, and $\Phi(y) = |y|^{\alpha-1} \text{sgn } y$ with $\alpha > 1$. (For an explanation, why we require $1/r \in C_{\text{rd}}(\mathbb{T})$ instead of $r \in C_{\text{rd}}(\mathbb{T})$, see, [31].) The terminology *half-linear* is motivated by the fact that the space of all solutions of (HL $^\Delta$ E) is homogeneous, but not generally additive. Thus, it has just “half of the properties” of a linear space. Equation (HL $^\Delta$ E) covers the half-linear differential equation (if $\mathbb{T} = \mathbb{R}$)

$$[r(t)\Phi(y')] + p(t)\Phi(y) = 0 \quad (\text{HLDE})$$

as well as the half-linear difference equation (if $\mathbb{T} = \mathbb{Z}$)

$$\Delta[r_k\Phi(\Delta y_k)] + p_k\Phi(y_{k+1}) = 0. \quad (\text{HL}\Delta\text{E})$$

Furthermore, the linear differential equation (frequently called as a Sturm-Liouville differential equation)

$$(r(t)y') + p(t)(y) = 0 \quad (\text{LDE})$$

is a special case of (HLDE) (when $\alpha = 2$). If $\Phi = \text{id}$ (i.e., $\alpha = 2$), then (HL Δ E) reduces to the linear (Sturm-Liouville) difference equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0. \quad (\text{L}\Delta\text{E})$$

Finally, the linear dynamic equation

$$(r(t)y^\Delta)^\Delta + p(t)y^\sigma = 0, \quad (\text{L}^\Delta\text{E})$$

which covers (LDE) and (L Δ E) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, is a special case of (HL $^\Delta$ E) (when $\alpha = 2$).

Oscillation and nonoscillation criteria have been established at first for equations (LDE) and (L Δ E), see, for example [1, 18, 38, 39], and later naturally extended on (HLDE), (HL Δ E), (L $^\Delta$ E) and (HL $^\Delta$ E), see, for example [6, 12, 13, 14, 31, 32, 33, 34, 35]. Because our investigation is concerned with equations (HL $^\Delta$ E) and (L $^\Delta$ E) and their special cases, we recall some concepts and facts about equation (HL $^\Delta$ E) as an equation including all other cases. The symbol Φ^{-1} denotes the inverse function of Φ . It can be shown that $\Phi^{-1}(y) = |y|^{\beta-1} \text{sgn } y$, where β is the conjugate number of α , i.e., $1/\alpha + 1/\beta = 1$. In this thesis, we will suppose that $\mathbb{T} = \mathcal{I}_a$, hence $p, 1/r \in C_{\text{rd}}(\mathcal{I}_a)$ with $r(t) \neq 0$. We say that $y \in C_{\text{rd}}^1(\mathcal{I}_a)$, is a solution of (HL $^\Delta$ E) provided

$$\{[r \Phi(y^\Delta)]^\Delta + p \Phi(y^\sigma)\}(t) = 0 \quad \text{holds for all } t \in \mathcal{I}_a.$$

Let us consider the initial value problem (IVP)

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0, \quad y(t_0) = A, \quad y^\Delta(t_0) = B \quad (2.11)$$

on \mathcal{I}_a , where $A, B \in \mathbb{R}$, $t_0 \in \mathcal{I}_a$.

Theorem 2.1 (Existence and Uniqueness [33, p. 380]). *Let p and $1/r$ are rd-continuous functions on \mathcal{I}_a . Then the IVP (2.11) has exactly one solution.*

Definition 2.8. We say that a nontrivial solution y of (HL $^\Delta$ E) has a *generalized zero* at t , if $r(t)y(t)y(\sigma(t)) \leq 0$. If $y(t) = 0$, we say that solution y has a *common zero* at t (common zero is a special case of generalized zero).

Definition 2.9. We say that a solution y of equation (HL $^\Delta$ E) is *nonoscillatory* on \mathcal{I}_a , if there exists $\tau \in \mathcal{I}_a$ such that does not exist any generalized zero at t for $t > \tau$. Otherwise, it is *oscillatory*. Oscillation may be equivalently defined as follows. A nontrivial solution y of (HL $^\Delta$ E) is called *oscillatory* on \mathcal{I}_a , if for every $\tau \in \mathcal{I}_a$ has y a generalized zero on \mathcal{I}_τ .

Theorem 2.2 (Sturm Type Separation Theorem [33, p. 388]). *Let x, y be two linearly independent solutions of $(HL^\Delta E)$. If there are $c_1, c_2 \in \mathcal{I}_a$, with $c_1 < c_2$, such that $(rxx^\sigma)(c_1) \leq 0$ and $(rxx^\sigma)(c_2) \leq 0$ (we exclude the case where $\sigma(c_1) = c_2$ and $y(c_2) = 0$), then there is $d \in [c_1, c_2]$ such that $(ryy^\sigma)(d) \leq 0$. Two nontrivial solutions of $(HL^\Delta E)$, which are not proportional, can not have a common zero.*

From the Sturm type separation theorem it is clear that if one solution of $(HL^\Delta E)$ is oscillatory (resp. nonoscillatory), then every solution of $(HL^\Delta E)$ is oscillatory (resp. nonoscillatory). Hence we can speak about *oscillation* or *nonoscillation* of equation $(HL^\Delta E)$.

Remark 2.4. In most literature one supposes only $r(t) > 0$, hence a generalized zero of a solution y is defined as a point $t \in \mathcal{I}_a$ such that only $y(t)y^\sigma(t) \leq 0$. This situation is common, in particular, in the continuous case (equation $(HLDE)$ and classic Sturm-Liouville differential equation), where the assumption of the continuity of $r(t)$ implies a preservation of $\text{sgn } r(t)$, hence (in this case) it is natural to suppose $r(t) > 0$ for all t . Thanks to Definition 2.8, which is designed just for the case $r(t) \neq 0$, it is guaranteed that all solutions of $(HL^\Delta E)$ are either oscillatory or nonoscillatory, thus every equation in the form of $(HL^\Delta E)$ can be classified as oscillatory or nonoscillatory. However, this assertion is not true, if we use a “more simple” definition of a generalized zero with $y(t)y^\sigma(t) \leq 0$. For more information about concept of generalized zero, see, e.g. [33] and references therein.

Theorem 2.3 (Sturm Type Comparison Theorem [33, p. 388]). *Consider the equation*

$$[R(t)\Phi(y^\Delta)]^\Delta + P(t)\Phi(y^\sigma) = 0, \quad (2.12)$$

and equation $(HL^\Delta E)$, where $R, r, P, p \in C_{rd}(\mathcal{I}_a)$ with $r(t), R(t) \neq 0$. Suppose that we have $R(t) \geq r(t)$ and $p(t) \geq P(t)$ for every $t \in \mathcal{I}_a$. If $(HL^\Delta E)$ is nonoscillatory on \mathcal{I}_a , then (2.12) is also nonoscillatory on \mathcal{I}_a .

Our approach to the oscillatory and nonoscillatory problems of $(HL^\Delta E)$ is based mainly on the application of the generalized Riccati dynamic equation

$$w^\Delta(t) + p(t) + \mathcal{S}[w, r, \mu](t) = 0, \quad (GR^\Delta E)$$

where

$$\mathcal{S}[w, r, \mu] = \lim_{\lambda \rightarrow \mu} \frac{w}{\lambda} \left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \lambda\Phi^{-1}(w))} \right). \quad (2.13)$$

It is not difficult to observe that

$$\mathcal{S}[w, r, \mu](t) = \begin{cases} \left\{ \frac{\alpha-1}{\Phi^{-1}(r)} |w|^\beta \right\} (t) & \text{at right-dense } t, \\ \left\{ \frac{w}{\mu} \left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \mu\Phi^{-1}(w))} \right) \right\} (t) & \text{at right-scattered } t. \end{cases}$$

The relation between (HL $^\Delta$ E) and (GR $^\Delta$ E) is following. If $y(t)$ is a solution of (HL $^\Delta$ E) with $y(t)y^\sigma(t) \neq 0$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$ we let

$$w(t) = \frac{r(t)\Phi(y^\Delta(t))}{\Phi(y(t))}. \quad (2.14)$$

Then for $t \in [t_1, t_2] \cap \mathcal{I}_a$, $w = w(t)$ satisfies equation (GR $^\Delta$ E). If $t \in [t_1, t_2] \cap \mathcal{I}_a$ is right-scattered, then from (2.10) and (GR $^\Delta$ E) we have

$$w(\sigma(t)) = \frac{r(t)w(t)}{\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))]} - \mu(t)p(t). \quad (2.15)$$

Note that (2.15) is trivially satisfied at a right-dense t . The following theorem can be understood as a central statement of the oscillation theory for equation (HL $^\Delta$ E).

Theorem 2.4 (Roundabout Theorem [33, p. 383]). *The following statements are equivalent*

- (i) *Every nontrivial solution of (HL $^\Delta$ E) has at most one generalized zero on \mathcal{I}_a .*
- (ii) *Equation (HL $^\Delta$ E) has a solution having no generalized zeros on \mathcal{I}_a .*
- (iii) *Equation (GR $^\Delta$ E) has a solution w with*

$$\{\Phi^{-1}(r) + \mu\Phi^{-1}(w)\}(t) > 0 \quad \text{for } t \in \mathcal{I}_a. \quad (2.16)$$

- (iv) *An α -degree functional \mathcal{F} ,*

$$\mathcal{F}(\xi; b, c) = \int_b^c \{r|\xi^\Delta|^\alpha - p|\xi^\sigma|^\alpha\}(t)\Delta t,$$

defined on $U(b, c) = \{\xi \in C_{\text{prd}}^1(\mathcal{I}_a) : \xi(b) = \xi(c) = 0\}$ is positive definite on U , i.e., $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U$ and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

The following theorem is a consequence of the Roundabout theorem and the Sturm type comparison theorem. The method of oscillation theory for (HL $^\Delta$ E), which uses the ideas of the following theorem, is usually referred to as the *Riccati technique*.

Theorem 2.5 (Riccati technique [33, p. 390]). *The following statements are equivalent*

- (i) *Equation (HL $^\Delta$ E) is nonoscillatory.*
- (ii) *There is $a \in \mathbb{T}$ and a function $w : \mathcal{I}_a \rightarrow \mathbb{R}$ such that (2.16) holds and $w(t)$ satisfies (GR $^\Delta$ E) for $t \in \mathcal{I}_a$.*

(iii) There is $a \in \mathbb{T}$ and a function $w : \mathcal{I}_a \rightarrow \mathbb{R}$ such that (2.16) holds and $w(t)$ satisfies

$$w^\Delta(t) + p(t) + \mathcal{S}[w, r, \mu](t) \leq 0 \quad \text{for } t \in \mathcal{I}_a.$$

Remark 2.5. If $\alpha = 2$, i.e., (HL $^\Delta$ E) reduces to (L $^\Delta$ E), then equation (GR $^\Delta$ E) reduces to the Riccati dynamic equation

$$w^\Delta(t) + p(t) + \frac{w^2(t)}{r(t) + \mu(t)w(t)} = 0 \quad (\text{R}^\Delta\text{E})$$

and Riccati substitution (2.14) can be written in a form

$$w(t) = \frac{r(t)y^\Delta(t)}{y(t)}. \quad (2.17)$$

Note that oscillatory theory of equation (HL $^\Delta$ E) will be needed in Chapter 5, while in Chapter 3 we will study nonoscillatory (precisely asymptotic) properties of solution of some special cases of equation (HL $^\Delta$ E), resp. (L $^\Delta$ E). For more information about equation (HL $^\Delta$ E), see [33]. In [6], we can find a detailed theory concerning equation (L $^\Delta$ E).

2.4 q -calculus and theory of q -difference equations

In this short section we briefly recall some important facts. We start with some preliminaries of q -calculus or *quantum calculus*, which is calculus on special lattice or time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$ and $\mu(t) = (q-1)t$. First note that the theory of q -calculus is a special case of the theory on time scales. However, for a comfort of the reader, we recall this theory in more detail.

Similarly as in general time scale case, by an interval $[a, b]_q$, where $a, b \in q^{\mathbb{N}_0}$, we will mean the set $\{t \in q^{\mathbb{N}_0} : a \leq t \leq b\}$. A symbol $[a, \infty)_q$ we will use for an infinite interval in $q^{\mathbb{N}_0}$, i.e., $[a, \infty)_q = \{a, aq, aq^2, \dots\}$ with $a \in q^{\mathbb{N}_0}$. The q -derivative of a function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

Here are some useful rules:

- (i) $D_q(fg)(t) = g(qt)D_q f(t) + f(t)D_q g(t) = f(qt)D_q g(t) + g(t)D_q f(t).$
- (ii) $D_q \left(\frac{f}{g} \right) (t) = \frac{g(t)D_q f(t) - f(t)D_q g(t)}{g(t)g(qt)}.$
- (iii) $f(qt) = f(t) + (q-1)tD_q f(t).$

The proper q -integral is defined by

$$\int_a^b f(t) d_q t = \begin{cases} (q-1) \sum_{t \in [a,b) \cap q^{\mathbb{N}_0}} t f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ (1-q) \sum_{t \in [b,a) \cap q^{\mathbb{N}_0}} t f(t) & \text{if } a > b, \end{cases}$$

$a, b \in q^{\mathbb{N}_0}$. The improper q -integral is defined by

$$\int_a^\infty f(t) d_q t = \lim_{b \rightarrow \infty} \int_a^b f(t) d_q t.$$

For $f \in \mathcal{R}(q^{\mathbb{N}_0})$ (i.e., for $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ satisfying $1 + (q-1)tf(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$) we get generalized exponential function

$$e_f(t, s) = \begin{cases} \prod_{\tau \in [s,t) \cap q^{\mathbb{N}_0}} [(q-1)\tau f(\tau) + 1] & \text{if } s < t \\ 1 & \text{if } s = t \\ 1 / \prod_{\tau \in [t,s) \cap q^{\mathbb{N}_0}} [(q-1)\tau f(\tau) + 1] & \text{if } s > t, \end{cases}$$

where $s, t \in q^{\mathbb{N}_0}$. Here are some useful properties of function $e_f(t, s)$:

(i) For $f \in \mathcal{R}(q^{\mathbb{N}_0})$, $e_f(\cdot, t_0)$ is a solution of the IVP

$$D_q y = f(t)y, \quad y(t_0) = 1, \quad t \in q^{\mathbb{N}_0}. \quad (\text{IVP}_q)$$

(ii) For $f \in \mathcal{R}^+(q^{\mathbb{N}_0})$ (i.e., for $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ satisfying $1 + (q-1)tf(t) > 0$ for all $t \in q^{\mathbb{N}_0}$), we have $e_f(t, s) > 0$ for all $s, t \in q^{\mathbb{N}_0}$.

(iii) If $f \in \mathcal{R}(q^{\mathbb{N}_0})$, then $e_f(t, \tau)e_f(\tau, s) = e_f(t, s)$ for all $s, t, \tau \in q^{\mathbb{N}_0}$.

(iv) If $f, g \in \mathcal{R}(q^{\mathbb{N}_0})$, then $e_f(t, s)e_g(t, s) = e_{f+g+t(q-1)f_g}(t, s)$ for all $s, t \in q^{\mathbb{N}_0}$.

For more details on this topic see [3, 9]. See also [6] for the calculus on time scales which contains q -calculus.

Now we recall some basic information about second order q -difference equations. We begin with half-linear q -difference equation

$$D_q[r(t)\Phi(D_q y(t))] + p(t)\Phi(y(qt)) = 0 \quad (\text{HL}q\text{E})$$

on $q^{\mathbb{N}_0}$ with $q > 1$, where $p, r : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ with $r(t) \neq 0$. The linear q -difference equation

$$D_q[r(t)D_q y(t)] + p(t)y(qt) = 0 \quad (\text{L}q\text{E})$$

is a special case of (HL q E) (when $\alpha = 2$). Of course, equations (HL q E) and (L q E) are the special cases of equations (HL Δ E) and (L Δ E), thus the theory established in previous section holds for these two equations too. Note that asymptotic properties of equations (L q E) and (HL q E) will be studied in Chapter 4. More detailed information about qualitative and quantitative properties of equations of type (L q E) we can find, e.g., in [5, 6, 7].

Regular and rapid variation on time scales with applications to dynamic equations

3

In following chapter we introduce the concept of regular and rapid variation on time scales, which extends and unifies the existing continuous and discrete theories. Later, we will use the established theory in applications, concretely, we will study asymptotic properties of solutions to half-linear (resp. linear) second order dynamic equations on time scales. The graininess of time scale \mathbb{T} is assumed to be $\mu(t) = o(t)$, in some cases we will assume that the graininess is $o(t) = O(t)$. In Chapter 4 we will study a special time scale case $\mathbb{T} = q^{\mathbb{N}_0}$ with $\mu(t) = (q-1)t$. The reasons why we consider only these two cases of graininess and why we study them separately, will be discussed in the end of this chapter in Section 3.5.

Throughout this chapter, \mathbb{T} is assumed to be unbounded above, i.e., considered on an interval of the form $\mathcal{I}_a = [a, \infty)$ with $a > 0$.

3.1 Theory of regular variation on time scales

Before we give the first definition, note that in some parts of this section the conditions on smoothness can be somehow relaxed. But we do not do it since our theory focuses on a generalization in the sense of a “domain of definition” rather than considering “badly behaving” functions. In [36] the concept of regular variation on \mathbb{T} was introduced in the following way.

Definition 3.1. A measurable function $f : \mathbb{T} \rightarrow (0, \infty)$ is said to be *regularly varying of index ϑ* , $\vartheta \in \mathbb{R}$, if there exists a positive rd-continuously Δ -differentiable function ω satisfying

$$f(t) \sim C\omega(t) \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{t\omega^\Delta(t)}{\omega(t)} = \vartheta, \quad (3.1)$$

C being a positive constant; we write $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *slowly varying*; we write $f \in \mathcal{SV}_{\mathbb{T}}$.

Using elementary properties of linear first order dynamic equations and generalized exponential functions $e_\delta(t, s)$, the following representation was established in [36].

Theorem 3.1. *A positive function $f \in C_{\text{rd}}(\mathcal{I}_a)$ belongs to $\mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if it has a representation*

$$f(t) = \varphi(t)e_{\delta}(t, a), \quad (3.2)$$

where $\varphi \in C_{\text{rd}}(\mathcal{I}_a)$ is a positive function tending to a positive constant and $\delta \in \mathcal{R}^+(\mathcal{I}_a)$ satisfies $\lim_{t \rightarrow \infty} t\delta(t) = \vartheta$.

Now we prove the representation theorem in the following form.

Theorem 3.2 (Representation theorem). *(i) Let $\mu(t) = o(t)$. It holds $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if it has a representation*

$$f(t) = \varphi(t)t^{\vartheta}e_{\eta}(t, a), \quad (3.3)$$

where φ is a positive measurable function tending to a positive constant and a function $\eta \in C_{\text{rd}}(\mathcal{I}_a)$ satisfies $\lim_{t \rightarrow \infty} t\eta(t) = 0$. If $\vartheta = 0$, then the condition $\mu(t) = o(t)$ can be omitted and (3.3) coincides with representation (3.2).

(ii) Let $\mu(t) = o(t)$. It holds $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if it has a representation

$$f(t) = \varphi(t)t^{\vartheta} \exp \left\{ \int_a^t \frac{\psi(s)}{s} \Delta s \right\}, \quad (3.4)$$

where φ is a positive measurable function tending to a positive constant and a function $\psi \in C_{\text{rd}}(\mathcal{I}_a)$ satisfies $\lim_{t \rightarrow \infty} \psi(t) = 0$. If $\vartheta = 0$, then the condition $\mu(t) = o(t)$ can be replaced by $\mu(t) = O(t)$.

Proof. We show the implications (3.2) \Rightarrow (3.3) \Rightarrow (3.4) \Rightarrow (3.1).

From (3.2), $f(t) \sim C_1 t^{\vartheta} L(t)$, where $C_1 > 0$ and $L(t) = e_{\delta}(t, a)t^{-\vartheta}$. Consequently,

$$\frac{tL^{\Delta}(t)}{L(t)} = t\delta(t) \left(\frac{t}{\sigma(t)} \right)^{\vartheta} - \vartheta \frac{t}{\sigma(t)} \left(\frac{\xi(t)}{\sigma(t)} \right)^{\vartheta-1} = o(1)$$

as $t \rightarrow \infty$, where $t \leq \xi(t) \leq \sigma(t)$, since $1 \leq \xi(t)/t \leq 1 + \mu(t)/t = 1 + o(1)$. Hence $L \in \mathcal{SV}_{\mathbb{T}}$, and so $L(t) = C_2 e_{\eta}(t, a)$ with $C_2 > 0$ and $\lim_{t \rightarrow \infty} t\eta(t) = 0$. This implies (3.3).

From (3.3) we have $f(t) \sim C_3 t^{\vartheta} \exp \left\{ \int_a^t \psi(s)/s \Delta s \right\} H(t)$, where $C_3 > 0$,

$$H(t) = e_{\frac{\psi}{t}}(t, a) \exp \left\{ - \int_a^t \frac{\psi(s)}{s} \Delta s \right\},$$

and $\lim_{t \rightarrow \infty} \psi(t) = 0$. We show that $\lim_{t \rightarrow \infty} H(t) = 1$. We have

$$e_{\frac{\psi}{t}}(t, a) = \exp \left\{ \int_a^t \xi_{\mu(s)} \frac{\psi(s)}{s} \Delta s \right\},$$

where

$$\xi_{\mu(t)}\left(\frac{\psi(t)}{t}\right) = \begin{cases} \ln(\mu(t)\psi(t)/t + 1)/\mu(t) & \text{for } \mu(t) > 0 \\ \psi(t)/t & \text{for } \mu(t) = 0. \end{cases}$$

In view of the equalities $\lim_{x \rightarrow 0} \ln(x + 1)/x = 1$ and $\lim_{t \rightarrow \infty} \mu(t)\psi(t)/t = 0$, we get $\xi_{\mu(t)}(\psi(t)/t) \sim \psi(t)/t$. Consequently, $\lim_{t \rightarrow \infty} H(t) = 1$, and so f has representation (3.4).

From (3.4) we have $f(t) = \varphi(t)\omega(t)$, where $\omega(t) = t^\vartheta \exp\left\{\int_a^t \psi(s)/s \Delta s\right\}$ with $\lim_{t \rightarrow \infty} \psi(t) = 0$. Then, at a right scattered t ,

$$\frac{t\omega^\Delta(t)}{\omega(t)} = \frac{(t^\vartheta)^\Delta}{t^{\vartheta-1}} + \left(\frac{\sigma(t)}{t}\right)^\vartheta \frac{\exp(\mu(t)\psi(t)/t) - 1}{\mu(t)\psi(t)/t} \psi(t),$$

while, at a right dense t , $t\omega^\Delta(t)/\omega(t) = \vartheta + \psi(t)$. Thanks to $\mu(t) = o(t)$ (resp. $\mu(t) = O(t)$ for $\vartheta = 0$) we get $\lim_{t \rightarrow \infty} t\omega^\Delta(t)/\omega(t) = \vartheta$, and so f satisfies (3.1). \square

Remark 3.1. From the last theorem it is clear that $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if $f(t) = t^\vartheta L(t)$, where $L \in \mathcal{SV}_{\mathbb{T}}$.

Next we prove that Definition 3.1 implies the following Karamata type definition.

Definition 3.2 (Karamata type definition). A measurable function $f : \mathbb{T} \rightarrow (0, \infty)$ satisfying

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \lambda^\vartheta \tag{3.5}$$

uniformly on each compact λ -set in $(0, \infty)$, where $\tau : \mathbb{R} \rightarrow \mathbb{T}$ is defined as $\tau(t) = \max\{s \in \mathbb{T} : s \leq t\}$, is said to be *regularly varying of index ϑ* ($\vartheta \in \mathbb{R}$) in the sense of Karamata. We write $f \in \mathcal{KR}\mathcal{V}_{\mathbb{T}}(\vartheta)$.

Theorem 3.3. Let $\mu(t) = o(t)$. If $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$, then $f \in \mathcal{KR}\mathcal{V}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then the condition $\mu(t) = o(t)$ can be replaced by $\mu(t) = O(t)$.

Proof. We want to show that (3.5) holds uniformly for λ from a compact subinterval $[c, d]$ of $(0, \infty)$. We confine our attention to $\lambda \geq 1$ (i.e., $1 \leq c < d$); the case $\lambda \in (0, 1)$ being handled similarly. From Theorem 3.2 we have (3.4). Hence,

$$\frac{f(\tau(\lambda t))}{f(t)} = \frac{\varphi(\tau(\lambda t))}{\varphi(t)} \left(\frac{\tau(\lambda t)}{t}\right)^\vartheta \exp\left\{\int_t^{\tau(\lambda t)} \frac{\psi(s)}{s} \Delta s\right\}. \tag{3.6}$$

Clearly, $\varphi(\tau(\lambda t))/\varphi(t) \rightarrow 1$ as $t \rightarrow \infty$ uniformly for $\lambda \in [c, d]$. To prove that $(\tau(\lambda t)/t)^\vartheta \rightarrow \lambda^\vartheta$ as $t \rightarrow \infty$ uniformly on the λ -set $[c, d]$, it is sufficient to show that $\sup_{\lambda \in [c, d]} |\lambda t/\tau(\lambda t) - 1| \rightarrow 0$ as $t \rightarrow \infty$. First note that for $x \in \mathbb{R}$, $x \geq a$,

$\tau(x) \leq x \leq \sigma(\tau(x))$, and thus $1 \leq x/\tau(x) \leq 1 + \mu(\tau(x))/\tau(x) = o(1)$ as $x \rightarrow \infty$. We have

$$\begin{aligned} \sup_{\lambda \in [c,d]} \left| \frac{\lambda t}{\tau(\lambda t)} - 1 \right| &\leq \sup_{\lambda \in [c,d]} \left(\frac{\sigma(\tau(\lambda t))}{\tau(\lambda t)} - 1 \right) \leq \sup_{\lambda \in [c,d]} \frac{\mu(\tau(\lambda t))}{\tau(\lambda t)} \leq \frac{\mu(\tau(\Lambda(t)t))}{\tau(ct)} \\ &= \frac{\mu(\tau(\Lambda(t)t))}{\tau(\Lambda(t)t)} \cdot \frac{\tau(\Lambda(t)t)}{\tau(ct)} \leq \frac{\mu(\tau(\Lambda(t)t))}{\tau(\Lambda(t)t)} \cdot \frac{\tau(dt)}{\tau(ct)} = o(1) \end{aligned}$$

as $t \rightarrow \infty$, where $\Lambda : \mathbb{T} \rightarrow [c, d]$ is a suitable function. The uniform convergence to 1 of the last term in (3.6) follows from

$$\begin{aligned} \sup_{\lambda \in [c,d]} \left| \int_t^{\tau(\lambda t)} \frac{\psi(s)}{s} \Delta s \right| &\leq \sup_{\lambda \in [c,d]} \int_t^{\tau(\lambda t)} \frac{|\psi(s)|}{s} \Delta s \leq \int_t^{\tau(dt)} \frac{|\psi(s)|}{s} \Delta s \\ &\leq (\tau(dt) - t) \sup_{s \geq t} \frac{|\psi(s)|}{s} \leq t(d-1) \sup_{s \geq t} \frac{|\psi(s)|}{s} \\ &\leq (d-1) \sup_{s \geq t} |\psi(s)| = o(1) \end{aligned}$$

as $t \rightarrow \infty$. □

Before showing that Karamata type definition makes an embedding possible, and, consequently, implies Definition 3.1, we prove a useful lemma.

Lemma 3.1. *Let $\mu(t) = O(t)$. If $f \in \mathcal{KR}\mathcal{V}_{\mathbb{T}}(\vartheta)$, then $f^\sigma(t)/f(t) \rightarrow 1$ as $t \rightarrow \infty$.*

Proof. If $\mu(t) = O(t)$, then $M \in \mathbb{N}$ exists such that $0 \leq \mu(t)/t \leq M - 1$ for all $t \in \mathbb{T}$. Hence $1 \leq (t + \mu(t))/t \leq M$ and thus

$$1 \leq \sigma(t)/t \leq M \quad \text{for all } t \in \mathbb{T} \quad (3.7)$$

and

$$1/M \leq \sqrt{t/\sigma(t)} \leq 1 \quad \text{for all } t \in \mathbb{T}. \quad (3.8)$$

We distinguish two cases.

(i) $\vartheta = 0$. By Definition 3.2

$$\lim_{t \rightarrow \infty} f(\tau(\Lambda(t)t))/f(t) = 1.$$

where $\Lambda : \mathbb{T} \rightarrow \mathbb{R}$ is a bounded function. Thanks to (3.7) we may take the function $\Lambda(t) = \sigma(t)/t$. Then we get

$$1 = \lim_{t \rightarrow \infty} \frac{f(\tau(t\sigma(t)/t))}{f(t)} = \lim_{t \rightarrow \infty} \frac{f^\sigma(t)}{f(t)}.$$

(ii) $\vartheta \neq 0$. From Definition 3.2 we have $\lim_{t \rightarrow \infty} f(\tau(t/\lambda))/f(t) = (1/\lambda)^\vartheta$ for all $\lambda > 0$ and thus we get

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\tau(\lambda t)/\lambda))}{f(\tau(\lambda t))} = \left(\frac{1}{\lambda}\right)^\vartheta \quad \text{for all } \lambda > 0. \quad (3.9)$$

Multiplying (3.5) by (3.9) and shifting t to $\sigma(t)$ we obtain

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\tau(\lambda \sigma(t))/\lambda))}{f^\sigma(t)} = 1 \quad \text{for all } \lambda > 0.$$

Hence,

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\tau(\Lambda(t)\sigma(t))/\Lambda(t)))}{f^\sigma(t)} = 1. \quad (3.10)$$

where $\Lambda : \mathbb{T} \rightarrow \mathbb{R}$ is a bounded function. Thanks to (3.8), we may take the function $\Lambda(t) = \sqrt{t/\sigma(t)}$. Since

$$\begin{aligned} \tau\left(\frac{\tau(\Lambda(t)\sigma(t))}{\Lambda(t)}\right) &= \tau\left(\frac{\tau\left(\sqrt{t/\sigma(t)}\sigma(t)\right)}{\sqrt{t/\sigma(t)}}\right) = \tau\left(\frac{\tau\left(\sqrt{t\sigma(t)}\right)}{\sqrt{t/\sigma(t)}}\right) \\ &= \tau\left(t\sqrt{\frac{\sigma(t)}{t}}\right) = \tau\left(\sqrt{t\sigma(t)}\right) = t \end{aligned}$$

we can rewrite (3.10) to $\lim_{t \rightarrow \infty} f(t)/f^\sigma(t) = 1$. \square

Remark 3.2. If $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$, then the property $\lim_{t \rightarrow \infty} f^\sigma(t)/f(t) = 1$ is almost immediate, however, only for $\mu(t) = o(t)$. Indeed, in view of (3.4) $f(t) = \varphi(t)\omega(t)$, where $\lim_{t \rightarrow \infty} \varphi(t) = C > 0$ and $\omega \in C_{\text{rd}}^1(\mathbb{T})$. Thanks to

$$\frac{\omega^\sigma(t)}{\omega(t)} = \frac{\omega(t) + \mu(t)\omega^\Delta(t)}{\omega(t)} = 1 + \frac{\mu(t)}{t} \cdot \frac{t\omega^\Delta(t)}{\omega(t)},$$

we have $\lim_{t \rightarrow \infty} f^\sigma(t)/f(t) = \lim_{t \rightarrow \infty} \omega^\sigma(t)/\omega(t) = 1 + 0 \cdot \vartheta = 1$.

Theorem 3.4 (Embedding theorem). *Assume that \mathbb{T} satisfies*

$$\begin{cases} \text{every large } t \in \mathbb{T} \text{ is isolated and } \mu \text{ is either bounded} \\ \text{or eventually nondecreasing with } \mu(t) = O(t) \text{ as } t \rightarrow \infty. \end{cases} \quad (3.11)$$

If $f \in \mathcal{KR}_{\mathbb{T}}(\vartheta)$, then the function $R : \mathbb{R} \rightarrow \mathbb{R}$ defined by $R(x) := f(\tau(x))$ satisfies $R \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$.

Proof. We can write

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{R(\lambda x)}{R(x)} &= \lim_{x \rightarrow \infty} \frac{f(\tau(\lambda x))}{f(\tau(x))} \cdot \frac{f(\tau(\lambda \tau(x)))}{f(\tau(\lambda \tau(x)))} = \lim_{x \rightarrow \infty} \frac{f(\tau(\lambda \tau(x)))}{f(\tau(x))} \cdot \frac{f(\tau(\lambda x))}{f(\tau(\lambda \tau(x)))} \\ &= \lambda^\vartheta \lim_{x \rightarrow \infty} \frac{f(\tau(\lambda x))}{f(\tau(\lambda \tau(x)))}. \end{aligned}$$

The theorem will be proved, if we show that

$$\lim_{x \rightarrow \infty} \frac{f(\tau(\lambda x))}{f(\tau(\lambda \tau(x)))} = 1 \quad \text{for all } \lambda > 0. \quad (3.12)$$

Due to [4, Theorem 1.4.3] it is enough to show that (3.12) holds for all λ in a set of positive measure. Next we show that for every λ from a suitably chosen set of positive measure there exists $A = A(\lambda) \in \mathbb{R}$ such that $\text{card}(\tau(\lambda \tau(x)), \tau(\lambda x)) \leq A$ for large $x \in \mathbb{R}$. If we show it, we can apply k -times ($k \leq A$) Lemma 3.2 (with the use of an obvious transitivity property) and hereby relation (3.12) will be verified.

(i) Let $\mu(t) < H$ ($H \in \mathbb{R}$) for large t . Then $x - \tau(x) < H$ and $\lambda x - \lambda \tau(x) < \lambda H$ for all $\lambda > 0$, hence $\tau(\lambda x) - \tau(\lambda \tau(x)) < (\lambda + 1)H < \infty$ for large x . Therefore, there exists $A \in \mathbb{R}$ such that $\text{card}(\tau(\lambda \tau(x)), \tau(\lambda x)) \leq A$ for large x and $\lambda > 0$.

(ii) Suppose that $\lim_{t \rightarrow \infty} \mu(t) = \infty$ and the function $\mu(t)$ is nondecreasing for large t . Let $\lambda > N$, where $N \in \mathbb{N}$ satisfies $\sigma(\tau(x))/\tau(x) \leq N$ for all x (this N exists, see (3.7)). Hence $\sigma(\tau(x))/\tau(x) \leq \lambda$ and therefore, $x < \lambda \tau(x)$ for all $x \in \mathbb{R}$. Using this inequality we can write

$$\begin{aligned} \tau(\lambda x) &\leq \tau(\lambda \sigma(\tau(x))) = \tau(\lambda \tau(x) + \lambda \mu(\tau(x))) \leq \lambda \tau(x) + \lambda \mu(\tau(x)) \\ &\leq \tau(\lambda \tau(x)) + \mu(\tau(\lambda \tau(x))) + \lambda \mu(\tau(x)) \\ &\leq \tau(\lambda \tau(x)) + \mu(\tau(\lambda \tau(x))) + \lambda \mu(\tau(\lambda \tau(x))) \\ &= \tau(\lambda \tau(x)) + (\lambda + 1)\mu(\tau(\lambda \tau(x))). \end{aligned}$$

Hence

$$\left(\tau(\lambda \tau(x)), \tau(\lambda x) \right) \subseteq \left(\tau(\lambda \tau(x)), \tau(\lambda \tau(x)) + (\lambda + 1)\mu(\tau(\lambda \tau(x))) \right)$$

and thus for $\tilde{x} \in \mathbb{R}$, $\tilde{x} := \tau(\lambda \tau(x))$, we get

$$\left(\tau(\lambda \tau(x)), \tau(\lambda x) \right) \subseteq \left(\tilde{x}, \tilde{x} + (\lambda + 1)\mu(\tilde{x}) \right).$$

It is easy to see that for large \tilde{x} there is $\text{card}(\tilde{x}, \tilde{x} + (\lambda + 1)\mu(\tilde{x})) < [\lambda + 1]$ (where $[\lambda + 1]$ denotes the integer part of number $\lambda + 1$), because the graininess at every point $\sigma(t)$ is greater (or the same) than the graininess at point t (for large t). \square

Later (in this section) we give comments to additional condition (3.11) on \mathbb{T} . In Section 3.5 we will discuss additional conditions, like $\mu(t) = o(t)$, or $\mu(t) = O(t)$.

The next result can be understood as a converse of the previous one, in view of Theorem 3.3. Condition (3.11) does not need to be assumed.

Theorem 3.5. *Let $\mu(t) = o(t)$. If $f : \mathbb{T} \rightarrow \mathbb{R}$ and $f(t) = R(t)$ for $t \in \mathbb{T}$, where $R \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$, then $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then the condition $\mu(t) = o(t)$ can be replaced by $\mu(t) = O(t)$.*

Proof. If $R \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$, then we have $R(x) = \varphi(x)x^\vartheta \exp\left\{\int_a^x \psi(s)/s ds\right\}$, where $\lim_{x \rightarrow \infty} \varphi(x) = C > 0$, $\lim_{x \rightarrow \infty} \psi(x) = 0$, ψ may be taken as continuous, and $a \in \mathbb{T}$, see e.g. [17]. Then, in view of (3.4), we get for $t \in \mathbb{T}$, $t \geq a$, $f(t) = \varphi(t)t^\vartheta \omega(t)$, where $\omega(t) = \exp\left\{\int_a^t \psi(s)/s ds\right\}$. Further $\omega^\Delta(t) = \exp(\eta(t))G(t)$, where

$$G(t) = \lim_{u \rightarrow t} \frac{\int_t^{\sigma(u)} \psi(s)/s ds}{(\sigma(u) - t)}$$

and

$$\int_a^t \frac{\psi(s)}{s} ds - \int_t^{\sigma(t)} \frac{|\psi(s)|}{s} ds \leq \eta(t) \leq \int_a^t \frac{\psi(s)}{s} ds + \int_t^{\sigma(t)} \frac{|\psi(s)|}{s} ds.$$

If t is right-scattered, then using the Mean Value Theorem,

$$G(t) = \frac{1}{\mu(t)} \int_t^{\sigma(t)} \frac{\psi(s)}{s} ds = \frac{\psi(\xi(t))}{\xi(t)},$$

where $t \leq \xi(t) \leq \sigma(t)$. If t is right-dense, then the L'Hospital rule yields $G(t) = \psi(t)/t$. Hence,

$$\frac{t\omega^\Delta(t)}{\omega(t)} = \frac{t\psi(\xi(t))}{\xi(t)} \cdot \frac{\exp(\eta(t))}{\exp\left\{\int_a^t \psi(s)/s ds\right\}}.$$

Since $1 \leq \xi(t)/t \leq 1 + \mu(t)/t$, we have that $t/\xi(t)$ is bounded. Moreover,

$$\int_t^{\sigma(t)} \frac{|\psi(s)|}{s} ds = |\psi(\zeta(t))| \ln(1 + \mu(t)/t) = o(1),$$

where $t \leq \zeta(t) \leq \sigma(t)$. Consequently, $\lim_{t \rightarrow \infty} t\omega^\Delta(t)/\omega(t) = 0$, and so $\omega \in \mathcal{SV}$. Hence we have $f \in \mathcal{RV}(\vartheta)$, in view of Theorem 3.2. \square

Remark 3.3. In the last proof, we may proceed in an alternative way, where we come to the regularly varying of index ϑ function f , which is represented by

$$f(t) = \varphi(t)t^\vartheta \exp\left\{\int_a^t \frac{\tilde{\psi}(s)}{s} \Delta s\right\}$$

with

$$\tilde{\psi}(t) = \lim_{u \rightarrow t} \frac{t}{\sigma(u) - t} \int_t^{\sigma(u)} \frac{\psi(s)}{s} ds = o(1).$$

Theorems 3.3, 3.4, and 3.5 imply the following equivalence between Definition 3.1 and Definition 3.2.

Theorem 3.6. *Let \mathbb{T} satisfy (3.11) with $\mu(t) = o(t)$. Then $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if $f \in \mathcal{KR}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then the condition $\mu(t) = o(t)$ can be replaced by $\mu(t) = O(t)$.*

Remark 3.4. Note that Theorem 3.4 (and hereby the if part of Theorem 3.6) requires an additional condition on the graininess, namely (3.11). This condition is not too restrictive regarding to practical purposes. Indeed, e.g., $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, or $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ with $H_0 = 0$, $H_n = \sum_{k=1}^n 1/k$ for $n \in \mathbb{N}$, or $\mathbb{T} = \mathbb{N}_0^\kappa = \{n^\kappa : n \in \mathbb{N}_0\}$ with $0 < \kappa < 1$ all have a bounded graininess, while $\mathbb{T} = \mathbb{N}_0^\chi$ with $\chi > 1$ has an unbounded increasing graininess satisfying $\mu(t) = o(t)$. On the other hand, we conjecture that (3.11) is not needed in Theorems 3.4 and 3.6, and can be simply relaxed to a natural condition $\mu(t) = o(t)$ (resp. $\mu(t) = O(t)$). Another improvement which we believe could work is an omission of the uniformity in Definition 3.2.

Now we introduce the concept of a normalized regular variation on time scales. For the concept of normalized regular variation in the continuous case see, e.g., [4, 23]. The concept of normalized regularly varying sequences was introduced in [29].

Definition 3.3. An rd-continuously Δ -differentiable function $f : \mathbb{T} \rightarrow (0, \infty)$ is said to be *normalized regularly varying of index ϑ* , $\vartheta \in \mathbb{R}$, if it satisfies

$$\lim_{t \rightarrow \infty} \frac{t f^\Delta(t)}{f(t)} = \vartheta; \quad (3.13)$$

we write $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *normalized slowly varying*; we write $f \in \mathcal{NSV}_{\mathbb{T}}$.

Remark 3.5. Note that every normalized regularly varying function f is regularly varying, but the converse proposition is not true even for $f \in C_{\text{rd}}^1(\mathbb{T})$. Take, e.g., sequence $f_k = k^\vartheta(1 + (-1)^k/k)$ ($\vartheta \in \mathbb{R}$) motivated by example given in [8, p. 96]. Then $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$, but $f \notin \mathcal{NRV}_{\mathbb{T}}(\vartheta)$ because of limit (3.13) does not exist. Indeed, $\limsup_{k \rightarrow \infty} k \Delta f_k / f_k = \vartheta + 2$, while $\liminf_{k \rightarrow \infty} k \Delta f_k / f_k = \vartheta - 2$.

In the end of this section we list an elementary properties of regularly (resp. normalized regularly) varying functions.

Proposition 3.1. *Let $\mu(t) = o(t)$, resp. $\mu(t) = O(t)$ if the index of regular variation is nonzero resp. zero. Then regularly (resp. normalized regularly) varying functions on \mathbb{T} have the following properties.*

- (i) For $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$ in representation formulae (3.2), (3.3), and (3.4), it holds $\varphi(t) \equiv \text{const} > 0$. Moreover, $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$ if and only if $f(t) = t^\vartheta L(t)$, where $L \in \mathcal{NSV}_{\mathbb{T}}$.
- (ii) Let $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. Then $\lim_{t \rightarrow \infty} \ln f(t) / \ln t = \vartheta$. This implies $\lim_{t \rightarrow \infty} f(t) = 0$ if $\vartheta < 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$ if $\vartheta > 0$.

- (iii) Let $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. Then $\lim_{t \rightarrow \infty} f(t)/t^{\vartheta-\varepsilon} = \infty$ and $\lim_{t \rightarrow \infty} f(t)/t^{\vartheta+\varepsilon} = 0$ for every $\varepsilon > 0$.
- (iv) Let $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta_1)$ and $g \in \mathcal{RV}_{\mathbb{T}}(\vartheta_2)$. Then $fg \in \mathcal{RV}_{\mathbb{T}}(\vartheta_1+\vartheta_2)$, $1/f \in \mathcal{RV}_{\mathbb{T}}(-\vartheta_1)$, and $f^\gamma \in \mathcal{RV}_{\mathbb{T}}(\gamma\vartheta)$. The same holds if $\mathcal{RV}_{\mathbb{T}}$ is replaced by $\mathcal{NRV}_{\mathbb{T}}$.
- (v) Let $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. If f is convex, then it is decreasing provided $\vartheta \leq 0$, and it is increasing provided $\vartheta > 0$. A concave f is increasing. If $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$, then it is decreasing provided $\vartheta < 0$ and it is increasing provided $\vartheta > 0$.
- (vi) Let $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$, $\vartheta \in \mathbb{R}$, be of the class C_{rd}^1 . If f is convex or concave, then $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$.
- (vii) (Zygmund type characterization) Let $\mu(t) = o(t)$. Let f be a positive function with $f \in C_{\text{rd}}^1$. Then $f \in \mathcal{NRV}_{\mathbb{T}}(\vartheta)$ if and only if $f(t)/t^\gamma$ is eventually increasing for each $\gamma < \vartheta$ and $f(t)/t^\zeta$ is eventually decreasing for each $\zeta > \vartheta$.

Proof. (i) We know that $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ can be written as $f(t) = t^\vartheta L(t)$, where $L \in \mathcal{SV}_{\mathbb{T}}$. The statement follows from

$$\frac{t f^\Delta(t)}{f(t)} = \frac{t(t^\vartheta)^\Delta}{t^\vartheta} + \left(\frac{\sigma(t)}{t}\right)^\vartheta \frac{t L^\Delta(t)}{L(t)}$$

since for $\vartheta \neq 0$ we have $t(t^\vartheta)^\Delta/t^\vartheta \rightarrow \vartheta$ and $(\sigma(t)/t)^\vartheta \rightarrow 1$ as $t \rightarrow \infty$.

(ii) From representation (3.4) we have

$$\frac{\ln f(t)}{\ln t} = \frac{\ln \varphi(t)}{\ln t} + \vartheta + \frac{\int_a^t \psi(s)/s \Delta s}{\ln t}.$$

We claim that the last term tends to zero as $t \rightarrow \infty$. This follows from the fact that $\psi(t) \rightarrow 0$ and $\ln t$ can be written as $\int^t (1 + |O(1)|)/s \Delta s$. Indeed, at a right-scattered t we have

$$(\ln t)^\Delta = \frac{1}{t} \cdot \frac{\ln(1 + \mu(t)/t)}{\mu(t)/t}.$$

(iii) This follows from representation (3.3) and part (ii) of this proposition.

(iv) This follows from representation (3.4).

(v) First note that the convexity of f implies clearly its eventual monotonicity. Similarly, the concavity implies that f is increasing; if f were decreasing then it cannot be eventually positive. Next we show that a convex $f \in \mathcal{SV}_{\mathbb{T}}$ is decreasing. By a contradiction assume that f is increasing. Thanks to convexity, we then have $f(t) \geq Mt$ for large t and for some $M > 0$. But now f cannot be slowly varying by (iii) of this proposition. Similarly we proceed when $\vartheta < 0$ and f is convex. If f is convex with $\vartheta > 0$, then it tends to ∞ and hence must be increasing. The claim for a normalized function follows from (i) of this proposition.

(vi) Let $f \in \mathcal{RV}(\vartheta)$. By (v) of this proposition, one of the conditions must eventually hold (a) f is convex, decreasing, or (b) f is concave, increasing, or (c) f is convex, increasing. Let (a) holds, i.e. f^Δ is nonpositive and nondecreasing. Then, from $f \in \mathcal{RV}(\vartheta)$ we have $f \in \mathcal{KR}\mathcal{V}_{\mathbb{T}}(\vartheta)$ by Theorem 3.3 and

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \lambda^\vartheta \quad \text{for all } \lambda > 0 \quad (3.14)$$

Let $\lambda \in (0, 1)$. Since $-f^\Delta$ is nonnegative and nonincreasing, we have

$$-f(t) + f(\tau(\lambda t)) = - \int_{\tau(\lambda t)}^t f^\Delta(s) \Delta s \geq -f^\Delta(t)[t - \tau(\lambda t)] \geq -f^\Delta(t)(1 - \lambda)t$$

for large t . This estimation and (3.14) imply

$$\limsup_{t \rightarrow \infty} \frac{-tf^\Delta(t)}{f(t)} \leq \limsup_{t \rightarrow \infty} \frac{1}{1 - \lambda} \left(\frac{f(\tau(\lambda t))}{f(t)} - 1 \right) = \frac{\lambda^\vartheta - 1}{1 - \lambda},$$

which holds for every $\lambda \in (0, 1)$. Taking now the limit as $\lambda \rightarrow 1^-$, we obtain

$$\limsup_{t \rightarrow \infty} \frac{-tf^\Delta(t)}{f(t)} \leq \lim_{\lambda \rightarrow 1^-} \frac{\lambda^\vartheta - 1}{1 - \lambda} = -\vartheta. \quad (3.15)$$

In view of (3.15) for $f \in \mathcal{SV}$, we may now restrict ourselves to $\vartheta \neq 0$. We have

$$-f(t) + f(\tau(\lambda t)) = - \int_{\tau(\lambda t)}^t f^\Delta(s) \Delta s \leq -f^\Delta(\tau(\lambda t))(t - \tau(\lambda t)).$$

This estimation, (3.14), and $\lambda \in (0, 1)$ imply

$$\begin{aligned} \lambda^\vartheta \liminf_{t \rightarrow \infty} \frac{-tf^\Delta(t)}{f(t)} &= \liminf_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} \cdot \frac{-\tau(\lambda t)f^\Delta(\tau(\lambda t))}{f(\tau(\lambda t))} \\ &\leq \liminf_{t \rightarrow \infty} \frac{\tau(\lambda t)[f(\tau(\lambda t))]}{f(t)[t - \tau(\lambda t)]} \\ &= \liminf_{t \rightarrow \infty} \frac{\tau(\lambda t)}{t - \tau(\lambda t)} \left(\frac{f(\tau(\lambda t))}{f(t)} - 1 \right). \end{aligned} \quad (3.16)$$

Since for $x \in \mathbb{R}$, $x \geq a$,

$$\tau(x) \leq x \leq \sigma(\tau(x)) = \tau(x) + \mu(\tau(x)),$$

we have

$$1 \leq \frac{x}{\tau(x)} \leq 1 + \frac{\mu(\tau(x))}{\tau(x)},$$

and so $\lim_{x \rightarrow \infty} (x/\tau(x)) = 1$. Consequently, in view of (3.14) and (3.16),

$$\lambda^\vartheta \liminf_{t \rightarrow \infty} \frac{-tf^\Delta(t)}{f(t)} \geq \liminf_{t \rightarrow \infty} \frac{\lambda}{\lambda t/\tau(\lambda t) - \lambda} \left(\frac{f(\tau(\lambda t))}{f(t)} - 1 \right) = \frac{\lambda}{1 - \lambda} (\lambda^\vartheta - 1)$$

for every $\lambda \in (0, 1)$. Hence,

$$\liminf_{t \rightarrow \infty} \frac{-tf^\Delta(t)}{f(t)} \geq \lim_{\lambda \rightarrow 1^-} \frac{\lambda^\vartheta - 1}{\lambda^{\vartheta-1} - \lambda^\vartheta} = -\vartheta. \quad (3.17)$$

From (3.15), and (3.17), we obtain $\lim_{t \rightarrow \infty} tf^\Delta(t)/f(t) = \vartheta$, $\vartheta \in \mathbb{R}$, which implies $f \in \mathcal{NRV}(\vartheta)$. Similarly, we can prove case (b). To prove (c) we use arguments also similar to (a). We use again that f satisfies (3.14). Now we take $\lambda > 1$. Using the equality

$$f(\tau(\lambda t)) - f(t) = \int_t^{\tau(\lambda t)} f^\Delta(s) \Delta s,$$

monotonicity properties of f^Δ , and (3.14) it is not difficult to show that

$$\limsup_{t \rightarrow \infty} \frac{tf^\Delta(t)}{f(t)} \leq \lim_{\lambda \rightarrow 1^+} \frac{\lambda^\vartheta - 1}{\lambda - 1} = \vartheta$$

and

$$\liminf_{t \rightarrow \infty} \frac{tf^\Delta(t)}{f(t)} \geq \lim_{\lambda \rightarrow 1^+} \frac{1 - \lambda^\vartheta}{1 - \lambda} = \vartheta.$$

(vii) By (iv) and (v) of this proposition $f(t)/t^\gamma \in \mathcal{NRV}_{\mathbb{T}}(\vartheta - \gamma)$ is increasing and $f(t)/t^\zeta \in \mathcal{NRV}_{\mathbb{T}}(\vartheta - \zeta)$ is decreasing. Conversely, from $(f(t)/t^\gamma)^\Delta > 0$ we get

$$\frac{tf^\Delta(t)}{f(t)} > \frac{(t^\gamma)^\Delta}{t^{\gamma-1}} = \gamma(1 + o(1)).$$

Similarly, $tf^\Delta(t)/f(t) < \zeta(1 + o(1))$. The statement follows by choosing γ and ζ arbitrarily close to ϑ . \square

3.2 Theory of rapid variation on time scales

In this section we establish the theory of rapid variation on time scales. Throughout this section, $\mu(t) = o(t)$ is assumed. This condition will be discussed later, in Section 3.5. As we show, if we want to obtain a reasonable theory, we cannot omit this additional requirement on the graininess.

Definition 3.4. Let c, d be the real constants such that $0 < c \leq d$ and $\vartheta \in \mathbb{R}$. A measurable function $f : \mathbb{T} \rightarrow (0, \infty)$ is said to be *rapidly varying of index* ∞ , resp. $-\infty$ if there exist function $\varphi : \mathbb{T} \rightarrow (0, \infty)$ satisfying $\varphi \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ or $c \leq \varphi(t) \leq d$ for large t and a positive rd-continuously Δ -differentiable function ω such that

$$f(t) = \varphi(t)\omega(t)$$

and

$$\lim_{t \rightarrow \infty} \frac{t\omega^\Delta(t)}{\omega(t)} = \infty, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{t\omega^\Delta(t)}{\omega(t)} = -\infty; \quad (3.18)$$

we write $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$, resp. $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. Moreover, the function ω is said to be *normalized rapidly varying of index ∞* , resp. *normalized rapidly varying of index $-\infty$* ; we write $\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$, resp. $\omega \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$.

Now we prove some important properties of (normalized) rapidly varying functions which will be needed later.

Proposition 3.2. (i) It holds $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ if and only if $1/f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$.

(ii) Let $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. Then for every $\vartheta \in [0, \infty)$ the function $f(t)/t^\vartheta$ is increasing for large t and $\lim_{t \rightarrow \infty} f(t)/t^\vartheta = \infty$.

(iii) Let $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. Then for every $\vartheta \in [0, \infty)$ the function $f(t)t^\vartheta$ is decreasing for large t and $\lim_{t \rightarrow \infty} f(t)t^\vartheta = 0$.

(iv) $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$ implies $f^\Delta(t) > 0$ for large t and $f(t)$ is increasing for large t , moreover f and f^Δ are tending to ∞ .

(v) $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$ implies $f^\Delta(t) < 0$ for large t and $f(t)$ is decreasing for large t , moreover f is tending to 0. If f is convex for large t or if there exists $h > 0$ such that $\mu(t) > h$ for large t , then f^Δ is tending to 0.

Proof. (i) Let $f = \varphi\omega$, $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$. First, we show that $\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty) \Leftrightarrow 1/\omega \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. Due to (3.18), $\omega^\Delta(t) > 0$ for large t . Therefore,

$$\begin{aligned} \omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty) &\Leftrightarrow \lim_{t \rightarrow \infty} \frac{\omega(t)}{t\omega^\Delta(t)} = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \frac{\omega^\sigma(t) - \mu(t)\omega^\Delta(t)}{t\omega^\Delta(t)} = 0 \\ &\Leftrightarrow \lim_{t \rightarrow \infty} \left(\frac{\omega^\sigma(t)}{t\omega^\Delta(t)} - \frac{\mu(t)}{t} \right) = 0 \Leftrightarrow \lim_{t \rightarrow \infty} \frac{\omega^\sigma(t)}{t\omega^\Delta(t)} = 0 \\ &\Leftrightarrow \lim_{t \rightarrow \infty} \frac{t\omega^\Delta(t)}{\omega^\sigma(t)} = \infty \Leftrightarrow \lim_{t \rightarrow \infty} \left(\frac{t}{1/\omega(t)} \cdot \frac{-\omega^\Delta(t)}{\omega(t)\omega^\sigma(t)} \right) = -\infty \\ &\Leftrightarrow \lim_{t \rightarrow \infty} \frac{t(1/\omega(t))^\Delta}{1/\omega(t)} = -\infty \Leftrightarrow \frac{1}{\omega} \in \mathcal{NRPV}_{\mathbb{T}}(-\infty). \end{aligned}$$

Now, since $1/\varphi \in \mathcal{RV}_{\mathbb{T}}(-\vartheta)$, see part (iv) of Proposition 3.1, or $0 < 1/d \leq 1/\varphi(t) \leq 1/c$ for large t , we have $1/f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. Similarly, $1/f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ implies $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$.

(ii) Let $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$ and $\vartheta \in [0, \infty)$. Then there exists a function $\xi(t)$, $t \leq \xi(t) \leq \sigma(t)$, such that

$$\left(\frac{f(t)}{t^\vartheta} \right)^\Delta = \frac{f^\Delta(t)t^\vartheta - f(t)(t^\vartheta)^\Delta}{t^\vartheta(\sigma(t))^\vartheta} = \frac{f^\Delta(t)t^\vartheta - \vartheta f(t)(\xi(t))^{\vartheta-1}}{t^\vartheta(\sigma(t))^\vartheta}. \quad (3.19)$$

In view of

$$\frac{t f^\Delta(t)}{f(t)} > \vartheta \left(\frac{\xi(t)}{t} \right)^{\vartheta-1} \quad \text{for large } t$$

(indeed, $tf^\Delta(t)/f(t) \rightarrow \infty$ as $t \rightarrow \infty$ and $\xi(t)/t \rightarrow 1$ as $t \rightarrow \infty$), which is equivalent to

$$f^\Delta(t)t^\vartheta > \vartheta f(t)(\xi(t))^{\vartheta-1} \quad \text{for large } t,$$

(3.19) is positive for large t and hence $f(t)/t^\vartheta$ is increasing for large t . By a contradiction, suppose that $\lim_{t \rightarrow \infty} f(t)/t^\vartheta = L$, $L \in (0, \infty)$ (note that a limit of this function exists for all $\vartheta \geq 0$ as a finite or infinite number, because the function $f(t)/t^\vartheta$ is increasing). Then $f(t) \sim Lt^\vartheta$ and hence $f \in \mathcal{RPV}_{\mathbb{T}}(\vartheta)$, which is contradiction with $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. Therefore, $\lim_{t \rightarrow \infty} f(t)/t^\vartheta = \infty$.

(iii) It follows from (i) and (ii).

(iv) Let $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. If we take $\vartheta = 0$ in (ii), we get $f(t)$ is increasing (thus $f^\Delta(t) > 0$) for large t and $\lim_{t \rightarrow \infty} f(t) = \infty$. To prove that $\lim_{t \rightarrow \infty} f^\Delta(t) = \infty$, it is enough to show that $\liminf_{t \rightarrow \infty} f^\Delta(t) = \infty$. We know that $f^\Delta(t) > 0$. Assume that $\liminf_{t \rightarrow \infty} f^\Delta(t) = c$, $c > 0$. Then, in view of $\lim_{t \rightarrow \infty} t/f(t) = 0$ (which follows from (ii)), $\liminf_{t \rightarrow \infty} tf^\Delta(t)/f(t) = 0$, contradiction with (3.18). So $\liminf_{t \rightarrow \infty} f^\Delta(t) = \infty$ and hence $\lim_{t \rightarrow \infty} f^\Delta(t) = \infty$.

(v) Analogously as in case (iv), we get (by using (iii) for $\vartheta = 0$) that $f(t)$ is decreasing (thus $f^\Delta(t) < 0$) for large t and $\lim_{t \rightarrow \infty} f(t) = 0$. Let f is convex for large t . Then $f^\Delta(t)$ increases for large t and $\lim_{t \rightarrow \infty} f^\Delta(t)$ exists as a nonpositive number. By a contradiction, assume that $\lim_{t \rightarrow \infty} f^\Delta(t) = k < 0$. Hence, $f^\Delta(t) \leq k$ for large t . By integration of last inequality from t_0 to t (where $t_0 \in \mathbb{T}$ is sufficiently large) we get $f(t) \leq kt + q$ ($q = kt_0 - f(t_0)$) for large t . Hence, $f(t) < 0$ for large t , contradiction. Let (for large t) $\mu(t)$ be bounded from below by a positive constant h . Then in a view that $f(t)$ is decreasing for large t

$$0 > f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)} > \frac{f^\sigma(t) - f(t)}{h} \quad \text{for large } t. \quad (3.20)$$

If $t \rightarrow \infty$ in (3.20), we get (by using $\lim_{t \rightarrow \infty} f(t) = 0$) $\lim_{t \rightarrow \infty} ((f^\sigma(t) - f(t))/h) = 0$, hence $\lim_{t \rightarrow \infty} f^\Delta(t) = 0$. □

Remark 3.6. (i) From the above proposition it is easy to see that the function $f(t) = a^t$ with $a > 1$ is a typical representative of the class $\mathcal{RPV}_{\mathbb{T}}(\infty)$, while the function $f(t) = a^t$ with $a \in (0, 1)$ is a typical representative of the class $\mathcal{RPV}_{\mathbb{T}}(-\infty)$. Of course, as we can see also from Definition 3.4, these classes are much wider. The rapidly varying function can be understood like a product of an exponential function and a function, which is regularly varying or bounded. However, the exact representation is not known for now. We conjecture that it could be somewhere near to this one: for $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$, resp. $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$,

$$f(t) = \varphi(t)a^{g(t)} \quad \text{for } a > 1, \quad (3.21)$$

resp.

$$f(t) = \varphi(t)a^{g(t)} \quad \text{for } a \in (0, 1), \quad (3.22)$$

where φ is a positive measurable function defined as in Definition 3.4 and $g(t) \geq h(t)$, $h \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ with $\vartheta > 0$. Observe that this “representation” is sufficiently wide and includes many various rapidly varying functions, e.g., $(\sin(t) + b)a^t$, $\ln(t)a^t$, $t^\gamma a^t$, a^{t^ϑ} and a^{bt} with $a \in (0, 1) \cup (1, \infty)$, $b > 1$, $\gamma \in (-\infty, \infty)$ and $\vartheta > 0$. The case $a \in (0, 1)$ stands for $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$, while the case $a \in (1, \infty)$ stands for $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$.

(ii) Case (ii), resp. (iii) (and of course (iv), resp. (v)) of the previous proposition does not hold generally for $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$, resp. $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. It is enough to take, e.g., a function $f(t) = a^{t-2\sin t}$ with $a > 1$, resp. $f(t) = a^{t-2\sin t}$ with $a < 1$. Note that $f(t) \in \mathcal{RPV}_{\mathbb{T}}(\pm\infty)$ in view of $a^{t-2\sin t} = a^{-2\sin t}a^t$ with bounded $a^{-2\sin t}$.

(iii) The assumption of convexity or existence $h > 0$ in the previous proposition in part (v) (unlike (iv)) is important, because without this condition only $\limsup_{t \rightarrow \infty} f^\Delta(t) = 0$ holds, as can we see in the following example. Let $n \in \mathbb{N}$ and consider function f defined on the discrete time scale $\mathbb{T} = \mathbb{N} \cup \{n + (1/2)^{n+2}\}$ such that

$$f(t) = \begin{cases} \left(\frac{1}{2}\right)^t & \text{for } t = n \\ \frac{3}{4} \left(\frac{1}{2}\right)^t & \text{for } t = n + \left(\frac{1}{2}\right)^{n+2}. \end{cases}$$

Then $f(t) \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$, $\liminf_{t \rightarrow \infty} f^\Delta(t) = -1$ and $\limsup_{t \rightarrow \infty} f^\Delta(t) = 0$.

Now we introduce Karamata type definition of rapid variation, see (2.4) - (2.7) and (3.5).

Definition 3.5 (Karamata type definition). Let $\tau : \mathbb{R} \rightarrow \mathbb{T}$ be defined as $\tau(t) = \max\{s \in \mathbb{T} : s \leq t\}$. A measurable function $f : \mathbb{T} \rightarrow (0, \infty)$ satisfying

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \begin{cases} \infty & \text{for } \lambda > 1 \\ 0 & \text{for } 0 < \lambda < 1, \end{cases} \quad (3.23)$$

is said to be *rapidly varying of index ∞ in the sense of Karamata*. We write $f \in \mathcal{KRPV}_{\mathbb{T}}(\infty)$. A measurable function $f : \mathbb{T} \rightarrow (0, \infty)$ satisfying

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \begin{cases} 0 & \text{for } \lambda > 1 \\ \infty & \text{for } 0 < \lambda < 1, \end{cases} \quad (3.24)$$

is said to be *rapidly varying of index $-\infty$ in the sense of Karamata*. We write $f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$.

Note that the classes $\mathcal{KRPV}_{\mathbb{T}}(\infty)$ and $\mathcal{KRPV}_{\mathbb{T}}(-\infty)$ can be described similarly as the classes $\mathcal{RPV}_{\mathbb{T}}(\infty)$ and $\mathcal{RPV}_{\mathbb{T}}(-\infty)$, see part (i) of Remark 3.6. Now

we prove some properties of rapidly varying functions in the sense of Karamata which will be needed later.

Proposition 3.3. (I) $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$ if and only if $1/f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$.

(II) Let $f : \mathbb{T} \rightarrow (0, \infty)$ be a measurable function, monotone for large t . Then

(i) $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$ implies f is increasing for large t and $\lim_{t \rightarrow \infty} f(t) = \infty$.

(ii) $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$ implies f is decreasing for large t and $\lim_{t \rightarrow \infty} f(t) = 0$.

(iii) $\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty$ ($\lambda > 1$) implies $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$.

(iv) $\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = 0$ ($\lambda > 1$) implies $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$.

Proof. (I) We have

$$\begin{aligned} f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty) &\Leftrightarrow \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \begin{cases} \infty & \text{for } \lambda > 1 \\ 0 & \text{for } 0 < \lambda < 1 \end{cases} \Leftrightarrow \\ &\Leftrightarrow \lim_{t \rightarrow \infty} \frac{\frac{1}{f(\tau(\lambda t))}}{\frac{1}{f(t)}} = \begin{cases} 0 & \text{for } \lambda > 1 \\ \infty & \text{for } 0 < \lambda < 1 \end{cases} \Leftrightarrow \frac{1}{f} \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty). \end{aligned}$$

(II) (i) Let $\lambda > 1$ and $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = \infty$ hold. Suppose that $f(t)$ is nonincreasing for large t . Then $\limsup_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) \leq 1$, contradiction. Similarly, if we suppose $\lim_{t \rightarrow \infty} f(t) = c < \infty$, we get $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = 1$, contradiction.

(ii) Let $\lambda > 1$ and $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = 0$ hold. Suppose that $f(t)$ is nondecreasing for large t . Then $\liminf_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) \geq 1$, contradiction. Similarly, if we suppose $\lim_{t \rightarrow \infty} f(t) = c > 0$, we get $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = 1$, contradiction.

(iii) Let $\lambda > 1$ and $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = \infty$ hold. From (i) we know that $f(t)$ is increasing for large t . Therefore,

$$\infty = \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda \tau(\frac{t}{\lambda})))}{f(\tau(\frac{t}{\lambda}))} \leq \lim_{t \rightarrow \infty} \frac{f(t)}{f(\tau(\frac{1}{\lambda}t))}$$

(due to $f(\tau(\lambda \tau(\frac{t}{\lambda}))) \leq f(t)$). Hence, $\lim_{t \rightarrow \infty} f(t)/f(\tau(\lambda t)) = \infty$ for $0 < \lambda < 1$ and thus $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = 0$ for $0 < \lambda < 1$. Therefore, $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$.

(iv) Let $\lambda > 1$ and $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = 0$ hold. From (ii) we know that $f(t)$ is decreasing for large t . Therefore,

$$0 = \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda \tau(\frac{t}{\lambda})))}{f(\tau(\frac{t}{\lambda}))} \geq \lim_{t \rightarrow \infty} \frac{f(t)}{f(\tau(\frac{1}{\lambda}t))} \geq 0$$

(due to $f(\tau(\lambda \tau(\frac{t}{\lambda}))) \geq f(t)$). Hence, $\lim_{t \rightarrow \infty} f(t)/f(\tau(\lambda t)) = 0$ for $0 < \lambda < 1$ and thus $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = \infty$ for $0 < \lambda < 1$. Therefore, $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$. \square

Remark 3.7. In view of (iii) and (iv) of part (II), we can naturally ask, whether the following condition

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty \text{ (resp. } 0) \quad \lambda > 1 \quad \Leftrightarrow \quad \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = 0 \text{ (resp. } \infty) \quad \lambda \in (0, 1)$$

holds as in the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We conjecture that if f is positive and monotone, then the condition holds. However, we are able to prove a missing implication (similarly as in the proof of Lemma 3.2) only on the assumption that f be a rd-continuously differentiable with $f^\Delta(t)$ increases for large t .

In the end of this section we answer a naturally question, whether Definition 3.5 is equivalent to Definition 3.4.

Lemma 3.2. *Let f be a positive rd-continuously differentiable function and let $f^\Delta(t)$ be increasing for large t . Then*

(i) $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$ if and only if $f \in \mathcal{RP}\mathcal{V}_{\mathbb{T}}(\infty)$ if and only if $f \in \mathcal{NR}\mathcal{PV}_{\mathbb{T}}(\infty)$.

(ii) $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$ if and only if $f \in \mathcal{RP}\mathcal{V}_{\mathbb{T}}(-\infty)$ if and only if $f \in \mathcal{NR}\mathcal{PV}_{\mathbb{T}}(-\infty)$.

Moreover, the assumption of convexity is not necessary in all if parts.

Proof. (i) We will proceed in the following way:

$$f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{NR}\mathcal{PV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{RP}\mathcal{V}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty).$$

Let $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$. First, observe that $f(t)$ is monotone for large t . Indeed, $f(t)$ is convex, so there exists t_0 such that $f(t)$ is monotone for $t > t_0$. Hence, $f(t)$ is increasing for large t due to Proposition 3.3. Now, for all $\lambda < 1$, we have

$$\begin{aligned} f(t) - f(\tau(\lambda t)) &= \int_{\tau(\lambda t)}^t f^\Delta(s) \Delta s \leq f^\Delta(t)[t - \tau(\lambda t)] \leq f^\Delta(t)[t - (\lambda t - \mu(\tau(\lambda t)))] \\ &= f^\Delta(t)[t(1 - \lambda) + \mu(\tau(\lambda t))]. \end{aligned}$$

Hence,

$$\frac{f^\Delta(t)[t(1 - \lambda) + \mu(\tau(\lambda t))]}{f(t)} \geq \frac{f(t) - f(\tau(\lambda t))}{f(t)}. \quad (3.25)$$

Note that $\mu(\tau(\lambda t))/f(t) \rightarrow 0$ as $t \rightarrow \infty$. Really, $f(t)$ is convex and increasing, so there exists $t_0 \in \mathbb{T}$ such that $f(t) > t$ for $t > t_0$ and hence,

$$0 = \lim_{t \rightarrow \infty} \frac{\mu(\tau(\lambda t))}{t} \geq \lim_{t \rightarrow \infty} \frac{\mu(\tau(\lambda t))}{f(t)} \geq 0.$$

Since $\lambda < 1$ is independent of t and can be chosen arbitrarily close to 1, in view of $\mu(\tau(\lambda t))/f(t) \rightarrow 0$ as $t \rightarrow \infty$ and $f(\tau(\lambda t))/f(t) \rightarrow 0$ as $t \rightarrow \infty$, from the inequality (3.25) we have

$$\liminf_{t \rightarrow \infty} \frac{t f^\Delta(t)}{f(t)} \geq \sup_{\lambda < 1} \frac{1}{1 - \lambda} = \infty \quad (3.26)$$

and thus $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. The part $f \in \mathcal{NRPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ holds trivially. Let $f \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ and take $\lambda > 1$. Then, by Definition 3.4

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \lim_{t \rightarrow \infty} \frac{\varphi(\lambda t)}{\varphi(t)} \cdot \frac{\omega(\lambda t)}{\omega(t)} = \lim_{t \rightarrow \infty} h_{\lambda}(t) \frac{\omega(\lambda t)}{\omega(t)}. \quad (3.27)$$

Let $\varphi \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. Hence, $\varphi \in \mathcal{KR}_{\mathbb{T}}(\vartheta)$ by Theorem 3.3, which implies that $h(t) \rightarrow \lambda^{\vartheta}$ as $t \rightarrow \infty$. Let φ is bounded, i.e., $0 < c \leq \varphi(t) \leq d$ for large t . Then,

$$\frac{c}{d} \leq \liminf_{t \rightarrow \infty} h_{\lambda}(t) \leq h_{\lambda}(t) \leq \limsup_{t \rightarrow \infty} h_{\lambda}(t) \leq \frac{d}{c}.$$

Together, $h_{\lambda}(t)$ is bounded both above and below for large t by the positive constants. Due to $\omega \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$, $\omega(t)$ is increasing for large t (thanks to Proposition 3.2). Now, for all $\lambda > 1$, we have

$$\begin{aligned} \omega(\tau(\lambda t)) &\geq \omega(\tau(\lambda t)) - \omega(t) = \int_t^{\tau(\lambda t)} \omega^{\Delta}(s) \Delta s \geq \omega^{\Delta}(t) [\tau(\lambda t) - t] \\ &\geq \omega^{\Delta}(t) [\lambda t - \mu(\tau(\lambda t)) - t] = \omega^{\Delta}(t) [t(\lambda - 1) - \mu(\tau(\lambda t))]. \end{aligned}$$

Hence,

$$\frac{\omega(\tau(\lambda t))}{\omega(t)} \geq \frac{\omega^{\Delta}(t) [t(\lambda - 1) - \mu(\tau(\lambda t))]}{\omega(t)}. \quad (3.28)$$

Since $\lambda > 1$, in view of $\mu(\tau(\lambda t))/\omega(t) \rightarrow 0$ as $t \rightarrow \infty$ (similar reasoning as before), from (3.27) and (3.28) we have

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} \geq \lim_{t \rightarrow \infty} h_{\lambda}(t) \frac{t\omega^{\Delta}(t)(\lambda - 1)}{\omega(t)} = \infty \quad (\lambda > 1),$$

and thus (thanks to Proposition 3.3) $f \in \mathcal{KR}_{\mathbb{T}}(\infty)$.

(ii) We will proceed analogically as in case (i). Let $f \in \mathcal{KR}_{\mathbb{T}}(-\infty)$. Similarly as in part (i), we get $f(t)$ is decreasing for large t due to Proposition 3.3. Now, for all $\lambda > 1$, we have

$$-f(\tau(\lambda t)) + f(t) = \int_t^{\tau(\lambda t)} (-f^{\Delta}(s)) \Delta s \leq -f^{\Delta}(t) (\tau(\lambda t) - t) \leq -f^{\Delta}(t) (\lambda - 1)t.$$

Hence,

$$-\frac{tf^{\Delta}(t)}{f(t)} \geq \frac{1}{\lambda - 1} \cdot \frac{-f(\tau(\lambda t)) + f(t)}{f(t)} = \frac{1}{\lambda - 1} \left(1 - \frac{f(\tau(\lambda t))}{f(t)} \right).$$

Since $\lambda > 1$ is independent of t and can be chosen arbitrarily close to 1, in view of $f(\tau(\lambda t))/f(t) \rightarrow 0$ as $t \rightarrow \infty$, from the above inequality we have

$$\liminf_{t \rightarrow \infty} -\frac{tf^{\Delta}(t)}{f(t)} \geq \sup_{\lambda > 1} \frac{1}{\lambda - 1} = \infty$$

and thus $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty)$. The part $f \in \mathcal{NRPV}_{\mathbb{T}}(-\infty) \Rightarrow f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ holds trivially. Let $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. By using Proposition 3.2, part (i) of this lemma and Proposition 3.3 we can successively write:

$$f \in \mathcal{RPV}_{\mathbb{T}}(-\infty) \Rightarrow \frac{1}{f} \in \mathcal{RPV}_{\mathbb{T}}(\infty) \Rightarrow \frac{1}{f} \in \mathcal{KRPV}_{\mathbb{T}}(\infty) \Rightarrow f \in \mathcal{KRPV}_{\mathbb{T}}(-\infty)$$

and hence, assertion follows. \square

Remark 3.8 (Important). (i) Note that the concept of normalized rapid variation is not known in the literature concerning the continuous (resp. discrete) theory and it seems that there is no reason to distinguish two cases of rapidly varying behavior in this situation. We conjecture that in this case, every positive differentiable function f (resp. every positive sequence), which is rapidly varying, is automatically normalized rapidly varying and it misses point to consider both definitions (specially, when we study asymptotic properties of differential or difference equations and deal with functions which are differentiable). However, the situation is different on general time scale case and the previous assertion is not true (only if f is convex and Δ -differentiable, then, in view of previous lemma, these two definitions are equivalent). Indeed, take, e.g., $\mathbb{T} = \mathbb{N} \cup \{n + 2^{-n}\}$, $n \in \mathbb{N}$, and $f, \varphi, \omega : \mathbb{T} \rightarrow \mathbb{R}$ satisfying the assumptions of Definition 3.4 such that

$$\varphi(t) = \begin{cases} 1 + 2^{-t} & \text{for } t = n, \\ 1 - 2^{-t} & \text{for } t = n + 2^{-n} \end{cases} \quad \text{and} \quad \omega(t) = 2^t \quad (t \in \mathbb{T}).$$

Then $\varphi(t) \rightarrow 1$ as $t \rightarrow \infty$ and $\omega(t) \in \mathcal{NRPV}_{\mathbb{T}}(\infty)$. Moreover,

$$f(t) = \varphi(t)\omega(t) = \begin{cases} 2^t + 1 & \text{for } t = n, \\ 2^t - 1 & \text{for } t = n + 2^{-n} \end{cases}$$

is of the class $C_{\text{rd}}^1(\mathbb{T})$. It is not difficult to verify that $f(t)$ is decreasing in each $t = n$, $n \in \mathbb{N}$. Hence, $f^\Delta(t)$ is negative for every $t = n$, thus $\liminf_{t \rightarrow \infty} t f^\Delta(t)/g(t) \leq 0$ and hence $f \notin \mathcal{NRPV}_{\mathbb{T}}(\infty)$.

(ii) Looking at Definition 3.4 and a condition on a function φ , the reader may ask why we require the function φ just in this form. The other eventualities are, e.g., to consider φ in the following forms:

(i) $\varphi(t) \sim C$, where $C > 0$ (less general form),

(ii) $t^c \leq \varphi(t) \leq t^d$, where $c, d \in \mathbb{R}$, $c \leq d$ (more general form).

However, the case (i) is less general then in our definition. Moreover, observe that the function φ from the previous example satisfies condition (i). The case (ii) is more general but not convenient since our theory focuses on a generalization in the sense of a “domain of definition” rather than considering “badly behaving” functions.

3.3 Applications to dynamic equations on time scales

In this section we apply the obtained theory from the previous two sections to the investigation of asymptotic behavior of solutions to linear and half-linear second order dynamic equations on time scale, which allows us to get a precise information about asymptotic varying behavior of positive solutions of mentioned equations. Consider the half-linear second order dynamic equation

$$[\Phi(y^\Delta)]^\Delta - p(t)\Phi(y^\sigma) = 0 \tag{HL}$$

and its special case, the linear second order dynamic equation (when $\alpha = 2$)

$$y^{\Delta\Delta} - p(t)y^\sigma = 0, \tag{L}$$

on unbounded time scale interval $\mathcal{I}_a = [a, \infty)$, where p is a positive rd-continuous function. Equations (HL), resp. (L) are the special cases of equations (HL $^\Delta$ E), resp. (L $^\Delta$ E) introduced in Section 2.3. Note that every solution y of (HL), resp. (L) is convex, i.e., y^Δ is nondecreasing. Along with (HL), consider the generalized Riccati dynamic equation

$$w^\Delta(t) - p(t) + S(t) = 0, \tag{GR}$$

where

$$S(t) = \lim_{\gamma(t) \rightarrow \mu(t)} \frac{w(t)}{\gamma(t)} \left(1 - \frac{1}{\Phi[1 + \gamma(t) \Phi^{-1}(w(t))]} \right). \tag{S}$$

Note that (GR) is special case of (GR $^\Delta$ E) and relation between (HL) and (GR) is analogical as in Section 2.3. Hence, $y(t)$ is a nonoscillatory solution of (HL) having no generalized zero on \mathcal{I}_a , i.e., $y(t)y^\sigma(t) > 0$ for $t \in \mathcal{I}_a$ if and only if $w(t) = \Phi(y^\Delta(t)/y(t))$ satisfies (GR) on \mathcal{I}_a with $1 + \mu(t) \Phi^{-1}(w(t)) > 0$ on \mathcal{I}_a .

We start with equation (HL) and establish necessary and sufficient conditions for all positive decreasing solutions of (HL) to be regularly varying. Note that this result generalizes the result established for equation (L), see [36]. For more related results for linear case in special settings see [26, 29, 30]. Specially, for half-linear differential case, see [20].

Theorem 3.7. *Let y be any positive decreasing solution of (HL) on \mathcal{I}_a .*

(i) *Let $\mu(t) = O(t)$. Then $y \in \mathcal{SV}$ if and only if*

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) \Delta s = 0. \tag{3.29}$$

Moreover, $y \in \mathcal{NSV}$.

(ii) Let $\mu(t) = o(t)$. Then $y \in \mathcal{RV}(\Phi^{-1}(\vartheta_0))$ if and only if

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) \Delta s = A > 0, \quad (3.30)$$

where ϑ_0 is the negative root of the algebraic equation

$$|\vartheta|^\beta - \vartheta - A = 0, \quad (3.31)$$

β is the conjugate number to α , i.e., $1/\alpha + 1/\beta = 1$.

Moreover, $y \in \mathcal{N}\mathcal{RV}(\Phi^{-1}(\vartheta_0))$.

Proof. (i) “Only if”: Let $y(t)$ be a slowly varying positive decreasing solution of (HL) on \mathcal{I}_a . Then $y^\Delta(t)$ is negative and nondecreasing on \mathcal{I}_a . Hence $y \in \mathcal{NSV}$ by Proposition 3.1, part (vi). Let $w(t) = \Phi(y^\Delta(t)/y(t))$. Then $w(t) < 0$ and satisfies (GR) with $1 + \mu(t) \Phi^{-1}(w(t)) > 0$ for $t \in \mathcal{I}_a$. Since $y \in \mathcal{NSV}$, we have

$$\lim_{t \rightarrow \infty} \frac{ty^\Delta(t)}{y(t)} = 0, \quad \text{thus} \quad \lim_{t \rightarrow \infty} t^{\alpha-1} \Phi \left(\frac{y^\Delta(t)}{y(t)} \right) = 0,$$

hence $\lim_{t \rightarrow \infty} t^{\alpha-1} w(t) = 0$ (and also $\lim_{t \rightarrow \infty} w(t) = 0$). Therefore,

$$\lim_{t \rightarrow \infty} (-1) |t^{\alpha-1} w(t)|^{\beta-1} = 0, \quad \text{thus} \quad \lim_{t \rightarrow \infty} t \Phi^{-1}(w(t)) = 0$$

and hence

$$\lim_{t \rightarrow \infty} Nt \Phi^{-1}(w(t)) = 0, \quad (3.32)$$

where $N > 0$ is an arbitrary real constant. In view of $\mu(t) = O(t)$, there exists positive N such that $\mu(t)/t \leq N$ for $t \in \mathcal{I}_a$, thus $\mu(t) \leq Nt$ for $t \in \mathcal{I}_a$. Therefore and from (3.32) we obtain $\lim_{t \rightarrow \infty} \mu(t) \Phi^{-1}(w(t)) = 0$. It is easy to show, that $S(t)$ defined by (S) is positive for $t \in \mathcal{I}_a$, provided that $w(t) < 0$ and $1 + \mu(t) \Phi^{-1}(w(t)) > 0$ for $t \in \mathcal{I}_a$. Applying the Lagrange mean value theorem, $S(t)$ can be alternatively written as

$$S(t) = \frac{(\alpha - 1)|w(t)|^\beta \xi^{\alpha-2}(t)}{[1 + \mu(t) \Phi^{-1}(w(t))]^{\alpha-1}}, \quad (3.33)$$

where $0 < 1 + \mu(t) \Phi^{-1}(w(t)) \leq \xi(t) \leq 1$. We show that $\int_t^\infty S(s) \Delta s < \infty$, which implies $\lim_{t \rightarrow \infty} \int_t^\infty S(s) \Delta s = 0$. Since $\lim_{t \rightarrow \infty} \mu(t) \Phi^{-1}(w(t)) = 0$, we get $\xi(t) \rightarrow 1$ as $t \rightarrow \infty$ and we have $S(t) \leq 2(\alpha - 1)|w(t)|^\beta$ for large t . Further, since $\lim_{t \rightarrow \infty} t^{1-\alpha} w(t) = 0$, there exists $M > 0$ such that $|w(t)| \leq Mt^{\alpha-1}$ for large t . Hence, for large t (with the use of validity $(Mt^{1-\alpha})^\beta = M^\beta t^{-\alpha}$)

$$\int_t^\infty S(s) \Delta s \leq 2(\alpha - 1) \int_t^\infty |w(s)|^\beta \Delta s \leq 2(\alpha - 1) M^\beta \int_t^\infty \frac{1}{s^\alpha} \Delta s < \infty. \quad (3.34)$$

Note that the integral $\int_t^\infty (1/s^\alpha)\Delta s$ is indeed convergent since $\mu(t) = O(t)$. By integration of (GR) from t to ∞ and multiplication by $t^{\alpha-1}$ yield

$$-t^{\alpha-1}w(t) + t^{\alpha-1} \int_t^\infty S(s)\Delta s = t^{\alpha-1} \int_t^\infty p(s)\Delta s. \quad (3.35)$$

Equality (3.33) and the time scale L'Hospital rule give

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty S(s)\Delta s = \lim_{t \rightarrow \infty} \frac{-(\alpha-1)|w(t)|^\beta \xi^{\alpha-2}(t)}{[1 + \mu(t)\Phi^{-1}(w(t))]^{\alpha-1} (t^{1-\alpha})^\Delta}.$$

Now differentiation of $t^{1-\alpha}$ and applying the Lagrange mean value theorem on this term with $t \leq \eta(t) \leq \sigma(t)$, we have (with the use of validity $\lim_{t \rightarrow \infty} \xi(t) = 1$ and $\eta(t)/t \leq \sigma(t)/t = 1 + \mu(t)/t \leq 1 + N$, $N \in \mathbb{R}$)

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty S(s)\Delta s &= \lim_{t \rightarrow \infty} \frac{-(\alpha-1)|w(t)|^\beta \xi^{\alpha-2}(t)}{(1-\alpha)\eta^{-\alpha}(t)[1 + \mu(t)\Phi^{-1}(w(t))]^{\alpha-1}} \\ &= \lim_{t \rightarrow \infty} \frac{(t^{\alpha-1})^\beta}{t^\alpha} \cdot \frac{\eta^\alpha(t)|w(t)|^\beta \xi^{\alpha-2}(t)}{[1 + \mu(t)\Phi^{-1}(w(t))]^{\alpha-1}} \\ &\leq (1+N)^\alpha \lim_{t \rightarrow \infty} |t^{\alpha-1}w(t)|^\beta = 0. \end{aligned}$$

Hence, from (3.35), we get (3.29).

“If”: Let $y > 0$ be a decreasing solution of (HL), then $\lim_{t \rightarrow \infty} y^\Delta(t) = 0$. Indeed, if not, then there is $K > 0$ such that $y^\Delta(t) \leq -K$ for $t \in \mathcal{I}_a$, and so $y(t) \leq y(a) - (t-a)K$. Letting $t \rightarrow \infty$ we have $\lim_{t \rightarrow \infty} y(t) = -\infty$, a contradiction with $y > 0$. Therefore, by integration of (HL) from t to ∞ yields $\Phi(y^\Delta(t)) = -\int_t^\infty p(s)\Phi(y^\sigma(s))\Delta s$. Multiplying this equality by $-t^{\alpha-1}/\Phi(y(t))$ we obtain

$$\begin{aligned} -\frac{t^{\alpha-1}\Phi(y^\Delta(t))}{\Phi(y(t))} &= \frac{t^{\alpha-1}}{\Phi(y(t))} \int_t^\infty p(s)\Phi(y^\sigma(s))\Delta s \\ &\leq \frac{t^{\alpha-1}\Phi(y(t))}{\Phi(y(t))} \int_t^\infty p(s)\Delta s = t^{\alpha-1} \int_t^\infty p(s)\Delta s. \end{aligned}$$

Hence $0 < -t^{\alpha-1}\Phi(y^\Delta(t)/y(t)) \rightarrow 0$, or $0 < -ty^\Delta(t)/y(t) \rightarrow 0$ as $t \rightarrow \infty$, in view of (3.29). Thus $y \in \mathcal{NSV}$.

(ii) “Only if”: Let $y \in \mathcal{RV}(\Phi^{-1}(\vartheta_0))$ be a positive decreasing solution of (HL) on \mathcal{I}_a . Then $y^\Delta(t)$ is negative and nondecreasing on \mathcal{I}_a . Thus $y \in \mathcal{NRV}(\Phi^{-1}(\vartheta_0))$ by Proposition 3.1, part (vi). Let $w = \Phi(y^\Delta(t)/y(t))$. Then $w(t)$ satisfies the equation (GR) with $1 + \mu(t)\Phi^{-1}(w(t)) > 0$ for $t \in \mathcal{I}_a$. Since $y \in \mathcal{NRV}(\Phi^{-1}(\vartheta_0))$, we have

$$\lim_{t \rightarrow \infty} \frac{ty^\Delta(t)}{y(t)} = \Phi^{-1}(\vartheta_0), \quad \text{hence} \quad \lim_{t \rightarrow \infty} t^{\alpha-1}\Phi\left(\frac{y^\Delta(t)}{y(t)}\right) = \Phi(\Phi^{-1}(\vartheta_0)),$$

thus $\lim_{t \rightarrow \infty} t^{\alpha-1} w(t) = \vartheta_0$ (and $\lim_{t \rightarrow \infty} w(t) = 0$). Therefore,

$$\lim_{t \rightarrow \infty} (-1) |t^{\alpha-1} w(t)|^{\beta-1} = -|\vartheta_0|^{\beta-1} \quad (3.36)$$

and hence

$$\lim_{t \rightarrow \infty} t \Phi^{-1}(w(t)) = -|\vartheta_0|^{\beta-1}. \quad (3.37)$$

In a view of $\mu(t) = o(t)$, $\lim_{t \rightarrow \infty} \mu(t)/t = 0$. Together with (3.37) we get

$$0 = \lim_{t \rightarrow \infty} \frac{\mu(t)}{t} \lim_{t \rightarrow \infty} t \Phi^{-1}(w(t)) = \lim_{t \rightarrow \infty} \mu(t) \Phi^{-1}(w(t)).$$

By integration of (GR) from t to ∞ and multiplication by $t^{\alpha-1}$ yield (3.35). The convergence of the series $\int_t^\infty S(s) \Delta s$ can be proved similarly as in the case “only if” of part (i), see (3.34). Further, as the same way as before, we can show that

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty S(s) \Delta s = \lim_{t \rightarrow \infty} |t^{\alpha-1} w(t)|^\beta = |\vartheta_0|^\beta,$$

where the last equality is the consequence of (3.36). Hence in view (3.31) and (3.35) we get (3.30).

“If”: Assume that (3.30) holds. Let y be a positive decreasing solution of (HL). Let $w_m(t) = t^{\alpha-1} \Phi(y^\Delta(t)/y(t))$. Similarly as in the case “if” of part (i), we have $\lim_{t \rightarrow \infty} y^\Delta(t) = 0$ and $0 < -w_m(t) \leq t^{\alpha-1} \int_t^\infty p(s) \Delta s$. Hence and due to (3.30), $-w_m(t)$ is bounded from above. We will show that $\lim_{t \rightarrow \infty} w_m(t) = \vartheta_0$, which implies $y \in \mathcal{NRV}(\Phi^{-1}(\vartheta_0))$. First observe that $w_m(t)$ satisfies the modified Riccati equation

$$\left(\frac{w_m(t)}{t^{\alpha-1}} \right)^\Delta - p(t) + F(t) = 0, \quad (3.38)$$

where

$$F(t) = \lim_{\gamma(t) \rightarrow \mu(t)} \frac{w_m(t)}{t^{\alpha-1} \gamma(t)} \left(1 - \frac{1}{\Phi[1 + \gamma(t) \Phi^{-1}(w_m(t)/t^{\alpha-1})]} \right),$$

with $1 + \mu(t) \Phi^{-1}(w_m(t)) > 0$ for $t \in \mathcal{I}_a$. Since $\lim_{t \rightarrow \infty} (w_m(t)/t^{\alpha-1}) = 0$, by integration of (3.38) from t to ∞ yields

$$-\frac{w_m(t)}{t^{\alpha-1}} = \int_t^\infty p(s) \Delta s - \int_t^\infty F(s) \Delta s. \quad (3.39)$$

If we write (3.30) as $t^{\alpha-1} \int_t^\infty p(s) \Delta s = A + \varepsilon_1(t) = |\vartheta_0|^\beta - \vartheta_0 + \varepsilon_1(t)$, where $\varepsilon_1(t)$ is some function satisfying $\lim_{t \rightarrow \infty} \varepsilon_1(t) = 0$, then multiplying (3.39) by $t^{\alpha-1}$ we obtain

$$-w_m(t) = |\vartheta_0|^\beta - \vartheta_0 - t^{\alpha-1} \int_t^\infty F(s) \Delta s + \varepsilon_1(t). \quad (3.40)$$

Applying the Lagrange mean value theorem, $F(t)$ can be written as

$$F(t) = \frac{(\alpha - 1)|w_m(t)/t^{\alpha-1}|^\beta \xi^{\alpha-2}(t)}{[1 + \mu(t) \Phi^{-1}(w_m(t)/t^{\alpha-1})]^{\alpha-1}},$$

where $0 < 1 + \mu(t) \Phi^{-1}(w_m(t)/t^{\alpha-1}) \leq \xi(t) \leq 1$. We show that we may write

$$t^{\alpha-1} \int_t^\infty F(s) \Delta s = t^{\alpha-1} \int_t^\infty \left[- (s^{1-\alpha})^\Delta \right] |w_m(s)|^\beta \Delta s + \varepsilon_2(t), \quad (3.41)$$

with some function $\varepsilon_2(t)$ satisfying $\lim_{t \rightarrow \infty} \varepsilon_2(t) = 0$. Denote

$$Q(t) = \frac{\xi^{\alpha-2}(t)}{[1 + \mu(t) \Phi^{-1}(w_m(t)/t^{\alpha-1})]^{\alpha-1}}.$$

Since $\lim_{t \rightarrow \infty} (w_m(t)/t^{\alpha-1}) = 0$, $\lim_{t \rightarrow \infty} \xi(t) = 1$ and so $\lim_{t \rightarrow \infty} Q(t) = 1$. We have

$$t^{\alpha-1} \int_t^\infty F(s) \Delta s = t^{\alpha-1} \int_t^\infty \left[- (s^{1-\alpha})^\Delta \right] |w_m(s)|^\beta \Delta s + t^{\alpha-1} \int_t^\infty H(s) \Delta s,$$

where

$$\begin{aligned} H(t) &= F(t) - \left[- (t^{1-\alpha})^\Delta \right] |w_m(t)|^\beta \\ &= \frac{(\alpha - 1)|w_m(t)|^\beta}{(t^{\alpha-1})^\beta} Q(t) - \left[- (t^{1-\alpha})^\Delta \right] |w_m(t)|^\beta \\ &= \frac{(\alpha - 1)|w_m(t)|^\beta}{(t^{\alpha-1})^\beta} Q(t) - \frac{\alpha - 1}{\gamma^\alpha(t)} |w_m(t)|^\beta, \end{aligned}$$

with $t \leq \gamma(t) \leq \sigma(t)$. Using the time scale L'Hospital rule and again the Lagrange mean value theorem on the term $t^{\alpha-1}$ with $t \leq \eta(t) \leq \sigma(t)$, we get

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty H(s) \Delta s &= \lim_{t \rightarrow \infty} \frac{-(\alpha - 1)|w_m(t)|^\beta (Q(t)/t^\alpha - 1/\gamma^\alpha(t))}{(1 - \alpha)/\eta^\alpha(t)} \\ &= \lim_{t \rightarrow \infty} |w_m(t)|^\beta \frac{(\gamma(t) \eta(t))^\alpha Q(t) - (t \eta(t))^\alpha}{(t \gamma(t))^\alpha} \\ &= \lim_{t \rightarrow \infty} |w_m(t)|^\beta \frac{(\gamma(t) \eta(t)/t^2)^\alpha - (t \eta(t)/t^2)^\alpha}{(t \gamma(t)/t^2)^\alpha} = 0, \end{aligned}$$

where we use the fact that $\lim_{t \rightarrow \infty} \gamma(t)/t = 1$ and $\lim_{t \rightarrow \infty} \eta(t)/t = 1$ following from $\mu(t) = o(t)$. Hence, $t^{\alpha-1} \int_t^\infty H(s) \Delta s = \varepsilon_2(t)$, with some $\varepsilon_2(t)$, where $\lim_{t \rightarrow \infty} \varepsilon_2(t) = 0$, and so (3.41) holds. In view of (3.41), from (3.40) we get

$$-w_m(t) = |\vartheta_0|^\beta - \vartheta_0 - t^{\alpha-1} \int_t^\infty \left[- (s^{1-\alpha})^\Delta \right] |w_m(s)|^\beta \Delta s + \varepsilon(t),$$

where $\varepsilon(t) = \varepsilon_1(t) - \varepsilon_2(t)$. Hence,

$$-w_m(t) = |\vartheta_0|^\beta - \vartheta_0 - t^{\alpha-1} G(t) \int_t^\infty \left[- (s^{1-\alpha})^\Delta \right] \Delta s + \varepsilon(t),$$

where $m \leq G(t) \leq M$ with $m = \inf_{t \in \mathcal{I}_a} |w_m(t)|^\beta$, $M = \sup_{t \in \mathcal{I}_a} |w_m(t)|^\beta$, or

$$G(t) - w_m(t) = |\vartheta_0|^\beta - \vartheta_0 + \varepsilon(t). \quad (3.42)$$

We show that $\lim_{t \rightarrow \infty} w_m(t) = \vartheta_0$. Recall that $-w_m(t) > 0$ is bounded from above. Assume that there exists $\lim_{t \rightarrow \infty} (-w_m(t)) = L \geq 0$. Then from (3.42) we get $L^\beta + L = |\vartheta_0|^\beta - \vartheta_0$. If $L > -\vartheta_0$, then $|\vartheta_0|^\beta = L^\beta + L + \vartheta_0 > L^\beta$, contradiction. Similarly we get contradiction if $L < -\vartheta_0$. Next we show that $\lim_{t \rightarrow \infty} (-w_m(t))$ exists. Assume that

$$\liminf_{t \rightarrow \infty} (-w_m(t)) = L_* < L^* = \limsup_{t \rightarrow \infty} (-w_m(t)).$$

Let L_1 be defined by $\liminf_{t \rightarrow \infty} G(t) = L_1^\beta$ and L_2 be defined by $\limsup_{t \rightarrow \infty} G(t) = L_2^\beta$. In general, $0 \leq L_* \leq L_1 \leq L_2 \leq L^*$. Assuming that at least one inequality is strict, which implies that at least one of the values is different from $-\vartheta_0$, we come to a contradiction, arguing similarly as in the case when L existed. All these observations prove that the limit $\lim_{t \rightarrow \infty} w_m(t)$ exists and is equal to ϑ_0 . \square

Remark 3.9. (i) The statements (i) and (ii) in the previous theorem could be unified, assuming $A \geq 0$ and $\vartheta_0 \leq 0$. However, the condition $\mu(t) = O(t)$ if $\vartheta_0 = 0$ or $\mu(t) = o(t)$ if $\vartheta_0 < 0$ has to be assumed, see concluding comments in Section 3.5.

(ii) It is easy to see that conditions (3.29) and (3.30) in the if parts of Theorem 3.7 can be replaced by the simpler ones $\lim_{t \rightarrow \infty} t^\alpha p(t) = 0$ and $\lim_{t \rightarrow \infty} t^\alpha p(t) = A$, respectively.

(iii) Observe that the condition y is decreasing in the last theorem does not need to be assumed. Indeed, we are actually dealing with all \mathcal{SV} or $\mathcal{RV}(\Phi^{-1}(\vartheta_0))$ solutions. Hence, $\vartheta_0 \leq 0$. Thanks to convexity of y , assertion now follows from part (v) of Proposition 3.1.

Necessary and sufficient conditions for all positive decreasing solutions of (L) to be regularly varying was established in [36]. Now we want to apply the above developed theory of regular variation, and complete the results from [36] in the sense of increasing solutions. Note that for decreasing solutions, the following theorem follows from the previous one.

Theorem 3.8. (i) Let $\mu(t) = O(t)$. Equation (L) has a fundamental set of solutions

$$u(t) = L(t) \in \mathcal{SV}_{\mathbb{T}} \quad \text{and} \quad v(t) = t\tilde{L}(t) \in \mathcal{RV}_{\mathbb{T}}(1) \quad (3.43)$$

if and only if

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) \Delta s = 0. \quad (3.44)$$

Moreover, $L, \tilde{L} \in \mathcal{NSV}_{\mathbb{T}}$ with $\tilde{L}(t) \sim 1/L(t)$. All positive decreasing solutions of (L) belong to $\mathcal{NSV}_{\mathbb{T}}$ and all positive increasing solutions of (L) belong to $\mathcal{NRV}_{\mathbb{T}}(1)$. Any of two conditions in (3.43) implies (3.44).

(ii) Let $\mu(t) = o(t)$. Equation (L) has a fundamental set of solutions

$$u(t) = t^{\vartheta_1} L(t) \in \mathcal{RV}_{\mathbb{T}}(\vartheta_1) \quad \text{and} \quad v(t) = t^{\vartheta_2} \tilde{L}(t) \in \mathcal{RV}(\vartheta_2) \quad (3.45)$$

if and only if

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) \Delta s = A > 0, \quad (3.46)$$

where $\vartheta_1 < 0$ and $\vartheta_2 = 1 - \vartheta_1$ are the roots of the equation $\vartheta^2 - \vartheta - A = 0$. Moreover $L, \tilde{L} \in \mathcal{NSV}_{\mathbb{T}}$ with $\tilde{L}(t) \sim 1/((1 - 2\vartheta_1)L(t))$. All positive decreasing solutions of (L) belong to $\mathcal{NRV}_{\mathbb{T}}(\vartheta_1)$ and all positive increasing solutions of (L) belong to $\mathcal{NRV}_{\mathbb{T}}(\vartheta_2)$. Any of two conditions in (3.45) implies (3.46).

Proof. Parts (i) and (ii) of the theorem will be proved simultaneously assuming $A \geq 0$ in (3.46) and, consequently, $\vartheta_1 \leq 0$ (resp. $\vartheta_2 \geq 1$).

“Only if parts”: In view of the convexity, a solution $u \in \mathcal{RV}_{\mathbb{T}}(\vartheta_1)$ necessarily decreases and a solution $v \in \mathcal{RV}_{\mathbb{T}}(\vartheta_2)$ necessarily increases by (v) of Proposition 3.1. For decreasing solution u , either see the proof in [36] or take $\alpha = 2$ in previous theorem. Using arguments similar to those in [36] we can show that if a positive increasing solution v of (L) belongs to $\mathcal{RV}_{\mathbb{T}}(\vartheta_2)$, then (3.46) holds. It means that (3.46) is necessary for any of two conditions in (3.45).

“If parts”: Let $\lim_{t \rightarrow \infty} t \int_t^\infty p(s) \Delta s = A$. From [36] (or from the previous theorem), if u is a positive decreasing solution (which always exists), then $u \in \mathcal{NRV}(\vartheta_1)$. Hence, $u(t) = t^{\vartheta_1} L(t)$, where $L \in \mathcal{NSV}$ by (i) of Proposition 3.1. Put $z = 1/u^2$. Then $z \in \mathcal{NRV}_{\mathbb{T}}(-2\vartheta_1)$ by (iv) of Proposition 3.1. Moreover, $z(t) \sim 1/(u(t)u^\sigma(t))$ as $t \rightarrow \infty$ by Lemma 3.1 or Remark 3.2. A second linearly independent solution v of (L) is given by $v(t) = u(t) \int_a^t 1/(u(s)u^\sigma(s)) \Delta s$. Taking into account that u is decreasing (recessive), it holds $\int_a^\infty 1/(u(s)u^\sigma(s)) \Delta s = \infty$. Further, $tz(t) \rightarrow \infty$ as $t \rightarrow \infty$ by (iv) and (ii) of Proposition 3.1. The time scale L'Hospital rule now yields

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{t/u(t)}{v(t)} &= \lim_{t \rightarrow \infty} \frac{tz(t)}{\int_a^t 1/(u(s)u^\sigma(s)) \Delta s} = \lim_{t \rightarrow \infty} \frac{z(t) + \sigma(t)z^\Delta(t)}{1/(u(t)u^\sigma(t))} \\ &= 1 + \lim_{t \rightarrow \infty} \frac{\sigma(t)z^\Delta(t)}{z(t)} = 1 - 2\vartheta_1. \end{aligned}$$

Hence $(1 - 2\vartheta_1)v(t) \sim t/u(t) = t^{1-\vartheta_1}/L(t)$. Consequently, $v(t) = t^{1-\vartheta_1} \tilde{L}(t)$, where $\tilde{L}(t) \sim 1/[(1 - \vartheta_1)L(t)]$ and $\tilde{L} \in \mathcal{SV}_{\mathbb{T}}$ by (iv) of Proposition 3.1. This implies

$v \in \mathcal{RV}_{\mathbb{T}}(\vartheta_2)$ by (i) of Proposition 3.1. Further, since (the solution) v is convex, it is increasing by (v) of Proposition 3.1, thus it is normalized by (vi) of Proposition 3.1, and therefore \tilde{L} is normalized too by (i) of Proposition 3.1. \square

Remark 3.10. (i) The condition $\mu(t) = O(t)$ if $\vartheta = 0$ or $\mu(t) = o(t)$ if $\vartheta \neq 0$ has to be assumed in Theorem 3.8, see concluding comments in Section 3.5.

(ii) Similarly as in half-linear case, see, (ii) in Remark 3.9, conditions (3.44) and (3.46) in the if parts of Theorem 3.8 can be replaced by the simpler ones $\lim_{t \rightarrow \infty} t^2 p(t) = 0$ and $\lim_{t \rightarrow \infty} t^2 p(t) = A$, respectively.

As an application of theory of rapid variation, we study asymptotic behavior of positive solutions of (HL). Recall that every positive solution of (HL) is convex.

Theorem 3.9. Let $\mu(t) = o(t)$. Equation (HL) has solutions $u \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$ and $v \in \mathcal{RPV}_{\mathbb{T}}(\infty)$ if and only if for all $\lambda > 1$

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s = \infty. \quad (3.47)$$

Moreover, all positive decreasing solutions of (HL) belong to $\mathcal{NRPV}_{\mathbb{T}}(-\infty)$ and all positive increasing solutions of (HL) belong to $\mathcal{NRPV}_{\mathbb{T}}(\infty)$.

Proof. “If”: Let u be a positive decreasing solution of (HL) and let (3.47) hold. By integration of equation (L) from t to $\tau(\sqrt{\lambda}t)$ ($\lambda > 1$) we get

$$\Phi(u^\Delta(\tau(\sqrt{\lambda}t))) - \Phi(u^\Delta(t)) = \int_t^{\tau(\sqrt{\lambda}t)} p(s) \Phi(u(\sigma(s))) \Delta s.$$

Since $u^\Delta < 0$ and u is positive decreasing with zero limit, we can write

$$-u^\Delta(t) \geq \Phi^{-1} \left(\int_t^{\tau(\sqrt{\lambda}t)} p(s) \Phi(u(\sigma(s))) \Delta s \right) \geq u(\tau(\sqrt{\lambda}t)) \Phi^{-1} \left(\int_t^{\tau(\sqrt{\lambda}t)} p(s) \Delta s \right). \quad (3.48)$$

In the last inequality we use the fact that

$$\int_a^b f^\sigma(t) g(t) \Delta t \geq f(b) \int_a^b g(t) \Delta t \quad (a, b \in \mathbb{T}; a < b)$$

holds for arbitrary positive decreasing function f and positive function g . This inequality follows from the time scales version of the second mean value theorem of integral calculus, see [33, Lemma 2.5]. By integration of (3.48) from t to $\tau(\sqrt{\lambda}t)$ ($\lambda > 1$) we get

$$u(t) - u(\tau(\sqrt{\lambda}t)) \geq \int_t^{\tau(\sqrt{\lambda}t)} u(\tau(\sqrt{\lambda}s)) \Phi^{-1} \left(\int_s^{\tau(\sqrt{\lambda}s)} p(r) \Delta r \right) \Delta s.$$

By using the same ideas as before, we get (with the use of $u(\tau(\sqrt{\lambda}\tau(\sqrt{\lambda}t))) \geq u(\tau(\lambda t))$)

$$u(t) \geq u(\tau(\lambda t)) \int_t^{\tau(\sqrt{\lambda}t)} \Phi^{-1} \left(\int_s^{\tau(\sqrt{\lambda}s)} p(r) \Delta r \right) \Delta s. \quad (3.49)$$

In view of (3.47) for any arbitrarily large constant $M > 0$ there exists t_0 sufficiently large such that

$$\int_t^{\tau(\sqrt{\lambda}t)} p(s) \Delta s \geq \frac{M}{t^{\alpha-1}}, \quad t > t_0. \quad (3.50)$$

Since u is positive, from (3.49) and (3.50) we get

$$\begin{aligned} \frac{u(t)}{u(\tau(\lambda t))} &\geq \Phi^{-1}(M) \int_t^{\tau(\sqrt{\lambda}t)} \Phi^{-1} \left(\frac{1}{s^{\alpha-1}} \right) \Delta s = \Phi^{-1}(M) \int_t^{\tau(\sqrt{\lambda}t)} \frac{1}{s} \Delta s \\ &\geq \Phi^{-1}(M) \int_t^{\tau(\sqrt{\lambda}t)} \frac{1}{s} ds = \Phi^{-1}(M) \ln \frac{\tau(\sqrt{\lambda}t)}{t} \\ &\geq \Phi^{-1}(M) \ln \frac{\sqrt{\lambda}t - \mu(\tau(\sqrt{\lambda}t))}{t} \\ &= \Phi^{-1}(M) \ln \left(\sqrt{\lambda} - \frac{\mu(\tau(\sqrt{\lambda}t))}{t} \right). \end{aligned}$$

where the inequality $\int_t^{\tau(\sqrt{\lambda}t)} (1/s) \Delta s \geq \int_t^{\tau(\sqrt{\lambda}t)} (1/s) ds$ (using also in further part of the proof of this theorem) follows from [34, Lemma 1.1]. Since $\mu(\tau(\sqrt{\lambda}t))/t \rightarrow 0$ as $t \rightarrow \infty$ and since M was arbitrarily large, this implies

$$\lim_{t \rightarrow \infty} \frac{u(t)}{u(\tau(\lambda t))} = \infty.$$

Consequently,

$$\lim_{t \rightarrow \infty} \frac{u(\tau(\lambda t))}{u(t)} = 0, \quad \lambda > 1,$$

which implies (due to Proposition 3.3) that $u \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$ and hence, due to Lemma 3.2 $u \in (\mathcal{N})\mathcal{RPV}_{\mathbb{T}}(-\infty)$.

Let v be a positive increasing solution of (HL) and let (3.47) hold. By integration of equation (HL) from $\tau(t/\sqrt{\lambda})$ to t ($\lambda > 1$), we get

$$\Phi(v^\Delta(t)) - \Phi \left(v^\Delta \left(\tau \left(\frac{t}{\sqrt{\lambda}} \right) \right) \right) = \int_{\tau(\frac{t}{\sqrt{\lambda}})}^t p(s) \Phi(v(\sigma(s))) \Delta s.$$

Since $v^\Delta > 0$ and v is positive increasing, we get

$$v^\Delta(t) \geq \Phi^{-1} \left(\int_{\tau(\frac{t}{\sqrt{\lambda}})}^t p(s) \Phi(v(\sigma(s))) \Delta s \right) \geq v \left(\tau \left(\frac{t}{\sqrt{\lambda}} \right) \right) \Phi^{-1} \left(\int_{\tau(\frac{t}{\sqrt{\lambda}})}^t p(s) \Delta s \right).$$

By integration of the last inequality from $\sigma(\tau(t/\sqrt{\lambda}))$ to t ($\lambda > 1$) we get

$$v(t) - v\left(\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)\right) \geq \int_{\sigma(\tau(\frac{t}{\sqrt{\lambda}}))}^t v\left(\tau\left(\frac{s}{\sqrt{\lambda}}\right)\right) \Phi^{-1}\left(\int_{\tau(\frac{s}{\sqrt{\lambda}})}^s p(r)\Delta r\right) \Delta s$$

By using the same ideas as before, we get

$$v(t) \geq v\left(\tau\left(\frac{t}{\lambda}\right)\right) \int_{\sigma(\tau(\frac{t}{\sqrt{\lambda}}))}^t \Phi^{-1}\left(\int_{\tau(\frac{s}{\sqrt{\lambda}})}^s p(r)\Delta r\right) \Delta s, \quad (3.51)$$

where we use

$$v\left(\tau\left(\frac{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}{\sqrt{\lambda}}\right)\right) \geq v\left(\tau\left(\frac{t}{\lambda}\right)\right).$$

Inequality (3.51) can be rewritten on the form

$$\frac{v(t)}{v\left(\tau\left(\frac{t}{\lambda}\right)\right)} \geq \int_{\sigma(\tau(\frac{t}{\sqrt{\lambda}}))}^t \Phi^{-1}\left(\int_{\tau(\frac{s}{\sqrt{\lambda}})}^s p(r)\Delta r\right) \Delta s, \quad (3.52)$$

In view of (3.47), which can be equivalently written with $\sqrt{\lambda}$ instead of λ , we have (due to $\{\tau(t/\sqrt{\lambda})\} \subseteq \mathbb{T}$ for large t)

$$\lim_{t \rightarrow \infty} \left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)^{\alpha-1} \int_{\tau(\frac{t}{\sqrt{\lambda}})}^{\tau(\sqrt{\lambda}\tau(\frac{t}{\sqrt{\lambda}}))} p(s)\Delta s = \infty.$$

Therefore, thanks to $\tau(t/\sqrt{\lambda}) \leq t/\sqrt{\lambda} < t$ and $\tau(\sqrt{\lambda}\tau(t/\sqrt{\lambda})) \leq t$, we get

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_{\tau(\frac{t}{\sqrt{\lambda}})}^t p(s)\Delta s = \infty,$$

which means that for arbitrarily large constant $M > 0$, there exists s_0 sufficiently large such that

$$\int_{\tau(\frac{s}{\sqrt{\lambda}})}^s p(r)\Delta r \geq \frac{M}{s^{\alpha-1}}, \quad s > s_0, \quad (3.53)$$

and since v is positive, then from (3.52) and (3.53), we get (by using the similar

calculations as in previous case for decreasing solution u)

$$\begin{aligned}
 \frac{v(t)}{v\left(\tau\left(\frac{t}{\lambda}\right)\right)} &\geq \Phi^{-1}(M) \int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^t \frac{1}{s} \Delta s \geq \Phi^{-1}(M) \int_{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}^t \frac{1}{s} ds \\
 &= \Phi^{-1}(M) \ln \frac{t}{\sigma\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)} \geq \Phi^{-1}(M) \ln \frac{t}{\frac{t}{\sqrt{\lambda}} + \mu\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)} \\
 &= \Phi^{-1}(M) \ln \frac{\sqrt{\lambda}}{1 + \frac{\mu\left(\tau\left(\frac{t}{\sqrt{\lambda}}\right)\right)}{\frac{t}{\sqrt{\lambda}}}}.
 \end{aligned}$$

Since $\mu(t) = o(t)$, $\mu(\tau(t/\sqrt{\lambda}))/(\tau(t/\sqrt{\lambda})) \rightarrow 0$ as $t \rightarrow \infty$ and since M was arbitrarily large, this yields

$$\lim_{t \rightarrow \infty} \frac{v(t)}{v\left(\tau\left(\frac{t}{\lambda}\right)\right)} \geq \Phi^{-1}(M) \ln \sqrt{\lambda} = \infty, \quad \lambda > 1,$$

i.e.,

$$\lim_{t \rightarrow \infty} \frac{v(\tau(\lambda t))}{v(t)} = 0, \quad \lambda < 1,$$

which implies, similarly as in the proof of Lemma 3.2, first implication of part (i) (indeed, the function v satisfies (3.25) and (3.26)), that $v \in (\mathcal{N})\mathcal{R}\mathcal{P}\mathcal{V}_{\mathbb{T}}(\infty)$.

“Only if”: Let u be a positive decreasing rapidly varying solution of (HL). Thanks to $u^{\Delta\Delta} > 0$ (see (HL)), we have u^{Δ} increases and due to Lemma 3.2, $u \in \mathcal{N}\mathcal{R}\mathcal{P}\mathcal{V}_{\mathbb{T}}(-\infty)$. Hence, $u^{\Delta}(t)$ is negative with zero limit and $u(t) \rightarrow 0$ as $t \rightarrow \infty$ (thanks to Proposition 3.2). Moreover, $-u^{\Delta}(t)$ decreases. For $\lambda > 1$ we have

$$\begin{aligned}
 -u^{\Delta}(\tau(\lambda t)) \tau(\lambda t) \left(1 - \frac{t}{\tau(\lambda t)}\right) &= -u^{\Delta}(\tau(\lambda t))(\tau(\lambda t) - t) \\
 &= -u^{\Delta}(\tau(\lambda t)) \int_t^{\tau(\lambda t)} \Delta s \\
 &\leq - \int_t^{\tau(\lambda t)} u^{\Delta}(s) \Delta s = u(t) - u(\tau(\lambda t)).
 \end{aligned} \tag{3.54}$$

From the fact that

$$\begin{aligned}
 1 - \frac{t}{\tau(\lambda t)} &\geq 1 - \frac{t}{\lambda t - \mu(\tau(\lambda t))} = 1 - \frac{1}{\lambda - \frac{\mu(\tau(\lambda t))}{t}} = 1 - \frac{1}{\lambda \left(1 - \frac{\mu(\tau(\lambda t))}{\lambda t}\right)} \\
 &\geq 1 - \frac{1}{\lambda \left(1 - \frac{\mu(\tau(\lambda t))}{\tau(\lambda t)}\right)},
 \end{aligned}$$

we have (due to $\mu(\tau(\lambda t))/\tau(\lambda t) \rightarrow 0$ as $t \rightarrow \infty$):

$$\lim_{t \rightarrow \infty} \left(1 - \frac{t}{\tau(\lambda t)} \right) \geq \lim_{t \rightarrow \infty} \left(1 - \frac{1}{\lambda \left(1 - \frac{\mu(\tau(\lambda t))}{\tau(\lambda t)} \right)} \right) = 1 - \frac{1}{\lambda} > 0.$$

Since $\lim_{t \rightarrow \infty} (u(t) - u(\tau(\lambda t))) = 0$, inequality (3.54) implies

$$\lim_{t \rightarrow \infty} \tau(\lambda t) u^\Delta(\tau(\lambda t)) = 0. \quad (3.55)$$

Due to $u^\Delta(t)$ is negative increasing,

$$\frac{u^\Delta(\tau(\lambda t))}{u^\Delta(t)} \leq 1.$$

Now we want to show that

$$\limsup_{t \rightarrow \infty} \frac{u^\Delta(\tau(\lambda t))}{u^\Delta(t)} < 1, \quad \lambda > 1. \quad (3.56)$$

By a contradiction, assume that there exist $\lambda_0 > 1$ and an unbounded sequence $\{t_k\}_{k=1}^\infty \subseteq \mathbb{T}$ such that

$$\lim_{t_k \rightarrow \infty} \frac{u^\Delta(\tau(\lambda_0^2 t_k))}{u^\Delta(t_k)} = 1. \quad (3.57)$$

Let y be a continuous positive decreasing function of a real variable, such that

$$\begin{aligned} y(t) &= -u^\Delta(t) \quad \text{for all } t \in \{t_k\} \quad \text{and} \\ y(t) &\geq -u^\Delta(t) \quad \text{for all } t \in \mathbb{T}. \end{aligned}$$

Thanks to $\mu(t)/t \rightarrow 0$ as $t \rightarrow \infty$, we have for large t

$$\frac{\mu(\tau(\lambda_0 t))}{\lambda_0 t} \leq \frac{\mu(\tau(\lambda_0 t))}{\tau(\lambda_0 t)} \leq \lambda_0 - 1$$

and therefore, we get

$$\mu(\tau(\lambda_0 t)) \leq \lambda_0^2 t - \lambda_0 t \leq \lambda_0^2 t - \tau(\lambda_0 t).$$

From the last inequality we have $\sigma(\tau(\lambda_0 t)) \leq \lambda_0^2 t$ for large t and hence

$$\lambda_0 t \leq \tau(\lambda_0^2 t). \quad (3.58)$$

From (3.57), (3.58) and thanks to y is decreasing we have

$$1 > \frac{y(\lambda_0 t_k)}{y(t_k)} \geq \frac{y(\tau(\lambda_0^2 t_k))}{y(t_k)} \geq \frac{u^\Delta(\tau(\lambda_0^2 t_k))}{u^\Delta(t_k)} \rightarrow 1,$$

as $t_k \rightarrow \infty$. Then (see the proof of [26, Theorem 1.3]) there exists a continuous positive decreasing function z of real variable, such that $z(t) = y(t)$ for every $t \in \mathbb{T}$ sufficiently large, and $\lim_{x \rightarrow \infty} (z(\lambda_0 x)/z(x)) = 1$. Since z is monotone, $\lim_{x \rightarrow \infty} (z(\lambda x)/z(x)) = 1$ holds for every $\lambda > 0$, see [4, Proposition 1.10.1] and this implies that z is slowly varying function, see [4]. Therefore, $\lim_{x \rightarrow \infty} xz(x) = \infty$. The contradiction follows by observing that

$$z(\tau(\lambda t)) = y(\tau(\lambda t)) = -u^\Delta(\tau(\lambda t)), \quad t \in \{t_k\}$$

and

$$\lim_{t \rightarrow \infty} -\tau(\lambda t) u^\Delta(\tau(\lambda t)) = 0, \quad t \in \{t_k\},$$

which holds due to (3.55). Hence, (3.56) holds. Therefore, there exists $N > 0$ such that

$$1 - \Phi \left(\frac{u^\Delta(\tau(\lambda t))}{u^\Delta(t)} \right) \geq N, \quad (3.59)$$

for every $\lambda > 1$ and t sufficiently large. By integration of (HL) from t to $\tau(\lambda t)$ we have

$$\Phi(u^\Delta(\tau(\lambda t))) - \Phi(u^\Delta(t)) = \int_t^{\tau(\lambda t)} p(s) \Phi(u(\sigma(s))) \Delta s \leq \Phi(u(t)) \int_t^{\tau(\lambda t)} p(s) \Delta s.$$

This implies

$$-\Phi(u^\Delta(t)) \left(1 - \frac{\Phi(u^\Delta(\tau(\lambda t)))}{\Phi(u^\Delta(t))} \right) \leq \Phi(u(t)) \int_t^{\tau(\lambda t)} p(s) \Delta s.$$

From (3.59) and by multiplying previous inequality by $t^{\alpha-1}$, we have

$$N \left(\frac{-tu^\Delta(t)}{u(t)} \right)^{\alpha-1} \leq t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s,$$

which (with $t \rightarrow \infty$) implies (3.47). □

Remark 3.11. Note that the previous theorem is new even for the linear case (when $\alpha = 2$), where u and v form a fundamental set of solutions of (L). A sufficiency part for increasing solution v is new also for the half-linear discrete case. For more information about the discrete case, see, e.g. [28, 30]. For continuous case, we refer to Marić's book [26] or to [27] for the corresponding results in the linear case. However, according to the best of our knowledge, the corresponding case of rapid variation in half-linear differential equations has not been processed in the literature. Finally note that a necessity part for increasing solutions has not been proved (even in linear case) in the differential (resp. difference or dynamic) equations setting yet.

3.4 \mathbb{M} -classification and Karamata functions

In this section we provide information about asymptotic behavior of all positive solutions of (L) and all positive decreasing solutions of (HL) as $t \rightarrow \infty$. First consider the linear second order dynamic equation (L). Note that all nontrivial solutions of (L) are nonoscillatory (i.e., of one sign for large t) and monotone for large t . Because of linearity, without loss of generality, we may consider just positive solutions of (L); we denote this set as \mathbb{M} . Thanks to the monotonicity, the set \mathbb{M} can be further split in the two classes \mathbb{M}^+ and \mathbb{M}^- , where

$$\begin{aligned} \mathbb{M}^+ &= \{y \in \mathbb{M} : \exists t_y \in \mathbb{T} \text{ such that } y(t) > 0, y^\Delta(t) > 0 \text{ for } t \geq t_y\}, \\ \mathbb{M}^- &= \{y \in \mathbb{M} : y(t) > 0, y^\Delta(t) < 0\}. \end{aligned}$$

These classes are always nonempty. To see it, the reader can follow the continuous ideas described, e.g., in [14, Chapter 4] or understand this equation as a special case of a more general quasi-linear dynamic equation; its asymptotic behavior is discussed, e.g., in [2].

A positive function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be *Karamata function*, if f is slowly or regularly or rapidly varying; we write $f \in \mathcal{KF}_{\mathbb{T}}$. In Theorem 3.8 we established necessary and sufficient conditions for all positive solutions of (L) to be regularly (resp. slowly) varying and in Theorem 3.9 (taking $\alpha = 2$) we completed this discussion for all positive solutions of (L) to be rapidly varying. Introduce the following notation:

$$\begin{aligned} \mathbb{M}_{SV}^- &= \mathbb{M}^- \cap \mathcal{NSV}_{\mathbb{T}}, \\ \mathbb{M}_{RV}^-(\vartheta_1) &= \mathbb{M}^- \cap \mathcal{NRV}_{\mathbb{T}}(\vartheta_1), \vartheta_1 < 0, \\ \mathbb{M}_{RV}^+(\vartheta_2) &= \mathbb{M}^+ \cap \mathcal{NRV}_{\mathbb{T}}(\vartheta_2), \vartheta_2 = 1 - \vartheta_1 > 1, \\ \mathbb{M}_{RPV}^-(-\infty) &= \mathbb{M}^- \cap \mathcal{NRPV}_{\mathbb{T}}(-\infty), \\ \mathbb{M}_{RPV}^+(\infty) &= \mathbb{M}^+ \cap \mathcal{NRPV}_{\mathbb{T}}(\infty), \\ \mathbb{M}_0^- &= \{y \in \mathbb{M}^- : \lim_{t \rightarrow \infty} y(t) = 0\}, \\ \mathbb{M}_\infty^+ &= \{y \in \mathbb{M}^+ : \lim_{t \rightarrow \infty} y(t) = \infty\} \end{aligned}$$

and distinguish three cases for behavior of the coefficient $p(t)$ from equation (L):

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) \Delta s = 0, \tag{3.60}$$

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) \Delta s = A > 0, \tag{3.61}$$

$$\lim_{t \rightarrow \infty} t \int_t^{\tau(\lambda t)} p(s) \Delta s = \infty \quad \forall \lambda > 1. \tag{3.62}$$

In view of (ii) of Proposition 3.1, Proposition 3.2, Theorem 3.8 and Theorem 3.9 we can claim

$$\begin{aligned} \mathbb{M}^- &= \mathbb{M}_{SV}^- \iff (3.60) \iff \mathbb{M}^+ = \mathbb{M}_{RV}^+(1) = \mathbb{M}_\infty^+, \\ \mathbb{M}^- &= \mathbb{M}_{RV}^-(\vartheta_1) = \mathbb{M}_0^- \iff (3.61) \iff \mathbb{M}^+ = \mathbb{M}_{RV}^+(\vartheta_2) = \mathbb{M}_\infty^+, \\ \mathbb{M}^- &= \mathbb{M}_{RPV}^-(-\infty) = \mathbb{M}_0^- \iff (3.62) \implies \mathbb{M}^+ = \mathbb{M}_{RPV}^+(\infty) = \mathbb{M}_\infty^+. \end{aligned} \quad (3.63)$$

Now consider the half-linear second order dynamic equation (HL). The space of all solutions is here more complicated than the space of all solution of equation (L). The reason is that we do not have a property of the linearity in this case. In Theorem 3.7 we established necessary and sufficient conditions for all positive decreasing solutions of (HL) to be regularly varying. By using Theorem 3.9 we completed this result in the sense of rapidly varying behavior. We distinguish three cases for behavior of coefficient $p(t)$ from equation (HL):

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) \Delta s = 0, \quad (3.64)$$

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) \Delta s = B > 0, \quad (3.65)$$

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s = \infty \quad \forall \lambda > 1. \quad (3.66)$$

In view of (ii) of Proposition 3.1, Proposition 3.2, Theorem 3.7 and Theorem 3.9 we can claim:

$$\begin{aligned} \mathbb{M}^- &= \mathbb{M}_{SV}^- \iff (3.64), \\ \mathbb{M}^- &= \mathbb{M}_{RV}^-(\vartheta_1) = \mathbb{M}_0^- \iff (3.65), \\ \mathbb{M}^- &= \mathbb{M}_{RPV}^-(-\infty) = \mathbb{M}_0^- \iff (3.66) \implies \mathbb{M}^+ = \mathbb{M}_{RPV}^+(\infty) = \mathbb{M}_\infty^+. \end{aligned} \quad (3.67)$$

The reader may wonder that we integrate from t to $\tau(\lambda t)$ in condition (3.62) (resp. (3.66)), while in conditions (3.60) and (3.61) (resp. (3.64) and (3.65)) we integrate from t to ∞ . In [28, Example 1], it is shown that there exists function $p : \mathbb{N} \rightarrow \mathbb{R}$ (so p is a sequence), which satisfies following condition

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) \Delta s = \infty, \quad \text{but} \quad \lim_{t \rightarrow \infty} t \int_t^{\tau(\lambda t)} p(s) \Delta s \neq \infty, \quad (3.68)$$

for some $\lambda > 1$. For simplicity, introduce the following notation

$$P = \lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) \Delta s, \quad P_\lambda = \lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s, \quad \lambda > 1.$$

In view of that example, $P = \infty$ does not imply $P_\lambda = \infty, \forall \lambda > 1$ (only the inverse implication holds, because $\int_t^{\tau(\lambda t)} p(s) \Delta s < \int_t^\infty p(s) \Delta s$). But if P is a finite (nonnegative) number, then P_λ is also a finite (nonnegative) number for all $\lambda > 1$ and contrariwise, if P_λ is finite (nonnegative) number for all $\lambda > 1$, then P is also finite (nonnegative) number. A relation between P and P_λ is shown in the following theorem.

Theorem 3.10. *It holds*

$$P = C \geq 0 \text{ if and only if } P_\lambda = \frac{C(\lambda^{\alpha-1} - 1)}{\lambda^{\alpha-1}}, \quad \forall \lambda > 1.$$

Proof. In this proof we will need a special sequence of reals. Take $\lambda > 1$ and $t \in \mathbb{T}$ sufficiently large and define sequence $\{r_n\}_{n=0}^\infty$ of reals such that $\lambda^{r_n t} = \tau(\lambda^{n t})$ for $n \in \mathbb{N} \cup \{0\}$. Note that $r_n = r_n(t)$. We show that r_n has following properties:

(i) $r_n < r_{n+1} \quad \forall n \in \mathbb{N}$

(ii) $r_0 = 0 < r_1 \leq 1 < r_2 \leq 2 < \dots < r_{n-1} \leq n-1 < r_n \leq n \quad \forall n \in \mathbb{N}$

(iii) $\tau(\lambda^{1+r_n t}) \leq \lambda^{r_{n+1} t} \quad \forall n \in \mathbb{N}$

(iv) $r_n(t) \rightarrow n \text{ as } t \rightarrow \infty \quad \forall n \in \mathbb{N}$

(i) Let $n \in \mathbb{N}$. First note that for $\tau(\lambda^{n t})$ right-dense $\lambda^{r_n t} = \tau(\lambda^{n t}) = \lambda^{n t} < \tau(\lambda^{n+1 t}) = \lambda^{r_{n+1} t}$ and (i) holds trivially. Now suppose that $\tau(\lambda^{n t})$ is right-scattered. Thanks to $\mu(t) = o(t), \mu(t) < (\lambda - 1)t$ for large t and we can write

$$\sigma(\tau(\lambda^{n t})) \leq \lambda^{n t} + \mu(\tau(\lambda^{n t})) < \lambda^{n t} + (\lambda - 1)\tau(\lambda^{n t}) \leq \lambda^{n t} + (\lambda - 1)\lambda^{n t} = \lambda^{n+1 t}.$$

Therefore, $\tau(\lambda^{n t}) < \sigma(\tau(\lambda^{n t})) \leq \tau(\lambda^{n+1 t})$. Hence, $\lambda^{r_n t} < \lambda^{r_{n+1} t}$ and (i) holds.

(ii) Note that $r_0 = 0$ holds trivially. Let $n \in \mathbb{N}$. By using (i) we can write $\tau(\lambda^{n-1 t}) \leq \lambda^{n-1 t} < \tau(\lambda^{n t}) = \lambda^{r_n t} \leq \lambda^{n t}$. Hence, $n-1 < r_n \leq n$.

(iii) Let $n \in \mathbb{N}$. It holds

$$\tau(\lambda^{1+r_n t}) = \tau(\lambda \lambda^{r_n t}) = \tau(\lambda(\tau(\lambda^{n t}))) \leq \tau(\lambda \lambda^{n t}) = \tau(\lambda^{n+1 t}) = \lambda^{r_{n+1} t}.$$

Hence, (iii) is fulfilled.

(iv) In view of

$$1 \geq \frac{\tau(\lambda^{n t})}{\lambda^{n t}} \geq \frac{\lambda^{n t} - \mu(\tau(\lambda^{n t}))}{\lambda^{n t}} = 1 - \frac{\mu(\tau(\lambda^{n t}))}{\lambda^{n t}} \rightarrow 1 \text{ as } t \rightarrow \infty,$$

we get $\lim_{t \rightarrow \infty} \tau(\lambda^{n t}) / (\lambda^{n t}) = 1$. Hence,

$$1 = \lim_{t \rightarrow \infty} \frac{\tau(\lambda^{n t})}{\lambda^{n t}} = \lim_{t \rightarrow \infty} \frac{\lambda^{r_n t}}{\lambda^{n t}} = \lambda^{r_n - n},$$

which implies $r_n(t) \rightarrow n$ as $t \rightarrow \infty$ for each $n \in \mathbb{N}$.

“If”: We wish to show that if there is $\lambda > 1$ such that $P_\lambda = L$, then $P = L\lambda^{\alpha-1}/(\lambda^{\alpha-1} - 1)$.

First suppose that there exist $\lambda > 1$ and $L_* > 0$ such that

$$\liminf_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s \geq L_*.$$

Let $\varepsilon > 0$ and take $t \in \mathbb{T}$ sufficiently large. Then by using the properties (i), (ii) and (iii) we get

$$(\lambda^n t)^{\alpha-1} \int_{\lambda^{r_n t}}^{\lambda^{r_{n+1} t}} p(s) \Delta s \geq (\lambda^{r_n t})^{\alpha-1} \int_{\lambda^{r_n t}}^{\tau(\lambda^{1+r_n t})} p(s) \Delta s \geq L_* - \varepsilon, \quad \forall n \in \mathbb{N} \cup \{0\},$$

hence,

$$t^{\alpha-1} \int_{\lambda^{r_n t}}^{\lambda^{r_{n+1} t}} p(s) \Delta s \geq \frac{L_* - \varepsilon}{(\lambda^{\alpha-1})^n}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Summing this inequality for n from 0 to ∞ we get

$$t^{\alpha-1} \int_t^\infty p(s) \Delta s \geq (L_* - \varepsilon) \sum_{n=0}^{\infty} \frac{1}{(\lambda^{\alpha-1})^n} = \frac{(L_* - \varepsilon)\lambda^{\alpha-1}}{\lambda^{\alpha-1} - 1},$$

which implies

$$\liminf_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) \Delta s \geq \frac{L_* \lambda^{\alpha-1}}{\lambda^{\alpha-1} - 1}.$$

Now suppose that there exist $\lambda > 1$ and $L^* > 0$ such that

$$\limsup_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s \leq L^*.$$

Let $\varepsilon > 0$. Take $t \in \mathbb{T}$ sufficiently large. Then

$$(\lambda^{r_n t})^{\alpha-1} \int_{\lambda^{r_n t}}^{\tau(\lambda^{1+r_n t})} p(s) \Delta s \leq L^* + \varepsilon, \quad \forall n \in \mathbb{N} \cup \{0\},$$

hence,

$$t^{\alpha-1} \int_{\lambda^{r_n t}}^{\tau(\lambda^{1+r_n t})} p(s) \Delta s \leq \frac{L^* + \varepsilon}{(\lambda^{\alpha-1})^{r_n}}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Summing this inequality for n from 0 to ∞ we get

$$t^{\alpha-1} \sum_{n=0}^{\infty} \int_{\lambda^{r_n t}}^{\tau(\lambda^{1+r_n t})} p(s) \Delta s \leq (L^* + \varepsilon) \sum_{n=0}^{\infty} \frac{1}{(\lambda^{\alpha-1})^{r_n}}. \quad (3.69)$$

In view of property (ii), it is clear that the series on the right-hand side of the inequality (3.69) can be majorized by the convergent series

$$\sum_{n=0}^{\infty} \frac{1}{(\lambda^{\alpha-1})^{n-1}}$$

for each sufficiently large t . Hence, using the property (iv), resp. $1 + r_n(t) \rightarrow r_{n+1}(t)$ as $t \rightarrow \infty$ following from (iv), (3.69) implies

$$\limsup_{t \rightarrow \infty} t^{\alpha-1} \sum_{n=0}^{\infty} \int_{\lambda^{r_n t}}^{\lambda^{r_{n+1} t}} p(s) \Delta s \leq L^* \sum_{n=0}^{\infty} \frac{1}{(\lambda^{\alpha-1})^n},$$

i.e.,

$$\limsup_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\infty} p(s) \Delta s \leq \frac{L^* \lambda^{\alpha-1}}{\lambda^{\alpha-1} - 1}.$$

Therefore, if $L = L_* = L^*$ part “If” follows.

“Only if”: Let $P = C$ and let $\lambda > 1$ be an arbitrary real number. Then

$$\begin{aligned} t^{\alpha-1} \int_t^{\infty} p(s) \Delta s &= t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s + t^{\alpha-1} \int_{\tau(\lambda t)}^{\infty} p(s) \Delta s \\ &= t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s + \frac{t^{\alpha-1}}{(\tau(\lambda t))^{\alpha-1}} (\tau(\lambda t))^{\alpha-1} \int_{\tau(\lambda t)}^{\infty} p(s) \Delta s. \end{aligned} \tag{3.70}$$

Since $(\tau(\lambda t))^{\alpha-1} \int_{\tau(\lambda t)}^{\infty} p(s) \Delta s \rightarrow A$ and $t/\tau(\lambda t) \rightarrow 1/\lambda$ as $t \rightarrow \infty$, from (3.70) we get

$$P_\lambda = \lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s = C - \frac{C}{\lambda^{\alpha-1}} = \frac{C(\lambda^{\alpha-1} - 1)}{\lambda^{\alpha-1}}.$$

□

In view of previous results, we get the following statement.

Corollary 3.1. *All positive solutions of (L) are Karamata functions if and only if for every $\lambda > 1$ there exists the (finite or infinite) limit*

$$\lim_{t \rightarrow \infty} t \int_t^{\tau(\lambda t)} p(s) \Delta s. \tag{3.71}$$

All positive decreasing solutions of (HL) are Karamata functions if and only if for every $\lambda > 1$ there exists the (finite or infinite) limit

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) \Delta s.$$

3.5 Concluding comments and open problems

In the first part of this section we give a few observations concerning graininess, which plays an important role in established theory of regular and rapid variation on time scales. The reader might wonder whether the condition $\mu(t) = o(t)$ (or $\mu(t) = O(t)$ in connection with the slowly varying functions), which repeatedly appears in our assumptions, can be omitted to obtain a general theory of regular and rapid variation which is applicable on any time scales; in particular, on time scales with a “large” graininess. For this purpose, distinguish three cases of the behavior of the graininess in theory of regularly (resp. slowly) varying functions of index ϑ , $\vartheta \in \mathbb{R}$, and of rapidly varying functions of index $\pm\infty$ on time scales:

- (i) $\mu(t) = o(t)$ (resp. $\mu(t) = O(t)$ if $\vartheta = 0$)

If we want to obtain a reasonable theory, which from a certain point of view corresponds with a continuous (or a discrete) theory, this condition on the graininess needs to be assumed and, moreover, cannot be improved. At first, recall that we want $f(t) = t^\vartheta$, $\vartheta \neq 0$, to be a typical example of a regularly varying function of index ϑ . However, for instance, with $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$, $\mu(t) = (q-1)t = O(t)$, see theory of q -calculus in Section 2.4, we have $f^\sigma(t)/f(t) \rightarrow q^\vartheta \neq 1$, and so the property from Lemma 3.1 fails to hold. Also

$$\lim_{t \rightarrow \infty} \frac{t f^\Delta(t)}{f(t)} = \frac{q^\vartheta - 1}{q - 1} \neq \vartheta.$$

Moreover, for $f(t) = t^\vartheta$, (3.5) holds only when $\lambda = q^j$, $j \in \mathbb{Z}$. Among others this means that even if Lemma 3.1 and Theorem 3.4 hold for $\mu(t) = O(t)$, it is sensible to assume $\mu(t) = o(t)$ when $\vartheta \neq 0$. Now, consider function $f(t) = (1/2)^t$ and take again $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. We expect that $f \in \mathcal{RPV}_{\mathbb{T}}(-\infty)$. But

$$\lim_{t \rightarrow \infty} \frac{t f^\Delta(t)}{f(t)} = \frac{1}{1 - q} \neq -\infty.$$

Note that for similar reasons the assumption $\mu(t) = o(t)$ cannot be omitted also in Proposition 3.2 and Lemma 3.2.

- (ii) $\mu(t) = Ct$, with $C > 0$

In the next chapter, we introduce a theory of q -regular and q -rapid variation, which means that considered functions are defined as in the q -calculus, i.e., on $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$ ($\mu(t) = (q-1)t$). The theory of q -regular and q -rapid variation was established using suitable modifications of “classical” theories. It is worthy to mention that the theory shows some interesting and surprising simplifications comparing with that on $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$,

since $\mathbb{T} = q^{\mathbb{N}_0}$ is somehow natural setting for a characterization of regularly (resp. rapidly) varying behavior.

(iii) Other cases

If the graininess is eventually “very big” (or a combination of “very big” and “small”), then the theory gives no proper results. Indeed, for instance, let $\mathbb{T} = 2^{p\mathbb{N}_0} = \{2^{pk} : k \in \mathbb{N}_0\}$ with $p > 1$. Note that $\mu(t) = t^p - t$, hence the condition $\mu(t) \neq O(t)$ is fulfilled. Take function $f(t) = t^\vartheta$ with $\vartheta \in \mathbb{R} \setminus \{0\}$ and use Definition 3.1. We expect that $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. But on this time scale we can observe that

$$\lim_{t \rightarrow \infty} \frac{tf^\Delta(t)}{f(t)} = \frac{t((t^p)^\vartheta - t^\vartheta)}{t^\vartheta(t^p - t)} = \frac{t^{\vartheta(p-1)} - 1}{t^{p-1} - 1} = \begin{cases} \infty & \text{for } \vartheta > 1, \\ 0 & \text{for } \vartheta < 1, \end{cases}$$

which is different from the expected value ϑ . Now use Definition 3.2 on function f for $\lambda = 1/2$, then, with $t = 2^{pk}$, $k \in \mathbb{N}_0$,

$$\frac{f(\tau(\lambda t))}{f(t)} = \left(\frac{\tau(2^{pk}/2)}{2^{pk}} \right)^\vartheta = \left(\frac{2^{p(k-1)}}{2^{pk}} \right)^\vartheta \rightarrow 0 \text{ as } t \rightarrow \infty \text{ resp. } k \rightarrow \infty.$$

It is not difficult to observe that also $f(\tau(\lambda t))/f(t) \rightarrow 0$ as $t \rightarrow \infty$ for any $\lambda \in (0, 1)$, which is again different from the expected value λ^ϑ . Moreover, in both cases the value of the limit is equal ∞ or 0 , which is related to the rapidly varying behavior. Now take function $f(t) = a^t$, $a \neq 1$, again on $\mathbb{T} = 2^{p\mathbb{N}_0}$ with $p > 1$. We expect that $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\infty)$ for $a > 1$ and $f \in \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(-\infty)$ for $a < 1$. But for $\lambda > 1$ we get $f(\tau(\lambda t))/f(t) \rightarrow 1$ as $t \rightarrow \infty$ (really, on this time scale for each $\lambda > 1$ there exists $t_0 \in \mathbb{T}$ such that $\tau(\lambda t) = t$ for $t > t_0$) and therefore $f \notin \mathcal{KR}\mathcal{PV}_{\mathbb{T}}(\pm\infty)$. It still remains to discuss whether the condition $\mu(t) = O(t)$ can be omitted (when $\vartheta = 0$). Again, let $\mathbb{T} = 2^{p\mathbb{N}_0}$ with $p > 1$. Take $f(t) = \ln t$. We expect that $f \in \mathcal{SV}$. Indeed, we have

$$\lim_{t \rightarrow \infty} \frac{tf^\Delta(t)}{f(t)} = \frac{t(\ln t^p - \ln t)}{(t^p - 1)\ln t} = \frac{t(p-1)}{t^p - 1} = 0.$$

On the other hand, with $\lambda \in (0, 1)$, $t = 2^k$, for sufficiently large t ,

$$\frac{f(\tau(\lambda t))}{f(t)} = \frac{\ln \tau(\lambda 2^{pk})}{\ln 2^{pk}} = \frac{\ln 2^{p(k-1)}}{\ln 2^{pk}} = \frac{p^{k-1}}{p^k} = \frac{1}{p} \neq 1,$$

where 1 is the expected value, in view of slow variation.

From the above observations, we conclude that it is advisable to distinguish and consider only the cases (i) and (ii) in the theory of regular and rapid variation on

time scales. Specially, concerning just a slow variation, it is sufficient to consider only one general case, namely $\mu(t) = O(t)$.

In the second part of this section we give a few information about open problems and perspectives related to equations (HL) or (L), their M-classifications and Karamata functions. Looking at relation (3.63) (resp. (3.67)), which can be alternatively rewritten on

$$M^- = M_{RPV}^-(-\infty) = M_0^- \text{ and } M^+ = M_{RPV}^+(\infty) = M_\infty^+ \iff (3.62) \text{ (resp. (3.66)),}$$

but it is not known (even in continuous and discrete case) whether

$$M^+ = M_{RPV}^+(\infty) = M_\infty^+ \implies (3.62) \text{ (resp. (3.66)).} \quad (3.72)$$

If the implication (3.72) were true, then the theory of asymptotic behavior of all solutions of equation (L) would be complete and we could claim (compare with Corollary 4.1) :

“There exists positive solution y of (L) such that $y \in \mathcal{KF}_\mathbb{T}$ (resp. $y \notin \mathcal{KF}_\mathbb{T}$) if and only if every positive solution y of (L) satisfies $y \in \mathcal{KF}_\mathbb{T}$ (resp. $y \notin \mathcal{KF}_\mathbb{T}$) if and only if the limit (3.71) exists (resp. does not exist). Specially, there exists positive decreasing solution u of (L) such that $u \in \mathcal{RPV}_\mathbb{T}(-\infty)$ if and only if there exists positive increasing solution v of (L) such that $v \in \mathcal{RPV}_\mathbb{T}(\infty)$ if and only if the limit (3.71) is equal ∞ . ”

On the other hand, thanks to the existence of a function p satisfying condition (3.68) we know that a positive decreasing “No-Karamata” solution $u \notin \mathcal{KF}_\mathbb{T}$ of (L) really exists. Indeed, it can be obtained as a decreasing solution of (L) with the mentioned coefficient p . However, the existence of an increasing solution v of (L) such that $v \notin \mathcal{KF}_\mathbb{T}$ has not been shown yet. From the above observations, there are three possibilities for fundamental set of rapidly varying solutions of equation (L):

- (i) $u \in \mathcal{RPV}_\mathbb{T}(-\infty), v \in \mathcal{RPV}_\mathbb{T}(\infty)$.
- (ii) $u \notin \mathcal{KF}_\mathbb{T}$ such that u is positive decreasing, $v \in \mathcal{RPV}_\mathbb{T}(\infty)$.
- (iii) $u \notin \mathcal{KF}_\mathbb{T}$ such that u is positive decreasing, $v \notin \mathcal{KF}_\mathbb{T}$ such that v is positive increasing.

Finally note that further possible research related to equation (HL) could be the following one - to establish necessary and sufficient conditions for all positive increasing solutions of (HL) to be regularly varying.

q -regular and q -rapid variation with applications to q -difference equations

4

This chapter is organized similarly as the previous one. At first, we introduce the concept of q -regularly and q -rapidly varying functions, i.e., the functions defined on the lattice $q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, $q > 1$. Recall that the theory of q -calculus was introduced in Section 2.4. Theory of q -regular, resp. q -rapid variation extends the existing related theories, see, Section 2.1. Later, we use the established theory in applications, concretely, in an asymptotic theory of second order linear and half-linear q -difference equations. The obtained results can be seen as q -versions of the existing ones in the linear (resp. half-linear) differential (resp. difference or dynamic) equation case. If we compare this and previous chapter, one can say that this “ q -version” of previous theory is “simpler”. The reason is mainly in structure of the set $q^{\mathbb{N}_0}$. Our results demonstrates that $q^{\mathbb{N}_0}$ is very natural setting for the theory of q -rapidly and q -regularly varying functions and its applications, and reveal some interesting phenomena, which are not known from the related theories. On the other hand, this fact allows us to prove some assertions in more general form unlike previous cases.

4.1 Theory of q -regular variation

In this section we establish the theory of q -regularly varying functions. Since the fraction $(q^a - 1)/(q - 1)$ appears quite frequently, let us introduce the notation

$$[a]_q := \frac{q^a - 1}{q - 1} \quad \text{for} \quad a \in \mathbb{R}. \quad (4.1)$$

Note that $\lim_{q \rightarrow 1} [a]_q = a$. Now we introduce the concept of q -regular variation.

Definition 4.1. A function $f : q^{\mathbb{N}_0} \rightarrow (0, \infty)$ is said to be q -regularly varying of index ϑ , $\vartheta \in \mathbb{R}$, if there exists a function $\omega : q^{\mathbb{N}_0} \rightarrow (0, \infty)$ satisfying

$$f(t) \sim C\omega(t) \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{tD_q\omega(t)}{\omega(t)} = [\vartheta]_q, \quad (4.2)$$

C being a positive constant; we write $f \in \mathcal{RV}_q(\vartheta)$. If $\vartheta = 0$, then f is said to be q -slowly varying; we write $f \in \mathcal{SV}_q$.

In fact, we have defined q -regular variation at infinity. If we consider a function $f : q^{\mathbb{Z}} \rightarrow (0, \infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(\cdot)$ is said to be q -regularly varying at zero if $f(1/t)$ is q -regularly varying at infinity. But it is apparent that it is sufficient to develop just the theory of q -regular variation at infinity. It is easy to see that the function t^ϑ is a typical representant of the class $\mathcal{RV}_q(\vartheta)$. Of course, this class is much wider as can be seen from the representations derived in the following theorem, where we also offer some other (simple) characterizations of q -regular variation and show a relation with the continuous theory.

Theorem 4.1. (i) (Simple characterization) For a positive function f , $f \in \mathcal{RV}_q(\vartheta)$ if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(qt)}{f(t)} = q^\vartheta. \quad (4.3)$$

Moreover, $f \in \mathcal{RV}_q(\vartheta)$ if and only if f satisfies just the second condition in (4.2), i.e.,

$$\lim_{t \rightarrow \infty} \frac{tD_q f(t)}{f(t)} = [\vartheta]_q. \quad (4.4)$$

(ii) (Zygmund type characterization) For a positive function f , $f \in \mathcal{RV}_q(\vartheta)$ if and only if $f(t)/t^\gamma$ is eventually increasing for each $\gamma < \vartheta$ and $f(t)/t^\eta$ is eventually decreasing for each $\eta > \vartheta$.

(iii) (Representation I) $f \in \mathcal{RV}_q(\vartheta)$ if and only if f has the representation

$$f(t) = \varphi(t)e_\delta(t, 1), \quad (4.5)$$

where $\varphi : q^{\mathbb{N}_0} \rightarrow (0, \infty)$ tends to a positive constant and $\delta : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ satisfies $\lim_{t \rightarrow \infty} t\delta(t) = [\vartheta]_q$ and $\delta \in \mathcal{R}^+$. Without loss of generality, in particular in the only if part, the function φ in (4.5) can be replaced by a positive constant.

(iv) (Representation II) $f \in \mathcal{RV}_q(\vartheta)$ if and only if f has the representation

$$f(t) = t^\vartheta \tilde{\varphi}(t)e_\psi(t, 1), \quad (4.6)$$

where $\tilde{\varphi} : q^{\mathbb{N}_0} \rightarrow (0, \infty)$ tends to a positive constant and $\psi : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ satisfies $\lim_{t \rightarrow \infty} t\psi(t) = 0$ and $\psi \in \mathcal{R}^+$. Without loss of generality, in particular in the only if part, the function $\tilde{\varphi}$ in (4.6) can be replaced by a positive constant.

(v) (Karamata type characterization) For a positive function f , $f \in \mathcal{RV}_q(\vartheta)$ if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = (\tau(\lambda))^\vartheta \quad \text{for } \lambda \geq 1 \quad (4.7)$$

where $\tau : [1, \infty) \rightarrow q^{\mathbb{N}_0}$ is defined as $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \leq x\}$.

(vi) (Imbeddability) If $f \in \mathcal{RV}_q(\vartheta)$ then $R \in \mathcal{RV}(\vartheta)$, where

$$R(x) = f(\tau(x)) \left(\frac{x}{\tau(x)} \right)^\vartheta \quad \text{for } x \in [1, \infty). \quad (4.8)$$

Conversely, if $R \in \mathcal{RV}(\vartheta)$, then $f \in \mathcal{RV}_q(\vartheta)$, where $f(t) = R(t)$ for $t \in q^{\mathbb{N}_0}$.

Proof. (i) If $f \in \mathcal{RV}_q(\vartheta)$, then with $\lim_{t \rightarrow \infty} \varphi(t) = C > 0$ we have

$$\lim_{t \rightarrow \infty} \frac{f(qt)}{f(t)} = \lim_{t \rightarrow \infty} \frac{\varphi(qt)\omega(qt)}{\varphi(t)\omega(t)} = \lim_{t \rightarrow \infty} \frac{\omega(t) + (q-1)tD_q\omega(t)}{\omega(t)} = 1 + (q-1)[\vartheta]_q = q^\vartheta,$$

which implies (4.3). Conversely,

$$\lim_{t \rightarrow \infty} \frac{tD_q f(t)}{f(t)} = \lim_{t \rightarrow \infty} \frac{t}{t(q-1)} \left(\frac{f(qt)}{f(t)} - 1 \right) = [\vartheta]_q$$

(ii) If $f \in \mathcal{RV}_q(\vartheta)$, then by (i)

$$\frac{f(qt)}{(qt)^\gamma} - \frac{f(t)}{t^\gamma} = \frac{f(t)}{(qt)^\gamma} \left(\frac{f(qt)}{f(t)} - q^\gamma \right) = \frac{f(t)}{(qt)^\gamma} (q^\vartheta - q^\gamma + o(1)).$$

The monotonicity for large t with η instead of γ follows similarly. Conversely, the monotonicities imply $(qt/t)^\gamma \leq f(qt)/f(t) \leq (qt/t)^\eta$ so that $q^\gamma \leq f(qt)/f(t) \leq q^\eta$. The statement follows by choosing γ and η arbitrarily close to ϑ and using (i).

Statements (iii) and (iv) follow from the implications $f \in \mathcal{RV}_q(\vartheta) \Rightarrow f$ satisfies (4.5) $\Rightarrow f$ satisfies (4.6) $\Rightarrow f \in \mathcal{RV}_q(\vartheta)$, which will be proved next. If $f \in \mathcal{RV}_q(\vartheta)$, then there is δ such that $D_q\omega(t) = \delta(t)\omega(t)$ and $\lim_{t \rightarrow \infty} t\delta(t) = [\vartheta]_q$. Since this is a first order q -difference equation and ω is its positive solution, it has the form $\omega(t) = \omega_0 e_\delta(t, 1)$ with $\omega_0 > 0$. Formula (4.5) now follows from the first condition in (4.2) and the fact that $e_\delta(t, 1) > 0$ implies $\delta \in \mathcal{R}^+$. If f satisfies (4.5), then we have $f(t) = \varphi(t)t^\vartheta L(t)$, where $L(t) = e_\delta(t, 1)/t^\vartheta > 0$ and $\lim_{t \rightarrow \infty} t\delta(t) = [\vartheta]_q$. We show that $\lim_{t \rightarrow \infty} tD_q L(t)/L(t) = 0$. Indeed, from

$$D_q L(t) = \frac{\delta(t)e_\delta(t, 1) - e_\delta(t, 1)[\vartheta]_q}{q^\vartheta t^\vartheta}$$

we get

$$\frac{tD_q L(t)}{L(t)} = \frac{t\delta(t)}{q^\vartheta} - \frac{[\vartheta]_q}{q^\vartheta} \rightarrow 0$$

as $t \rightarrow \infty$. Hence, arguing as in the previous part, there is ψ such that $L(t) = \psi_0 e_\psi(t, 1) > 0$ where $\psi_0 > 0$ and $\lim_{t \rightarrow \infty} t\psi(t) = 0$. Thus f can be written in the form (4.6). If f satisfies (4.6), we have $f(t) = \tilde{\varphi}(t)\omega(t)$, where $\omega(t) = t^\vartheta e_\psi(t, 1) > 0$ and $\lim_{t \rightarrow \infty} t\psi(t) = 0$. Similarly, as in the previous part, it is easy to show that $\lim_{t \rightarrow \infty} tD_q\omega(t)/\omega(t) = [\vartheta]_q$. The fact that φ and $\tilde{\varphi}$ can be replaced by a constant follows from (i).

(v) The if part trivially follows from (i). Conversely, assume that $f \in \mathcal{RV}_q(\vartheta)$. Then (4.6) holds. First observe that if $t \in q^{\mathbb{N}_0}$ and $\lambda \geq 1$, then $t = q^n$ and $\lambda \in [q^j, q^{j+1})$ for some $j, n \in \mathbb{N}_0$. Hence, $\tau(\lambda t)/t = q^{n+j}/q^j = \tau(\lambda)$.

From (4.6) we have

$$\frac{f(\tau(\lambda t))}{f(t)} = \frac{\tilde{\varphi}(\tau(\lambda t))}{\tilde{\varphi}(t)} \left(\frac{\tau(\lambda t)}{t} \right)^\vartheta e_\psi(\tau(\lambda t), t).$$

Hence,

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = (\tau(\lambda))^\vartheta \lim_{t \rightarrow \infty} e_\psi(\tau(\lambda t), t).$$

Set $q^n = t$, $q^{n+m} = \tau(\lambda t)$, $m, n \in \mathbb{N}_0$. Note that $\tau(\lambda) = q^m$, where m is fixed since $\tau(\lambda t) = \tau(\lambda)t$. We have

$$e_\psi(\tau(\lambda t), t) = \prod_{j=n}^{n+m-1} [(q-1)q^j \psi(q^j) + 1].$$

Since $q^j \psi(q^j) \rightarrow 0$ as $j \rightarrow \infty$, we obtain $\lim_{t \rightarrow \infty} e_\psi(\tau(\lambda t), t) = 1$. Hence (4.7) holds for $\lambda \geq 1$.

(vi) First we show that if f satisfies (4.7), then $R : [1, \infty) \rightarrow (0, \infty)$ given by (4.8) satisfies $R \in \mathcal{RV}(\vartheta)$. Note that $R(t) = f(t)$ for $t \in q^{\mathbb{N}_0}$. We have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{R(\lambda x)}{R(x)} &= \lim_{x \rightarrow \infty} \frac{f(\tau(\lambda x))}{f(\tau(x))} \left(\frac{\lambda x}{\tau(\lambda x)} \right)^\vartheta \left(\frac{\tau(x)}{x} \right)^\vartheta \\ &= \lambda^\vartheta \lim_{x \rightarrow \infty} \frac{f(\tau(\lambda \tau(x)))}{f(\tau(x))} \Omega(x, \lambda) \\ &= \lambda^\vartheta (\tau(\lambda))^\vartheta \lim_{x \rightarrow \infty} \Omega(x, \lambda), \end{aligned}$$

where

$$\Omega(x, \lambda) = \left(\frac{\tau(x)}{\tau(\lambda x)} \right)^\vartheta \frac{f(\tau(\lambda x))}{f(\tau(\lambda \tau(x)))}.$$

Since for each $\lambda, x \geq 1$, there are $m, n \in \mathbb{N}_0$ such that $\lambda \in [q^m, q^{m+1})$ and $x \in [q^n, q^{n+1})$, we have $\lambda x \in [q^{m+n}, q^{m+n+2})$, and so either (I) $\tau(\lambda x) = q^{m+n} = \tau(\lambda)\tau(x)$ or (II) $\tau(\lambda x) = q^{m+n+1} = q\tau(\lambda)\tau(x)$. Recall $\tau(\lambda \tau(x)) = \tau(\lambda)\tau(x)$. In case (I) we obtain $\Omega(x, \lambda) = (\tau(\lambda))^{-\vartheta}$, while in case (II)

$$\Omega(x, \lambda) = (q\tau(\lambda))^{-\vartheta} \frac{f(q\tau(\lambda)\tau(x))}{f(\tau(\lambda)\tau(x))}.$$

Since $\lim_{t \rightarrow \infty} f(qt)/f(t) = q^\vartheta$, from (I) and (II) we get $\lim_{x \rightarrow \infty} \Omega(x, \lambda) = (\tau(\lambda))^{-\vartheta}$. Hence, $\lim_{x \rightarrow \infty} R(\lambda x)/R(x) = \lambda^\vartheta$ for all $\lambda > 1$ and so by [4, Theorem 1.4.1], for

all $\lambda > 0$. Consequently, $R \in \mathcal{RV}(\vartheta)$. Conversely, if $R \in \mathcal{RV}(\vartheta)$, then by [4, Theorem 1.3.1, Theorem 1.4.1],

$$R(x) = \Phi(x)x^\vartheta \exp \left\{ \int_1^x \frac{\Psi(s)}{s} ds \right\},$$

where Φ, Ψ are bounded measurable functions on $[1, \infty)$ such that $\lim_{x \rightarrow \infty} \Phi(x) = C > 0$ and $\lim_{x \rightarrow \infty} \Psi(x) = 0$ (Ψ may be taken as continuous). Hence for $t \in q^{\mathbb{N}_0}$ we have

$$f(t) = \Phi(t)t^\vartheta \exp \left\{ \int_1^t \frac{\Psi(s)}{s} ds \right\}.$$

Then

$$\frac{f(qt)}{f(t)} = \frac{\Phi(qt)}{\Phi(t)} q^\vartheta \exp \left\{ \int_t^{qt} \frac{\Psi(s)}{s} ds \right\}.$$

Using the mean value theorem,

$$\int_t^{qt} \frac{\Psi(s)}{s} ds = \Psi(\xi(t)) \ln q \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $t \leq \xi(t) \leq qt$. Consequently $\lim_{t \rightarrow \infty} f(qt)/f(t) = q^\vartheta$, and the statement follows from (i). \square

Remark 4.1. (i) (Important) The so-called normalized regularly varying functions of index ϑ can be defined as a function satisfying (4.4) or, equivalently, as those having representation (4.5) or (4.6) with a constant instead of $\varphi(t)$ or $\tilde{\varphi}(t)$, respectively. However, in contrast to the classical continuous or discrete case, owing to (i) and (ii) of Theorem 4.1, the distinction between normalized (or Zygmund) and ordinary regular variation disappears in q -calculus. Therefore, we do not need to introduce the concept of normalized regular variation. Moreover, in the q -calculus case we have another property not known in the classical theories: A Karamata type characterization (4.7) can be substantially simplified to (4.3). Note that for the discrete case, an analog of (4.7) is $f([\lambda t])/f(t) \rightarrow \lambda^\vartheta$ and an analog for (4.3) can be seen as $f(t+1)/f(t) \rightarrow 1$. However, the latter one is just necessary for regular variation on \mathbb{Z} . Altogether we see that regularly varying functions in q -calculus can be defined very simply by (4.3) or by (4.4), and that $\varphi(t)$ and $\tilde{\varphi}(t)$ in representations (4.5) and (4.6), respectively, can be replaced by a positive constant without loss of generality. The reason for this simplification may be that regular variation can be based on a product characterization which is very natural for the q -calculus case.

(ii) A suitable extension of the operator τ enables to have formula (4.7) also for $\lambda \in (0, 1)$.

(iii) Observe how the above (but also subsequent) results nicely resembles continuous results as $q \rightarrow 1$.

Regularly varying functions on $q^{\mathbb{N}_0}$ possess a number of properties. We list the following ones, which will be needed later.

Proposition 4.1. *Regularly varying functions have the following properties:*

- (i) It holds $f \in \mathcal{RV}_q(\vartheta)$ if and only if $f(t) = t^\vartheta L(t)$, where $L \in \mathcal{SV}_q$.
- (ii) Let $f \in \mathcal{RV}_q(\vartheta)$. Then $\lim_{t \rightarrow \infty} \log f(t) / \log t = \vartheta$. This implies $\lim_{t \rightarrow \infty} f(t) = 0$ if $\vartheta < 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$ if $\vartheta > 0$.
- (iii) Let $f \in \mathcal{RV}_q(\vartheta)$. Then $\lim_{t \rightarrow \infty} f(t) / t^{\vartheta - \varepsilon} = \infty$ and $\lim_{t \rightarrow \infty} f(t) / t^{\vartheta + \varepsilon} = 0$ for every $\varepsilon > 0$.
- (iv) Let $f \in \mathcal{RV}_q(\vartheta)$. Then $f^\gamma \in \mathcal{RV}_q(\gamma\vartheta)$.
- (v) Let $f \in \mathcal{RV}_q(\vartheta_1)$ and $g \in \mathcal{RV}_q(\vartheta_2)$. Then $fg \in \mathcal{RV}_q(\vartheta_1 + \vartheta_2)$ and $1/f \in \mathcal{RV}_q(-\vartheta_1)$.
- (vi) Let $f \in \mathcal{RV}_q(\vartheta)$. Then f is decreasing provided $\vartheta < 0$, and it is increasing provided $\vartheta > 0$. A concave f is increasing. If $f \in \mathcal{SV}_q$ is convex, then it is decreasing.

Proof. (i), (iv), (v) The proofs of these parts are trivial.

(ii) From (4.5), using the q -L'Hospital rule, we have

$$\lim_{t \rightarrow \infty} \frac{\log f(t)}{\log t} = \lim_{t \rightarrow \infty} \frac{\sum_{s \in [1, t) \cap q^{\mathbb{N}_0}} \log[(q-1)s\delta(s) + 1]}{\log t} = \lim_{t \rightarrow \infty} \frac{\log[(q-1)t\delta(t) + 1]}{\log q} = \vartheta.$$

Alternatively we can see it from the imbedding result.

(iii) Follows from (4.6) and (ii) of this proposition.

(vi) The part for $\vartheta \neq 0$ is simple. For $\vartheta = 0$, i.e., $f \in \mathcal{SV}_q$, first we show that $D_q^2 f(t) > 0$ implies eventual monotonicity of f . Indeed, either we have $D_q f(t) < 0$ for all $t \in q^{\mathbb{N}_0}$, or if there is $t_0 \in q^{\mathbb{N}_0}$ such that $D_q f(t_0) \geq 0$, then $0 \leq D_q f(t_0) < D_q f(qt_0) < \dots$, hence $D_q f(t) > 0$ for all $t \in (t_0, \infty) \cap q^{\mathbb{N}_0}$. By a contradiction assume that $D_q f(t) \geq 0$. Thanks to the convexity we have $D_q f(t) \geq M > 0$ for large $t \in q^{\mathbb{N}_0}$ and for some $M > 0$. Integrating from s to t we obtain $f(t) \geq f(s) + (t-s)M$. But now f cannot be slowly varying by (iii) of this proposition. \square

4.2 Theory of q -rapid variation

In this section we extend the previous theory of q -regularly varying functions to q -rapidly varying functions and establish analogical theory as in Section 3.2 on $q^{\mathbb{N}_0}$. In view of (4.1), it is natural to extend this notation to

$$[\infty]_q := \lim_{t \rightarrow \infty} \frac{q^t - 1}{q - 1} = \infty \quad \text{and} \quad [-\infty]_q := \lim_{t \rightarrow -\infty} \frac{q^t - 1}{q - 1} = \frac{1}{1 - q}.$$

Looking at the values on the right hand sides of (4.4) and (4.3) we will be interested in situations, where these values attain their extremal values, i.e., $[\infty]_q$ and $[-\infty]_q$ in (4.4) and ∞ and 0 in (4.3). This leads to the concept of q -rapid variation.

Definition 4.2. A function $f : q^{\mathbb{N}_0} \rightarrow (0, \infty)$ is said to be q -rapidly varying of index ∞ , resp. of index $-\infty$ if

$$\lim_{t \rightarrow \infty} \frac{tD_q f(t)}{f(t)} = [\infty]_q, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{tD_q f(t)}{f(t)} = [-\infty]_q;$$

we write $f \in \mathcal{RPV}_q(\infty)$, resp. $f \in \mathcal{RPV}_q(-\infty)$.

In fact, we have defined q -rapid variation at infinity. If we consider a function $f : q^{\mathbb{Z}} \rightarrow (0, \infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(t)$ is said to be q -rapidly varying at zero if $f(1/t)$ is q -rapidly varying at infinity. But it is apparent that it is sufficient to develop just the theory of q -rapid variation at infinity. It is easy to see that the function $f(t) = a^t$ with $a > 1$ is a typical representative of the class $\mathcal{RPV}_q(\infty)$, while the function $f(t) = a^t$ with $a \in (0, 1)$ is a typical representative of the class $\mathcal{RPV}_q(-\infty)$. Of course, these classes are much wider as can be seen also from the simple representations derived in the following proposition, where we present important properties of q -rapidly varying functions.

Proposition 4.2. (i) (Simple characterization) For a function $f \in q^{\mathbb{N}_0} \rightarrow (0, \infty)$, $f \in \mathcal{RPV}_q(\infty)$, resp. $f \in \mathcal{RPV}_q(-\infty)$, if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(qt)}{f(t)} = \infty, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{f(qt)}{f(t)} = 0.$$

(ii) (Karamata type definition) Define $\tau : [1, \infty) \rightarrow q^{\mathbb{N}_0}$ by $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \leq x\}$. For a function $f \in q^{\mathbb{N}_0} \rightarrow (0, \infty)$, $f \in \mathcal{RPV}_q(\infty)$, resp. $f \in \mathcal{RPV}_q(-\infty)$, if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \text{for some } \lambda \in [q, \infty), \quad (4.9)$$

which holds if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \text{for every } \lambda \in [q, \infty),$$

which holds if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \text{for some } \lambda \in (0, 1),$$

which holds if and only if f satisfies

$$\lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \text{resp.} \quad \lim_{t \rightarrow \infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \text{for every } \lambda \in (0, 1).$$

- (iii) It holds $f \in \mathcal{RPV}_q(\infty)$ if and only if $1/f \in \mathcal{RPV}_q(-\infty)$.
- (iv) If $f \in \mathcal{RPV}_q(\infty)$, then for each $\vartheta \in [0, \infty)$ the function $f(t)/t^\vartheta$ is eventually increasing and $\lim_{t \rightarrow \infty} f(t)/t^\vartheta = \infty$.
- (v) If $f \in \mathcal{RPV}_q(-\infty)$, then for each $\vartheta \in [0, \infty)$ the function $f(t)t^\vartheta$ is eventually decreasing and $\lim_{t \rightarrow \infty} f(t)t^\vartheta = 0$.

Proof. (i) It follows from the identity

$$\frac{tD_q f(t)}{f(t)} = \frac{1}{q-1} \left(\frac{f(qt)}{f(t)} - 1 \right).$$

(ii) We prove that $f \in \mathcal{RPV}_q(\infty)$ if and only if the first condition in (4.9) holds. Other cases follow similarly. First assume that $f \in \mathcal{RPV}_q(\infty)$. As shown in (iv), f is eventually increasing. Hence, $f(\tau(\lambda t))/f(t) \geq f(qt)/f(t)$ for large t and $\lambda \in [q, \infty)$. Hence, thanks to (i), the result follows. Conversely assume that $\lim_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) = \infty$ for some $\lambda \in [q, \infty)$. Let $m \in \mathbb{N}$ be such that $\lambda \in [q^m, q^{m+1})$. Then

$$\frac{f(\tau(\lambda t))}{f(t)} = \frac{f(q^m t)}{f(t)} = \frac{f(q^m t)}{f(q^{m-1} t)} \cdot \frac{f(q^{m-1} t)}{f(q^{m-2} t)} \cdots \frac{f(qt)}{f(t)}. \tag{4.10}$$

Assume by a contradiction $f \notin \mathcal{RPV}_q(\infty)$, i.e., by (i), $\liminf_{t \rightarrow \infty} f(qt)/f(t) < \infty$. Then $\liminf_{t \rightarrow \infty} f(q^i t)/f(q^{i-1} t) < \infty$ for all $i \in \mathbb{N}$. Hence, in view of (4.10), we get $\liminf_{t \rightarrow \infty} f(\tau(\lambda t))/f(t) < \infty$, which is a contradiction.

(iii) In view of (i), the proof is trivial.

(iv) We have

$$\begin{aligned} D_q \left(\frac{f(t)}{t^\vartheta} \right) &= \frac{D_q f(t) t^\vartheta - f(t) \frac{(qt)^\vartheta - t^\vartheta}{(q-1)t}}{t^\vartheta (qt)^\vartheta} \\ &= \frac{D_q f(t) - \frac{f(t)}{t} [\vartheta]_q}{(qt)^\vartheta}. \end{aligned} \tag{4.11}$$

Since $f \in \mathcal{RPV}_q(\infty)$, for each $M > 0$ there exists $t_0 \in q^{\mathbb{N}_0}$ such that $tD_q f(t)/f(t) > M$ for $t \geq t_0$. Hence, $D_q f(t) > f(t)[\vartheta]_q/t$ for large t , and so $D_q(f(t)/t^\vartheta)$ is eventually positive in view of (4.11). Consequently, $f(t)/t^\vartheta$ is eventually increasing, and so its limit, as $t \rightarrow \infty$, must exist (finite positive or infinite). By a contradiction assume that $\lim_{t \rightarrow \infty} f(t)/t^\vartheta = K \in (0, \infty)$. Then $f(t) \sim Kt^\vartheta$, which implies $\lim_{t \rightarrow \infty} f(qt)/f(t) = q^\vartheta \neq \infty$, i.e., $f \notin \mathcal{RPV}_q(\infty)$, contradiction.

(v) It follows from (iii) and (iv). □

Remark 4.2. (i) In contrast to the general theory on time scales (see Definition 3.5 in Section 3.2) or classical continuous and discrete theories (see Section 2.1) we can observe that the Karamata type definition is substantially simpler, see (ii)

and also (i) of Proposition 4.2. Actually, here we can restrict our consideration just to one value of the parameter λ . The reason for this simply looking condition may be that rapid variation can be based on a product in the argument characterization which is quite natural for the q -calculus case. Observe that we do not consider the values of λ in the sets $(0, 1)$ and $(1, \infty)$ as in general time scales (resp. classical continuous or discrete) theory, but in the sets $(0, 1)$ and $[q, \infty)$. It is because $\tau(\lambda t) = t$ for $\lambda \in (1, q)$.

(ii) Another simplification in comparison with general time scale theory (resp. classical theories) is that for showing the equivalence between the Karamata type definition (or the simple characterization (i) of Proposition 4.2) and Definition 4.2 we do not need additional assumptions like convexity, see Lemma 3.2 in Section 3.2 (resp. Section 2.1, namely assumptions of validity of conditions (2.8) and (2.9)).

(iii) In view of (iv) and (v) of Proposition 4.2, $\mathcal{RPV}_q(\infty)$ functions are always eventually increasing to ∞ , while $\mathcal{RPV}_q(-\infty)$ functions are always eventually decreasing to zero.

(iv) Similarly as in theory of q -regular variation, it is not difficult to see that the concept of normalized q -rapid variation misses point. Indeed, let “normalized” q -rapidly varying functions are defined as in Definition 4.2. Let us define the concept of q -rapid variation in a seemingly more general way: A function $g : q^{\mathbb{N}_0} \rightarrow (0, \infty)$ is q -rapidly varying of index ∞ , resp. of index $-\infty$ if there are positive functions φ and ω satisfying $g(t) = \varphi(t)\omega(t)$, $\lim_{t \rightarrow \infty} \varphi(t) = C \in (0, \infty)$, $\lim_{t \rightarrow \infty} tD_q\omega(t)/\omega(t) = [\infty]_q$ resp. $[-\infty]_q$. By Proposition 4.2 we then get

$$\lim_{t \rightarrow \infty} \frac{g(qt)}{g(t)} = \frac{\varphi(qt)}{\varphi(t)} \cdot \frac{\omega(qt)}{\omega(t)} = \infty \text{ resp. } 0,$$

and so g is also “normalized” q -rapidly varying of index ∞ , resp. $-\infty$. Note that these observations remain valid even under another generalization, where the condition $\lim_{t \rightarrow \infty} \varphi(t) = C \in (0, \infty)$ is relaxed to $0 < C_1 \leq \varphi(t) \leq C_2 < \infty$ for large t .

4.3 Applications to q -difference equations

In this section we apply the theory from the previous two sections to the investigation of asymptotic behavior of solutions to linear and half-linear second order q -difference equations, which allows us to get a precise information about asymptotic varying behavior of positive solutions of mentioned equations. Consider the half-linear second order q -difference equation

$$D_q[\Phi(D_q y(t))] - p(t)\Phi(y(qt)) = 0, \tag{HLq}$$

on $q^{\mathbb{N}_0}$ with $q > 1$, where $p : q^{\mathbb{N}_0} \rightarrow (0, \infty)$. The linear q -difference equation

$$D_q^2 y(t) - p(t)y(qt) = 0 \tag{Lq}$$

is a special case of (HL q) (when $\alpha = 2$). Equations (HL q), resp. (L q) are the special cases of equations (HL q E), resp. (L q E) introduced in Section 2.4.

In this section we use two methods of the proofs. First method uses the classical tool, namely the Riccati type transformation. This type of method we also used in the proofs in Section 3.3. Second method of the proof is suitable for investigating of (HL q) and (L q). This method is new and designed just for the q -calculus case. It also turns out to be elegant and powerful tool also for the examination of asymptotic behavior to many other q -difference equations, which then may serve to predict how their (trickily detectable) continuous counterparts look like (we simply take, formally, the limit as $q \rightarrow 1^+$). Note that second method is more practical and allows us to get more general results than the previous one.

In the following theorem, by using the first method, we establish necessary and sufficient conditions for all positive solutions of (L q) to be q -regularly varying. Note that all nontrivial solutions of (L q) are nonoscillatory (i.e., are eventually of one sign) and eventually monotone and convex, i.e., $D_q y$ is nondecreasing. Because of linearity, without loss of generality, it is sufficient to consider just positive solutions of (L q).

Theorem 4.2. (i) Equation (L q) has a fundamental set of solutions

$$u(t) = L(t) \in \mathcal{SV}_q, \quad v(t) = t\tilde{L}(t) \in \mathcal{RV}_q(1) \tag{4.12}$$

if and only if

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) d_q s = 0. \tag{4.13}$$

Moreover, $\tilde{L} \in \mathcal{SV}_q$ with $\tilde{L}(t) \sim 1/L(t)$. All positive decreasing solutions of (L q) belong to \mathcal{SV}_q and all positive increasing solutions of (L q) belong to $\mathcal{RV}_q(1)$. Any of two conditions in (4.12) implies (4.13).

(ii) Equation (L q) has a fundamental set of solutions

$$u(t) = t^{\vartheta_1} L(t) \in \mathcal{RV}_q(\vartheta_1), \quad v(t) = t^{\vartheta_2} \tilde{L}(t) \in \mathcal{RV}_q(\vartheta_2) \tag{4.14}$$

if and only if

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) d_q s = A > 0, \tag{4.15}$$

where $\vartheta_i = \log_q[(q-1)\lambda_i + 1]$, $i = 1, 2$, $\lambda_1 < 0 < \lambda_2$ are the roots of the equation

$$\lambda^2 - [A(q-1) + 1]\lambda - A = 0.$$

It holds $\vartheta_1 < 0 < \vartheta_2$,

$$\lambda_2 = [\vartheta_2]_q = A(q-1) + 1 - [\vartheta_1]_q = A(q-1) + 1 - \lambda_1,$$

and $\vartheta_2 = 1 - \vartheta_1$. Moreover, $L, \tilde{L} \in \mathcal{SV}_q$ with

$$\tilde{L}(t) \sim 1/(q^{\vartheta_1}[1 - 2\vartheta_1]_q L(t)).$$

All positive decreasing solutions of (Lq) belong to $\mathcal{RV}_q(\vartheta_1)$ and all positive increasing solutions of (Lq) belong to $\mathcal{RV}_q(\vartheta_2)$. Any of two conditions in (4.14) implies (4.15).

Proof. Parts (i) and (ii) will be proved simultaneously assuming $A \geq 0$ in (4.15) and, consequently, $\lambda_1 \leq 0$ or $\vartheta_1 \leq 0$, if it is not said otherwise. Recall that $[a, \infty)_q = \{a, aq, aq^2, \dots\} \subseteq q^{\mathbb{N}_0}$.

“Only if parts”: Let $u \in \mathcal{RV}_q(\vartheta_1)$ be a positive decreasing solution of (Lq) on $[a, \infty)_q$. Set $w = D_q u/u$. Then $w(t) < 0$ and satisfies the Riccati type q -difference equation

$$D_q w(t) - p(t) + \frac{w^2(t)}{1 + (q-1)tw(t)} = 0 \tag{4.16}$$

with $w \in \mathcal{R}^+$ on $[a, \infty)_q$. We have $\lim_{t \rightarrow \infty} tw(t) = [\vartheta_1]_q$ and so $\lim_{t \rightarrow \infty} w(t) = 0$. We show that

$$\int_a^\infty \frac{w^2(t)}{1 + (q-1)tw(t)} d_q t < \infty.$$

Since $1 + (q-1)tw(t) \rightarrow q^{\vartheta_1}$, we have $1 + (q-1)tw(t) > q^{\vartheta_1}/2$ for large t . Moreover, there is $N > 0$ such that $|w(t)| \leq N/t$ for large t . Without loss of generality, these large t 's can be taken as $t \in [a, \infty)_q$. Then

$$\int_a^\infty \frac{w^2(t)}{1 + (q-1)tw(t)} d_q t \leq \frac{2N^2q}{q^{\vartheta_1}} \int_a^\infty \frac{d_q t}{qt^2} = \frac{2N^2q}{aq^{\vartheta_1}},$$

since $D_q(1/t) = -1/(qt^2)$. Integration of (4.16) and multiplication by t yield

$$-tw(t) + t \int_t^\infty \frac{w^2(s)}{1 + (q-1)sw(s)} d_q s = t \int_t^\infty p(s) d_q s. \tag{4.17}$$

The q -L'Hospital rule gives

$$\lim_{t \rightarrow \infty} t \int_t^\infty \frac{w^2(s)}{1 + (q-1)sw(s)} d_q s = \lim_{t \rightarrow \infty} \frac{qt^2 w^2(t)}{1 + (q-1)tw(t)} = \frac{q[\vartheta_1]_q}{1 + (q-1)[\vartheta_1]_q}.$$

Hence, from (4.17) we get

$$\lim_{t \rightarrow \infty} t \int_t^\infty p(s) d_q s = \frac{[\vartheta_1]_q^2 - [\vartheta_1]_q}{1 + (q-1)[\vartheta_1]_q} = A.$$

Similar arguments show that also $v \in \mathcal{RV}_q(\vartheta_2)$ being a positive increasing solution of (L q) implies (4.15).

Note that even without assuming monotonicity, a solution $u \in \mathcal{RV}_q(\vartheta_1)$ necessarily decreases while a solution $v \in \mathcal{RV}_q(\vartheta_2)$ necessarily increases by (vi) of Proposition 4.1.

“If parts”: Let u be a positive decreasing solution of (L q) on $[a, \infty)_q$. Then $\lim_{t \rightarrow \infty} D_q u(t) = 0$. Indeed, if not, then there is $K > 0$ such that $D_q u(t) \leq -K$ for $t \in [a, \infty)_q$ since $D_q u$ is negative increasing. Hence $u(t) \leq u(a) - (t - a)K$. Letting $t \rightarrow \infty$ we have $\lim_{t \rightarrow \infty} u(t) = -\infty$, a contradiction with positivity of u . Integration of (L q) from t to ∞ yields $D_q u(t) = -\int_t^\infty p(s)u(qs) d_qs$. Hence,

$$0 < \frac{-tD_q u(t)}{u(t)} = \frac{t}{u(t)} \int_t^\infty p(s)u(qs) d_qs \leq t \int_t^\infty p(s) d_qs. \quad (4.18)$$

If (4.13) holds, then we are done since (4.18) implies $\lim_{t \rightarrow \infty} tD_q u(t)/u(t) = 0$, and so $u \in \mathcal{SV}_q$. Next we assume (4.15) with $A > 0$. Set $\eta(t) = tD_q u(t)/u(t)$. From (4.18), $0 < -\eta(t) \leq t \int_t^\infty p(s) d_qs$, and so η is bounded. Further, η satisfies the modified Riccati q -difference equation

$$D_q \left(\frac{\eta(t)}{t} \right) - p(t) + \frac{\eta^2(t)/t^2}{1 + (q-1)\eta(t)} = 0 \quad (4.19)$$

with $\eta/t \in \mathcal{R}^+$ on $[a, \infty)_q$. Since η is bounded, we have $\lim_{t \rightarrow \infty} \eta(t)/t = 0$ and so integration of (4.19) from t to ∞ yields

$$-\frac{\eta(t)}{t} = \int_t^\infty p(s) d_qs - \int_t^\infty \frac{\eta^2(s)/s^2}{1 + (q-1)\eta(s)} d_qs. \quad (4.20)$$

Let us write condition (4.15) as $t \int_t^\infty p(s) d_qs = A + \varepsilon(t)$, where $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$. Further, let us write

$$\int_t^\infty \frac{\eta^2(s)/s^2}{1 + (q-1)\eta(s)} d_qs = G(t) \int_t^\infty \frac{d_qs}{qs^2} = \frac{G(t)}{t},$$

where $m(t) \leq G(t) \leq M(t)$ with

$$m(t) = \inf_{s \geq t} \frac{q\eta^2(s)}{1 + (q-1)\eta(s)} \quad \text{and} \quad M(t) = \sup_{s \geq t} \frac{q\eta^2(s)}{1 + (q-1)\eta(s)}.$$

With these equalities, multiplication of (4.20) by t yields

$$G(t) - \eta(t) = A + \varepsilon(t). \quad (4.21)$$

We claim that $\lim_{t \rightarrow \infty} \eta(t) = [\vartheta_1]_q$. Recall that η is bounded and denote $K_* = \liminf_{t \rightarrow \infty} (-\eta(t))$, $K^* = \limsup_{t \rightarrow \infty} (-\eta(t))$. Observe the monotone properties of the function $f(x) = qx^2/(1 + (q-1)x)$ which occurs in the formula for G . Recall

that our “admissible” x ’s are just the nonpositive ones satisfying $1 + (q - 1)x > 0$. The function G is bounded. Define K_1 and K_2 by

$$\liminf_{t \rightarrow \infty} G(t) = \frac{qK_1^2}{1 - (q - 1)K_1} \quad \text{and} \quad \limsup_{t \rightarrow \infty} G(t) = \frac{qK_2^2}{1 - (q - 1)K_2}.$$

Thanks to monotonicity of f and boundedness of η we have $0 \leq K_* \leq K_1 \leq K_2 \leq K^* < 1/(q - 1)$. Now we distinguish several cases which lead to a contradiction, and altogether show that $\lim_{t \rightarrow \infty} \eta(t)$ exists and is equal to $[\vartheta_1]_q$. Assume, for instance, $K_1 < -[\vartheta_1]_q$. Then $K_* < -[\vartheta_1]_q$. Noticing that

$$A = \frac{[\vartheta_1]_q^2 - [\vartheta_1]_q}{1 + (q - 1)[\vartheta_1]_q} = \frac{q[\vartheta_1]_q^2}{1 - (q - 1)(-[\vartheta_1]_q)} + (-[\vartheta_1]_q)$$

and taking \liminf as $t \rightarrow \infty$ in (4.21) we get

$$\frac{qK_1^2}{1 - (q - 1)K_1} + K_* = \frac{q[\vartheta_1]_q^2}{1 - (q - 1)(-[\vartheta_1]_q)} + (-[\vartheta_1]_q).$$

Thanks to monotonicity of f , from the last equation we have $K_* = B + (-[\vartheta_1]_q)$, where $B = f([\vartheta_1]_q) - f(K_1)$ is positive. Hence, $K_* > -[\vartheta_1]_q$, a contradiction. In a similar manner we obtain a contradiction when $K_* < -[\vartheta_1]_q$ and $K_1 = -[\vartheta_1]_q$. If $K_1 > -[\vartheta_1]_q$, then $K^* \geq K_2 > -[\vartheta_1]_q$ and a contradiction is obtained by taking \limsup as $t \rightarrow \infty$ in (4.21). This proves that $\lim_{t \rightarrow \infty} \eta(t) = [\vartheta_1]_q$, and so $u(t) = t^{\vartheta_1} L(t) \in \mathcal{RV}_q(\vartheta_1)$, where u is a positive decreasing solution of (Lq) and $L \in \mathcal{SV}_q$. Now consider a linearly independent solution v of (Lq), which is given by $v(t) = u(t) \int_a^t (1/(u(s)u(qs))) d_qs$. Put $z = 1/u^2$. Then $z \in \mathcal{RV}_q(-2\vartheta_1)$ by (v) of Proposition 4.1. Since $\int_a^\infty (1/(u(s)u(qs))) d_qs = \infty$, the q -L’Hospital rule and Theorem 4.1 yield

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{t/u(t)}{v(t)} &= \lim_{t \rightarrow \infty} \frac{tz(t)}{\int_a^t (1/(u(s)u(qs))) d_qs} = \lim_{t \rightarrow \infty} \frac{z(t) + qtD_q z(t)}{1/(u(t)u(qt))} \\ &= \lim_{t \rightarrow \infty} \left(\frac{u(t)u(qt)}{u^2(t)} + \frac{qu(t)u(qt)}{u^2(t)} \cdot \frac{tD_q z(t)}{z(t)} \right) = q^{\vartheta_1} + q^{\vartheta_1+1}[-2\vartheta_1]_q =: \gamma. \end{aligned}$$

Hence, $\gamma v(t) \sim t/u(t) = t^{1-\vartheta_1}/L(t)$. Consequently, $v(t) = t^{\vartheta_2} \tilde{L}(t)$, where $\tilde{L}(t) \sim 1/(\gamma L(t))$, $\tilde{L} \in \mathcal{SV}_q$, and so $v \in \mathcal{RV}_q(\vartheta_2)$ by (v) of Proposition 4.1 since $\vartheta_2 = 1 - \vartheta_1$. The last equality follows from

$$\begin{aligned} \vartheta_2 &= \log_q[(q - 1)\lambda_2 + 1] = \log_q[(q - 1)(A(q - 1) + 1 - \lambda_1) + 1] \\ &= \log_q \left[(q - 1) \left(\frac{(q - 1)(\lambda_1^2 - \lambda_1)}{1 + (q - 1)\lambda_1} + 1 - \lambda_1 \right) + 1 \right] \\ &= \log_q \frac{q}{1 + (q - 1)\lambda_1} \\ &= \log_q q - \log_q[(q - 1)\lambda_1 + 1] \\ &= 1 - \vartheta_1. \end{aligned}$$

The solution v increases by (vi) of Proposition 4.1. For the quantity γ we have

$$\gamma = q^{\vartheta_1} \left(1 + \frac{q^{1-2\vartheta_1} - q}{q - 1} \right) = q^{\vartheta_1} \frac{q^{1-2\vartheta_1} - 1}{q - 1} = q^{\vartheta_1} [1 - 2\vartheta_1]_q.$$

The theorem is proved. □

Remark 4.3. For related results concerning linear differential and difference equations case see [26] and [30], respectively. Observe how the constants (indices of regular variation) ϑ_1, ϑ_2 in Theorem 4.2 differ from those in the continuous case. On the other hand, note how Theorem 4.2 resembles the continuous result as $q \rightarrow 1$.

Before we extend previous result on equation (HL q), we prove a few important lemmas, which will be needed later in new method of the proof. The following lemmas will play the important roles in showing q -regularly and q -rapidly varying behavior of solutions to (HL q).

Lemma 4.1. Define the functions $f, g : (0, \infty) \rightarrow \mathbb{R}$ by

$$f(x) = \Phi \left(\frac{x}{q} - \frac{1}{q} \right) - \Phi \left(1 - \frac{1}{x} \right) \quad \text{and} \quad g(x) = \Phi \left(\frac{x}{q} - \frac{1}{q} \right) + \Phi \left(1 - \frac{1}{x} \right).$$

Then $x \mapsto f(x)$ is strictly increasing for $x > q^{1-1/\alpha}$ and strictly decreasing for $0 < x < q^{1-1/\alpha}$, and $x \mapsto g(x)$ is strictly increasing for $x > 0$. Moreover, $f(1) = 0$ and $f(q^\vartheta) > 0$ provided $\vartheta > 1$.

Proof. The part concerning the monotonicity follows from the equalities

$$f'(x) = \frac{\alpha - 1}{(x - 1)^2} \left(q \left| \frac{x - 1}{q} \right|^\alpha - \left| \frac{x - 1}{x} \right|^\alpha \right).$$

and

$$g'(x) = \frac{\alpha - 1}{(x - 1)^2} \left(q \left| \frac{x - 1}{q} \right|^\alpha + \left| \frac{x - 1}{x} \right|^\alpha \right).$$

Further, with $\vartheta > 1$, $f(q^\vartheta) > 0$ if and only if $q^\vartheta/q - 1/q > 1 - 1/q^\vartheta$ if and only if $q^\vartheta(q^\vartheta - 1) > q(q^\vartheta - 1)$ if and only if $q^\vartheta > q$. □

The next lemma shows that (HL q) can be viewed in terms of fractions, which appear in characterization of q -regular or q -rapid variation.

Lemma 4.2. Define the operator \mathcal{L} by

$$\mathcal{L}[y](t) = \Phi \left(\frac{y(q^2t)}{qy(qt)} - \frac{1}{q} \right) - \Phi \left(1 - \frac{y(t)}{y(qt)} \right)$$

for $y \neq 0$. Then

$$D_q(\Phi(D_q y(t))) = \frac{\Phi(y(qt))}{(q - 1)^{\alpha t}} \mathcal{L}[y](t)$$

and equation (HL q) can be written as $\mathcal{L}[y](t) = (q - 1)^{\alpha t} p(t)$ for $y \neq 0$.

Proof. The statement is an easy consequence of the formula for q -derivative. Indeed,

$$\begin{aligned} D_q(\Phi(D_q y(t))) &= D_q \left(\Phi \left(\frac{y(qt) - y(t)}{(q-1)t} \right) \right) \\ &= \frac{1}{(q-1)t} \left(\Phi \left(\frac{y(q^2t) - y(qt)}{(q-1)qt} \right) - \Phi \left(\frac{y(qt) - y(t)}{(q-1)t} \right) \right) \\ &= \frac{1}{(q-1)^\alpha t^\alpha} \left(\Phi \left(\frac{y(qt)}{q} \left(\frac{y(q^2t)}{y(qt)} - 1 \right) \right) - \Phi \left(y(qt) \left(1 - \frac{y(t)}{y(qt)} \right) \right) \right) \\ &= \frac{\Phi(y(qt))}{(q-1)^\alpha t^\alpha} \mathcal{L}[y](t) \end{aligned}$$

□

The following lemma will play an important role when dealing with indices of regular variation of solutions to (HL q).

Lemma 4.3. Define the function $h_q : (\Phi(1/(1-q)), \infty) \rightarrow \mathbb{R}$ by

$$h_q(x) = \frac{x}{1 - q^{1-\alpha}} \left(1 - (1 + (q-1)\Phi^{-1}(x))^{1-\alpha} \right).$$

Then the graph of $x \mapsto h_q(x)$ is a parabola like curve with the minimum at the origin. If $C > 0$, then the equation $h_q(x) - x - C = 0$ has two real roots $x_1 < 0$ and $x_2 > 1$ on $(\Phi(1/(1-q)), \infty)$. If $C = 0$, then the algebraic equation has the roots $x_1 = 0$ and $x_2 = 1$. Taking the limit in h_q as $q \rightarrow 1^+$ it holds $h_1(x) = |x|^\beta$.

Proof. The shape of the curve follows from the facts that

$$h'_q(x) \operatorname{sgn}(x) = \frac{1}{1 - q^{1-\alpha}} (1 - (1 + (q-1)\Phi^{-1}(x))^{-\alpha}) \operatorname{sgn}(x) > 0$$

and

$$h''_q(x) = \frac{\beta(q-1)}{1 - q^{1-\alpha}} |x|^{\beta-2} (1 + (q-1)\Phi^{-1}(x))^{-\alpha-1} > 0$$

for admissible x , $x \neq 0$, and $h_q(0) = h'_q(0) = h''_q(0) = 0$. The statement concerning the roots x_1, x_2 then easily follows from observing the intersections of the graphs of the line $x \mapsto x$ and the function $x \mapsto h_q(x) - C$, view of $h_q(1) = 1$. The equality $h_1(x) = |x|^\beta$ follows either by direct using the L'Hospital rule to h_q with the respect to q or from the identity (4.22), in view of the fact that $[a]_q$ tends to $[a]_1 = a$ as $q \rightarrow 1^+$. □

We will need to rewrite the expression in the algebraic equation from the previous lemma in other terms; such a relation is described in the next statement.

Lemma 4.4. For $\vartheta \in \mathbb{R}$ it holds

$$\Phi([\vartheta]_q)[1 - \vartheta]_{q^{\alpha-1}} = \Phi([\vartheta]_q) - h_q(\Phi([\vartheta]_q)), \quad (4.22)$$

where $1 + (q - 1)\Phi^{-1}(\Phi([\vartheta]_q)) > 0$.

Proof. We have $1 + (q - 1)\Phi^{-1}(\Phi([\vartheta]_q)) = 1 + (q - 1)[\vartheta]_q = q^\vartheta > 0$. Further,

$$\begin{aligned} \Phi([\vartheta]_q) - h_q(\Phi([\vartheta]_q)) &= \Phi([\vartheta]_q) \left(1 - \frac{1}{1 - q^{1-\alpha}} (1 - q^{\vartheta(1-\alpha)}) \right) \\ &= \Phi([\vartheta]_q) \frac{q^{\vartheta(1-\alpha)} - q^{1-\alpha}}{1 - q^{1-\alpha}} \cdot \frac{q^{\alpha-1}}{q^{\alpha-1}} \\ &= \Phi([\vartheta]_q) \frac{q^{(\alpha-1)(1-\vartheta)} - 1}{q^{\alpha-1} - 1} \\ &= \Phi([\vartheta]_q)[1 - \vartheta]_{q^{\alpha-1}}. \end{aligned}$$

□

Next is described an important relation between the expression from the previous lemma and the function f from Lemma 4.1.

Lemma 4.5. For $\vartheta \in \mathbb{R}$ it holds

$$(q - 1)^{-\alpha} f(q^\vartheta) = \Phi([\vartheta]_q)[1 - \vartheta]_{q^{\alpha-1}}[1 - \alpha]_q. \quad (4.23)$$

Proof. In view of $(q^{1-\alpha} - 1)/(q^{\alpha-1} - 1) = -1/q^{\alpha-1}$, we have

$$\begin{aligned} \frac{\Phi([\vartheta]_q)[1 - \vartheta]_{q^{\alpha-1}}[1 - \alpha]_q}{(q - 1)^{-\alpha}} &= \Phi \left(\frac{q^\vartheta - 1}{q - 1} \right) \frac{(q^{(\alpha-1)(1-\vartheta)} - 1)(q - 1)^\alpha (q^{1-\alpha} - 1)}{(q^{\alpha-1} - 1)(q - 1)} \\ &= \Phi(q^\vartheta - 1) \frac{1 - q^{(\alpha-1)(1-\vartheta)}}{q^{\alpha-1}} \\ &= \frac{\Phi(q^\vartheta - 1)}{q^{\alpha-1}} - \frac{\Phi(q^\vartheta - 1)}{q^{(\alpha-1) - (\alpha-1)(1-\vartheta)}} \\ &= \Phi \left(\frac{q^\vartheta - 1}{q} \right) - \Phi \left(\frac{q^\vartheta - 1}{q^\vartheta} \right) \\ &= f(q^\vartheta). \end{aligned}$$

□

Now we are ready to use the mentioned method, which is designed just for the q -calculus case and turns out to be more effective than the previous method using the integral conditions. The following theorem can be understood as a generalization of the “linear” results from Theorem 4.2. Recall that all positive solutions of (HL q) are eventually monotone and convex.

Theorem 4.3. (i) Equation (HL $_q$) has solutions

$$u \in \mathcal{SV}_q \quad \text{and} \quad v \in \mathcal{RV}_q(1) \tag{4.24}$$

if and only if

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = 0. \tag{4.25}$$

All positive decreasing solutions of (HL $_q$) belong to \mathcal{SV}_q and all positive increasing solutions of (HL $_q$) belong to $\mathcal{RV}_q(1)$. Any of two conditions in (4.24) implies (4.25).

(ii) Equation (HL $_q$) has solutions

$$u \in \mathcal{RV}_q(\vartheta_1) \quad \text{and} \quad v \in \mathcal{RV}_q(\vartheta_2) \tag{4.26}$$

if and only if

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = B > 0, \tag{4.27}$$

where $\vartheta_i = \log_q[(q-1)\Phi^{-1}(\lambda_i) + 1]$, $i = 1, 2$, $\lambda_1 < \lambda_2$ being the roots of the equation $h_q(\lambda) - \lambda + B/[1-\alpha]_q = 0$; these roots satisfy $\lambda_1 \in (\Phi(1/(1-q)), 0)$, $\lambda_2 > 1$, and ϑ_1, ϑ_2 satisfy $\vartheta_1 \in (-\infty, 0)$ and $\vartheta_2 > 1$. All positive decreasing solutions of (HL $_q$) belong to $\mathcal{RV}_q(\vartheta_1)$ and all positive increasing solutions of (HL $_q$) belong to $\mathcal{RV}_q(\vartheta_2)$. Any of two conditions in (4.26) implies (4.27).

Proof. First note that the intervals of allowed values for λ_1 and λ_2 follows from Lemma 4.3. The intervals for ϑ_1, ϑ_2 are then consequences of the relations $q^{\vartheta_i} = (q-1)\Phi^{-1}(\lambda_i) + 1$, $i = 1, 2$.

Parts (i) and (ii) will be proved simultaneously, assuming $B \geq 0$ in (4.27) and, consequently, having $\lambda_1 \in (\Phi(1/(1-q)), 0]$ and $\lambda_2 \geq 1$.

“Only if parts”: Assume $u \in \mathcal{RV}_q(\vartheta_1)$. Using Lemmas 4.2, 4.4, and 4.5, we get

$$\begin{aligned} \lim_{t \rightarrow \infty} t^\alpha p(t) &= (q-1)^{-\alpha} \lim_{t \rightarrow \infty} \mathcal{L}[u](t) \\ &= (q-1)^{-\alpha} \left(\Phi \left(\frac{q^{\vartheta_1}}{q} - \frac{1}{q} \right) - \Phi \left(1 - \frac{1}{q^{\vartheta_1}} \right) \right) \\ &= (q-1)^{-\alpha} f(q^{\vartheta_1}) \\ &= \Phi([\vartheta_1]_q) [1 - \vartheta_1]_{q^{\alpha-1}} [1 - \alpha]_q \\ &= [1 - \alpha]_q (\Phi([\vartheta_1]_q) - g_q(\Phi([\vartheta_1]_q))) \\ &= [1 - \alpha]_q (\lambda_1 - g_q(\lambda_1)) \\ &= [1 - \alpha]_q \frac{B}{[1 - \alpha]_q} \\ &= B. \end{aligned}$$

The same arguments work for $v \in \mathcal{RV}_q(\vartheta_2)$.

In view of Proposition 4.1, solutions in $\mathcal{RV}_q(\vartheta_1)$ necessarily decrease (this includes also \mathcal{SV}_q solutions because of their convexity) and solutions in $\mathcal{RV}_q(\vartheta_1)$ necessarily increase.

“If parts”: Assume $\lim_{t \rightarrow \infty} t^\alpha p(t) = B \geq 0$ and u is a positive decreasing solution of (HL q) on $[a, \infty)_q$. Let us write B as $B = [1 - \alpha]_q(\Phi([\vartheta_1]_q) - g_q(\Phi([\vartheta_1]_q)))$. In view of Lemma 4.2, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathcal{L}[u](t) &= (q - 1)^\alpha \lim_{t \rightarrow \infty} t^\alpha p(t) = (q - 1)^\alpha B \\ &= (q - 1)^\alpha [1 - \alpha]_q(\Phi([\vartheta_1]_q) - g_q(\Phi([\vartheta_1]_q))). \end{aligned}$$

Now, using Lemma 4.4 and Lemma 4.5 in the last equality, we get

$$\lim_{t \rightarrow \infty} \mathcal{L}[u](t) = f(q^{\vartheta_1}). \tag{4.28}$$

We will show that $\lim_{t \rightarrow \infty} u(qt)/u(t) = q^{\vartheta_1}$. Denote $M_* = \liminf_{t \rightarrow \infty} u(qt)/u(t)$ and $M^* = \limsup_{t \rightarrow \infty} u(qt)/u(t)$. First note that the case $M_* = 0$ cannot happen. Indeed, if $M_* = 0$, then $\limsup_{t \rightarrow \infty} \mathcal{L}[u](t) = \infty$, which is in a contradiction with a real value of $f(q^{\vartheta_1})$, in view of (4.28). We also have u decreasing, and hence $M_*, M^* \in (0, 1]$. In view of the above observations, taking the \liminf as $t \rightarrow \infty$ in $\mathcal{L}[u](t) = (q - 1)^\alpha t^\alpha p(t)$, we get

$$\Phi\left(\frac{M_*}{q} - \frac{1}{q}\right) - \Phi\left(1 - \frac{1}{M^*}\right) = f(q^{\vartheta_1}).$$

Similarly, the \limsup yields

$$\Phi\left(\frac{M^*}{q} - \frac{1}{q}\right) - \Phi\left(1 - \frac{1}{M_*}\right) = f(q^{\vartheta_1}).$$

Subtracting these equations we obtain $g(M_*) = g(M^*)$. In view of monotone properties of g (see Lemma 4.1), we get $M := M_* = M^*$. Hence, from (4.28), $f(M) = f(q^{\vartheta_1})$. We claim that $M = q^{\vartheta_1}$. Since $M, q^{\vartheta_1} \in (0, 1]$, we work here with f on the interval $(0, 1]$, where it is strictly decreasing, see Lemma 4.1. Hence, $M \neq q^{\vartheta_1}$ would lead to a contradiction. Thus we get $\lim_{t \rightarrow \infty} u(qt)/u(t) = M = q^{\vartheta_1}$, which implies $u \in \mathcal{RV}_q(\vartheta_1)$. Similarly we proceed with a positive increasing solution v of (HL q). However, certain additional steps need to be shown. First note that for $N_* = \liminf_{t \rightarrow \infty} v(qt)/v(t)$ and $N^* = \limsup_{t \rightarrow \infty} v(qt)/v(t)$ we have $N_*, N^* \in [1, \infty)$. Because of monotone properties of g we get $\lim_{t \rightarrow \infty} v(qt)/v(t) = N \in (1, \infty)$. The limit value 1 is excluded since it would lead to $v \in \mathcal{SV}_q$, and the solution v then decreases, (in view of convexity, see Proposition 4.1 (vi)), which is a contradiction. Notice that another argument is $0 = f(1) = f(N) = f(q^{\vartheta_2}) > 0$ by Lemma 4.1. We claim that $N = q^{\vartheta_2}$. We have that $f(N) = f(q^{\vartheta_2})$ and recall that $\vartheta_2 > 1$. If $N \geq q^{1-1/\alpha}$ and we assume $N \neq q^{\vartheta_2}$, then we immediately get a

contradiction since $x \mapsto f(x)$ is strictly increasing on $[q^{1-1/\alpha}, \infty)$, see Lemma 4.1. Assume now $N \in (1, q^{1-1/\alpha})$. We know that $x \mapsto f(x)$ is strictly decreasing on $(1, q^{1-1/\alpha})$, see Lemma 4.1. Moreover, $f(1) = 0$. Hence, $f(N) < f(1) = 0$. From Lemma 4.1 we also know that $f(q^{\vartheta_2}) > 0$ for $\vartheta_2 > 1$, and so we get $f(N) < f(q^{\vartheta_2})$, a contradiction. Thus we obtain $\lim_{t \rightarrow \infty} v(qt)/v(t) = N = q^{\vartheta_2}$, which implies $v \in \mathcal{RV}_q(\vartheta_2)$. \square

In the end of this section we establish nonintegral conditions guaranteeing that positive solutions of equation (HL q) are q -rapidly varying. We use the analogical method as before.

Theorem 4.4. *Equation (HL q) has solutions*

$$u \in \mathcal{RPV}_q(-\infty) \quad \text{and} \quad v \in \mathcal{RPV}_q(\infty) \quad (4.29)$$

if and only if

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = \infty. \quad (4.30)$$

All positive decreasing solutions of (HL q) belong to $\mathcal{RPV}_q(-\infty)$ and all positive increasing solutions of (HL q) belong to $\mathcal{RPV}_q(\infty)$. Any of two conditions in (4.29) implies (4.30)

Proof. “Only if”: Let u be a solution of (HL q) such that $u \in \mathcal{RPV}_q(-\infty)$. Then u is eventually decreasing (towards zero) and $\lim_{t \rightarrow \infty} u(qt)/u(t) = 0$ by Proposition 4.2. Hence, $\lim_{t \rightarrow \infty} u(q^2t)/u(qt) = 0$ and $\lim_{t \rightarrow \infty} u(t)/u(qt) = \infty$, and so

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = \lim_{t \rightarrow \infty} (q-1)^{-\alpha} \mathcal{L}[u](t) = \infty,$$

in view of Lemma 4.2. Similarly, for a solution v of (HL q) with $v \in \mathcal{RPV}_q(\infty)$, we have v is eventually increasing (towards ∞) and $\lim_{t \rightarrow \infty} v(qt)/v(t) = \infty$, and so

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = \lim_{t \rightarrow \infty} (q-1)^{-\alpha} \mathcal{L}[v](t) = \infty.$$

“If”: Let (4.30) hold and u be a positive decreasing solution of (HL q) on $[a, \infty)_q$. Since u is decreasing, we have $u(qt) \leq u(t)$, and in view of Lemma 4.2,

$$\infty = \lim_{t \rightarrow \infty} t^\alpha p(t) = \lim_{t \rightarrow \infty} (q-1)^{-\alpha} \mathcal{L}[u](t) \leq -(q-1)^{-\alpha} \Phi(1 - u(t)/u(qt)).$$

Consequently, $\lim_{t \rightarrow \infty} u(t)/u(qt) = \infty$, and so $u \in \mathcal{RPV}_q(-\infty)$ by Proposition 3.3. Now assume that v is a positive increasing solution of (HL q) on $q^{\mathbb{N}_0}$. Then $v(qt) \geq v(t)$, and, similarly as above,

$$\infty = \lim_{t \rightarrow \infty} t^\alpha p(t) \leq (q-1)^{-\alpha} \Phi(v(q^2t)/v(qt) - 1/q),$$

which implies $\lim_{t \rightarrow \infty} v(q^2t)/v(qt) = \infty$, and, consequently, $v \in \mathcal{RPV}_q(\infty)$. \square

Remark 4.4. The results presented in Theorem 4.4, except of the necessity part for an increasing solution, are q -versions of results presented for the equation (HL) on general time scale, see Theorem 3.9. Recall that a necessity part for increasing solutions has not been proved in the differential (resp. difference or dynamic) equations setting yet.

4.4 Concluding comments and \mathbb{M}_q -classification

In the first part of this section we discuss the form of conditions guaranteeing q -regular resp. q -rapid variation of solutions. We compare the linear results (written in terms of integral conditions, see Theorem 4.2) with half-linear ones (written in terms of nonintegral condition, see Theorem 4.3 and Theorem 4.4) and with the results from the differential equations case. The following questions may come in our minds: 1. Are the half-linear results on q -regularly varying solutions really extensions of the linear ones? 2. How are these conditions related? The next lemma shows that in the case of existence of a proper limit, integral and nonintegral conditions are actually equivalent. Hence the half-linear results really extend the linear ones. Moreover, from that lemma it follows that while the integral condition is not suitable for a unified characterization of rapid or regularly varying behavior of solutions to (HL q), this can be done via the nonintegral condition, see also Corollary 4.1. Later we show that these relations are specific just for q -calculus and differ from what is known in the continuous case. We stress that there is no sign condition on p in the next lemma.

Lemma 4.6. *Let $p : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ and $\alpha > 1$. It holds*

$$\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) d_q s = C \in \mathbb{R} \quad \text{if and only} \quad \lim_{t \rightarrow \infty} t^\alpha p(t) = -[1 - \alpha]_q C \in \mathbb{R}.$$

Moreover,

$$\text{if } \lim_{t \rightarrow \infty} t^\alpha p(t) = \pm\infty, \text{ then } \lim_{t \rightarrow \infty} t \int_t^\infty p(s) d_q s = \pm\infty,$$

but the opposite implication does not hold in general.

Proof. If. Assume $\lim_{t \rightarrow \infty} t^\alpha p(t) = -[1 - \alpha]_q C$, where $C \in \mathbb{R} \cup \{\pm\infty\}$. Using the q -L'Hospital rule, we get

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) d_q s &= \lim_{t \rightarrow \infty} \frac{-p(t)}{((qt)^{1-\alpha} - t^{1-\alpha}) / ((q-1)t)} \\ &= \frac{t^\alpha p(t)}{-[1 - \alpha]_q} \\ &= C. \end{aligned}$$

Only if. Assume $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) d_q s = C \in \mathbb{R}$. We have

$$\begin{aligned} t^{\alpha-1} \int_t^\infty p(s) d_q s &= t^{\alpha-1} \left(\int_t^{qt} p(s) d_q s + \int_{qt}^\infty p(s) d_q s \right) \\ &= (q-1)t^\alpha p(t) + \frac{1}{q^{\alpha-1}} (qt)^{\alpha-1} \int_{qt}^\infty p(s) d_q s. \end{aligned}$$

Hence,

$$\begin{aligned} t^\alpha p(t) &= \frac{1}{q-1} \left(t^{\alpha-1} \int_t^\infty p(s) d_q s - \frac{1}{q^{\alpha-1}} (qt)^{\alpha-1} \int_{qt}^\infty p(s) d_q s \right) \\ &\rightarrow \frac{1}{q-1} \left(C - \frac{C}{q^{\alpha-1}} \right) \\ &= -C[1 - \alpha]_q. \end{aligned}$$

as $t \rightarrow \infty$. It remains to find a function p such that $\lim_{t \rightarrow \infty} t \int_t^\infty p(s) d_q s = \infty$, but $\lim_{t \rightarrow \infty} t^\alpha p(t) \neq \infty$. For simplicity we present an example corresponding with the case $\alpha = 2$. Define the function

$$p(t) = \begin{cases} t^{-2}/q & \text{for } t = q^{2n}, \\ t^{-2}/q + t^{-3/2} & \text{for } t = q^{2n+1}, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$. Then

$$t^2 p(t) = \begin{cases} 1/q & \text{for } t = q^{2n}, \\ \sqrt{t} + 1/q & \text{for } t = q^{2n+1}. \end{cases}$$

Thus we see that $\liminf_{t \rightarrow \infty} t^2 p(t) = 1/q < \infty = \limsup_{t \rightarrow \infty} t^2 p(t)$. Further, with $t = q^n$, we have $\int_t^\infty p(s) d_q s = (q-1) \sum_{j=n}^\infty q^j p(q^j)$. Hence, summing appropriate geometric series, we obtain

$$t \int_t^\infty p(s) d_q s = \begin{cases} q^{n/2} \sqrt{q} + 1 = \sqrt{qt} + 1 & \text{for } t = q^{2n}, \\ q^{n/2} q + 1 = q\sqrt{t} + 1 & \text{for } t = q^{2n+1}. \end{cases}$$

Consequently, $\lim_{t \rightarrow \infty} t \int_t^\infty p(s) d_q s \geq \lim_{t \rightarrow \infty} \sqrt{qt} = \infty$. □

To see an interesting specific character of the results in q -calculus case, let us recall some of their continuous counterparts, see [20, 26] or modify the results from Chapter 3 on continuous case. Positive solutions of the equation

$$(\Phi(y'(t)))' - p(t)\Phi(y(t)) = 0, \tag{4.31}$$

$p(t) > 0$, are regularly varying if and only if $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) ds = A$ exists as a finite number, and are rapidly varying if and only if $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\lambda t} p(s) ds = \infty$

for all $\lambda > 1$. The indices of regular variation are given here by $\Phi^{-1}(\vartheta_i)$, $i = 1, 2$, where $\vartheta_1 < \vartheta_2$ are the roots of $|\vartheta|^\beta = \vartheta - A = 0$; with the use of Lemma 4.3, observe how this result matches the one from Theorem 4.3 as $q \rightarrow 1^+$. Further recall $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) ds = C_1$ exists finite if and only $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\lambda t} p(s) ds = C_2(\lambda)$ exists finite for all $\lambda > 1$ with $C_2(\lambda) = C_1(\lambda^{\alpha-1} - 1)/\lambda^{\alpha-1}$; therefore all positive solutions of (4.31) are rapidly or regularly varying if and only if for every $\lambda > 1$ $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\lambda t} p(s) ds$ exists finite or infinite. The expression

$$t^{\alpha-1} \int_t^{\tau(\lambda t)} p(s) d_q s, \tag{4.32}$$

considered on $q^{\mathbb{N}_0}$, can be understood as a q -version of $t^{\alpha-1} \int_t^{\lambda t} p(s) ds$. Further note that the expression $t^\alpha p(t)$ in q -calculus can be viewed in two ways: First, simply as a nonintegral expression. Second, up to certain constant multiple, as $t^{\alpha-1} \int_t^{qt} p(s) d_q s$, which is equal to (4.32) where $\lambda = q$. While the existence of a (finite or infinite) limit $\lim_{t \rightarrow \infty} t^\alpha p(t)$ clearly cannot serve to guarantee regularly or rapidly varying behavior of solutions to (4.31) (in the sense of sufficiency and necessity), this is possible in q -calculus case. It is because in q -calculus there are “closer” relations among the limits $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^\infty p(s) d_q s$ and $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{qt} p(s) d_q s$ and $\lim_{t \rightarrow \infty} t^\alpha p(t)$, which may not hold in classical calculus. Also note that while in the continuous case we need the existence of the limit $\lim_{t \rightarrow \infty} t^{\alpha-1} \int_t^{\lambda t} p(s) ds$ for all parameters $\lambda > 1$, in q -calculus case we require its existence just for one parameter $\lambda = q$, compare also with Karamata type definitions in cases of both calculi. Finally note that the situation in the classical discrete case, see [29], or in a general time scale case (with a graininess μ such that $\mu(t) = o(t)$ as $t \rightarrow \infty$), see the results from Section 3.4 and 3.5, is similar to that in the continuous case, and so the q -calculus case is really exceptional.

In the second part of this section we provide information about asymptotic behavior of all positive solutions of (HL $_q$) as $t \rightarrow \infty$ and we establish so-called \mathbb{M}_q -classification for equation (HL $_q$) can be understood as a q -version of \mathbb{M} -classification for equation (HL), see section 3.4. Note that all nontrivial solutions of (HL $_q$) are nonoscillatory (i.e., of one sign for large t) and monotone for large t . Note that the solution space of (HL $_q$) has just one half of the properties which characterize linearity, namely homogeneity (but not additivity). Just because of homogeneity, without loss of generality, we may restrict our consideration only to positive solutions of (HL $_q$); we denote this set as \mathbb{M}_q . Thanks to the monotonicity, the set \mathbb{M}_q can be further split in the two classes \mathbb{M}_q^+ and \mathbb{M}_q^- , where

$$\begin{aligned} \mathbb{M}_q^+ &= \{y \in \mathbb{M}_q : \exists t_y \in q^{\mathbb{N}_0} \text{ such that } y(t) > 0, D_q y(t) > 0 \text{ for } t \geq t_y\}, \\ \mathbb{M}_q^- &= \{y \in \mathbb{M}_q : y(t) > 0, D_q y(t) < 0\}. \end{aligned}$$

It is not difficult to see that these classes are always nonempty. The reason is similar as in section 3.4 concerning \mathbb{M} -classification.

A positive function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is said to be a q -Karamata function, if f is q -slowly or q -regularly or q -rapidly varying; we write $f \in \mathcal{KF}_q$. We introduce the following notation:

$$\begin{aligned} \mathbb{M}_{qSV}^- &= \mathbb{M}_q^- \cap \mathcal{SV}_q, \\ \mathbb{M}_{qRV}^-(\vartheta_1) &= \mathbb{M}_q^- \cap \mathcal{RV}_q(\vartheta_1), \vartheta_1 < 0, \\ \mathbb{M}_{qRV}^+(\vartheta_2) &= \mathbb{M}_q^+ \cap \mathcal{RV}_q(\vartheta_2), \vartheta_2 \geq 1, \\ \mathbb{M}_{qRPV}^-(-\infty) &= \mathbb{M}_q^- \cap \mathcal{RPV}_q(-\infty), \\ \mathbb{M}_{qRPV}^+(\infty) &= \mathbb{M}_q^+ \cap \mathcal{RPV}_q(\infty), \\ \mathbb{M}_{q0}^- &= \{y \in \mathbb{M}_q^- : \lim_{t \rightarrow \infty} y(t) = 0\}, \\ \mathbb{M}_{q\infty}^+ &= \{y \in \mathbb{M}_q^+ : \lim_{t \rightarrow \infty} y(t) = \infty\} \end{aligned}$$

We distinguish three cases for behavior of the coefficient $p(t)$ from equation (HL q):

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = 0, \tag{4.33}$$

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = B > 0, \tag{4.34}$$

$$\lim_{t \rightarrow \infty} t^\alpha p(t) = \infty. \tag{4.35}$$

With the use of the results of this paper we can claim:

$$\begin{aligned} \mathbb{M}_q^- &= \mathbb{M}_{qSV}^- \iff (4.33) \iff \mathbb{M}_q^+ = \mathbb{M}_{qRV}^+(1) = \mathbb{M}_{q\infty}^+, \\ \mathbb{M}_q^- &= \mathbb{M}_{qRV}^-(\vartheta_1) = \mathbb{M}_{q0}^- \iff (4.34) \iff \mathbb{M}_q^+ = \mathbb{M}_{qRV}^+(\vartheta_2) = \mathbb{M}_{q\infty}^+, \\ \mathbb{M}_q^- &= \mathbb{M}_{qRPV}^-(-\infty) = \mathbb{M}_{q0}^- \iff (4.35) \iff \mathbb{M}_q^+ = \mathbb{M}_{qRPV}^+(\infty) = \mathbb{M}_{q\infty}^+. \end{aligned}$$

In view of previous results, we get the following statement.

Corollary 4.1. *The following statements are equivalent:*

- $\exists u \in \mathbb{M}_q : u \in \mathcal{KF}_q$.
- $\forall u \in \mathbb{M}_q : u \in \mathcal{KF}_q$.
- *There exists the (finite or infinite) limit*

$$\lim_{t \rightarrow \infty} t^\alpha p(t). \tag{4.36}$$

Because of this corollary and due to example from the proof of Lemma 4.6, equation (HL q) may possess a positive solution, which is not in \mathcal{KF}_q . In fact, such a case happens if and only if the limit (4.36) does not exist, and then necessarily no positive solution of (HL q) is an element of \mathcal{KF}_q .

The above relations between the \mathbb{M}_q -classification and Karamata like behavior of solutions to (HL $_q$) could be refined, provided more detailed information on the existence in all subclasses (in the sense of effective conditions) would be at disposal. Or, possibly, all observations can be extended to equations with no sign condition on p or to some other q -difference equations. This can be understood as another direction for a future research.

Telescoping principle for oscillation of half-linear dynamic equations

5

In this chapter, we establish the so-called “*telescoping principle*” for oscillation of the second order half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0 \quad (\text{HL}^\Delta\text{E})$$

on (an infinite) time scale interval \mathcal{I}_a . Throughout this chapter, we suppose that $1/r, p \in C_{\text{rd}}(\mathcal{I}_a)$ with $r(t) \neq 0$. Recall that this equation was in detail introduced in Section 2.3, where we recall the basic information about oscillation theory for this equation.

5.1 Introduction to oscillatory problems

Many oscillation criteria concerning equation (HL $^\Delta$ E) require to know the properties of the Δ -integral of coefficient $p(t)$ on the whole interval \mathcal{I}_a . According to the behavior of this integral, we can sometime decide whether our equation is oscillatory or nonoscillatory. On the other hand, from the Sturm separation theorem, it is clear that oscillation can be taken as an interval property. Consider equation (HL $^\Delta$ E). If there exists a sequence of subsets $[a_i, b_i] \cap \mathbb{T}$ of \mathcal{I}_a , $a_i \rightarrow \infty$ as $i \rightarrow \infty$, such that for each i there is a nontrivial solution of equation (HL $^\Delta$ E) which has at least two zeros (resp. generalized zeros) in $[a_i, b_i]$, then every solution of (HL $^\Delta$ E) is oscillatory with at least one zero (resp. generalized zero) in each $[a_i, b_i]$. From the above observation, the oscillation of (HL $^\Delta$ E) can be studied on suitable intervals $[a_i, b_i]$, precisely, any oscillation criterion can be (successfully) founded from a behavior of coefficients $p(t)$ and $r(t)$ on the suitable chosen intervals $[a_i, b_i]$.

M. K. Kwong and A. Zettl [25] applied this idea to oscillation of equation (LDE) and constructed so-called “*telescoping principle*” which allows to trim off “problem” parts of $\int_0^t p(s)ds$ and use any known oscillation criterion to the remaining “good” parts. Q. Kong and A. Zettl [24] came up with an analogical telescoping principle for equation (L Δ E) and used it to obtain some new oscillation results for difference equations. P. Řehák [32] extended this telescoping principle to equation (HL Δ E). Finally, L. H. Erbe, L. Kong and Q. Kong [15] unified and

generalized the telescoping principle for equations (LDE) and (L Δ E) into the only one telescoping principle on time scales for equation (L Δ E) and found many new oscillation results for this equation.

Our aim is to extend, modify and generalize the results of previous papers to equation (HL Δ E) and make an analogical telescoping principle for half-linear time scale case. Unlike previous works we formulate the telescoping principle under the weaker assumption $r(t) \neq 0$ (instead $r(t) > 0$), which is new even in the linear case.

5.2 Telescoping principle

In this section we establish the telescoping principle for oscillation of equation (HL Δ E). Consider the following set

$$J = \left(\bigcup_{i=1}^{\infty} J_i \right) \cap \mathcal{I}_a, \quad J_i = (a_i, \sigma(b_i)), \quad i \in \mathbb{N},$$

where $a_i, b_i \in \mathcal{I}_a$ with $a < a_i < b_i < a_{i+1}$ and if $\mu(a_i) = 0$ then $\mu(\sigma(b_i)) = 0$ for all $i \in \mathbb{N}$. We call J an *interval shrinking set* in \mathcal{I}_a . With the help of the set J we define following “shrinking” transformation on the time scale \mathcal{I}_a .

At first, we define a new time scale $\widehat{\mathcal{I}}_a$ by:

$$\begin{aligned} \widehat{\mathcal{I}}_a := & \{s \in \mathcal{I}_a : s \leq a_1\} \\ & \cup \left\{ \bigcup_{j=1}^{\infty} \left\{ s = t - \sum_{i=1}^j (\sigma(b_i) - \sigma(a_i)) : t \in [\sigma(b_j), a_{j+1}] \cap \mathcal{I}_a \right\} \right\}, \end{aligned} \quad (5.1)$$

which is, anyway, the set \mathcal{I}_a without the set J . More precisely, it is the set \mathcal{I}_a , where we trim off the time scales intervals $(\sigma(a_i), \sigma(b_i))$ (or if one wants, it is the set \mathcal{I}_a , where each (time scale) subinterval $(a_i, \sigma(b_i))$ is collapsed to its left point).

Now we define an interval shrinking transformation $\mathcal{T} : \mathcal{I}_a \rightarrow \widehat{\mathcal{I}}_a$ by:

$$s = \mathcal{T}t = \begin{cases} t & t \in [a, a_1] \cap \mathcal{I}_a, \\ a_1 & t \in (a_1, \sigma(b_1)) \cap \mathcal{I}_a, \\ a_{j+1} - \sum_{i=1}^j (\sigma(b_i) - \sigma(a_i)) & t \in (a_{j+1}, \sigma(b_{j+1})) \cap \mathcal{I}_a, \\ t - \sum_{i=1}^j (\sigma(b_i) - \sigma(a_i)) & t \in [\sigma(b_j), a_{j+1}] \cap \mathcal{I}_a, \end{cases}$$

where $j \in \mathbb{N}$. For $s \in \widehat{\mathcal{I}}_a$ we define an inverse transformation $\mathcal{T}^{-1} : \widehat{\mathcal{I}}_a \rightarrow \mathcal{I}_a$ by:

$$\mathcal{T}^{-1}s = \inf \{t \in \mathcal{I}_a : \mathcal{T}t = s\}.$$

Note, that the condition if $\mu(a_i) = 0$ then $\mu(\sigma(b_i)) = 0$ implies $\widehat{\mu}(s) = \mu(t)$ for all $t = \mathcal{T}^{-1}s$, where $\widehat{\mu}$ denotes the graininess in $\widehat{\mathcal{I}}_a$.

Lemma 5.1. *A solution y of equation (HL $^\Delta$ E) satisfies $r(t)y(t)y(\sigma(t)) > 0$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$ if and only if the corresponding solution $w(t)$ of the generalized Riccati equation (GR $^\Delta$ E) satisfies $-\mu^{\alpha-1}(t)w(t) < r(t)$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$.*

Proof. From Theorem 2.4 (Roundabout Theorem) it is clear that equation (HL $^\Delta$ E) has a solution y satisfying $r(t)y(t)y(\sigma(t)) > 0$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$ if and only if the corresponding solution $w(t)$ of equation (GR $^\Delta$ E) satisfies

$$\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t)) > 0, \tag{5.2}$$

for $t \in [t_1, t_2] \cap \mathcal{I}_a$. However, (5.2) is equivalent to $-\mu^{\alpha-1}(t)w(t) < r(t)$, so lemma holds. \square

Let $\widehat{\mathcal{I}}_a$ be defined by (5.1) and consider the telescoped equation of (HL $^\Delta$ E)

$$[\widehat{r}(s)\Phi(x^\Delta)]^\Delta + \widehat{p}(s)\Phi(x^{\widehat{\sigma}}) = 0, \quad s \in \widehat{\mathcal{I}}_a, \tag{HL $^\Delta$ E}$$

where $\widehat{r}(s) = r(t)$, $\widehat{p}(s) = p(t)$, for $t = \mathcal{T}^{-1}s$, and where $\widehat{\sigma}$ denotes the forward jump operator in $\widehat{\mathcal{I}}_a$. The following theorem is similar to the comparison type result. In simple terms, it says that if a certain solution $x(s)$ of the telescoped equation (HL $^\Delta$ E) has a generalized zero in $[a, b] \cap \widehat{\mathcal{I}}_a$, then a corresponding solution $y(t)$ of the original equation (HL $^\Delta$ E) has a generalized zero in $[\mathcal{T}^{-1}a, \mathcal{T}^{-1}b] \cap \mathcal{I}_a$.

Theorem 5.1. *Assume*

$$\int_{\sigma(a_i)}^{\sigma(b_i)} p(t)\Delta t \geq 0, \quad i \in \mathbb{N}, \tag{5.3}$$

and let $d \in \widehat{\mathcal{I}}_a$ be such that $d > a$. Suppose that x be a solution of (HL $^\Delta$ E) with $\widehat{r}(s)x(s)x(\widehat{\sigma}(s)) > 0$ for $s \in [a, d] \cap \widehat{\mathcal{I}}_a$ and $\widehat{r}(d)x(d)x(\widehat{\sigma}(d)) \leq 0$. Let y be a solution of (HL $^\Delta$ E) with $y(a) \neq 0$,

$$\frac{r(a)\Phi(y^\Delta(a))}{\Phi(y(a))} \leq \frac{\widehat{r}(a)\Phi(x^\Delta(a))}{\Phi(x(a))}.$$

Then there exists $e \leq \mathcal{T}^{-1}d$ such that $r(e)y(e)y(\sigma(e)) \leq 0$. More precisely, if $d \leq \mathcal{T}a_i$, then there exists $e \leq a_i$ such that $r(e)y(e)y(\sigma(e)) \leq 0$.

Proof. In this proof, by $w \not\leq v$ we mean either $w \geq v$ or w does not exist. The proof is by induction with respect to the location of the point $d \in \widehat{\mathcal{I}}_a$. Assume the contrary. Then $w(t)$ defined by (2.14) satisfies the generalized Riccati equation (GR $^\Delta$ E) and (2.15), and by Lemma 5.1 $-\mu^{\alpha-1}(t)w(t) < r(t)$ holds for $t \in [a, \mathcal{T}^{-1}d] \cap \mathcal{I}_a$. For $s \in \widehat{\mathcal{I}}_a$, let

$$v(s) = \frac{\widehat{r}(s)\Phi(x^\Delta(s))}{\Phi(x(s))}.$$

Then it follows that v satisfies the telescoped generalized Riccati equation

$$v^\Delta(s) + \widehat{p}(s) + \mathcal{S}[v, \widehat{r}, \widehat{\mu}](s) = 0, \quad s \in \widehat{\mathcal{I}}_a, \quad (\widehat{\text{GR}}^\Delta\text{E})$$

and

$$v(\widehat{\sigma}(s)) = \frac{\widehat{r}(s)v(s)}{\Phi[\Phi^{-1}(\widehat{r}(s)) + \widehat{\mu}(s)\Phi^{-1}(v(s))]} - \widehat{\mu}(s)\widehat{p}(s), \quad s \in \widehat{\mathcal{I}}_a. \quad (5.4)$$

By Lemma 5.1,

$$-\widehat{\mu}^{\alpha-1}(s)v(s) < \widehat{r}(s)$$

for $s \in [a, d) \cap \widehat{\mathcal{I}}_a$ and moreover

$$-\widehat{\mu}^{\alpha-1}(d)v(d) \not\leq \widehat{r}(d).$$

(i) Assume that $d \leq \mathcal{T}a_1 = a_1$, then $t = \mathcal{T}^{-1}s = s$ for $s \in [a, d] \cap \widehat{\mathcal{I}}_a$, so we have $\widehat{r}(t) = r(t)$, $\widehat{p}(t) = p(t)$, and equations (GR $^\Delta$ E) and ($\widehat{\text{GR}}^\Delta\text{E}$) are the same on $[a, d] \cap \mathcal{I}_a$. We wish to show that

$$w(t) \leq v(t), \quad t \in [a, d] \cap \mathcal{I}_a. \quad (5.5)$$

From Theorem 2.1 (Existence and Uniqueness), it follows that the initial value problem

$$w_n^\Delta(t) + p(t) + \mathcal{S}[w_n, r, \mu](t) + \frac{1}{n} = 0, \quad w_n(a) = w(a), \quad (5.6)$$

has a unique solution $w_n(t)$ on $t \in [a, d] \cap \mathcal{I}_a$. It is clear that $w_n(t) \rightarrow w(t)$ as $n \rightarrow \infty$ for $t \in [a, d] \cap \mathcal{I}_a$. We want to show that for all large $n \in \mathbb{N}$,

$$w_n(t) \leq v(t), \quad t \in [a, d] \cap \mathcal{I}_a. \quad (5.7)$$

Assume the contrary. Since $w_n(a) \leq v(a)$, suppose that there exist points t_* , $t^* \in \mathcal{I}_a$ with $a \leq t_* < t^* \leq d$ such that

$$w_n(t) \leq v(t), \quad t \in (a, t_*] \cap \mathcal{I}_a \quad \text{and} \quad w_n(t) > v(t), \quad t \in (t_*, t^*] \cap \mathcal{I}_a. \quad (5.8)$$

If t_* is right-scattered in \mathcal{I}_a (thus in $\widehat{\mathcal{I}}_a$), then from (5.6) and (2.15)

$$w_n(\sigma(t_*)) = \frac{r(t_*)w_n(t_*)}{\Phi[\Phi^{-1}(r(t_*)) + \mu(t_*)\Phi^{-1}(w_n(t_*))]} - \mu(t_*)p(t_*) - \frac{\mu(t_*)}{n}. \quad (5.9)$$

Let the function $\tilde{\mathcal{S}}(w, r, \mu)$ represent the first term of the right-hand side of equations (5.9) and (5.4), so

$$\tilde{\mathcal{S}}(w, r, \mu) = \frac{rw}{\Phi[\Phi^{-1}(r) + \mu\Phi^{-1}(w)]}.$$

Then from the continuity of $\tilde{\mathcal{S}}$ with respect to the first variable and from the positivity of term $[\Phi^{-1}(r) + \mu\Phi^{-1}(w)]$ (see, (5.2)), we obtain

$$\begin{aligned} \frac{\partial \tilde{\mathcal{S}}}{\partial w} &= r \left(\frac{[\Phi^{-1}(r) + \mu\Phi^{-1}(w)]^{\alpha-1} - [\Phi^{-1}(r) + \mu\Phi^{-1}(w)]^{\alpha-2} \mu\Phi^{-1}(w)}{[\Phi^{-1}(r) + \mu\Phi^{-1}(w)]^{2\alpha-2}} \right) \\ &= r \left(\frac{[\Phi^{-1}(r) + \mu\Phi^{-1}(w)] - \mu\Phi^{-1}(w)}{[\Phi^{-1}(r) + \mu\Phi^{-1}(w)]^\alpha} \right) \\ &= \frac{|r|^\beta}{[\Phi^{-1}(r) + \mu\Phi^{-1}(w)]^\alpha}. \end{aligned}$$

Hence

$$\frac{\partial \tilde{\mathcal{S}}(w, r, \mu)}{\partial w} > 0,$$

which means that the function $\tilde{\mathcal{S}}$ is increasing with respect to w . If we compare (5.4) and (5.9), we obtain a contradiction to (5.8). If t_* is right-dense in \mathcal{I}_a (thus in $\widehat{\mathcal{I}}_a$), then $w_n(t_*) = v(t_*)$ and moreover $w_n(t) > v(t)$ for $t \in (t_*, t^*] \cap \mathcal{I}_a$. From (GR $^\Delta$ E) and (5.6), $w_n^\Delta(t_*) < v^\Delta(t_*)$, so there exists $\bar{t} \in (t_*, t^*] \cap \mathcal{I}_a$ such that $w_n(\bar{t}) < v(\bar{t})$, which is a contradiction to (5.8). Hence (5.7) holds. Therefore, from $w_n(t) \rightarrow w(t)$ as $n \rightarrow \infty$ for $t \in [a, d] \cap \mathcal{I}_a$, we get (5.5), and so letting $t = d$ in (5.5), we have (with the use of the validity of $\widehat{\mu}(s) = \mu(t)$ for all $t = \mathcal{T}^{-1}s$)

$$-\mu^{\alpha-1}(d)w(d) \geq -\widehat{\mu}^{\alpha-1}(d)v(d) \not\leq \widehat{r}(d) = r(d).$$

Hence (with use the fact $d = \mathcal{T}^{-1}d$), $-\mu^{\alpha-1}(\mathcal{T}^{-1}d)w(\mathcal{T}^{-1}d) \not\leq r(\mathcal{T}^{-1}d)$, which is the contradiction to assumption.

(ii) Assume that $\mathcal{T}a_1 < d \leq \mathcal{T}a_2$, then arguing as in the first part above, we see that $w(\sigma(a_1)) \leq v(\widehat{\sigma}(a_1)) = v(\widehat{\sigma}(\mathcal{T}a_1))$. Now we integrate (GR $^\Delta$ E) from $\sigma(a_1)$ to $\sigma(b_1)$ and obtain

$$w(\sigma(b_1)) - w(\sigma(a_1)) = - \int_{\sigma(a_1)}^{\sigma(b_1)} \mathcal{S}[w, r, \mu](t) \Delta t - \int_{\sigma(a_1)}^{\sigma(b_1)} p(t) \Delta t. \quad (5.10)$$

We wish to show, that the right-hand side of (5.10) is nonpositive and so the relation

$$w(\sigma(b_1)) \leq w(\sigma(a_1)) \quad (5.11)$$

holds. Due to (5.3) is it enough to show that the function \mathcal{S} is nonnegative. Under the assumption that $-\mu^{\alpha-1}(t)w(t) < r(t)$ for $t \in [a, \mathcal{T}^{-1}d] \cap \mathcal{I}_a$, which is equivalent to (5.2), it is enough to show that function \mathcal{S} satisfies

$$\mathcal{S}(w, r, \mu) \geq 0 \quad \text{for} \quad \Phi^{-1}(r) + \mu\Phi^{-1}(w) > 0. \quad (5.12)$$

The statement (3.17) is obvious if $\mu = 0$. Hence, suppose $\mu > 0$. It is not difficult to compute (similar like in case of the function $\tilde{\mathcal{S}}$) that

$$\frac{\partial \mathcal{S}(w, r, \mu)}{\partial w} = \frac{[\Phi^{-1}(r) + \mu \Phi^{-1}(w)]^\alpha - |\Phi^{-1}(r)|^\alpha}{\mu [\Phi^{-1}(r) + \mu \Phi^{-1}(w)]^\alpha}.$$

For the case $r > 0$, function $\partial \mathcal{S}(w, r, \mu)/\partial w$ is negative for $w < 0$ and positive for $w > 0$. Hence for $w = 0$ has function $\mathcal{S}(w, r, \mu)$ minimum, which is 0 for every $r > 0$, thus (5.12) holds. One can observe that function $\mathcal{S}(w, r, \mu)$ with arbitrary fixed $r < 0$ and $[\Phi^{-1}(r) + \mu \Phi^{-1}(w)] > 0$ is increasing with the respect to the variable w for $w > (2/\mu)^{\alpha-1}|r|$, decreasing for $|r| < w < (2/\mu)^{\alpha-1}|r|$ and $\mathcal{S}((2/\mu)^{\alpha-1}|r|, r, \mu)$ is positive. The statement (5.12) now follows from the continuity of \mathcal{S} . The case $r, w < 0$ is excluded due to $[\Phi^{-1}(r) + \mu \Phi^{-1}(w)] > 0$. Hence (5.12) holds and thus (5.11) holds too. Because of (5.11),

$$w(\sigma(b_1)) \leq w(\sigma(a_1)) \leq v(\widehat{\sigma}(\mathcal{T}a_1)) = v(\mathcal{T}\sigma(b_1)). \quad (5.13)$$

Now since $\mathcal{T}\sigma(b_1) = \widehat{\sigma}(\mathcal{T}a_1)$, it follows that $w(t)$ and $v(s)$ satisfy the same generalized Riccati equation for $\sigma(b_1) \leq t \leq \mathcal{T}^{-1}d$ and $\mathcal{T}\sigma(b_1) \leq s \leq d$, respectively, and also from (5.13), $w(\sigma(b_1)) \leq v(\mathcal{T}\sigma(b_1))$ holds. As before, we see that

$$-\mu^{\alpha-1}(\mathcal{T}^{-1}d)w(\mathcal{T}^{-1}d) \geq -\widehat{\mu}^{\alpha-1}(d)v(d) \not\leq \widehat{r}(d) = r(\mathcal{T}^{-1}d). \quad (5.14)$$

This implies that $-\mu^{\alpha-1}(\mathcal{T}^{-1}d)w(\mathcal{T}^{-1}d) \not\leq r(\mathcal{T}^{-1}d)$, which is the contradiction to assumption.

The proof of the induction step from i to $i + 1$ is similar and hence is omitted. \square

Theorem 5.2 (Telescoping Principle). *Under the same conditions and with the same notation of Theorem 5.1, if the telescoped equation $(\widehat{\text{HL}}^\Delta \text{E})$ is oscillatory, then $(\text{HL}^\Delta \text{E})$ is oscillatory too.*

Proof. Let $y(t)$ be a solution of $(\text{HL}^\Delta \text{E})$ with $y(a) \neq 0$ and let $x_1(s)$ be a solution of $(\widehat{\text{HL}}^\Delta \text{E})$ with $x_1(a) \neq 0$ satisfying

$$\frac{r(a)\Phi(y^\Delta(a))}{\Phi(y(a))} \leq \frac{\widehat{r}(a)\Phi(x_1^\Delta(a))}{\Phi(x_1(a))}.$$

Since $x_1(s)$ is oscillatory, there exists a smallest $d_1 > a$ in $\widehat{\mathcal{I}}_a$, which satisfies $\widehat{r}(s)x_1(s)x_1(\widehat{\sigma}(s)) > 0$ for $s \in [a, d_1) \cap \widehat{\mathcal{I}}_a$ and $\widehat{r}(d_1)x_1(d_1)(x_1(\widehat{\sigma}(d_1))) \leq 0$. By Theorem 5.1, there exists $e_1 \leq \mathcal{T}^{-1}d_1$ in \mathcal{I}_a with $r(e_1)y(e_1)(y(\sigma(e_1))) \leq 0$. Now, we will be interested in behavior of the solution $y(t)$ for $t > e_1$. Let $f_1 \in \mathcal{I}_a$ with $f_1 \geq e_1$ satisfies $y(f_1) \neq 0$. Let $x_2(s)$ be a solution of $(\widehat{\text{HL}}^\Delta \text{E})$ with $x_2(f_1) \neq 0$ satisfying

$$\frac{r(f_1)\Phi(y^\Delta(f_1))}{\Phi(y(f_1))} \leq \frac{\widehat{r}(\mathcal{T}f_1)\Phi(x_1^\Delta(\mathcal{T}f_1))}{\Phi(x_1(\mathcal{T}f_1))}.$$

Proceeding as before, we show that there exists $e_2 \in \mathcal{T}_a$ with $e_2 > e_1$ such that $r(e_2)y(e_2)y(\sigma(e_2)) \leq 0$. Continuing this process leads to the conclusion that y is oscillatory and therefore the equation (HL $^\Delta$ E) is oscillatory too. \square

This principle can be applied to get many new examples of oscillatory equations. We use a process which is the reverse of the construction in Theorem 5.2. We begin with any known oscillatory equation (HL $^\Delta$ E). We choose a sequence of numbers $s_i \in \widehat{\mathcal{T}}_a$ such that $s_i \rightarrow \infty$. Now we cut $\widehat{\mathcal{T}}_a$ at each s_i and pull the two halves of $\widehat{\mathcal{T}}_a$ apart to form a gap of arbitrary new (bounded) time scale interval. Now we define a function r on the new time scale interval and create an arbitrary function p , whose integral over the new time scale interval is nonnegative. When we do it at each s_i and relabel the so-constructed new coefficient functions by $r(t)$ and $p(t)$, then we obtain equation (HL $^\Delta$ E), which is oscillatory.

Conclusions

The central idea of this thesis was to establish a theory of regular and rapid variation on time scales and then to apply the obtained theory to an investigation of asymptotic behavior of solutions of linear and half-linear second order dynamic equations on time scale.

In Chapter 3 we established the theory which is valid on any time scale with “sufficiently” small graininess. We show that the condition on graininess is necessary in the case we want to obtain a reasonable theory. Note that in application, we prove many new asymptotic properties, particular in case of rapid variation our results are new even in linear and even in continuous and discrete case.

In Chapter 4 we study analogically q -regular variation and q -rapid variation considered on a special lattice $q^{\mathbb{N}_0}$. The situation required this case with a “bigger” graininess – important and frequently used – to be studied separately, because the asymptotic behavior of functions in these cases was different. In applications, we completed results concerning the equation $(HL)_q$ and showed a new method for the proofs convenient exactly for q -difference equations. This method can be used for examination of asymptotic behavior of many other q -difference equations and sometimes may serve to predict how their continuous counterparts look like.

Chapter 5 which is devoted to so-called “telescoping principle”, is not connected with theory of regular and rapid variation, however strong connecting tool is the half-linear dynamic equation and studying its oscillatory properties. Note that results established in this chapter are new even in the linear case.

Many opened problems mentioned in the thesis, e.g., to prove some results in theory of regular variation without additional conditions, to find a convenient representation for the rapidly varying function, to establish the necessary and sufficient conditions for all positive increasing solutions of (HL) to be regularly varying, await further examination.

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