MASARYK UNIVERSITY FACULTY OF SCIENCE

# Constructions of Almost Periodic Sequences and Functions and Homogeneous Linear Difference and Differential Systems

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# Preface

Many phenomena in nature have oscillatory character and their mathematical models have led to the introduction of certain classes of functions to describe them. Such a class form almost periodic sequences and functions with applications in theoretical mechanics, electronics, biology, celestial mechanics, astrodynamics, geophysics, and so on.

In this work we consider only a small part of the notion of almost periodicity combined with the theory of linear difference and differential equations. More precisely, we use special constructions of almost periodic sequences (Chapter 1) and functions (Chapter 3) to analyse non-almost periodic solutions of almost periodic homogeneous linear difference (Chapter 2) and differential (Chapter 4) systems respectively. Our aim is to find systems all of whose solutions can be almost periodic and prove that in any neighbourhood of such a system there exists a system which does not have an almost periodic solution other than the trivial one.

All results presented in this work are due to the author and are embodied in [160–163]. The corresponding results about almost periodic solutions was partially investigated in the diploma work of the author (see [165]). Thus they are omitted. The motivation and history of our topic are included at the beginnings of chapters; less important comments and historical notes are mentioned in footnotes (see the ends of both of parts); and used notations and basic definitions are in sections called Preliminaries. Note that definitions, theorems, lemmas, corollaries, examples, and remarks are numbered consecutively within each chapter.

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# Contents

# PART I

# CONSTRUCTIONS OF ALMOST PERIODIC SEQUENCES AND HOMOGENEOUS LINEAR DIFFERENCE SYSTEMS

# Abstracts of Part I

**Chapter 1:** We define almost periodic sequences with values in a pseudometric space $\mathcal{X}$ and we modify the Bochner definition of almost periodicity so that it remains equivalent with the Bohr definition. We present one (easily modifiable) method for constructing almost periodic sequences in $\mathcal{X}$. Using such a construction, we find almost periodic sequences with prescribed values. Then we apply the method to construct almost periodic homogeneous linear difference systems which do not have any nontrivial almost periodic solution. We treat this problem in a general setting where we suppose that entries of matrices in linear systems belong to a ring with a unit.

**Chapter 2:** We consider almost periodic homogeneous linear difference systems. We suppose that the coefficient matrices belong to a group. The goal is to find such groups that the systems having no nontrivial almost periodic solution form a dense subset of the set of all considered systems. A closer examination of the used methods reveals that the problem can be treated in such a generality that the entries of coefficient matrices are allowed to belong to any complete metric field. The concepts of transformable and strongly transformable groups of matrices are introduced and these concepts enable us to derive efficient conditions for determining what matrix groups have the required property.

# Chapter 1

# Constructions of almost periodic sequences with given properties

First of all we mention the article [63] by K. Fan which considers almost periodic sequences of elements of a metric space[1] and the article [159] by H. Tornehave about almost periodic functions of the real variable with values in a metric space. In these papers, it is shown that many theorems that are valid for complex valued sequences and functions are no longer true. For continuous functions, it was observed that the important property is the local connection by arcs of the space of values and also its completeness. However, we will not use their results or other theorems and we will define the notion of the almost periodicity of sequences in pseudometric spaces without any additional restrictions, i.e., the definition is similar to the classical definition of H. Bohr, the modulus being replaced by the distance. We also refer to [80], [117], [118], [130], [173], [176]. We add that the concept of almost periodic functions of several variables with respect to Hausdorff metrics can be found in [149] which is an extension of [54] (see also [55], [131]).

In Banach spaces, a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if any sequence of translates of $\{\varphi_k\}$ has a subsequence which converges and its convergence is uniform with respect to $k$ in the sense of the norm. In 1933, the continuous case of the previous result was proved by S. Bochner in [17], where the fundamental theorems of the theory of almost periodic functions with values in a Banach space are proved too—see, e.g., [5], [6, pp. 3–25] or [101], where the theorems have been redemonstrated by the methods of the functional analysis. We remark that the discrete version of this result can be proved similarly as in [17]. We also mention directly the papers [138] and [166].

In pseudometric spaces, the above result is not generally true. Nevertheless, by a modification of the Bochner proof of this result, we will prove that a necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be almost periodic is that any sequence of translates of $\{\varphi_k\}$ has a subsequence satisfying the Cauchy condition, uniformly with respect to $k$.

We will also analyse systems of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z} \quad (\text{or } k \in \mathbb{N}_0),$$

where $\{A_k\}$ is almost periodic. We want to prove that there exists a system of the above form which does not have an almost periodic solution other than the trivial one. (See

Theorem 1.26.) A closer examination of the methods used in constructions reveals that the problem can be treated in possibly the most general setting:

almost periodic sequences attain values in a pseudometric space;

the entries of almost periodic matrices are elements of an infinite ring with a unit.

We note that many theorems about the existence of almost periodic solutions of almost periodic difference systems of general forms are published in [16], [76], [79], [145], [168], [173], [174], [176] and several these existence theorems are proved there in terms of discrete Lyapunov functions. Here, we can also refer to the monograph [171] and [177, Theorems 3.6, 3.7, 3.8]. For linear systems with $k \in \mathbb{N}_0$, see [4], [150].

This chapter is organized as follows. Section 1.2 presents the definition of almost periodic sequences in a pseudometric space, the above necessary and sufficient condition for the almost periodicity of a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$, and some basic properties of almost periodic sequences in pseudometric spaces.

In Section 1.3, we show the way one can construct almost periodic sequences in pseudometric spaces. We remark that our process is comprehensible and easily modifiable and that methods of generating almost periodic sequences are mentioned in [123, Section 4] as well.

The goal of Section 1.4 is to find almost periodic sequences whose ranges consist of arbitrarily given sets applying a construction from the previous section. More precisely, for any totally bounded countable set $X$, it is proved the existence of an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ such that $\{\psi_k;\, k \in \mathbb{Z}\} = X$ and $\psi_k = \psi_{k+lq(k)}$, $l \in \mathbb{Z}$ for all $k$ and some $q(k) \in \mathbb{N}$ which depends on $k$.

Finally, in Section 1.5, we use results from the second and the third section to obtain a theorem which will play important role in Chapter 2, where it is proved that the almost periodic homogeneous linear difference systems which do not have any nonzero almost periodic solution form a dense subset of the set of all considered systems. Using our constructions, we will get generalizations of statements from [158] and [164], where unitary (and orthogonal) systems are studied.

## 1.1 Preliminaries

As usual, $\mathbb{R}^+$ denotes the set of all positive reals, $\mathbb{R}_0^+$ the set of all nonnegative real numbers, $\mathbb{N}_0$ the set of all natural numbers including the zero.

Let $\mathcal{X} \neq \emptyset$ be an arbitrary set and let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ have these properties:

(a) $d(x, x) = 0$ for all $x \in \mathcal{X}$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$,

(c) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{X}$.

We say that $d$ is *a pseudometric* on $\mathcal{X}$ and $(\mathcal{X}, d)$ *a pseudometric space*.

For given $\varepsilon > 0$, $x \in \mathcal{X}$, in the same way as in metric spaces, we define *the $\varepsilon$-neighbourhood* of $x$ in $\mathcal{X}$ as the set $\{y \in \mathcal{X};\, d(x, y) < \varepsilon\}$. It will be denoted by $\mathcal{O}_\varepsilon(x)$. We recall that, same as in metric spaces, the function $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$ is *continuous* in the pseudometric spaces $(\mathcal{X}_1, d_1)$ and $(\mathcal{X}_2, d_2)$ if

$$(\forall x \in \mathcal{X}_1)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in \mathcal{O}_\delta(x) \subseteq \mathcal{X}_1) \implies (d_2(\Phi(x), \Phi(y)) < \varepsilon),$$

and it is *uniformly continuous* if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathcal{X}_1)(\forall y \in \mathcal{O}_\delta(x) \subseteq \mathcal{X}_1) \implies (d_2(\Phi(x), \Phi(y)) < \varepsilon).$$

Sequences, which we will consider, will be subsets of $\mathcal{X}$. The scalar (and vector) valued sequences will be denoted by the lower-case letters, the matrix valued sequences by the capital letters ($\mathcal{X}$ is a set of matrices in this case), and each one of the scalar and matrix valued sequences by the symbols $\{\varphi_k\}$, $\{\psi_k\}$, $\{\chi_k\}$.

## 1.2 Almost periodic sequences in pseudometric spaces

Now we introduce a "natural" generalization of the almost periodicity. We remark that our approach is very general and that the theory of almost periodic sequences presented here does not distinguish between $x \in \mathcal{X}$ and $y \in \mathcal{X}$ if $d(x, y) = 0$.

### 1.2.1 The Bohr definition

**Definition 1.1.** A sequence $\{\varphi_k\}$ is called *almost periodic* if for any $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that any set consisting of $p(\varepsilon)$ consecutive integers (nonnegative integers if $k \in \mathbb{N}_0$) contains at least one integer $l$ with the property that

$$d(\varphi_{k+l}, \varphi_k) < \varepsilon, \quad k \in \mathbb{Z} \quad (\text{or } k \in \mathbb{N}_0).$$

In the above definition, $l$ is called an *$\varepsilon$-translation number* of $\{\varphi_k\}$.

Consider again $\varepsilon > 0$. Henceforward, the set of all $\varepsilon$-translation numbers of a sequence $\{\varphi_k\}$ will be denoted by $T(\{\varphi_k\}, \varepsilon)$.

**Remark 1.2.** If $\mathcal{X}$ is a Banach space ($d(x, y)$ is given by $\|x - y\|$), then a necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be almost periodic is it to be *normal*; i.e., $\{\varphi_k\}$ is almost periodic if and only if any sequence of translates of $\{\varphi_k\}$ has a subsequence, uniformly convergent for $k \in \mathbb{Z}$ in the sense of the norm. This statement and the below given Theorem 1.3 are not valid if $\{\varphi_k\}$ is defined for $k \in \mathbb{N}_0$ and if we consider only translates to the right—see the example $\mathcal{X} = \mathbb{R}$, $\varphi_0 = 1$, and $\varphi_k = 0$, $k \in \mathbb{N}$. But, if we consider translates to the left, then both of results are valid for $k \in \mathbb{N}_0$ as well.

### 1.2.2 The Bochner definition

It is seen that the result mentioned in Remark 1.2 is no longer valid if the space of values fails to be complete. Especially, in a pseudometric space $(\mathcal{X}, d)$, it is not generally true that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if it is normal. Nevertheless, applying the methods from any one of the proofs of the results [6, Statement $(\zeta)$], [39, Theorem 1.10], and [69, Theorem 1.14], one can easily prove that every normal sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic. Further, we can prove the next theorem (a generalization of the theorem called the Bochner definition) which we will need later.[2] We add that its proof is a modification of the proof of [39, Theorem 1.26].

**Theorem 1.3.** *Let $\{\varphi_k\}_{k\in\mathbb{Z}}$ be given. For an arbitrary sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$, there exists a subsequence $\{\tilde{h}_n\}_{n\in\mathbb{N}} \subseteq \{h_n\}_{n\in\mathbb{N}}$ with the Cauchy property with respect to $\{\varphi_k\}$; i.e., for any $\varepsilon > 0$, there exists $M = M(\varepsilon) \in \mathbb{N}$ for which the inequality*

$$d\big(\varphi_{k+\tilde{h}_i}, \varphi_{k+\tilde{h}_j}\big) < \varepsilon$$

*holds for all $i, j, k \in \mathbb{Z}$, $i, j > M$, if and only if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic.*

*Proof.* If any sequence of translates of $\{\varphi_k\}$ has a subsequence which has the Cauchy property, then $\{\varphi_k\}$ is almost periodic. It can be proved similarly as [39, Theorem 1.10], where it is not used that $\mathcal{X}$ is complete.[3] To prove the opposite implication, we will assume that $\{\varphi_k\}$ is almost periodic, and we will use the known method of the diagonal extraction.

Let $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ and $\vartheta > 0$ be arbitrary. By Definition 1.1, there exists a positive integer $p$ such that, in any set $\{h_n - p, h_n - p + 1, \dots, h_n\}$, there exists a $\vartheta$-translation number $l_n$. We know that $0 \le h_n - l_n \le p$ for all $n \in \mathbb{N}$. We put $k_n := h_n - l_n$, $n \in \mathbb{N}$. Clearly, $k_n = c = \text{const.}$ (a constant value from $\{0, 1, \dots, p\}$) for infinitely many values of $n$. Since

$$d\left(\varphi_{k+h_n}, \varphi_{k+k_n}\right) = d\left(\varphi_{(k+h_n-l_n)+l_n}, \varphi_{k+h_n-l_n}\right) < \vartheta, \quad k \in \mathbb{Z},$$

there exists a subsequence $\{h_n^1\}$ of $\{h_n\}$ and an integer $c_1$ such that

$$d\left(\varphi_{k+h_n^1}, \varphi_{k+c_1}\right) < \vartheta, \quad k \in \mathbb{Z},\ n \in \mathbb{N}. \tag{1.1}$$

Consider now a sequence of positive numbers $\vartheta_1 > \vartheta_2 > \cdots > \vartheta_n > \cdots$ converging to 0. We extract from the sequence $\{\varphi_{k+h_n}\}$ a subsequence $\{\varphi_{k+h_n^1}\}$ which satisfies (1.1) for $\vartheta = \vartheta_1$. From this sequence we extract a subsequence $\{\varphi_{k+h_n^2}\}$ for which an inequality analogous to (1.1) is valid. Of course, $c$ will not be the same, but will depend on the subsequence. We proceed further in the same way. Next, we form the sequence $\{\varphi_{k+h_n^n}\}_{n\in\mathbb{N}}$. Assume that $\varepsilon > 0$ is given and that we have $2\vartheta_m < \varepsilon$ for $m \in \mathbb{N}$. As a result, for $i, j > m$, $i, j \in \mathbb{N}$, we obtain

$$d\left(\varphi_{k+h_i^i}, \varphi_{k+h_j^j}\right) \le d\left(\varphi_{k+h_i^i}, \varphi_{k+c_m}\right) + d\left(\varphi_{k+c_m}, \varphi_{k+h_j^j}\right) < \varepsilon, \quad k \in \mathbb{Z},$$

where $c_m$ is the number corresponding to the sequence $\{\varphi_{k+h_n^m}\}_{n\in\mathbb{N}}$ and $\vartheta_m$. □

In Chapter 2, we will consider almost periodic sequences in complete metric spaces. Thus, we will also use the following version of the so-called Bochner definition:

**Corollary 1.4.** *An arbitrary sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ in a complete metric space is almost periodic if and only if from any sequence of the form $\{\{\varphi_{k+h_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$, where $\{h_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$, one can extract a subsequence converging uniformly for $k \in \mathbb{Z}$.*

### 1.2.3 Properties of almost periodic sequences

Note that many statements for almost periodic sequences with values in $\mathbb{C}$ extend to sequences with values in a complete metric space (or in a pseudometric space). We mention the following results which we will need later and which can be easily proved using methods from the classical theory of almost periodic functions (see [6], [39] for the classical cases and, e.g., [11] for generalizations). We also refer to [120], [176].

**Theorem 1.5.** *Let $\mathcal{X}_1, \mathcal{X}_2$ be arbitrary pseudometric spaces and $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$ be a uniformly continuous map. If $\{\varphi_k\} \subseteq \mathcal{X}_1$ is almost periodic, then the sequence $\{\Phi(\varphi_k)\}$ is almost periodic too.*

*Proof.* Taking $\varepsilon > 0$ arbitrarily, let $\delta(\varepsilon) > 0$ be the number corresponding to $\varepsilon$ from the definition of the uniform continuity of $\Phi$. Now, Theorem 1.5 follows from the fact that the set of all $\varepsilon$-translation numbers of $\{\Phi(\varphi_k)\}$ contains the set of all $\delta(\varepsilon)$-translation numbers of $\{\varphi_k\}$, i.e., from the inclusion

$$T\left(\{\varphi_k\}, \delta(\varepsilon)\right) \subseteq T\left(\{\Phi(\varphi_k)\}, \varepsilon\right).$$

$\square$

**Theorem 1.6.** *For every sequence of almost periodic sequences*

$$\{\varphi_k^1\}, \dots, \{\varphi_k^i\}, \dots,$$

*the sequence of $\lim\limits_{i\to\infty} \varphi_k^i$ is almost periodic if the convergence is uniform with respect to $k$.*

*Proof.* The proof can be easily obtained by a modification of the proof of [39, Theorem 6.4]. $\square$

**Theorem 1.7.** *Let $(\mathcal{X}, d)$ be a complete metric space. For an almost periodic sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ and an arbitrary sequence of integers $h_1, \dots, h_i, \dots,$ there exists a subsequence $\{\tilde{h}_i\}_{i\in\mathbb{N}}$ of $\{h_i\}_{i\in\mathbb{N}}$ such that*

$$\lim_{j\to\infty}\left(\lim_{i\to\infty} \varphi_{k+\tilde{h}_i-\tilde{h}_j}\right) = \varphi_k.$$

*Proof.* Since $\{\varphi_k\}$ is normal, we know that there exists $\{\bar{h}_i\}_{i\in\mathbb{N}} \subseteq \{h_i\}_{i\in\mathbb{N}}$ for which the sequence $\{\{\varphi_{k+\bar{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$ converges uniformly to an almost periodic sequence (see Theorem 1.6), denoted as $\{\psi_k\}$. Applying Corollary 1.4 again, we obtain a subsequence $\{\tilde{h}_i\}_{i\in\mathbb{N}} \subseteq \{\bar{h}_i\}_{i\in\mathbb{N}}$ with the property that the sequence $\{\{\psi_{k-\tilde{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$ is uniformly convergent. We will denote the limit as $\{\chi_k\}$.

Now we choose $\varepsilon > 0$ arbitrarily. We have

$$\varrho\left(\psi_{k-\tilde{h}_i}, \chi_k\right) < \frac{\varepsilon}{2}, \quad \varrho\left(\psi_k, \varphi_{k+\tilde{h}_j}\right) < \frac{\varepsilon}{2}, \quad k \in \mathbb{Z}$$

if $i, j > n$ $(i, j \in \mathbb{N})$ for some sufficiently large $n = n(\varepsilon) \in \mathbb{N}$. Thus, for all $k \in \mathbb{Z}$, it is true

$$\varrho\left(\varphi_k, \chi_k\right) \le \varrho\left(\varphi_k, \psi_{k-\tilde{h}_i}\right) + \varrho\left(\psi_{k-\tilde{h}_i}, \chi_k\right) < \varepsilon.$$

Because of the arbitrariness of $\varepsilon > 0$, we get the identity $\{\varphi_k\} \equiv \{\chi_k\}$. $\square$

**Remark 1.8.** It is possible to prove that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if every pair of sequences $\{h_i\}_{i\in\mathbb{N}}, \{l_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ have common subsequences $\{\tilde{h}_i\}_{i\in\mathbb{N}}, \{\tilde{l}_i\}_{i\in\mathbb{N}}$ with the property that

$$\lim_{j\to\infty}\left(\lim_{i\to\infty}\varphi_{k+\tilde{h}_i+\tilde{l}_j}\right) = \lim_{i\to\infty}\varphi_{k+\tilde{h}_i+\tilde{l}_i}, \qquad \text{pointwise for } k \in \mathbb{Z}. \tag{1.2}$$

In fact, condition (1.2) is necessary and sufficient in any one of the two modes of the convergence; i.e., in the strongest version, this condition is necessary in the uniform sense and sufficient in the pointwise sense. For almost periodic functions defined on $\mathbb{R}$ with values in $\mathbb{C}$, the above result is due to S. Bochner and it can be found in [18]. The proof from the paper can be generalized for complete metric spaces (see [124], [125]). If this necessary and sufficient condition is applied only to the case $\{l_i\} \equiv \{-h_i\}$ (as in Theorem 1.7), then one gets a different class of sequences called almost automorphic sequences—see [31], [132].

Taking $n \in \mathbb{N}$ and using Theorem 1.3 (and Remark 1.2) $n$-times, one can easily prove:

**Corollary 1.9.** *Let sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ be given. Then, the sequence $\{\psi_k\}$ which is defined by*

$$\psi_k := \varphi_j^{i+1} \quad \textit{for all considered } k,$$

*where $k = jn + i$, $j \in \mathbb{Z}$, $i \in \{0, \dots, n-1\}$, is almost periodic if and only if all sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ are almost periodic.*

**Corollary 1.10.** *Let $(\mathcal{X}_1, d_1), \dots, (\mathcal{X}_n, d_n)$ be pseudometric spaces and $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ be arbitrary sequences with values in $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. The sequence $\{\psi_k\}$, with values in $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ given by*

$$\psi_k := (\varphi_k^1, \dots, \varphi_k^n) \quad \textit{for all considered } k,$$

*is almost periodic if and only if any one of sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ is almost periodic.*

**Corollary 1.11.** *Let $\varepsilon > 0$ be arbitrary and let the sequences $\{\varphi_k^1\}_{k\in\mathbb{Z}}, \dots, \{\varphi_k^n\}_{k\in\mathbb{Z}}$ be almost periodic. Then, the set*

$$T(\{\varphi_k^1\}, \varepsilon) \cap \cdots \cap T(\{\varphi_k^n\}, \varepsilon)$$

*is relative dense in $\mathbb{Z}$.*

We remark that it is possible to use the above results to obtain more general versions of the below given Theorems 1.12, 1.16, 1.18.

## 1.3 Constructions of almost periodic sequences

Now we prove several theorems which facilitate to find almost periodic sequences having certain specific properties. In Theorem 1.12, we consider almost periodic sequences for $k \in \mathbb{N}_0$; in Theorem 1.14 and Corollary 1.15, sequences for $k \in \mathbb{Z}$ obtained from almost periodic sequences for $k \in \mathbb{N}_0$; and, in Theorems 1.16 and 1.18, sequences for $k \in \mathbb{Z}$.

**Theorem 1.12.** *Let $\varphi_0, \dots, \varphi_m \in \mathcal{X}$ and $j \in \mathbb{N}$ be arbitrarily given. Let $\{r_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of nonnegative real numbers such that*

$$\sum_{n=1}^{\infty} r_n < \infty. \tag{1.3}$$

*Then, any sequence $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$, where*

$$\begin{gathered}
\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-(m+1)}\right), \quad k \in \{m+1, \dots, 2m+1\}, \\
\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-2(m+1)}\right), \quad k \in \{2(m+1), \dots, 3(m+1)-1\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-j(m+1)}\right), \quad k \in \{j(m+1), \dots, (j+1)(m+1)-1\}, \\
\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k-(j+1)(m+1)}\right), \quad k \in \{(j+1)(m+1), \dots, 2(j+1)(m+1)-1\}, \\
\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k-2(j+1)(m+1)}\right), \quad k \in \{2(j+1)(m+1), \dots, 3(j+1)(m+1)-1\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k-j(j+1)(m+1)}\right), \quad k \in \{j(j+1)(m+1), \dots, (j+1)^2(m+1)-1\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_n}\left(\varphi_{k-(j+1)^{n-1}(m+1)}\right), \quad k \in \big\{(j+1)^{n-1}(m+1), \\
\dots, 2(j+1)^{n-1}(m+1)-1\big\}, \\
\varphi_k \in \mathcal{O}_{r_n}\left(\varphi_{k-2(j+1)^{n-1}(m+1)}\right), \quad k \in \big\{2(j+1)^{n-1}(m+1), \\
\dots, 3(j+1)^{n-1}(m+1)-1\big\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_n}\left(\varphi_{k-j(j+1)^{n-1}(m+1)}\right), \quad k \in \big\{j(j+1)^{n-1}(m+1), \\
\dots, (j+1)^{n}(m+1)-1\big\}, \\
\vdots
\end{gathered}$$

*are arbitrary too, is almost periodic.*

*Proof.* Consider an arbitrary $\varepsilon > 0$. We need to prove that the set of all $\varepsilon$-translation numbers of $\{\varphi_k\}$ is relative dense in $\mathbb{N}_0$. Using (1.3), one can find $n(\varepsilon)$ for which

$$\sum_{n=n(\varepsilon)}^{\infty} r_n < \frac{\varepsilon}{2}. \tag{1.4}$$

We see that

$$\begin{aligned}
&\varphi_{k+(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k),\\
&\varphi_{k+2(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k),\\
&\qquad\vdots\\
&\varphi_{k+j(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k)
\end{aligned} \tag{1.5}$$

if

$$0 \le k < (j+1)^{n(\varepsilon)-1}(m+1).$$

Next, from (c) and (1.5) it follows ($i \in \{(j+1)^n, \dots, (j+1)^{n+1}-1\}$, $n \in \mathbb{N}$)

$$\begin{aligned}
&\varphi_{k+(j+1)(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}}(\varphi_k),\\
&\varphi_{k+((j+1)^2-1)(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}}(\varphi_k),\\
&\qquad\vdots\\
&\varphi_{k+i(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}+\dots+r_{n(\varepsilon)+n}}(\varphi_k),\\
&\qquad\vdots
\end{aligned}$$

for $k \in \{0, \dots, (j+1)^{n(\varepsilon)-1}(m+1)-1\}$. Therefore (consider (1.4)), we have

$$\varphi_{k+l(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{\frac{\varepsilon}{2}}(\varphi_k), \quad 0 \le k < (j+1)^{n(\varepsilon)-1}(m+1),\ l \in \mathbb{N}_0. \tag{1.6}$$

We put

$$q(\varepsilon) := (j+1)^{n(\varepsilon)-1}(m+1). \tag{1.7}$$

Any $p \in \mathbb{N}_0$ can be expressed uniquely in the form

$$p = k(p) + l(p)q(\varepsilon) \quad \text{for some } k(p) \in \{0, \dots, q(\varepsilon)-1\} \text{ and } l(p) \in \mathbb{N}_0.$$

Applying (1.6), we obtain

$$\begin{aligned}
d\left(\varphi_p, \varphi_{p+lq(\varepsilon)}\right) &= d\left(\varphi_{k(p)+l(p)q(\varepsilon)}, \varphi_{k(p)+l(p)q(\varepsilon)+lq(\varepsilon)}\right)\\
&\le d\left(\varphi_{k(p)+l(p)q(\varepsilon)}, \varphi_{k(p)}\right) + d\left(\varphi_{k(p)}, \varphi_{k(p)+(l+l(p))q(\varepsilon)}\right)\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned} \tag{1.8}$$

where $p, l \in \mathbb{N}_0$ are arbitrary; i.e., $lq(\varepsilon)$ is an $\varepsilon$-translation number of $\{\varphi_k\}$ for all $l \in \mathbb{N}_0$. The fact that the set $\{lq(\varepsilon); l \in \mathbb{N}_0\}$ is relative dense in $\mathbb{N}_0$ proves the theorem. □

**Remark 1.13.** From the proof of Theorem 1.12 (see (1.7) and (1.8)), for any $\varepsilon > 0$ and any sequence $\{\varphi_k\}$ considered there, we get the existence of $n(\varepsilon) \in \mathbb{N}$ such that the set of all $\varepsilon$-translation numbers of $\{\varphi_k\}$ contains $\{l(j+1)^{n(\varepsilon)-1}(m+1); l \in \mathbb{N}\}$; i.e., we have

$$T(\{\varphi_k\}, n(\varepsilon)) := \{l(j+1)^{n(\varepsilon)-1}(m+1);\ l \in \mathbb{N}\} \subseteq T(\{\varphi_k\}, \varepsilon) \tag{1.9}$$

for every $\varepsilon > 0$.

**Theorem 1.14.** *Let $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be an almost periodic sequence and let $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ and $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be such that*

$$r_n l_n \to 0 \quad as \quad n \to \infty. \tag{1.10}$$

*If for all $n$, there exists a set $T(r_n)$ of some $r_n$-translation numbers of $\{\varphi_k\}$ which is relative dense in $\mathbb{N}_0$ and, for every nonzero $l = l(r_n) \in T(r_n)$, there exists $i = i(l) \in \{1, \dots, l_n + 1\}$ with the property that*

$$\varphi_{(i-1)l+k} \in \mathcal{O}_{r_n l_n}(\varphi_{il-k}), \quad k \in \{0, \dots, l\}, \tag{1.11}$$

*then the sequence $\{\psi_k\}_{k\in\mathbb{Z}}$, given by the formula*

$$\psi_k := \varphi_k \text{ for } k \in \mathbb{N}_0 \quad and \quad \psi_k := \varphi_{-k} \text{ for } k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{1.12}$$

*is almost periodic.*

*If for all $n$, there exists a set $\widetilde{T}(r_n)$ of some $r_n$-translation numbers of $\{\varphi_k\}$ which is relative dense in $\mathbb{N}_0$ and, for every nonzero $m = m(r_n) \in \widetilde{T}(r_n)$, there exists $i = i(m) \in \{1, \dots, l_n + 1\}$ with the property that*

$$\varphi_{(i-1)m+k} \in \mathcal{O}_{r_n l_n}(\varphi_{im-k-1}), \quad k \in \{0, \dots, m-1\}, \tag{1.13}$$

*then the sequence $\{\chi_k\}_{k\in\mathbb{Z}}$, given by the formula*

$$\chi_k := \varphi_k \text{ for } k \in \mathbb{N}_0 \quad and \quad \chi_k := \varphi_{-(k+1)} \text{ for } k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{1.14}$$

*is almost periodic.*

*Proof.* We will prove only the first part of Theorem 1.14. The proof of the second case (the almost periodicity of $\{\chi_k\}$) is analogic. Let $\varepsilon > 0$ be arbitrarily small. Consider $n \in \mathbb{N}$ satisfying (see (1.10))

$$r_n l_n < \frac{\varepsilon}{3}. \tag{1.15}$$

We will prove that the set $T(\{\psi_k\}, \varepsilon)$ of all $\varepsilon$-translation numbers of $\{\psi_k\}$ contains the numbers $\{\pm l; l \in T(r_n)\}$; i.e., we will get the inequality

$$d(\psi_k, \psi_{k\pm l}) < \varepsilon, \quad l \in T(r_n),\ k \in \mathbb{Z} \tag{1.16}$$

which proves the theorem because $\{\pm l;\ l \in T(r_n)\}$ is relative dense in $\mathbb{Z}$.

First of all we choose arbitrary $l \in T(r_n)$. From the theorem, we have $i = i(l)$. Without loss of the generality, we can consider only $+l$. (For $-l$, we can proceed similarly.) Because of $l_n \in \mathbb{N}$ and $l \in T(r_n)$, from (1.12) and (1.15) it follows

$$d(\psi_k, \psi_{k+l}) < \frac{\varepsilon}{3}, \quad k \notin \{-l, \dots, -1\},\ k \in \mathbb{Z}. \tag{1.17}$$

Let $k \in \{-l, \dots, -1\}$ be also arbitrarily chosen. Evidently, we have

$$k + (1-i)l \in \{-il, \dots, -(i-1)l - 1\}$$

and

$$\begin{aligned} d(\psi_k, \psi_{k+l}) &\le d\left(\psi_k, \psi_{k+(1-i)l}\right) + d\left(\psi_{k+(1-i)l}, \psi_{k+l}\right) \\ &= d\left(\varphi_{-k}, \varphi_{(i-1)l-k}\right) + d\left(\varphi_{(i-1)l-k}, \varphi_{l+k}\right). \end{aligned} \tag{1.18}$$

The number $(i-1)l$ is an $(\varepsilon/3)$-translation number of $\{\varphi_k\}$. It follows from (c), (1.15), and from $i \le l_n + 1$. Therefore, we have

$$d\left(\varphi_{-k}, \varphi_{(i-1)l-k}\right) < \frac{\varepsilon}{3}. \tag{1.19}$$

Using (1.11) and (1.15), we get

$$d\left(\varphi_{(i-1)l-k}, \varphi_{il+k}\right) < r_n l_n < \frac{\varepsilon}{3}.$$

Thus, it holds

$$d\left(\varphi_{(i-1)l-k}, \varphi_{l+k}\right) < \frac{2\varepsilon}{3}. \tag{1.20}$$

Indeed, $(i-1)l$ is an $(\varepsilon/3)$-translation number of $\{\varphi_k\}$ (consider again (c), (1.15), and the inequality $i-1 \le l_n$).

Altogether, from (1.18), (1.19), and (1.20), we obtain

$$d\left(\psi_k, \psi_{k+l}\right) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon. \tag{1.21}$$

Since the choice of $k$, $l$ was arbitrary (see (1.17)), (1.21) gives (1.16). □

**Corollary 1.15.** *Let $m \in \mathbb{N}_0$, $j \in \mathbb{N}$, and the sequence $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be from Theorem 1.12 and $M > 0$ be arbitrary. If for all $n > M$, $n \in \mathbb{N}$, there exists at least one $i \in \{1, \dots, j\}$ satisfying*

$$\varphi_{i(j+1)^n(m+1)+k} = \varphi_{(i+1)(j+1)^n(m+1)-k}, \quad k \in \{0, \dots, (j+1)^n(m+1)\}, \tag{1.22}$$

*then the sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ given by (1.12) is almost periodic. If for all $n > M$, $n \in \mathbb{N}$, there exists at least one $i \in \{1, \dots, j\}$ satisfying*

$$\varphi_{i(j+1)^n(m+1)+k} = \varphi_{(i+1)(j+1)^n(m+1)-k-1}, \quad k \in \{0, \dots, (j+1)^n(m+1)-1\}, \tag{1.23}$$

*then the sequence $\{\chi_k\}_{k\in\mathbb{Z}}$ given by (1.14) is almost periodic.*

*Proof.* We put

$$r_n := \frac{1}{n}, \quad l_n := 1, \quad T(r_n) := T\left(\{\varphi_k\}, n\left(\frac{r_n}{2}\right)\right) \quad \text{for all } n \in \mathbb{N},$$

where $T\left(\{\varphi_k\}, n(\varepsilon)\right)$ is defined by (1.9). Since we can assume that $n\,(1/2) > M-1$, it suffices to consider Theorem 1.14 and Remark 1.13 (from (c), using (1.22) and (1.23), we get (1.11) and (1.13), respectively). □

**Theorem 1.16.** *Let $\varphi_0, \dots, \varphi_n \in \mathcal{X}$ and $j \in \mathbb{N}$ be given and $\{r_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ be arbitrary so that*

$$\sum_{i=1}^{\infty} r_i < \infty \tag{1.24}$$

*holds. Then, every sequence $\{\varphi_k\}$ for which it is true*

$$\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-(n+1)}\right), \quad k \in \{n+1, \dots, 2(n+1)-1\},$$

$$\vdots$$

$$\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-j(n+1)}\right), \quad k \in \{j(n+1), \dots, (j+1)(n+1)-1\},$$

$$\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k+(j+1)(n+1)}\right), \quad k \in \{-(j+1)(n+1), \dots, -1\},$$

$$\vdots$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k+j(j+1)(n+1)}\right),\\ &\quad k \in \{-j(j+1)(n+1), \dots, -(j-1)(j+1)(n+1)-1\},\end{aligned}$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_3}\left(\varphi_{k-(j+1)^2(n+1)}\right),\\ &\quad k \in \{(j+1)(n+1), \dots, (j+1)(n+1)+(j+1)^2(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_3}\left(\varphi_{k-j(j+1)^2(n+1)}\right),\\ &\quad k \in \{(j+1)(n+1)+(j-1)(j+1)^2(n+1), \dots,\\ &\qquad (j+1)(n+1)+j(j+1)^2(n+1)-1\},\end{aligned}$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_4}\left(\varphi_{k+(j+1)^3(n+1)}\right),\\ &\quad k \in \{-(j+1)^3(n+1)-j(j+1)(n+1), \dots, -j(j+1)(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_4}\left(\varphi_{k+j(j+1)^3(n+1)}\right),\\ &\quad k \in \{-j(j+1)^3(n+1)-j(j+1)(n+1), \dots,\\ &\qquad -(j-1)(j+1)^3(n+1)-j(j+1)(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_{2i}}\left(\varphi_{k+(j+1)^{2i-1}(n+1)}\right),\\ &\quad k \in \{-((j+1)^{2i-1}+\dots+j(j+1)^3+j(j+1))(n+1), \dots,\\ &\qquad -(j(j+1)^{2i-3}+\dots+j(j+1)^3+j(j+1))(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_{2i}}\left(\varphi_{k+j(j+1)^{2i-1}(n+1)}\right),\\ &\quad k \in \{-(j(j+1)^{2i-1}+\dots+j(j+1)^3+j(j+1))(n+1), \dots,\\ &\qquad -((j-1)(j+1)^{2i-1}+\dots+j(j+1)^3+j(j+1))(n+1)-1\},\end{aligned}$$

$$\begin{aligned}&\varphi_k \in \mathcal{O}_{r_{2i+1}}\left(\varphi_{k-(j+1)^{2i}(n+1)}\right),\\ &\quad k \in \{(j+1)(n+1)+j(j+1)^2(n+1)+\dots+j(j+1)^{2i-2}(n+1), \dots,\\ &\qquad (j+1)(n+1)+j(j+1)^2(n+1)+\cdots\\ &\qquad\quad +j(j+1)^{2i-2}(n+1)+(j+1)^{2i}(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned} &\varphi_k \in \mathcal{O}_{r_{2i+1}}\left(\varphi_{k-j(j+1)^{2i}(n+1)}\right), \\ &\quad k \in \{(j+1)(n+1) + j(j+1)^2(n+1) + \cdots \\ &\qquad + j(j+1)^{2i-2}(n+1) + (j-1)(j+1)^{2i}(n+1), \dots, \\ &\qquad\quad (j+1)(n+1) + j(j+1)^2(n+1) + \cdots + j(j+1)^{2i}(n+1) - 1\}, \\ &\qquad\qquad \vdots \end{aligned}$$

*is almost periodic.*

*Proof.* Let $\varepsilon > 0$ be arbitrarily given and let the number $i(\varepsilon) \in \mathbb{N}$ satisfy the condition (see (1.24))

$$\sum_{i=i(\varepsilon)}^{\infty} r_i < \frac{\varepsilon}{2}.$$

One can show

$$\left\{l(j+1)^{i(\varepsilon)-1}(n+1);\, l \in \mathbb{Z}\right\} \subseteq T\left(\{\varphi_k\}, \varepsilon\right).$$

The fact that the above set is relative dense in $\mathbb{Z}$ proves the theorem. □

For $n = 0$, $j = 1$, we get the most important case of Theorem 1.16:

**Corollary 1.17.** *Let $\psi_0 \in \mathcal{X}$ and $\{\varepsilon_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_0^+$ satisfying*

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty \tag{1.25}$$

*be arbitrary. Then, every sequence $\{\psi_k\}_{k \in \mathbb{Z}}$ for which it is valid*

$$\begin{aligned} \psi_k &\in \mathcal{O}_{\varepsilon_1}\left(\psi_{k-2^0}\right), \quad k \in \{1\} = \{2-1\}, \\ \psi_k &\in \mathcal{O}_{\varepsilon_2}\left(\psi_{k+2^1}\right), \quad k \in \{-2, -1\}, \\ \psi_k &\in \mathcal{O}_{\varepsilon_3}\left(\psi_{k-2^2}\right), \quad k \in \{2, \dots, 2+2^2-1\}, \\ \psi_k &\in \mathcal{O}_{\varepsilon_4}\left(\psi_{k+2^3}\right), \quad k \in \{-2^3-2, \dots, -2-1\}, \\ \psi_k &\in \mathcal{O}_{\varepsilon_5}\left(\psi_{k-2^4}\right), \quad k \in \{2+2^2, \dots, 2+2^2+2^4-1\}, \\ &\vdots \\ \psi_k &\in \mathcal{O}_{\varepsilon_{2i}}\left(\psi_{k+2^{2i-1}}\right), \quad k \in \{-2^{2i-1} - \cdots - 2^3 - 2, \cdots, -2^{2i-3} - \cdots - 2^3 - 2 - 1\}, \\ \psi_k &\in \mathcal{O}_{\varepsilon_{2i+1}}\left(\psi_{k-2^{2i}}\right), \quad k \in \{2+2^2+\cdots+2^{2i-2}, \dots, 2+2^2+\cdots+2^{2i-2}+2^{2i}-1\}, \\ &\vdots \end{aligned}$$

*is almost periodic.*

**Theorem 1.18.** *Let $\varphi_0, \dots, \varphi_m \in \mathcal{X}$ be given, $\{r_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$, $\{j_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, and $n \in \mathbb{N}_0$ be arbitrary such that $m+n$ is even and*

$$\sum_{i=1}^{\infty} r_i\, j_i < \infty. \tag{1.26}$$

*For any $\varphi_{m+1}, \dots, \varphi_{m+n}$, if we set*

$$\psi_k := \varphi_{k+\frac{m+n}{2}}, \quad k \in \left\{-\frac{m+n}{2}, \dots, \frac{m+n}{2}\right\},$$

$$M := \frac{m+n}{2}, \qquad N := m+n$$

*and we choose arbitrarily*

$$\begin{gathered}
\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k+N+1}\right), \quad k \in \{-N-M-1, \dots, -M-1\},\\
\vdots\\
\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k+N+1}\right), \quad k \in \{-j_1N-M-1, \dots, -(j_1-1)N-M-1\},\\
\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k-N-1}\right), \quad k \in \{M+1, \dots, N+M+1\},\\
\vdots\\
\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k-N-1}\right), \quad k \in \{(j_1-1)N+M+1, \dots, j_1N+M+1\},\\
\vdots\\
\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k+p_i}\right), \quad k \in \{-p_i-p_{i-1}-\dots-p_1, \dots, -p_{i-1}-\dots-p_1\},\\
\vdots\\
\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k+p_i}\right), \quad k \in \{-j_ip_i-p_{i-1}-\dots-p_1, \dots, -(j_i-1)p_i-p_{i-1}-\dots-p_1\},\\
\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k-p_i}\right), \quad k \in \{p_{i-1}+\dots+p_1, \dots, p_i+p_{i-1}+\dots+p_1\},\\
\vdots\\
\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k-p_i}\right), \quad k \in \{(j_i-1)p_i+p_{i-1}+\dots+p_1, \dots, j_ip_i+p_{i-1}+\dots+p_1\},\\
\vdots
\end{gathered}$$

*where*

$$\begin{gathered}
p_1 := (j_1N+M+1)+1, \qquad p_2 := 2(j_1N+M+1)+1,\\
p_3 := (2j_2+1)p_2, \quad \dots \quad p_i := (2j_{i-1}+1)p_{i-1}, \quad \dots,
\end{gathered}$$

*then the resulting sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ is almost periodic.*

*Proof.* Consider arbitrary $\varepsilon > 0$ and a positive integer $n(\varepsilon) \geq 2$ for which (see (1.26))

$$\sum_{i=n(\varepsilon)}^{\infty} r_i\, j_i < \frac{\varepsilon}{4}.$$

One can show that

$$\{lp_{n(\varepsilon)};\, l \in \mathbb{Z}\} \subseteq T(\{\psi_k\}, \varepsilon)$$

which completes the proof. ∎

## 1.4 Almost periodic sequences with given values

Now we will construct almost periodic sequences whose ranges consist of arbitrarily given sets satisfying only necessary conditions. We are motivated by the paper [70] where a similar problem is investigated for real valued sequences. In that paper, using an explicit construction, it is shown that, for any bounded countable set of real numbers, there exists an almost periodic sequence whose range is this set and which attains each value in this set periodically.[4] We will extend this result to sequences attaining values in $\mathcal{X}$.

Concerning almost periodic sequences with indices $k \in \mathbb{N}$ (or asymptotically almost periodic sequences), we refer to [92] where it is proved that, for any precompact sequence $\{x_k\}_{k\in\mathbb{N}}$ in a metric space $\mathcal{X}$, there exists a permutation $P$ of the set of positive integers such that the sequence $\{x_{P(k)}\}_{k\in\mathbb{N}}$ is almost periodic. Let us point out that the definition of *the asymptotic almost periodicity* in [92] is based on the Bochner concept; i.e., a bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathcal{X}$ is called almost periodic if the set of sequences $\{x_{k+p}\}_{k\in\mathbb{N}}$, $p \in \mathbb{N}$, is precompact in the space of all bounded sequences in $\mathcal{X}$. It is known that, for sequences with values in complete metric spaces, the Bochner definition is equivalent with the Bohr definition which we prefer. Moreover, we know that these definitions remain also equivalent in an arbitrary pseudometric space if one replaces the convergence in the Bochner definition by the Cauchy property (see Theorem 1.3). But, it is seen that the result of [92] for the almost periodicity on $\mathbb{N}$ cannot be true for the almost periodicity on $\mathbb{Z}$ or $\mathbb{R}$ (see also Remark 1.2).

In a Banach space, an other important necessary and sufficient condition for a function to be almost periodic is that it has *the approximation property*; i.e., a function is almost periodic if and only if there exists a sequence of trigonometric polynomials which converges uniformly to the function on the whole real line in the norm topology (see [39, Theorems 6.8, 6.15]). There exist generalizations of this result (see [33], [159]). For example, it is proved in [11] that an almost periodic function with fuzzy real numbers as values can be uniformly approximated by a sequence of generalized trigonometric polynomials. We add that fuzzy real numbers form a complete metric space. One shows that the approximation theorem remains generally unvalid if one does not require the completeness of the space of values. Thus, we cannot use this idea in our constructions for general pseudometric spaces.

We prove that, for a countable subset of $\mathcal{X}$, there exists an almost periodic sequence whose range is exactly this set. Since the range of any almost periodic sequence is totally bounded, this requirement on the set is necessary. Now we prove that the condition is sufficient as well.

**Theorem 1.19.** *Let any countable and totally bounded set $X \subseteq \mathcal{X}$ be given. There exists an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying*

$$\{\psi_k;\ k \in \mathbb{Z}\} = X \tag{1.27}$$

*with the property that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ such that*

$$\psi_l = \psi_{l+jq(l)}, \quad j \in \mathbb{Z}. \tag{1.28}$$

*Proof.* Let us put

$$X = \{\varphi_k;\ k \in \mathbb{N}\}.$$

Without loss of the generality we can assume that the set $\{\varphi_k;\ k \in \mathbb{N}\}$ is infinite because, for only finitely many different $\varphi_k$, we can define $\{\psi_k\}$ as periodic. Since $\{\varphi_k;\ k \in \mathbb{N}\}$ is totally bounded, for any $\varepsilon > 0$, it can be imbedded into a finite number of spheres of radius $\varepsilon$. Let us denote by $x_1^i, \dots, x_{m(i)}^i$ the centres of the spheres of radius $2^{-i}$ which cover the set for all $i \in \mathbb{N}$. Evidently, we can also assume that

$$x_1^i, \dots, x_{m(i)}^i \in \{\varphi_k;\ k \in \mathbb{N}\}, \quad i \in \mathbb{N}, \tag{1.29}$$

and that

$$x_1^i = \varphi_i, \quad i \in \mathbb{N}. \tag{1.30}$$

We will construct $\{\psi_k\}$ applying Corollary 1.17. We choose arbitrary $n(1) \in \mathbb{N}$ for which $2^{2n(1)} > m(1)$. We put

$$\psi_0 := x_1^1,\ \psi_1 := x_2^1,\ \dots,\ \psi_{m(1)-1} := x_{m(1)}^1,$$

$$\psi_k := x_1^1,\ k \in \{-2^{2n(1)-1} - \dots - 2^3 - 2, \dots, -1\} \cup \{m(1), \dots, 2 + 2^2 + \dots + 2^{2n(1)} - 1\}$$

and

$$\varepsilon_k := L, \quad k \in \{1, \dots, 2n(1) + 1\}, \tag{1.31}$$

where

$$L := \max_{i,j \in \{1,\dots,m(1)\}} d\left(x_i^1, x_j^1\right) + 1.$$

In the second step, we choose $n(2) > n(1) + m(2)$ ($n(2) \in \mathbb{N}$). We define

$$\psi_k := \psi_{k+2^{2n(1)+1}}, \quad k \in \{-2^{2n(1)+1} - \dots - 2^3 - 2, \cdots, -2^{2n(1)-1} - \dots - 2^3 - 2 - 1\},$$

$$\psi_k := \psi_{k-2^{2n(1)+2}}, \quad k \in \{2 + 2^2 + \dots + 2^{2n(1)}, \dots, 2 + 2^2 + \dots + 2^{2n(1)+2} - 1\},$$

$$\vdots$$

$$\psi_k := \psi_{k+2^{2n(2)-1}}, \quad k \in \{-2^{2n(2)-1} - \dots - 2^3 - 2, \cdots, -2^{2n(2)-3} - \dots - 2^3 - 2 - 1\}$$

and we put

$$\varepsilon_k := 0,\ k \in \{2n(1) + 2, \dots, 2n(2)\}, \quad \varepsilon_{2n(2)+1} := 2^{-1}. \tag{1.32}$$

Since $n(2) > n(1) + m(2)$, from the above definition of $\psi_k$, it follows that, for each $j \in \{1, \dots, m(1)\}$, there exist at least $2m(2) + 2$ integers

$$l \in \{-2^{2n(2)-1} - \dots - 2^3 - 2, \dots, 2^{2n(2)-2} + \dots + 2^2 + 2 - 1\}$$

such that $\psi_l = x_j^1$. Thus, we can define

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(2)+1}}\left(\psi_{k-2^{2n(2)}}\right),\ k \in \{2 + 2^2 + \dots + 2^{2n(2)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(2)} - 1\}$$

with the property that

$$\{\psi_k;\ k \in \{2 + 2^2 + \dots + 2^{2n(2)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(2)-2} + 2^{2n(2)} - 1\}\} =$$

$$= \{x_1^1, \dots, x_{m(1)}^1, x_1^2, \dots, x_{m(2)}^2\}.$$

In addition, we can put
$$\psi_{2^{2n(2)}} := \psi_0 = x_1^1 \tag{1.33}$$
and we can assume that
$$\psi_k = x_1^1 \text{ for some } k \in \{2 + \dots + 2^{2n(2)-2}, \dots, 2 + \dots + 2^{2n(2)} - 1\} \setminus \{2^{2n(2)}\}.$$

In the third step, we choose $n(3) > n(2)+m(3)$ ($n(3) \in \mathbb{N}$) and we proceed analogously. We construct $\{\psi_k\}$ for
$$k \in \{-2^{2n(2)+1} - \dots - 2^3 - 2, \dots, -2^{2n(2)-1} - \dots - 2^3 - 2 - 1\},$$
$$\vdots$$
$$k \in \{-2^{2n(3)-1} - \dots - 2^3 - 2, \dots, -2^{2n(3)-3} - \dots - 2^3 - 2 - 1\}$$
as in the $2(n(2)+1)$-th, …, $2n(3)$-th steps of the process (mentioned in Corollary 1.17) for
$$\varepsilon_k := 0, \quad k \in \{2n(2)+2, \dots, 2n(3)\}. \tag{1.34}$$
Especially, we have
$$\begin{aligned} &\psi_k = x_1^1, \quad k \in J_0^3, \\ &J_0^3 := \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{-2^{2n(3)-1} - \dots - 2, \dots, 2 + \dots + 2^{2n(3)-2} - 1\}. \end{aligned} \tag{1.35}$$

As in the second step, for all $j(1) \in \{1,2\}$ and $j(2) \in \{1, \dots, m(j(1))\}$, there exist at least $2m(3)+2$ integers
$$l \in \{-2^{2n(3)-1} - \dots - 2^3 - 2, \dots, 2 + 2^2 + \dots + 2^{2n(3)-2} - 1\} \setminus \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\}$$
such that $\psi_l = x_{j(2)}^{j(1)}$. It is seen that, to get
$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(3)+1}}\left(\psi_{k-2^{2n(3)}}\right), \ k \in \{2 + 2^2 + \dots + 2^{2n(3)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(3)} - 1\},$$
where
$$\varepsilon_{2n(3)+1} := 2^{-2}, \tag{1.36}$$
satisfying
$$\begin{aligned} \{\psi_k;\ k \in \{2 + 2^2 + \dots + 2^{2n(3)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(3)-2} + 2^{2n(3)} - 1\}\} = \\ = \{x_1^1, \dots, x_{m(1)}^1, \dots, x_1^3, \dots, x_{m(3)}^3\}, \end{aligned}$$
we need less than (or equal to) $m(3)+1$ such integers $l$. Thus, we can define these $\psi_k$ so that
$$\begin{aligned} &\psi_k = x_1^1, \quad k \in I_0^3, \\ &I_0^3 := \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(3)-2}, \dots, 2 + \dots + 2^{2n(3)} - 1\}, \end{aligned} \tag{1.37}$$
$$\psi_{2^{2n(3)}+1} = \psi_1 = x_2^1, \quad \psi_{2^{2n(3)}-1} = \psi_{-1} = x_1^1, \tag{1.38}$$
$$\psi_k = \psi_1 \text{ for some } k \in \{2 + \dots + 2^{2n(3)-2}, \dots, 2 + \dots + 2^{2n(3)} - 1\} \setminus \{2^{2n(3)} + 1\},$$

$$\psi_k = \psi_{-1} \text{ for some } k \in \{2 + \cdots + 2^{2n(3)-2}, \dots, 2 + \cdots + 2^{2n(3)} - 1\} \setminus \{2^{2n(3)} - 1\}.$$

We proceed further in the same way. In the $i$-th step, we have $n(i) > n(i-1) + m(i)$ ($n(i) \in \mathbb{N}$) and

$$\psi_k := \psi_{k+2^{2n(i-1)+1}},\ k \in \{-2^{2n(i-1)+1} - \cdots - 2, \cdots, -2^{2n(i-1)-1} - \cdots - 2 - 1\},$$

$$\vdots$$

$$\psi_k := \psi_{k+2^{2n(i)-1}},\ k \in \{-2^{2n(i)-1} - \cdots - 2, \cdots, -2^{2n(i)-3} - \cdots - 2 - 1\}$$

and we denote

$$\varepsilon_k := 0,\ k \in \{2n(i-1)+2, \dots, 2n(i)\}, \quad \varepsilon_{2n(i)+1} := 2^{-i+1}. \tag{1.39}$$

We have also

$$\begin{aligned} &\psi_k = \psi_0, \quad k \in J_0^i, \\ &J_0^i := \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{-2^{2n(i)-1} - \cdots - 2, \dots, 2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned} \tag{1.40}$$

$$\begin{aligned} &\psi_k = \psi_1, \quad k \in J_1^i, \\ &J_1^i := \{1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\} \cap \{-2^{2n(i)-1} - \cdots - 2, \dots, 2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned} \tag{1.41}$$

$$\begin{aligned} &\psi_k = \psi_{-1}, \quad k \in J_{-1}^i, \\ &J_{-1}^i := \{-1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned} \tag{1.42}$$

$$\vdots$$

$$\begin{aligned} &\psi_k = \psi_{i-3}, \quad k \in J_{i-3}^i, \\ &J_{i-3}^i := \{i - 3 + j\, 2^{2n(i-1)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap\ \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \\ &\psi_k = \psi_{-i+3}, \quad k \in J_{-i+3}^i, \\ &J_{-i+3}^i := \{-i + 3 + j\, 2^{2n(i-1)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap\ \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\} \end{aligned}$$

if $i - 3 < 2^{2n(2)}$. If $2^{2n(2)} \le i - 3 < 2^{2n(2)+1}$, we have

$$\vdots$$

$$\begin{aligned} &\psi_k = \psi_{-2^{2n(2)}+1}, \quad k \in J_{-2^{2n(2)}+1}^i, \\ &J_{-2^{2n(2)}+1}^i := \{-2^{2n(2)} + 1 + j\, 2^{2n(2^{2n(2)}+1)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap\ \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \\ &\psi_k = \psi_{2^{2n(2)}+1}, \quad k \in J_{2^{2n(2)}+1}^i, \\ &J_{2^{2n(2)}+1}^i := \{2^{2n(2)} + 1 + j\, 2^{2n(2^{2n(2)}+2)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap\ \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned}$$

$$\vdots$$

If $2^{2n(2)+1} \leq i-3$, then we omit the values $\psi_{j\,2^{2n(2)}}, \psi_{1+j\,2^{2n(3)}}, \psi_{-1+j\,2^{2n(3)}}, \dots$ For simplicity, let $i-2 < 2^{2n(2)}$.

Considering the construction, for all $j(1) \in \{1, \dots, i-1\}$, $j(2) \in \{1, \dots, m(j(1))\}$, there exist at least $2m(i)+2$ integers

$$l \in \{-2^{2n(i)-1} - \dots - 2^3 - 2, \dots, 2 + 2^2 + \dots + 2^{2n(i)-2} - 1\} \setminus (J_0^i \cup \dots \cup J_{-i+3}^i)$$

such that $\psi_l = x_{j(2)}^{j(1)}$. Evidently (similarly as in the third step), we can obtain

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(i)+1}}(\psi_{k-2^{2n(i)}}),\ k \in \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\}$$

for which

$$\begin{aligned}\{\psi_k;\ k \in \{2 + \dots + 2^{2n(i)-2}, \dots, &2 + \dots + 2^{2n(i)-2} + 2^{2n(i)} - 1\}\} \\ &= \{x_1^1, \dots, x_{m(1)}^1, \dots, x_1^i, \dots, x_{m(i)}^i\},\end{aligned} \tag{1.43}$$

and, in addition, we have

$$\begin{aligned}&\psi_k = \psi_0, \quad k \in I_0^i, \\ &I_0^i := \{j\,2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\},\end{aligned} \tag{1.44}$$

$$\begin{aligned}&\psi_k = \psi_1, \quad k \in I_1^i, \\ &I_1^i := \{1 + j\,2^{2n(3)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\},\end{aligned} \tag{1.45}$$

$$\begin{aligned}&\psi_k = \psi_{-1}, \quad k \in I_{-1}^i, \\ &I_{-1}^i := \{-1 + j\,2^{2n(3)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\},\end{aligned} \tag{1.46}$$

$$\vdots$$

$$\begin{aligned}&\psi_k = \psi_{i-3}, \quad k \in I_{i-3}^i, \\ &I_{i-3}^i := \{i - 3 + j\,2^{2n(i-1)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\}, \\ &\psi_k = \psi_{-i+3}, \quad k \in I_{-i+3}^i, \\ &I_{-i+3}^i := \{-i + 3 + j\,2^{2n(i-1)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\},\end{aligned}$$

and

$$\psi_k = \psi_{i-2},\ k = 2^{2n(i)} + i - 2, \quad \psi_k = \psi_{-i+2},\ k = 2^{2n(i)} - i + 2,$$

$$\psi_k = \psi_{i-2} \text{ for some } k \in \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\} \setminus \{2^{2n(i)} + i - 2\},$$

$$\psi_k = \psi_{-i+2} \text{ for some } k \in \{2 + \dots + 2^{2n(i)-2}, \dots, 2 + \dots + 2^{2n(i)} - 1\} \setminus \{2^{2n(i)} - i + 2\}.$$

Using this construction, we get the sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ with the property that (see (1.33), (1.35), (1.37), (1.40), (1.44))

$$\psi_k = \psi_0, \quad k \in \{j\,2^{2n(2)};\ j \in \mathbb{Z}\}$$

and that (see (1.38), (1.41), (1.42), (1.45), (1.46))

$$\psi_k = \psi_1, \quad k \in \{1 + j\,2^{2n(3)};\ j \in \mathbb{Z}\},$$

$$\psi_k = \psi_{-1}, \quad k \in \{-1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\},$$

and so on; i.e., for any $l \in \mathbb{Z}$, there exists $i(l) \in \mathbb{N}$ satisfying

$$\psi_k = \psi_l, \quad k \in \{l + j\, 2^{2n(i(l))};\ j \in \mathbb{Z}\}. \tag{1.47}$$

Now it suffices to show that the sequence $\{\psi_k\}$ is almost periodic. Indeed, (1.27) follows from the process, (1.29), (1.30), and (1.43); (1.28) follows from (1.47) for $q(l) = 2^{2n(i(l))}$. Since we construct $\{\psi_k\}$ using Corollary 1.17, $\{\psi_k\}$ is almost periodic if (1.25) is satisfied. Immediately, see (1.31), (1.32), (1.34), (1.36), (1.39), we have

$$\sum_{i=1}^{\infty} \varepsilon_i = L\,(2n(1) + 1) + 1 \tag{1.48}$$

which completes the proof. □

## 1.5 An application

Let $m \in \mathbb{N}$ be arbitrarily given. We will analyse almost periodic systems of $m$ homogeneous linear difference equations of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0), \tag{1.49}$$

where $\{A_k\}$ is almost periodic. Let $\mathfrak{X}$ denote *the set of all systems* (1.49).

An important characteristic property of linear difference systems, which makes them simple to treat, is the well-known superposition principle (see [1], [99], [106]). In particular, since we study homogeneous systems, we obtain that every solution of a system $\mathfrak{S} \in \mathfrak{X}$ can be expressed as a *right* linear combination of $m$ solutions of $\mathfrak{S}$; i.e., any solution $\{x_k\}$ of $\mathfrak{S}$ can be written as

$$x_k = P_k \cdot x_l, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0) \tag{1.50}$$

for some matrix valued sequence $\{P_k\}$ and some $l \in \mathbb{Z}$ ($l \in \mathbb{N}_0$). Conversely, for any considered $r_1, \dots, r_m$, the sequence $\{x_k\}$ defined by the formula

$$x_k := P_k \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0)$$

is a solution of $\mathfrak{S}$. For given $\mathfrak{S} \in \mathfrak{X}$ determined by $\{A_k\}$, the sequence $\{P_k\}$ is called *the principal fundamental matrix* if $P_0$ is the identity matrix. We get immediately

$$\begin{aligned} P_k &= \prod_{i=0}^{k-1} A_{k-i-1} && \text{for all } k > 0; \\ P_k &= \prod_{i=k}^{-1} A_i^{-1} && \text{for all } k < 0 \text{ if } k \in \mathbb{Z}. \end{aligned} \tag{1.51}$$

Our aim is to study the existence of a system $\widehat{\mathfrak{S}} \in \mathfrak{X}$ which does not have any nontrivial almost periodic solutions. We are going to treat this problem in a very general setting and this motivates our requirements on the set of values of matrices $A_k$.

We need the set of entries of $A_k$ to be a subset of a set $R$ with two operations and unit elements such that $R$ with them is a ring because the multiplication of matrices $A_k$ has to be associative (consider the natural expression of solutions of (1.49), i.e., consider (1.50) and (1.51)). We also need the set of all considered $A_k$ to form a set $X$ which has the below given properties (1.55), and we need that there exists at least one of the below mentioned functions $F_1$, $F_2 : [-1, 1] \to X$, see (1.56), (1.57), respectively. The conditions (1.56) are common, natural, and simple. However, the main theorem of this chapter (the existence of the above system $\widehat{\mathfrak{S}} \in \mathfrak{X}$) is true for many subsets of the set of all unitary or orthogonal matrices which contain of matrices that have eigenvalue $\lambda = 1$. Thus, we will also consider the existence of $F_2$.

We remark that it is possible to obtain results about the nonexistence of nontrivial almost periodic solutions using different methods than those presented here. For example, if the zero solution of a system $\mathfrak{S}$ of the form (1.49) is asymptotically (or even exponentially) stable, then it is obviously that we can choose $\widehat{\mathfrak{S}} := \mathfrak{S}$. See [100] and more general [12], [52], [89], [148], and [177], where the method of Lyapunov function(al)s is used.

Let $R = (R, \oplus, \odot)$ be an infinite ring with a unit and a zero denoted as $e_1$ and $e_0$, respectively. The symbol $\mathcal{M}(R, m)$ will denote the set of all $m \times m$ matrices with elements from $R$. If we consider the $i$-th column of $U \in \mathcal{M}(R, m)$, then we write $U_i$; and $R^m$ if we consider the set of all $m \times 1$ vectors with entries attaining values from $R$. As usual, we define the multiplication $\cdot$ of matrices from $\mathcal{M}(R, m)$ (and $U \cdot v$, $U \in \mathcal{M}(R, m)$, $v \in R^m$) by $\oplus$ and $\odot$. Let $d$ be a pseudometric on $R$ and suppose that

$$\text{the operations } \oplus \text{ and } \odot \text{ are continuous with respect to } d. \tag{1.52}$$

It gives the pseudometrics in $R^m$ and $\mathcal{M}(R, m)$ because $\mathcal{M}(R, m)$ can be expressed as $R^{m \times m}$; i.e., $d$ in $R^m$ and $\mathcal{M}(R, m)$ is the sum of $m$ and $m^2$ nonnegative numbers given by $d$ in $R$, respectively. For simplicity, we will also denote these pseudometrics as $d$.

The vector $v \in R^m$ is called *nonzero* (or *nontrivial*) if $d\left(v, (e_0, \dots, e_0)^T\right) > 0$. We say that a nonzero vector $(r_1, \dots, r_m)^T$, where $r_1, \dots, r_m \in R$, is an $e_1$-*eigenvector* of $U \in \mathcal{M}(R, m)$ if

$$d\left(U \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}\right) = 0,$$

and that $V \in \mathcal{M}(R, m)$ is *regular for a nonzero vector* $(r_1, \dots, r_m)^T \in R^m$ if

$$d\left(V \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \begin{pmatrix} e_0 \\ \vdots \\ e_0 \end{pmatrix}\right) > 0. \tag{1.53}$$

Next, we set

$$\mathcal{I} := \begin{pmatrix} e_1 & e_0 & \dots & e_0 \\ e_0 & e_1 & \dots & e_0 \\ \vdots & \vdots & \ddots & \vdots \\ e_0 & e_0 & \dots & e_1 \end{pmatrix} \in \mathcal{M}(R, m).$$

If for given $U \in \mathcal{M}(R,m)$ and $X \subseteq \mathcal{M}(R,m)$, there exists the unique matrix $V \in X$ (we put $V = W$ if $d(V,W) = 0$) for which

$$U \cdot V = V \cdot U = \mathcal{I},$$

then we define $U^{-1} := V$ and we say that $V$ is *the inverse matrix* of $U$ in $X$.

For any function $H : [a,b] \to X$ ($a \leq 0 < b$, $a, b \in \mathbb{R}$) and $s \in \mathbb{R}$, we extend its domain of definition as follows

$$H(s) := \begin{cases} H(\sigma) \cdot (H(b))^l & \text{for } s \geq 0, \\ (H(a))^l \cdot H(\sigma) & \text{for } s < 0 \text{ if } a < 0, \end{cases} \tag{1.54}$$

where $s = lb + \sigma$ for $l \in \mathbb{N}_0$, $\sigma \in [0,b)$ or $s = la + \sigma$ for $l \in \mathbb{N}_0$, $\sigma \in (a,0]$. Hereafter, we will restrict coefficients $A_k$ in (1.49) to be elements of an infinite set $X \subseteq \mathcal{M}(R,m)$ with the following properties:

$$\mathcal{I} \in X; \quad U, V \in X \implies U \cdot V \in X,\ U^{-1} \text{ exists in } X; \tag{1.55}$$

and either[5]

> there exists a continuous function $F_1 : [-1,1] \to X$ satisfying
>
> $$F_1(0) = \mathcal{I}; \quad F_1(t) = F_1^{-1}(-t), \quad t \in [0,1]; \tag{1.56}$$
>
> and matrix $F_1(1)$ has no $e_1$-eigenvector

or[6]

> there exist continuous $F_2 : [-1,1] \to X$, $t_1, \dots, t_q \in (0,1]$, $\delta > 0$ such that
>
> $$F_2(0) = \mathcal{I}; \quad F_2\left(\sum_{i=1}^{p} s_i\right) = \prod_{i=1}^{p} F_2(s_i), \quad s_1, \dots, s_p \in [-1,1]; \tag{1.57}$$
>
> and, for any $v \in R^m$, one can find $j \in \{1, \dots, q\}$ for which $v$ is not an $e_1$-eigenvector of $F_2(t)$, $t \in (\max\{0, t_j - \delta\}, \min\{t_j + \delta, 1\})$.

We recall that, for $U_1, \dots, U_p \in X$ ($p \in \mathbb{N}$), we define

$$\prod_{i=1}^{p} U_i := U_1 \cdot U_2 \cdots U_p, \quad \prod_{i=p}^{1} U_i := U_p \cdot U_{p-1} \cdots U_1.$$

For the above function $H$, we also use the conventional notation

$$(H(s))^0 := \mathcal{I}, \quad H^{-1}(s) := (H(s))^{-1} \quad \text{for all considered } s \in \mathbb{R}.$$

Actually, a closer examination of our process reveals that the pseudometric $d$ can be defined "only" on the set

$$\{F_j(s_1) \cdots F_j(s_n) \cdot v;\ v \in R^m, s_1, \dots, s_n \in [-1,1]\}$$

and the set $\{F_j(t);\, t \in [-1,1]\}$ can be countable for both of $j \in \{1,2\}$.[7]

We need a sequence $\{a_k\}_{k\in\mathbb{N}_0}$ of real numbers, which has special properties (mentioned in the below given Lemmas 1.20–1.23), to prove the main theorem of this chapter. We define the sequence $\{a_k\}_{k\in\mathbb{N}_0}$ by the recurrent formula

$$a_0 := 1, \quad a_1 := 0, \quad a_{2^n+k} := a_k - \frac{1}{2^n}, \quad k = 0, \dots, 2^n - 1,\ n \in \mathbb{N}. \tag{1.58}$$

For this sequence, we have the following results:

**Lemma 1.20.** *The sequence $\{a_k\}$ is almost periodic.*

The above lemma follows from Theorem 1.12 where we set $\varphi_k = a_k$ $(k \in \mathbb{N})$ and

$$\mathcal{X} = \mathbb{R}, \quad m = 0, \quad j = 1, \quad \varphi_0 = 1, \quad r_n = \frac{4}{2^n},\ n \in \mathbb{N}.$$

**Lemma 1.21.** *The following holds*

$$a_{2^{n+2}-1-i} = -a_{2^{n+1}+i} \tag{1.59}$$

*for any $n \in \mathbb{N}_0$ and $i \in \{0, \dots, 2^n - 1\}$; i.e., $i \in \{0, \dots, 2^{n+1} - 1\}$.*

Before proving this statement, observe that (1.59) is equivalent to

$$\sum_{k=2^{n+1}+i}^{2^{n+2}-1-i} a_k = 0, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\};$$

i.e., to

$$\sum_{k=0}^{2^{n+1}-1+i} a_k = \sum_{k=0}^{2^{n+2}-1-i} a_k, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\}.$$

*Proof of Lemma 1.21.* Obviously, (1.59) is true for $n \in \{0,1\}$ because

$$a_2 = -a_3 = \frac{1}{2}, \quad a_4 = -a_7 = \frac{3}{4}, \quad a_5 = -a_6 = -\frac{1}{4};$$

i.e.,

$$\sum_{k=0}^{1} a_k = \sum_{k=0}^{3} a_k = \sum_{k=0}^{7} a_k = 1, \quad \sum_{k=0}^{4} a_k = \sum_{k=0}^{6} a_k = \frac{7}{4}.$$

Suppose that (1.59) is true also for $2, \dots, n-1$. We choose $i \in \{0, \dots, 2^n - 1\}$ arbitrarily. (We have $2^{n+2} - 1 - i \geq 2^{n+1} + 2^n$.) From (1.58) and the induction hypothesis it follows

$$a_{2^{n+2}-1-i} + a_{2^{n+1}+2^n+i} = -\frac{1}{2^n}, \quad a_{2^{n+1}+i} - a_{2^{n+1}+2^n+i} = \frac{1}{2^n}.$$

Summing the above equalities, we get (1.59). □

**Lemma 1.22.** *We have*

$$\sum_{k=0}^{n} a_k \geq 1, \quad n \in \mathbb{N}_0. \tag{1.60}$$

*Proof.* Evidently, $a_0 = a_0 + a_1 = 1$. It means that (1.60) is true for $n = 0$ and $n = 1 = 2^1 - 1$. Let it be valid for arbitrarily given $2^p - 1$ and all $n < 2^p - 1$, i.e., let

$$\sum_{k=0}^{n} a_k \geq 1, \quad n \leq 2^p - 1,\ n \in \mathbb{N}_0.$$

Considering the definition of $\{a_k\}$, we obtain

$$\sum_{k=0}^{2^p+j-1} a_k = \sum_{k=0}^{2^p-1} a_k + \sum_{k=2^p}^{2^p+j-1} a_k \geq 1 + \sum_{k=0}^{j-1} a_k - j\frac{1}{2^p} \geq 1 + 1 - 1 = 1$$

for any $j \in \{1, \ldots, 2^p\}$. Lemma 1.22 now follows by the induction. □

**Lemma 1.23.** *We have*

$$\sum_{k=0}^{2^n-1} a_k = 1, \tag{1.61}$$

$$\sum_{k=0}^{2^{n+i}+2^n-1} a_k = 2 - \frac{1}{2^i}, \tag{1.62}$$

*where* $n \in \mathbb{N}_0$, $i \in \mathbb{N}$.

*Proof.* It is possible to prove this result by means of Lemma 1.21, but we prove it directly using (1.58) and the induction principle. We have

$$a_0 = 1, \quad a_0 + a_1 = 1, \quad a_0 + a_1 + a_2 + a_3 = 1.$$

If we assume that

$$\sum_{k=0}^{2^{n-1}-1} a_k = 1,$$

then we get (see (1.58))

$$\begin{aligned}\sum_{k=0}^{2^n-1} a_k &= \sum_{k=0}^{2^{n-1}-1} a_k + \sum_{k=2^{n-1}}^{2^n-1} a_k \\ &= \sum_{k=0}^{2^{n-1}-1} a_k + \sum_{k=0}^{2^{n-1}-1} \left(a_k - \frac{1}{2^{n-1}}\right) \\ &= 2\sum_{k=0}^{2^{n-1}-1} a_k - 1 = 1.\end{aligned}$$

Therefore, (1.61) is proved. Analogously, applying (1.58) and (1.61), one can compute

$$\begin{aligned}\sum_{k=0}^{2^{n+i}+2^n-1} a_k &= \sum_{k=0}^{2^{n+i}-1} a_k + \sum_{k=2^{n+i}}^{2^{n+i}+2^n-1} a_k \\ &= 1 + \sum_{k=0}^{2^n-1} \left(a_k - \frac{1}{2^{n+i}}\right) = 1 + \left(1 - \frac{1}{2^i}\right)\end{aligned}$$

what gives (1.62). □

Applying matrix valued functions $F_1$, $F_2$, we obtain the next lemma.

**Lemma 1.24.** *For each $j \in \{1,2\}$, any $n \in \mathbb{N}_0$, and each $i \in \{0,\dots,2^n-1\}$, it holds*

$$F_j(a_{2^{n+2}-1-i}) = F_j^{-1}(a_{2^{n+1}+i})$$

*and, consequently,*

$$\prod_{k=2^{n+1}+i}^{2^{n+2}-1-i} F_j(a_k) = \prod_{k=2^{n+2}-1-i}^{2^{n+1}+i} F_j(a_k) = \mathcal{I}.$$

*Proof.* Clearly, this is a corollary of Lemma 1.21. Consider (1.56), (1.57), and the fact that the multiplication of matrices (in $\mathcal{M}(R,m)$) is associative. ☐

Immediately, from Lemma 1.23 (see (1.57)), we have the following formulas for the function $F_2$:

**Lemma 1.25.** *The equalities*

$$\prod_{k=0}^{2^n-1} F_2\left(a_k\right) = F_2(1), \qquad \prod_{k=0}^{2^{n+i}+2^n-1} F_2(a_k) = F_2\left(2-\frac{1}{2^i}\right)$$

*hold for all $n \in \mathbb{N}_0$ and $i \in \mathbb{N}$.*

Now we can prove the main statement of Chapter 1.

**Theorem 1.26.** *There exists a system of the form* (1.49) *that does not possess a nonzero almost periodic solution.*

*Proof.* First we suppose that the coefficients $A_k$ belong to $X$ so that there exists a function $F_1$ from (1.56). Using Theorem 1.5, we get the almost periodicity of the sequence $\{F_1(a_k)\}_{k\in\mathbb{N}_0}$, where $\{a_k\}$ is given by (1.58). We want to show that all nonzero solutions of the system $\mathfrak{S}_1 \in \mathfrak{X}$ determined by $\{F_1(a_k)\}$ are not almost periodic.

By contradiction, suppose that there exist $c_1,\dots,c_m \in R$ such that the vector valued sequence

$$\{f_k\} := \left\{ P_k \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}, \quad k \in \mathbb{N}_0, \tag{1.63}$$

where $\{P_k\}_{k\in\mathbb{N}_0}$ is the principal fundamental matrix of $\mathfrak{S}_1$, is nontrivial and almost periodic; i.e., suppose that $\mathfrak{S}_1$ has a nontrivial almost periodic solution $\{f_k\}$. Since $\{f_k\}$ is almost periodic, $(c_1,\dots,c_m)^T$ is nonzero, and, because of $a_0 = 1$, it is valid

$$f_i = U_i \cdot F_1(1) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \text{for any } i \in \mathbb{N} \text{ and some } U_i \in X, \tag{1.64}$$

we know that (see (1.53))

$$F_1(1) \text{ is regular for } c := (c_1,\dots,c_m)^T. \tag{1.65}$$

Considering (1.58), the uniform continuity of $F_1$ and the continuity of the multiplication of matrices (see (1.52)), (c), Lemma 1.24, and (1.63), from the first part of Theorem 1.14 (see the proof of Corollary 1.15 and again Lemma 1.24), one can obtain that the sequence $\{g_k\}_{k\in\mathbb{Z}}$, where

$$g_k := f_k, \quad k \in \mathbb{N}_0, \quad g_k := f_{-k}, \quad k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{1.66}$$

is almost periodic too. Now we use Theorem 1.3 for $\{\varphi_k\} \equiv \{g_k\}$ and $\{h_n\}_{n\in\mathbb{N}} \equiv \{2^n\}_{n\in\mathbb{N}}$ (we can also consider directly $\{\varphi_k\} \equiv \{f_k\}$ and use Remark 1.2). This theorem implies that, for any $\varepsilon > 0$, there exists an infinite set $N(\varepsilon) \subseteq \mathbb{N}$ such that the inequality

$$d\left(g_{k+2^{n_1}}, g_{k+2^{n_2}}\right) < \varepsilon, \quad k \in \mathbb{Z} \tag{1.67}$$

holds for all $n_1, n_2 \in N(\varepsilon)$.

Using (1.56) (twice), we get $d\left(c, F_1(1)\, c\right) > 0$, and consequently (consider (1.65))

$$\vartheta := d\left(F_1(1) \cdot c, F_1(1) \cdot F_1(1) \cdot c\right) > 0. \tag{1.68}$$

From Lemma 1.24 (for $i = 0$), (1.63), and (1.66) (see also (1.64)), we have

$$g_0 = c, \quad g_1 = F_1(1) \cdot c, \quad \dots, \quad g_{2^n} = F_1(1) \cdot c, \tag{1.69}$$

where $n \in \mathbb{N}$ is arbitrary, and hence, considering (1.58), it holds

$$d\left(g_{2^i+2^n}, F_1(1) \cdot F_1(1) \cdot c\right) \to 0 \quad \text{as} \quad n \to \infty \tag{1.70}$$

for every $i \in \mathbb{N}$ because $F_1$ is uniformly continuous and the multiplication of matrices is continuous. We also have

$$d\left(g_{2^{n_2}+2^{n_1}}, F_1(1) \cdot c\right) < \frac{\vartheta}{2} \tag{1.71}$$

for all $n_1, n_2 \in N(\vartheta/2)$. Indeed, put $k = 2^{n_2}$ in (1.67) and consider (1.69) for $n = n_2 + 1$. If we choose $n_1 \in N(\vartheta/2)$ and put $i = n_1$ in (1.70), then there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, it holds

$$d\left(g_{2^{n_1}+2^n}, F_1(1) \cdot F_1(1) \cdot c\right) < \frac{\vartheta}{2}.$$

Thus, for arbitrarily given $n_2 \geq n_0$, $n_2 \in N(\vartheta/2)$, we get

$$d\left(g_{2^{n_2}+2^{n_1}}, F_1(1) \cdot F_1(1) \cdot c\right) < \frac{\vartheta}{2}. \tag{1.72}$$

Finally, applying (1.68), (c), (1.71), and (1.72), we have

$$\vartheta \leq d\left(F_1(1) \cdot c, g_{2^{n_2}+2^{n_1}}\right) + d\left(g_{2^{n_2}+2^{n_1}}, F_1(1) \cdot F_1(1) \cdot c\right) < \vartheta.$$

This contradiction gives the proof when we consider (1.56) for $k \in \mathbb{N}_0$.

Let $k \in \mathbb{Z}$. Then, we can consider the system $\widetilde{\mathfrak{S}}_1$ determined by the sequence

$$B_k := F_1(a_k), \quad k \in \mathbb{N}_0, \quad B_k := F_1(-a_{-k-1}), \quad k \in \mathbb{Z} \setminus \mathbb{N}_0. \tag{1.73}$$

Since the sequence $\{|\, a_k \,|\}_{k\in\mathbb{N}_0}$ is almost periodic (see Theorem 1.5) and has the form of $\{\varphi_k\}_{k\in\mathbb{N}_0}$ from Theorem 1.12 and since it is valid (see (1.59))

$$|\, a_{2^{n+2}-1-i} \,| = |\, a_{2^{n+1}+i} \,|, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\},$$

the fact that $\{B_k\}$ is almost periodic follows from the second part of Corollary 1.15, from Corollary 1.9, and Theorem 1.5. Next, the process is the same as for $k \in \mathbb{N}_0$. Let $\{P_k\}_{k\in\mathbb{Z}}$ be the principal fundamental matrix of $\widetilde{\mathfrak{S}}_1$ and $g_k := f_k$, $k \in \mathbb{Z}$. Also now we have (1.67), and consequently we get the same contradiction.

Let the coefficients $A_k$ belong to $X$ such that there exists a function $F_2$ from (1.57). Consider the numbers $t_1, \ldots, t_q \in (0,1]$ and $\delta > 0$ from (1.57). Without loss of the generality, we can assume

$$\delta < t_1 < \cdots < t_q \quad \text{and} \quad t_q < 1 - \delta. \tag{1.74}$$

Indeed, if $t_j = 1$, then we can put $t_j := 1 - \frac{\delta}{2}$ and redefine $\delta$. We repeat that any vector $v \in R^m$ determines some $j \in \{1, \ldots, q\}$ (see again (1.57)) such that $v$ is not an $e_1$-eigenvector of $F_2(t)$ for $t \in (t_j - \delta, t_j + \delta)$.

From Theorem 1.5 it follows that the sequence $\{F_2(a_k)\}_{k\in\mathbb{N}_0}$ is almost periodic. Thus, it determines a system of the form (1.49). We will denote it as $\mathfrak{S}_2$. Suppose that $\mathfrak{S}_2$ has a nontrivial almost periodic solution $\{x_k\}_{k\in\mathbb{N}_0}$. For the principal fundamental matrix $\{P_k\}$ of the system $\mathfrak{S}_2$, we have

$$x_k = P_k \cdot x_0, \quad k \in \mathbb{N}_0,$$

where the vector $x_0$ is nonzero. Using this fact and taking into account Lemma 1.22 and (1.57), we obtain

$$x_n = F_2(t) \cdot F_2^i(1) \cdot x_0 \quad \text{for some } i \in \mathbb{N},\ t \in [0,1), \tag{1.75}$$

and for arbitrary $n \in \mathbb{N}$. From Lemma 1.25, we also get

$$x_{2^n} = F_2(1) \cdot x_0 \quad \text{for all } n \in \mathbb{N}_0 \tag{1.76}$$

and

$$x_{2^{n+i}+2^n} = F_2\left(1 - \frac{1}{2^i}\right) \cdot F_2(1) \cdot x_0 \quad \text{for all } n \in \mathbb{N}_0,\ i \in \mathbb{N}. \tag{1.77}$$

Analogously as for $\{f_k\}$, one can extend $\{x_k\}_{k\in\mathbb{N}_0}$ by the formula

$$x_k := x_{-k}, \quad k \in \mathbb{Z} \setminus \mathbb{N}_0$$

for all $k \in \mathbb{Z}$ so that the sequence $\{x_k\}_{k\in\mathbb{Z}}$ is almost periodic too. Now we apply Theorem 1.3 for the sequences $\{x_k\}_{k\in\mathbb{Z}}$ and $\{2^n\}_{n\in\mathbb{N}}$. For any $\varepsilon > 0$, there exists an infinite set $M(\varepsilon) \subseteq \mathbb{N}$ such that, for any $n_1, n_2 \in M(\varepsilon)$, we have

$$d\left(x_{k+2^{n_1}}, x_{k+2^{n_2}}\right) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1.78}$$

Since $F_2$ is uniformly continuous and the multiplication of matrices is continuous, for arbitrary $i \in \mathbb{N}$ and $\varepsilon > 0$, we have from (1.58) and (1.76) that

$$d\left(x_{2^i+2^n}, F_2(1) \cdot F_2(1) \cdot x_0\right) < \varepsilon \quad \text{for sufficiently large } n \in \mathbb{N}. \tag{1.79}$$

Because of the almost periodicity of $\{x_k\}$ and (1.75), the matrix $F_2(1)$ has to be regular for $x_0$. Let $\varepsilon > 0$ be arbitrarily small and $n_1 \in M(\varepsilon)$ arbitrarily large. From (1.78) and

(1.79), where we choose $k = 2^{n_1-j}$ and $i = 2^{n_1-j}$ for $j \in \{0, \dots, n_1\}$, it follows that, for given $n_1$, there exists sufficiently large $n_2 \in M(\varepsilon)$ for which

$$\begin{aligned} & d\left(x_{2^{n_1-j}+2^{n_1}}, F_2(1) \cdot F_2(1) \cdot x_0\right) \leq \\ & d\left(x_{2^{n_1-j}+2^{n_1}}, x_{2^{n_1-j}+2^{n_2}}\right) + d\left(x_{2^{n_1-j}+2^{n_2}}, F_2(1) \cdot F_2(1) \cdot x_0\right) < 2\varepsilon. \end{aligned} \tag{1.80}$$

Since $\varepsilon$ (in (1.80)) is arbitrarily small, choosing $j = 0$, we get

$$d\left(x_{2^{n_1+1}}, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0$$

which gives (see (1.76)) that $F_2(1)\, x_0$ is an $e_1$-eigenvector of $F_2(1)$, i.e., we have

$$d\left(F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0. \tag{1.81}$$

If we choose $j = 1$, then we obtain (consider (1.77))

$$d\left(F_2\left(\frac{1}{2}\right) \cdot F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0.$$

Analogously, for any $j$ (the number $n_1$ is arbitrarily large), we get

$$d\left(F_2\left(1 - \frac{1}{2^j}\right) \cdot F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0.$$

Thus,

$$d\left(F_2\left(2 - \frac{1}{2^j}\right) \cdot x_0, F_2(2) \cdot x_0\right) = 0, \quad j \in \mathbb{N}. \tag{1.82}$$

Hence, we have

$$d\left(F_2\left(2 - \frac{1}{2^j}\right) \cdot x_0, F_2\left(2 - \frac{1}{2^{j-1}}\right) \cdot x_0\right) = 0, \quad j \in \mathbb{N}. \tag{1.83}$$

Because of

$$F_2\left(2 - \frac{1}{2^j}\right) = F_2\left(\frac{1}{2^j} + 2 - \frac{1}{2^{j-1}}\right) = F_2\left(\frac{1}{2^j}\right) \cdot F_2\left(2 - \frac{1}{2^{j-1}}\right)$$

and (see (1.81) and (1.82))

$$d\left(F_2\left(2 - \frac{1}{2^{j-1}}\right) \cdot x_0, F_2(1) \cdot x_0\right) = 0,$$

from (1.83) it follows

$$d\left(F_2\left(\frac{1}{2^j}\right) \cdot F_2(1) \cdot x_0, F_2(1) \cdot x_0\right) = 0 \quad \text{for all } j \in \mathbb{N},$$

i.e., $F_2(1)\, x_0$ is an $e_1$-eigenvector of $F_2(2^{-j})$ for all $j \in \mathbb{N}$.

Since any number $t \in [0, 1]$ can be expressed in the form

$$\sum_{i=1}^{\infty} \frac{a_i}{2^i}, \quad \text{where } a_i \in \{0, 1\},$$

for considered $\delta > 0$, there exists $n \in \mathbb{N}$ such that, for every $t \in [0,1]$, there exist $a_1, \ldots, a_n \in \{0,1\}$ satisfying
$$\left| t - \sum_{i=1}^{n} \frac{a_i}{2^i} \right| < \delta.$$
Thus, $F_2(1)\, x_0$ is an $e_1$-eigenvector of $F_2(t_j + s_j)$ for some $|\, s_j \,| < \delta$ and any $j \in \{1, \ldots, q\}$ which cannot be true. This contradiction shows that $\{x_k\}_{k \in \mathbb{N}_0}$ is not almost periodic.

If one considers the system $\widetilde{\mathfrak{S}}_2$ obtained from $\mathfrak{S}_2$ as in (1.73) (after replacing $\mathfrak{S}_1$ by $\mathfrak{S}_2$), then, similarly as for $F_1$ and $k \in \mathbb{N}_0$, one can prove that $\widetilde{\mathfrak{S}}_2 \in \mathfrak{X}$ and that any its nontrivial solution $\{x_k\}_{k \in \mathbb{Z}}$ is not almost periodic. □

**Remark 1.27.** Let a nonzero $F_1(1)\, v \in R^m$ not be an $e_1$-eigenvector of matrix $F_1(1)$ from (1.56); i.e., the condition (1.56) be weakened in this way. Then, from the first part of the proof of Theorem 1.26, we obtain that the sequence $\{f_k\}$, given by (1.63), is not almost periodic for $(c_1, \ldots, c_m)^T = v$. It means that there exists a system $\mathfrak{S}^1 \in \mathfrak{X}$ with the principal fundamental matrix $\{P_k^1\}$ such that the sequence $\{P_k^1\, v\}_{k \in \mathbb{N}_0}$ or $\{P_k^1\, v\}_{k \in \mathbb{Z}}$ is not almost periodic.

Analogously, it is seen: If one requires in (1.57) only that, for a nonzero vector $v \in R^m$, there exists $t \in (0,1]$ for which $F_2(1)\, v$ is not an $e_1$-eigenvector of $F_2(t)$, then there exists a system $\mathfrak{S}^2 \in \mathfrak{X}$ satisfying that the sequence $\{P_k^2\, v\}_{k \in \mathbb{Z}}$ (or $\{P_k^2\, v\}_{k \in \mathbb{N}_0}$), where $\{P_k^2\}_{k \in \mathbb{Z}}$ (or $\{P_k^2\}_{k \in \mathbb{N}_0}$) is the principal fundamental matrix of $\mathfrak{S}^2$, is not almost periodic.

The condition
$$F_2\left(\sum_{i=1}^{p} s_i\right) = \prod_{i=1}^{p} F_2\,(s_i), \quad s_1, \ldots, s_p \in [-1,1],\ p \in \mathbb{N} \tag{1.84}$$
in (1.57) is "strong". For example, from it follows that the multiplication of matrices from the set $\{F_2(t);\ t \in \mathbb{R}\}$ is commutative. At the same time, we say that, for many subsets of unitary or orthogonal matrices, it is not a limitation and that the method in the proof of Theorem 1.26 can be simplified in many cases. We will show it in two important special cases.

**Example 1.28.** If for any nontrivial vector $v \in R^m$, there exists $\varepsilon(v) > 0$ with the property that
$$F_2(t) \cdot v \notin \mathcal{O}_{\varepsilon(v)}(v) \quad \text{for all } t \geq 1 \text{ (see (1.54))},$$
then the fact, that the systems $\mathfrak{S}_2$ and $\widetilde{\mathfrak{S}}_2$ from the proof of Theorem 1.26 do not have nontrivial almost periodic solutions, follows directly from Lemma 1.22 and (1.84). Indeed, the set $T(\{x_k\}, \varepsilon(x_0)) \setminus \{0\}$ is empty for any nonzero solution $\{x_k\}$.

**Example 1.29.** Let the function $F_2$, in addition to (1.57), satisfy
$$F_2(s) = F_2(0) = \mathcal{I} \tag{1.85}$$
for some positive irrational number $s$, (1.74) hold, and $p \in \mathbb{N}$ be arbitrary. Then, the system $\mathfrak{S}$ determined by the sequence
$$\{A_k\} := \{F_2(k/p)\},$$

where $k \in \mathbb{N}_0$ or $k \in \mathbb{Z}$, has no nontrivial almost periodic solutions.

The function $F_2(t/p)$, $t \in \mathbb{R}$, is continuous and periodic with a period $ps$ (see (1.84), (1.85)). Using the compactness of the interval $[0, ps]$, (1.84), and Theorem 1.3, we get that $\{F_2(k/p)\}_{k\in\mathbb{Z}}$ is almost periodic. The almost periodicity of $\{F_2(k/p)\}_{k\in\mathbb{N}_0}$ is now obvious.

Suppose, by contradiction, that $\{x_k\} \equiv \{P_k\, x_0\}$ is a nontrivial almost periodic solution of $\mathfrak{S}$. We mention that there exists $\delta > 0$ satisfying that, for any nonzero $v \in R^m$, one can find $j \in \mathbb{N}$ such that there exists a positive number $\vartheta(v)$ for which

$$\vartheta(v) \leq d\left(F_2\left(\frac{j}{p} + t\right) \cdot v, v\right), \quad t \in (-\delta, \delta), \tag{1.86}$$

because

$$\{F_2(k/p); k \in \mathbb{N}\} \text{ is dense in } \{F_2(t); t \in \mathbb{R}\} \tag{1.87}$$

which is proved (for a continuous periodic function $F_2$ satisfying (1.84) with the smallest period $s > 0$ that is an irrational number) in detail, e.g., in [165, pp. 44–46]. Evidently, (1.87) gives that

$$\{F_2(k/p); k \in N\} \text{ is dense in } \{F_2(t); t \in \mathbb{R}\} \tag{1.88}$$

for any set $N$ what is relative dense in $\mathbb{N}$.

Since the multiplication of matrices is continuous, there exists $\varepsilon > 0$ which satisfies that every vector $u$ with the property $d(u, x_0) < \varepsilon$ determines the same $j$ in (1.86) as $x_0$ and one can find

$$\vartheta(u) \geq \frac{\vartheta(x_0)}{2}. \tag{1.89}$$

From (1.84), we see that

$$x_k = F_2\left(\sum_{i=0}^{k-1} \frac{i}{p}\right) \cdot x_0, \quad k \in \mathbb{N}. \tag{1.90}$$

Let $l$ be an arbitrary positive $(\vartheta(x_0)/2)$-translation number of $\{x_k\}$, thus, let

$$d(x_{k+l}, x_k) < \frac{\vartheta(x_0)}{2} \quad \text{for all } k \in \mathbb{N}, \tag{1.91}$$

and let $N$ be the set of all positive $\varepsilon$-translation numbers of $\{x_k\}$. Since

$$\sum_{i=0}^{k+l-1} \frac{i}{p} = \sum_{i=0}^{k-1} \frac{i}{p} + \frac{kl}{p} + \frac{l(l-1)}{2p}, \quad k \in \mathbb{N},$$

for all $k \in N$, we have (see again (1.84))

$$d(x_{k+l}, x_k) = d\left(F_2\left(\frac{kl}{p} + \frac{l(l-1)}{2p}\right) \cdot F_2\left(\sum_{i=0}^{k-1} \frac{i}{p}\right) \cdot x_0, F_2\left(\sum_{i=0}^{k-1} \frac{i}{p}\right) \cdot x_0\right). \tag{1.92}$$

From (1.88), if we replace $1/p$ by $l/p$, we get the choice of $k \in N$ such that

$$\left|\frac{j}{p} - \frac{kl}{p} - \frac{l(l-1)}{2p}\right| < \delta \pmod{s} \tag{1.93}$$

for $j$ in (1.86) determined by $x_0$. From (1.86), (1.89) (consider the definition of $\varepsilon$), (1.90), (1.92), and (1.93), we have

$$d(x_{k+l}, x_k) \geq \frac{\vartheta(x_0)}{2}$$

for at least one $k \in \mathbb{N}$. But, at the same time, we have (1.91). This contradiction gives that $\{x_k\}$ cannot be almost periodic. See also the proof of the first part of [158, Proposition 2], where almost periodic unitary systems are studied.

At the end, we remark that the last considered system $\mathfrak{S}$ (in Example 1.29) has no physical interpretations in any technical applications if we consider directly the sequence $\{k/p\}$; in contrast to $\mathfrak{S}_2$ and $\widetilde{\mathfrak{S}}_2$ (the sequence $\{a_k\}$). In applications, the following can be utilized: Let $\{u_k\}_{k\in\mathbb{Z}}$ (or $\{u_k\}_{k\in\mathbb{N}_0}$) be a sequence of arbitrary values and let the below considered function $\varphi$ be defined on the set $\{u_k; k \in \mathbb{Z}\}$ or $\{u_k; k \in \mathbb{N}_0\}$. If we extend the definition of the discrete almost periodicity so that $\varphi$ is *almost periodic* if for every $\varepsilon > 0$, one can find $p\,(\varepsilon) \in \mathbb{N}_0$ with the property that any set, in the form $\{k_0, \dots, k_0 + p\,(\varepsilon)\}$, $k_0 \in \mathbb{Z}$ (or $k_0 \in \mathbb{N}_0$), contains a number $l$ satisfying the inequality

$$d\,(\varphi(u_{k+l}), \varphi(u_k)) < \varepsilon$$

for all $k \in \mathbb{Z}$ (or $k \in \mathbb{N}_0$), then all results (mentioned in this chapter) about almost periodic sequences are still valid.

In this chapter, we considered almost periodic sequences, their main properties (especially, the Bochner condition), and almost periodic solutions of almost periodic difference equations in pseudometric spaces. Analogously, the Bohr and the Bochner almost periodic random sequences are defined, their properties investigated, and almost periodic solutions of almost periodic random difference equations are discussed in [80].

# Chapter 2

# Almost periodic homogeneous linear difference systems

We will consider almost periodic solutions of almost periodic linear difference systems. Our aim is to analyse the systems which have no nontrivial almost periodic solution. We are motivated by the paper [158], where unitary systems (determined by unitary matrices) are studied. One of the main statements of [158] says that the systems whose solutions are not almost periodic form an everywhere dense subset in the space of all considered unitary systems. We also note that important partial cases of the theorem and the process are mentioned in [151] and [57], [96], [154], respectively.

In the proof of this result, it is substantially used that the group of considered matrices is not commutative. Thus, e.g., the dimension of the systems has to be at least two. We will use methods based on our general constructions because we want to generalize the result also for commutative groups of matrices (especially, for the scalar case). It implies that we can treat the problem in a general setting. Scalar sequences will attain values in a complete metric space on an infinite field with continuous operations with respect to the metric similarly as scalar discrete processes in [38], where the main results are proved for real or complex entries (see the below given Theorem 2.14).

The almost periodicity of solutions of almost periodic linear difference equations is also studied in [4] and [175] (nonhomogeneous systems). We can refer to the known article [38] again. Explicit almost periodic solutions are obtained for a class of these equations in [76]. For difference systems of general forms, criteria of the existence of almost periodic solutions are presented in [21], [144], [173], [176]. The existence of an almost periodic sequence of solutions for an almost periodic difference equation is discussed in [83] (and [79] as in [176]). Concerning the existence theorems for almost periodic solutions of almost periodic delay difference systems, see [68] or [177] (methods and techniques from that paper are similarly used and developed in [178]).

This chapter is organized as follows. We begin with notations which are used throughout this chapter. Then we introduce general homogeneous linear difference systems and a metric in the space of all these almost periodic systems.

Since every nontrivial almost periodic solution of a homogeneous linear difference system is bounded and does not have a subsequence converging to the zero vector (see the below given Lemma 2.18), it is interesting to consider only groups of matrices with eigenvalues having absolute value 1 in the complex case and corresponding generalizations. To formulate our results in a simple and consistent form, we introduce the concepts of transformable and then strongly transformable groups of matrices. One of the conditions in the definition of transformable groups means that it is possible to transform any matrix into any other using finitely many arbitrarily small "jumps" in the complex case (for usual metrics on the group of considered matrices). We remark that the group has to be infinite because we study solutions which are not periodic.

In the second section, there is also shown that the group of all unitary matrices, the group of all orthogonal matrices of a dimension at least two with determinant 1, and some their subgroups are strongly transformable. In addition, the subgroups from Examples 2.4 and 2.6 are commutative.

In Section 2.3, we present conditions for strongly transformable and then transformable groups ensuring that, in any neighbourhood of every considered system, there exists a system which does not possess an almost periodic solution other than the trivial one. Especially, it is proved as Theorem 2.11 (consider also Theorem 2.20) that, for a strongly transformable group, this condition is the existence of matrices $M_1, \dots, M_j, \dots$ from the group with the property that, for any nonzero vector $x$, one can find $j \in \mathbb{N}$ such that $M_j\, x \neq x$. In each case, the corresponding corollary about the Cauchy problem is explicitly formulated.

## 2.1 Preliminaries

We will use the following notations: $\mathbb{N}_0$ for the set of positive integers including the zero; $\mathbb{R}^+$ for the set of all positive reals; $\mathbb{R}_0^+$ for the set of all nonnegative real numbers; and "i" for the imaginary unit. Let $F = (F, \oplus, \odot)$ be an infinite field with a unit and a zero denoted as $e_1$ and $e_0$, respectively, and let $m \in \mathbb{N}$ be arbitrarily given. Hereafter, we will consider $m$ as *the dimension* of difference systems under consideration.

The symbol $\mathcal{M}at(F, m)$ will denote the set of all $m \times m$ matrices with elements from $F$ and $F^m$ the set of all $m \times 1$ vectors with entries attaining values from $F$. As usual, we define *the identity matrix* $I$ and *the zero matrix* $O$. Analogously, for *the trivial vector*, we put $o := [e_0, e_0, \dots, e_0]^T \in F^m$. Since $F$ is a field, we have the notion of the nonsingular matrices from $\mathcal{M}at(F, m)$. For any invertible matrix $U$, we denote *the inverse matrix* as $U^{-1}$. For arbitrary $U_j, \dots, U_{j+n} \in \mathcal{M}at(F, m)$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$, we define

$$\prod_{i=j}^{j+n} U_i := U_j \cdot U_{j+1} \cdots U_{j+n}, \qquad \prod_{i=j+n}^{j} U_i := U_{j+n} \cdot U_{j+n-1} \cdots U_j.$$

Let $\varrho$ be a metric on $F$ and suppose that the operations $\oplus$ and $\odot$ are continuous with respect to $\varrho$ and that the metric space $(F, \varrho)$ is complete. The metric $\varrho$ induces the metrics in $F^m$ and $\mathcal{M}at(F, m)$ as the sum of $m$ and $m^2$ nonnegative numbers given by $\varrho$ in $F$, respectively. We will also denote these metrics as $\varrho$. For any $\varepsilon > 0$ and $\alpha$ from a metric space, the $\varepsilon$-neighbourhood of $\alpha$ will be denoted by $\mathcal{O}_\varepsilon^\varrho(\alpha)$. Note that the continuity of $\oplus$ and $\odot$ implies that the multiplication $\cdot$ of matrices from $\mathcal{M}at(F, m)$ (and $U \cdot v$, $U \in \mathcal{M}at(F, m)$, $v \in F^m$) is continuous.

All sequences, which we will consider, will be defined for $k \in \mathbb{Z}$ (or $i, j \in \mathbb{N}$) and will attain values in one of the metric spaces $F$, $F^m$, $\mathcal{M}at(F, m)$ (or $\mathbb{C}$). The vector (and scalar) valued sequences will be denoted by the lower-case letters, the matrix valued sequences by the capital letters, and each one of the scalar, vector, and matrix valued sequences by the symbols $\{\varphi_k\}$, $\{\psi_k\}$, $\{\chi_k\}$.

## 2.2 General homogeneous linear difference systems

We will consider $m$-dimensional homogeneous linear difference equations of the form

$$x_{k+1} = A_k \cdot x_k, \qquad k \in \mathbb{Z}, \tag{2.1}$$

where $\{A_k\}$ is an almost periodic sequence of *nonsingular* matrices from a given infinite set $\mathcal{X} \subset \mathcal{M}at(F, m)$. We need the set of all considered $A_k$ to form the set $\mathcal{X}$ which has the below given properties. The set of all these almost periodic systems will be denoted by the symbol $\mathcal{AP}(\mathcal{X})$.

We will identify the sequence $\{A_k\}$ with the system (2.1) which is determined by $\{A_k\}$. In the space $\mathcal{AP}(\mathcal{X})$, we introduce the metric

$$\sigma(\{A_k\}, \{B_k\}) := \sup_{k\in\mathbb{Z}} \varrho(A_k, B_k), \quad \{A_k\}, \{B_k\} \in \mathcal{AP}(\mathcal{X}).$$

For $\varepsilon > 0$, the symbol $\mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ denotes the *$\varepsilon$-neighbourhood* of $\{A_k\}$ in $\mathcal{AP}(\mathcal{X})$.

### 2.2.1 Transformable groups

**Definition 2.1.** We say that the infinite set $\mathcal{X} \subset \mathcal{M}at(F, m)$ is *transformable* if the following conditions are fulfilled:

(i) For all $U, V \in \mathcal{X}$, it holds

$$U \cdot V \in \mathcal{X}, \quad U^{-1} \in \mathcal{X}.$$

(ii) For any $L \in \mathbb{R}^+$ and $\varepsilon > 0$, there exists $p = p(L, \varepsilon) \in \mathbb{N}$ with the property that, for any $n \geq p$ $(n \in \mathbb{N})$ and any sequence $\{C_0, C_1, \dots, C_n\} \subset \mathcal{X}$, $L \leq \varrho(C_i, O)$, $i \in \{0, 1, \dots, n\}$, one can find a sequence $\{D_1, \dots, D_n\} \subset \mathcal{X}$ for which

$$D_i \in \mathcal{O}^{\varrho}_{\varepsilon}(C_i), \quad 1 \leq i \leq n, \qquad D_n \cdots D_2 \cdot D_1 = C_0.$$

(iii) The multiplication of matrices is uniformly continuous on $\mathcal{X}$ and has the Lipschitz property on a neighbourhood of $I$ in $\mathcal{X}$. Especially, for every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that

$$C \cdot D,\ D \cdot C \in \mathcal{O}^{\varrho}_{\varepsilon}(C) \quad \text{if} \quad C \in \mathcal{X},\ D \in \mathcal{O}^{\varrho}_{\eta}(I) \cap \mathcal{X};$$

and there exist $\zeta > 0$ and $P \in \mathbb{R}^+$ such that

$$C \cdot D,\ D \cdot C \in \mathcal{O}^{\varrho}_{\varepsilon P}(C) \quad \text{if} \quad C \in \mathcal{O}^{\varrho}_{\zeta}(I) \cap \mathcal{X},\ D \in \mathcal{O}^{\varrho}_{\varepsilon}(I) \cap \mathcal{X},\ \varepsilon \in (0, \zeta).$$

(iv) For any $L \in \mathbb{R}^+$, there exists $Q = Q(L) \in \mathbb{R}^+$ with the property that, for every $\varepsilon > 0$ and $C, D \in \mathcal{X} \setminus \mathcal{O}^{\varrho}_{L}(O)$ satisfying $C \in \mathcal{O}^{\varrho}_{\varepsilon}(D)$, it is valid that

$$C^{-1} \cdot D,\ D \cdot C^{-1} \in \mathcal{O}^{\varrho}_{\varepsilon Q}(I).$$

For simplicity, in the below mentioned examples, we will consider only the complex or real case and we will speak about the classical case. Henceforth, we will use known results of matrix analysis which can be found, e.g., in [74], [75], [84], [98].

If

$$((F, \oplus, \odot), \varrho(\cdot,\cdot)) = ((\mathbb{C}, +, \cdot), |\cdot - \cdot|) \text{ or } ((F, \oplus, \odot), \varrho(\cdot,\cdot)) = ((\mathbb{R}, +, \cdot), |\cdot - \cdot|)),$$

then, evidently, the multiplication of matrices satisfies the Lipschitz condition on any set $\mathcal{O}^{\varrho}_{K}(O)$. For an arbitrary matrix norm[8] (especially, for the $l_1$ norm) denoted by $||\cdot||$, we have

$$||\, A^{-1} - (A+E)^{-1} \,|| \leq \frac{||\, A^{-1} \cdot E \,||}{1 - ||\, A^{-1} \cdot E \,||} \, ||\, A^{-1} \,|| \tag{2.2}$$

for any matrices $A, E$ such that $A$ is invertible and $||\, A^{-1}\, E \,|| < 1$. If we have a bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C}, m)$, then from (2.2) it follows that the map $C \mapsto C^{-1}$, $C \in \mathcal{X}$ has the Lipschitz property too. Hence, the condition (iv) is satisfied.

Finally, the conditions (iii) and (iv) are fulfilled for any bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C}, m)$. Further, for any bounded group $\mathcal{X}$, there exists $\varepsilon > 0$ for which $\mathcal{X} \cap \mathcal{O}^{\varrho}_{\varepsilon}(O) = \emptyset$. At the same time, from the condition (ii), we know that $\mathcal{X} \cap \mathcal{O}^{\varrho}_{1}(O) = \emptyset$ for any transformable set $\mathcal{X} \subset \mathcal{M}at(\mathbb{C}, m)$. Indeed, it suffices to consider $C_0 = I$ and the constant sequence $\{C_1, \ldots, C_n\}$ given by a matrix $C$ such that $||\, C \,|| < 1$.

Let $\varepsilon > 0$, a bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C}, m)$, and $C_0, C_1, \ldots, C_n \in \mathcal{X}$ be arbitrarily given. The uniform continuity of the multiplication of matrices on $\mathcal{X}$ implies the existence of $\eta = \eta(\varepsilon) > 0$ such that $C\,D, D\,C \in \mathcal{O}^{\varrho}_{\varepsilon}(C)$ if $D \in \mathcal{O}^{\varrho}_{\eta}(I) \cap \mathcal{X}$, $C \in \mathcal{X}$ (see (ii)). We define the maps $H_1$, $H_2$ on $\mathcal{X} \times \mathcal{X}$ by

$$H_1\left((C, D)\right) := C \cdot D \cdot C^{-1}, \quad H_2\left((C, D)\right) := C^{-1} \cdot D \cdot C.$$

Since $H_1$, $H_2$ satisfy the Lipschitz condition, there exists $R \in \mathbb{R}^{+}$ such that the images of $\{C\} \times \mathcal{O}^{\varrho}_{\eta/R}(I) \cap \mathcal{X}$ in both of $H_1$ and $H_2$ are subsets of $\mathcal{O}^{\varrho}_{\eta}(I)$ for all $C \in \mathcal{X}$.

If we replace

$$\prod_{i_1=n}^{1} F_{i_1} \cdot \prod_{i_2=1}^{n} C_{i_2} \qquad \text{by} \qquad \prod_{i=1}^{n} E_i \cdot C_i,$$

where

$$F_1 = H_1\left((I, E_1)\right), \qquad F_2 = H_1\left((E_1 \cdot C_1, E_2)\right),$$

$$\ldots \quad F_n = H_1\left((E_1 \cdot C_1 \cdots E_{n-1} \cdot C_{n-1}, E_n)\right),$$

we see that $F_i \in \mathcal{O}^{\varrho}_{\eta/R}(I) \cap \mathcal{X}$, $i \in \{1, \ldots, n\}$, implies $E_i \in \mathcal{O}^{\varrho}_{\eta}(I)$, $i \in \{1, \ldots, n\}$. Thus, from the existence of matrices $F_1, \ldots, F_n \in \mathcal{O}^{\varrho}_{\eta/R}(I) \cap \mathcal{X}$ for which

$$\prod_{i_1=n}^{1} F_{i_1} \cdot \prod_{i_2=1}^{n} C_{i_2} = C_0,$$

it follows the existence of matrices $D_1, \ldots, D_n \in \mathcal{X}$ satisfying

$$D_i \in \mathcal{O}^{\varrho}_{\varepsilon}(C_i), \quad 1 \leq i \leq n, \qquad D_n \cdots D_2 \cdot D_1 = C_0.$$

It means that a bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C}, m)$ is transformable if for any sufficiently small $\varepsilon > 0$, there exists $p(\varepsilon) \in \mathbb{N}$ such that, for all $n \geq p(\varepsilon)$ $(n \in \mathbb{N})$, any matrix from $\mathcal{X}$ can be expressed as a product of $n$ matrices from $\mathcal{O}^{\varrho}_{\varepsilon}(I) \cap \mathcal{X}$. We remark that several processes in the proofs of the below given results can be simplified in the classical case. For example, in the proofs of Lemma 2.10 and Lemma 2.18, one can use that, for any $\varepsilon > 0$, $K \in \mathbb{R}^{+}$, and $n \in \mathbb{N}$, there exists $\xi = \xi(\varepsilon, K, n) > 0$ for which

$$\varrho(M_1 \cdot M_2 \cdots M_n, O) < \varepsilon, \quad M_1, M_2, \ldots, M_n \in \mathcal{O}^{\varrho}_{K}(O)$$

and

$$\varrho(M_1 \cdots M_n \cdot u, o) < \varepsilon, \quad M_1, \ldots, M_n \in \mathcal{O}^{\varrho}_{K}(O),\ u \in \mathcal{O}^{\varrho}_{K}(o)$$

if we have $M_i \in \mathcal{O}^{\varrho}_{\xi}(O)$ for at least one $i \in \{1, \ldots, n\}$ and $u \in \mathcal{O}^{\varrho}_{\xi}(o)$, respectively.

### 2.2.2 Examples of transformable groups

Now we mention the most important examples of transformable groups:

**Example 2.2.** The group of all unitary matrices is transformable. Obviously, it suffices to show that, for every $\varepsilon > 0$, any unitary matrix can be obtained as the $n$-th power of some unitary matrix from the $\varepsilon$-neighbourhood of $I$ for all sufficiently large $n \in \mathbb{N}$. To show this, let $\varepsilon > 0$, $n \in \mathbb{N}$, and a $m \times m$ unitary matrix $U$ with eigenvalues $\exp(\mathrm{i}\lambda_1), \ldots, \exp(\mathrm{i}\lambda_m)$, where $\lambda_1, \ldots, \lambda_m \in [-\pi, \pi)$, be arbitrarily given. We have

$$U = W \cdot J \cdot W^{*} \quad \text{for some unitary matrix } W = W(U),$$

where $J = \mathrm{diag}\,[\exp(\mathrm{i}\lambda_1), \ldots, \exp(\mathrm{i}\lambda_m)]$ and $W^{*}$ denotes the conjugate transpose of $W$. We find a unitary matrix $V$ for which $V^n = U$. By

$$W^{*} \cdot V^{n} \cdot W = (W^{*} \cdot V \cdot W)^{n} = J,$$

we obtain

$$V = W \cdot \mathrm{diag}\,[\exp(\mathrm{i}\lambda_1/n), \ldots, \exp(\mathrm{i}\lambda_m/n)] \cdot W^{*}.$$

Since the multiplication of matrices is uniformly continuous on the set of all unitary matrices, it remains to consider sufficiently large $n \in \mathbb{N}$.

**Example 2.3.** Let $m \geq 2$ and $F = \mathbb{R}$. Now we will show that the group of $m \times m$ orthogonal matrices with determinant 1 is transformable. Analogously as for unitary matrices, it is enough to prove that any orthogonal matrix $U$ for which $\det U = 1$ is products of $n \geq p(\varepsilon)$, $n \in \mathbb{N}$ orthogonal matrices from the $\varepsilon$-neighbourhood of $I$ for arbitrary $\varepsilon > 0$ and some $p(\varepsilon) \in \mathbb{N}$. Indeed, it is seen that there exists a neighbourhood of $I$ which contains only orthogonal matrices with determinant 1.

Let $m = 2$. Observe that a two-dimensional orthogonal matrix has the form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

where $\alpha \in [-\pi, \pi)$, if and only if its determinant is 1. It can be easily computed that

$$\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{bmatrix} = \begin{bmatrix} \cos(\alpha_1 + \alpha_2) & -\sin(\alpha_1 + \alpha_2) \\ \sin(\alpha_1 + \alpha_2) & \cos(\alpha_1 + \alpha_2) \end{bmatrix}$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$ and that, consequently, any this matrix (for some $\alpha \in [-\pi, \pi)$) can be obtained as the $n$-th power of the orthogonal matrix of this type given by the argument $\alpha/n$ for all $n \in \mathbb{N}$.

Now we use the induction principle with respect to $m$. Assume that the statement is true for $m - 1 \geq 2$ and prove it for $m$. Let $U$ be an orthogonal $m \times m$ matrix which is not in any one of the forms

$$\begin{bmatrix} 1 & o^T \\ o & V \end{bmatrix}, \qquad \begin{bmatrix} V & o \\ o^T & 1 \end{bmatrix}, \tag{2.3}$$

where $V$ is an orthogonal matrix of dimension $m - 1$, $o \in F^{m-1}$, and suppose that $U$ has the element on the position $(1, m)$ different from 0 (in the second case, we put $U_2 := U$ in the below given process). We multiply $U$ from the left by an orthogonal matrix $U_1$ which is in the second form from (2.3) and satisfies that $U_2 := U_1 \cdot U$ has 0 on the position $(1, m)$. For $U_2$, we define an orthogonal matrix $U_3$ so that the $m$-th row of $U_3$ is the last column of $U_2$ and so that the first column and the first row of $U_3$ are zero except the number 1 on the position $(1, 1)$. Obviously, the product $U_4 := U_3 \cdot U_2$ is equal to a matrix which has the second form from (2.3). Summarizing, we get $U = U_1^T \cdot U_3^T \cdot U_4$. Thus, one can express any orthogonal matrix $U$ as a product of at most three matrices of the forms given in (2.3). Further, the matrices of this product can be evidently chosen so that the determinant of all of them is 1 if the determinant of the given matrix is 1 too. Now the induction hypothesis gives the validity of the above statement.

In the complex case, i.e., for $m \geq 2$ and $F = \mathbb{C}$, the group of all $m \times m$ unitary matrices with determinant 1 is transformable as well. It suffices to consider Example 2.2 and diagonalizations of unitary matrices.

**Example 2.4.** Let a unitary matrix $S$ be given. Let $\mathcal{X}_S$ be the set of the unitary matrices which are simultaneously diagonalizable for the single similarity matrix $S$, i.e., let

$$\mathcal{X}_S = \left\{ S^{-1} \cdot \operatorname{diag}\left[\exp\left(\mathrm{i}\lambda_1\right), \ldots, \exp\left(\mathrm{i}\lambda_m\right)\right] \cdot S;\ \lambda_1, \ldots, \lambda_m \in [-\pi, \pi) \right\}.$$

Obviously, $\mathcal{X}_S$ is a subgroup of the $m \times m$ unitary group (different from the group if $m \geq 2$). Since diagonalizable (normal) matrices are simultaneously (unitarily) diagonalizable if and only if they commute under multiplication, $\mathcal{X}_S$ is a commutative group. Analogously as in Example 2.2, one can show that $\mathcal{X}_S$ is transformable. Further, $\mathcal{X}_S$ is transformable also for arbitrary nonsingular matrix $S$. Especially, a transformable set does not need to be a subgroup of the $m \times m$ unitary group.

**Example 2.5.** Now we consider the set of the unitary matrices with the determinant in the form $\exp(\mathrm{i}r)$, $r \in \mathbb{Q}$ or $r \in \mathbb{Z}$. Evidently, these matrices form a group as well. Considering diagonalizations of unitary matrices and the uniform continuity of the multiplication of unitary matrices, we get that this group is dense in the group of all unitary matrices. Thus (see Example 2.2), it satisfies the condition (ii). Finally, it is transformable. In general, any dense subgroup of a transformable set is transformable as well.

**Example 2.6.** Let a unitary matrix $S$ be given. Analogously as in Example 2.4 and Example 2.5, we can show that the group

$$\{S^* \cdot \operatorname{diag}\left[\exp\left(\mathrm{i}\lambda_1\right), \ldots, \exp\left(\mathrm{i}\lambda_m\right)\right] \cdot S;\ \lambda_1, \ldots, \lambda_m \in \mathbb{Q}\}$$

is transformable. In general, the matrices with eigenvalues in the form $\exp(\mathrm{i}r)$, where $r \in \mathbb{Q}$ or $r \in \mathbb{Z}$, from a given commutative transformable subgroup of the $m \times m$ unitary group form transformable set if it is infinite. Indeed, if complex matrices $A$, $B$ commute and have eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_m$, respectively, then the eigenvalues of $A\,B$ are $\lambda_1\mu_{j_1}, \lambda_2\mu_{j_2}, \ldots, \lambda_m\mu_{j_m}$ for some permutation $j_1, \ldots, j_m$ of the indices $1, \ldots, m$.

### 2.2.3 Strongly transformable groups

**Lemma 2.7.** *If $\mathcal{X}$ is transformable, then, for any $\{L_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ and $j \geq 2$, $j \in \mathbb{N}$, one can find $\{\varepsilon_i\}_{i\in\mathbb{N}} \equiv \{\varepsilon_i(\{L_i\}, j)\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ satisfying*

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty,$$

$$j^i \geq p\left(L_{g(i)}, \varepsilon_i\right) \quad \text{for infinitely many } i \in \mathbb{N}, \text{ some } g(i) \in \mathbb{N}, \tag{2.4}$$

*where $g(i) \to \infty$ as $i \to \infty$.*

*Proof.* The lemma follows directly from (ii). Indeed, one can put

$$\begin{aligned} \varepsilon_i &:= 2^{-k}, \quad i = f(k) \text{ for some } k \in \mathbb{N};\\ \varepsilon_i &:= 2^{-i}, \quad i \notin \{f(k);\, k \in \mathbb{N}\},\, i \in \mathbb{N} \end{aligned}$$

for arbitrarily given increasing discrete function $f : \mathbb{N} \to \mathbb{N}$ with the property that

$$j^{f(k)} \geq p\left(L_k, 2^{-k}\right), \quad k \in \mathbb{N},$$

whose inverse function is considered in (2.4) as $g$. □

Of course, the inequality (2.4) does not need to be true for all $i \in \mathbb{N}$ or for a set of $i$ which is relative dense in $\mathbb{N}$. This fact motivates the next definition.

**Definition 2.8.** The set $\mathcal{X}$ is *strongly transformable* if it is transformable and if for any $L \in \mathbb{R}^+$, there exist $j = j(L) \in \mathbb{N}$ and a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}} \equiv \{\varepsilon_i(L)\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty, \tag{2.5}$$

$$j^i \geq p\left(L, \varepsilon_i\right) \quad \text{for all } i \in \mathbb{N}. \tag{2.6}$$

**Example 2.9.** Since we considered only maps which satisfy the Lipschitz condition in the above examples (in the classical case) and since we can choose

$$p\left(L, \varepsilon, m+1\right) \leq 3p\left(L, \varepsilon, m\right)$$

in (ii) when using the induction principle with respect to $m$ (see Example 2.3), all transformable sets of matrices mentioned in Examples 2.2–2.6 are actually strongly transformable.

In the next section, we will construct almost periodic sequences by Theorem 1.16 and we will use (2.4) or (2.6) in the constructions.[9]

## 2.3 Systems without almost periodic solutions

Now we can consider the gist of this chapter.

**Lemma 2.10.** *If an almost periodic sequence of nonsingular $A_k \in \mathcal{M}at(F,m)$ is such that, for any $\varepsilon > 0$, there exists $i = i(\varepsilon) \in \mathbb{Z}$ for which $\varrho(O, A_i) < \varepsilon$, then the system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$ does not have a nontrivial almost periodic solution.*

*Proof.* By contradiction, suppose that we have an almost periodic sequence $\{A_k\}$, a sequence $\{h_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ satisfying

$$\varrho\,(O, A_{h_i}) < \frac{1}{i}, \quad i \in \mathbb{N}, \tag{2.7}$$

and a nontrivial almost periodic solution $\{x_k\}$ of the system $x_{k+1} = A_k\, x_k$. Using Corollary 1.4, we obtain uniformly convergent common subsequences

$$\{\{A_{k+\tilde{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}} \quad \text{and} \quad \{\{x_{k+\tilde{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$$

of the sequences $\{\{A_{k+h_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$ and $\{\{x_{k+h_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$, respectively. The limits will be denoted as $\{B_k\}$ and $\{y_k\}$.

We put $\varepsilon := \varrho\,(x_0, o)\,/2 > 0$. Because of the almost periodicity of $\{x_k\}$, there exists some $p\,(\varepsilon)$ from Definition 1.1. We will consider the sets $N_i := \{i+1, i+2, \dots, i+p\,(\varepsilon)\}$ for $i \in \mathbb{Z}$. Any one of the sets $N_i$ contains a number $l \in T(\{x_k\}, \varepsilon)$. Thus,

$$x_l \notin \mathcal{O}^{\varrho}_{\varepsilon}(o). \tag{2.8}$$

From (2.7) it follows that $B_0 = O$. Since the multiplication of matrices is continuous, one can find $\vartheta > 0$ for which

$$C_j \cdots C_0 \cdot y \in \mathcal{O}^{\varrho}_{\varepsilon}(o), \quad j = 0, 1, \dots, p\,(\varepsilon) - 1$$

if $y \in \mathcal{O}^{\varrho}_{\vartheta}(y_0)$ and $C_i \in \mathcal{O}^{\varrho}_{\vartheta}(B_i)$, $i \in \{0, \dots, j\}$. There exists $i \in \mathbb{N}$ such that

$$\varrho\left(A_{k+\tilde{h}_i}, B_k\right) < \vartheta, \quad \varrho\left(x_{k+\tilde{h}_i}, y_k\right) < \vartheta, \quad k \in \mathbb{Z}.$$

Therefore,

$$x_{j+\tilde{h}_i} \in \mathcal{O}^{\varrho}_{\varepsilon}(o), \quad j = 1, \dots, p\,(\varepsilon)\,. \tag{2.9}$$

Indeed, it is valid

$$x_{j+\tilde{h}_i} = A_{j+\tilde{h}_i-1} \cdots A_{\tilde{h}_i} \cdot x_{\tilde{h}_i}, \quad j = 1, \dots, p\,(\varepsilon)\,.$$

This contradiction (compare (2.8) with (2.9)) gives the proof. □

**Theorem 2.11.** *Let $\mathcal{X}$ be strongly transformable. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. If there exist $L \in \mathbb{R}^+$ and $\{M_i\}_{i\in\mathbb{N}}$ such that*

$$M_i,\, M_i^{-1} \in \mathcal{X} \smallsetminus \mathcal{O}^{\varrho}_{L}(O), \quad i \in \mathbb{N} \tag{2.10}$$

*and that, for any nonzero vector $u \in F^m$, one can find $i = i(u) \in \mathbb{N}$ with the property that $M_i\, u \neq u$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not possess a nontrivial almost periodic solution.*

*Proof.* If $\{A_k\}$ has a nontrivial almost periodic solution, then there exists $K \in \mathbb{R}^+$ such that $\varrho(A_k, O) > K$ for all $k$. Indeed, it follows from Lemma 2.10. Since it suffices to consider only very small $\varepsilon > 0$, we can assume without loss of the generality that

$$L + \varepsilon < \varrho(I, O), \quad L + \varepsilon < \varrho(A_k, O), \quad k \in \mathbb{Z}; \tag{2.11}$$

otherwise we can put $B_k := A_k$, $k \in \mathbb{Z}$.

Let $\eta = \eta(\varepsilon/2)$, $\zeta$, $P$ and $Q = Q(L)$ be from (iii) and (iv), respectively. Further, let $\eta < \varepsilon < \zeta$ and let $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$, $n \in \mathbb{N}$, and $j \geq 2$ ($j \in \mathbb{N}$) satisfy

$$\sum_{i=1}^{\infty} \varepsilon_i < \frac{\eta}{PQ}, \tag{2.12}$$

$$j^i(n+1) \geq p(L, \varepsilon_i) \quad \text{for all } i \in \mathbb{N}. \tag{2.13}$$

The inequality (2.12) follows from (2.5) if we omit finitely many values of $\varepsilon_i$, and (2.13) from the fact that $n$ and $j$ can be arbitrarily large and from (2.6). We remark that $P, Q \geq 1$.

We put

$$B_k := A_k, \quad C_k := I \quad \text{for } k \in \{0, 1, \dots, n\}$$

and we choose

$$B_k = A_k \cdot C_k \quad \text{for some } C_k \in \mathcal{O}^{\varrho}_{\varepsilon_2 Q}\left(C_{k-(n+1)}\right) \cap \mathcal{X},\ k \in \{n+1, \dots, 2(n+1)-1\},$$

$$\vdots$$

$$B_k = A_k \cdot C_k,\ C_k \in \mathcal{O}^{\varrho}_{\varepsilon_2 Q}\left(C_{k-j^4(n+1)}\right) \cap \mathcal{X},\ k \in \{j^4(n+1), \dots, (j^4+1)(n+1)-1\}$$

arbitrarily such that

$$\prod_{k=(j^2+1)(n+1)-1}^{n+1} B_k = M_1, \qquad \prod_{k=(j^3+j^2)(n+1)-1}^{n+1} B_k = \prod_{k=(j^4+1)(n+1)-1}^{n+1} B_k = I.$$

For

$$C_1, \dots, C_n, D_1, \dots, D_n \in \mathcal{X}, \quad D_i \in \mathcal{O}^{\varrho}_{\vartheta}(C_i),$$

where $\vartheta > 0$, $L + \vartheta \leq \varrho(C_i, O)$, $i \in \{1, \dots, n\}$, and $n \geq p(L, \vartheta)$, we can express

$$D_n \cdots D_2 \cdot D_1 = C_n \cdot \left(C_n^{-1} \cdot D_n\right) \cdots C_1 \cdot \left(C_1^{-1} \cdot D_1\right),$$

where

$$(C_i^{-1} \cdot D_i) \in \mathcal{O}^{\varrho}_{\vartheta Q}(I), \quad i \in \{1, \dots, n\}.$$

Using this fact and considering (2.10), (2.11), and (2.13), we get the existence of the above matrices $C_k$.

In the second step, we put

$$B_k := A_k \cdot C_{k+(j^4+1)(n+1)}, \quad k \in \{-(j^4+1)(n+1), \dots, -1\},$$

$$\vdots$$

$$B_k := A_k \cdot C_{k+j^4(j^4+1)(n+1)},\ k \in \{-j^4(j^4+1)(n+1), \dots, -(j^4-1)(j^4+1)(n+1)-1\}$$

and we denote

$$C_k := A_k^{-1} \cdot B_k, \quad k \in \{-j^4(j^4+1)(n+1), \dots, -1\}.$$

Now we choose

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-(j^4+1)^2(n+1)}\right) \cap \mathcal{X},$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1) + (j^4+1)^2(n+1) - 1\},$$

$$\vdots$$

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-j^4(j^4+1)^2(n+1)}\right) \cap \mathcal{X},$$
$$k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1) - 1\}$$

arbitrarily such that

$$\prod_{k=j^7(n+1)-1}^{(j^4+1)(n+1)} B_k = M_1, \qquad \prod_{k=(j^7+j^6-j^0)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=j^8\,(n+1)-1}^{(j^4+1)(n+1)} B_k = M_1, \qquad \prod_{k=(j^8+j^6-j^3)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=j^9\,(n+1)-1}^{(j^4+1)(n+1)} B_k = M_2, \qquad \prod_{k=(j^9+j^6-j^0)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=j^{10}(n+1)-1}^{(j^4+1)(n+1)} B_k = M_2, \qquad \prod_{k=(j^{10}+j^6-j^3)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=(j^4+1)(n+1)+j^4(j^4+1)^2(n+1)-1}^{(j^4+1)(n+1)} B_k = I.$$

Such matrices $C_k$ exist. Indeed, we can transform

$$\tilde{B}_k := A_k \cdot C_{k-(j^4+1)^2(n+1)},$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1) + (j^4+1)^2(n+1) - 1\},$$

$$\vdots$$

$$\tilde{B}_k := A_k \cdot C_{k-j^4(j^4+1)^2(n+1)},$$
$$k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1) - 1\}$$

into $B_k$ by

$$B_k = A_k \cdot C_{k-(j^4+1)^2(n+1)} \cdot \tilde{C}_k,$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1) + (j^4+1)^2(n+1) - 1\},$$

$$\vdots$$

$$\begin{aligned}&B_k = A_k \cdot C_{k-j^4(j^4+1)^2(n+1)} \cdot \tilde{C}_k,\\ &k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\},\end{aligned}$$

where $\tilde{C}_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 Q}(I)$ for all considered $k$ (see the condition (iv)), and hence we have (see the condition (iii) and also the below given (2.17) and (2.18))

$$\begin{aligned}&C_{k-(j^4+1)^2(n+1)} \cdot \tilde{C}_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-(j^4+1)^2(n+1)}\right),\\ &k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1)+(j^4+1)^2(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&C_{k-j^4(j^4+1)^2(n+1)} \cdot \tilde{C}_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-j^4(j^4+1)^2(n+1)}\right),\\ &k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\}.\end{aligned}$$

Thus, we can obtain $C_k$ from the previous step and from the above $\tilde{C}_k$.

We put

$$\begin{aligned}&B_k := A_k \cdot C_{k+(j^4+1)^3(n+1)},\\ &k \in \{-(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, -j^4(j^4+1)(n+1)-1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&B_k := A_k \cdot C_{k+j^4(j^4+1)^3(n+1)},\\ &k \in \{-j^4(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots,\\ &\qquad -(j^4-1)(j^4+1)^3(n+1) - j^4(j^4+1)(n+1) - 1\}\end{aligned}$$

and we denote

$$\begin{aligned}&C_k := A_k^{-1} \cdot B_k,\\ &k \in \{-j^4(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, -j^4(j^4+1)(n+1)-1\}.\end{aligned}$$

We proceed further in the same way. In the $(2i-1)$-th step, we choose

$$\begin{aligned}&B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_{2i} PQ}\left(C_{k-(j^4+1)^{2i-2}(n+1)}\right) \cap \mathcal{X},\\ &k \in \{(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1),\\ &\qquad \dots, (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) +\\ &\qquad\qquad \dots + j^4(j^4+1)^{2i-4}(n+1) + (j^4+1)^{2i-2}(n+1) - 1\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_{2i} PQ}\left(C_{k-j^4(j^4+1)^{2i-2}(n+1)}\right) \cap \mathcal{X},\\ &k \in \{(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1)+\\ &\qquad (j^4-1)(j^4+1)^{2i-2}(n+1), \dots, (j^4+1)(n+1) + j^4(j^4+1)^2(n+1)+\\ &\qquad\qquad \dots + j^4(j^4+1)^{2i-4}(n+1) + j^4(j^4+1)^{2i-2}(n+1) - 1\}\end{aligned}$$

such that

$$\prod_{k=p_1^0-1}^{q(i)} B_k = M_1, \qquad \prod_{k=p_1^0+(j^{3i}-j^0)(n+1)-1}^{q(i)} B_k = I,$$

$$\prod_{k=p_1^1-1}^{q(i)} B_k = M_1, \qquad \prod_{k=p_1^1+(j^{3i}-j^3)(n+1)-1}^{q(i)} B_k = I,$$

$$\vdots$$

$$\prod_{k=p_1^{i-1}-1}^{q(i)} B_k = M_1, \qquad \prod_{k=p_1^{i-1}+(j^{3i}-j^{3(i-1)})(n+1)-1}^{q(i)} B_k = I,$$

$$\vdots$$

$$\prod_{k=p_i^0-1}^{q(i)} B_k = M_i, \qquad \prod_{k=p_i^0+(j^{3i}-j^0)(n+1)-1}^{q(i)} B_k = I,$$

$$\prod_{k=p_i^1-1}^{q(i)} B_k = M_i, \qquad \prod_{k=p_i^1+(j^{3i}-j^3)(n+1)-1}^{q(i)} B_k = I,$$

$$\vdots$$

$$\prod_{k=p_i^{i-1}-1}^{q(i)} B_k = M_i, \qquad \prod_{k=p_i^{i-1}+(j^{3i}-j^{3(i-1)})(n+1)-1}^{q(i)} B_k = I,$$

and

$$\prod_{k=p(i)}^{q(i)} B_k = I,$$

where $p_1^0, \dots, p_1^{i-1}, \dots, p_i^0, \dots, p_i^{i-1}$ are arbitrary positive integers for which

$$q(i) + j^{2i}\,(n+1) \le p_1^0$$

and

$$p_1^0 + (j^{2i} + j^{3i})(n+1) \le p_1^1, \quad \dots \quad p_1^{i-2} + (j^{2i} + j^{3i})(n+1) \le p_1^{i-1},$$
$$p_1^{i-1} + (j^{2i} + j^{3i})(n+1) \le p_2^0,$$

$$\vdots$$

$$p_i^0 + (j^{2i} + j^{3i})(n+1) \le p_i^1, \quad \dots \quad p_i^{i-2} + (j^{2i} + j^{3i})(n+1) \le p_i^{i-1},$$
$$p_i^{i-1} + (j^{2i} + j^{3i})(n+1) \le p(i)$$

if

$$q(i) = (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1),$$
$$p(i) = (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-2}(n+1) - 1.$$

The existence of these numbers follows from

$$p(i) - q(i) = j^4(j^4+1)^{2i-2}(n+1) - 1 \ge (j^{2i} + j^{3i})(i^2+1)(n+1),$$
$$i, j \ge 2\,(i, j \in \mathbb{N}), \ n \in \mathbb{N}$$

and the existence of the above matrices $B_k$ follows from (2.13) and from

$$j^{3i} - j^{3(i-k)} \geq j^{2i}, \quad k \in \{1, \dots, i\},\ i \in \mathbb{N},\ j \geq 2\,(j \in \mathbb{N}).$$

In the $2i$-th step, we put

$$\begin{aligned}
&B_k := A_k \cdot C_{k+(j^4+1)^{2i-1}(n+1)}, \\
&k \in \{-((j^4+1)^{2i-1} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1), \\
&\qquad \dots, -(j^4(j^4+1)^{2i-3} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\}, \\
&\qquad\qquad \vdots \\
&B_k := A_k \cdot C_{k+j^4(j^4+1)^{2i-1}(n+1)}, \\
&k \in \{-j^4((j^4+1)^{2i-1} + \dots + (j^4+1)^3 + (j^4+1))(n+1), \\
&\qquad \dots, -((j^4-1)(j^4+1)^{2i-1} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\}
\end{aligned}$$

and we denote

$$\begin{aligned}
C_k := A_k^{-1} \cdot B_k, \quad &k \in \{-(j^4(j^4+1)^{2i-1} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1), \\
&\dots, -(j^4(j^4+1)^{2i-3} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\}.
\end{aligned}$$

Using this construction, we obtain the sequence $\{B_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$.

We will consider the system

$$x_{k+1} = B_k \cdot x_k, \qquad k \in \mathbb{Z}. \tag{2.14}$$

Suppose that there exists a nonzero vector $u \in F^m$ for which the solution $\{x_k\}$ of (2.14) satisfying $x_{n+1} = u$ is almost periodic. We know that

$$x_k = \prod_{i=k-1}^{n+1} B_i \cdot u \quad \text{for } k > n+1,\ k \in \mathbb{N}. \tag{2.15}$$

If we choose $\{h_i\}_{i\in\mathbb{N}} \equiv \{j^{3(i-1)}(n+1)\}_{i\in\mathbb{N}}$ for $\{\varphi_k\} \equiv \{x_k\}$ in Corollary 1.4 (see also Theorem 1.3), then, for any $\vartheta > 0$, we get the existence of an infinite set $N = N(\vartheta) \subseteq \mathbb{N}_0$ such that

$$\varrho\left(x_{k+j^{3i_1}(n+1)}, x_{k+j^{3i_2}(n+1)}\right) < \vartheta, \quad k \in \mathbb{Z},\ i_1, i_2 \in N. \tag{2.16}$$

Thus, for every $\vartheta > 0$, there exist infinitely many $\vartheta$-translation numbers in the form $(j^{3i_1} - j^{3i_2})(n+1)$, where $i_1 > i_2$ $(i_1, i_2 \in \mathbb{N})$. For some $i \in \mathbb{N}$ with the property that $M_i\, u \neq u$, we choose $\vartheta < \varrho\,(M_i\, u, u)$ and the above $i_1 > i_2 > i$ $(i_1, i_2 \in \mathbb{N})$ arbitrarily. We have (see (2.16))

$$\varrho\left(x_{k+(j^{3i_1}-j^{3i_2})(n+1)}, x_k\right) < \vartheta, \quad k \in \mathbb{Z}.$$

From (2.15) and the construction of $\{B_k\}$, we obtain

$$\varrho\left(x_{k+(j^{3i_1}-j^{3i_2})(n+1)}, x_k\right) > \vartheta \quad \text{for at least one } k \in \mathbb{N}.$$

This contradiction gives that $\{x_k\}$ cannot be almost periodic. It means that system (2.14) does not have a nontrivial almost periodic solution.

Now it suffices to show that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$; i.e., that $B_k \in \mathcal{O}^{\varrho}_{\tilde{\varepsilon}}(A_k)$ for all $k$ and some $\tilde{\varepsilon} \in (0, \varepsilon)$ and that $\{B_k\}$ is almost periodic. It is seen that

$$C_k \in \mathcal{O}^{\varrho}_{\varepsilon_2 Q}(I), \qquad k \in \{-j^4(j^4+1)(n+1), \dots, 0, \dots, (j^4+1)(n+1)-1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4)PQ}(I),$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4)PQ}(I),$$
$$k \in \{-j^4(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, -j^4(j^4+1)(n+1)-1\}$$

and that, for all $i \geq 3$ ($i \in \mathbb{N}$), it is valid

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4+\dots+\varepsilon_{2i})PQ}(I),$$
$$k \in \{(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1), \dots,$$
$$(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-2}(n+1) - 1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4+\dots+\varepsilon_{2i})PQ}(I),$$
$$k \in \{-(j^4(j^4+1)^{2i-1} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1), \dots,$$
$$-(j^4(j^4+1)^{2i-3} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\}.$$

Thus, we have (see (2.12))

$$C_k \in \mathcal{O}^{\varrho}_{\eta}(I), \quad k \in \mathbb{Z}, \tag{2.17}$$

and hence (see (iii))

$$B_k \in \mathcal{O}^{\varrho}_{\varepsilon/2}(A_k), \quad k \in \mathbb{Z}. \tag{2.18}$$

Indeed,

$$B_k = A_k \cdot C_k \quad \text{for all } k \in \mathbb{Z}. \tag{2.19}$$

From Theorem 1.16 it follows that the sequence $\{C_k\}$ is almost periodic. Using Corollary 1.11 and the almost periodicity of $\{A_k\}$, we see that the set

$$T(\{A_k\}, \delta) \cap T(\{C_k\}, \delta) \text{ is relative dense in } \mathbb{Z} \tag{2.20}$$

for any $\delta > 0$. Since the multiplication of matrices is uniformly continuous on $\mathcal{X}$, considering (2.19), we have

$$T(\{A_k\}, \delta(\vartheta)) \cap T(\{C_k\}, \delta(\vartheta)) \subseteq T(\{B_k\}, \vartheta) \tag{2.21}$$

for arbitrary $\vartheta > 0$, where $\delta(\vartheta) > 0$ is the number corresponding to $\vartheta$ from the definition of the uniform continuity of the multiplication of two matrices. Finally, (2.20) and (2.21) give the almost periodicity of $\{B_k\}$ which completes the proof. $\square$

**Example 2.12.** All groups of matrices from Examples 2.2–2.6 (except the general case in Example 2.6) satisfy the requirements of Theorem 2.11.

From the proof of the above theorem, we get the following result:

**Corollary 2.13.** *Let $\mathcal{X}$ be strongly transformable. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$, $\varepsilon > 0$, and $u \neq o$, $u \in F^m$ be given. If there exists a matrix $M \in \mathcal{X}$ such that $M\,u \neq u$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ for which the solution of*

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

*is not almost periodic.*

Note that, in Theorem 2.11, the condition (2.10) can be omitted, i.e., we can put $L = 0$. We will obtain this fact from the below given Theorem 2.20. Now, for the later comparison, let us recall a statement from [38], see also [1, Theorem 2.10.1], and one of its known consequences.[10]

**Theorem 2.14.** *Let $(F, \varrho(\cdot,\cdot)) = (\mathbb{C}, |\cdot - \cdot|)$. If a vector valued sequence $\{b_k\}$ is almost periodic and a matrix $A \in \mathcal{M}at(\mathbb{C}, m)$ nonsingular, then a solution of*

$$x_{k+1} = A \cdot x_k + b_k, \quad k \in \mathbb{Z}$$

*is almost periodic if and only if it is bounded.*

**Corollary 2.15.** *Let $(F, \varrho(\cdot,\cdot)) = (\mathbb{C}, |\cdot - \cdot|)$. Let a periodic sequence $\{A_k\}$ of $m \times m$ nonsingular matrices with complex elements be given. Then, a solution of the system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$ is almost periodic if and only if it is bounded.*

*Proof.* Every almost periodic sequence is bounded, and hence we need only to show that the boundedness of a solution implies its almost periodicity. Assume that we have a periodic system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$ and its bounded solution $\{x_k\}$. Let $n \in \mathbb{N}$ be a period of $\{A_k\}$. Applying Theorem 2.14, we get that the sequence $\{y_k^1\} \equiv \{x_{nk}\}$; i.e.,

$$y_0^1 = x_0, \quad y_k^1 = \prod_{i=nk-1}^{0} A_i \cdot x_0, \quad k \in \mathbb{N}, \quad y_k^1 = \prod_{i=nk}^{-1} A_i^{-1} \cdot x_0, \quad k \in \mathbb{Z} \smallsetminus \mathbb{N};$$

is almost periodic. Indeed, $\{y_k^1\}$ is a bounded solution of the constant system

$$y_{k+1} = A_{n-1} \cdots A_1 \cdot A_0 \cdot y_k, \quad k \in \mathbb{Z}.$$

Analogously, one can show that the sequences $\{y_k^j\} \equiv \{x_{nk+j-1}\}$, $j \in \{2, 3, \ldots, n\}$ are almost periodic as well. The almost periodicity of $\{x_k\}$ follows from Corollary 1.9. □

**Remark 2.16.** We add that it is possible to obtain several modifications of Corollary 2.15 for nonhomogeneous systems if the nonhomogeneousness is almost periodic. Let mention at least the most important one—the continuous version for differential systems. If a complex matrix valued function $A(t)$, $t \in \mathbb{R}$, is periodic and a complex vector valued function $b(t)$, $t \in \mathbb{R}$, is almost periodic, then any solution of $x'(t) = A(t)\, x(t) + b(t)$, $t \in \mathbb{R}$ is almost periodic if and only if it is bounded. See introduction of Chapter 4 or, e.g., [69, Corollary 6.5], and [115] for generalizations and supplements.

**Example 2.17.** Consider again $(F, \varrho(\cdot,\cdot)) = (\mathbb{C}, |\cdot - \cdot|)$. We want to document that Corollary 2.15 is no longer true if $\{A_k\}$ is only almost periodic. It was shown (see Lemma 1.20 and consider the second part of Corollary 1.15) that the real sequence $\{a_k\}$ defined by the recurrent formula

$$a_0 := 1, \quad a_1 := 0, \quad a_{2^n+k} := a_k - \frac{1}{2^n}, \quad n \in \mathbb{N},\ k = 0, \ldots, 2^n - 1 \tag{2.22}$$

on $\mathbb{N}_0$ and by the prescription

$$a_k := -a_{-k-1} \quad \text{for } k \in \mathbb{Z} \smallsetminus \mathbb{N}_0 \tag{2.23}$$

is almost periodic and that it satisfies (see Lemma 1.23)

$$\sum_{k=0}^{2^n-1} a_k = 1, \qquad \sum_{k=0}^{2^{n+j}+2^n-1} a_k = 2 - \frac{1}{2^j}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}. \tag{2.24}$$

Let $\mathcal{X}$ be the set of all $m \times m$ diagonal matrices with numbers on the diagonal which has absolute value 1. (It is easily seen that, in this case, $\mathcal{X}$ is strongly transformable. See also Examples 2.4 and 2.9.) All solutions of the system of the form (2.1) given by the almost periodic (see Theorem 1.5) sequence $\{A_k\} \equiv \operatorname{diag}[\exp(\mathrm{i}a_k), \dots, \exp(\mathrm{i}a_k)]$ are obviously bounded but we will show that they are not almost periodic (except the trivial one).

It suffices to consider the scalar case, i.e., $m = 1$. Assume that the system has an almost periodic solution $\{x_k\}$. We have

$$x_k = \exp\left(\mathrm{i}\sum_{j=0}^{k-1} a_j\right) \cdot x_0, \quad k \in \mathbb{N}.$$

Especially (see (2.24)),

$$x_{2^n} = \exp(\mathrm{i}) \cdot x_0, \quad x_{2^{n+j}+2^n} = \exp\left(2\mathrm{i} - 2^{-j}\mathrm{i}\right) \cdot x_0, \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}. \tag{2.25}$$

Using Corollary 1.4 (or Theorem 1.3) for the sequence $\{2^j\}_{j\in\mathbb{N}}$, for any $\varepsilon > 0$, we get an infinite set $N = N(\varepsilon) \subseteq \mathbb{N}$ such that

$$|\, x_{k+2^{j(1)}} - x_{k+2^{j(2)}} \,| < \varepsilon \quad \text{for all } k \in \mathbb{Z},\ j(1), j(2) \in N.$$

For some $j(1) \in N$, the choice $k = 2^{j(1)}$ and (2.25) give

$$\left|\exp(\mathrm{i}) - \exp\left(2\mathrm{i} - 2^{j(1)-j(2)}\mathrm{i}\right)\right| \cdot |\, x_0 \,| < \varepsilon, \quad j(1) < j(2),\ j(1), j(2) \in N.$$

Since $\varepsilon$ can be arbitrarily small and $j(2) > j(1)$ can be found for every $\varepsilon > 0$, we obtain $x_0 = 0$. Thus, the system does not have a nontrivial almost periodic solution.

In the above example, we see that the boundedness is necessary to the almost periodicity of solutions of considered almost periodic systems but not sufficient. Now we prove a more important necessary (also not sufficient, see again Example 2.17) condition about the limitation of almost periodic solutions in the next lemma.[11]

**Lemma 2.18.** *Let an almost periodic sequence of nonsingular $A_k \in \mathcal{M}at(F, m)$ be given. Let $\{x_k\}$ be an almost periodic solution of the system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$. Then, it is valid either $x_k = o$, $k \in \mathbb{Z}$ or*

$$\inf_{k\in\mathbb{Z}} \varrho(x_k, o) > 0.$$

*Proof.* Suppose that an almost periodic solution $\{x_k\}$ of a system satisfies $\inf_{k\in\mathbb{Z}} \varrho(x_k, o) = 0$. Let $\{h_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ be such that

$$\lim_{i\to\infty} \varrho(x_{h_i}, o) = 0. \tag{2.26}$$

Considering Corollary 1.4 and Theorems 1.6 and 1.7, we get a subsequence $\{\tilde{h}_i\}$ of $\{h_i\}$ for which there exist almost periodic sequences $\{B_k\}$, $\{y_k\}$ satisfying

$$\lim_{i\to\infty} A_{k+\tilde{h}_i} = B_k, \quad \lim_{i\to\infty} B_{k-\tilde{h}_i} = A_k, \quad \lim_{i\to\infty} x_{k+\tilde{h}_i} = y_k, \quad \lim_{i\to\infty} y_{k-\tilde{h}_i} = x_k,$$

where the convergences can be uniform with respect to $k \in \mathbb{Z}$ (see Remark 1.8). We have $y_{k+1} = B_k\, y_k$, $k \in \mathbb{Z}$ and $y_0 = o$ (see (2.26)). Thus, $\{y_k\} \equiv \{o\}$. Consequently, $x_k = \lim_{i\to\infty} y_{k-\tilde{h}_i} = o$ for $k \in \mathbb{Z}$. $\square$

**Example 2.19.** Applying Theorem 1.16 for $n = 0$, $\varphi_0 = 2$, $j = 1$, and $r_i = 3/2^i$, $i \in \mathbb{N}$, we construct the everywhere nonzero almost periodic sequence

$$b_0 := 2, \quad b_1 := 2 - 1, \quad b_{-2} := 2 - \tfrac{1}{2}, \quad b_{-1} := 1 - \tfrac{1}{2},$$

$$\vdots$$

$$b_k := b_{k+2^{2i-1}} - \tfrac{1}{2^{2i-1}}, \quad k \in \{-2^{2i-1} - \cdots - 2^3 - 2, \dots, -2^{2i-3} - \cdots - 2^3 - 2 - 1\},$$

$$b_k := b_{k-2^{2i}} - \tfrac{1}{2^{2i}}, \quad k \in \{2 + 2^2 + \cdots + 2^{2i-2}, \dots, 2 + 2^2 + \cdots + 2^{2i-2} + 2^{2i} - 1\},$$

$$\vdots$$

in the space $(\mathbb{R}, |\cdot - \cdot|)$. Since

$$\lim_{i\to\infty} b_{2^0-2^1+2^2-2^3+\cdots+(-2)^i} = 0,$$

the equation $x_{k+1} = b_k\, x_k$, $k \in \mathbb{Z}$ does not have a nontrivial almost periodic solution (see Lemma 2.10) and the vector valued sequence $\{b_k\, u\}$, where $u \neq o$, $u \in \mathbb{R}^m$, is not a solution of an almost periodic homogeneous linear difference system.

Moreover, for any bounded countable set of real numbers, it is shown in [70] that there exists an almost periodic sequence whose range is the set. It means that there exists a large class of almost periodic sequences which cannot be solutions of any almost periodic system (2.1).

**Theorem 2.20.** *Let $\mathcal{X}$ be transformable and let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrary. If there exists a matrix $M(\vartheta) \in \mathcal{O}^{\varrho}_{\vartheta}(O) \cap \mathcal{X}$ for any $\vartheta > 0$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not have an almost periodic solution other than the trivial one.*[12]

*Proof.* We put $L_i := \varrho\,(M_i, O)$ for matrices $M_i \in \mathcal{X}$, $i \in \mathbb{N}$ such that

$$\lim_{i\to\infty} L_i = 0, \qquad L_{i+1} < L_i, \quad i \in \mathbb{N}. \tag{2.27}$$

Let $\eta = \eta(\varepsilon/2)$, $\zeta$, and $P$ and $Q = Q(L_1)$ be from the conditions (iii) and (iv), respectively. As in the proof of Theorem 2.11, we can assume (or choose $\{B_k\} \equiv \{A_k\}$) that

$$\eta < \varepsilon < \zeta, \qquad L_1 + \varepsilon < \varrho\,(A_k, O)\,, \quad k \in \mathbb{Z}$$

and that (see also Lemma 2.7) we have $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$, $j \geq 2$ ($j \in \mathbb{N}$), and $n \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{\infty} \varepsilon_i < \frac{\eta}{PQ},$$

$$j^i(n+1) \geq p\left(L_{g(i)}, \varepsilon_i\right) \quad \text{for infinitely many odd } i \in \mathbb{N}, \tag{2.28}$$

where $g(i)$ is from Lemma 2.7 too.

The set of all $i \neq 1$ ($i \in \mathbb{N}$), which are not divisible by 2 and for which (2.28) is valid, will be denoted by $N$. Let $N = \{i_1, i_2, \dots, i_l, \dots\}$, where $i_l < i_{l+1}$, $l \in \mathbb{N}$. Since we can redefine $L_i$ (choose other $M_i$), we can also assume that $g(i_l) \geq l$, $l \in \mathbb{N}$. We will construct sequences $\{B_k\}$ and $\{C_k\}$ as in the proof of Theorem 2.11 for $j^4$ replaced by $j$. First of all we put

$$p_l := (j + 1 + j(j+1)^2 + \cdots + j(j+1)^{i_l-3})(n+1), \quad l \in \mathbb{N},$$
$$q_l := (j + 1 + j(j+1)^2 + \cdots + j(j+1)^{i_l-1})(n+1) - 1, \quad l \in \mathbb{N}.$$

Before the $i_1$-th step (for $k \leq p_1 - 1$), we choose the matrices $C_k$ (consequently $B_k$) arbitrarily. We will obtain $B_k$ and define

$$J_1 := \prod_{k=p_1}^{0} B_k, \qquad J_2 := \prod_{k=p_2}^{0} B_k, \qquad \dots$$

In the $i_1$-th step, we choose the matrices $C_k$ arbitrarily if $J_1 \in \mathcal{O}^{\varrho}_{L_1}(O)$, and so that

$$\prod_{k=q_1}^{0} B_k = M_1 \quad \text{if} \quad J_1 \notin \mathcal{O}^{\varrho}_{L_1}(O).$$

Between the $i_1$-th step and the $i_2$-th step, we choose them again arbitrarily. In the $i_2$-th step, we choose them arbitrarily if $J_2 \in \mathcal{O}^{\varrho}_{L_2}(O)$, and so that

$$\prod_{k=q_2}^{0} B_k = M_2 \quad \text{if} \quad J_2 \notin \mathcal{O}^{\varrho}_{L_2}(O).$$

If we proceed further in the same way, then we get matrices $C_k$, $B_k$ for all $k \in \mathbb{Z}$. Analogously as in the proof of Theorem 2.11, we can prove that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$. We have

$$\varrho\left(\prod_{k=r_l}^{0} B_k, O\right) \leq L_l \quad \text{for } r_l \in \{p_l, q_l\},\ l \in \mathbb{N}. \tag{2.29}$$

Evidently, for any $u \in F^m$ and $\mu > 0$, there exists $\delta = \delta\,(u, \mu) > 0$ with the property that $\varrho\,(C\,u, o) < \mu$ if $C \in \mathcal{O}^{\varrho}_{\delta}(O)$. Using this, from (2.27), (2.29), and Lemma 2.18, we get that all nontrivial solutions of the system of the form (2.1) given by $\{B_k\}$ are not almost periodic. □

**Corollary 2.21.** *Let $\mathcal{X}$ be transformable, $\{A_k\} \in \mathcal{AP}(\mathcal{X})$, nonzero $u \in F^m$, and $\varepsilon > 0$ be arbitrary. If there exists a sequence $\{M_i\}_{i \in \mathbb{N}} \subseteq \mathcal{X}$ with the property that $\lim_{i\to\infty} \varrho\,(M_i\,u, o) = 0$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ for which the solution of*

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

*is not almost periodic.*

*Proof.* It suffices to consider Lemma 2.18, the construction from the proof of Theorem 2.11, and the sequence $\{B_k\}$ from the proof of Theorem 2.20. □

Combining of Theorems 2.11 and 2.20, we obtain:

**Theorem 2.22.** *Let $\mathcal{X}$ be strongly transformable and have a dense countable subset. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. If for any vector $u \neq o$, $u \in F^m$, there exists $M(u) \in \mathcal{X}$ for which*

$$M(u) \cdot u \neq u,$$

*then there exists $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\})$ which does not have almost periodic solutions.*

It remains to examine the case for transformable $\mathcal{X}$ when there exists $L \in \mathbb{R}^+$ for which $\mathcal{X} \cap \mathcal{O}^\varrho_L(O) = \emptyset$.

**Theorem 2.23.** *Let $\mathcal{X}$ be transformable and let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrary. If there exist a precompact transformable set $\mathcal{X}_0 \subseteq \mathcal{X}$ and $M \in \mathcal{X}_0$ for which $\varrho\,(M\,u, u) > 0$ for all $u \neq o$, $u \in F^m$, and if there exists $\{u_i\}_{i \in \mathbb{N}} \subseteq F^m$ such that, for any $\vartheta > 0$ and $u \neq o$, $u \in F^m$, it is possible to find $\delta = \delta(u, \vartheta) > 0$ and $i = i(u) \in \mathbb{N}$ with the property that, for every $C \in \mathcal{X} \smallsetminus \mathcal{O}^\varrho_\vartheta(I)$, one can choose $N(u_i, C) \in \mathcal{X}$ satisfying*

$$\varrho\,(N(u_i, C) \cdot u, C \cdot N(u_i, C) \cdot u) > \delta, \tag{2.30}$$

*then there exists $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\})$ which does not possess an almost periodic solution other than the trivial one.*

*Proof.* From Theorem 2.20 it follows that, without loss of the generality, we can assume the existence of $L \in \mathbb{R}^+$ such that $\mathcal{X} \cap \mathcal{O}^\varrho_L(O) = \emptyset$. From (ii) for $C_0 = M$, $C_i = I$, $i \in \{1, \dots, n\}$ and (iii), we get that, for every $\vartheta(1) > 0$, there exist matrices $D_1(1), \dots, D_{n(1)}(1) \in \mathcal{X}_0$ satisfying

$$D_1(1) \in \mathcal{O}^\varrho_{\vartheta(1)}(I), \qquad D_{n(1)}(1) \in \mathcal{O}^\varrho_{\vartheta(1)}(M),$$

$$D_i(1) \in \mathcal{O}^\varrho_{\vartheta(1)}(D_{i+1}(1)), \quad i \in \{1, \dots, n(1) - 1\}.$$

We put $D_0(1) := I$, $D_{n(1)+1}(1) := M$. For every number $\vartheta(2) \in (0, \vartheta(1))$ and each $i \in \{1, \dots, n(1) + 1\}$, analogously (consider $C_0 = D_i(1)$, $C_1 = D_{i-1}(1)$, $C_l = I$ for each $l \in \{2, \dots, n\}$), there exist matrices $D_1(2, i), \dots, D_{n(2)}(2, i) \in \mathcal{X}_0$ satisfying

$$D_1(2, i) \in \mathcal{O}^\varrho_{\vartheta(2)}(D_{i-1}(1)), \qquad D_{n(2)}(2, i) \in \mathcal{O}^\varrho_{\vartheta(2)}(D_i(1)),$$

$$D_l(2, i) \in \mathcal{O}^\varrho_{\vartheta(2)}(D_{l+1}(2, i)), \quad l \in \{1, \dots, n(2) - 1\}.$$

Since we can proceed in the same way for a sequence of positive numbers $\vartheta_1 > \vartheta_2 > \cdots > \vartheta_n > \cdots$ converging to 0, $\mathcal{X}_0$ is precompact, and since the space $\mathcal{M}at(F, m)$ is complete, there exists a continuous map $F : [0, 1] \to \mathcal{M}at(F, m)$ such that

$$F(0) = I, \quad F(1) = M, \quad \text{the set } F([0, 1]) \cap \mathcal{X} \text{ is dense in } F([0, 1]). \tag{2.31}$$

Let $T$ denote the set of all $t \in [-1, 1]$ for which $F(|\,t\,|) \in \mathcal{X}$. If we extend the domain of definition of $F$ by the formula $F(-t) := (F(t))^{-1}$, $t \in T \cap (0, 1]$, then we get continuous $F : T \to \mathcal{X}$. Indeed, the map $C \mapsto C^{-1}$, $C \in \mathcal{X}$ is uniformly continuous—consider the conditions (iii) and (iv) and the composition $C \mapsto C^{-1}\,D \mapsto C^{-1}\,D\,D^{-1}$ for given $D \in \mathcal{X}$ from a neighbourhood of $C$.

Let the almost periodic sequence $\{a_k\}$ be from Example 2.17. Directly from (2.22) and (2.23), it is seen that $\{a_k; k \in \mathbb{Z}\} \subset [-1,1]$. We can assume that $a_k \in T$ for all $k$ (consider (2.31)). From Theorem 1.26, we know that the system $\{F(a_k)\}$ does not have a nonzero almost periodic solution. (We remark that the almost periodicity of $\{F(a_k)\}$ follows from Theorem 1.5.)

Suppose that there exists $p \in \mathbb{N}$ satisfying $A_{k+p-1} \cdots A_{k+1} \cdot A_k = I$ for all $k$. Let $\tilde{n} \in \mathbb{N}$ be such that $F(a) \in \mathcal{O}^{\varrho}_{\varepsilon/2}(F(b))$ if $|\, a-b \,| \leq 2^{1-\tilde{n}}$, $a, b \in T$. From the condition (ii), we obtain the existence of a number $l \in \mathbb{N}$ for which we can construct periodic $\{\tilde{A}_k\} \in \mathcal{O}^{\sigma}_{\varepsilon/2}(\{A_k\})$ with a period $2^{\tilde{n}} lp$ and with the property that

$$\prod_{k=lp-1}^{0} \tilde{A}_k = F(a_0), \qquad \prod_{k=2lp-1}^{lp} \tilde{A}_k = F(a_1), \qquad \dots \qquad \prod_{k=2^{\tilde{n}}lp-1}^{(2^{\tilde{n}}-1)lp} \tilde{A}_k = F(a_{2^{\tilde{n}}-1}).$$

Now we change the matrices $\tilde{A}_{klp-1}$ to $B_{klp-1}$ for $k \in \mathbb{Z}$ in order that

$$\prod_{k=ilp-1}^{(i-1)lp} B_k = F(a_{i-1}), \quad i \in \mathbb{Z}.$$

We define $B_k := \tilde{A}_k$ for the other numbers $k$. Considering (2.22) and (2.23), consequently $|\, a_k - a_{k+i2^{\tilde{n}}} \,| \leq 2^{1-\tilde{n}}$, $i, k \in \mathbb{Z}$, we have

$$B_k \in \mathcal{O}^{\varrho}_{\varepsilon/2}\left(\tilde{A}_k\right), \quad k \in \mathbb{Z}.$$

Thus, $B_k \in \mathcal{O}^{\varrho}_{\tilde{\varepsilon}}(A_k)$ for all $k$ and some $\tilde{\varepsilon} < \varepsilon$. From Corollaries 1.9 and 1.11, the almost periodicity of $\{F(a_k)\}$ and $\{\tilde{A}_k\}$, and the conditions (iii) and (iv), we get the almost periodicity of $\{B_k\}$. We emphasize that $\{F(a_k)\}$ does not have a nonzero almost periodic solution. Finally, $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ and this system cannot have nontrivial almost periodic solutions.

We put $\vartheta := \eta(\varepsilon/2)/Q(L)$. If $A_{k+p-1} \cdots A_{k+1}\, A_k \in \mathcal{O}^{\varrho}_{\vartheta}(I)$, $k \in \mathbb{Z}$ for some $p \in \mathbb{N}$, it suffices to replace matrices $A_{kp-1}$ by $B_{kp-1}$ for $k \in \mathbb{Z}$ so that

$$B_{kp-1} \cdot A_{kp-2} \cdots A_{kp-p} = I, \quad k \in \mathbb{Z}.$$

Indeed, considering the conditions (iii), (iv) and

$$B_{kp-1} = A^{-1}_{kp-p} \cdots A^{-1}_{kp-2} \cdot A^{-1}_{kp-1} \cdot A_{kp-1}, \quad k \in \mathbb{Z}, \tag{2.32}$$

we have that $B_{kp-1} \in \mathcal{O}^{\varrho}_{\varepsilon/2}(A_{kp-1})$, $k \in \mathbb{Z}$; the almost periodicity of $\{A^{-1}_{kp-p} \cdots A^{-1}_{kp-2}\, A^{-1}_{kp-1}\}$ and, consequently (see also (2.32)), the almost periodicity of the obtained sequence follow from Corollaries 1.9 and 1.11, the almost periodicity of $\{A_k\}$, the uniform continuity of the multiplication of matrices on $\mathcal{X}$ and the map $C \mapsto C^{-1}$, $C \in \mathcal{X}$, and from Theorem 1.5.

It remains to consider the case that $A_{k+p-1} \cdots A_{k+1}\, A_k \notin \mathcal{O}^{\varrho}_{\vartheta/2}(I)$ for all $p \in \mathbb{N}$ and infinitely many $k \in \mathbb{N}$ which depend on $p$. (It has to be valid for infinitely many $k$, not finitely many only, because sequence $\{A_k\}$ is almost periodic.) Now we can construct $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon/3}(\{A_k\})$ as in the proof of Theorem 2.11. Since $\mathcal{X}$ is only transformable, we can get matrices $B_k$ having certain properties generally in infinitely many steps as in the proof of Theorem 2.20 (see also Lemma 2.7) for arbitrarily given $j \geq 2$ ($j \in \mathbb{N}$) and $n \in \mathbb{N}$. If we obtain $B_{k+p-1} \cdots B_{k+1} B_k \in \mathcal{O}^{\varrho}_{\vartheta}(I)$ for all $k$ and some $p$ after finitely many steps of the process, then we can use the above mentioned construction. Thus, we can assume without loss of the generality that we can choose matrices $B_k$ in odd steps $i_1, \dots, i_l, \dots$ so that

$$\prod_{k=p_1^1-1}^{0} B_k = N(u_1, C_1^1), \qquad B_{p_1^1} = C_1^1 \notin \mathcal{O}^{\varrho}_{\vartheta/2}(I),$$

$$\vdots$$

$$\prod_{k=p_l^1-1}^{0} B_k = N(u_1, C_l^1), \qquad \prod_{k=p_l^1+l-1}^{p_l^1} B_k = C_l^1 \notin \mathcal{O}^{\varrho}_{\vartheta/2}(I),$$

$$\vdots$$

$$\prod_{k=p_l^l-1}^{0} B_k = N(u_l, C_l^l), \qquad \prod_{k=p_l^l+l-1}^{p_l^l} B_k = C_l^l \notin \mathcal{O}^{\varrho}_{\vartheta/2}(I),$$

$$\vdots$$

where $p_l^1, \dots, p_l^l$ are positive integers for which

$$p_l^1 + l + j^{i_l}(n+1) \leq p_l^2, \; \dots, \; p_l^{l-1} + l + j^{i_l}(n+1) \leq p_l^l.$$

Suppose now that the solution $\{x_k\}$ of $x_{k+1} = B_k\, x_k$, $k \in \mathbb{Z}$, $x_0 = u$ is almost periodic for some $u \neq o$, $u \in F^m$. Let $\delta$ and $i$ be the numbers corresponding to $u$ from the statement of the theorem. Immediately, from the above construction and from (2.30) it follows that the set $T(\{x_k\}, \delta)$ contains of numbers less than $i$. This contradiction proves the theorem. □

**Example 2.24.** Obviously, the $m \times m$ unitary group and the group of all orthogonal matrices of dimension $m \geq 2$ with determinant 1 satisfy the conditions of Theorem 2.23 (as any dense transformable subset of one of them). The groups from Examples 2.4 and 2.6 do not satisfy the conditions for $m \geq 2$; consider, e.g., $S = I$,

$$u = \left[1, o^T\right]^T, \quad C = \begin{bmatrix} I & o \\ o^T & \exp(\mathrm{i}) \end{bmatrix},$$

where $o$ and $I$ have dimension $m-1$. We add that we can obtain examples of transformable sets which are not strongly transformable using changes of the metric in Examples 2.2–2.6.

**Corollary 2.25.** *Let $\mathcal{X}$ be transformable and such that $\mathcal{X} \cap \mathcal{O}^{\varrho}_{L}(O) = \emptyset$ for some $L \in \mathbb{R}^+$, and let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$, $u \neq o$, $u \in F^m$, and $\varepsilon > 0$ be arbitrary. If there exist a precompact transformable set $\mathcal{X}_0 \subseteq \mathcal{X}$ and $M \in \mathcal{X}_0$ satisfying $\varrho(M\,u, u) > 0$ and if there exists $\delta > 0$ and, for any $C \in \mathcal{X} \setminus \mathcal{O}^{\varrho}_{\eta(\varepsilon/2)/2Q(L)}(I)$, there exists $N(C) \in \mathcal{X}$ with the property that*

$$\varrho\,(N(C) \cdot u, C \cdot N(C) \cdot u) > \delta,$$

*then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ for which the solution of*

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

*is not almost periodic.*

*Proof.* It follows from the proof of Theorem 2.23 and from Remark 1.27. □

At the end, we say that it is possible to obtain various generalizations and modifications of results presented in this section. For simplicity, we considered only sufficiently general and, at the same time, more important cases. Especially, in Chapter 1, the constructions are used for a ring with a pseudometric. For almost periodic sequences defined for $k \in \mathbb{N}$ (or $k \in \mathbb{N}_0$), it suffices to replace Corollary 1.4 by Remark 1.2 (or to apply [92]) and Theorem 1.16 by Theorem 1.12. The basic theory of almost periodic sequences on $\mathbb{N}$ is established, e.g., in [51].

Again for simplicity, we required in the condition (ii) that matrices $D_1, \dots, D_n$ with the given property exist for all $n \geq p$, not only for an infinite set of $n$ (relative dense in $\mathbb{N}$). Hence, the group of all real orthogonal matrices of dimension $m \geq 2$ is not transformable. Indeed, one can find $\varepsilon > 0$ such that, in the $\varepsilon$-neighbourhood of an orthogonal matrix, there are only orthogonal matrices with the same determinant. Let an almost periodic sequence of orthogonal matrices $A_k$ be given. Using the fact that the sequence of det $A_k$ is periodic, we can modify the proof of Theorem 2.11 and get the statement of this theorem.

In this chapter, we studied non-almost periodic solutions of almost periodic difference equations. We considered homogeneous linear difference systems all of whose solutions can be almost periodic (e.g., unitary systems) in a general setting when coefficients belong to a complete metric field. We found classes of these systems such that, in any neighbourhood of an arbitrary system, there exists a system from the same class which does not possess any nontrivial almost periodic solution. Note that an analogous general statement for differential equations is not known.

# Footnotes to Part I

1 In fact, asymptotically almost periodic sequences are considered in [63] based on the Fréchet concept from [71], [72]. Note that, in Banach spaces, a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is *asymptotically almost periodic* if and only if it is the sum of an almost periodic sequence and a sequence which approaches zero as $n \to \infty$; a function defined for $t \in \mathbb{R}$ is *asymptotically almost periodic* if and only if it can be expressed as the sum of an almost periodic function and a continuous function which approaches zero as $t \to +\infty$. One finds that these representations are unique. Here, we can also refer to [136], [170].

2 Using a combination of the methods mentioned in the proofs of Theorems 1.3 and 3.4, it is possible to prove that, for a continuous function $f : G \to \mathcal{X}$ with $G$ an abelian topological group and $\mathcal{X}$ a complete metric space, the definitions of almost periodicity in the sense of Bohr and Bochner are equivalent. For other equivalent definitions (e.g., the von Neumann and the Maak definition) of almost periodicity, see [108], [114]. For the first time, almost periodic functions on groups with values in Banach spaces were studied by S. Bochner and J. von Neumann in [19], [20].

We warn that the so-called Levitan definition is not equivalent with the Bohr one (see [109]). Every Bohr almost periodic sequence or function is Levitan almost periodic and, moreover, every almost automorphic sequence or function (see Remark 1.8) is Levitan almost periodic (see [9]). Generally speaking, the inverses are not true because Levitan almost periodic sequences and functions may be unbounded.

3 This implication follows from the corresponding part of the proof of Theorem 3.4 (mentioned in Footnote 5 in Part II) as well. We have to consider only that the argument of the considered function $\psi$ is no longer a real number, but an integer.

4 It is also proved in [70] that, for any bounded countable set of real numbers which is dense in itself, there exists a one-to-one function from $\mathbb{Z}$ onto the set with the property of being an almost periodic sequence.

5 The function $F_1$ exists if, e.g., the pseudometric $d : X \times X \to \mathbb{R}_0^+$ is such that the map $U \mapsto U^{-1}$ is continuous on $X$, and there exists a continuous function $G : [0, 1] \to X$ which satisfies that at least one of matrices $G^{-1}(0)\, G(1)$, $G(1)\, G^{-1}(0)$ does not have an $e_1$-eigenvector.

6 The conditions in (1.57) are realized if, e.g., the map $U \mapsto U^{-1}$ is continuous on $X$ and there exist continuous functions $G_1, \ldots, G_q : [0,1] \to X$ such that

$$G_j^{-1}(0) \cdot G_j\left(\sum_{i=1}^{p} s_i\right) = \prod_{i=1}^{p} G_j^{-1}(0) \cdot G_j(s_i)$$

and

$$\left(G_j\left(\sum_{i=1}^{p} s_i\right) \cdot G_j(0)\right)^{-1} = \prod_{i=1}^{p} G_j^{-1}(0) \cdot G_j^{-1}(s_i)$$

or

$$G_j\left(\sum_{i=1}^{p} s_i\right) \cdot G_j^{-1}(0) = \prod_{i=1}^{p} G_j(s_i) \cdot G_j^{-1}(0)$$

and

$$\left(G_j(0) \cdot G_j\left(\sum_{i=1}^{p} s_i\right)\right)^{-1} = \prod_{i=1}^{p} G_j^{-1}(s_i) \cdot G_j^{-1}(0),$$

where

$$j \in \{1, \ldots, q\}, \ p \in \mathbb{N}, \ s_1, \ldots, s_p \in [0,1];$$

for all $j_1, j_2 \in \{1, \ldots, q\}$, one can find $r = r(j_1, j_2) \in (0,1]$ with at least one property from

$$G_{j_1}^{-1}(0) \cdot G_{j_1}(1) = G_{j_2}^{-1}(0) \cdot G_{j_2}(r), \quad G_{j_2}^{-1}(0) \cdot G_{j_2}(1) = G_{j_1}^{-1}(0) \cdot G_{j_1}(r)$$

or

$$G_{j_1}(1) \cdot G_{j_1}^{-1}(0) = G_{j_2}(r) \cdot G_{j_2}^{-1}(0), \quad G_{j_2}(1) \cdot G_{j_2}^{-1}(0) = G_{j_1}(r) \cdot G_{j_1}^{-1}(0);$$

and the condition on arbitrary $v \in R^m$ is the same as in (1.57), where

$$F_2(t), \quad t \in (\max\{0, t_j - \delta\}, \min\{t_j + \delta, 1\})$$

is replaced by

$$G_j^{-1}(0) \cdot G_j(1) \quad \text{or} \quad G_j(1) \cdot G_j^{-1}(0).$$

7 Here we comment our assumptions on $R$ and $X$: Because of (1.52), the requirement for the existence of $\delta > 0$ (in (1.57)) can be dropped. Note that $R$ does not need to be commutative. Thus, the set of all solutions of (1.49) is not generally a modulus over $R$ with the scalar multiplication given by

$$r\begin{pmatrix} x_k^1 \\ \vdots \\ x_k^m \end{pmatrix} := \begin{pmatrix} r \odot x_k^1 \\ \vdots \\ r \odot x_k^m \end{pmatrix},$$

where $\{(x_k^1, \ldots, x_k^m)^T\}$ is a solution of (1.49), $r \in R$, $k \in \mathbb{Z}$ ($k \in \mathbb{N}_0$). Indeed, the following does not need to hold

$$P_k \cdot \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^m \end{pmatrix} = x_0^1 \cdot (P_k)_1 + \cdots + x_0^m \cdot (P_k)_m$$

for some considered $k$ and a solution $\{x_k\}$ of (1.49) (see (1.50)).

For the main requirements, consider two results concerning the existence, the uniqueness (and the uniform asymptotic stability) of an almost periodic solution of the almost periodic *real* (non)homogeneous linear system (1.49) for $k \in \mathbb{Z}$ in the article [175] or directly the simple example: Let $R := \mathbb{R}$, $m := 2$,

$$X_1 := \left\{ \begin{pmatrix} 0 & 10^l \\ 10^{-l} & 0 \end{pmatrix}; \, l \in \mathbb{Z} \right\},$$
$$X_2 := \left\{ \begin{pmatrix} 0 & 10^l \\ 10^{-l} & 0 \end{pmatrix}; \, l \in \mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} 10^l & 0 \\ 0 & 10^{-l} \end{pmatrix}; \, l \in \mathbb{Z} \right\}$$

with the usual metric on $\mathbb{R}$. For $X_1$, every $\mathfrak{S} \in \mathfrak{X}$ has all solutions almost periodic and, at the same time, for $X_2$, it is easy to find a system from $\mathfrak{X}$ which has only one almost periodic solution—the trivial one.

8 The $l_1$ norm is the corresponding matrix norm to the absolute norm

$$\| x \|_1 := \sum_{j=1}^{m} | x_j |, \quad x = (x_1, \ldots, x_m)^T \in \mathbb{C}^m.$$

See also Footnote 13 in Part II.

9 In concrete computations, it is useful to consider Theorem 1.18 and sequences

$$\{\varepsilon_i\}_{i\in\mathbb{N}} \equiv \{\varepsilon_i(L)\}_{i\in\mathbb{N}} \subset \mathbb{R}^+, \quad \{j_i\}_{i\in\mathbb{N}} \equiv \{j_i(L)\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$$

for which

$$\sum_{i=1}^{\infty} \varepsilon_i \, j_i < \infty$$

and

$$q_i \geq p\,(L, \varepsilon_i) \quad \text{for infinitely many } i \in \mathbb{N}$$

if $\mathcal{X}$ is transformable, or for all sufficiently large $i \in \mathbb{N}$ if $\mathcal{X}$ is strongly transformable, where

$$q_1 := 1, \quad q_{i+1} := (2j_i + 1)\, q_i, \ i \in \mathbb{N}.$$

10 Several modifications and generalizations of Theorem 2.14 are known. The first theorem of the type as Theorem 2.14 was established by E. Esclangon (in [61]) for quasiperiodic (see Footnote 12 in Part II) solutions of linear differential equations of higher orders. It was extended by H. Bohr and O. Neugebauer (in [26]) to the form mentioned in Remark 2.16. In [133], Theorem 2.14 is proved if $A \in \mathcal{M}at(\mathbb{R}, m)$ and $\{b_k\}$ is almost periodic in various metrics.

11 Similarly as for Corollary 2.15, modifications of Lemma 2.18 can be proved. For example, it is possible to prove the almost automorphic version of the result (consider Remark 1.8); i.e., if $\{x_k\}$ is almost automorphic, then the conclusion of Lemma 2.18 is true.

12 The condition of Theorem 2.20 cannot be satisfied if $\varrho$ (in $\mathcal{M}at(F, m)$) is given by a matrix norm (see (ii)). Of course, there exist transformable sets which satisfy the condition. It is seen if, e.g., one considers "large" fields as the field of all meromorphic functions on a connected open set or the field of all rational functions on a variety.

# PART II

# CONSTRUCTIONS OF ALMOST PERIODIC FUNCTIONS AND HOMOGENEOUS LINEAR DIFFERENTIAL SYSTEMS

# Abstracts of Part II

**Chapter 3:** We define almost periodic functions with values in a pseudometric space $\mathcal{X}$. We mention the Bohr and the Bochner definition of almost periodicity and properties of almost periodic functions. We present one modifiable method for constructing almost periodic functions in $\mathcal{X}$. Using this method, we find almost periodic functions whose ranges contain or consist of given subsets of $\mathcal{X}$.

**Chapter 4:** Applying a construction from Chapter 3, we prove that, in any neighbourhood of an almost periodic skew-Hermitian linear differential system, there exists an almost periodic skew-Hermitian system which does not possess a nontrivial almost periodic solution.

# Chapter 3

# Constructions of almost periodic functions with given properties

This chapter is analogous to Chapter 1 where almost periodic *sequences* are considered. Here we will consider almost periodic *functions*. Our aim is to show a way one can generate almost periodic functions with several prescribed properties. Since our process can be used for generalizations of classical (complex valued) almost periodic functions, we introduce the almost periodicity in pseudometric spaces and we present our method for almost periodic functions with values in a pseudometric space $\mathcal{X}$ as in Chapter 1.

Note that we obtain the most important case if $\mathcal{X}$ is a Banach space, and that the theory of almost periodic functions of real variable with values in a Banach space, given by S. Bochner in [17], is in its essential lines similar to the theory of classical almost periodic functions which is due to H. Bohr in [24], [25]. We introduce almost periodic functions in pseudometric spaces using a trivial extension of the Bohr concept, where the modulus is replaced by the distance. In the classical case, we refer to the monographs [13], [69], and [114]; for functions with values in Banach spaces, to [6], [39], [108]; for other extensions[1], to [8], [10], [14], [15], [28], [73], [77], [91], [167]; for modifications, to [39], [78] and the references cited therein[2]; for applications[3], to [29], [40], [141].

Necessary and sufficient conditions for a continuous function with values in a Banach space to be almost periodic may be no longer valid for continuous functions in general metric spaces. For the approximation condition, it is seen that the completeness of the space of values is necessary and H. Tornehave (in [159]) also required the local connection by arcs of the space of values. In the Bochner condition, it suffices to replace the convergence by the Cauchy condition. Since we need the Bochner concept as well, we recall that the Bochner condition means that any sequence of translates of a given continuous function has a subsequence which converges, uniformly on the domain of the function. The fact, that this condition is equivalent with the Bohr definition of almost periodicity in Banach spaces, was proved by S. Bochner in [17].

The above mentioned Bohr definition and Bochner condition are formulated in Section 3.2 (with some basic properties of almost periodic functions). In this section, processes from [39] are generalized. Analogously, the theory of almost periodic functions of real variable with fuzzy real numbers as values is developed in [11] (see also [139]). We add that fuzzy real numbers form a complete metric space.

In Section 3.3, we mention the way one can construct almost periodic functions with prescribed properties in a pseudometric space. We present it in Theorems 3.12, 3.13, 3.15. Note that it is possible to obtain many modifications and generalizations of our process. A special construction of almost periodic functions with given properties is published (and applied) in [93]. Finally, in Section 3.4, we use Theorem 3.13 to construct almost periodic functions with prescribed values.

## 3.1 Preliminaries

Let $\mathcal{X}$ be an arbitrary pseudometric space with a pseudometric $\varrho$; i.e., let (see Section 1.1)

$$\varrho\,(x, x) = 0, \quad \varrho\,(x, y) = \varrho\,(y, x) \geq 0, \quad \varrho\,(x, y) \leq \varrho\,(x, z) + \varrho\,(z, y)$$

for all $x, y, z \in \mathcal{X}$. The symbol $\mathcal{O}_\varepsilon(x)$ will denote the $\varepsilon$-neighbourhood of $x$ in $\mathcal{X}$ for arbitrary $\varepsilon > 0$, $x \in \mathcal{X}$. The set of all nonnegative real numbers will be denoted by $\mathbb{R}_0^+$.

## 3.2 Almost periodic functions in pseudometric spaces

We introduce the almost periodicity in $\mathcal{X}$. Observe that we are not able to distinguish between $x \in \mathcal{X}$ and $y \in \mathcal{X}$ if $\varrho\,(x, y) = 0$.

### 3.2.1 The Bohr definition

**Definition 3.1.** A continuous function $\psi : \mathbb{R} \to \mathcal{X}$ is *almost periodic* if for any $\varepsilon > 0$, there exists a number $p\,(\varepsilon) > 0$ with the property that any interval of length $p\,(\varepsilon)$ of the real line contains at least one point $s$ such that

$$\varrho\,(\psi(t + s), \psi(t)) < \varepsilon, \quad t \in \mathbb{R}.$$

The number $s$ is called an *$\varepsilon$-translation number* and the set of all $\varepsilon$-translation numbers of $\psi$ is denoted by $T(\psi, \varepsilon)$.

### 3.2.2 The Bochner definition

If $\mathcal{X}$ is a Banach space, then a continuous function $\psi$ is almost periodic if and only if any set of translates of $\psi$ has a subsequence, uniformly convergent on $\mathbb{R}$ in the sense of the norm. See, e.g., [39, Theorem 6.6]. Evidently, this result cannot be longer valid if the space of values is not complete. Nevertheless, we prove the below given Theorem 3.4, where the convergence is replaced by the Cauchy condition. Before proving this statement, we mention two simple lemmas. Their proofs can be easily obtained by modifying the proofs of [39, Theorem 6.2] and [39, Theorem 6.5], respectively.[4]

**Lemma 3.2.** *An almost periodic function with values in $\mathcal{X}$ is uniformly continuous on the real line.*

**Lemma 3.3.** *The set of all values of an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ is totally bounded in $\mathcal{X}$.*

**Theorem 3.4.** *Let $\psi : \mathbb{R} \to \mathcal{X}$ be a continuous function. Then, $\psi$ is almost periodic if and only if from any sequence of the form $\{\psi(t + s_n)\}_{n\in\mathbb{N}}$, where $s_n$ are real numbers, one can extract a subsequence $\{\psi(t + r_n)\}_{n\in\mathbb{N}}$ satisfying the Cauchy uniform convergence condition on $\mathbb{R}$, i.e., for any $\varepsilon > 0$, there exists $l(\varepsilon) \in \mathbb{N}$ with the property that*

$$\varrho\left(\psi(t + r_i), \psi(t + r_j)\right) < \varepsilon, \quad t \in \mathbb{R}$$

*for all $i, j > l(\varepsilon)$, $i, j \in \mathbb{N}$.*

*Proof.* The sufficiency of the condition can be proved using a simple extension of the argument used in the proof of [39, Theorem 1.10]. In that proof, it is only supposed, by contradiction, that any sequence of translates of $\psi$ has a subsequence which satisfies the Cauchy uniform convergence condition, and that $\psi$ is not almost periodic. Thus, it suffices to replace the modulus by the distance in the proof of [39, Theorem 1.10].[5]

To prove the converse implication, we will assume that $\psi$ is an almost periodic function, and we will apply the well-known method of diagonal extraction and modify the proof of [39, Theorem 6.6].

Let $\{t_n;\, n \in \mathbb{N}\}$ be a dense subset of $\mathbb{R}$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be an arbitrarily given sequence. From the sequence $\{\psi(t_1 + s_n)\}_{n\in\mathbb{N}}$, using Lemma 3.3, we choose a subsequence $\{\psi(t_1 + r_n^1)\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $l_1(\varepsilon) \in \mathbb{N}$ with the property that

$$\varrho\left(\psi\left(t_1 + r_i^1\right), \psi(t_1 + r_j^1)\right) < \varepsilon, \quad i, j > l_1(\varepsilon),\, i, j \in \mathbb{N}.$$

Such a subsequence exists because infinitely many values $\psi(t_1 + s_n)$ is in a neighbourhood of radius $2^{-i}$ for all $i \in \mathbb{N}$ (consider the method of diagonal extraction). Analogously, from the sequence $\{\psi(t_2 + r_n^1)\}_{n\in\mathbb{N}}$, we get $\{\psi(t_2 + r_n^2)\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $l_2(\varepsilon) \in \mathbb{N}$ for which

$$\varrho\left(\psi(t_2 + r_i^2), \psi(t_2 + r_j^2)\right) < \varepsilon, \quad i, j > l_2(\varepsilon),\, i, j \in \mathbb{N}.$$

We proceed further in the same way. We obtain $\{r_n^k\} \subseteq \cdots \subseteq \{r_n^1\}$, $k \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrarily given, $p = p\,(\varepsilon/5)$ be from Definition 3.1, $\delta = \delta\,(\varepsilon/5)$ correspond to $\varepsilon/5$ from the definition of the uniform continuity of $\psi$ (see Lemma 3.2) and let a finite set $\{t_1, \dots, t_j\} \subset \{t_n;\, n \in \mathbb{N}\}$ satisfy

$$\min_{i\in\{1,\dots,j\}} |\, t_i - t \,| < \delta, \quad t \in [0, p].$$

Obviously, there exists $l \in \mathbb{N}$ such that, for all integers $n_1, n_2 > l$, it is valid

$$\varrho\left(\psi(t_i + r_{n_1}^{n_1}), \psi(t_i + r_{n_2}^{n_2})\right) < \frac{\varepsilon}{5}, \quad i \in \{1, \dots, j\}.$$

Let $t \in \mathbb{R}$ be given, $s = s(t) \in [-t, -t + p]$ be an $(\varepsilon/5)$-translation number of $\psi$, and $t_i = t_i(s) \in \{t_1, \dots, t_j\}$ be such that $|\, t + s - t_i \,| < \delta$. Finally, we have

$$\begin{aligned} \varrho\left(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_2}^{n_2})\right) \;&\le\; \varrho\left(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_1}^{n_1} + s)\right) \\ &+\; \varrho\left(\psi(t + r_{n_1}^{n_1} + s), \psi(t_i + r_{n_1}^{n_1})\right) \;+\; \varrho\left(\psi(t_i + r_{n_1}^{n_1}), \psi(t_i + r_{n_2}^{n_2})\right) \\ &+\; \varrho\left(\psi(t_i + r_{n_2}^{n_2}), \psi(t + r_{n_2}^{n_2} + s)\right) \;+\; \varrho\left(\psi(t + r_{n_2}^{n_2} + s), \psi(t + r_{n_2}^{n_2})\right). \end{aligned}$$

Thus, we obtain

$$\varrho\left(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_2}^{n_2})\right) < \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon \tag{3.1}$$

for all $t \in \mathbb{R}$, $n_1, n_2 > l$, $n_1, n_2 \in \mathbb{N}$. Evidently, (3.1) completes the proof of the theorem if we put $r_n := r_n^n$, $n \in \mathbb{N}$. □

### 3.2.3 Properties of almost periodic functions

Analogously as for complex valued almost periodic functions or almost periodic sequences in Chapter 1, one can prove many properties of almost periodic functions with values in a pseudometric space.

**Theorem 3.5.** *Let $\mathcal{X}_1, \mathcal{X}_2$ be pseudometric spaces and $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$ be a uniformly continuous map. If $\psi : \mathbb{R} \to \mathcal{X}_1$ is almost periodic, then $\Phi \circ \psi$ is almost periodic as well.*

*Proof.* We can proceed similarly as in the proof of Theorem 1.5. If $\delta(\varepsilon) > 0$ is the number corresponding to arbitrary $\varepsilon > 0$ from the definition of the uniform continuity of $\Phi$, then it is valid

$$T(\psi, \delta(\varepsilon)) \subseteq T(\Phi \circ \psi, \varepsilon)$$

which proves the theorem. □

**Theorem 3.6.** *The limit of a uniformly convergent sequence of almost periodic functions is almost periodic.*

*Proof.* It is possible to prove the theorem using the process from the proof of [39, Theorem 6.4]. □

Directly from Theorem 3.4, we obtain the following corollaries.

**Corollary 3.7.** *Let $\mathcal{X}$ be a Banach space. The sum of two almost periodic functions with values in $\mathcal{X}$ is an almost periodic function.*

**Corollary 3.8.** *If $\mathcal{X}_1, \dots, X_n$ are pseudometric spaces and $\psi_1, \dots, \psi_n$ are arbitrary almost periodic functions with values in $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively, then the function $\psi$, with values in $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ given by $\psi := (\psi_1, \dots, \psi_n)$, is almost periodic.*

We add that one can use Corollary 3.8 to obtain simple modifications of the below presented method of constructions of almost periodic functions. Moreover, from Corollary 3.8, we get:

**Corollary 3.9.** *The set*

$$T(\psi_1, \varepsilon) \cap T(\psi_2, \varepsilon) \cap \cdots \cap T(\psi_n, \varepsilon)$$

*is relative dense in $\mathbb{R}$ for arbitrary almost periodic functions $\psi_1, \psi_2, \dots, \psi_n$ and any $\varepsilon > 0$.*[6]

To conclude this section we establish theorems which show how almost periodic functions can be characterized by almost periodic sequences.[7]

**Theorem 3.10.** *A uniformly continuous function $\psi : \mathbb{R} \to \mathcal{X}$ is almost periodic if and only if there exists a sequence of positive numbers $r_n$, $n \in \mathbb{N}$, satisfying $r_n \to 0$ as $n \to \infty$, such that the sequence $\{\psi(r_n k)\}_{k \in \mathbb{Z}}$ is almost periodic for all $n \in \mathbb{N}$.*

*Proof.* One can prove the theorem using a corresponding extension of the proof of [39, Theorem 1.29]. □

**Theorem 3.11.** *Let $\mathcal{X}$ be a Banach space. A necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ to be almost periodic is the existence of an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ for which $\psi(k) = \varphi_k$, $k \in \mathbb{Z}$.*

*Proof.* The sufficiency of the condition follows directly from Definitions 1.1 and 3.1. Conversely, assume that an almost periodic sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is given. We define

$$\psi(t) := \varphi_k + (t-k)(\varphi_{k+1} - \varphi_k), \quad k \le t < k+1, \ k \in \mathbb{Z}. \tag{3.2}$$

Evidently, $\psi : \mathbb{R} \to \mathcal{X}$ is continuous and $\psi(k) = \varphi_k$, $k \in \mathbb{Z}$. The almost periodicity of $\psi$ follows from

$$T\left(\{\varphi_k\}, \frac{\varepsilon}{3}\right) \subseteq T(\psi, \varepsilon)$$

which can be proved using (3.2). □

## 3.3 Constructions of almost periodic functions

Now we present the way one can generate almost periodic functions with given properties in the next theorem.

**Theorem 3.12.** *For arbitrary $a > 0$, any continuous function $\psi : \mathbb{R} \to \mathcal{X}$ such that*

$$\psi(t) \in \mathcal{O}_a\left(\psi(t-1)\right), \quad t \in (1,2],$$

$$\begin{gathered}
\psi(t) \in \mathcal{O}_a\left(\psi(t+2)\right), \quad t \in (-2,0], \\
\psi(t) \in \mathcal{O}_{a/2}\left(\psi(t-4)\right), \quad t \in (2,6], \\
\psi(t) \in \mathcal{O}_{a/2}\left(\psi(t+8)\right), \quad t \in (-10,-2], \\
\psi(t) \in \mathcal{O}_{a/4}\left(\psi(t-2^4)\right), \quad t \in (2+2^2, 2+2^2+2^4], \\
\psi(t) \in \mathcal{O}_{a/4}\left(\psi(t+2^5)\right), \quad t \in (-2^5-2^3-2, -2^3-2], \\
\vdots \\
\psi(t) \in \mathcal{O}_{a\,2^{-n}}\left(\psi(t-2^{2n})\right), \quad t \in (2+2^2+\cdots+2^{2n-2}, 2+2^2+\cdots+2^{2n-2}+2^{2n}], \\
\psi(t) \in \mathcal{O}_{a\,2^{-n}}\left(\psi(t+2^{2n+1})\right), \quad t \in (-2^{2n+1}-\cdots-2^3-2, -2^{2n-1}-\cdots-2^3-2], \\
\vdots
\end{gathered}$$

*is almost periodic.*

*Proof.* Let $\varepsilon > 0$ be arbitrary and $k = k(\varepsilon) \in \mathbb{N}$ be such that $2^k > 8a/\varepsilon$. We have to prove that the set of all $\varepsilon$-translation numbers of $\psi$ is relative dense in $\mathbb{R}$. We will obtain this from the fact that $l\,2^{2k}$ is an $\varepsilon$-translation number of $\psi$ for any integer $l$.

First we define

$$\varphi(t) := \psi(t), \quad t \in [-2^{2k-1} - \cdots - 2^3 - 2, 2+2^2+\cdots+2^{2k-2}].$$

We see that

$$\begin{aligned}
&\psi(t) \in \mathcal{O}_{\varepsilon/8}\left(\psi(t-2^{2k})\right), \quad t \in [2+2^2+\cdots+2^{2k-2}, 2+2^2+\cdots+2^{2k}],\\
&\psi(t) \in \mathcal{O}_{\varepsilon/8}\left(\varphi(t-2^{2k})\right), \quad t \in [2+2^2+\cdots+2^{2k-2}, 2+2^2+\cdots+2^{2k}],\\
&\psi(t) \in \mathcal{O}_{\varepsilon/8}\left(\psi(t+2^{2k+1})\right), \quad t \in [-2^{2k+1}-\cdots-2^3-2, -2^{2k-1}-\cdots-2^3-2],\\
&\psi(t) \in \mathcal{O}_{\varepsilon/16}\left(\psi(t-2^{2k+2})\right), \quad t \in [2+2^2+\cdots+2^{2k}, 2+2^2+\cdots+2^{2k+2}],\\
&\psi(t) \in \mathcal{O}_{\varepsilon/16}\left(\psi(t+2^{2k+3})\right), \quad t \in [-2^{2k+3}-\cdots-2^3-2, -2^{2k+1}-\cdots-2^3-2],\\
&\qquad\vdots
\end{aligned}$$

In a pseudometric space $\mathcal{X}$, it implies

$$\begin{aligned}
&\psi(t+2^{2k}) \in \mathcal{O}_{\varepsilon/8}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\psi(t-2^{2k+1}) \in \mathcal{O}_{\varepsilon/8}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\psi(t-2^{2k}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/8}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\psi(t+2^{2k+1}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\psi(t+3\,2^{2k}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/8+\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\psi(t+2^{2k+2}) \in \mathcal{O}_{\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\psi(t+2^{2k}+2^{2k+2}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&\qquad\vdots
\end{aligned}$$

Since

$$\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{16}+\frac{\varepsilon}{16}+\frac{\varepsilon}{32}+\frac{\varepsilon}{32}+\cdots=\frac{\varepsilon}{2},$$

we have

$$\begin{aligned}
&\psi(t+l\,2^{2k}) \in \mathcal{O}_{\varepsilon/2}\left(\varphi(t)\right),\\
&t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\ l \in \mathbb{Z}.
\end{aligned} \tag{3.3}$$

We express any $t \in \mathbb{R}$ as the sum of numbers $p(t)$ and $q(t)$ for which

$$\begin{aligned}
&p(t) \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\\
&q(t) \in \mathbb{Z} \quad \text{and} \quad q(t) = j2^{2k} \quad \text{for some } j \in \mathbb{Z}.
\end{aligned}$$

Using (3.3), we obtain

$$\begin{aligned}
\varrho\left(\psi(t), \psi(t+l\,2^{2k})\right) &\le \varrho\left(\psi(p(t)+q(t)), \varphi(p(t))\right)\\
&\quad + \varrho\left(\varphi(p(t)), \psi(p(t)+(j+l)\,2^{2k})\right) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned} \tag{3.4}$$

for any $t \in \mathbb{R}$, $l \in \mathbb{Z}$, which terminates the proof. ☐

The process mentioned in the previous theorem is easily modifiable. We illustrate this fact by the following two theorems.

**Theorem 3.13.** *Let $M > 0$, $x_0 \in \mathcal{X}$, and $j \in \mathbb{N}$ be given. Let $\varphi : [0, M] \to \mathcal{X}$ satisfy $\varphi(0) = \varphi(M) = x_0$. If $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ has the property that*

$$\sum_{n=1}^{\infty} r_n < \infty, \tag{3.5}$$

*then an arbitrary continuous function $\psi : \mathbb{R} \to \mathcal{X}$, $\psi|_{[0,M]} \equiv \varphi$ for which*

$$\begin{aligned}\psi(t) = x_0, \quad & t \in \{iM,\, 2 \le i \le j+1\} \cup \{-i(j+1)M,\, 1 \le i \le j\}\\ & \bigcup_{n=1}^{\infty} \{((j+1) + \cdots + j(j+1)^{2n-2} + i(j+1)^{2n})M;\ 1 \le i \le j\}\\ & \bigcup_{n=1}^{\infty} \{-((j+1) + \cdots + j(j+1)^{2n-1} + i(j+1)^{2n+1})M;\ 1 \le i \le j\}\end{aligned}$$

*and, at the same time, for which it is valid*

$$\psi(t) \in \mathcal{O}_{r_1}(\psi(t-M)), \quad t \in (M, 2M),$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_1}(\psi(t-jM)), \quad t \in (jM, (j+1)M),$$
$$\psi(t) \in \mathcal{O}_{r_2}(\psi(t+(j+1)M)), \quad t \in (-(j+1)M, 0),$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_2}(\psi(t+j(j+1)M)), \quad t \in (-j(j+1)M, -(j-1)(j+1)M),$$
$$\psi(t) \in \mathcal{O}_{r_3}\big(\psi(t-(j+1)^2M)\big), \quad t \in \big((j+1)M, ((j+1)+(j+1)^2)M\big),$$

$$\vdots$$

$$\begin{aligned}&\psi(t) \in \mathcal{O}_{r_3}\big(\psi(t-j(j+1)^2M)\big),\\ &\quad t \in \big(((j+1)+(j-1)(j+1)^2)M, ((j+1)+j(j+1)^2)M\big),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\psi(t) \in \mathcal{O}_{r_{2n}}\big(\psi(t+(j+1)^{2n-1}M)\big),\\ &\quad t \in \big(-((j+1)^{2n-1} + j(j+1)^{2n-3} + \cdots + j(j+1)^3 + j(j+1))M,\\ &\qquad\qquad -(j(j+1)^{2n-3} + \cdots + j(j+1)^3 + j(j+1))M\big),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\psi(t) \in \mathcal{O}_{r_{2n}}\big(\psi(t+j(j+1)^{2n-1}M)\big),\\ &\quad t \in \big(-(j(j+1)^{2n-1} + j(j+1)^{2n-3} + \cdots + j(j+1)^3 + j(j+1))M,\\ &\qquad -((j-1)(j+1)^{2n-1} + j(j+1)^{2n-3} + \cdots + j(j+1)^3 + j(j+1))M\big),\end{aligned}$$

$$
\begin{aligned}
&\psi(t) \in \mathcal{O}_{r_{2n+1}}\left(\psi(t-(j+1)^{2n}M)\right), \\
&\quad t \in \big(((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2})M, \\
&\qquad ((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2}+(j+1)^{2n})M\big), \\
&\qquad\qquad \vdots \\
&\psi(t) \in \mathcal{O}_{r_{2n+1}}\left(\psi(t-j(j+1)^{2n}M)\right), \\
&\quad t \in \big(((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2}+(j-1)(j+1)^{2n})M, \\
&\qquad ((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2}+j(j+1)^{2n})M\big), \\
&\qquad\qquad \vdots
\end{aligned}
$$

*is almost periodic.*

*Proof.* We can prove this theorem analogously as Theorem 3.12. Let $\varepsilon$ be a positive number and let an odd integer $n(\varepsilon) \geq 2$ have the property (see (3.5)) that

$$\sum_{n=n(\varepsilon)}^{\infty} r_n < \frac{\varepsilon}{2}. \tag{3.6}$$

We will prove that $l(j+1)^{n(\varepsilon)-1}M$ is an $\varepsilon$-translation number of $\psi$ for all $l \in \mathbb{Z}$. Arbitrarily choosing $l \in \mathbb{Z}$ and $t \in \mathbb{R}$, if we put

$$s := l(j+1)^{n(\varepsilon)-1}M, \tag{3.7}$$

then it suffices to show that the inequality

$$d\left(\psi(t), \psi(t+s)\right) < \varepsilon \tag{3.8}$$

holds; i.e., this inequality proves the theorem.

We can write $t$ as the sum of numbers $t_1$ and $t_2$, where

$$
\begin{aligned}
t_1 &\geq -\left(j(j+1)^{n(\varepsilon)-2}+\cdots+j(j+1)^3+j(j+1)\right)M, \\
t_1 &\leq \left(j+1+j(j+1)^2+\cdots+j(j+1)^{n(\varepsilon)-3}\right)M
\end{aligned}
\tag{3.9}
$$

and

$$t_2 = i(j+1)^{n(\varepsilon)-1}M \qquad \text{for some } i \in \mathbb{Z}. \tag{3.10}$$

Now we have (see (3.9) and the proof of Theorem 3.12)

$$
\begin{aligned}
\varrho\left(\psi(t), \psi(t+s)\right) &\leq \varrho\left(\psi(t_1+t_2), \psi(t_1)\right) + \varrho\left(\psi(t_1), \psi(t_1+t_2+s)\right) \\
&< \sum_{n=n(\varepsilon)}^{n(\varepsilon)+p-1} r_n + \sum_{n=n(\varepsilon)}^{n(\varepsilon)+q-1} r_n.
\end{aligned}
\tag{3.11}
$$

Indeed, we can express (consider (3.7) and (3.10))

$$t_2 \quad = \left(i_1(j+1)^{n(\varepsilon)-1} + i_2(j+1)^{n(\varepsilon)} + \cdots + i_p(j+1)^{n(\varepsilon)+p-1}\right)(m+1),$$

$$t_2 + s = \left( l_1(j+1)^{n(\varepsilon)-1} + l_2(j+1)^{n(\varepsilon)} + \cdots + l_q(j+1)^{n(\varepsilon)+q-1} \right)(m+1),$$

where $i_1, \ldots, i_p, l_1, \ldots, l_q \subseteq \{-j, \ldots, 0, \ldots, j\}$ satisfy

$$i_1 \geq 0, \quad i_2 \leq 0, \quad \cdots \quad (-1)^p\, i_p \leq 0, \qquad l_1 \geq 0, \quad l_2 \leq 0, \quad \cdots \quad (-1)^q\, l_q \leq 0.$$

It is sure that (3.6) and (3.11) give (3.8). □

For $j = 1$, we get the most important case of Theorem 3.13:

**Corollary 3.14.** *Let $M > 0$ and $x_0 \in \mathcal{X}$ be given and let $\varphi : [0, M] \to \mathcal{X}$ be such that*

$$\varphi(0) = \varphi(M) = x_0.$$

*If $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ satisfies*

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty, \tag{3.12}$$

*then any continuous function $\psi : \mathbb{R} \to \mathcal{X}$, $\psi|_{[0,M]} \equiv \varphi$ for which*

$$\begin{aligned} \psi\,(t) = x_0, \quad t \in \{2M, -2M\} &\cup \{(2 + 2^2 + \cdots + 2^{2(i-1)} + 2^{2i})M;\, i \in \mathbb{N}\} \\ &\cup \{-(2 + 2^3 + \cdots + 2^{2i-1} + 2^{2i+1})M;\, i \in \mathbb{N}\} \end{aligned} \tag{3.13}$$

*and, at the same time, for which it is valid*

$$\psi(t) \in \mathcal{O}_{\varepsilon_1}\left(\psi(t - M)\right), \quad t \in (M, 2M),$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_2}\left(\psi(t + 2M)\right), \quad t \in (-2M, 0),$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_3}\left(\psi(t - 2^2M)\right), \quad t \in (2M, (2 + 2^2)M),$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_4}\left(\psi(t + 2^3M)\right), \quad t \in (-(2^3 + 2)M, -2M),$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_5}\left(\psi(t - 2^4M)\right), \quad t \in ((2 + 2^2)M, (2 + 2^2 + 2^4)M),$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2i}}\left(\psi(t + 2^{2i-1}M)\right),\ t \in (-(2^{2i-1} + \cdots + 2)M, -(2^{2i-3} + \cdots + 2)M),$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2i+1}}\left(\psi(t - 2^{2i}M)\right),\ t \in ((2 + 2^2 + \cdots + 2^{2i-2})M, (2 + 2^2 + \cdots + 2^{2i})M),$$

$$\vdots$$

*is almost periodic.*

**Theorem 3.15.** *Let $\varphi : (-r, r] \to \mathcal{X}$, $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$, and $\{j_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be arbitrary such that*

$$\sum_{n=1}^{\infty} r_n j_n < \infty \tag{3.14}$$

*holds, and let a function $\psi : \mathbb{R} \to \mathcal{X}$ satisfy $\psi|_{(-r,r]} \equiv \varphi$ and*

$$\psi(t) \in \mathcal{O}_{r_1}\left(\varphi(t - 2r)\right), \quad t \in (r, r + 2r],$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_1}\left(\varphi(t-2r)\right), \quad t \in (r + (j_1 - 1)2r, r + j_1 2r]\,,$$

$$\psi(t) \in \mathcal{O}_{r_1}\left(\varphi(t+2r)\right), \quad t \in (-2r - r, -r]\,,$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_1}\left(\varphi(t+2r)\right), \quad t \in (-j_1 2r - r, -(j_1 - 1)2r - r]\,,$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_n}\left(\varphi(t-p_n)\right), \quad t \in (p_1 + \dots + p_{n-1}, p_1 + \dots + p_{n-1} + p_n]\,,$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_n}\left(\varphi(t-p_n)\right), \quad t \in (p_1 + \dots + p_{n-1} + (j_n - 1)p_n, p_1 + \dots + p_{n-1} + j_n p_n]\,,$$

$$\psi(t) \in \mathcal{O}_{r_n}\left(\varphi(t+p_n)\right), \quad t \in (-p_n - p_{n-1} - \dots - p_1, -p_{n-1} - \dots - p_1]\,,$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_n}\left(\varphi(t+p_n)\right), \quad t \in (-j_n p_n - p_{n-1} - \dots - p_1, -(j_n - 1)p_n - p_{n-1} - \dots - p_1]\,,$$

$$\vdots$$

*where*

$$p_1 := r + j_1 2r, \qquad p_2 := 2(r + j_1 2r),$$

$$p_3 := (2j_2 + 1)p_2, \quad \dots \quad p_n := (2j_{n-1} + 1)p_{n-1}, \quad \dots$$

*If $\psi$ is continuous on $\mathbb{R}$, then it is almost periodic.*

*Proof.* It is not difficult to prove Theorem 3.15 analogously as Theorems 3.12 and 3.13. For given $\varepsilon > 0$, let an integer $n(\varepsilon) \geq 2$ satisfy

$$\sum_{n=n(\varepsilon)}^{\infty} r_n\, j_n < \frac{\varepsilon}{4}.$$

One can prove the inclusion

$$\{l\, p_{n(\varepsilon)};\ l \in \mathbb{Z}\} \subseteq T(\psi, \varepsilon) \tag{3.15}$$

which yields the almost periodicity of $\psi$. □

**Remark 3.16.** From the proofs of Theorems 3.12, 3.13, 3.15 (see (3.4), (3.7) and (3.8), (3.15)), we get an important property of the set of all $\varepsilon$-translation numbers of the resulting function $\psi$. For any $\varepsilon > 0$, there exists nonzero $c \in \mathbb{R}$ for which

$$\{l\, c;\ l \in \mathbb{Z}\} \subseteq T(\psi, \varepsilon).$$

Hence, applying the method from the above theorems, one cannot construct almost periodic functions without this property.[8]

## 3.4 Almost periodic functions with given values

We prove two theorems for almost periodic functions corresponding to Theorem 1.19, where it is proved that, for any countable and totally bounded set, there exists an almost periodic sequence whose range is the set. In the first theorem, we need that the totally bounded set is the range of a uniformly continuous function $\varphi$ for which the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite. We also use the result for sequences to construct an almost periodic function whose range contains an arbitrarily given totally bounded sequence if one requires the connection by arcs of the space of values (see the below mentioned Theorem 3.19).

Concerning a continuous counterpart of Theorem 1.19 (or Definition 3.1 and Lemma 3.3), the given set of values has to be the totally bounded graph of a continuous function. In addition, any almost periodic function is uniformly continuous (see Lemma 3.2). Considering these facts, we formulate the following theorem.

**Theorem 3.17.** *Let $\varphi : \mathbb{R} \to \mathcal{X}$ be any uniformly continuous function such that the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite and the set $\{\varphi(t);\, t \in \mathbb{R}\}$ is totally bounded. There exists an almost periodic function $\psi$ with the property that*

$$\{\psi(k);\, k \in \mathbb{Z}\} = \{\varphi(k);\, k \in \mathbb{Z}\}, \quad \{\psi(t);\, t \in \mathbb{R}\} = \{\varphi(t);\, t \in \mathbb{R}\} \tag{3.16}$$

*and that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ for which*

$$\psi(l+s) = \psi(l+s+jq(l)), \quad j \in \mathbb{Z},\ s \in [0,1). \tag{3.17}$$

*Proof.* We will construct $\psi : \mathbb{R} \to \mathcal{X}$ applying Corollary 3.14 similarly as $\{\psi_k\}$ applying Corollary 1.17 in the proof of Theorem 1.19. Considering that the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite, let sufficiently large $M, N \in \mathbb{Z}$ have the property that $\varphi(M) = \varphi(N)$ and that, for any $l \in \mathbb{Z}$, there exists $j(l) \in \{N, N+1, \dots, M-1\}$ for which

$$\varphi(l) = \varphi(j(l)), \quad \varphi(l+1) = \varphi(j(l)+1). \tag{3.18}$$

Without loss of the generality, we can assume that $N = 0$ because, if $N < 0$, then we can redefine finitely many the below given $\varepsilon_i$ and put $\psi \equiv \varphi$ on a sufficiently large interval.

Since $\varphi$ is uniformly continuous with totally bounded range (see also (3.18)), for arbitrarily small $\varepsilon > 0$, there exist $l_1(\varepsilon), \dots, l_{m(\varepsilon)}(\varepsilon) \in \mathbb{Z}$ such that, for any $l \in \mathbb{Z}$, we have

$$\varrho\left(\varphi(l+s), \varphi(l_i+s)\right) < \varepsilon, \quad s \in [0,1]$$

for at least one integer $l_i \in \{l_1(\varepsilon), \dots, l_{m(\varepsilon)}(\varepsilon)\}$. We put $\varepsilon_i := 2^{-i}$, $i \in \mathbb{N}$, i.e.,

$$l_1^i := l_1(2^{-i}), \dots, l_{m(i)}^i := l_{m(2^{-i})}\big(2^{-i}\big), \quad i \in \mathbb{N}.$$

In addition, we will assume that

$$\{l_j^i;\, j \in \{1, \dots, m(i)\}, i \in \mathbb{N}\} = \mathbb{Z}. \tag{3.19}$$

First we define

$$\psi(t) := \varphi(t), \quad t \in [0, M]. \tag{3.20}$$

We choose arbitrary $n(1) \in \mathbb{N}$ for which $2^{2n(1)}M > m(1)$. There exist (see (3.18))

$$j_1^1, j_2^1, \dots, j_{m(1)}^1 \in \{0, 1, \dots, M-1\}$$

such that

$$\begin{aligned}\varphi(l_1^1) = \psi(j_1^1), \quad \varphi(l_1^1+1) = \psi(j_1^1+1),\\ \varphi(l_2^1) = \psi(j_2^1), \quad \varphi(l_2^1+1) = \psi(j_2^1+1),\\ \vdots \\ \varphi(l_{m(1)}^1) = \psi(j_{m(1)}^1), \quad \varphi(l_{m(1)}^1+1) = \psi(j_{m(1)}^1+1).\end{aligned}$$

We define

$$\begin{gathered}\psi(s+M+j_1^1) := \varphi(s+l_1^1), \quad s \in [0,1],\\ \psi(t) := \psi(t-M), \quad t \in (M, 2M] \setminus [M+j_1^1, M+j_1^1+1],\\ \psi(s+2M+j_2^1) := \varphi(s+l_2^1), \quad s \in [0,1],\\ \psi(t) := \psi(t-2M), \quad t \in (2M, 3M] \setminus [2M+j_2^1, 2M+j_2^1+1],\\ \vdots \\ \psi(s+m(1)M+j_{m(1)}^1) := \varphi(s+l_{m(1)}^1), \quad s \in [0,1],\end{gathered}$$

$$\psi(t) := \psi(t-m(1)M),\ t \in (m(1)M, (m(1)+1)M] \setminus [m(1)M+j_{m(1)}^1, m(1)M+j_{m(1)}^1+1]$$

and we define $\psi$ as periodic with period $M$ on

$$[-(2^{2n(1)-1} + \dots + 2^3 + 2)M, (2 + 2^2 + \dots + 2^{2n(1)})M] \setminus (M, (m(1)+1)M).$$

It is easily to see that we construct $\psi$ as in Corollary 3.14 for

$$\varepsilon_i := L, \quad i \in \{1, \dots, 2n(1)+1\} \tag{3.21}$$

if $L > 0$ is sufficiently large.

In the second step, we choose $n(2) > n(1) + m(2)$ ($n(2) \in \mathbb{N}$) and we put

$$\begin{gathered}\psi(t) := \psi(t + 2^{2n(1)+1}M),\ t \in [-(2^{2n(1)+1} + \dots + 2)M, \cdots, -(2^{2n(1)-1} + \dots + 2)M),\\ \psi(t) := \psi(t - 2^{2n(1)+2}M),\ t \in ((2 + \dots + 2^{2n(1)})M, \dots, (2 + \dots + 2^{2n(1)+2})M],\\ \vdots \\ \psi(t) := \psi(t + 2^{2n(2)-1}M),\ t \in [-(2^{2n(2)-1} + \dots + 2)M, \cdots, -(2^{2n(2)-3} + \dots + 2)M)\end{gathered}$$

and

$$\varepsilon_i := 0,\ i \in \{2n(1)+2, \dots, 2n(2)\}, \quad \varepsilon_{2n(2)+1} := 2^{-1}. \tag{3.22}$$

From $n(2) > n(1) + m(2)$ and the above construction, we see that, for each integer $j$, $1 \le j \le m(1)$, there exist at least $2m(2)+2$ intervals of the form

$$[a, a+1] \subset [-(2^{2n(2)-1} + \dots + 2)M, \dots, (2^{2n(2)-2} + \dots + 2)M]$$

such that $a \in \mathbb{Z}$ and

$$\psi|_{[a,a+1]} \equiv \varphi|_{[l_j^1, l_j^1+1]}, \quad \text{i.e.,} \quad \psi(s+a) = \varphi(s+l_j^1), \quad s \in [0,1].$$

It implies that we can define continuous

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2n(2)+1}}\left(\psi(t - 2^{2n(2)}M)\right), \ t \in ((2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M]$$

for which

$$\psi|_{[2^{2n(2)}M, 2^{2n(2)}M+1]} \equiv \psi|_{[0,1]},$$

$$\begin{aligned}&\psi|_{[k,k+1]} \equiv \psi|_{[0,1]} \text{ for some } k,\\ &k \in \{(2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M - 1\} \setminus \{2^{2n(2)}M\}\end{aligned}$$

and

$$\begin{aligned}\psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]}, \ & l \in \{(2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M - 1\},\\ & \text{some } j(l) \in \{0, \ldots, M-1, l_1^1, \ldots, l_{m(1)}^1, l_1^2, \ldots, l_{m(2)}^2\},\end{aligned}$$

$$\begin{aligned}\{\varphi(t);\, t \in [l_1^1, l_1^1 + 1] \cup \cdots \cup [l_{m(1)}^1, l_{m(1)}^1 + 1] \cup [l_1^2, l_1^2 + 1] \cup \cdots \cup [l_{m(2)}^2, l_{m(2)}^2 + 1]\}\\ \subseteq \{\psi(t);\, t \in [(2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M]\}.\end{aligned}$$

In the third step, we choose $n(3) > n(2) + m(3)$ ($n(3) \in \mathbb{N}$) and we construct $\psi$ for

$$\varepsilon_i := 0, \quad i \in \{2n(2)+2, \ldots, 2n(3)\}, \quad \varepsilon_{2n(3)+1} := 2^{-2}. \tag{3.23}$$

We have continuous

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2n(3)+1}}\left(\psi(t - 2^{2n(3)}M)\right), \ t \in ((2 + \cdots + 2^{2n(3)-2})M, \ldots, (2 + \cdots + 2^{2n(3)})M]$$

satisfying

$$\begin{aligned}\psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]}, \ & l \in \{(2 + \cdots + 2^{2n(3)-2})M, \ldots, (2 + \cdots + 2^{2n(3)})M - 1\},\\ & \text{at least one } j(l) \in \{0, \ldots, M-1, l_1^1, l_2^1, \ldots, l_{m(3)}^3\},\end{aligned}$$

$$\begin{aligned}\{\varphi(t); t \in [l_1^1, l_1^1 + 1] \cup [l_2^1, l_2^1 + 1] \cup \cdots \cup [l_{m(3)}^3, l_{m(3)}^3 + 1]\}\\ \subseteq \{\psi(t);\, t \in [(2 + \cdots + 2^{2n(3)-2})M, \ldots, (2 + \cdots + 2^{2n(3)})M]\}.\end{aligned}$$

In addition, we have

$$\begin{aligned}\psi|_{[l,l+1]} \equiv \psi|_{[0,1]}, \quad & l \in \{j\, 2^{2n(2)}M;\, j \in \mathbb{Z}\}\\ & \cap \{-(2^{2n(3)-1} + \cdots + 2)M, \ldots, (2 + \cdots + 2^{2n(3)})M - 1\},\end{aligned}$$

$$\psi|_{[2^{2n(3)}M+1, 2^{2n(3)}M+2]} \equiv \psi|_{[1,2]}, \qquad \psi|_{[2^{2n(3)}M-1, 2^{2n(3)}M]} \equiv \psi|_{[-1,0]},$$

$$\begin{aligned}&\psi|_{[k,k+1]} \equiv \psi|_{[1,2]} \text{ for some } k,\\ &k \in \{(2 + \cdots + 2^{2n(3)-2})M, \ldots, (2 + \cdots + 2^{2n(3)})M - 1\} \setminus \{2^{2n(3)}M + 1\}\end{aligned}$$

$$\begin{aligned}&\psi|_{[k,k+1]} \equiv \psi|_{[-1,0]} \text{ for some } k,\\ &k \in \{(2 + \cdots + 2^{2n(3)-2})M, \ldots, (2 + \cdots + 2^{2n(3)})M - 1\} \setminus \{2^{2n(3)}M - 1\}.\end{aligned}$$

Continuing in the same manner, in the $i$-th step, we choose $n(i) > n(i-1) + m(i)$ ($n(i) \in \mathbb{N}$) and we construct $\psi$ for

$$\varepsilon_k := 0,\ k \in \{2n(i-1)+2, \dots, 2n(i)\}, \quad \varepsilon_{2n(i)+1} := 2^{-i+1}. \tag{3.24}$$

For simplicity, let $i - 2 < 2^{2n(2)}M$ (see also the proof of Theorem 1.19 for $j\,2^{2n(2)}$ replaced by $[j\,2^{2n(2)}M, j\,2^{2n(2)}M+1]$, $1 + j\,2^{2n(3)}$ by $[1 + j\,2^{2n(3)}M, 1 + j\,2^{2n(3)}M + 1]$, and so on). Again, for each $j(1) \in \{1, \dots, i-1\}$, $j(2) \in \{1, \dots, m(j(1))\}$, there exist at least $2m(i)+2$ integers

$$\begin{aligned} l \in\, &\{-(2^{2n(i)-1} + \cdots + 2)M, \dots, (2 + \cdots + 2^{2n(i)-2})M - 1\} \\ &\setminus (\{j\,2^{2n(2)}M;\ j \in \mathbb{Z}\} \cup \{1 + j\,2^{2n(3)}M;\ j \in \mathbb{Z}\} \cup \{-1 + j\,2^{2n(3)}M;\ j \in \mathbb{Z}\} \cup \\ &\qquad \cdots \cup \{i - 3 + j\,2^{2n(i-1)}M;\ j \in \mathbb{Z}\} \cup \{3 - i + j\,2^{2n(i-1)}M;\ j \in \mathbb{Z}\}) \end{aligned}$$

such that

$$\psi|_{[l,l+1]} \equiv \varphi|_{\left[l^{j(1)}_{j(2)}, l^{j(1)}_{j(2)}+1\right]}.$$

Thus, we can define continuous

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2n(i)+1}}\left(\psi(t - 2^{2n(i)}M)\right),\ t \in ((2 + \cdots + 2^{2n(i)-2})M, \dots, (2 + \cdots + 2^{2n(i)})M]$$

satisfying

$$\begin{aligned} &\psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]},\ l \in \{(2 + \cdots + 2^{2n(i)-2})M, \dots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ &\qquad\qquad\qquad \text{at least one } j(l) \in \{0, \dots, M-1, l^1_1, l^1_2, \dots, l^i_{m(i)}\}, \end{aligned}$$

$$\begin{aligned} &\{\varphi(t); t \in [l^1_1, l^1_1 + 1] \cup [l^1_2, l^1_2 + 1] \cup \cdots \cup [l^i_{m(i)}, l^i_{m(i)} + 1]\} \\ &\qquad \subseteq \{\psi(t);\ t \in [(2 + \cdots + 2^{2n(i)-2})M, \dots, (2 + \cdots + 2^{2n(i)})M]\}. \end{aligned} \tag{3.25}$$

In addition, we can define $\psi$ so that

$$\begin{aligned} &\psi|_{[l,l+1]} \equiv \psi|_{[0,1]},\ l \in \{j\,2^{2n(2)}M;\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \dots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ &\psi|_{[l,l+1]} \equiv \psi|_{[1,2]},\ l \in \{1 + j\,2^{2n(3)}M;\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \dots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ &\psi|_{[l,l+1]} \equiv \psi|_{[-1,0]},\ l \in \{-1 + j\,2^{2n(3)}M;\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \dots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ &\qquad\qquad\qquad \vdots \\ &\psi|_{[l,l+1]} \equiv \psi|_{[i-3,i-2]},\ l \in \{i - 3 + j\,2^{2n(i-1)}M;\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \dots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ &\psi|_{[l,l+1]} \equiv \psi|_{[3-i,4-i]},\ l \in \{3 - i + j\,2^{2n(i-1)}M;\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \dots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ &\psi|_{\left[2^{2n(i)}M+i-2, 2^{2n(i)}M+i-1\right]} \equiv \psi|_{[i-2,i-1]}, \quad \psi|_{\left[2^{2n(i)}M+2-i, 2^{2n(i)}M+3-i\right]} \equiv \psi|_{[2-i,3-i]}, \end{aligned}$$

$$\psi|_{[k,k+1]} \equiv \psi|_{[i-2,i-1]} \text{ for some } k,$$
$$k \in \{(2+\cdots+2^{2n(i)-2})M,\ldots,(2+\cdots+2^{2n(i)})M-1\} \setminus \{2^{2n(i)}M+i-2\},$$
$$\psi|_{[k,k+1]} \equiv \psi|_{[2-i,3-i]} \text{ for some } k,$$
$$k \in \{(2+\cdots+2^{2n(i)-2})M,\ldots,(2+\cdots+2^{2n(i)})M-1\} \setminus \{2^{2n(i)}M+2-i\}.$$

Evidently, it is valid

$$\varrho\left(\varphi(i),\varphi(j)\right) \geq 2^{-K} \quad \text{or} \quad \varrho\left(\varphi(i),\varphi(j)\right) = 0$$

for all $i, j \in \mathbb{Z}$ and some $K \in \mathbb{N}$. If we begin the construction by

$$l_1^K := l_1\big(2^{-K}\big),\ldots,l_{m(K)}^K := l_{m(2^{-K})}\big(2^{-K}\big),$$

then we have to obtain

$$\psi(k) = \psi(k+M), \quad k \in \mathbb{Z}.$$

Hence, we can construct the above $\psi$ in order that the sequence $\{\psi(k)\}_{k\in\mathbb{Z}}$ is periodic with period $M$ which gives (3.13) and the continuity of $\psi$. We construct $\psi$ using the process from Corollary 3.14 for all $i \in \mathbb{N}$ and we obtain an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$. Indeed, we have (3.20) and, summarizing (3.21), (3.22), (3.23), ..., (3.24), ..., we get (3.12) (see also (1.48)). For periodic $\{\psi(k)\}_{k\in\mathbb{Z}}$, the first identity in (3.16) follows from (3.18) and (3.20) and the second one from the construction, (3.19), and (3.25). As in the proof of Theorem 1.19, we see that, for any $l \in \mathbb{Z}$, there exists $i(l) \in \mathbb{N}$ satisfying

$$\psi|_{[k,k+1]} \equiv \psi|_{[l,l+1]}, \quad k \in \{l + j\, 2^{2n(i(l))}M;\ j \in \mathbb{Z}\}.$$

It gives (3.17) for $q(l) = 2^{2n(i(l))}M$. □

As an example which illustrates the previous theorem, we mention the following statement:

**Corollary 3.18.** *For any continuous function $F : [0,1] \to \mathcal{X}$, there exists an almost periodic function $\psi$ with the property that*

$$\{\psi(t);\ t \in \mathbb{R}\} = \{F(t);\ t \in (0,1)\}.$$

*Proof.* It suffices to show that there exists a uniformly continuous function $\varphi : \mathbb{R} \to \mathcal{X}$ for which $\{\varphi(k);\ k \in \mathbb{Z}\} = \{F(1/2)\}$ and $\{\varphi(t);\ t \in \mathbb{R}\} = \{F(t);\ t \in (0,1)\}$, and to apply Theorem 3.17. For example, one can put

$$\varphi(k+s) := F\left(\frac{1}{2}+s\right), \quad k \in \mathbb{N},\ s \in \left[0, \frac{k}{2k+1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2}+\frac{k}{2k+1}\right), \quad k \in \mathbb{N},\ s \in \left[\frac{k}{2k+1}, 1-\frac{k}{2k+1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2}+1-s\right), \quad k \in \mathbb{N},\ s \in \left[1-\frac{k}{2k+1}, 1\right);$$

$$\varphi(k+s) := F\left(\frac{1}{2}-s\right), \quad k \in \mathbb{Z} \setminus \mathbb{N},\ s \in \left[0, \frac{k}{2k-1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2} - \frac{k}{2k-1}\right), \quad k \in \mathbb{Z} \setminus \mathbb{N},\ s \in \left[\frac{k}{2k-1}, 1 - \frac{k}{2k-1}\right),$$
$$\varphi(k+s) := F\left(\frac{1}{2} + s - 1\right), \quad k \in \mathbb{Z} \setminus \mathbb{N},\ s \in \left[1 - \frac{k}{2k-1}, 1\right).$$

□

In Theorem 3.17, we have constructed an almost periodic function $\psi$ for which the set $\{\psi(k);\ k \in \mathbb{Z}\}$ has to be finite. Now we use Theorem 1.19 to obtain an almost periodic function with infinitely many given values on $\mathbb{Z}$. We proved as Theorem 3.11 that, for any almost periodic sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ in a Banach space $\mathcal{X}$, there exists an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ for which $\psi(k) = \varphi_k$, $k \in \mathbb{Z}$. Since Theorem 3.11 does not need to be true in a metric space, we require a condition about the local connection by arcs of given values.[9]

**Theorem 3.19.** *Let any countable and totally bounded set $X \subseteq \mathcal{X}$ be given. If all $x, y \in X$ can be connected in $\mathcal{X}$ by continuous curves which depend uniformly continuously on $x$ and $y$, then there exists an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ such that*

$$\{\psi(k);\ k \in \mathbb{Z}\} = X \tag{3.26}$$

*and that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ for which*

$$\psi(l+s) = \psi(l+s+jq(l)), \quad j \in \mathbb{Z},\ s \in [0,1).$$

*Proof.* Using Theorem 1.19, we get an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying (1.27). Let continuous functions $f_k : [0,1] \to \mathcal{X}$, $k \in \mathbb{Z}$ for which $f_k(0) = \psi_k$, $f_k(1) = \psi_{k+1}$ be from the statement of the theorem. Obviously, the function

$$\psi(k+s) := f_k(s), \quad k \in \mathbb{Z},\ s \in [0,1)$$

defined on $\mathbb{R}$ is continuous and (3.26) is satisfied. From the proof of Theorem 1.19, it follows (see (1.47)) that, for any $l \in \mathbb{Z}$, there exist $r(l,1), r(l,2) \in \mathbb{N}$ with the property that

$$\psi_l = \psi_{l+j\,2^{r(l,1)}}, \quad \psi_{l+1} = \psi_{l+1+j\,2^{r(l,2)}}, \quad j \in \mathbb{Z}.$$

Thus, for any $l \in \mathbb{Z}$, there exists $r(l) \in \mathbb{N}$ such that

$$\psi(l) = \psi(l+j\,2^{r(l)}), \quad \psi(l+1) = \psi(l+1+j\,2^{r(l)}), \quad j \in \mathbb{Z}$$

which implies

$$\psi(l+s) = \psi(l+s+j\,2^{r(l)}), \quad j \in \mathbb{Z},\ s \in [0,1].$$

It remains to show that $\psi$ is almost periodic. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be the number corresponding to $\varepsilon$ from the definition of the uniform continuity of the connections of the values $\varphi_i$, $i \in \mathbb{N}$. Let $l \in \mathbb{Z}$ be a $\delta$-translation number of $\{\psi_k\}$, i.e., let

$$\varrho\left(\psi_{k+l}, \psi_k\right) < \delta, \quad k \in \mathbb{Z}. \tag{3.27}$$

By the definition of the function $\psi$, we have

$$\varrho\left(\psi(t+l), \psi(t)\right) < \varepsilon, \quad t \in \mathbb{R}.$$

Indeed, it suffices to consider (3.27) for $k$ and $k+1$. Since any $\delta$-translation number of $\{\psi_k\}$ is an $\varepsilon$-translation number of $\psi : \mathbb{R} \to \mathcal{X}$ and since the set of all $\delta$-translation numbers of almost periodic $\{\psi_k\}$ is relative dense in $\mathbb{Z}$, function $\psi$ is almost periodic as well. □

From Theorem 3.19, one can easily obtain the following result which also follows from the approximation theorem (mentioned in introduction of this chapter) or from Theorems 1.19 and 3.11 (see also the proof of Theorem 3.11).

**Corollary 3.20.** *Let $\mathcal{X}$ be a Banach space. If $X \subseteq \mathcal{X}$ is countable and totally bounded, then there exists an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ such that*

$$\{\psi(k);\ k \in \mathbb{Z}\} = X$$

*and that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ for which*

$$\psi(l + s) = \psi(l + s + jq(l)), \quad j \in \mathbb{Z},\ s \in [0, 1).$$

Note that interesting open problems about general almost periodic functions are mentioned in [43] (see also [69]).

# Chapter 4

# Almost periodic homogeneous linear differential systems

We will analyse almost periodic solutions of almost periodic linear differential systems. Sometimes this field is called the Favard theory what is based on the Favard contributions in [65] (see also [31, Theorem 1.2], [40, Chapter 5], [69, Theorem 6.3] or [127, Theorem 1]; for homogeneous case, see [36], [64]).[10] In this context, sufficient conditions for the existence of almost periodic solutions are mentioned in [35], [47], [88] (for generalizations, see [41], [42], [67], [82], [85], [86], [90], [97], [107], [110], [121], [140], [142], [143]; for other extensions and supplements of the Favard theorem, see [2], [17], [31], [32], [44], [45], [46], [48], [49], [69] with the references cited therein, [87], [111]). Certain sufficient conditions, under which *homogeneous* systems that have nontrivial bounded solutions also have nontrivial almost periodic solutions, are given in [128].

It is a corollary of the Favard (and the Floquet) theory that any bounded solution of an almost periodic linear differential system is almost periodic if the matrix valued function, which determines the system, is periodic (see [69, Corollary 6.5]; for a generalization in the homogeneous case, see [81]). This result is no longer valid for systems with almost periodic coefficients. There exist systems for which all solutions are bounded, but none of them is almost periodic (see [94], [95], [127], [137]). Homogeneous systems have the zero solution which is almost periodic, but do not need to have an other almost periodic solution. We note that the existence of a homogeneous system, which has bounded solutions (separated from zero) and, at the same time, all systems from some neighbourhood of it do not have any nontrivial almost periodic solution, is proved in [153].

We will consider the set of all almost periodic skew-Hermitian differential systems with the uniform topology of matrix functions on the real axis.[11] In [152], it is proved that the systems, all of whose solutions are almost periodic, form a dense subset of the set of all considered systems. We add that special cases of this result are proved in [104], [105].[12] For systems whose solutions are not almost periodic, we refer to [154].

Using the method for constructing almost periodic functions from Chapter 3, we will prove that, in any neighbourhood of a system, there exists a system which does not possess an almost periodic solution other than the trivial one, not only with a fundamental matrix which is not almost periodic as in [154]. It means that, applying our method, we will get a stronger version of a statement from [154]. We remark that constructions of almost periodic linear differential systems with given properties are used in [103], [112], [113].

## 4.1 Preliminaries

Let $m \in \mathbb{N}$ be arbitrarily given. In this chapter, we will use the following notations: $\mathcal{I}m(\varphi)$ for the range of a function $\varphi$, $\mathcal{M}at(\mathbb{C}, m)$ for the set of all $m \times m$ matrices with complex elements, $U(m) \subset \mathcal{M}at(\mathbb{C}, m)$ for the group of all unitary matrices of dimension $m$, $A^*$ for the conjugate transpose of $A \in \mathcal{M}at(\mathbb{C}, m)$, $I$ for the identity matrix, $O$ for the zero matrix, and "i" for the imaginary unit.

## 4.2 Skew-Hermitian systems without almost periodic solutions

We will study systems of $m$ homogeneous linear differential equations of the form

$$x'(t) = A(t) \cdot x(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where $A$ is an almost periodic function with $\mathcal{I}m(A) \subset \mathcal{M}at(\mathbb{C}, m)$ and with the property that $A(t) + A^*(t) = O$ for any $t \in \mathbb{R}$, i.e., $A : \mathbb{R} \to \mathcal{M}at(\mathbb{C}, m)$ is an almost periodic function of *skew-Hermitian* (*skew-adjoint*) matrices. Let $\mathcal{S}$ be *the set of all systems* (4.1). We will identify the function $A$ with the system (4.1) which is determined by $A$. Especially, we will write $A \in \mathcal{S}$ and $O \in \mathcal{S}$ will denote the system (4.1) given by $A(t) = O$, $t \in \mathbb{R}$.

In the vector space $\mathbb{C}^m$, we will consider the absolute norm $||\cdot||_1$ (one can also consider the Euclidean norm or the maximum norm). Let $||\cdot||$ be the corresponding matrix norm.[13] Considering that every almost periodic function is bounded (see Lemma 3.3), the distance between two systems $A, B \in \mathcal{S}$ is defined by the norm of the matrix valued functions $A$, $B$, uniformly on $\mathbb{R}$; i.e., we introduce the metric

$$\sigma(A, B) := \sup_{t \in \mathbb{R}} ||\, A(t) - B(t)\, ||, \quad A, B \in \mathcal{S}.$$

For $\varepsilon > 0$, the symbol $\mathcal{O}^{\sigma}_{\varepsilon}(A)$ will denote the $\varepsilon$-*neighbourhood* of $A$ in $\mathcal{S}$.

Now we recall the notion of the frequency module and its rational hull which can be introduced for all almost periodic function with values in a Banach space.[14] *The frequency module* $\mathcal{F}$ of an almost periodic function $A : \mathbb{R} \to \mathcal{M}at(\mathbb{C}, m)$ is the $\mathbb{Z}$-module of the real numbers, generated by the numbers $\lambda$ such that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{2\pi \mathrm{i} \lambda t} A(t)\, dt \neq O.$$

*The rational hull of* $\mathcal{F}$ is the set

$$\{\lambda / l;\ \lambda \in \mathcal{F},\ l \in \mathbb{Z}\}.$$

For the frequency modules of almost periodic linear differential systems and their solutions, we refer to [69, Chapters 4, 6], [127].

In [152], it is proved that, in any neighbourhood of a system (4.1) with frequency module $\mathcal{F}$, there exists a system with a frequency module contained in the rational hull of $\mathcal{F}$ possessing all almost periodic solutions with frequencies belonging to the rational hull of $\mathcal{F}$ as well. From [156, Theorem 1] it follows that there exists a system (4.1) which cannot be approximated by the so-called reducible systems[15] with frequency module $\mathcal{F}$ (there exists an open set of irreducible systems with a fixed frequency module—see [155] for the real case); i.e., a neighbourhood of a system (4.1) with frequency module $\mathcal{F}$ may not contain a system with almost periodic solutions and frequency module $\mathcal{F}$. In this case, see also [56] and [157] for reducible constant systems and systems reducing to diagonal form by Lyapunov transformation with frequency module $\mathcal{F}$, respectively.

In addition, for $k \in \mathbb{N}$, it is proved in [154] that the systems with $k$-dimensional frequency basis of $A$, having solutions which are not almost periodic, form a subset of the second category of the space of all considered systems with $k$-dimensional frequency basis of $A$. Thus, it is known (see also [152, Corollary 1]) that the systems with $k$-dimensional frequency basis of $A$ and with an almost periodic fundamental matrix form a dense set of the first category in the space of all systems (4.1) with $k$-dimensional frequency basis.

In this context, we formulate the following result that the systems having no nontrivial almost periodic solution form a dense subset of $\mathcal{S}$.

**Theorem 4.1.** *For any $A \in \mathcal{S}$ and $\varepsilon > 0$, there exists $B \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ which does not have an almost periodic solution other than the trivial one.*

*Proof.* Let $A, C \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary. Since the sum of skew-Hermitian matrices is also skew-Hermitian and since the sum of two almost periodic functions is almost periodic (see Corollary 3.7), we have that $A + C \in \mathcal{S}$. Let $X_A(t)$, $t \in \mathbb{R}$ and $X_C(t)$, $t \in \mathbb{R}$ be the principal (i.e., $X_A(0) = X_C(0) = I$) fundamental matrices of $A \in \mathcal{S}$ and $C \in \mathcal{S}$, respectively. If the matrices $C(t)$, $X_A(t)$ commute for all $t \in \mathbb{R}$, then the matrix valued function $X_A(t)\, X_C(t)$, $t \in \mathbb{R}$ is the principal fundamental matrix of $A + C \in \mathcal{S}$. Indeed, from $X'_A(t) = A(t)\, X_A(t)$, $X'_C(t) = C(t)\, X_C(t)$, $t \in \mathbb{R}$, we obtain

$$
\begin{aligned}
&(X_A(t) \cdot X_C(t))' = A(t) \cdot X_A(t) \cdot X_C(t) + X_A(t) \cdot C(t) \cdot X_C(t) = \\
&A(t) \cdot X_A(t) \cdot X_C(t) + C(t) \cdot X_A(t) \cdot X_C(t) = (A + C)(t) \cdot X_A(t) \cdot X_C(t), \quad t \in \mathbb{R}.
\end{aligned}
$$

It gives that it suffices to find $C \in \mathcal{O}^{\sigma}_{\varepsilon}(O)$ for which all matrices $C(t)$, $t \in \mathbb{R}$ have the form $\operatorname{diag}[\mathrm{i}a, \dots, \mathrm{i}a]$, $a \in \mathbb{R}$ and for which the vector valued function $X_A(t)\, X_C(t)\, u$, $t \in \mathbb{R}$ is not almost periodic for any vector $u \in \mathbb{C}^m$, $\|u\|_1 = 1$.

We will construct such an almost periodic function $C$ using Theorem 3.12 for $a = \varepsilon/4$. First of all we put

$$C(t) \equiv O, \quad t \in [0, 1].$$

Then, in the first step of our construction, we define $C$ on $(1, 2]$ arbitrarily so that it is constant on $[1+1/4, 1+3/4]$ and $\|C(t)\| < \varepsilon/4$ for $t$ from this interval, $C(2) := C(1) = O$, and it is linear between values $O$, $C(3/2)$ on $[1, 1+1/4]$ and $[1+3/4, 2]$.

In the second step, we define continuous $C$ satisfying $\|C(t) - C(t+2)\| < \varepsilon/4$ for $t \in [-2, 0)$ arbitrarily so that it is constant on

$$[-2 + 1/16, -2 + 1 - 1/16], \quad [-2 + 1 + 1/4 + 1/16, -2 + 1 + 3/4 - 1/16];$$

at the same time, we put

$$C(-2) := C(0) = O, \quad C(-1+1/4) := C(1+1/4) = C(3/2),$$

$$C(-1) := C(1) = O, \quad C(-1/4) := C(2-1/4) = C(3/2),$$

and

$$C(t) \equiv C(3/2)/2, \quad t \in [-1+1/16, -1+1/4-1/16] \cup [-1/4+1/16, -1/16]$$

and we define $C$ so that it is linear on

$$[-2, -2+1/16], \quad [-1-1/16, -1], \quad [-1, -1+1/16],$$

$$[-1+1/4-1/16, -1+1/4], \quad [-1+1/4, -1+1/4+1/16],$$
$$[-1/4-1/16, -1/4], \quad [-1/4, -1/4+1/16], \quad [-1/16, 0].$$

Analogously, in the third step, we get $C$ on $(2, 6]$ for which we can choose constant values on

$$[4-2+1/16+8^{-1}/16, 4-2+1-1/16-8^{-1}/16],$$
$$[4-2+1+1/4+1/16+8^{-1}/16, 4-2+1+3/4-1/16-8^{-1}/16],$$
$$[4-1+1/16+8^{-1}/16, 4-1+1/4-1/16-8^{-1}/16],$$
$$[4-1/4+1/16+8^{-1}/16, 4-1/16-8^{-1}/16],$$
$$[4+8^{-1}/16, 4+1-8^{-1}/16], \quad [4+1+1/4+8^{-1}/16, 4+1+3/4-8^{-1}/16]$$

arbitrarily so that $\|\, C(t) - C(t-4) \,\| < \varepsilon/8$, $t \in (2, 6]$; at the same time, we put

$$C(4-2+1/16) := C(-2+1/16) = C(-3/2),$$

$$C(4-2+1-1/16) := C(-1-1/16) = C(-3/2),$$
$$C(4-1) := C(-1) = O,$$
$$C(4-1+1/16) := C(-1+1/16) = C(3/2)/2,$$
$$C(4-1+1/4-1/16) := C(-1+1/4-1/16) = C(3/2)/2,$$
$$C(4-1+1/4) := C(-1+1/4) = C(3/2),$$
$$C(4-2+1+1/4+1/16) := C(-2+1+1/4+1/16) = C(-1/2),$$
$$C(4-2+1+3/4-1/16) := C(-2+1+3/4-1/16) = C(-1/2),$$
$$C(4-1/4) := C(-1/4) = C(3/2),$$
$$C(4-1/4+1/16) := C(-1/4+1/16) = C(3/2)/2,$$
$$C(4-1/16) := C(-1/16) = C(3/2)/2,$$
$$C(4) := C(0), \quad C(4+1) := C(1),$$
$$C(4+1+1/4) := C(1+1/4) = C(3/2),$$
$$C(4+1+3/4) := C(1+3/4) = C(3/2),$$

$$
\begin{aligned}
C(4+2) &:= C(2) = C(0) = O,\\
C(t) &\equiv C(-3/2)/2, \quad t \in [4-2+8^{-1}/16, 4-2+1/16-8^{-1}/16]\\
&\qquad\qquad \cup [4-1-1/16+8^{-1}/16, 4-1-8^{-1}/16],\\
C(t) &\equiv C(3/2)/4, \quad t \in [4-1+8^{-1}/16, 4-1+1/16-8^{-1}/16]\\
&\qquad\qquad \cup [4-1/16+8^{-1}/16, 4-8^{-1}/16],\\
C(t) &\equiv 3\,C(3/2)/4, \quad t \in [4-1+1/4-1/16+8^{-1}/16, 4-1+1/4-8^{-1}/16]\\
&\qquad\qquad \cup [4-1/4+8^{-1}/16, 4-1/4+1/16-8^{-1}/16],\\
C(t) &\equiv (C(3/2)+C(-1/2))/2, \quad t \in [4-1+1/4+8^{-1}/16, 4-1+1/4+1/16-8^{-1}/16]\\
&\qquad\qquad \cup [4-1/4-1/16+8^{-1}/16, 4-1/4-8^{-1}/16],\\
C(t) &\equiv (8\,C(4+1)+1\,C(4+1+1/4))/9,\\
&\qquad t \in [4+1+8^{-1}/16, 4+1+8^{-1}/16\cdot 3],\\
C(t) &\equiv (7\,C(4+1)+2\,C(4+1+1/4))/9,\\
&\qquad t \in [4+1+8^{-1}/16\cdot 5, 4+1+8^{-1}/16\cdot 7],\\
&\vdots\\
C(t) &\equiv (1\,C(4+1)+8\,C(4+1+1/4))/9,\\
&\qquad t \in [4+1+8^{-1}/16\cdot 29, 4+1+8^{-1}/16\cdot 31],\\
C(t) &\equiv (8\,C(4+1+3/4)+1\,C(4+2))/9,\\
&\qquad t \in [4+1+3/4+8^{-1}/16, 4+1+3/4+8^{-1}/16\cdot 3],\\
C(t) &\equiv (7\,C(4+1+3/4)+2\,C(4+2))/9,\\
&\qquad t \in [4+1+3/4+8^{-1}/16\cdot 5, 4+1+3/4+8^{-1}/16\cdot 7],\\
&\vdots\\
C(t) &\equiv (1\,C(4+1+3/4)+8\,C(4+2))/9,\\
&\qquad t \in [4+1+3/4+8^{-1}/16\cdot 29, 4+1+3/4+8^{-1}/16\cdot 31]
\end{aligned}
$$

and we define continuous $C$ so that it is linear on the rest of subintervals.

If we denote

$$a_1^1 := 0, \quad b_1^1 := 0, \quad c_1^1 := 1,$$

$$a_2^1 := 1, \quad b_2^1 := 1+1/4, \quad c_2^1 := 1+3/4, \quad a_3^1 := 2$$

and (compare with the situation after the second step)

$$a_1^2 := -2, \quad b_1^2 := -2, \quad c_1^2 := -2,$$

$$
\begin{gathered}
a_2^2 := -2, \quad b_2^2 := -2+1/16, \quad c_2^2 := -1-1/16,\\
a_3^2 := -1, \quad b_3^2 := -1, \quad c_3^2 := -1,\\
a_4^2 := -1, \quad b_4^2 := -1+1/16, \quad c_4^2 := -1+1/4-1/16,\\
a_5^2 := -1+1/4, \quad b_5^2 := -1+1/4+1/16, \quad c_5^2 := -1+3/4-1/16,\\
a_6^2 := -1+3/4, \quad b_6^2 := -1+3/4+1/16, \quad c_6^2 := -1/16,
\end{gathered}
$$

we see that $C$ does not need to be constant only on

$$[a_j^1 - 2, a_j^1 - 2 + 4^{-2}], \quad [b_2^1 - 2 - 4^{-2}, b_2^1 - 2], \quad [b_j^1 - 2, b_j^1 - 2 + 4^{-2}],$$

$$[c_j^1 - 2 - 4^{-2}, c_j^1 - 2], \quad [c_j^1 - 2, c_j^1 - 2 + 4^{-2}], \quad [a_{j+1}^1 - 2 - 4^{-2}, a_{j+1}^1 - 2]$$

for $j \in \{1, 2\}$, i.e., on

$$[a_j^2, b_j^2], \ j \in \{1, \dots, 6\}, \quad [c_j^2, a_{j+1}^2], \ j \in \{1, \dots, 5\}, \quad [c_6^2, 0],$$

and it has to be constant on each one of the intervals

$$[a_2^1 - 2 + 4^{-2}, b_2^1 - 2 - 4^{-2}], \quad [c_2^1 - 2 + 4^{-2}, a_3^1 - 2 - 4^{-2}],$$

$$[b_j^1 - 2 + 4^{-2}, c_j^1 - 2 - 4^{-2}], \quad j \in \{1, 2\},$$

i.e., on

$$[b_j^2, c_j^2], \quad j \in \{1, \dots, 6\}.$$

It is also seen that

$$a_1^2 = d_1^1, \quad b_1^2 = d_2^1, \quad c_1^2 = d_3^1, \quad a_2^2 = d_4^1, \quad \cdots \quad c_6^2 = d_{18}^1,$$

where $d_1^1, d_2^1, \dots, d_{18}^1$ is the nondecreasing sequence of all numbers

$$a_j^1 - 2, \quad b_j^1 - 2, \quad c_j^1 - 2,$$

$$\min\{a_j^1 - 2 + 4^{-2}, b_j^1 - 2\}, \quad \max\{a_j^1 - 2, b_j^1 - 2 - 4^{-2}\},$$
$$\min\{c_j^1 - 2, b_j^1 - 2 + 4^{-2}\}, \quad \max\{c_j^1 - 2 - 4^{-2}, b_j^1 - 2\},$$
$$\min\{c_j^1 - 2 + 4^{-2}, a_{j+1}^1 - 2\}, \quad \max\{c_j^1 - 2, a_{j+1}^1 - 2 - 4^{-2}\}$$

for $j \in \{1, 2\}$. We put $a_7^2 := 0$.

Let $d_1^2, d_2^2, \dots, d_{168}^2$ be the nondecreasing sequence of all numbers

$$b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4,$$

$$b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4, \quad b_1^1 + 4,$$
$$c_1^1 + 4, \quad \min\{c_1^1 + 4, b_1^1 + 4 + 8^{-1}/16\}, \quad \max\{c_1^1 + 4 - 8^{-1}/16, b_1^1 + 4\},$$
$$c_2^1 + 4, \quad \min\{c_2^1 + 4, b_2^1 + 4 + 8^{-1}/16\}, \quad \max\{c_2^1 + 4 - 8^{-1}/16, b_2^1 + 4\},$$
$$a_1^1 + (4k+1)(b_1^1 - a_1^1)/32 + 4, \quad a_1^1 + (4k+3)(b_1^1 - a_1^1)/32 + 4,$$
$$a_1^1 + (4k+4)(b_1^1 - a_1^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$
$$c_1^1 + (4k+1)(a_2^1 - c_1^1)/32 + 4, \quad c_1^1 + (4k+3)(a_2^1 - c_1^1)/32 + 4,$$
$$c_1^1 + (4k+4)(a_2^1 - c_1^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$
$$a_2^1 + (4k+1)(b_2^1 - a_2^1)/32 + 4, \quad a_2^1 + (4k+3)(b_2^1 - a_2^1)/32 + 4,$$
$$a_2^1 + (4k+4)(b_2^1 - a_2^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$
$$c_2^1 + (4k+1)(a_3^1 - c_2^1)/32 + 4, \quad c_2^1 + (4k+3)(a_3^1 - c_2^1)/32 + 4,$$

$$c_2^1 + (4k+4)(a_3^1 - c_2^1)/32 + 4, \quad k \in \{0, 1, \dots, 7\}$$

and

$$\begin{gathered} a_{j+1}^2 + 4, \quad b_j^2 + 4, \quad c_j^2 + 4, \\ \min\{a_j^2 + 4 + 8^{-1}/16, b_j^2 + 4\}, \quad \max\{a_j^2 + 4, b_j^2 + 4 - 8^{-1}/16\}, \\ \min\{c_j^2 + 4, b_j^2 + 4 + 8^{-1}/16\}, \quad \max\{c_j^2 + 4 - 8^{-1}/16, b_j^2 + 4\}, \\ \min\{c_j^2 + 4 + 8^{-1}/16, a_{j+1}^2 + 4\}, \quad \max\{c_j^2 + 4, a_{j+1}^2 + 4 - 8^{-1}/16\} \end{gathered}$$

for $j \in \{1, \dots, 6\}$. We denote

$$a_1^3 := 2, \quad b_1^3 := d_1^2, \quad c_1^3 := d_2^2, \quad a_2^3 := d_3^2, \quad \cdots \quad a_{57}^3 := d_{168}^2.$$

We remark that, in the sequences of $d_j^l$, $l \in \mathbb{N}$, values are a number of time.

In the fourth step, we define $C$ so that

$$\big|\big|\, C(t) - C(t + 2^3)\,\big|\big| < \frac{\varepsilon}{2^3}, \quad t \in [-2^3 - 2, -2).$$

We consider the nondecreasing sequence $d_1^3, d_2^3, \dots, d_{21 \cdot 8^2}^3$ of

$$\begin{gathered} a_j^3 - 2^3, \quad b_j^3 - 2^3, \quad c_j^3 - 2^3, \\ \min\{a_j^3 - 2^3 + 8^{-2}/16, b_j^3 - 2^3\}, \quad \max\{a_j^3 - 2^3, b_j^3 - 2^3 - 8^{-2}/16\}, \\ \min\{c_j^3 - 2^3, b_j^3 - 2^3 + 8^{-2}/16\}, \quad \max\{c_j^3 - 2^3 - 8^{-2}/16, b_j^3 - 2^3\}, \\ \min\{c_j^3 - 2^3 + 8^{-2}/16, a_{j+1}^3 - 2^3\}, \quad \max\{c_j^3 - 2^3, a_{j+1}^3 - 2^3 - 8^{-2}/16\} \end{gathered}$$

for $j \in \{1, \dots, 7 \cdot 8\}$, 144 numbers $b_1^1 - 2^3$, and

$$\begin{gathered} c_1^1 - 2^3, \quad \min\{c_1^1 - 2^3, b_1^1 - 2^3 + 8^{-2}/16\}, \quad \max\{c_1^1 - 2^3 - 8^{-2}/16, b_1^1 - 2^3\}, \\ c_2^1 - 2^3, \quad \min\{c_2^1 - 2^3, b_2^1 - 2^3 + 8^{-2}/16\}, \quad \max\{c_2^1 - 2^3 - 8^{-2}/16, b_2^1 - 2^3\}, \\ \min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3\}, \\ \max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\ a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4) - 2^3, \qquad k \in \{1, \dots, 8 \cdot 4\}, \\ \min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3\}, \\ \max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\ c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4) - 2^3, \qquad k \in \{1, \dots, 8 \cdot 4\}, \\ \min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3\}, \\ \max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\ a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4) - 2^3, \qquad k \in \{1, \dots, 8 \cdot 4\}, \\ \min\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3 + 8^{-2}/16, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3\}, \\ \max\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3 - 8^{-2}/16\}, \\ c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4) - 2^3, \qquad k \in \{1, \dots, 8 \cdot 4\}, \end{gathered}$$

$$c_1^2 - 2^3, \quad \min\{c_1^2 - 2^3, b_1^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_1^2 - 2^3 - 8^{-2}/16, b_1^2 - 2^3\},$$
$$c_2^2 - 2^3, \quad \min\{c_2^2 - 2^3, b_2^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_2^2 - 2^3 - 8^{-2}/16, b_2^2 - 2^3\},$$
$$c_3^2 - 2^3, \quad \min\{c_3^2 - 2^3, b_3^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_3^2 - 2^3 - 8^{-2}/16, b_3^2 - 2^3\},$$
$$c_4^2 - 2^3, \quad \min\{c_4^2 - 2^3, b_4^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_4^2 - 2^3 - 8^{-2}/16, b_4^2 - 2^3\},$$
$$c_5^2 - 2^3, \quad \min\{c_5^2 - 2^3, b_5^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_5^2 - 2^3 - 8^{-2}/16, b_5^2 - 2^3\},$$
$$c_6^2 - 2^3, \quad \min\{c_6^2 - 2^3, b_6^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_6^2 - 2^3 - 8^{-2}/16, b_6^2 - 2^3\},$$
$$\min\{a_1^2 + (k-1)(b_1^2 - a_1^2)/8 - 2^3 + 8^{-2}/16, a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3\},$$
$$\max\{a_1^2 + (k-1)(b_1^2 - a_1^2)/8 - 2^3, a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3 - 8^{-2}/16\},$$
$$a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3, \qquad k \in \{1, \dots, 8\},$$
$$\min\{c_1^2 + (k-1)(a_2^2 - c_1^2)/8 - 2^3 + 8^{-2}/16, c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3\},$$
$$\max\{c_1^2 + (k-1)(a_2^2 - c_1^2)/8 - 2^3, c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3 - 8^{-2}/16\},$$
$$c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3, \qquad k \in \{1, \dots, 8\},$$
$$\vdots$$
$$\min\{a_6^2 + (k-1)(b_6^2 - a_6^2)/8 - 2^3 + 8^{-2}/16, a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3\},$$
$$\max\{a_6^2 + (k-1)(b_6^2 - a_6^2)/8 - 2^3, a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3 - 8^{-2}/16\},$$
$$a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3, \qquad k \in \{1, \dots, 8\},$$
$$\min\{c_6^2 + (k-1)(a_7^2 - c_6^2)/8 - 2^3 + 8^{-2}/16, c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3\},$$
$$\max\{c_6^2 + (k-1)(a_7^2 - c_6^2)/8 - 2^3, c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3 - 8^{-2}/16\},$$
$$c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3, \qquad k \in \{1, \dots, 8\}.$$

We put

$$a_1^4 := d_1^3, \quad b_1^4 := d_2^3, \quad c_1^4 := d_3^3, \quad \cdots \quad c_{7\cdot 8^2}^4 := d_{21\cdot 8^2}^3, \quad a_{7\cdot 8^2+1}^4 := -2.$$

We recall that $C$ can be increasing or decreasing only on

$$[a_j^4, b_j^4], \quad [c_j^4, a_{j+1}^4], \quad j \in \{1, \dots, 7 \cdot 8^2\}.$$

We proceed further in the same way (as in the third and the fourth step). In the $2n$-th step, we define continuous $C$ so that

$$\left|\left| \, C(t) - C(t + 2^{2n-1}) \, \right|\right| < \frac{\varepsilon}{2^{n+1}}, \quad t \in [-2^{2n-1} - \cdots - 2, -2^{2n-3} - \cdots - 2).$$

We get the nondecreasing sequence $\{d_l^{2n-1}\}$ from

$$a_j^{2n-1} - 2^{2n-1}, \quad b_j^{2n-1} - 2^{2n-1}, \quad c_j^{2n-1} - 2^{2n-1},$$

$$\min\{a_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16, b_j^{2n-1} - 2^{2n-1}\}, \ \max\{a_j^{2n-1} - 2^{2n-1}, b_j^{2n-1} - 2^{2n-1} - 8^{2-2n}/16\},$$
$$\min\{c_j^{2n-1} - 2^{2n-1}, b_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16\}, \ \max\{c_j^{2n-1} - 2^{2n-1} - 8^{2-2n}/16, b_j^{2n-1} - 2^{2n-1}\},$$

$\min\{c_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16, a_{j+1}^{2n-1} - 2^{2n-1}\},\ \max\{c_j^{2n-1} - 2^{2n-1}, a_{j+1}^{2n-1} - 2^{2n-1} - 8^{2-2n}/16\}$ for $j \in \{1, \dots, 7 \cdot 8^{2n-3}\}$, from

$$
\begin{gathered}
c_1^1 - 2^{2n-1},\ \min\{c_1^1 - 2^{2n-1}, b_1^1 - 2^{2n-1} + 8^{2-2n}/16\},\ \max\{c_1^1 - 2^{2n-1} - 8^{2-2n}/16, b_1^1 - 2^{2n-1}\},\\
c_2^1 - 2^{2n-1},\ \min\{c_2^1 - 2^{2n-1}, b_2^1 - 2^{2n-1} + 8^{2-2n}/16\},\ \max\{c_2^1 - 2^{2n-1} - 8^{2-2n}/16, b_2^1 - 2^{2n-1}\},\\
\min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},\\
\min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},\\
\min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},\\
\min\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, c_2^1 + k(a_3^1 - c_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}\},\\
\max\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, c_2^1 + k(a_3^1 - c_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\},\\
c_2^1 + k(a_3^1 - c_2^1)/(8\cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-3}\},\\
c_1^2 - 2^{2n-1},\ \min\{c_1^2 - 2^{2n-1}, b_1^2 - 2^{2n-1} + 8^{2-2n}/16\},\ \max\{c_1^2 - 2^{2n-1} - 8^{2-2n}/16, b_1^2 - 2^{2n-1}\},\\
\vdots\\
c_6^2 - 2^{2n-1},\ \min\{c_6^2 - 2^{2n-1}, b_6^2 - 2^{2n-1} + 8^{2-2n}/16\},\ \max\{c_6^2 - 2^{2n-1} - 8^{2-2n}/16, b_6^2 - 2^{2n-1}\},\\
\min\{a_1^2 + (k-1)(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, a_1^2 + k(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},\\
\max\{a_1^2 + (k-1)(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, a_1^2 + k(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},\\
a_1^2 + k(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},\\
\min\{c_1^2 + (k-1)(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, c_1^2 + k(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},\\
\max\{c_1^2 + (k-1)(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, c_1^2 + k(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},\\
c_1^2 + k(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},\\
\vdots\\
\min\{a_6^2 + (k-1)(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, a_6^2 + k(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},\\
\max\{a_6^2 + (k-1)(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, a_6^2 + k(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},\\
a_6^2 + k(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},\\
\min\{c_6^2 + (k-1)(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, c_6^2 + k(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},\\
\max\{c_6^2 + (k-1)(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, c_6^2 + k(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},\\
c_6^2 + k(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},
\end{gathered}
$$

$$\vdots$$

$$c_1^{2n-2} - 2^{2n-1}, \quad \min\{c_1^{2n-2} - 2^{2n-1}, b_1^{2n-2} - 2^{2n-1} + 8^{2-2n}/16\},$$
$$\max\{c_1^{2n-2} - 2^{2n-1} - 8^{2-2n}/16, b_1^{2n-2} - 2^{2n-1}\},$$

$$\vdots$$

$$c_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1}, \quad \min\{c_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1}, b_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1} + 8^{2-2n}\},$$
$$\max\{c_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1} - 8^{2-2n}/16, b_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1}\},$$
$$\min\{a_1^{2n-2}+(k-1)(b_1^{2n-2}-a_1^{2n-2})/8-2^{2n-1}+8^{2-2n}/16, a_1^{2n-2}+k(b_1^{2n-2}-a_1^{2n-2})/8-2^{2n-1}\},$$
$$\max\{a_1^{2n-2}+(k-1)(b_1^{2n-2}-a_1^{2n-2})/8-2^{2n-1}, a_1^{2n-2}+k(b_1^{2n-2}-a_1^{2n-2})/8-2^{2n-1}-8^{2-2n}/16\},$$
$$a_1^{2n-2} + k(b_1^{2n-2} - a_1^{2n-2})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$
$$\min\{c_1^{2n-2}+(k-1)(a_2^{2n-2}-c_1^{2n-2})/8-2^{2n-1}+8^{2-2n}/16, c_1^{2n-2}+k(a_2^{2n-2}-c_1^{2n-2})/8-2^{2n-1}\},$$
$$\max\{c_1^{2n-2}+(k-1)(a_2^{2n-2}-c_1^{2n-2})/8-2^{2n-1}, c_1^{2n-2}+k(a_2^{2n-2}-c_1^{2n-2})/8-2^{2n-1}-8^{2-2n}/16\},$$
$$c_1^{2n-2} + k(a_2^{2n-2} - c_1^{2n-2})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$

$$\vdots$$

$$\min\{a_{7\cdot 8^{2n-4}}^{2n-2} + (k-1)(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1} + 8^{2-2n}/16,$$
$$a_{7\cdot 8^{2n-4}}^{2n-2} + k(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1}\},$$
$$\max\{a_{7\cdot 8^{2n-4}}^{2n-2} + (k-1)(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1},$$
$$a_{7\cdot 8^{2n-4}}^{2n-2} + k(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1} - 8^{2-2n}/16\},$$
$$a_{7\cdot 8^{2n-4}}^{2n-2} + k(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$
$$\min\{c_{7\cdot 8^{2n-4}}^{2n-2} + (k-1)(a_{7\cdot 8^{2n-4}+1}^{2n-2} - c_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1} + 8^{2-2n}/16,$$
$$c_{7\cdot 8^{2n-4}}^{2n-2} + k(a_{7\cdot 8^{2n-4}+1}^{2n-2} - c_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1}\},$$
$$\max\{c_{7\cdot 8^{2n-4}}^{2n-2} + (k-1)(a_{7\cdot 8^{2n-4}+1}^{2n-2} - c_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1},$$
$$c_{7\cdot 8^{2n-4}}^{2n-2} + k(a_{7\cdot 8^{2n-4}+1}^{2n-2} - c_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1} - 8^{2-2n}/16\},$$
$$c_{7\cdot 8^{2n-4}}^{2n-2} + k(a_{7\cdot 8^{2n-4}+1}^{2n-2} - c_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$

and from a number of $b_1^1 - 2^{2n-1}$ such that the total number of $d_l^{2n-1}$ is $21 \cdot 8^{2n-2}$. We denote

$$a_1^{2n} := d_1^{2n-1}, \quad b_1^{2n} := d_2^{2n-1}, \quad c_1^{2n} := d_3^{2n-1}, \quad \cdots$$
$$c_{7\cdot 8^{2n-2}}^{2n-1} := d_{21\cdot 8^{2n-2}}^{3}, \quad a_{7\cdot 8^{2n-2}+1}^{2n-1} := -2^{2n-3} - \cdots - 2.$$

In the $(2n+1)$-th step, we define continuous $C$ so that

$$\left|\left| C(t) - C(t - 2^{2n}) \right|\right| < \frac{\varepsilon}{2^{n+2}}, \quad t \in (2 + \cdots + 2^{2n-2}, 2 + \cdots + 2^{2n}].$$

Now $C$ has constant values on $[b_j^{2n+1}, c_j^{2n+1}]$, $j \in \{1, \dots, 7 \cdot 8^{2n-1}\}$, where we put

$$a_1^{2n+1} := 2 + 2^2 + \cdots + 2^{2n-2}$$

and we obtain

$$b_1^{2n+1}, \quad c_1^{2n+1}, \quad a_2^{2n+1}, \quad \cdots \quad c_{7\cdot 8^{2n-1}}^{2n+1}, \quad a_{7\cdot 8^{2n-1}+1}^{2n+1}$$

from the nondecreasing sequence of

$$a_{j+1}^{2n} + 2^{2n}, \quad b_j^{2n} + 2^{2n}, \quad c_j^{2n} + 2^{2n},$$
$$\min\{a_j^{2n} + 2^{2n} + 8^{1-2n}/16, b_j^{2n} + 2^{2n}\}, \quad \max\{a_j^{2n} + 2^{2n}, b_j^{2n} + 2^{2n} - 8^{1-2n}/16\},$$
$$\min\{c_j^{2n} + 2^{2n}, b_j^{2n} + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c_j^{2n} + 2^{2n} - 8^{1-2n}/16, b_j^{2n} + 2^{2n}\},$$
$$\min\{c_j^{2n} + 2^{2n} + 8^{1-2n}/16, a_{j+1}^{2n} + 2^{2n}\}, \quad \max\{c_j^{2n} + 2^{2n}, a_{j+1}^{2n} + 2^{2n} - 8^{1-2n}/16\}$$

for $j \in \{1, \ldots, 7 \cdot 8^{2n-2}\}$ and

$$c_1^1 + 2^{2n}, \quad \min\{c_1^1 + 2^{2n}, b_1^1 + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c_1^1 + 2^{2n} - 8^{1-2n}/16, b_1^1 + 2^{2n}\},$$
$$c_2^1 + 2^{2n}, \quad \min\{c_2^1 + 2^{2n}, b_2^1 + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c_2^1 + 2^{2n} - 8^{1-2n}/16, b_2^1 + 2^{2n}\},$$
$$\min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n} + 8^{1-2n}/16, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n}\},$$
$$\max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n} - 8^{1-2n}/16\},$$
$$a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, \quad k \in \{1, \ldots, 8 \cdot 4^{2n-2}\},$$
$$\min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n} + 8^{1-2n}/16, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n}\},$$
$$\max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n} - 8^{1-2n}/16\},$$
$$c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, \quad k \in \{1, \ldots, 8 \cdot 4^{2n-2}\},$$
$$\min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n} + 8^{1-2n}/16, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n}\},$$
$$\max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n} - 8^{1-2n}/16\},$$
$$a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, \quad k \in \{1, \ldots, 8 \cdot 4^{2n-2}\},$$
$$\min\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n} + 8^{1-2n}/16, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n}\},$$
$$\max\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n} - 8^{1-2n}/16\},$$
$$c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4^{2n-2}) + 2^{2n}, \quad k \in \{1, \ldots, 8 \cdot 4^{2n-2}\},$$
$$c_1^2 + 2^{2n}, \quad \min\{c_1^2 + 2^{2n}, b_1^2 + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c_1^2 + 2^{2n} - 8^{1-2n}/16, b_1^2 + 2^{2n}\},$$
$$\vdots$$
$$c_6^2 + 2^{2n}, \quad \min\{c_6^2 + 2^{2n}, b_6^2 + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c_6^2 + 2^{2n} - 8^{1-2n}/16, b_6^2 + 2^{2n}\},$$
$$\min\{a_1^2 + (k-1)(b_1^2 - a_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n} + 8^{1-2n}/16, a_1^2 + k(b_1^2 - a_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n}\},$$
$$\max\{a_1^2 + (k-1)(b_1^2 - a_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n}, a_1^2 + k(b_1^2 - a_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n} - 8^{1-2n}/16\},$$
$$a_1^2 + k(b_1^2 - a_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n}, \quad k \in \{1, \ldots, 8 \cdot 4^{2n-3}\},$$
$$\min\{c_1^2 + (k-1)(a_2^2 - c_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n} + 8^{1-2n}/16, c_1^2 + k(a_2^2 - c_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n}\},$$
$$\max\{c_1^2 + (k-1)(a_2^2 - c_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n}, c_1^2 + k(a_2^2 - c_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n} - 8^{1-2n}/16\},$$
$$c_1^2 + k(a_2^2 - c_1^2)/(8 \cdot 4^{2n-3}) + 2^{2n}, \quad k \in \{1, \ldots, 8 \cdot 4^{2n-3}\},$$

$$\vdots$$

$$\min\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16, a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}\},$$
$$\max\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}, a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},$$
$$a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-3}\},$$
$$\min\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16, c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}\},$$
$$\max\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}, c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},$$
$$c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-3}\},$$

$$\vdots$$

$$c_1^{2n-1}+2^{2n}, \quad \min\{c_1^{2n-1}+2^{2n}, b_1^{2n-1}+2^{2n}+8^{1-2n}/16\},$$
$$\max\{c_1^{2n-1}+2^{2n}-8^{1-2n}/16, b_1^{2n-1}+2^{2n}\},$$

$$\vdots$$

$$c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}, \quad \min\{c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}, b_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}+8^{1-2n}/16\},$$
$$\max\{c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}-8^{1-2n}/16, b_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}\},$$
$$\min\{a_1^{2n-1}+(k-1)(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}+8^{1-2n}/16, a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}\},$$
$$\max\{a_1^{2n-1}+(k-1)(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}, a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$
$$\min\{c_1^{2n-1}+(k-1)(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}+8^{1-2n}/16, c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}\},$$
$$\max\{c_1^{2n-1}+(k-1)(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}, c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$

$$\vdots$$

$$\min\{a_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}+8^{1-2n}/16,$$
$$a_{7\cdot 8^{2n-3}}^{2n-1}+k(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}\},$$
$$\max\{a_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n},$$
$$a_{7\cdot 8^{2n-3}}^{2n-1}+k(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$a_{7\cdot 8^{2n-3}}^{2n-1}+k(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$
$$\min\{c_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}+8^{1-2n}/16,$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+k(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}\},$$
$$\max\{c_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n},$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+k(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+k(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$

and the corresponding number of $b_1^1+2^{2n}$.

Using this construction, we get a continuous function $C$ on $\mathbb{R}$. From Theorem 3.12 it follows that $C$ is almost periodic. Since

$$||\,C(t)\,|| = 0,\ t \in [0,1], \quad ||\,C(t)\,|| < \varepsilon/4,\ t \in (1,2],$$

$$||\,C(t) - C(t+2)\,|| < \varepsilon/4,\ t \in [-2,0), \quad ||\,C(t) - C(t-4)\,|| < \varepsilon/8,\ t \in (2,6],$$

$$\vdots$$

$$\big|\big|\,C(t) - C(t+2^{2n-1})\,\big|\big| < \varepsilon/2^{n+1}, \quad t \in [-2^{2n-1} - \cdots - 2^3 - 2, -2^{2n-3} - \cdots - 2^3 - 2),$$

$$\big|\big|\,C(t) - C(t-2^{2n})\,\big|\big| < \varepsilon/2^{n+2}, \quad t \in (2 + 2^2 + \cdots + 2^{2n-2}, 2 + 2^2 + \cdots + 2^{2n}],$$

we see that

$$||\,C(t)\,|| < \sum_{j=1}^{\infty} \frac{2\varepsilon}{2^{j+1}} = \varepsilon, \quad t \in \mathbb{R}.$$

We denote

$$I_n := [2 + 2^2 + \cdots + 2^{2n-2}, 2 + 2^2 + \cdots + 2^{2n}].$$

We will prove that we can choose constant values of $C(t)$, $t \in I_n$ on subintervals with the total length denoted by $r_{2n+1}$ which is grater than $2^{2n-1}$ for all $n \in \mathbb{N}$. We can choose values of $C$ on

$$[4 - 2 + 1/16 + 8^{-1}/16, 4 - 2 + 1 - 1/16 - 8^{-1}/16] \subset [2,6],$$

$$[4 - 2 + 1 + 1/4 + 1/16 + 8^{-1}/16, 4 - 2 + 1 + 3/4 - 1/16 - 8^{-1}/16] \subset [2,6],$$

$$[4 + 8^{-1}/16, 4 + 1 - 8^{-1}/16],\ [4 + 1 + 1/4 + 8^{-1}/16, 4 + 1 + 3/4 - 8^{-1}/16] \subset [2,6].$$

Hence,

$$r_3 \geq 55/64 + 23/64 + 63/64 + 31/64 = 43/16, \tag{4.2}$$

i.e., the statement is valid for $n = 1$. We use the induction principle with respect to $n$. Assume that the statement is true for $1, 2, \ldots, n-1$ and prove it for $n$. Without loss of the generality (consider the below given process), we can also assume that the estimation $r_{2j} > 2^{2(j-1)}$ is valid for $j \in \{1, \ldots, n\}$ (note that $r_2 = 5/4 > 2^0$) if we use analogous notation.

In view of the construction, we see that we can choose $C$ on any interval

$$[s + 2^{2n} + 8^{1-2n}/16, t + 2^{2n} - 8^{1-2n}/16]$$

if we can choose $C$ on $[s,t]$, where $s = b_j^l < c_j^l = t$, $l < 2n+1$. Especially, we can choose function $C$ on

$$[2^{2n} + 8^{1-2n}/16, 1 + 2^{2n} - 8^{1-2n}/16],$$

$$[1 + 1/4 + 2^{2n} + 8^{1-2n}/16, 1 + 3/4 + 2^{2n} - 8^{1-2n}/16],$$

$$[-2 + 1/16 + 2^{2n} + 8^{1-2n}/16, -2 + 1 - 1/16 + 2^{2n} - 8^{1-2n}/16],$$

$$[-2 + 1 + 1/4 + 1/16 + 2^{2n} + 8^{1-2n}/16, -2 + 1 + 3/4 - 1/16 + 2^{2n} - 8^{1-2n}/16]$$

and on less than $7 \cdot 8^{2n-1} - 4$ subintervals of $I_n$. Expressing

$$\begin{aligned} I_n = & [0 + 2^{2n}, 1 + 2^{2n}] \cup [1 + 2^{2n}, 2 + 2^{2n}] \cup [-2 + 2^{2n}, 0 + 2^{2n}] \cup \cdots \\ & \cup [2 + 2^2 + \cdots + 2^{2n-4} + 2^{2n}, 2 + 2^2 + \cdots + 2^{2n-2} + 2^{2n}] \\ & \cup [-2^{2n-1} - \cdots - 2^3 - 2 + 2^{2n}, -2^{2n-3} - \cdots - 2^3 - 2 + 2^{2n}] \end{aligned}$$

and using the induction hypothesis, the construction, and (4.2), we obtain that we can choose $C$ on intervals of the lengths grater than or equal to

$$1 - 2 \cdot 8^{1-2n}/16, \quad 1/2 - 2 \cdot 8^{1-2n}/16,$$

$$1 - 1/8 - 2 \cdot 8^{1-2n}/16, \quad 1/2 - 1/8 - 2 \cdot 8^{1-2n}/16,$$

$$43/16 + 2^2 + 2^3 + \cdots + 2^{2n-3} + 2^{2n-2} - 2 \cdot 8^{1-2n}/16 \cdot \left(7 \cdot 8^{2n-1} - 4\right).$$

Summing, we get

$$r_{2n+1} \geq 1 + \frac{1}{2} + \frac{7}{8} + \frac{3}{8} + \frac{11}{16} + 2^{2n-1} - 2 - \frac{7}{8} > 2^{2n-1}, \tag{4.3}$$

which is the above statement. Analogously, we can prove

$$r_{2n} > 2^{2n-2}, \quad n \in \mathbb{N}. \tag{4.4}$$

Now we describe the principal fundamental matrix $X_C$ on $I_n$ for arbitrary $n \in \mathbb{N}$. Since $C$ is constant a has the form diag $[\mathrm{i}a, \mathrm{i}a, \ldots, \mathrm{i}a]$ for some $a \in \mathbb{R}$ on each interval $[b_j^{2n+1}, c_j^{2n+1}]$, $j \in \{1, \ldots, 6 \cdot 4^{2n-1}\}$, from

$$X_C(t_2) - X_C(t_1) = \int_{t_1}^{t_2} C(\tau) \cdot X_C(\tau)\, d\tau, \quad t_1, t_2 \in \mathbb{R},$$

we obtain

$$\begin{aligned} & \left|\left| X_C(t) - X_C^{2n+1}(t) \right|\right| \leq \\ & \sum_{j=1}^{k} \left( \int_{a_j^{2n+1}}^{b_j^{2n+1}} || C(\tau) \cdot X_C(\tau) ||\, d\tau + \int_{c_j^{2n+1}}^{a_{j+1}^{2n+1}} || C(\tau) \cdot X_C(\tau) ||\, d\tau \right) \end{aligned} \tag{4.5}$$

if $t \leq a_{k+1}^{2n+1}$, $t \in I_n$, where

$$X_C^{2n+1}(t) := X_C(2 + 2^2 + \cdots + 2^{2n-2}), \quad t \in [2 + 2^2 + \cdots + 2^{2n-2}, b_1^{2n+1}],$$

$$X_C^{2n+1}(t) := \exp\left(C(b_1^{2n+1})(t - b_1^{2n+1})\right) \cdot X_C^{2n+1}(b_1^{2n+1}),\ t \in (b_1^{2n+1}, c_1^{2n+1}],$$

$$X_C^{2n+1}(t) := {X_C}^{2n+1}(c_1^{2n+1}), \quad t \in (c_1^{2n+1}, b_2^{2n+1}],$$

$$\vdots$$

$$X_C^{2n+1}(t) := \exp\left(C(b_{7 \cdot 8^{2n-1}}^{2n+1})(t - b_{7 \cdot 8^{2n-1}}^{2n+1})\right) \cdot X_C^{2n+1}(b_{7 \cdot 8^{2n-1}}^{2n+1}),\ t \in (b_{7 \cdot 8^{2n-1}}^{2n+1}, c_{7 \cdot 8^{2n-1}}^{2n+1}],$$

$$X_C^{2n+1}(t) := X_C^{2n+1}(c_{7 \cdot 8^{2n-1}}^{2n+1}), \quad t \in (c_{7 \cdot 8^{2n-1}}^{2n+1}, 2 + 2^2 + \cdots + 2^{2n}].$$

It is seen that $X_C$ is bounded (see also the below given, where it is shown that $X_C(t) \in U(m)$ for all $t$) as almost periodic $C$. Any interval

$$[2 + \cdots + 2^{2n-2} + l - 1, 2 + \cdots + 2^{2n-2} + l], \quad l \in \{1, \dots, 2^{2n}\}, n \in \mathbb{N}$$

contains at most $4^{2n+1}$ subintervals where $C$ can be linear. Indeed, it suffices to consider the construction. We repeat that the length of each one of the considered subintervals is $8^{1-2n}/16$ which implies that the total length of them on

$$J_n^l := [2 + 2^2 + \cdots + 2^{2n-2}, 2 + 2^2 + \cdots + 2^{2n-2} + 2^{2n-l}], \quad l \in \{1, \dots, n\}$$

is less than $2^{1-l}$. Thus (consider also (4.5)), there exists $K \in \mathbb{R}$ such that

$$\left|\left| X_C(t) - X_C^{2n+1}(t) \right|\right| \leq \frac{K}{2^l}, \quad t \in J_n^l, \, l \in \{1, \dots, n\}, \, n \in \mathbb{N}. \tag{4.6}$$

From the form diag $[\mathrm{i}a(t), \dots, \mathrm{i}a(t)]$ of all matrices $C(t)$, we see that

$$||\, C(t)\, || = |\, a(t)\, |, \quad t \in \mathbb{R}.$$

For simplicity, let $a(t) \geq 0$, $t \in \mathbb{R}$. Let $a_j^n \in \mathbb{R}$, $j \in \{1, \dots, n\}$ be arbitrarily chosen. Considering the construction and combining (4.3) and (4.4), we get that we can choose constant values of

$$C(t), \quad t \in [2 + \cdots + 2^{2n-2} + (l-1)\, 2^n, 2 + \cdots + 2^{2n-2} + l\, 2^n]$$

on subintervals with the total length grater than $2^{n-2}$ for each $l \in \{1, \dots, 2^n\}$ and all sufficiently large $n \in \mathbb{N}$. Since we choose $C$ only so that

$$\left|\left| C(t) - C(t - 2^{2n}) \right|\right| < \frac{\varepsilon}{2^{n+2}}, \quad t \in I_n,$$

we see that we can obtain

$$X_C^{2n+1}(t_j^n) = \mathrm{diag}\, \left[\exp\left(\mathrm{i}a_j^n\right), \dots, \exp\left(\mathrm{i}a_j^n\right)\right]$$

for arbitrary $t_j^n$ such that

$$\begin{array}{l} t_1^n \geq 2 + 2^2 + 2^4 + \cdots + 2^{2n-2} + 3^n - 3^0, \quad t_2^n \geq t_1^n + 3^n - 3^1, \\ \cdots \quad t_n^n \geq t_{n-1}^n + 3^n - 3^{n-1}, \quad 2 + 2^2 + 2^4 + \cdots + 2^{2n} \geq t_n^n \end{array} \tag{4.7}$$

because we have

$$4^n > n(3^n - 3^0) > 3^n - 3^0 > \cdots > 3^n - 3^{n-1} > 2^{2n-k+1}$$

for sufficiently large $n \in \mathbb{N}$ and some $k = k(n) \in \{1, \dots, n\}$ satisfying

$$2^{2n-k-2} \cdot \varepsilon \cdot 2^{-n-2} > 2\pi.$$

We recall that we need to prove the existence of such $C$, given by the above construction, for which the vector valued function $X_A(t)\, X_C(t)\, u$, $t \in \mathbb{R}$ is not almost periodic for any $u \in \mathbb{C}^m$, $||\, u\, ||_1 = 1$. Since

$$(X_A(t) \cdot X_A^*(t))' = A(t) \cdot X_A(t) \cdot X_A^*(t) - X_A(t) \cdot X_A^*(t) \cdot A(t), \quad t \in \mathbb{R}$$

and since the constant function given by $I$ is a solution of $X' = A\,X - X\,A$, $X(0) = I$, we have $X_A(t) \in U(m)$ for all $t$. Thus, $X_C(t) \in U(m)$, $t \in \mathbb{R}$ as well. We add that $X_A(t)\, X_A^*(t) = I$, $t \in \mathbb{R}$ implies $A^*(t) + A(t) = O$, $t \in \mathbb{R}$.

Let $c \in \mathbb{C}$, $|\, c\,| = 1$, and $N \in U(m)$ be arbitrarily given. Obviously, for any $M \in U(m)$, we can choose a number $a(M, c) \in [0, 2\pi)$ in order that all eigenvalues of matrix

$$P := M \cdot \mathrm{diag}\,[\exp(\mathrm{i}a(M,c)), \ldots, \exp(\mathrm{i}a(M,c))]$$

are not in the neighbourhood of $c$ with a given radius which depends only on dimension $m$. Indeed, if $M$ has eigenvalues $\lambda_1, \ldots, \lambda_m$, then the eigenvalues of $P$ are $\lambda_1 \exp(\mathrm{i}a(M,c))$, $\ldots, \lambda_m \exp(\mathrm{i}a(M,c))$. Considering $P\,u - N\,u$ and expressing vectors $u \in \mathbb{C}^m$, $||\,u\,||_1 = 1$, as linear combinations of the eigenvectors of $P$, we see that $P\,u$ cannot be in a neighbourhood of $N\,u$ for some $c \in \mathbb{C}$, $|\,c\,| = 1$. Thus (the considered multiplication of matrices and vectors is uniformly continuous), there exist $\vartheta > 0$ and $\xi > 0$ such that, for any matrices $M, N \in U(m)$, one can find $a(M, N) \in (0, 2\pi)$ satisfying

$$\begin{aligned} &||\,M \cdot \mathrm{diag}\,[\exp(\mathrm{i}\tilde{a}), \ldots, \exp(\mathrm{i}\tilde{a})] \cdot u - N \cdot u\,||_1 > \vartheta, \\ &u \in \mathbb{C}^m,\ ||\,u\,||_1 = 1, \quad \tilde{a} \in (a(M,N) - \xi, a(M,N) + \xi)\,. \end{aligned} \tag{4.8}$$

We showed that we can construct $C$ so that we obtain

$$X_C^{2n+1}(t_j^n) = \mathrm{diag}\,\left[\exp\left(\mathrm{i}a_j^n\right), \ldots, \exp\left(\mathrm{i}a_j^n\right)\right]$$

for arbitrarily given $a_j^n \in [0, 2\pi)$ and any $t_j^n$ satisfying (4.7) if $n \in \mathbb{N}$ is sufficiently large and $j \in \{1, \ldots, n\}$. Especially, for sufficiently large $n \in \mathbb{N}$ and for

$$\begin{aligned} &t_1^n := 2 + 2^2 + 2^4 + \cdots + 2^{2n-2} + 3^n - 3^0, \\ &t_2^n := t_1^n + 3^n - 3^1, \quad \cdots \quad t_n^n := t_{n-1}^n + 3^n - 3^{n-1}, \end{aligned} \tag{4.9}$$

we can choose all $X_C^{2n+1}(t_j^n)$ in the form without any conditions. Hence, we obtain diagonal matrices $X_C^{2n+1}(t_j^n)$, $j \in \{1, \ldots, n\}$, determined by numbers

$$\exp\left(\mathrm{i}a\left(X_A(t_j^n), X_A(t_j^n - 3^n + 3^{j-1}) \cdot X_C(t_j^n - 3^n + 3^{j-1})\right)\right)$$

on their diagonals.

It is seen from (4.9) that each

$$t_j^n \in [2 + 2^2 + \cdots + 2^{2n-2}, 2 + 2^2 + \cdots + 2^{2n-2} + n3^n].$$

Thus (see (4.6)), for any $\eta > 0$, we have

$$\left|\left|\, X_C(t_j^n) - X_C^{2n+1}(t_j^n)\,\right|\right| < \eta \tag{4.10}$$

for sufficiently large $n = n(\eta) \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$. From (4.8) and (4.10) it follows that

$$\left|\left|\, X_A(t_j^n) \cdot X_C(t_j^n) \cdot u - X_A(t_j^n - 3^n + 3^{j-1}) \cdot X_C(t_j^n - 3^n + 3^{j-1}) \cdot u\,\right|\right|_1 > \vartheta \tag{4.11}$$

for any $u \in \mathbb{C}^m$, $||\,u\,||_1 = 1$, sufficiently large $n \in \mathbb{N}$, and $j \in \{1, \ldots, n\}$.

By contradiction, suppose that there exists $u \in \mathbb{C}^m$, $\|u\|_1 = 1$, with the property that $X_A(t)\,X_C(t)\,u$, $t \in \mathbb{R}$ is almost periodic. Applying Theorem 3.4 for

$$\psi(t) = X_A(t) \cdot X_C(t) \cdot u,\ t \in \mathbb{R}, \quad s_n = 3^n,\ n \in \mathbb{N}, \quad \varepsilon = \vartheta,$$

we get

$$\|\, X_A(t + 3^{n_1}) \cdot X_C(t + 3^{n_1}) \cdot u \, - X_A(t + 3^{n_2}) \cdot X_C(t + 3^{n_2}) \cdot u \,\|_1 < \vartheta, \quad t \in \mathbb{R} \tag{4.12}$$

for all $n_1, n_2$ from an infinite subset of $\mathbb{N}$. If we rewrite (4.12) as follows

$$\|\, X_A(t) \cdot X_C(t) \cdot u \, - X_A(t + 3^{n_2} - 3^{n_1}) \cdot X_C(t + 3^{n_2} - 3^{n_1}) \cdot u \,\|_1 < \vartheta, \quad t \in \mathbb{R},$$

then it is easy to see that (4.11) is not valid for infinitely many $n \in \mathbb{N}$. This contradiction proves the theorem. □

In Chapters 3 and 4, we studied the Bohr almost periodic functions, we modified the Bochner definition, we mentioned main properties of almost periodic functions and we considered almost periodic solutions of almost periodic skew-Hermitian differential equations. Analogously, it is possible to obtain (by simple modifications of our processes) the corresponding results for pseudo almost periodic functions (see [50], [53]). Note that a function, defined for $t \in \mathbb{R}$ with values in a Banach space, is *pseudo almost periodic* if and only if it can be represented as the sum of an almost periodic function and a continuous function $f$ satisfying

$$\lim_{T \to +\infty} \frac{1}{T} \int_{-T}^{T} \|\, f(t) \,\|\, dt = 0.$$

# Footnotes to Part II

1 There exist definitions of almost periodic functions defined on various sets. For almost periodic functions defined on the torus (on the annuloid), see [122], [135]; on a tube, see [66]; on a circle, see [27].

2 The bibliography of 137 items can be useful for many readers. We add that it includes 18 papers by A. Kovanko whose important results about (the Besicovitch, the Muckenhoupt, the Stepanov, and the Weyl) types of almost periodic functions are not mentioned in this work. See also bibliographical notes in [39] (704 items) and directly [7], [108]. Other interesting types of almost periodic functions (e.g., asymptotic, the Eberlein, pseudo, and weakly almost periodic functions) are investigated in [172]. Very special types can be found in [27], [146].

3 Theory of almost periodic functions has been one of the most interesting topics in analysis for its significance in the physical sciences (see, e.g., [147]). There exist many remarkable books (see [37], [69], [171] and the papers cited therein as well) concerning almost periodic solutions of ordinary (or functional) differential equations (in applications).

4 Since the Bochner definition is very important (in this work), we mention the proofs of Lemmas 3.2 and 3.3 in the full version:

*Proof of Lemma 3.2.* Let an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ be given and let $p = p\,(\varepsilon/3)$, where $\varepsilon > 0$ is arbitrary, be from Definition 3.1. Since $\psi$ is uniformly continuous on the interval $I := [-1, 1+p]$, there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$\varrho\,(\psi(t_1), \psi(t_2)) < \frac{\varepsilon}{3}, \quad t_1, t_2 \in I,\ |\, t_1 - t_2\,| < \delta.$$

Let $t_1, t_2 \in \mathbb{R}$ satisfying $|\, t_1 - t_2\,| < \delta$ be arbitrary and $s = s(t_1, \delta) \in [-t_1, -t_1 + p]$ be an $(\varepsilon/3)$-translation number of $\psi$. Evidently, $t_1 + s \in I$, $t_2 + s \in I$. Finally, we have

$$\begin{aligned}\varrho\,(\psi(t_1), \psi(t_2)) &\le \varrho\,(\psi(t_1), \psi(t_1+s)) + \varrho\,(\psi(t_1+s), \psi(t_2+s)) \\ &\quad + \varrho\,(\psi(t_2+s), \psi(t_2)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

which terminates the proof. □

*Proof of Lemma 3.3.* Let $p = p\,(\varepsilon/2)$ be from Definition 3.1 for arbitrarily given $\varepsilon > 0$. Obviously, the set of all values of $\psi$ on $[0,p]$ is a subset of a finite number of neighbourhoods of radius $\varepsilon/2$. Let us denote by $x_1, x_2, \dots, x_q$ the centres of these neighbourhoods that cover the set $\{\psi(t);\, t \in [0,p]\}$. For an arbitrary $t \in \mathbb{R}$, we take an $(\varepsilon/2)$-translation number $s = s(t) \in [-t, -t+p]$ of $\psi$. Thus, $t + s \in [0,p]$. Let $x(t) \in \{x_1, x_2, \dots, x_q\}$ be the centre of the neighbourhood of radius $\varepsilon/2$ which contains $\psi(t+s)$. We get

$$\varrho\,(x(t), \psi(t)) \leq \varrho\,(x(t), \psi(t+s)) + \varrho\,(\psi(t+s), \psi(t)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It shows that, for any $\varepsilon > 0$, the set of all values of $\psi$ is covered by a finite number of neighbourhoods of radius $\varepsilon$. □

5 Now we will prove that any Bochner almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ is Bohr almost periodic:

Assume, on the contrary, that $\psi$ is not Bohr almost periodic. Then, there exists a number $\varepsilon > 0$ such that, for any $p \in \mathbb{N}$, one can find an interval of length $p$ which does not contain any $\varepsilon$-translation number of $\psi$. Consider an arbitrary number $l_1 \in \mathbb{N}$ and an interval $(a_1, b_1) \subseteq \mathbb{R}$ of the length greater than $2(l_1 + 1)$ which contains no $\varepsilon$-translation number of $\psi$. We choose $l_2 \in \mathbb{Z}$ such that $l_2 - l_1 \in (a_1, b_1)$. Thus, $l_2 - l_1$ is not an $\varepsilon$-translation number of $\psi$. Next, there exists an interval $(a_2, b_2) \subseteq \mathbb{R}$ of the length greater than $2(l_1 + l_2 + 1)$ such that there exists no $\varepsilon$-translation number of $\psi$ in $(a_2, b_2)$. We can also find $l_3 \in \mathbb{Z}$ for which $l_3 - l_1, l_3 - l_2 \in (a_2, b_2)$, and hence $l_3 - l_1, l_3 - l_2$ cannot be $\varepsilon$-translation numbers of $\psi$.

Proceeding in a similar way, we get a sequence $\{l_n\}_{n\in\mathbb{N}}$ satisfying that none of the numbers $l_{n_1} - l_{n_2}$, where $n_1 \neq n_2$ $(n_1, n_2 \in \mathbb{N})$, is an $\varepsilon$-translation number of $\psi$. Therefore, we obtain

$$\varrho\,(\psi(t + l_{n_1} - l_{n_2}), \psi(t)) \geq \varepsilon$$

for all $n_1 \neq n_2$ $(n_1, n_2 \in \mathbb{N})$ and at least one $t \in \mathbb{R}$. This contradiction proves that $\psi$ is Bohr almost periodic.

6 For the first time, Corollary 3.9 was proved for almost periodic functions with values in an arbitrary metric space in [134].

7 Such theorems (which show how almost periodic functions can be characterized by almost periodic sequences) are also used to study almost periodic solutions of differential equations. Especially, general examples of differential equations, for which a solution $x(t)$ defined for $t \in \mathbb{R}$ is almost periodic if and only if $\{x(k)\}_{k\in\mathbb{Z}}$ is an almost periodic sequence, are mentioned in [3], [119], [126].

8 An important class of almost periodic functions is the class of *limit-periodic* functions. To this class belong the uniform limits of sequences of periodic continuous functions (in general, having different periods). Note that these functions can be defined by their Fourier exponents (see [25]). It is seen that, using the constructions mentioned in Section 3.3, we obtain limit-periodic functions.

9 We remark that the first interesting generalization of the approximation theorem for a complete metric space is due to H. Tornehave and it can be found in [159]. It is required there that, for every compact subset $S$, there exists a positive number $d$ such that any points $x, y \in S$ with distance less than $d$ can be connected by a continuous curve which depends continuously on $x$ and $y$ and which reduces to $x$ for $x = y$. This requirement motivates the main condition of Theorem 3.19.

10 The first important result about almost periodic solutions of linear equations, in the general case when each coefficient can be almost periodic, was proved by R. Cameron (in [30]). But R. Cameron considered only the scalar case. His result was extended to linear (and quasi-linear) systems by J. Massera (in [116]).

11 The importance of skew-Hermitian systems may be illustrated by the well-known Cameron-Johnson theorem states that any almost periodic homogeneous linear differential system can be reduced by a Lyapunov transformation to a skew-symmetric system if all solutions of the given system and all of its limit equations are bounded in $\mathbb{R}$ (for this result and its generalizations, see [34]).

12 In [104] and [105], the result is proved for systems which have a frequency basis of dimension two or three (see introduction of Section 4.2). We remark that an almost periodic function with a finite frequency basis is called *quasiperiodic* (see also [49], [154]) and that the quasiperiodic functions as a special class of functions were studied by P. Bohl and E. Esclangon (see [22], [58], [60], [62]). They also showed the applications of these functions to the theory of differential equations before H. Bohr introduced the classical almost periodic functions (see [23], [59]).

13 In the vector space $\mathbb{C}^m$, the following three norms are in common use: *the Euclidean norm*

$$||\, x\,||_2 := \sqrt{\sum_{j=1}^{m} |\, x^j\,|^2};$$

*the absolute norm*

$$||\, x\,||_1 := \sum_{j=1}^{m} |\, x^j\,|;$$

and *the maximum norm*

$$||\, x\,||_\infty := \max_{1\le j\le m} |\, x^j\,|,$$

where $x = (x^1, \ldots, x^m)^T \in \mathbb{C}^m$. The space $\mathcal{M}at(\mathbb{C}, m)$ can be considered as equivalents to $\mathbb{C}^{m^2}$. Thus, a matrix norm $||\cdot||$ should satisfy the usual three conditions and, in addition, we require

$$||\, A \cdot B\,|| \le ||\, A\,||\,||\, B\,||, \quad A, B \in \mathcal{M}at(\mathbb{C}, m)$$

and the compatibility with $||\cdot||_p$, $p \in \{1, 2, \infty\}$, i.e.,

$$||\, A \cdot x\,||_p \le ||\, A\,||\;||\, x\,||_p$$

for all $x \in \mathbb{C}^m$ and any $A \in \mathcal{M}at(\mathbb{C}, m)$. Once, in $\mathbb{C}^m$, a norm $||\cdot||_p$ ($p \in \{1, 2, \infty\}$) is defined, then the corresponding matrix norm $||\cdot||$ is given by

$$|| A || := \sup_{x \neq 0} \frac{|| A \cdot x ||_p}{|| x ||_p}, \quad A \in \mathcal{M}at(\mathbb{C}, m).$$

14 The definition of the frequency module (and, consequently, its rational hull) for almost periodic sequences and functions in Banach spaces is introduced in [169].

15 Reducible systems and irreducible systems (in this context) are introduced and investigated in [56], [102], [129].

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