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On constructions of left determined model structures

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Abstrakt

Prezentujeme dvě metody pro rozšíření kofibrantně generovaného slabého faktorizačního systému $(\mathcal{L}, \mathcal{R})$ v lokálně presentovatelné kategorii na modelovou strukturu (s $\mathcal{L}$ jako kofibracemi) tak, že třída slabých ekvivalencí je co nejmenší. Tato modelová struktura je zleva určená ve smyslu Rosický a Tholen [*Left-determined model categories and universal homotopy theories* Trans. Amer. Math. Soc. 355 (2003), no. 9].

Je-li každý objekt kofibrantní, každý kartézský cylindr (vhodná funktoriální faktorizace $X + X \xrightarrow{\gamma_X} \mathrm{C}X \xrightarrow{\sigma_X} X$ kodiagonály taková, že $\gamma_X \in \mathcal{L}$) a každá podmnožina $S \subseteq \mathcal{L}$ dávají modelovou strukturu, jejíž třída slabé ekvivalence je nejmenší lokalizátor. Naopak, nejmenší lokalizátor (vůči $\mathcal{L}$) který obsahuje množinu morfismů lze získat tímto způsobem právě tehdy, pokud obsahuje všechny σ_X některých kartézských cylindrů. Toto rozšíří odpovídající výsledky Cisinského [*Théories homotopiques dans les topos* J. Pure Appl. Algebra 174 (2002) no. 1] pro (Grothendieckovy) toposy a monomorfismy jako kofibrace.

Předpokládejme, že je množina $I = \{s^n : S^{n-1} \to B^n \mid n \in \mathbb{N}\}$ generujících kofibrací, získaných z morfismů $b_n^0, b_n^1 : B_n \to B_{n+1}$ $(n \in \mathbb{N})$ které splňují $b_n^i b_{n+1}^j = b_n^i b_{n+1}^k$ prostřednictvím postupných fibrovaných sum od $S^{-1} = 0 \to B^0$. Je-li relativní homotopie mezi paralelními buňkami (morfismy $x, y : B^n \to X$ s $s_n x = s_n y$) tranzitivní, a pokud existuje fibrantní kocylindr (morfismy $\pi_X^0, \pi_X^1 : \Gamma X \to X$ ve $\mathcal{R}$ se společnou sekcí), který má "homotopickou výměnnou vlastnost", pak existuje modelová struktura, kde slabé ekvivalence jsou morfismy, které mají relativní homotopickou liftovací vlastnost vzhledem ke všem morfismům v I. Toto zobecňuje výsledky Lafonta, Métayera a Worytkiewicze [*A folk model structure on omega-cat*] pro $\omega\mathbf{Cat}$, t.j., kategorii (striktní) ω-kategorií.

Obě konstrukce mohou být indukovány na vhodných reflektivních podkategoriích. Jsou uvedeny některé příklady.

Abstract

We present two methods for extending a cofibrantly generated weak factorization system $(\mathcal{L}, \mathcal{R})$ in a locally presentable category to a model structure (with $\mathcal{L}$ as cofibrations) where the class of weak equivalences is as small as possible. The model structure is then left determined in the sense of Rosický and Tholen [*Left-determined model categories and universal homotopy theories* Trans. Amer. Math. Soc. 355 (2003), no. 9].

If every object is cofibrant, then any cartesian cylinder (a suitable functorial factorization $X + X \xrightarrow{\gamma_X} \mathrm{C}X \xrightarrow{\sigma_X} X$ of codiagonals with $\gamma_X \in \mathcal{L}$) and any subset $S \subseteq \mathcal{L}$ give a set S' with $S \subseteq S' \subseteq \mathcal{L}$ and a model structure whose class of weak equivalences is the smallest localizer containing S'. Conversely, the smallest localizer (with respect to $\mathcal{L}$) containing a set of maps can be obtained in this way iff it contains all σ_X of some cartesian cylinder. This extends corresponding results of Cisinski [*Théories homotopiques dans les topos* J. Pure Appl. Algebra 174 (2002) no. 1] for Grothendieck toposes and monomorphisms as cofibrations.

Suppose there is a set $I = \{s^n : S^{n-1} \to B^n \mid n \in \mathbb{N}\}$ of generating cofibrations, obtained from maps $b_n^0, b_n^1 : B_n \to B_{n+1}$ $(n \in \mathbb{N})$ satisfying $b_n^i b_{n+1}^j = b_n^i b_{n+1}^k$ via successive pushouts starting from $S^{-1} = 0 \to B^0$. If relative homotopy between parallel cells (maps $x, y : B^n \to X$ with $s_n x = s_n y$) is transitive and if there is a fibrant cocylinder (natural maps $\pi_X^0, \pi_X^1 : \Gamma X \to X$ in $\mathcal{R}$ with a common section) that has the "homotopy exchange property", then there is a model structure where the weak equivalences are the maps that have the relative homotopy lifting property w.r.t. all maps in I. This generalizes results of Lafont, Métayer and Worytkiewicz [*A folk model structure on omega-cat*] for $\omega\mathbf{Cat}$, the category of (strict) ω-categories.

Both constructions can be induced on suitable reflective subcategories. Several examples are provided.

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Preface

The subject of this thesis is the construction of (closed) Quillen model structures from a class of maps that can serve as cofibrations, such that the resulting class of weak equivalences is the smallest possible choice. Model structures where the cofibrations determine the weak equivalences in this way have been considered by J. Rosický and W. Tholen [26] under the name "left determined".

For Grothendieck toposes such a construction has been given by D.C. Cisinski [4] with all monomorphisms as cofibrations in spe. His construction also allows to specify a set of monomorphisms that are meant to become trivial cofibrations. He showed that the resulting class of weak equivalences is the smallest class of maps that satisfies some closure conditions and contains a certain set of trivial cofibrations produced by this construction. Moreover, every such class of maps (a "smallest localizer") arises in this way.

The current work extends Cisinski's construction and results to locally presentable categories. It also allows for more general classes of prospective cofibrations than just monomorphisms.

Recently, Y. Lafont, F. Métayer and K. Worytkiewicz [14] have constructed a left determined model structure on the category of (strict) ω-categories by a different method. This construction can also be extended to locally presentable categories.

After a short introduction, providing some context and overview, the necessary background material about model categories is assembled in the first chapter.

The main result on Cisinski's construction is given in Chapter 2. We describe the construction and show that, under certain assumptions, it produces a cofibrantly generated model structure. In the original case of monomorphisms in a Grothendieck topos the assumptions we develop are equivalent to the ones Cisinski uses. We then compare the weak equivalences of the resulting model structures with smallest localizers and identify conditions under which a smallest localizer can be obtained through the construction.

Chapter 3 contains an abstract version of the construction given by Lafont, Métayer and Worytkiewicz, which is suitable for locally presentable categories. We also consider the situation, where both constructions can be applied and give conditions for the resulting model structures to coincide.

Several examples are given in Chapter 4, where the emphasis is on locally presentable categories that are not toposes. In order to construct these, we describe methods for inducing both constructions on certain reflective subcategories. The limitations of both methods are also discussed. The central ingredient of Cisinski's construction is the notion of a cartesian cylinder for the class of cofibrations in spe, and we characterize these completely for the case of monomorphisms in module categories.

The main results in Chapters 2 and 4 will appear in *Applied Categorical Structures*. For this thesis, I included the new material of Chapter 3 and the results relating the two constructions. Accordingly, I also rearranged and expanded the presentation in Chapters 1 and 4, corrected some misprints of the article version and probably introduced new ones.

The notation is mostly standard. But we write composition in reading order (i.e. from left to right) and denote identity morphisms by the names of their objects.

I want to thank Professor Jiří Rosický, who introduced me to this interesting area of research. His mathematical insights, patience and optimism have been an important help. I am also grateful for the welcoming and friendly atmosphere created by the department's faculty and staff. Last not least the support under GACR grant #201/05/H005 has been vital.

Contents

Introduction

Model categories were introduced by Quillen [20] as an abstract framework for homotopy theory. A model category is a complete and cocomplete category equipped with a model structure, i.e. three distinguished classes of maps, called cofibrations, fibrations and weak equivalences, satisfying certain axioms. The axioms are in fact those of a closed model category, but the adjective "closed" is usually dropped nowadays.

Given such a model structure on a category, any two of the three classes of maps involved determine the remaining one and hence the whole model structure. Going one step further, one can ask for model structures where already one of the classes determines the other two.

Rosický and Tholen [26] introduced the notion of a left determined model category, where the class $\mathcal{W}$ of weak equivalences is determined by the class $\mathcal{C}$ of cofibrations as the smallest class of maps satisfying some closure conditions. For such a model category, $\mathcal{W}$ is then the smallest possible class of weak equivalences such that $\mathcal{C}$ and $\mathcal{W}$ yield a model structure.

Independently, Cisinski [4] considered classes of maps (under the name localizer) that satisfy (almost) the same closure conditions for the case where the underlying category is a (Grothendieck) topos and $\mathcal{C}$ is the class of monomorphisms.

For this case he gave an explicit construction of model structures from a given set S of monomorphisms and showed that the resulting class of weak equivalences is the smallest localizer (with respect to monomorphisms) containing some $S' \supseteq S$. Conversely, given any set of maps (not necessarily monomorphisms) the smallest localizer containing these can be realized in this way. In particular, for the empty set, the smallest localizer is part of a model structure, which is then left determined.

For the construction one starts with a suitable natural cylinder (a functorial factorization of codiagonals) and the associated homotopy relation, specifies a set S of monomorphisms that are meant to become weak equivalences, identifies the fibrant objects of the model structure in spe, and with these finally defines the weak equivalences. The set $S' \supseteq S$ is produced during the construction.

We want to extend this construction and the corresponding results to a more general context, where the class of cofibrations may not be the monomorphisms and where the underlying category is not necessarily a topos.

To see why such a construction is desirable in general, consider the homotopy category $\mathrm{Ho}(\mathcal{K})$ of a model category $\mathcal{K}$. It can be defined as $\mathcal{K}[\mathcal{W}^{-1}]$ (i.e. by formally inverting weak equivalences). But it is also equivalent to the quotient category $\mathcal{K}_{cf}/\!\sim$ with respect to some homotopy relation, where $\mathcal{K}_{cf}$ is the full subcategory of cofibrant and fibrant objects. It turns out that there are several different ways of defining a homotopy relation in $\mathcal{K}$ which all agree on $\mathcal{K}_{cf}$. Among them is one which uses only knowledge of the cofibrations. Consequently, the weak equivalences are only needed to identify the fibrant objects — and whenever Cisinski's construction is available, this can be done without having to know all weak equivalences in advance.

For such a generalization to work, one first needs to inspect the conditions of the original situation and extract those parts that can serve as an axiomatic starting point in the general case. These fall into three sorts which we now describe:

First there are conditions on the class of cofibrations in spe. We assume that these are already part of cofibrantly generated weak factorization system and that every object is cofibrant. Here a weak factorization system $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ consists of two classes of maps such that $\mathcal{K} = \mathcal{L}\mathcal{R}$ holds and $\mathcal{L}$ and $\mathcal{R}$ mutually determine each other via a certain lifting condition in the following sense: write $f \mathrel{\Box} g$ if every solid square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd} \tag{$*$}$$

has a (dotted) diagonal. It is then required that $\mathcal{L} = {}^{\Box}\mathcal{R} := \{f \in \mathcal{K} \mid \forall r \in \mathcal{R}\colon f \mathrel{\Box} r\}$ and $\mathcal{R} = \mathcal{L}^{\Box} := \{g \in \mathcal{K} \mid \forall \ell \in \mathcal{L}\colon \ell \mathrel{\Box} g\}$. To be cofibrantly generated means that there is a subset $I \subset \mathcal{L}$ which already determines $\mathcal{R}$ as $I^{\Box}$ above. Then $\mathcal{L} = {}^{\Box}(I^{\Box})$ and I is called a set of generating cofibrations.

Weak factorization systems are the building blocks of model structures because the axioms for a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ require in particular that both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems. Therefore any class $\mathcal{L}$ of possible cofibrations must already be part of a weak factorization system.

Moreover, in the original case of monomorphisms in a Grothendieck topos, the corresponding weak factorization system is cofibrantly generated. Also every map $0 \to X$ from an initial object is a monomorphism. Therefore we include these properties in our assumptions for the general case and call such weak factorization system cofibrant for short.

Then there are general conditions on the underlying category when the restriction to Grothendieck toposes is to be lifted. We assume, that the underlying category is locally presentable. Locally presentable categories, first defined by Gabriel and Ulmer [7], include many categories from everyday mathematics like categories of universal algebra, Banach spaces with linear contractions, small categories and functors, Grothendieck abelian categories and Grothendieck toposes. At the same time they still have nice properties: they are complete and cocomplete (hence suitable as model categories), are closed under several categorical constructions and allow application of (the dual form of) Freyd's Special Adjoint Functor Theorem. Moreover, any set of maps in a locally presentable category gives rise to a weak factorization system which makes it easier to assert the existence of certain factorizations of maps.

Finally there are conditions on the cylinder used for the construction. As in the original case, a cylinder is a functorial factorization of codiagonals $X + X \to X$ into $\gamma_X : X + X \to \mathrm{C}X$ and $\sigma_X : \mathrm{C}X \to X$, and we require that the γ_X are cofibrations. The additional conditions will be discussed later. Among other things, we explicitly require that the cylinder functor C is a left adjoint, whereas in [4] it is only required that it preserves colimits. These two conditions are equivalent for locally presentable categories, but explicit use of the adjointness seems to clarify some arguments. In particular it explains why the construction of the model structure does not depend on the choice of the generating set of cofibrations. We call such cylinders cartesian.

Once these conditions are taken care of, almost all steps in Cisinski's original proof can be adapted. Given a cofibrant weak factorization system $(\mathcal{L}, \mathcal{R})$ and a cartesian cylinder $(\mathrm{C}, \gamma, \sigma)$ for $(\mathcal{L}, \mathcal{R})$, the construction produces from any set $S \subseteq \mathcal{L}$ a set $S' = \Lambda(\mathrm{C}, S)$ and a cofibrantly generated model structure such that the class of weak equivalences is the smallest localizer (with respect to $\mathcal{L}$) containing S'. Moreover, all the maps $\sigma_X : \mathrm{C}X \to X$ are weak equivalences.

Conversely, given a set of maps, the smallest localizer containing these can be realized by this construction if and only if it already contains all the σ_X for some cartesian cylinder.

In the case of Grothendieck toposes with monomorphisms as cofibrations, such cylinders always exist. In fact there are cartesian cylinders such that the maps σ_X lie in every localizer. In more general cases this might not hold.

For the category $\omega\mathbf{Cat}$ of (strict) ω-categories, Lafont, Métayer and Worytkiewicz [14] have recently constructed a left determined model structure. We briefly sketch the relevant notions. Let $\mathcal{B}$ be the category with objects $n \in \mathbb{N}$ and morphisms $\mathcal{B}(n, n) = \{id_n\}$, $\mathcal{B}(n, n+1) = \{d^0_n, d^1_n\}$ and $\mathcal{B}(m, n) = \emptyset$ for $m > n$

$$0 \overset{d^1_0}{\underset{d^0_0}{\rightrightarrows}} 1 \overset{d^1_1}{\underset{d^0_1}{\rightrightarrows}} 2 \overset{d^1_2}{\underset{d^0_2}{\rightrightarrows}} 3 \overset{d^1_3}{\underset{d^0_3}{\rightrightarrows}} \cdots$$

such that $d^i_n d^j_{n+1} = d^i_n d^k_{n+1}$ always holds, i.e. the composition of nontrivial maps only depends on the first map. It then follows that $|\mathcal{B}(n, m)| = 2$ for $n < m$. An ω-graph, also known as globular set, is an object of the presheaf category $\mathbf{Set}^{\mathcal{B}^{op}}$. Given such an ω-graph X the maps $X(d^0_n)$ and $X(d^1_n)$, together with their various compositions, give maps $\mathrm{dom}_{n,k}, \mathrm{cod}_{n,k} : X_n \to X_k$ for $n > k$. The elements of X_n are called n-cells. Two n-cells $x, y \in X_n$ are parallel if $n = 0$ or if $\mathrm{dom}_{n,n-1}(x) = \mathrm{dom}_{n,n-1}(y)$ and $\mathrm{cod}_{n,n-1}(x) = \mathrm{cod}_{n,n-1}(y)$. They are k-composable (for $n > k$) if $\mathrm{cod}_{n,k}(x) = \mathrm{dom}_{n,k}(y)$. An ω-category is an ω-graph X equipped with identity cells and a multiplication of composable cells such that (together with the above domain and codomain maps) each tripel X_n, X_k, X_l with $n > k > l$ is a 2-category.

To construct the model structure in [14], the authors start with the free ω-categories generated by the representable ω-graphs and build the set of generating cofibrations in $\omega\mathbf{Cat}$ from $\emptyset \to B^0$ via successive pushouts. They then introduce an equivalence relation $\sim$ between parallel cells and finally define the weak equivalences via a lifting property with respect to the generating cofibrations, which is almost the same as the relation $\square$ from above, except that the right lower triangle in $(*)$ is only required to commute up to $\sim$.

As before, we want to extend this construction from $\omega\mathbf{Cat}$ to arbitrary locally presentable categories. However, extracting suitable ingredients for an abstract version of this construction is less straightforward than for Cisinski's construction. The main obstacle is, that ω-categories have internal structure which is used throughout the constructions and proofs in [14], and that this structure is not available for objects of a category in general. In particular, one needs to describe the construction without mentioning the composition.

We now describe the ingredients of the abstract setup for the construction of [14] in an arbitrary locally presentable category $\mathcal{K}$:

First we need to carry out the construction of the set of generating cofibrations. For this it is enough to assume the existence of a functor $B\colon \mathcal{B} \to \mathcal{K}$, which we call a "system of balls". This assumption also provides an abstract notion of cell: an n-cell of an object K is just a map from $B(n)$ to K.

The relation of ω-equivalence between parallel cells in [14] is defined via composition and it is immediately proved that this gives an equivalence relation. It is later proved that ω-equivalence can equivalently be expressed in terms of a relative homotopy relation via suitable cylinder objects. Such cylinder objects always exist, so we simply take these as starting point. However, we need to assume that the resulting homotopy relation between parallel cells is transitive.

We assume the existence of a cocylinder (a functorial factorization of diagonals) with certain properties. In [14], such a cocylinder is explicitely constructed, which we cannot imitate because of lack of composition.

Under these assumption we can then prove that the construction from [14] produces a left determined model structure on $\mathcal{K}$ in which every object is fibrant.

Having two different constructions available, we can then consider situations where a weak factorization system is obtained from a system of balls and has a cartesian cylinder. Because the right adjoint of the cylinder can be given a cocylinder structure, the necessary conditions for the construction of Lafont, Métayer and Worytkiewicz can be expressed in terms of the cylinder. If Cisinski's construction can also be applied, we obtain another condition for both constructions to give the same result.

1 Model categories

1.1 Weak factorization systems and model structures

We follow Adámek, Herrlich, Rosický, Tholen [1] in introducing model structures via the notion of a weak factorization system. Beside the original introduction by Quillen [20], other sources include the article of Beke [3] and the books of Hirschhorn [8] and Hovey [9]. Most definitions do not need the underlying category to be complete and cocomplete as is usually assumed when working with model structures. For now we tacitly assume that the relevant limits and colimits exist for the various statements to make sense.

1.1.1 Definition. We say that two maps f and g are in **diagonal relation** and write $f \mathbin{\square} g$ if for every solid square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

the (dotted) diagonal exists. For a class $\mathcal{H}$ of maps we set:

$$\mathcal{H}^{\square} = \{g \in \mathcal{K} \mid \forall h \in \mathcal{H} \colon h \mathbin{\square} g\} \quad \text{and} \quad {}^{\square}\mathcal{H} = \{f \in \mathcal{K} \mid \forall h \in \mathcal{H} \colon f \mathbin{\square} h\}$$

A **weak factorization system** in a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps such that the following two conditions are satisfied:

(1) $\mathcal{L} = {}^{\square}\mathcal{R}$ and $\mathcal{L}^{\square} = \mathcal{R}$.

(2) Every map f has a factorization as $f = \ell r$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$.

The weak factorization system $(\mathcal{L}, \mathcal{R})$ is **cofibrantly generated** if $\mathcal{L} = {}^{\square}(I^{\square})$ for some subset $I \subseteq \mathcal{L}$. It is **functorial** if there is a functor $F \colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ together with natural maps $\lambda \colon \mathrm{dom} \to F$ and $\rho \colon F \to \mathrm{cod}$ such that $\lambda_f \in \mathcal{L}$, $\rho_f \in \mathcal{R}$ and $f = \lambda_f \rho_f$ for all $f \in \mathcal{K}^{\mathbf{2}}$.

1.1.2 Remark. We first list some useful properties of the diagonal relation:

(a) Any class of the form ${}^{\square}\mathcal{H}$ is stable under pushouts, retracts in $\mathcal{K}^{\mathbf{2}}$ and transfinite compositions of smooth chains, where a smooth chain is a colimit preserving functor $D \colon \alpha \to \mathcal{K}$ from some ordinal and its transfinite composition is the induced map from D_0 to $\mathrm{colim}_{\beta<\alpha} D_\beta$. The dual results hold for classes of the form $\mathcal{H}^{\square}$.

We write $\mathrm{cell}(\mathcal{H})$ for the class of those maps that are transfinite compositions of pushouts of maps from $\mathcal{H}$. Hence the above observation in particular gives $\mathrm{cell}(\mathcal{H}) \subseteq {}^{\square}(\mathcal{H}^{\square})$.

(b) (The "retract argument") Suppose $f = xy$. If $f \,\square\, y$, then by redrawing

$$\begin{tikzcd} \bullet \arrow[r, "x"] \arrow[d, "f"'] & \bullet \arrow[d, "y"] \\ \bullet \arrow[r, equal] \arrow[ur, "d"] & \bullet \end{tikzcd} \quad \text{as} \quad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "f"'] & \bullet \arrow[r, equal] \arrow[d, "x"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d"'] & \bullet \arrow[r, "y"'] & \bullet \end{tikzcd}$$

one obtains f as a retract of x. Dually, if $x \,\square\, f$ then f is a retract of y.

(c) If a pair $(\mathcal{L}, \mathcal{R})$ satisfies condition (2) of Definition 1.1.1, then condition (1) can be replaced by the two conditions

(i) $\forall \ell \in \mathcal{L}$, $r \in \mathcal{R}$: $\ell \,\square\, r$.

(ii) Both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts in $\mathcal{K}^{\mathbf{2}}$.

The inclusions $\mathcal{L} \subseteq {}^{\square}\mathcal{R}$ and $\mathcal{L}^{\square} \supseteq \mathcal{R}$ follow from (i) and the inclusions $\mathcal{L} \supseteq {}^{\square}\mathcal{R}$ and $\mathcal{L}^{\square} \subseteq \mathcal{R}$ are obtained by application of the retract argument.

(d) If $F : \mathcal{X} \rightleftarrows \mathcal{A} : G$ are functors with F left adjoint to G then for any $f \in \mathcal{X}$ and $g \in \mathcal{A}$ we have $F(f) \,\square\, g \iff f \,\square\, G(g)$.

Ringel [22] has investigated pairs $(\mathcal{L}, \mathcal{R})$ that satisfy only condition (1) of Definition 1.1.1. The diagonal relation $\square$ gives a Galois-connection on classes of maps, i.e. we always have $\mathcal{L} \subseteq {}^{\square}\mathcal{R} \iff \mathcal{L}^{\square} \supseteq \mathcal{R}$. Therefore any pair of the form $({}^{\square}(\mathcal{H}^{\square}), \mathcal{H}^{\square})$ satisfies that condition. The relevance of the special case where $\mathcal{H}$ is a set comes from the result that under certain smallness assumptions condition (2) is automatically satisfied.

1.1.3 Definition. Given a class $\mathcal{H}$ of maps in a cocomplete category $\mathcal{K}$, we call a smooth chain $D\colon \alpha \to \mathcal{K}$ an **α-chain in $\mathcal{H}$** if for all $\beta < \beta + 1 \leq \alpha$ the map $D_\beta \to D_{\beta+1}$ is in $\mathcal{H}$

An object X is **λ-small with respect to $\mathcal{H}$** if the functor $\mathcal{K}(X, -)\colon \mathcal{K} \to \mathbf{Set}$ preserves colimits of all λ'-chains in $\mathcal{H}$ for every regular cardinal $\lambda' \geq \lambda$. It is **small with respect to $\mathcal{H}$** if it is λ-small for some λ. A set I of maps is **small with respect to $\mathcal{H}$** if the domain of each map is.

1.1.4 Lemma (the "small object argument")**.** *Let I be a set of maps in a cocomplete category $\mathcal{K}$ and suppose that I is small with respect to* $\mathrm{cell}(I)$*. Then every map f can be factored as $f = xy$ with $x \in \mathrm{cell}(I)$ and $y \in I^{\square}$ and this factorization can be made functorial. In particular $({}^{\square}(I^{\square}), I^{\square})$ is a functorial factorization system. Moreover, every map in ${}^{\square}(I^{\square})$ is a retract of a map in* $\mathrm{cell}(I)$*.*

Proof. See e.g. [9, Theorem 2.1.14], [8, Proposition 10.5.16] or [3, Proposition 1.3]. $\square$

1.1.5 Remark. Remark 1.1.2 and the Lemma imply that for a such a set I the maps in ${}^{\square}(I^{\square})$ are exactly the retracts of maps in $\mathrm{cell}(I)$.

1.1.6 Definition. A **model structure** $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{K}$ consists of three classes of maps $\mathcal{C}$ (cofibrations), $\mathcal{F}$ (fibrations) and $\mathcal{W}$ (weak equivalences) such that the following conditions are satisfied:

(1) $\mathcal{W}$ is closed under retracts in $\mathcal{K}^{\mathbf{2}}$ and has the **2-3 property**: if in $f = gh$ two of the maps lie in $\mathcal{W}$ then so does the third.

(2) Both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Observe that (as for weak factorization systems) if $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure on $\mathcal{K}$, then $(\mathcal{F}, \mathcal{W}, \mathcal{C})$ is a model structure on $\mathcal{K}^{op}$. The classes $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{W} \cap \mathcal{F}$ are called **trivial cofibrations** and **trivial fibrations** respectively. The model structure is **cofibrantly generated** or **functorial** if the two weak factorization systems in (2) are. An object X is called **cofibrant** if the map $(0 \to X)$ from the initial object is a cofibration and **fibrant** if the map $(X \to 1)$ to the terminal object is a fibration. For a functorial model structure, one obtains the **cofibrant replacement functor** and the **fibrant replacement functor** by restricting the two functorial factorizations to $(0{\downarrow}\mathcal{K})$ and $(\mathcal{K}{\downarrow}1)$ respectively.

1.1.7 Remark. Any weak factorization system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ gives a model structure with $\mathcal{C} = \mathcal{L}$, $\mathcal{F} = \mathcal{R}$ and $\mathcal{W} = \mathcal{K}$ for which Definition 1.1.1 and Definition 1.1.6 produce the same notions of "cofibrantly generated" and "functorial". Any notion about model structures in general (like e.g. "(co)fibrant objects" or "(co)fibrant replacement functor" from above) can be applied to weak factorization systems by considering this special model structure.

1.2 Cylinders and homotopy

1.2.1 Definition. Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a model structure on a category $\mathcal{K}$.

(a) A **(compatible) cylinder object** for X is an object $\mathrm{C}X$ together with a factorization of the codiagonal $(X|X)\colon X + X \to X$ as $X + X \xrightarrow{\gamma_X} \mathrm{C}X \xrightarrow{\sigma_X} X$ with $\sigma_X \in \mathcal{W}$. Together with the coproduct inclusions one then obtains maps as in the diagram below:

$$\begin{tikzcd}
& X + X \arrow[d, "\gamma_X"] & \\
X \arrow[ur, bend left, "\iota^0_X"] \arrow[r, "\gamma^0_X"] \arrow[dr, equal, bend right] & \mathrm{C}X \arrow[d, "\sigma_X"] & X \arrow[ul, bend right, "\iota^1_X"'] \arrow[l, "\gamma^1_X"'] \arrow[dl, equal, bend left] \\
& X &
\end{tikzcd} \tag{1.2.1}$$

The cylinder object is **cofibrant** if $\gamma_X^0, \gamma_X^1 \in \mathcal{C}$. It is called **good** if $\gamma_X \in \mathcal{C}$. A good cylinder object is called **final** if also $\sigma_X \in \mathcal{C}^{\square}$. Given two cylinder objects $\mathrm{C}X$ and $\mathrm{C}'X$ for X, we call $\mathrm{C}'X$ **finer than** $\mathrm{C}X$ if there is an $f\colon \mathrm{C}X \to \mathrm{C}'X$ making the following diagram commutative:

$$\begin{tikzcd}
 & X+X \arrow[dl, bend right, "\gamma_X"'] \arrow[dr, bend left, "\gamma'_X"] & \\
\mathrm{C}X \arrow[rr, "f"] \arrow[dr, bend right, "\sigma_X"'] & & \mathrm{C}'X \arrow[dl, bend left, "\sigma'_X"] \\
 & X &
\end{tikzcd}$$

(b) A **(functorial) cylinder** $(\mathrm{C}, \gamma, \sigma)$ for $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a functor $\mathrm{C}\colon \mathcal{K} \to \mathcal{K}$ together with natural maps γ and σ whose X-components γ_X and σ_X make $\mathrm{C}X$ into a compatible cylinder object as in (a). The cylinder is **cofibrant**, **good** or **final** respectively if all cylinder objects $(\mathrm{C}X, \gamma_X, \sigma_X)$ are. A cylinder $(\mathrm{C}', \gamma', \sigma')$ is **finer than** $(\mathrm{C}, \gamma, \sigma)$ iff $\mathrm{C}'X$ is finer than $\mathrm{C}X$ for all X.

(c) Given a cylinder object $(\mathrm{C}X, \gamma_X, \sigma_X)$, two maps $f, g\colon X \to Y$ are **homotopic** if the induced map $(f|g)\colon X + X \to Y$ factors through $\gamma_X\colon X + X \to \mathrm{C}X$. This will be written as $f \sim g$ or $f \sim g \pmod{\mathrm{C}X}$. Often the cylinder objects will be part of a cylinder $(\mathrm{C}, \gamma, \sigma)$ and in this case we write $f \sim g \pmod{\mathrm{C}}$. A map $h\colon \mathrm{C}X \to Y$ with $(f|g) = \gamma_X h$ will be called a **homotopy** from f to g.

(d) The symmetric transitive closure of $\sim$ is written as $\approx$. If the homotopy relation comes from a cylinder as in (b), then $\sim$ is reflexive and compatible with composition and therefore $\approx$ is a congruence relation. The quotient category will be denoted by $\mathcal{K}/\!\approx$. A map $f\colon X \to Y$ is a **homotopy equivalence**, if its image in $\mathcal{K}/\!\approx$ is an isomorphism, or equivalently, if there exists a $g\colon Y \to X$ with $fg \approx X$ and $gf \approx Y$.

(e) A **cocylinder object** for X is a cylinder object in the opposite category $\mathcal{K}^{op}$. Here $\mathcal{K}^{op}$ is equipped with the dual model structure $(\mathcal{F}, \mathcal{W}, \mathcal{C})$. In terms of the original $\mathcal{K}$ this means an object ΓX together with a factorization of the diagonal $(X, X)\colon X \to X \times X$ as $X \xrightarrow{\tau_X} \Gamma X \xrightarrow{\pi_X} X \times X$ with $\tau_X \in \mathcal{W}$. We let $\pi_X^0, \pi_X^1\colon \Gamma X \to X$ be the maps induced by the product projections. The definitions in (a)-(c) above are dualized as well, except that we call the cocylinder object **fibrant** if $\pi_X^0, \pi_X^1 \in \mathcal{F}$.

For a weak factorization system $(\mathcal{L}, \mathcal{R})$, cylinder objects, functorial cylinders and homotopy are defined as those for the trivial model structure $(\mathcal{L}, \mathcal{K}, \mathcal{R})$.

1.2.2 Remark. Our terminology "good" is borrowed from Dwyer and Spaliński [6, Definition 4.2]. However, we use "final" instead of their "very good".

(a) Suppose that in diagram (1.2.1) the object X is cofibrant. Then the coproduct injections ι_X^0 and ι_X^1 are cofibrations, being pushouts of the map $(0 \to X)$. Consequently, if γ_X is a cofibration then γ_X^0 and γ_X^1 are also.

(b) For a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ the $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorizations of codiagonals provide enough final cylinder objects and every good cylinder object $\mathrm{C}X$ can be refined to a final one by a $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorization of $\sigma_X \colon \mathrm{C}X \to X$. Also every final cylinder object is a finest one among the good cylinder objects: if $\mathrm{C}'X$ is final and $\mathrm{C}X$ is good, then $\gamma_X \mathbin{\square} \sigma'_X$ will give a diagonal in

$$\begin{tikzcd}
X+X \arrow[r, "\gamma'_X"] \arrow[d, "\gamma_X"'] & \mathrm{C}'X \arrow[d, "\sigma'_X"] \\
\mathrm{C}X \arrow[r, "\sigma_X"'] \arrow[ur] & X
\end{tikzcd}$$

so that $\mathrm{C}'X$ is finer than $\mathrm{C}X$.

If $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is functorial, then one always has enough final cylinders and every good cylinder $(\mathrm{C}, \gamma, \sigma)$ can be refined to a final cylinder through a functorial factorization of σ.

(c) If $(\mathrm{C}'X, \gamma'_X, \sigma'_X)$ is finer than $(\mathrm{C}X, \gamma_X, \sigma_X)$ then the implication

$$f \sim g \pmod{\mathrm{C}'X} \implies f \sim g \pmod{\mathrm{C}X}$$

holds for any two maps $f, g \colon X \to Y$. In particular, any two final cylinder objects determine the same homotopy relation. The same holds for cylinders.

(d) We have built compatibility with the given model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{K}$ into the definition of cylinder objects by insisting on $(\mathcal{K}, \mathcal{W})$-factorizations of codiagonals. The notion of a cylinder object still makes sense without considering any model structures, and this can be reproduced within our definition by using the model structure $(\mathcal{K}, \mathcal{K}, \mathrm{Iso})$, which exists for any category $\mathcal{K}$ and does not impose any restrictions on cylinder objects. In the absence of coproducts, one can still define cylinder objects and homotopy by rephrasing Definition 1.2.1 in terms of γ_X^0 and γ_X^1 instead of γ_X. Similar for functorial cylinders.

Kamps and Porter [12] use this notion of (functorial) cylinder as a starting point for abstract homotopy.

(e) Given a weak factorization system $(\mathcal{L}, \mathcal{R})$, consider a square

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\ell"'] & X \arrow[d, "r"] \\
L \arrow[r] & Y
\end{tikzcd}$$

with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. Then any two diagonals $d, d' \colon L \to X$ are homotopic via a final cylinder object $\mathrm{C}L$

$$\begin{tikzcd} L + L \arrow[rr, "(d|d')"] \arrow[d, "\gamma_L"'] & & X \arrow[d, "r"] \\ \mathrm{C}L \arrow[urr, "h"] \arrow[r, "\sigma_X"'] & L \arrow[r] & Y \end{tikzcd}$$

where the homotopy $h \colon \mathrm{C}L \to X$ comes from $\gamma_L \,\square\, r$. This suggests that one may consider such pairs (d, d') of common diagonals and define a homotopy relation as the smallest congruence relation generated by those pairs. Such a definition does not use cylinder objects and also works when the pair $(\mathcal{L}, \mathcal{R})$ satisfies only condition (1) of Definition 1.1.1. This approach has been developed by Ringel [23].

1.2.3. In the context of model categories, there are two other methods of defining homotopy, which do not use (functorial) cylinders but only cylinder objects and therefore also work for model structures that are not functorial.

(a) Two maps $f, g \colon X \to Y$ are homotopic if $(f|g)$ factors through some cylinder object (diagram adapted from [20, p 1.4]):

$$\begin{tikzcd} X + X \arrow[r, "(f|g)"] \arrow[d, "(X|X)"'] \arrow[dr, "c"] & Y \\ X & P \arrow[l, "w"] \arrow[u, "h"'] \end{tikzcd} \tag{1.2.2}$$

This is known as "left homotopy" in the literature on model categories. Whether one insists on good cylinder objects (as e.g. in [20, Definition 4 (p 1.5)], [8, Definition 7.3.2] or [9, Definition 1.2.4]) or not (as in [20, Definition 3 (p 1.4)]) does not change the homotopy relation.

(b) One uses Definition 1.2.1(c) for a fixed (possibly nonfunctorial) choice of final cylinder objects. The existence of certain diagonals then works as a substitute for the missing naturality. The homotopy relation with respect to such a choice will always be reflexive, symmetric and compatible with composition. Moreover, by Remark 1.2.2(c) it does not depend on the choice of cylinder objects. This approach was introduced by Kurz and Rosický [13].

Since we will only work with functorial model structures, we will usually use functorial cylinders. However, in showing that the homotopy category $\mathrm{Ho}(\mathcal{K})$ of a model category $\mathcal{K}$ is equivalent to the quotient category $\mathcal{K}_{cf}/\!\approx$ one uses the homotopy relation in (a) and it is not immediately clear whether the result is the same if our method of defining homotopy (or the one in (b)) is used instead, because the three methods will in general produce different homotopy relations on the $\mathcal{K}(X, Y)$.

We recall the construction of $\mathcal{K}_{cf}/\approx$ (for details, see e.g. [9, Section 1.2]): one first considers the full subcategory $\mathcal{K}_{cf}$ of those objects that are both cofibrant and fibrant. On this subcategory, left-homotopy from (a) is an equivalence relation which is also compatible with composition. The homotopy category $\mathrm{Ho}(\mathcal{K})$ is then equivalent to $\mathcal{K}_{cf}/\sim$.

The following Lemma ensures that all three notions of homotopy agree on $\mathcal{K}(X, Y)$ whenever Y is fibrant and therefore $\mathcal{K}_{cf}/\sim$ can be constructed with any of these.

1.2.4 Lemma. *Let* $(\mathrm{C}X, \gamma_X, \sigma_X)$ *be a good cylinder object for a model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *and let* $f, g\colon X \to Y$ *be maps with* Y *fibrant. Suppose there are maps* $c\colon X + X \to P$ *and* $w\colon P \to X$ *with* $w \in \mathcal{W}$ *as in diagram* (1.2.2). *Then* $f \sim g \pmod{\mathrm{C}X}$.

Proof. First factor w as $w = c'r$ with $c' \in \mathcal{C}$ and $r \in \mathcal{C}^{\Box}$. Then $w, r \in \mathcal{W}$ forces $c' \in \mathcal{C} \cap \mathcal{W}$ by the 2-3-property. We then have the diagram

$$\begin{tikzcd}
X+X \arrow[r, "c"] \arrow[d, "\gamma_X"'] & P \arrow[d, "c'"] \arrow[dr, "h"] & \\
\mathrm{C}X \arrow[r, "d"] \arrow[dr, "\sigma_X"'] & \bullet \arrow[r, "k"'] \arrow[d, "r"] & Y \\
& X &
\end{tikzcd}$$

where d exists because $\gamma_X \ \Box\ r$ and k exists because $c' \ \Box\ (Y \to 1)$. The equation $\gamma_X dk = ch = (f|g)$ gives $f \sim g$. $\Box$

We will also need a relative notion of cylinder objects and homotopy.

1.2.5 Definition. Let $\mathcal{K}$ be a category with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

(a) For an object A of $\mathcal{K}$ consider the comma category $(A{\downarrow}\mathcal{K})$ together with the projection $\mathrm{cod}\colon (A{\downarrow}\mathcal{K}) \to \mathcal{K}$ which assigns to any $s\colon A \to X$ its codomain X and likewise for maps.

We say that a map f in $(A{\downarrow}\mathcal{K})$ is a cofibration, fibration or weak equivalence if its underlying map $\mathrm{cod}(f)$ in $\mathcal{K}$ is. To see that this indeed gives a model structure, one can use Remark 1.1.2(c): factorizations of maps and diagonals for squares in $(A{\downarrow}\mathcal{K})$ can be constructed in $\mathcal{K}$ and cod preserves retracts, like any functor. This gives the two weak factorization systems in Definition 1.1.6(2) and the closure of weak equivalences under retracts. Finally, cod reflects the 2-3 property and hence the weak equivalences have it.

(b) Let $s\colon A \to X$ be a fixed map. A **cylinder object relative to** s is a cylinder object for the object s in the comma-category $(A{\downarrow}\mathcal{K})$, equipped with the model structure in (a). In terms of the original $\mathcal{K}$ this is the same as an ordinary cylinder object $(\mathrm{C}_X, \gamma_X, \sigma_X)$ for X with the additional requirement of $s\gamma_X^0 = s\gamma_X^1$. In this situation $\gamma_X \colon X + X \to \mathrm{C}X$ factors through the canonical map $X + X \to X +_A X$ to the pushout of s with itself (i.e. the coproduct in $(A{\downarrow}\mathcal{K})$). We also write γ_X for the induced map $(\gamma_X^0|_A\gamma_X^1) \colon X +_A X \to \mathrm{C}X$. In particular, one obtains diagram (1.2.1) with $X +_A X$ in place of $X + X$.

Expanding Definition 1.2.1(c) in $(A{\downarrow}\mathcal{K})$, one obtains that two maps $f, g \colon X \to Y$ with $sf = sg$ are **homotopic relative to** s if the induced map $(f|g) \colon X +_A X \to Y$ factors through $\gamma_X \colon X +_A X \to \mathrm{C}X$. This will also be written as $f \sim g$ or sometimes as $f \overset{s}{\sim} g$ if the map s is not clear from the context.

1.2.6 Remark. The name "relative homotopy" is suggested by situations where relative cylinders can be obtained from ordinary ones. Consider the following diagram.

$$\begin{tikzcd}
A + A \arrow[r, "\gamma_A"] \arrow[d, "s+s"'] & \mathrm{C}A \arrow[r, "\sigma_A"] \arrow[d, "j"] & A \arrow[d, "t"] \\
X + X \arrow[r] \arrow[dr, "\gamma_X"'] & Q \arrow[r, "i"] \arrow[d, "(\gamma_X|\mathrm{C}s)"] & X +_A X \arrow[d, "\gamma'_X"] \\
& \mathrm{C}X \arrow[r, "u"] \arrow[dr, "\sigma_X"'] & P \arrow[d, "\sigma'_X"] \\
& & X
\end{tikzcd}$$

where the squares are pushouts and $\sigma'_X = (\sigma_X|s)$. Suppose that $\mathrm{C}s \colon \mathrm{C}A \to \mathrm{C}X$ is a cofibration and that weak equivalences are preserved by taking pushouts along cofibrations. Then $\sigma_A \in \mathcal{W}$ gives $u \in \mathcal{W}$. From the 2-3 property we obtain also $\sigma'_X \in \mathcal{W}$, so that $(P, \gamma'_X, \sigma'_X)$ is a cylinder object for $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ relative to s.

Now consider two $f, g \colon X \to Y$ with $sf = gf$. Chasing back through the pushout squares, we see that $(f|_A g) \colon X +_A X \to Y$ can be extended along γ'_X iff there is a $h \colon \mathrm{C}X \to Y$ with $\gamma_X h = (f|g)$ and $(\mathrm{C}s)h = \sigma_A t(f|_A g) = \sigma_A sf$. In other words, f and g are homotopic relative to s iff they are homotopic relative to A in the ordinary topological sense.

1.3 Locally presentable categories

We now turn to accessible and locally presentable categories. The main source for this material is the book of Adámek and Rosický [2].

1.3.1 Definition. Let λ denote a regular cardinal.

(a) an object X in a category $\mathcal{K}$ is **λ-presentable** if the functor $\mathcal{K}(X, -) \colon \mathcal{K} \to \mathbf{Set}$ preserves λ-directed colimits. It is **presentable** if it is λ-presentable for some λ.

(b) A category $\mathcal{K}$ is **λ-accessible** if it satisfies the following two conditions:

(1) $\mathcal{K}$ has λ-directed colimits.

(2) there is a set $\mathcal{A}$ of λ-presentable objects of $\mathcal{K}$ such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

It is **accessible** if it is λ-accessible for some λ.

(c) A category $\mathcal{K}$ is **locally λ-presentable** if it is λ-accessible and cocomplete. It then follows that it is also complete, see e.g. [2, Corollary 1.28]. It is **locally presentable** if it is locally λ-presentable for some λ.

(d) A functor $F\colon \mathcal{K} \to \mathcal{L}$ is λ-accessible if both $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible and F preserves λ-directed colimits. It is **accessible** if it is λ-accessible for some regular cardinal λ.

(e) A full subcategory $\mathcal{K}$ of $\mathcal{L}$ is **accessibly embedded** if it is closed under λ-directed colimits for some regular cardinal λ.

1.3.2 Remark. Because every regular cardinal $\lambda' \geq \lambda$ is in particular λ-directed, a λ-presentable object is λ-small. More generally, a λ-presentable object is also λ'-presentable for all regular $\lambda' \geq \lambda$.

Also every object K of a locally presentable category $\mathcal{K}$ is presentable: suppose $\mathcal{K}$ is locally κ-presentable and write K as a directed colimit $K = \operatorname{colim}_i A_i$ of κ-presentable objects. Choose some regular cardinal $\lambda \geq \kappa$ such that the diagram for the colimit has less than λ morphisms. Then the A_i are also λ-presentable and therefore K is λ-presentable by [2, Proposition 1.16].

Consequently, every object of a locally presentable category $\mathcal{K}$ is a small object and therefore every set of maps in $\mathcal{K}$ permits application of the small object argument of Lemma 1.1.4.

1.3.3. Let $F\colon \mathcal{A} \to \mathcal{B}$ be any functor. We write $F\mathcal{A}$ for the **full image** of $\mathcal{A}$ under F, i.e. the full subcategory of $\mathcal{B}$ determined by all objects FX ($X \in \mathcal{A}$). If $\mathcal{K}$ is a full subcategory of $\mathcal{B}$, we write $F^{-1}\mathcal{K}$ for its **full preimage** under F, i.e. the full subcategory of $\mathcal{A}$ determined by all those objects $X \in \mathcal{A}$ with $FX \in \mathcal{K}$.

1.3.4 Lemma. *Let $F\colon \mathcal{A} \to \mathcal{C}$ be an accessible functor and let $\mathcal{K}$ be a full subcategory of $\mathcal{C}$.*

(a) If $\mathcal{K}$ is accessible and accessibly embedded in $\mathcal{C}$ then $F^{-1}\mathcal{K}$ is also accessible and accessibly embedded in $\mathcal{A}$.

(b) If $\mathcal{K}$ is the full image of an accessible functor and also isomorphism-closed in $\mathcal{C}$ then the same holds for $F^{-1}\mathcal{K}$.

Proof. Part (a) is [2, Remark 2.50]. For part (b), let $G\colon \mathcal{B} \to \mathcal{C}$ be an accessible functor with $\mathcal{K} = G\mathcal{B}$.

(1) The comma category $(F{\downarrow}G)$ is accessible and the projection $(F{\downarrow}G) \to \mathcal{A}$ is accessible by [2, Theorem 2.43].

(2) We obtain an accessible functor $H\colon (F{\downarrow}G) \to \mathcal{C}^{\mathbf{2}}$ via $H(A, B, u\colon FA \to GB) = u$. Since the full subcategory of $\mathcal{C}^{\mathbf{2}}$ given by isomorphisms is accessible and accessibly embedded in $\mathcal{C}^{\mathbf{2}}$, the same holds for its preimage under H, by part (a). This preimage is the full subcategory $\mathrm{Iso}(F, G)$ of $(F{\downarrow}G)$ whose objects are those $(A, B, u\colon FA \to GB)$ for which u is an isomorphism.

(3) $F^{-1}(G\mathcal{B})$ is the full image of the composite $\mathrm{Iso}(F, G) \hookrightarrow (F{\downarrow}G) \to \mathcal{A}$. □

We now turn to weak factorization systems in locally presentable categories. The following theorem should indicate, why these categories are a convenient setting.

1.3.5 Theorem. *Let $\mathcal{K}$ be a locally presentable category and I a set of maps in $\mathcal{K}$.*

(a) Every map f can be factored as $f = xy$ with $x \in \mathrm{cell}(I)$ and $y \in I^{\square}$. Moreover this factorization can be made functorial. In particular $({}^{\square}(I^{\square}), I^{\square})$ is a functorial factorization system.

(b) In the situation of (a), the factorization functor $\mathcal{K}^{\mathbf{2}} \to \mathcal{K}$ is accessible.

(c) The full subcategory of $\mathcal{K}^{\mathbf{2}}$ given by the homotopy equivalences with respect to a final cylinder is the full image of an accessible functor.

Proof. Part (a) is a special case of Lemma 1.1.4 by Remark 1.3.2. Part (b) is due to J.H. Smith; for a published proof see e.g. Rosický [24, Proposition 3.1]. The statements therein are phrased for model structures but apply to weak factorization systems via Remark 1.1.7. Part (c) is [25, Proposition 3.8]. □

Our main tool will be a theorem of Smith which describes conditions under which two classes $\mathcal{C}$ and $\mathcal{W}$ of maps in a locally presentable category are part of a cofibrantly generated model structure.

1.3.6 Definition. A functor $F\colon \mathcal{A} \to \mathcal{B}$ satisfies

(a) the **solution set condition at an object** B of $\mathcal{B}$ if there is a set of maps $\{f_i\colon B \to FA_i \mid i \in I\}$ such that every map $f\colon B \to FA$ factors as $f = f_i(Fu)$ for some f_i and $u\colon A_i \to A$.

(b) the **solution set condition at a class of objects**, if it satisfies the solution set condition at every element of that class.

(c) the **solution set condition**, if it satisfies the solution set condition at all objects of $\mathcal{B}$.

A full subcategory $\mathcal{K}$ of $\mathcal{B}$ satisfies the conditions above if its inclusion functor does.

1.3.7 Lemma. *Every accessible functor $F\colon \mathcal{K} \to \mathcal{L}$ (and hence its full image) satisfies the solution set condition.*

Proof. [2, Corollary 2.45] $\square$

1.3.8 Remark. When $\mathcal{B}$ has only sets $\mathcal{B}(B, B')$ of maps between objects (as we will always assume), it is sufficient to specify a set of objects FA_i in part (a) of Definition 1.3.6. For the special case where a class $\mathcal{A}$ of maps in a category $\mathcal{K}$ is regarded as a full subcategory of $\mathcal{K}^{\mathbf{2}}$, Definition 1.3.6 has a more user friendly description: The class $\mathcal{A}$ satisfies the solution set condition at a map m if there is a set $\mathcal{A}_m \subseteq \mathcal{A}$ of maps (the solution set), such that every map in $\mathcal{K}^{\mathbf{2}}(m, a)$ with $a \in \mathcal{A}$ factors though some $u \in \mathcal{A}_m$:

$$\begin{tikzcd}
\bullet \arrow[r, dotted] \arrow[rr, bend left] \arrow[d, "m"'] & \bullet \arrow[r, dotted] \arrow[d, "u"] & \bullet \arrow[d, "a"] \\
\bullet \arrow[r, dotted] \arrow[rr, bend right] & \bullet \arrow[r, dotted] & \bullet
\end{tikzcd}$$

1.3.9 Theorem (Smith's Theorem)**.** *Let $\mathcal{K}$ be a locally presentable category, I a set of maps and $\mathcal{W}$ a class of maps in $\mathcal{K}$. Suppose that the following conditions are satisfied:*

(1) $\mathcal{W}$ has the 2-3 property and is closed under retracts in $\mathcal{K}^{\mathbf{2}}$.

(2) $I^{\square} \subseteq \mathcal{W}$

(3) ${}^{\square}(I^{\square}) \cap \mathcal{W}$ is closed under pushouts and transfinite composition.

(4) $\mathcal{W}$ satisfies the solution set condition at I.

Then setting $\mathcal{C} := {}^{\square}(I^{\square})$ and $\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\square}$ gives a cofibrantly generated model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{K}$.

Proof. ([3, Theorem 1.7]) We only sketch the steps in the proof as far as needed for later use.

(1) [3, Lemma 1.8] Suppose that $J \subseteq {}^{\square}(I^{\square}) \cap \mathcal{W}$ is a class of maps such that in the arrow category $\mathcal{K}^{\mathbf{2}}$ any map from an $i \in I$ to a $w \in \mathcal{W}$ factors through some $j \in J$:

$$\begin{tikzcd}
\bullet \arrow[r, dotted] \arrow[rr, bend left] \arrow[d, "i"'] & \bullet \arrow[r, dotted] \arrow[d, "j"] & \bullet \arrow[d, "w"] \\
\bullet \arrow[r, dotted] \arrow[rr, bend right] & \bullet \arrow[r, dotted] & \bullet
\end{tikzcd}$$

Then every $f \in \mathcal{W}$ has a $(\mathrm{cell}(J), I^{\square})$-factorization.

(2) [3, Lemma 1.9] There exists a set J with the property in (1). This J can be constructed from the solution sets $\mathcal{W}_i$ $(i \in I)$ in the following way: for each map $(u, v)\colon i \to w$ with $i \in I$ and $w \in \mathcal{W}_i$ form the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "i'"] \arrow[rr, bend left, "w"] & & Y \\
\bullet \arrow[r] \arrow[rrr, bend right, "v"'] & P \arrow[r, "p"] & Q \arrow[r, "q"] & Y
\end{tikzcd}$$

where the left square is a pushout and pq is a $(\mathrm{cell}(I), I^{\Box})$-factorization of $(v|w)\colon P \to Q$. Then the composition $i'p$ is the resulting element of J.

Now for any class J as in (1), one has $\mathcal{C} \cap \mathcal{W} \subseteq {}^{\Box}(J^{\Box})$ because for any $f \in \mathcal{C} \cap \mathcal{W}$, a factorization $f = xy$ with $x \in \mathrm{cell}(J)$ and $y \in I^{\Box}$ will exhibit f as a retract of x. Also one has ${}^{\Box}(J^{\Box}) \subseteq {}^{\Box}((\mathcal{C} \cap \mathcal{W})^{\Box})$.

If J happens to be a set, every element of ${}^{\Box}(J^{\Box})$ is a retract of a map in $\mathrm{cell}(J)$. Because $\mathcal{C} \cap \mathcal{W}$ is stable under the operations involved, the last inclusion can be sharpened to ${}^{\Box}(J^{\Box}) \subseteq \mathcal{C} \cap \mathcal{W}$ and hence equality.

It remains to check that the resulting cofibrantly generated weak factorization systems $(\mathcal{C}, \mathcal{C}^{\Box})$ and $(\mathcal{C} \cap \mathcal{W}, (\mathcal{C} \cap \mathcal{W})^{\Box})$ satisfy $\mathcal{C}^{\Box} = \mathcal{W} \cap ((\mathcal{C} \cap \mathcal{W})^{\Box})$. The inclusion "$\subseteq$" already holds by condition (ii). Conversely, assume $f \in \mathcal{W} \cap ((\mathcal{C} \cap \mathcal{W})^{\Box})$. To establish $i \mathbin{\Box} f$ for $i \in I$, factor any square

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\quad \text{as} \quad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[r] \arrow[d, "j"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, dotted] & \bullet
\end{tikzcd}$$

and use $j \mathbin{\Box} f$ to obtain a diagonal of the right square. $\Box$

1.3.10 Corollary (of proof)**.** *The class $\mathcal{W}$ has solution sets consisting of trivial cofibrations.* $\Box$

1.3.11 Remark. Conditions (1)–(3) in the above Theorem are necessary for any cofibrantly generated model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ with I being the set of generating cofibrations. Rosický [25, Theorem 4.3] has recently shown that condition (4) is also necessary.

1.3.12. If $\mathcal{K}$ is only assumed to be cocomplete but not necessarily locally presentable, one has to pay attention to the use of the small object argument in the steps of the proof.

(a) If I is small w.r.t. $\mathrm{cell}(I)$, then all needed $(\mathrm{cell}(I), I^{\Box})$-factorizations exist.

(b) Let J be a class as in step (1) of the above proof. If I is small w.r.t. $\mathrm{cell}(J)$, then the construction of the $(\mathrm{cell}(J), I^{\Box})$-factorization still works.

(c) The set J provided in step (2) of the above proof satisfies $J \subseteq \mathrm{cell}(I)$ and hence $\mathrm{cell}(J) \subseteq \mathrm{cell}(I)$. Consequently, the condition in (b) holds whenever the condition in (a) holds. Moreover, the domains of maps in J are domains of maps from the solution sets $\mathcal{W}_i$ $(i \in I)$. Consequently, if all $\mathcal{W}_i$ are small w.r.t. $\mathrm{cell}(I)$, then all needed $(\mathrm{cell}(J), J^{\square})$-factorizations exist.

As remarked by Beke [3, Remark 1.11], the solution sets $\mathcal{W}_i$ are usually not given in some canonical way, so that it is best to assume that every object is small, which holds for locally presentable categories. Nevertheless, the construction in Chapter 3 will produce such solution sets. Therefore we record the needed requirements in the following version of Smith's Theorem.

1.3.13 Theorem (Smith's Theorem, second variant)**.** *Let $\mathcal{K}$ be a cocomplete category, I a set of maps and $\mathcal{W}$ a class of maps in $\mathcal{K}$. Suppose that in addition to (1)–(4) of Theorem 1.3.9, the following conditions are satisfied:*

(5) I is small with respect to $\mathrm{cell}(I)$*.*

(6) All $\mathcal{W}_i$ $(i \in I)$ are small with respect to $\mathrm{cell}(I)$*.*

Then setting $\mathcal{C} := {}^{\square}(I^{\square})$ and $\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\square}$ gives a cofibrantly generated model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{K}$.

We now look at some of the conditions in Theorem 1.3.9.

1.3.14 Definition. Let $\mathcal{C}$ be a fixed class of maps. A class $\mathcal{W}$ of maps is a **localizer** for $\mathcal{C}$ if it satisfies the following conditions:

(i) $\mathcal{W}$ has the 2-3 property.

(ii) $\mathcal{C}^{\square} \subseteq \mathcal{W}$.

(iii) $\mathcal{C} \cap \mathcal{W}$ is closed under pushouts and transfinite composition.

We say that a localizer $\mathcal{W}$ is **split** if $\mathcal{W}$ is also closed under retracts in $\mathcal{K}^{\mathbf{2}}$. For a given class S of maps we write $\mathcal{W}_{\mathcal{C}}(S)$ and $\mathcal{W}^{s}_{\mathcal{C}}(S)$ for the smallest localizer and the smallest split localizer respectively that contains S. In particular, $\mathcal{W}_{\mathcal{C}}(\emptyset)$ is the smallest localizer for $\mathcal{C}$.

1.3.15 Definition. A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is **left determined** if $\mathcal{W} = \mathcal{W}^{s}_{\mathcal{C}}(\emptyset)$.

1.3.16 Remark. The notion of a localizer was given by Cisinski [4, Définition 3.4] for the special case where $\mathcal{K}$ is a (Grothendieck) topos and $\mathcal{C}$ is the class of all monomorphisms. Definition 1.3.15 was given by Rosický and Tholen [26, Definition 2.1]. We use the notion of a split localizer in order to state it in a more compact form. The terminology itself is suggested by the correspondence between retracts and idempotents.

Observe that each condition in Definition 1.3.14 is stable under intersections, i.e. if it is satisfied by every $\mathcal{W}_t$ in some (possibly large) family $\mathcal{W}_t$ $(t \in T)$, then it is also satisfied by their intersection. Therefore the smallest (split) localizer containing a given class S always exists and the definitions of $\mathcal{W}_{\mathcal{C}}(S)$ and $\mathcal{W}^s_{\mathcal{C}}(S)$ make sense. We always have $\mathcal{W}_{\mathcal{C}}(S) \subseteq \mathcal{W}^s_{\mathcal{C}}(S)$. Also, if $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure with $S \subseteq \mathcal{W}$ then $\mathcal{W}^s_{\mathcal{C}}(S) \subseteq \mathcal{W}$. In particular, whenever $\mathcal{C}$ and $\mathcal{W}_{\mathcal{C}}(S)$ give a model structure then $\mathcal{W}_{\mathcal{C}}(S) = \mathcal{W}^s_{\mathcal{C}}(S)$.

2 Cisinski's construction

We now present the construction of a cofibrantly generated model structure from a suitable cofibrantly generated weak factorization system and cylinder. As in the original case, we need additional conditions on the cylinder used. Our conditions in Definition 2.1.9 are different from those of Cisinski [4, Définition 2.3]. Nevertheless, they are equivalent in the case of $(\mathrm{Mono}, \mathrm{Mono}^{\square})$ in a Grothendieck topos. This is discussed in Section 1. In Section 2 we show that this construction gives a model structure and in the last section we describe conditions under which the class of weak equivalences of this model structure is minimal in an appropriate sense. In this chapter all cylinders are understood to be good cylinders. We sometimes insert the word "good" just for emphasis.

2.1 Cartesian cylinders

We first look at one particular ingredient of the construction in a more general setting.

2.1.1 Definition. Let $\mathcal{A}$ be a category with pushouts. Given a natural map $\alpha\colon F \to F'\colon \mathcal{X} \to \mathcal{A}$ and a map $f\colon X \to Y$ let $f \star \alpha$ be the map in the diagram below:

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & F'X \arrow[d] \arrow[ddr, bend left, "F'f"] & \\
FY \arrow[r] \arrow[drr, bend right, "\alpha_Y"'] & FY \underset{FX}{+} F'X \arrow[dr, "f\star\alpha"] & \\
 & & F'Y
\end{tikzcd}$$

Dually, let $\mathcal{X}$ be a category with pullbacks. Given a natural map $\beta\colon G' \to G\colon \mathcal{A} \to \mathcal{X}$ and a map $g\colon A \to B$ let $\beta \star g$ be the map in the diagram below:

$$\begin{tikzcd}
G'A \arrow[dr, "\beta\star g"] \arrow[drr, bend left, "\beta_A"] \arrow[ddr, bend right, "G'g"'] & & \\
 & G'B \underset{GB}{\times} GA \arrow[r] \arrow[d] & GA \arrow[d, "Gg"] \\
 & G'B \arrow[r, "\beta_B"'] & GB
\end{tikzcd}$$

For a class I of maps, we write $I \star \alpha$ for $\{f \star \alpha \mid f \in I\}$ and $\beta \star I$ for $\{\beta \star f \mid f \in I\}$.

2.1.2. For the next Lemma, recall the notion of a conjugate pair of natural maps between two adjunctions from e.g. Mac Lane [17, IV-7]: given two adjunctions $F : \mathcal{X} \rightleftarrows \mathcal{A} : G$ and $F' : \mathcal{X} \rightleftarrows \mathcal{A} : G'$, two natural maps $\alpha\colon F \to F'$ and $\beta\colon G' \to G$ are conjugate if the diagram

$$\begin{tikzcd}
\mathcal{A}(F'X, A) \arrow[r, "\cong"] \arrow[d, "\mathcal{A}(\alpha_X, A)"'] & \mathcal{X}(X, G'A) \arrow[d, "\mathcal{X}(X, \beta_A)"] \\
\mathcal{A}(FX, A) \arrow[r, "\cong"] & \mathcal{X}(X, GA)
\end{tikzcd}$$

commutes for all $X \in \mathcal{X}$ and $A \in \mathcal{A}$.

2.1.3 Lemma. *Suppose $\alpha\colon F \to F'$ and $\beta\colon G' \to G$ are two conjugate natural maps. Then for all $f\colon X \to Y$ and $g\colon A \to B$ one has*

$$(f \star \alpha) \,\square\, g \qquad \Longleftrightarrow \qquad f \,\square\, (\beta \star g)$$

Proof. We will show the direction "$\Rightarrow$". The opposite direction then follows by duality. So assume $(f \star \alpha) \,\square\, g$ and consider any diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & G'A \arrow[d, "\beta\star g"] \arrow[dr, "\beta_A"] & \\
Y \arrow[r, "v"'] \arrow[dr, "vp"'] & P \arrow[r, "q"'] \arrow[d, "p"] & GA \arrow[d, "Gg"] \\
& G'B \arrow[r, "\beta_B"'] & GB
\end{tikzcd}$$

where P is the pullback of β_B and Gg. We need a diagonal for the left upper square. Switching via the adjunctions (indicated by $\widehat{(\ \ )}$ in both directions) gives the solid arrows of the diagram

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & F'X \arrow[d, "j"'] \arrow[dr, "\widehat{u}"] & \\
FY \arrow[r, "i"] \arrow[dr, "\alpha_Y"'] \arrow[rr, bend right, "\widehat{vq}"'] & Q \arrow[r, dotted, "r"] \arrow[d, "f\star\alpha"] & A \arrow[d, "g"] \\
& F'Y \arrow[r, "\widehat{vp}"'] \arrow[ur, dotted, "d"'] & B
\end{tikzcd}$$

where Q is the pushout of Ff and α_X. Now $r\colon Q \to A$ is induced by $\widehat{vq}$ and $\widehat{u}$. Testing against i and j yields the commutativity of the right lower square (i.e. $rg = (f \star \alpha)\widehat{vp}$), which therefore has a diagonal $d\colon F'Y \to A$. Switching back via the adjunction gives

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & G'A \arrow[d, "\beta\star g"] \arrow[dr, "\beta_A"] & \\
Y \arrow[ur, "\widehat{d}"] \arrow[r, "v"'] \arrow[dr, "vp"'] & P \arrow[r, "q"'] \arrow[d, "p"] & GA \arrow[d, "Gg"] \\
& G'B \arrow[r, "\beta_B"'] & GB
\end{tikzcd}$$

where the equality $\widehat{d}(\beta \star g) = v$ can be verified by testing against p and q. Hence $\widehat{d}\colon Y \to G'A$ is the desired diagonal. ☐

2.1.4 Corollary. *In the situation of the previous Lemma, let I be a class of maps in $\mathcal{X}$ and J be a class of maps in $\mathcal{A}$. Then*

$$I \star \alpha \subseteq J \implies ({}^{\square}(I^{\square})) \star \alpha \subseteq {}^{\square}(J^{\square})$$

Proof.

$$\begin{aligned}
I \star \alpha \subseteq J &\implies I \star \alpha \subseteq {}^{\square}(J^{\square}) \\
&\implies I^{\square} \supseteq \beta \star (J^{\square}) \\
&\implies ({}^{\square}(I^{\square}))^{\square} \supseteq \beta \star (J^{\square}) \\
&\implies ({}^{\square}(I^{\square})) \star \alpha \subseteq {}^{\square}(J^{\square})
\end{aligned}$$ ☐

2.1.5 Remark. Corollary 2.1.4 applies to any natural map between left adjoints (assuming that the necessary pushouts and pullbacks exist) because such a map uniquely determines a conjugate map between the respective right adjoints (see e.g. Mac Lane [17, IV-7, Theorem 2]).

2.1.6 Definition. Let $(\mathcal{L}, \mathcal{R})$ be a cofibrantly generated weak factorization system in a locally presentable category $\mathcal{K}$. For a functorial cylinder $(\mathrm{C}, \gamma, \sigma)$, a generating set I and a subset $S \subseteq {}^{\square}(I^{\square})$ define $\Lambda(\mathrm{C}, S, I)$ via the following construction:

$$\Lambda^0(\mathrm{C}, S, I) := S \cup (I \star \gamma^0) \cup (I \star \gamma^1) \tag{2.1.1}$$

$$\Lambda^{n+1}(\mathrm{C}, S, I) := \Lambda^n(\mathrm{C}, S, I) \star \gamma \tag{2.1.2}$$

$$\Lambda(\mathrm{C}, S, I) := \bigcup_{n \geq 0} \Lambda^n(\mathrm{C}, S, I) \tag{2.1.3}$$

2.1.7 Lemma. *Suppose a cylinder functor* C *for $(\mathcal{L}, \mathcal{R})$ is a left adjoint. Then for any two generating subsets $I, J \subseteq \mathcal{L}$ one has*

$${}^{\square}(\Lambda(\mathrm{C}, S, I)^{\square}) = {}^{\square}(\Lambda(\mathrm{C}, S, J)^{\square})$$

Proof. We will drop C and S from the notation for Λ and show $\Lambda^n(I) \subseteq {}^{\square}(\Lambda(J)^{\square})$ for all $n \geq 0$.

(1) We have $J \star \gamma^k \subseteq \Lambda(J)$ (for $k = 0, 1$). Corollary 2.1.4 then gives $\mathcal{L} \star \gamma^k \subseteq {}^{\square}(\Lambda(J)^{\square})$. So in particular $\Lambda^0(I) \subseteq {}^{\square}(\Lambda(J)^{\square})$.

(2) Assume $\Lambda^n(I) \subseteq {}^{\square}(\Lambda(J)^{\square})$. Corollary 2.1.4 then gives

$$\Lambda^{n+1}(I) = \Lambda^n(I) \star \gamma \subseteq {}^{\square}(\Lambda(J)^{\square})$$

□

2.1.8 Remark. In general one cannot expect $\Lambda(\mathrm{C}, S, I) \subseteq \mathcal{L}$ without any further assumptions. However, if C is a left adjoint, Lemma 2.1.7 shows, that this property does not depend on the choice of the generating subset. This motivates the following definition.

2.1.9 Definition. Let $(\mathcal{L}, \mathcal{R})$ be weak factorization system in a category $\mathcal{K}$. A good functorial cylinder $(\mathrm{C}, \gamma, \sigma)$ for $(\mathcal{L}, \mathcal{R})$ is **cartesian** if

(a) The cylinder functor $\mathrm{C} \colon \mathcal{K} \to \mathcal{K}$ is a left adjoint

(b) $\mathcal{L} \star \gamma \subseteq \mathcal{L}$ and $\mathcal{L} \star \gamma^k \subseteq \mathcal{L} \quad (k = 0, 1)$

2.1.10 Remark. Condition (a) allows using Lemma 2.1.3 and Corollary 2.1.4. In particular, if $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated by some subset $I \subseteq \mathcal{L}$, Condition (b) already holds whenever $I \star \gamma^0$, $I \star \gamma^1$ and $I \star \gamma$ lie in $\mathcal{L}$. Also for any $f \in \mathcal{L}$ we have $\mathrm{C}f = f'(f \star \gamma^0)$ where f' is a pushout of f, so that $\mathrm{C}f$ is again in $\mathcal{L}$.

We now compare Definition 2.1.9 with [4, Définition 2.3]. Let $\mathcal{E}$ be a Grothendieck topos. Recall the following properties:

(1) Colimits in $\mathcal{E}$ are universal: given a colimit cocone $x_i \colon X_i \to X$ and a map $f \colon Y \to X$, the induced maps $f^*(x_i) \colon f^*(X_i) \to Y$ obtained from pulling back the x_i along f again form a colimit cocone. This is [10, Lemma 1.51].

(2) $\mathcal{E}$ is locally presentable. This follows from [2, Theorem 1.46] together with the fact that the sheaves with respect to a site form a small orthogonality class (in the sense of [2, Definition 1.35]) inside the respective presheaf topos.

(3) Whenever one has a diagram

$$\begin{tikzcd}
P \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d] \arrow[ddr, bend left, "y"] & \\
A \arrow[r] \arrow[drr, bend right, "x"'] & Q \arrow[dr, "x \vee y"] & \\
& & X
\end{tikzcd} \tag{2.1.4}$$

where x and y are monomorphisms, P is the pullback of x and y, and Q is the pushout of a and b, then the induced map $x \vee y \colon Q \to X$ is also a monomorphism. This follows from [10, Proposition 1.55].

(4) Monomorphisms are closed under transfinite composition. This follows from repeated application of [2, Corollary 1.60].

From the last three items above, it follows by [3, Proposition 1.12] that $(\mathrm{Mono}, \mathrm{Mono}^{\square})$ is a cofibrantly generated weak factorization system. Now suppose $(\mathrm{C}, \gamma, \sigma)$ is a cylinder for $(\mathrm{Mono}, \mathrm{Mono}^{\square})$ and consider the following conditions:

DH1 The functor C preserves monomorphisms and all colimits.

DH2 If $f \colon X \to Y$ is a monomorphism then

$$\begin{tikzcd} X \arrow[r, "\gamma^k_X"] \arrow[d, "f"'] & \mathrm{C}X \arrow[d, "\mathrm{C}f"] \\ Y \arrow[r, "\gamma^k_Y"'] & \mathrm{C}Y \end{tikzcd} \tag{2.1.5}$$

are pullback squares $(k = 0, 1)$.

DH3 If $f \colon X \to Y$ is a monomorphism then

$$\begin{tikzcd} X + X \arrow[r, "\gamma_X"] \arrow[d, "f+f"'] & \mathrm{C}X \arrow[d, "\mathrm{C}f"] \\ Y + Y \arrow[r, "\gamma_Y"'] & \mathrm{C}Y \end{tikzcd} \tag{2.1.6}$$

is a pullback square.

Conditions DH1 and DH2 were introduced by Cisinski [4, Définition 2.3]. We first observe, that it is enough to restrict attention to DH1:

2.1.11 Lemma. *Given a cylinder* $(\mathrm{C}, \gamma, \sigma)$ *for* $(\mathrm{Mono}, \mathrm{Mono}^{\square})$, *one has the implications DH1 $\implies$ DH2 $\implies$ DH3.*

Proof. Assume that the cylinder satisfies DH1. For every $f \colon X \to Y$, the outer rectangle in the diagram

$$\begin{tikzcd} X \arrow[r, "\gamma^k_X"] \arrow[d, "f"'] & \mathrm{C}X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{C}f"] & X \arrow[d, "f"] \\ Y \arrow[r, "\gamma^k_Y"'] & \mathrm{C}Y \arrow[r, "\sigma_Y"'] & Y \end{tikzcd}$$

is always a pullback. If f is a monomorphism then so is $\mathrm{C}f$ and hence the left square is also a pullback. So the cylinder satisfies DH2.

Assume that the cylinder satisfies DH2. Given a monomorphism $f\colon X \to Y$, consider for $k = 0, 1$ the diagrams

$$\begin{tikzcd} X \arrow[r, "p^k"] \arrow[d, "f"'] & P \arrow[r, "g"] \arrow[d, "h"] & \mathrm{C}X \arrow[d, "\mathrm{C}f"] \\ Y \arrow[r, "\iota_Y^k"'] & Y + Y \arrow[r, "\gamma_Y"'] & \mathrm{C}Y \end{tikzcd}$$

where the right square is a pullback and p^k is induced by the maps $f\iota_Y^k$ and γ_X^k. By DH2 the outer rectangle is also pullback and hence the left square is a pullback too. Because coproducts are universal, the maps p^0 and p^1 make P into a coproduct of X and X. The canonical isomorphism $u\colon X + X \to P$ with $\iota_X^k u = p^k$ then satisfies $uh = f + f$ and $ug = \gamma_X$. So the cylinder satisfies DH3. □

2.1.12 Corollary. *In a Grothendieck topos a good cylinder for* $(\mathrm{Mono}, \mathrm{Mono}^{\square})$ *is cartesian iff it satisfies DH1 (and hence DH2 and DH3) above.*

Proof. Let $(\mathrm{C}, \gamma, \sigma)$ be a cylinder.

Suppose it is cartesian. Then the left adjoint C preserves all colimits and we already noted in Remark 2.1.10 that $f \in \mathrm{Mono}$ implies $\mathrm{C}f \in \mathrm{Mono}$. Therefore condition DH1 is satisfied, as well as conditions DH2 and DH3.

Conversely, suppose that condition DH1 is satisfied. Now, any locally presentable category is cocomplete (by definition), co-wellpowered (by [2, Theorem 1.58]) and has a (small) generator (by [2, Theorem 1.20]). Therefore it satisfies the dual form of the conditions in Freyd's Special Adjoint Functor Theorem, and the colimit preserving functor C is indeed a left adjoint.

To check that Mono is stable under the $(-) \star \gamma^k$ and $(-) \star \gamma$, match diagram (2.1.4) above with the diagrams (2.1.5) and (2.1.6). More precisely, for a monomorphism $f\colon X \to Y$ let $a = f$, $b = \gamma_X^k$, $x = \gamma_Y^k$, $y = \mathrm{C}f$ in diagram (2.1.4). Then $f \star \gamma^k$ coincides (up to isomorphism) with $x \vee y$ and because condition DH2 is satisfied, $x \vee y$ is a monomorphism. Similarly, conditions DH3 gives that $f \star \gamma$ is a monomorphism. □

2.2 The model structure

2.2.1 Definition. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system, cofibrantly generated by a subset $I \subseteq \mathcal{L}$. Let $(\mathrm{C}, \gamma, \sigma)$ be a cylinder for $(\mathcal{L}, \mathcal{R})$ and $S \subseteq \mathcal{L}$ be any subset. Define $\mathcal{W}(\mathrm{C}, S, I)$ as the class of all those maps $f\colon X \to Y$ such that for all objects T with $(T \to 1) \in \Lambda(\mathrm{C}, S, I)^{\square}$ the induced map $f^*\colon \mathcal{K}(Y, T)/{\approx} \to \mathcal{K}(X, T)/{\approx}$ is bijective.

2.2.2 Remark. Clearly $\mathcal{W}(\mathrm{C}, S, I)$ contains all isomorphisms, has the 2-3 property and is closed under retracts in $\mathcal{K}^{\mathbf{2}}$. Furthermore, whenever fg and gf lie in $\mathcal{W}(\mathrm{C}, S, I)$, then so do f and g. All these properties follow from the corresponding properties of bijections. Also note, that for $f \sim g$, one has $f \in \mathcal{W}(\mathrm{C}, S, I) \iff g \in \mathcal{W}(\mathrm{C}, S, I)$ because the induced maps $f^*, g^* \colon \mathcal{K}(Y, T)/{\approx} \to \mathcal{K}(X, T)/{\approx}$ coincide.

Besides being cofibrantly generated, the weak factorization system $(\mathrm{Mono}, \mathrm{Mono}^{\square})$ in a Grothendieck topos has the property that each object is cofibrant, i.e. that each map $(0 \to X)$ is in $\mathcal{L}$. For convenience, we combine these two properties into one definition:

2.2.3 Definition. A model structure (weak factorization system) is **cofibrant** if it is cofibrantly generated and every object is cofibrant.

2.2.4 Lemma. *Let $(\mathcal{L}, \mathcal{R})$ be a cofibrant weak factorization system, let $(\mathrm{C}, \gamma, \sigma)$ be a cartesian cylinder and let $\Lambda := \Lambda(\mathrm{C}, S, I)$ as in Definition 2.1.6. Then the natural maps γ^0 and γ^1 have their components in ${}^{\square}(\Lambda^{\square})$.*

Proof. Application of Corollary 2.1.4 to $I \star \gamma^k \subseteq \Lambda$ gives $\mathcal{L} \star \gamma^k \subseteq {}^{\square}(\Lambda^{\square})$. Because the left adjoint C must preserve the initial object, γ^k_X differs from $(0 \to X) \star \gamma^k$ only by composition with some isomorphism (due to the choice involved in Definition 2.1.1). Hence $\gamma^k_X \in {}^{\square}(\Lambda^{\square})$. $\square$

We are now ready to state the main result of the section.

2.2.5 Theorem. *Let $\mathcal{K}$ be a locally presentable category and $(\mathcal{L}, \mathcal{R})$ a cofibrant weak factorization system generated by a set $I \subseteq \mathcal{L}$. Let $(\mathrm{C}, \gamma, \sigma)$ be a cartesian cylinder for $(\mathcal{L}, \mathcal{R})$ and $S \subseteq \mathcal{L}$ an arbitrary subset. Then, setting*

$$\mathcal{C} := \mathcal{L} \qquad \mathcal{W} := \mathcal{W}(\mathrm{C}, S, I) \qquad \mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\square} \tag{2.2.1}$$

gives a cofibrant model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{K}$. Moreover, $(\mathrm{C}, \gamma, \sigma)$ is also a cylinder for this model structure.

2.2.6 Remark. Theorem 2.2.5 does not remain valid if "cofibrant" is weakened to "cofibrantly generated" in its statement. Let $\mathcal{G}$ be a (small) generator in $\mathcal{K}$ and consider the set of codiagonal maps $I := \{(G|G) \colon G + G \to G \mid G \in \mathcal{G}\}$.

(1) $I^{\square}$ is the class Mono of monomorphisms and ${}^{\square}(I^{\square})$ is the class StrEpi of strong epimorphisms.

(2) The $(\mathrm{StrEpi}, \mathrm{Mono})$-factorization of every codiagonal $(X|X)$ as

$$X + X \xrightarrow{(X|X)} X \xrightarrow{X} X$$

gives a good cylinder $(\mathrm{C}, \gamma, \sigma)$ where C and σ are the identity and $\gamma_X = (X|X)$. In particular, C is a left adjoint and the homotopy relation is equality.

(3) If $f\colon X \to Y$ is a strong epimorphism, then $f \star \gamma^0$, $f \star \gamma^1$ and $f \star \gamma$ are also strong epimorphisms. This is clear for γ^0 and γ^1 because they are identity transformations. In the case of γ, it is enough to observe that $f = g(f \star \gamma)$, where g is the pushout of $f + f$ along γ_X. (Alternatively one can consider the conjugate map $\overline{\gamma}$, check that $\overline{\gamma} \star (-)$ preserves monomorphisms and apply Lemma 2.1.3).

Altogether, $(\mathrm{StrEpi}, \mathrm{Mono})$ is cofibrantly generated and $(\mathrm{C}, \gamma, \sigma)$ is cartesian. Going through the construction of $\Lambda = \Lambda(\emptyset, I)$ in this case, one obtains that Λ^0 consists only of isomorphisms and therefore all Λ^n consist only of isomorphisms. Consequently, every object X satisfies $(X \to 1) \in \Lambda^\square$ and $\mathcal{W}(\emptyset, I)$ is the class of isomorphisms. In particular $\mathrm{StrEpi}^\square$ is not included in $\mathcal{W}(\emptyset, I)$.

The rest of this section will consist of the proof of Theorem 2.2.5 via Smith's Theorem 1.3.9. It turns out that almost all steps in the proof of [4, Théorème 2.13] can be reused with only minor modifications to verify conditions (1)–(3) of Theorem 1.3.9. However, in verifying condition (4) we will depart from [4] and use Part (c) of Theorem 1.3.5 (i.e. [25, Proposition 3.8]). Condition (1) already already holds by Remark 2.2.2. We now turn to condition 1.3.9(2).

2.2.7. By Lemma 2.1.7, $\Lambda(\mathrm{C}, S, I)^\square$ and hence $\mathcal{W}(\mathrm{C}, S, I)$ do not depend on I. While they do depend on C and S (it will turn out that S is contained in $\mathcal{C} \cap \mathcal{W}$ and the components of σ lie in $\mathcal{W}$), the particular choices of C and S do not play any role in the proof. Therefore we will simply write Λ for $\Lambda(\mathrm{C}, S, I)$ and $\mathcal{W}$ for $\mathcal{W}(\mathrm{C}, S, I)$. We call an object X **fibrant** if $(X \to 1) \in \Lambda^\square$. In Lemma 2.2.20 we will show that these objects coincide with the fibrant objects of the resulting model structure, so that the terminology is justified.

2.2.8 Definition ([4, Définition 2.15]). A map $f\colon X \to Y$ is a **dual strong deformation retract** if there exist maps $g\colon Y \to X$ and $h\colon \mathrm{C}X \to X$ such that the following diagram commutes

$$\begin{tikzcd}
X+X \arrow[rr, "(X|fg)"] \arrow[d, "\gamma_X"'] & & X \arrow[d, "f"] & Y \arrow[l, "g"'] \arrow[dl, equal] \\
\mathrm{C}X \arrow[r, "\sigma_X"'] \arrow[urr, "h"] & X \arrow[r, "f"'] & Y &
\end{tikzcd} \tag{2.2.2}$$

2.2.9 Lemma. *Every element of $\mathcal{C}^\square$ is a dual strong deformation retract.*

Proof. Let $f\colon X \to Y \in \mathcal{C}^\square$. Because every object is cofibrant, f is a retraction, so there is a $g\colon Y \to X$ such that the right triangle in diagram (2.2.2) commutes. Because of $(X|fg)f = (f|f) = (X|X)f = \gamma_X \sigma_X f$ the left square of that diagram also commutes. Now $\gamma_X \,\square\, f$ gives the desired diagonal $h\colon \mathrm{C}X \to X$. $\square$

2.2.10 Corollary. $\mathcal{C}^\square \subseteq \mathcal{W}$.

Proof. By the previous Lemma, it is enough to check that every dual strong deformation retract is in $\mathcal{W}$. If f and g are as in Diagram (2.2.2), then $X \sim fg$ and $Y = gf$. Using Remark 2.2.2, one obtains that fg and gf are in $\mathcal{W}$ and hence $f \in \mathcal{W}$. $\square$

2.2.11 Remark. In fact, one has $\mathcal{C}^\square = (\mathcal{C} \cap \mathcal{W})^\square \cap \mathcal{W}$. For the direction not covered by the Corollary, factor a given $f \in (\mathcal{C} \cap \mathcal{W})^\square \cap \mathcal{W}$ as $f = \ell r$ with $\ell \in \mathcal{C}$ and $r \in \mathcal{C}^\square$. Then $r \in \mathcal{W}$ and hence $\ell \in \mathcal{C} \cap \mathcal{W}$. Therefore $\ell \,\square\, f$ and f is a retract of r. So in the language of model structures, the "trivial fibrations are indeed those fibrations that are trivial".

Condition 1.3.9(2) holds by Corollary 2.2.10. Verifying condition 1.3.9(3) will occupy us until Corollary 2.2.21.

2.2.12 Lemma. *Let X and T be objects with T fibrant. Then the homotopy relation $\sim$ is an equivalence relation on $\mathcal{K}(X, T)$.*

Proof. The relation is clearly reflexive. For symmetry and transitivity let $u, v, w \in \mathcal{K}(X, T)$ and suppose $v \sim u$ and $v \sim w$ via maps $h, k\colon \mathrm{C}X \to X$ with $\gamma_X h = (v|u)$ and $\gamma_X k = (v|w)$. This gives the solid arrows in the following diagram

$$\begin{tikzcd}
X + X \arrow[r, "\gamma_X"] \arrow[d, "\gamma^0_X + \gamma^0_X"'] & \mathrm{C}X \arrow[rr, "\sigma_X"] \arrow[d, "p"'] \arrow[dr, "\mathrm{C}(\gamma^0_X)"] & & X \arrow[dd, "v"] \\
\mathrm{C}X + \mathrm{C}X \arrow[r] \arrow[drrr, bend right, "(h|k)"'] & Q \arrow[r, "\gamma^0_X \star \gamma"] \arrow[drr, "t"] & \mathrm{CC}X \arrow[dr, dotted, "d"] & \\
 & & & T
\end{tikzcd}$$

where Q is the pushout of $\gamma^0_X + \gamma^0_X$ and γ_X and where t is induced by the commuting outer rectangle. By Lemma 2.2.4 we have $\gamma^0_X \in {}^\square(\Lambda^\square)$. Applying Corollary 2.1.4 to $\Lambda \star \gamma \subseteq \Lambda$ gives $\gamma^0_X \star \gamma \in {}^\square(\Lambda^\square)$. Hence $(\gamma^0_X \star \gamma) \,\square\, (T \to 1)$ and $d\colon \mathrm{CC}X \to T$ exists. Therefore the following diagram commutes

$$\begin{tikzcd}
X + X \arrow[r, "\gamma^1_X + \gamma^1_X"'] \arrow[d, "\gamma_X"'] & \mathrm{C}X + \mathrm{C}X \arrow[d, "\gamma_{\mathrm{C}X}"] \arrow[dr, "(h|k)"] & \\
\mathrm{C}X \arrow[r, "\mathrm{C}(\gamma^1_X)"'] & \mathrm{CC}K \arrow[r, "d"'] & T
\end{tikzcd}$$

exhibiting a homotopy from u to w. $\square$

2.2.13 Remark. With the previous Lemma, the condition for $f\colon X \to Y$ to be in $\mathcal{W}$ can be rephrased in terms of the homotopy relation instead of its transitive closure: for any given $t\colon X \to T$ with T fibrant there is a $u\colon Y \to T$ with $t \sim fu$ and such a u is determined up to homotopy. In particular one obtains the following description for maps between fibrant objects:

2.2.14 Corollary. *Suppose X and Y are fibrant. Then $f\colon X \to Y$ is in $\mathcal{W}$ if and only if there exist a $g\colon Y \to X$ with $X \sim fg$ and $Y \sim gf$.*

Proof. One direction is clear. If $f\colon X \to Y$ is in $\mathcal{W}$ then using the remark with $t = X\colon X \to X$ gives a $g\colon Y \to X$ with $X \sim fg$. Therefore $f \sim fgf$ and using the remark with $t = f\colon X \to Y$ yields $gf \sim Y$. $\square$

2.2.15 Lemma. ${}^{\square}(\Lambda^{\square}) \subseteq \mathcal{W}$

Proof. Suppose $f\colon X \to Y$ is in ${}^{\square}(\Lambda^{\square})$ and let $t\colon X \to T$ be a map with T fibrant.

(1) *Existence:* Because $f \mathbin{\square} (T \to 1)$, there exists a $u\colon Y \to T$ with $t = fu$, so in particular $t \sim fu$.

(2) *Uniqueness:* Assume $u, v\colon Y \to T$ with $t \sim fu$ and $t \sim fv$. By Lemma 2.2.12, $fu \sim fv$ and there is some $h\colon \mathrm{C}X \to X$ with $\gamma_X h = (fu|fv) = (f+f)(u|v)$. Therefore one has the following diagram

$$\begin{tikzcd}
X+X \arrow[r, "\gamma_X"] \arrow[d, "f+f"'] & \mathrm{C}X \arrow[d] \arrow[ddr, bend left, "h"] & \\
Y+Y \arrow[r] \arrow[ddrr, phantom] \arrow[drr, bend right, "(u|v)"'] & Y+Y \underset{X+X}{+} \mathrm{C}X \arrow[dr, "r"] & \\
& & T
\end{tikzcd}$$

where r is the induced map from the pushout. By Corollary 2.1.4 $f \star \gamma \in {}^{\square}(\Lambda^{\square})$ and hence $(f \star \gamma) \mathbin{\square} (T \to 1)$, so that r factors through $f \star \gamma$ via some $d\colon \mathrm{C}Y \to T$. Therefore $(u|v) = \gamma_Y d$ and $u \sim v$. $\square$

2.2.16 Corollary. *The natural maps γ^0 and γ^1 have their components in $\mathcal{C} \cap \mathcal{W}$. The natural map σ has its components in $\mathcal{W}$.*

Proof. For any object X we have $\gamma_X^k \in {}^{\square}(\Lambda^{\square}) \subseteq \mathcal{C} \cap \mathcal{W}$ by Lemma 2.2.4 and Lemma 2.2.15. The 2-3 property of $\mathcal{W}$ then implies $\sigma_X \in \mathcal{W}$. $\square$

The two implications obtained in Lemma 2.2.9 and in Corollary 2.2.10 can be strengthened to equivalences under some conditions.

2.2.17 Lemma. *Suppose $f \in \Lambda^{\square}$. Then*

$$f \in \mathcal{C}^{\square} \iff f \textit{ is a dual strong deformation retract}$$

Proof. The direction "$\Rightarrow$" is Lemma 2.2.9. For the direction "$\Leftarrow$", assume $f\colon X \to Y$ to be a strong dual deformation retract with maps $g\colon Y \to X$ and $h\colon CX \to X$ as in diagram (2.2.2), i.e. $gf = X$, $(X|fg) = \gamma_X h$ and $hf = \sigma_X f$. Any commutative square

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "c"'] & X \arrow[d, "f"] \\
L \arrow[r, "v"'] & Y
\end{tikzcd}$$

with $c \in \mathcal{C}$ gives rise to the following diagram

$$\begin{tikzcd}
 & X \arrow[r, "\gamma^1_X"] & \mathrm{C}X \arrow[ddr, "h"] & & \\
K \arrow[ur, "u"] \arrow[r, "\gamma^1_K"] \arrow[d, "c"'] & \mathrm{C}K \arrow[ur, "\mathrm{C}u"'] \arrow[d, "p"] & & & \\
L \arrow[r, "q"'] \arrow[dr, "\gamma^1_L"'] & P \arrow[rr, "x"] & & X \arrow[dr, "f"] & \\
 & \mathrm{C}L \arrow[r, "\sigma_L"'] & L \arrow[r, "v"'] & Y \arrow[u, "g"] \arrow[r, equal] & Y
\end{tikzcd}$$

where P is the pushout of c and γ^1_K and $x\colon P \to X$ is induced by $\gamma^1_K(\mathrm{C}u)h = u\gamma^1_X h = ufg = cvg$. Testing against p and q gives the commutativity of the lower right square in

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[dr, "\gamma^0_K"] \arrow[dd, "c"'] & X \arrow[r, "\gamma^0_X"] & \mathrm{C}X \arrow[drr, "h"] & & \\
 & \mathrm{C}K \arrow[ur, "\mathrm{C}u"'] \arrow[r, "p"'] & P \arrow[rr, "x"] \arrow[d, "c\star\gamma^1"] & & X \arrow[d, "f"] \\
L \arrow[rr, "\gamma^0_L"'] & & \mathrm{C}L \arrow[urr, dotted, "d"] \arrow[r, "\sigma_L"'] & L \arrow[r, "v"'] & Y
\end{tikzcd}$$

and hence $(c\star\gamma^1) \boxslash f$ gives a diagonal $d\colon \mathrm{C}L \to X$. The outer diagram then shows that $d' := \gamma^0_L d\colon L \to Y$ is the desired diagonal. $\square$

2.2.18 Lemma. *Suppose $f \in \Lambda^{\boxslash}$ with fibrant codomain. Then*

$$f \in \mathcal{C}^{\boxslash} \iff f \in \mathcal{W}$$

Proof. The direction "$\Rightarrow$" is Corollary 2.2.10. For the direction "$\Leftarrow$", assume $f\colon X \to Y \in \mathcal{W}$ and Y fibrant. By Lemma 2.2.17, it is sufficient to show that f is a dual strong deformation retract. We will construct $g\colon Y \to X$ and $h\colon \mathrm{C}X \to X$, such that the equations in diagram (2.2.2) are satisfied.

Because f and $(Y \to 1)$ are in $\Lambda^{\boxslash}$, the same holds for $(X \to 1)$. By Corollary 2.2.14 there exists a $g\colon Y \to X$ with $X \sim fg$ and $Y \sim gf$. Let $k\colon \mathrm{C}X \to X$ be the homotopy from X to fg.

(1) *One may assume $Y = gf$.* Consider the following diagram

$$
\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[d, "\gamma^1_Y"'] & X \arrow[d, "f"] \\
Y \arrow[r, "\gamma^0_Y"] & \mathrm{C}Y \arrow[r] \arrow[ur, dotted, "d"] & Y
\end{tikzcd}
$$

where the right square comes from $Y \sim gf$. The diagonal $d\colon \mathrm{C}Y \to X$ exists because $\gamma^1_Y \in {}^{\square}(\Lambda^{\square})$ by Lemma 2.2.4. Let $g' := \gamma^0_Y d$. Then $g'f = Y$ and $(g'|g) = \gamma_Y d$. Hence $X \sim fg \sim fg'$ and by Lemma 2.2.12 we have $X \sim fg'$ via some homotopy k'. Now replace g and k by g' and k'.

(2) *There are maps $x\colon \mathrm{C}X + \mathrm{C}X \underset{X+X}{+} \mathrm{C}X \to X$ and $d\colon \mathrm{CC}X \to X$ such that the following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{C}X + \mathrm{C}X \underset{X+X}{+} \mathrm{C}X \arrow[rrr, "x"] \arrow[d, "\gamma^1_X \star \gamma_X"'] & & & X \arrow[d, "f"] \\
\mathrm{CC}X \arrow[r, "\sigma_{\mathrm{C}X}"'] \arrow[urrr, "d"] & \mathrm{C}X \arrow[r, "k"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}
\tag{2.2.3}
$$

The equation

$$
\begin{aligned}
(\gamma^1_X + \gamma^1_X)(k|kfg) &= (\gamma^1_X k|\gamma^1_X kfg) \\
&= (fg|fgfg) \\
&= (X|X)fg \\
&= \gamma_X \sigma_X fg
\end{aligned}
$$

induces x in the following diagram

$$
\begin{tikzcd}
X + X \arrow[r, "\gamma_X"] \arrow[d, "\gamma^1_X + \gamma^1_X"'] & \mathrm{C}X \arrow[r, "\sigma_X"] \arrow[d, "j"'] & X \arrow[d, "f"] \\
\mathrm{C}X + \mathrm{C}X \arrow[r, "i"] \arrow[drr, bend right, "(k|kfg)"'] & Q \arrow[dr, "x"] & Y \arrow[d, "g"] \\
 & & X
\end{tikzcd}
$$

where Q is the pushout of $\gamma^1_X + \gamma^1_X$ and γ_X with coprojections $i\colon \mathrm{C}X + \mathrm{C}X \to Q$ and $j\colon \mathrm{C}X \to Q$. The commutativity of the outer rectangle of diagram (2.2.3) now follows from the following two equations

$$\begin{aligned} ixf &= (k|kfg)f \\ &= (kf|kf) \\ &= (\mathrm{C}X|\mathrm{C}X)kf \\ &= \gamma_{\mathrm{C}X}\sigma_{\mathrm{C}X}kf \\ &= i(\gamma^1_X \star \gamma_X)\sigma_{\mathrm{C}X}kf \end{aligned}$$

$$\begin{aligned} jxf &= \sigma_X fgf \\ &= \sigma_X \gamma^1_X kf \\ &= \mathrm{C}(\gamma^1_X)\sigma_{\mathrm{C}X}kf \\ &= j(\gamma^1_X \star \gamma_X)\sigma_{\mathrm{C}X}kf \end{aligned}$$

Finally the existence of the diagonal d in diagram (2.2.3) follows from $\gamma^1_X \star \gamma_X \in {}^{\square}(\Lambda^{\square})$.

(3) *With x and d as in (2), let $h := (\mathrm{C}\gamma^0_X)d\colon \mathrm{C}X \to X$. Then the following diagram commutes:*

$$\begin{tikzcd} X+X \arrow[rr, "(X|fg)"] \arrow[d, "\gamma_X"'] & & X \arrow[d, "f"] \\ \mathrm{C}X \arrow[urr, "h"] \arrow[r, "\sigma_X"'] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

The lower triangle is the equation

$$(\mathrm{C}\gamma^0_X)df = (\mathrm{C}\gamma^0_X)\sigma_{\mathrm{C}X}kf = \sigma_X\gamma^0_X kf = \sigma_X f$$

The upper triangle is the equation

$$\begin{aligned} \gamma_X(\mathrm{C}\gamma^0_X)d &= (\gamma^0_X + \gamma^0_X)\gamma_{\mathrm{C}X}d \\ &= (\gamma^0_X + \gamma^0_X)i(\gamma^1_X \star \gamma_X)d \\ &= (\gamma^0_X + \gamma^0_X)ix \\ &= (\gamma^0_X + \gamma^0_X)(k|kfg) \\ &= (\gamma^0_X k|\gamma^0_X kfg) \\ &= (X|fg) \end{aligned}$$

Altogether, h and g satisfy the equations in diagram (2.2.2). □

2.2.19 Corollary. *Let $f\colon X \to Y \in \mathcal{C}$ with fibrant codomain. Then*

$$f \in \mathcal{W} \iff f \in {}^{\square}(\Lambda^{\square})$$

Proof. The direction "$\Leftarrow$" is Lemma 2.2.15. For the direction "$\Rightarrow$", suppose $f \in \mathcal{W}$. Factor f as ip with $i \in {}^{\square}(\Lambda^{\square})$ and $p \in \Lambda^{\square}$. Then p satisfies the condition of the previous Lemma and hence

$$f \in \mathcal{W} \iff p \in \mathcal{W} \iff p \in \mathcal{C}^{\square}$$

so that in particular $f \mathbin{\square} p$. Therefore f is a retract of i and lies in ${}^{\square}(\Lambda^{\square})$. $\square$

2.2.20 Lemma. *Let $\mathcal{N} = \{p \in \Lambda^{\square} \mid p$ has a fibrant codomain$\}$. Then*

(a) $\mathcal{C} \cap \mathcal{W} = \mathcal{C} \cap {}^{\square}\mathcal{N}$.

(b) $\mathcal{N} \subseteq (\mathcal{C} \cap \mathcal{W})^{\square}$

(c) $(X \to 1) \in \Lambda^{\square} \iff (X \to 1) \in (\mathcal{C} \cap \mathcal{W})^{\square}$

Proof. First observe that because of ${}^{\square}(\Lambda^{\square}) \subseteq \mathcal{C} \cap \mathcal{W}$ (Lemma 2.2.15 together with condition (b) of Definition 2.1.9) we have $\Lambda^{\square} \supseteq (\mathcal{C} \cap \mathcal{W})^{\square}$ and hence the implication "$\Leftarrow$" in (c) always holds. The implication "$\Rightarrow$" in (c) follows from (b). Moreover, (a) implies (b) via $\mathcal{C} \cap {}^{\square}\mathcal{N} \subseteq {}^{\square}\mathcal{N}$. So it is enough to show (a). Let $c\colon K \to L$ be any map in $\mathcal{C}$. Factor $(L \to 1)$ through some $u\colon L \to L'$ with $u \in {}^{\square}(\Lambda^{\square})$ and L' fibrant. Then in particular $u \in \mathcal{C}$ with fibrant codomain and hence $u \in \mathcal{W}$ by Corollary 2.2.19. Therefore

$$c \in \mathcal{W} \iff cu \in \mathcal{W} \iff cu \in {}^{\square}(\Lambda^{\square}) \tag{$*$}$$

where the second equivalence again results from Corollary 2.2.19.

(1) Suppose $c \in \mathcal{W}$. Consider any $p \in \mathcal{N}$ and maps $x\colon K \to X$ and $y\colon L \to Y$ as in the following diagram:

$$\begin{tikzcd} K \arrow[r, "c"] \arrow[d, "x"'] & L \arrow[r, "u"] \arrow[d, "y"] & L' \arrow[dll, dotted, "d"'] \arrow[dl, dotted, "y'"] \\ X \arrow[r, "p"'] & Y & \end{tikzcd}$$

Then $y'\colon L' \to Y$ exists because $u \mathbin{\square} (Y \to 1)$ and $d\colon L' \to X$ exists because of the above $(*)$. The equations $cud = x$ and $udp = uy' = y$ then exhibit $ud\colon L \to X$ as the desired diagonal.

(2) Suppose $c \in {}^{\square}\mathcal{N}$. Factor cu as $cu = xp$ with $x \in {}^{\square}(\Lambda^{\square})$ and $p \in \Lambda^{\square}$. Because u has fibrant codomain, the same holds for p and hence $p \in \mathcal{N}$. Because $u \in {}^{\square}(\Lambda^{\square}) \subseteq {}^{\square}\mathcal{N}$, also $cu \in {}^{\square}\mathcal{N}$. Therefore cu is a retract of p and hence $cu \in {}^{\square}(\Lambda^{\square}) \subseteq \mathcal{W}$. Now by $(*)$ above, $c \in \mathcal{W}$. $\square$

2.2.21 Corollary. *$\mathcal{C} \cap \mathcal{W}$ is stable under pushouts, transfinite composition and retracts.*

Proof. By part (a) of the previous Lemma, $\mathcal{C} \cap \mathcal{W}$ can be expressed as the intersection of two classes, each of which is stable under these operations. $\square$

It now remains to verify condition 1.3.9(4). We want to express $\mathcal{W}$ as the full preimage (under some accessible functor) of the class of homotopy equivalences with respect to some final cylinder. Observe that the cylinder used in the construction may not be final.

2.2.22 Lemma. *There is a final refinement* $(\mathrm{C}', \gamma', \sigma')$ *of* $(\mathrm{C}, \gamma, \sigma)$ *such that for any two maps* $f, g\colon X \to Y$ *with fibrant codomain we have*

$$f \sim g \quad (\mathrm{mod}\ \mathrm{C}') \iff f \sim g \quad (\mathrm{mod}\ \mathrm{C})$$

In particular, the two cylinders agree on the notion of homotopy equivalences between fibrant objects.

Proof. Let $\sigma = \lambda\rho$ be a functorial $(\mathcal{C}, \mathcal{C}^{\square})$-factorization of σ and for each object X set $\mathrm{C}'X = \mathrm{cod}(\lambda_X)$, $\gamma'_X = \gamma_X\lambda_X$ and $\sigma'_X = \rho_X$. Then $(\mathrm{C}', \gamma', \sigma')$ is a final refinement of $(\mathrm{C}, \gamma, \sigma)$ and the direction "$\Rightarrow$" was already noted in part (c) of Remark 1.2.2.

For the direction "$\Leftarrow$", the argument is similar to the one in Lemma 1.2.4. Assume $f \sim g$ (mod C) for maps $f, g\colon X \to Y$ with Y fibrant. Let $h\colon \mathrm{C}X \to Y$ be a homotopy from f to g and consider the square:

$$\begin{tikzcd} \mathrm{C}X \arrow[r, "h"] \arrow[d, "\lambda_X"'] & Y \arrow[d] \\ \mathrm{C}'X \arrow[r] & 1 \end{tikzcd}$$

Corollary 2.2.10 gives $\rho_X \in \mathcal{C}^{\square} \subseteq \mathcal{W}$ and Corollary 2.2.16 gives $\lambda_X\rho_X = \sigma_X \in \mathcal{W}$. Therefore the 2-3 property of $\mathcal{W}$ forces $\lambda_X \in \mathcal{W}$ and hence $\lambda_X \in \mathcal{C} \cap \mathcal{W}$. By part (c) of Lemma 2.2.20 we have $(Y \to 1) \in (\mathcal{C} \cap \mathcal{W})^{\square}$. This gives the desired diagonal $d\colon \mathrm{C}'X \to Y$ of the above square, establishing $f \sim g$ (mod C'). □

2.2.23 Corollary. *The class $\mathcal{W}$ satisfies the solution set condition.*

Proof. By Lemma 1.3.7, it is sufficient to exhibit $\mathcal{W}$ as the full image of some accessible functor. Let $L\colon \mathcal{K} \to \mathcal{K}$ be the fibrant replacement functor given by the weak factorization system $({}^{\square}(\Lambda^{\square}), \Lambda^{\square})$, which is accessible by part (b) of Theorem 1.3.5. Via composition, L induces a functor $L_*\colon \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{2}}$, which is also accessible because colimits in $\mathcal{K}^{\mathbf{2}}$ are calculated pointwise.

Let $f\colon X \to Y$ be any map.

(1) $f \in \mathcal{W} \iff Lf \in \mathcal{W}$

Consider the square

$$\begin{tikzcd} X \arrow[r, "\ell_X"] \arrow[d, "f"'] & LX \arrow[d, "Lf"] \\ Y \arrow[r, "\ell_Y"] & LY \end{tikzcd}$$

where $\ell_X, \ell_Y \in {}^{\square}(\Lambda^{\square})$ are given by the functorial factorization. By Lemma 2.2.15 ℓ_X and ℓ_X lie in $\mathcal{W}$. Now the 2-3 property of $\mathcal{W}$ gives the above equivalence.

(2) $Lf \in \mathcal{W} \iff Lf$ *is a homotopy equivalence* (mod C)

By construction, Lf has fibrant domain and codomain. The equivalence now follows from Corollary 2.2.14.

Let $(\mathrm{C}', \gamma', \sigma')$ be a final refinement of $(\mathrm{C}, \gamma, \sigma)$ as in the previous Lemma. Then point (2) still remains valid with C' in place of C. Therefore $\mathcal{W}$ is the preimage, under the accessible functor L_*, of the class of homotopy equivalences determined by C'. By part (c) of Theorem 1.3.5 that class is the full image of an accessible functor. It is also isomorphism-closed. Hence the same holds for $\mathcal{W}$ by Lemma 1.3.4. □

Proof of Theorem 2.2.5. By Remark 2.2.2, Corollary 2.2.10, Corollary 2.2.21 and Lemma 2.2.23, the classes $\mathcal{C}$ and $\mathcal{W}$ satisfy the conditions of Smith's Theorem 1.3.9. □

2.3 Left determination

The following Lemma and Theorem are adapted from [4, Proposition 3.8] and [4, Théorème 3.9]. They provide the connection between the weak equivalences produced by Cisinski's construction and the smallest localizers containing a given set of maps.

2.3.1 Lemma. *Let $(\mathcal{C}, \mathcal{C}^\square)$ be a cofibrant weak factorization system in $\mathcal{K}$, generated by a subset $I \subseteq \mathcal{C}$. Let $(\mathrm{C}, \gamma, \sigma)$ be a cartesian cylinder and let $S \subseteq \mathcal{C}$ be a set of maps. Then $\mathcal{W}(\mathrm{C}, S, I) = \mathcal{W}_\mathcal{C}(\Lambda(\mathrm{C}, S, I))$.*

Proof. We will again write Λ for $\Lambda(\mathrm{C}, S, I)$ and $\mathcal{W}$ for $\mathcal{W}(\mathrm{C}, S, I)$. The inclusion $\mathcal{W}_\mathcal{C}(\Lambda) \subseteq \mathcal{W}$ holds because $\Lambda \subseteq \mathcal{W}$ by Lemma 2.2.15.

Now given any $f \colon X \to Y \in \mathcal{W}$, use $(\mathrm{cell}(\Lambda), \Lambda^\square)$-factorizations of $(X \to 1)$ and $(Y \to 1)$ to obtain a diagram

$$\begin{tikzcd}
X \arrow[r, "\ell_X"] \arrow[dd, "f"'] & X' \arrow[dd, "f'"'] \arrow[dr, "z"] & \\
& & Z \arrow[dl, "y"] \\
Y \arrow[r, "\ell_Y"'] & Y' &
\end{tikzcd}$$

where ℓ_X and ℓ_Y are in $\mathrm{cell}(\Lambda)$, X' and Y' are fibrant, f' is induced by this factorization and $f' = zy$ is in turn a factorization with $z \in \mathrm{cell}(\Lambda)$ and $y \in \Lambda^\square$. In particular ℓ_X, ℓ_Y and z are in $\mathcal{W}_\mathcal{C}(\Lambda)$. Then the 2-3 property gives

$$f \in \mathcal{W} \implies y \in \mathcal{W} \iff y \in \mathcal{C}^\square \implies y \in \mathcal{W}_\mathcal{C}(\Lambda) \implies f \in \mathcal{W}_\mathcal{C}(\Lambda)$$

where the equivalence in the middle is given by Lemma 2.2.18 □

2.3.2 Theorem. *Let $(\mathcal{C}, \mathcal{C}^{\square})$ be a cofibrant weak factorization system in $\mathcal{K}$ and S be an arbitrary set of maps (not necessarily included in $\mathcal{C}$). Suppose that $(\mathrm{C}, \gamma, \sigma)$ is a cartesian cylinder such that all components of σ lie in $\mathcal{W}_{\mathcal{C}}(S)$. Then, setting $\mathcal{W} := \mathcal{W}_{\mathcal{C}}(S)$ and $\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\square}$ gives a cofibrant model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{K}$. Also $\mathcal{W}_{\mathcal{C}}(S) = \mathcal{W}^{s}_{\mathcal{C}}(S)$.*

Proof. First observe, that one may assume $S \subseteq \mathcal{C}$: factor each $s \in S$ as $s = c_s r_s$ with $c_s \in \mathcal{C}$ and $r_s \in \mathcal{C}^{\square}$ and consider $S' := \{c_s \mid s \in S\}$. Any given localizer contains S if and only if it contains S', because all the r_s lie in it. Therefore $\mathcal{W}_{\mathcal{C}}(S') = \mathcal{W}_{\mathcal{C}}(S)$.

Now assume $S \subseteq \mathcal{C}$. Let I be some generating subset of $\mathcal{C}$. By the previous Lemma, it is enough to show $\mathcal{W}_{\mathcal{C}}(\Lambda(\mathrm{C}, S, I)) = \mathcal{W}_{\mathcal{C}}(S)$. We will write $\Lambda(S)$ for $\Lambda(\mathrm{C}, S, I)$.

The inclusion $S \subseteq \Lambda(S)$ already forces $\mathcal{W}_{\mathcal{C}}(S) \subseteq \mathcal{W}_{\mathcal{C}}(\Lambda(S))$ and therefore it remains to show $\Lambda(S) \subseteq \mathcal{W}_{\mathcal{C}}(S)$.

By assumption, the components of σ lie in $\mathcal{W}_{\mathcal{C}}(S)$. Consequently the components of γ^0 and γ^1 lie in $\mathcal{C} \cap \mathcal{W}_{\mathcal{C}}(S)$. We will now show $\Lambda^n(S) \subseteq \mathcal{W}_{\mathcal{C}}(S)$ for all $n \geq 0$.

(1) We already have $S \subseteq \mathcal{W}_{\mathcal{C}}(S)$. Let $f\colon X \to Y$ be in $\mathcal{C}$ and consider the following diagram used for the definition of $f \star \gamma^0$

$$\begin{tikzcd}
X \arrow[r, "\gamma^0_X"] \arrow[d, "f"'] & \mathrm{C}X \arrow[d, "p"] \arrow[ddr, bend left, "\mathrm{C}f"] & \\
Y \arrow[r, "q"'] \arrow[drr, bend right, "\gamma^0_Y"'] & Q \arrow[dr, "f\star\gamma^0"] & \\
 & & \mathrm{C}Y
\end{tikzcd}$$

where Q is the pushout of f and γ^0_X. Because $\gamma^0_X \in \mathcal{C} \cap \mathcal{W}_{\mathcal{C}}(S)$ we have $q \in \mathcal{W}_{\mathcal{C}}(S)$. Together with $\gamma^0_Y \in \mathcal{W}_{\mathcal{C}}(S)$ this gives $f \star \gamma^0 \in \mathcal{W}_{\mathcal{C}}(S)$. In the same way $f \star \gamma^1 \in \mathcal{W}_{\mathcal{C}}(S)$. Hence $I \star \gamma^0$ and $I \star \gamma^1$ are contained in $\mathcal{W}_{\mathcal{C}}(S)$

(2) Assume $\Lambda^n(S) \subseteq \mathcal{W}_{\mathcal{C}}(S)$ and let $f\colon X \to Y$ be in $\Lambda^n(S)$. By assumption $f \in \mathcal{W}_{\mathcal{C}}(S)$ and hence f lies in $\mathcal{C} \cap \mathcal{W}_{\mathcal{C}}(S)$. Then the same holds for $f + X$ and $Y + f$ (being pushouts of f), as for their composition $f + f = (f + X)(Y + f)$. Moreover $f \in \mathcal{C} \cap \mathcal{W}_{\mathcal{C}}(S)$ together with $\gamma^0_X, \gamma^0_Y \in \mathcal{W}_{\mathcal{C}}(S)$ force $\mathrm{C}f \in \mathcal{W}_{\mathcal{C}}(S)$ by the 2-3 property. Altogether, in the following diagram used for the definition of $f \star \gamma$

$$\begin{tikzcd}
X+X \arrow[r, "\gamma_X"] \arrow[d, "f+f"'] & \mathrm{C}X \arrow[d, "r"] \arrow[ddr, bend left, "\mathrm{C}f"] & \\
Y+Y \arrow[r] \arrow[drr, bend right, "\gamma_Y"'] & Q \arrow[dr, "f\star\gamma"] & \\
 & & \mathrm{C}Y
\end{tikzcd}$$

both maps r and $\mathrm{C}f$ lie in $\mathcal{W}_{\mathcal{C}}(S)$, and hence $f \star \gamma \in \mathcal{W}_{\mathcal{C}}(S)$. $\square$

In view of Corollary 2.2.16 it is clear that the condition of σ having its components in $\mathcal{W}_{\mathcal{C}}(S)$ cannot be omitted from the Theorem. This condition will always be satisfied (regardless of the $\mathcal{W}_{\mathcal{C}}(S)$ in question) whenever the cylinder is final, i.e. when σ has its components in $\mathcal{C}^{\square}$.

2.3.3 Corollary. *Let $(\mathcal{C}, \mathcal{C}^{\square})$ be a cofibrant weak factorization system in $\mathcal{K}$ and suppose that there is a final cartesian cylinder for $(\mathcal{C}, \mathcal{C}^{\square})$. Then $\mathcal{C}$, $\mathcal{W}_{\mathcal{C}}(S)$ and $(\mathcal{C} \cap \mathcal{W}_{\mathcal{C}}(S))^{\square}$ form a cofibrantly generated model structure. In particular for $S = \emptyset$, the construction of Theorem 2.2.5 gives a left determined model structure.*

2.3.4 Remark. The above results also show, that the construction of the model structure from $(\mathcal{C}, \mathcal{C}^{\square})$ and S does not depend on the choice of the cylinder $(\mathrm{C}, \gamma, \sigma)$ as long as $\sigma_X \in \mathcal{W}_{\mathcal{C}}(S)$ is satisfied. For example, if the underlying category is distributive and if the class $\mathcal{C}$ is stable under pullbacks along product projections, then any factorization of the codiagonal $(1|1)\colon 2 = 1 + 1 \to 1$ as a composition of some $g\colon 2 \to V$ and $s\colon V \to 1$ with $g \in \mathcal{C}$ and $s \in \mathcal{C}^{\square}$ will provide a final cylinder with $\mathrm{C} = (-) \times V$, $\gamma = (-) \times g$ and $\sigma = (-) \times s$. If V is exponentiable then C is a left adjoint.

2.3.5 Example. Let $\top\colon 1 \to \Omega$ be the subobject classifier of a Grothendieck topos $\mathcal{E}$ and let $\perp\colon 1 \to \Omega$ be the characteristic map of $0 \to 1$, which means that $\perp$ is the uniquely determined map in the pullback:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\ 1 \arrow[r, "\perp"'] & \Omega \end{tikzcd}$$

Then the induced map $(\perp|\top)\colon 1 + 1 \to \Omega$ is a monomorphism (this is just another instance of Diagram (2.1.4)). Since Ω is injective, this gives a $(\mathrm{Mono}, \mathrm{Mono}^{\square})$-factorization of the codiagonal $(1|1)\colon 1 + 1 \to 1$. Therefore $(-) \times \Omega$ gives a final cylinder and the natural map γ is given as $(-) \times (\perp, \top)$.

Because $\mathcal{E}$ is cartesian closed, $(-) \times \Omega$ is a left adjoint and it clearly preserves monomorphisms. By Corollary 2.1.12, the resulting cylinder is cartesian.

3 Model structures from balls

In this chapter we present an abstract version of the construction given by Lafont, Métayer and Worytkiewicz [14] as outlined in the introduction. The generating cofibrations and the weak equivalences in spe are introduced in Section 1, together with properties that hold without any further assumptions. In Section 2 we describe the conditions on cocylinders that are needed for the proof that these classes indeed form a model structure. The proof itself is carried out in Section 3. Finally, we reconnect this model structure with Cisinski's construction in Section 4.

Familiarity with ω-categories is not necessary to understand the contents of this chapter, but some pointers to the corresponding notions in [14][1] are provided.

3.1 Balls and relative homotopy lifting

The following Definition is modelled after [14, Section 4, Introduction].

3.1.1 Definition. A **system of balls** is a family of objects B^n ($n \in \mathbb{N}$) and maps $b_n^0, b_n^1 \colon B^n \to B^{n+1}$ satisfying the relations $b_n^i b_{n+1}^j = b_n^i b_{n+1}^k$. Given such a system, we inductively define a family of **spheres** S^{n-1} together with maps $s_n \colon S^{n-1} \to B^n$ satisfying $s_n b_n^0 = s_n b_n^1$:

(i) For $n = -1$, set $S^{\text{-}1} = 0$ and let $s_n \colon 0 \to B^0$ be the unique map from the initial object. Then $s_0 b_0^0 = s_0 b_0^1$ holds.

(ii) Assume that s_n is constructed and satisfies $s_n b_n^0 = s_n b_n^1$. Let S^n be the pushout of s_n with itself and let s_{n+1} be the map $(b_n^0|b_n^1) \colon S^n \to B^{n+1}$ induced by b_n^0 and b_n^1. In the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "s_n"] \arrow[d, "s_n"'] & B^n \arrow[d, "\iota_n^1"'] \arrow[dr, "b_n^1"] & \\
B^n \arrow[r, "\iota_n^0"] \arrow[dr, "b_n^0"'] & S^n \arrow[r, "s_{n+1}"] \arrow[d, "s_{n+1}"] & B^{n+1} \arrow[d, "b_{n+1}^1"] \\
& B^{n+1} \arrow[r, "b_{n+1}^0"'] & B^{n+2}
\end{tikzcd}$$

[1] at the time of writing (10/2009) [14] is still in preprint form, so the exact locations may be subject to change.

the commutativity of the lower right square can be checked against the pushout injections ι_n^0 and ι_n^1. So again $s_{n+1}b_{n+1}^0 = s_{n+1}b_{n+1}^1$.

A map $x\colon B^n \to X$ is also called an n-**cell** of X. Two cells $x, y\colon B^n \to X$ are said to be **parallel** (written as $x \mid y$) if $s_n x = s_n y$. Observe that any two 0-cells are parallel. By construction of S^n, such a pair corresponds to the induced map $(x|y)\colon S^n \to X$ which we also call a parallel pair of cells.

We now assume that a system of balls is given such that all balls B^n are small with respect to cell(I), where $I = \{s_n\colon S^{n-1} \to B^n \mid n \geq 0\}$ (observe that this is automatically satisfied in a locally presentable category). We will call such a system a **system of small balls** for short. It then follows that the S^n are also small with respect to cell(I). Therefore setting $\mathcal{C} = {}^{\square}(I^{\square})$ gives a cofibrantly generated weak factorization system $(\mathcal{C}, \mathcal{C}^{\square})$ in which moreover all balls and spheres are cofibrant.

For each $n \geq 0$, we can chose some $(P^n, \gamma_n, \sigma_n)$ as final cylinder object for B^n relative to $s_n\colon S^{n-1} \to B^n$ as described in Definition 1.2.5. By Remark 1.2.2(a) these cylinder objects are also cofibrant. Also note, that by Remark 1.2.2(c) the resulting relative homotopy relation between parallel cells does not depend on the choice of the cylinder objects.

For lighter notation we will often drop subscripts from the maps s_n, b_n^i, γ_n and σ_n.

3.1.2 Remark. There is an automorphism $t_n\colon S^n \to S^n$ that interchanges the pushout injections ι_n^0 and ι_n^1 and hence satisfies $t(B^n|B^n) = (B^n|B^n)$ and $t(x|y) = (y|x)$ for parallel cells x and y. Consequently, the homotopy relation is also symmetric.

3.1.3 Lemma. *For a commutative square*

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ B^n \arrow[r, "v"'] & Y \end{tikzcd} \tag{3.1.1}$$

the following conditions are equivalent

(1) There is a map $d\colon B^n \to X$ that extends u along s such that $df \sim v$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ B^n \arrow[r, "v"'] \arrow[ur, "d"] \arrow[ur, phantom, "\sim", shift right=3ex] & Y \end{tikzcd} \tag{3.1.2}$$

(2) There is a map $d\colon B^n \to X$ that extends u along s and a map $h\colon P^n \to Y$ that extends $(df|v)\colon S^n \to Y$ along $\gamma\colon S^n \to P^n$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & X \\ B^n \arrow[ur, "d"'] & \end{tikzcd} \qquad \begin{tikzcd} S^n \arrow[r, "(df|v)"] \arrow[d, "\gamma"'] & Y \\ P^n \arrow[ur, "h"'] & \end{tikzcd} \tag{3.1.3}$$

(3) *There are maps $d\colon B^n \to X$ and $h\colon P^n \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} S^{n-1} \arrow[r, "s"'] \arrow[d, "s"'] \arrow[rr, bend left, "u"] & B^n \arrow[r, "d"'] \arrow[d, "\gamma^0"] & X \arrow[d, "f"] \\ B^n \arrow[r, "\gamma^1"] \arrow[rr, bend right, "v"'] & P^n \arrow[r, "h"] & Y \end{tikzcd} \tag{3.1.4}$$

In this case the square (3.1.1) *is said to have a* ***relative homotopy lifting****.*

Proof. Conditions (1) and (2) differ only in that the homotopy h is explicitly mentioned in (2). Condition (3) is an encoding of the two diagrams of (2) into one diagram. $\square$

3.1.4 Remark. Dugger and Isaksen [5, Definition 3.1] have introduced this notion of relative homotopy lifting for simplicial sets. There is also a version of Lemma 3.1.3 for the transitive closure $\approx$ of $\sim$. For $k \geq 1$, let $\mathcal{D}_k$ be the set $\{0, \dots, 2k\}$ equipped with the order relation $d < d' \iff (2|d \wedge |d-d'| = 1)$:

$$\begin{tikzcd} 0 \arrow[dr] & & 2 \arrow[dl] & \cdots & 2k-2 \arrow[dr] & & 2k \arrow[dl] \\ & 1 & & \cdots & & 2k-1 & \end{tikzcd} \tag{3.1.5}$$

For $n \geq 0$ let $F\colon \mathcal{D}_k \to \mathcal{K}$ and $G\colon \mathcal{D}_k \to \mathcal{K}$ be the functors with

$$F(d) = \begin{cases} B^n & \text{, if } d \text{ is even} \\ S^n & \text{, if } d \text{ is odd} \end{cases} \qquad G(d) = \begin{cases} B^n & \text{, if } d \text{ is even} \\ P^n & \text{, if } d \text{ is odd} \end{cases}$$

on objects and

$$\begin{aligned} F(d, d+1) &= \iota^0 & \qquad G(d, d+1) &= \gamma^0 \\ F(d+1, d) &= \iota^1 & \qquad G(d+1, d) &= \gamma^1 \end{aligned}$$

on morphisms. Set $S^n(k) = \operatorname{colim} F$ and $P^n(k) = \operatorname{colim} G$. Let $\varphi\colon F \to G$ be the natural map with $\varphi_d = B^n$ for even d and $\varphi_d = \gamma$ for odd d and define $\gamma(k)\colon S^n(k) \to P^n(k)$ as the map induced between the colimits.

Then one can show that maps from $S^n(k)$ to X correspond to sequences $x_0|\dots|x_k$ of $k+1$ parallel n-cells of X, that maps from $P^n(k)$ to X correspond to sequences $x_0 \sim \dots \sim x_k$ of $k+1$ parallel n-cells of X such that consecutive cells are homotopic, and that for two parallel n-cells $x, x'\colon B^n \to X$ we have $x \approx y$ iff there is some sequence $(x|\dots|x')\colon S^n(k) \to X$ that factors through $\gamma(k)\colon S^n(k) \to P^n(k)$.

Conditions (2) and (3) in Lemma 3.1.3 can then be rephrased with $\gamma(k)$ in place of γ, where γ^1 and γ^0 are replaced by the leftmost and rightmost injections from B^n into the colimit $P^n(k)$.

3.1.5 Definition. We say that a map $f\colon X \to Y$ has the **relative homotopy lifting property** with respect to $s\colon S^{n-1} \to B^n$ if the above square (3.1.1) has a relative homotopy lifting for every choice of u, v. We let $\mathcal{W}$ be the class of those maps that have the relative homotopy lifting property with respect to all maps in $I = \{s_n\colon S^{n-1} \to B^n \mid n \geq 0\}$.

3.1.6 Remark. In [14], the class $\mathcal{W}$ in the above definition is introduced via a different method: first, the notion of ω-equivalence of parallel cells is defined [14, Subsection 4.2, Definition 6]. The resulting relation $\sim$ is then used to define $\mathcal{W}$ as in part (1) of Lemma 3.1.3 [14, Subsection 4.2, Definition 8]. Only later it is proved that this relation coincides with relative homotopy given by final relative cylinder objects [14, Subsection 4.7, Lemma 18]. The relative homotopy lifting property then appears in [14, Subsection 4.2, Proposition 8].

3.1.7 Lemma. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps and suppose $g \in \mathcal{C}^{\Box}$. Then $fg \in \mathcal{W} \iff f \in \mathcal{W}$*

Proof. For the direction "$\Rightarrow$", consider diagram (3.1.1) in Lemma 3.1.3. Because of $fg \in W$ there is an extension $d\colon B^n \to X$ of u along s, such that $dfg \sim vg$ via some map $h\colon P^n \to Z$ as in diagram (3.1.3) (with fg and vg in place of f and v respectively). The resulting square

$$\begin{tikzcd} S^n \arrow[r, "(df|v)"] \arrow[d, "\gamma"'] & X \arrow[d, "g"] \\ P^n \arrow[r, "h"'] & Z \end{tikzcd}$$

has a diagonal and hence $df \sim v$.

For the direction "$\Leftarrow$" the maps v' and d in the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ B^n \arrow[r, "v'"'] \arrow[ur, "d"] \arrow[dr, "v"'] & Y \arrow[d, "g"] \arrow[ul, phantom, "\sim" near start] \\ & Z \end{tikzcd}$$

are obtained by first lifting v through g, using $s \,\Box\, g$, and then applying $f \in \mathcal{W}$ to the resulting square (3.1.1). Because homotopy is preserved by postcomposition, $df \sim v'$ gives $dfg \sim v'g = v$. $\square$

3.1.8 Corollary. *The maps $\gamma^0, \gamma^1\colon B^n \to P^n$ lie in $\mathcal{C} \cap \mathcal{W}$.* $\square$

3.2 Cocylinders with homotopy exchange

We keep $I = \{s_n : S^{n-1} \to B^n \mid n \geq 0\}$, $\mathcal{C} = {}^{\square}(I^{\square})$ and $\mathcal{W}$ from the previous section. The following definitions are modelled after [14, Subsection 4.4] where a fibrant cocylinder for $(\mathcal{C}, \mathcal{C}^{\square})$ is constructed for $\omega\mathbf{Cat}$. The notion of homotopy exchange is taken from [14, Subsection 4.4, Lemma 13].

3.2.1 Definition. Let (Γ, π, τ) be a cocylinder. A map $g : Z \to \Gamma X$ is called **trivial** if it factors through $\tau_X : X \to \Gamma X$. A cell $g : B^n \to \Gamma X$ is called **degenerate** if both $b^0 g$ and $b^1 g$ are trivial or equivalently, if sg is trivial. More general, given a $c \in \mathcal{C}$ with codomain Z, a map $g : Z \to \Gamma X$ is called **trivial on** c if cg is trivial.

3.2.2 Definition. A cocylinder (Γ, π, τ) has the **homotopy exchange property** at X if it satisfies the following condition:

Let $\{i, j\} = \{0, 1\}$ and suppose that there are maps $g : B^n \to \Gamma X$ and $x : B^n \to X$ with $g\pi^j_X \sim x$. Then there exists a $g' : B^n \to \Gamma X$ which is parallel to g and satisfies $g\pi^i_X = g'\pi^i_X$ and $g'\pi^j_X = x$.

A cocylinder has the **homotopy exchange property** if it satisfies the above condition at each object X.

3.2.3 Remark. Whether a cocylinder has the homotopy exchange property does not depend on the choice of the final cylinder objects by Remark 1.2.2(c).

In terms of the homotopy relations given by Γ and the P^n, the homotopy exchange property (at X) means that for any three parallel n-cells $x, y, z : B^n \to X$ the implications

$$(x \sim y \bmod \Gamma) \wedge (y \overset{s}{\sim} z \bmod P^n) \implies (x \sim z \bmod \Gamma)$$
$$(y \sim x \bmod \Gamma) \wedge (z \overset{s}{\sim} y \bmod P^n) \implies (z \sim x \bmod \Gamma)$$

hold, with the additional requirement that the homotopy between x and z can be taken to be parallel to the one between x and y.

3.2.4 Lemma. *Suppose a fibrant cocylinder (Γ, π, τ) for $(\mathcal{C}, \mathcal{C}^{\square})$ has the homotopy exchange property. Then for any two parallel cells $x, y : B^n \to \Gamma X$ the following are equivalent:*

(1) There is a degenerate $g : B^n \to \Gamma X$ with $x = g\pi^0_X$ and $y = g\pi^1_X$.

(2) $x \approx y$.

Proof. For the direction "(1)⇒(2)", observe that by Lemma 3.1.7 the map $\tau : X \to \Gamma X$ lies in $\mathcal{W}$. This gives the map $d : B^n \to X$ in

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "s"'] & X \arrow[d, "\tau_X"] \\ B^n \arrow[r, "g"'] \arrow[ur, "d"] & \Gamma X \arrow[ul, phantom, "\sim"] \end{tikzcd}$$

and hence $x = g\pi^0_X \sim d\tau\pi^0_X = d = d\tau\pi^1_X \sim g\pi^1_X = y$.

For the direction "(2)⇐(1)" we proceed by induction on the length of a chain $x = x_1 \sim \cdots \sim x_k = y$.

For $k = 1$, the trivial cell $x\tau_X \colon B^n \to \Gamma X$ will do. For $k > 1$, let $g \colon B^n \to \Gamma X$ be degenerate with $g\pi^0_X = x$ and $g\pi^1_X = x_{k-1}$. Then homotopy exchange gives a $g' \colon B^n \to \Gamma X$ with $g'\pi^0_X = x$ and $g'\pi^1_X = x_k$. Because g and g' are parallel, g' is also degenerate. □

3.2.5 Definition ([14, Subsection 4.5, Definition 13])**.** Let (Γ, π, τ) be a cocylinder. For a given $f \colon X \to Y$ consider the following diagram

$$\begin{tikzcd}
 & \Gamma X \arrow[d, dashed, "{(\Gamma f, \pi^0_X)}"] \arrow[dr, bend left, "\pi^0_X"] & \\
X \arrow[ur, bend left, "\tau_X"] \arrow[r, "\tilde f"] \arrow[d, "f"] & \Pi(f) \arrow[r, "\pi_f"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y \arrow[r, "\tau_Y"] \arrow[dr, equal] & \Gamma Y \arrow[r, "\pi^0_Y"] \arrow[d, "\pi^1_Y"] & Y \\
 & Y &
\end{tikzcd} \tag{3.2.1}$$

where $\Pi(f)$ is the pullback of π^0_Y and f and where $\tilde f = (f\tau_Y, X)$ is induced by the equation $f = f\tau_X\pi^0_X$. Let $\hat f = f'\pi^1_Y$ and define the **gluing factorization** as $f = \tilde f \hat f$.

For a given map f we will always use f', $\tilde f$ and $\hat f$ as in diagram (3.2.1).

3.2.6 Lemma ([14, Subsection 4.5, Proposition 7])**.** *Suppose that the cocylinder (Γ, π, τ) is fibrant and has the homotopy exchange property. Then, in the factorization $f = \tilde f \hat f$ we have $f \in \mathcal{W} \iff \hat f \in \mathcal{C}^\Box$.*

Proof. For the direction "⇐", observe that in diagram (3.2.1) the map π_f is in $\mathcal{C}^\Box$ and hence by Lemma 3.1.7 the map $\tilde f$ is in $\mathcal{W}$. Consequently, $\hat f \in \mathcal{C}^\Box$ gives $f \in \mathcal{W}$, again by Lemma 3.1.7.

For the direction "⇒", assume $f \in \mathcal{W}$. We need to show $s \mathbin{\Box} \hat f$ for the maps $s \colon S^{n-1} \to B^n$.

Let $u \colon S^{n-1} \to \Pi(f)$ and $y \colon B^n \to Y$ be maps with $sy = u\hat f$ as in the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "u"] \arrow[d, "s"] & \Pi(f) \arrow[r, "\pi_f"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
B^n \arrow[r, "g"] \arrow[dr, "y"'] & \Gamma Y \arrow[r, "\pi^0_Y"] \arrow[d, "\pi^1_Y"] & Y \\
 & Y &
\end{tikzcd}$$

where $g\colon B^n \to \Gamma Y$ is obtained from $s \mathbin{\Box} \pi^1_Y$. By assumption there is a $d\colon B^n \to X$ with $u\pi_f = sd$ and $df \sim g\pi^0_y$. By homotopy exchange, the g in the above diagram can be replaced with a $g'\colon B^n \to \Gamma Y$ such that $g'\pi^1_Y = y$ and $g'\pi^0_Y = df$. The latter equation induces a map $d'\colon B^n \to \Pi(f)$ to the pullback

$$\begin{tikzcd}
S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & \Pi(f) \arrow[r, "\pi_f"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
B^n \arrow[ur, "d'"] \arrow[r, "g'"] \arrow[dr, "y"'] & \Gamma Y \arrow[r, "\pi^0_Y"] \arrow[d, "\pi^1_Y"] & Y \\
& Y &
\end{tikzcd}$$

which gives the desired diagonal. ☐

3.2.7 Corollary. *Given a commuting square*

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, "c"'] & X \arrow[d, "f"] \\
D \arrow[r, "y"'] & Y
\end{tikzcd}$$

with $c \in \mathcal{C}$ and $f \in \mathcal{W}$, there is an extension $d\colon D \to X$ of x along c and a $h\colon D \to \Gamma Y$ with $h\pi^0_Y = df$ and $h\pi^1_Y = y$. Moreover, h is trivial on c.

Proof. The previous Lemma gives the diagonal $d'\colon D \to \Pi(f)$ in the diagram

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[dd, "c"'] & X \arrow[r, "\tau_X"] & \Gamma X \arrow[r, "u"] & \Pi(f) \arrow[r, "\pi_f"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& & & \Gamma Y \arrow[r, "\pi^0_Y"] \arrow[d, "\pi^1_Y"] & Y \\
D \arrow[rrr, "y"'] \arrow[uurrr, "d'"] & & & Y &
\end{tikzcd}$$

where $u = (\Gamma f, \pi^0_X)$. Now set $d = d'\pi_f$ and $h = d'f'$. ☐

3.2.8 Corollary. *Suppose that $\sim$ is transitive and let $f\colon X \to Y$ be a map in $\mathcal{W}$. Then $xf \sim yf \implies x \sim y$ for any two parallel cells $x, y\colon B^n \to X$.*

Proof. Consider the square

$$\begin{tikzcd}
S^n \arrow[r, "(x|y)"] \arrow[d, "\gamma"'] & X \arrow[d, "f"] \\
P^n \arrow[r] & Y
\end{tikzcd}$$

and use the previous corollary. ☐

A map $h\colon X \to \Gamma Y$ is a homotopy with respect to the cocylinder (Γ, π, τ) and if h is degenerate on $c\colon A \to X$ then h is a relative homotopy. This gives two ways of describing homotopy between parallel cells, which are related via Lemma 3.2.4 in case of fibrant cocylinders with homotopy exchange. Likewise one can extend Definition 3.1.5.

3.2.9 Definition. Given maps $c\colon C \to D$ and $f: X \to Y$, we say that f has the **relative homotopy lifting property** with respect to c if for any given square

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "c"'] & X \arrow[d, "f"] \\ D \arrow[r, "v"'] & Y \end{tikzcd}$$

there is a $d\colon D \to X$ with $u = cd$ and a homotopy $h\colon D \to \Gamma Y$ from df to v (i.e. $d\pi_Y^0 = df$ and $d\pi_Y^1 = v$) which is trivial on c.

3.2.10 Remark. By Lemma 3.2.4 the two definitions Definitions 3.1.5 and 3.2.9 agree in the special case of $c = s\colon S^{n-1} \to B^n$ whenever the relative final homotopy relation $\sim$ is transitive. Corollary 3.2.7 then says that the maps in $\mathcal{W}$ have the relative homotopy lifting property with respect to all maps in $\mathcal{C}$, not just those from I.

3.3 The model structure

3.3.1 Theorem. *Let $\{B^n \mid n \geq 0\}$ be a system of small balls in a complete and cocomplete category $\mathcal{K}$ with sphere maps $I = \{s_n\colon S^{n-1} \to B^n \mid n \geq 0\}$. Suppose that the final relative homotopy relation between parallel cells is transitive and that $(^{\square}(I^{\square}), I^{\square})$ has a fibrant cocylinder (Γ, π, τ) with the homotopy exchange property.*

Then there is a cofibrantly generated model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ where $\mathcal{C} = {}^{\square}(I^{\square})$ and $\mathcal{W}$ is the class of those maps that have the relative homotopy lifting property with respect to all maps in I. This model structure has the following properties:

(a) every object is fibrant.

(b) the trivial cofibrations are generated by the set $J = \{\gamma_n^0\colon B^n \to P^n \mid n \geq 0\}$. obtained from final cylinder objects $(P^n, \gamma_n, \sigma_n)$ for B^n relative to s_n.

(c) $\mathcal{W} = \mathcal{W}_{\mathcal{C}}(\emptyset)$. In particular, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is left determined.

3.3.2 Remark. It is necessary to assume that homotopy between parallel cells is transitive: two such cells $x, y\colon B^n \to X$ can be seen as maps from $s_n\colon S^{n-1} \to B^n$ to $s_n x = s_n y\colon S^{n-1} \to X$ in $(S^{n-1}{\downarrow}\mathcal{K})$ equipped with the model structures from Definition 1.2.5(a). By Lemma 1.2.4 the homotopy via $(P^n, \gamma_n, \sigma_n)$ agrees with the usual left homotopy on $(S^{n-1}{\downarrow}\mathcal{K})(s_n, s_n x)$ because $s_n x$ is fibrant. Because s_n is cofibrant, left homotopy is transitive on $(S^{n-1}{\downarrow}\mathcal{K})(s_n, s_n x)$ (see e.g. [9, Proposition 1.2.5(iii)]).

For the proof we will verify the conditions in Smith's Theorem 1.3.9 and 1.3.13.

The smallness condition 1.3.13(5) was already built into the assumptions. Part (1) in Lemma 3.1.3 immediately gives condition 1.3.9(2).

3.3.3 Lemma. $\mathcal{C}^{\square} \subseteq \mathcal{W}$ $\square$

The solution set condition 1.3.9(4) together with condition 1.3.13(6) is provided by the following lemma.

3.3.4 Lemma. *For any $s_n\colon S^{n-1} \to B^n$, the singleton set $\mathcal{W}_{s_n} = \{\gamma_n^0\colon B^n \to P^n\}$ is a solution set at s_n. The set $J = \{\gamma_n^0\colon B^n \to P^n \mid n \geq 0\}$ generates the trivial cofibrations.*

Proof. We already remarked that the S^{n-1} are small with respect to cell(I). By Corollary 3.1.8 the maps γ_n^0 lie in $\mathcal{C} \cap \mathcal{W}$. Satisfaction of the solution set condition as expressed in Remark 1.3.8 is given by condition (3) of Lemma 3.1.3. That ${}^{\square}(J^{\square}) = \mathcal{C} \cap \mathcal{W}$ holds, was already noted in the proof of Theorem 1.3.9. $\square$

The following two Lemmas give condition 1.3.9(1).

3.3.5 Lemma. *The class $\mathcal{W}$ is closed under retracts in the arrow category $\mathcal{K}^{\mathbf{2}}$*

Proof. Let $g \in \mathcal{W}$ and consider a diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "x"] \arrow[d, "s"'] & X \arrow[r, "i"] \arrow[d, "f"] & U \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\
B^n \arrow[r, "y"'] & Y \arrow[r, "j"'] & V \arrow[r, "q"'] & Y
\end{tikzcd}$$

where $ip = X$ and $jq = Y$. Let $d\colon B^n \to U$ be a map such that $xi = sd$ and $dg \sim yj$. Then $dp\colon B^n \to X$ satisfies $x = sdp$ and $dpf = dgq \sim y$. $\square$

3.3.6 Lemma. *The class $\mathcal{W}$ has the 2-3 property.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two maps.

We first show that in case of $g \in \mathcal{W}$ the equivalence $fg \in \mathcal{W} \iff f \in \mathcal{W}$ holds. The arguments are similar to those in the proof of Lemma 3.1.7. Consider the following diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & X \arrow[d, "f"] \\
B^n \arrow[r, "v'"'] \arrow[ur, "d"] \arrow[dr, "v"'] & Y \arrow[d, "g"] \\
 & Z
\end{tikzcd}$$

For the direction "$\Rightarrow$" assume that only u and v' are given and let $v = v'g$. The map d with $sd = u$ and $dfg \sim v'g$ exists because $fg \in \mathcal{W}$. By Corollary 3.2.8 one obtains $df \sim v'$.

For the direction "$\Leftarrow$" assume that only u and v are given. The map v' with $sv' = uf$ and $v'g \sim v$ is obtained from $g \in \mathcal{W}$. Applying $f \in \mathcal{W}$ to the resulting upper square gives the map d with $sd = u$ and $df \sim v'$. Together we have $dfg \sim v'g \sim v$.

It remains to show the implication $f, fg \in \mathcal{W} \implies g \in \mathcal{W}$. Given a square

$$\begin{tikzcd}
S^{n-1} \arrow[r, "y"] \arrow[d, "s"'] & Y \arrow[d, "g"] \\
B^n \arrow[r, "z"'] & Z
\end{tikzcd}$$

first observe that, because S^{n-1} is cofibrant, Corollary 3.2.7 gives a map $x\colon S^{n-1} \to X$ such that $xf \sim y$ via some homotopy $t\colon S^{n-1} \to \Gamma Y$. This gives the following diagram

$$\begin{tikzcd}
& X \arrow[r, "\tau_X"] & \Gamma X \arrow[d, "{(\Gamma(fg),\pi_X^0)}"] \arrow[dr, bend left, "\pi_X^0"] & \\
S^{n-1} \arrow[ur, "x"] \arrow[r] \arrow[dr, "t"] \arrow[dddd, "s"'] & \Pi(f) \arrow[r] \arrow[d] & \Pi(fg) \arrow[r, "\pi_{fg}"'] \arrow[d] & X \arrow[d, "f"] \\
& \Gamma Y \arrow[r] \arrow[dr, "\Gamma g"'] & \Pi(g) \arrow[r, "\pi_g"'] \arrow[d] & Y \arrow[d, "g"] \\
& & \Gamma Z \arrow[r, "\pi_Z^0"'] \arrow[d, "\pi_Z^1"] & Z \\
B^n \arrow[rr, "z"'] & & Z &
\end{tikzcd}$$

where the three squares are pullbacks and $S^{n-1} \to \Pi(f)$ is induced by t and $(x\tau_X\Gamma(fg), x)$. Observe that $t(\Gamma g)$ is trivial. From that diagram we extract the square

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "s"'] & \Pi(fg) \arrow[d, "\widehat{fg}"] \\
B^n \arrow[r, "z"'] & Z
\end{tikzcd}$$

which has a diagonal $p\colon B^n \to \Pi(fg)$ because $s \,\square\, \widehat{fg}$ by Lemma 3.2.6. Set $u = p(fg)'\colon B^n \to \Gamma Z$. Because $t(\Gamma g)$ is trivial, u is trivial on s, so that $u\pi_Z^0 \sim u\pi_Z^1 = z$. Now consider the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[rrr, "t"] \arrow[d, "s"'] & & & \Gamma Y \arrow[d, "\pi_Y^0"] \\
B^n \arrow[urrr, "v"] \arrow[r, "p"'] \arrow[drrr, bend right, "u\pi_Z^0"'] & \Pi(fg) \arrow[r, "\pi_{fg}"] & X \arrow[r, "f"] & Y \arrow[d, "g"] \\
& & & Z
\end{tikzcd}$$

where the diagonal comes from $s \,\square\, \pi^0_Y$. Again, $v(\Gamma g)$ is trivial on s and therefore $v(\Gamma g)\pi^0_Z \sim v(\Gamma g)\pi^1_Z$. Set $d = v\pi^1_Y \colon B^n \to Y$. Then we have $sd = sv\pi^1_Y = t\pi^1_Y = y$. Also

$$dg = v\pi^1_Y g = v(\Gamma g)\pi^1_Z \sim v(\Gamma g)\pi^0_Z = v\pi^0_Y g = u\pi^0_Z \sim u\pi^1_Z = z$$

and hence $dg \sim z$. $\square$

It remains to verify condition 1.3.9(3). For this we introduce a useful class of maps between $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{W}$.

3.3.7 Definition ([14, Subsection 4.6, Definition 14])**.** A map $f\colon X \to Y$ is called an **immersion** if the square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dotted, "\sim"'] & Y \end{tikzcd}$$

has a relative homotopy lifting. This means that there are maps $g\colon Y \to X$ and $h\colon Y \to \Gamma Y$ with

(i) $fg = X$.

(ii) $h\pi^0_Y = gf$ and $h\pi^1_Y = Y$.

(iii) $fh = f\tau_Y$.

Here the last equation stems from the observation that if fh factors through τ_Y then it must necessarily factor via f. We write $\mathcal{Z}$ for the class of all immersions.

3.3.8 Lemma ([14, Subsection 4.6, Lemma 15])**.** *A map $f\colon X \to Y$ is an immersion iff the square*

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "f"'] & \Pi(f) \arrow[d, "\hat{f}"] \\ Y \arrow[r, equal] \arrow[ur, dotted] & Y \end{tikzcd} \tag{3.3.1}$$

has a diagonal.

Proof. First assume that the above square has a diagonal $k\colon Y \to \Pi(f)$. Then $g = k\pi_f\colon Y \to X$ and $h = kf'\colon Y \to \Gamma Y$ establish $f \in \mathcal{Z}$.

Conversely, assume $f \in \mathcal{Z}$ via maps $g\colon Y \to X$ and $h\colon Y \to \Gamma Y$. Then $h\pi^0_Y = gf$ gives a map $k = (h, g)\colon Y \to \Pi(f)$ to the pullback. Consequently $(h, g)\hat{f} = h\pi^1_Y = Y$ and

$$f(h, g) = (fh, X) = (f\tau_Y, X) = \tilde{f}$$

so that k is the desired diagonal. $\square$

3.3.9 Corollary ([14, Subsection 4.6, Corollary 4, Lemma 16])**.** $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{Z} \subseteq \mathcal{W}$.

Proof. The inclusion $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{Z}$ is a special case of Corollary 3.2.7.

Now assume $f \in \mathcal{Z}$. Given the leftmost square in the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "u"] \arrow[d, "s"'] & X \arrow[r, "\tilde{f}"] \arrow[d, "f"] & \Pi(f) \arrow[r, "\pi_f"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
B^n \arrow[r, "y"'] & Y \arrow[ur, "k"'] \arrow[dr, equal] & \Gamma Y \arrow[r, "\pi^0_Y"'] \arrow[d, "\pi^1_Y"] & Y \\
& & Y &
\end{tikzcd}$$

set $d = yk\pi_f\colon B^n \to X$ and $h = ykf'\colon B^n \to \Gamma Y$ where k is a diagonal of the square (3.3.1). Then $sd = u$, $h\pi^0_Y = df$ and $h\pi^1_Y = y$. Moreover, $sh = u\tilde{f}f' = uf\tau_Y$ is trivial. □

3.3.10 Lemma. *$\mathcal{C} \cap \mathcal{W}$ is closed under pushouts.*

Proof. Because of Corollary 3.3.9 it is enough to show that $\mathcal{Z}$ is closed under pushouts.

Let $f\colon X \to Y$ be in $\mathcal{Z}$ with $g\colon Y \to X$ and $h\colon Y \to \Gamma Y$ as in Definition 3.3.7 and consider a pushout diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Z \arrow[d, "j"] \\
Y \arrow[r, "i"'] & Q
\end{tikzcd}$$

where we want to show $j \in \mathcal{Z}$.

Define the maps $p = (gu|Z)\colon Q \to Z$ and $k = (h\Gamma i|j\tau_Q)\colon Q \to \Gamma Q$ from the pushout. Then p and k exhibit j as an immersion: $jp = Z$ and $jk = j\tau_Q$ are immediate from the definition and moreover we have $k\pi^\epsilon_Q = (h(\Gamma i)\pi^\epsilon_Q|j) = (h\pi^\epsilon_Y i|j)$ and therefore
$k\pi^0_Q = (gfi|j) = (guj|j) = pj$ and $k\pi^1_Q = (i|j) = Q$. □

3.3.11 Lemma. *$\mathcal{C} \cap \mathcal{W}$ is closed under transfinite composition.*

Proof. Let $X\colon \lambda \to \mathcal{K}$ be a smooth chain starting at $X = X_0$, with transition maps $f_{\alpha,\beta}\colon X_\alpha \to X_\beta$, its colimit $Y = \operatorname{colim}_{\beta<\lambda} X_\beta$, and maps $f_\alpha\colon X_\alpha \to Y$ to the colimit. Suppose that all maps $f_{\alpha,\alpha+1}$ are in $\mathcal{C} \cap \mathcal{W}$. The map $f_0 = f\colon X \to Y$ is already in $\mathcal{C}$. It is therefore enough to construct a diagonal for the square

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] \arrow[d, "f"'] & \Pi(f) \arrow[d, "\hat{f}"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

because Lemma 3.3.8 then yields $f \in \mathcal{Z} \subseteq \mathcal{W}$, where the inclusion is given by Corollary 3.3.9.

Fix notation according to the following diagram (for all $\alpha \leq \beta < \lambda$)

$$\begin{tikzcd}
X \arrow[r, "\tau_X"] & \Gamma X \arrow[r, "u_{0,\alpha}"] \arrow[dr, "\Gamma f_{0,\alpha}"'] & \Pi(f_{0,\alpha}) \arrow[r, "u_{\alpha,\beta}"] \arrow[d, "f'_{0,\alpha}"] & \Pi(f_{0,\beta}) \arrow[r, "u_\beta"] \arrow[d, "f'_{0,\beta}"] & \Pi(f) \arrow[r, "\pi_f"] \arrow[d] & X \arrow[d, "f_{0,\alpha}"] \\
& & \Gamma X_\alpha \arrow[r] \arrow[d, "\pi^1_{X_\alpha}"'] \arrow[dr, "\Gamma f_{\alpha,\beta}"'] & \Pi(f_{\alpha,\beta}) \arrow[r] \arrow[d, "f'_{\alpha,\beta}"] & \Pi(f_\alpha) \arrow[r, "\pi_{f_\alpha}"] \arrow[d] & X_\alpha \arrow[d, "f_{\alpha,\beta}"] \\
& & X_\alpha \arrow[dr, "f_{\alpha,\beta}"'] & \Gamma X_\beta \arrow[r] \arrow[d, "\pi^1_{X_\beta}"'] \arrow[dr, "\Gamma f_\beta"'] & \Pi(f_\beta) \arrow[r, "\pi_{f_\beta}"] \arrow[d, "f'_\beta"] & X_\beta \arrow[d, "f_\beta"] \\
& & & X_\beta \arrow[dr, "f_\beta"'] & \Gamma Y \arrow[r, "\pi^0_Y"'] \arrow[d, "\pi^1_Y"] & Y \\
& & & & Y &
\end{tikzcd}$$

where the squares are pullbacks and the horizontal compositions $\Gamma Z \to Z$ are the projections π^0_Z. Observe that $\Pi(f_{0,0}) = \Gamma X$ and also $\tilde{f} = \tau_X u_{0,\alpha} u_\alpha$ and $\tilde{f}_{0,\alpha} = \tau_X u_{0,\alpha}$.

We will first construct a sequence of maps $c_\beta \colon X_\beta \to \Pi(f_{0,\beta})$ such that

$$\begin{tikzcd}
X_\alpha \arrow[r, "c_\alpha"] \arrow[d, "f_{\alpha,\beta}"'] & \Pi(f_{0,\alpha}) \arrow[r, "u_{\alpha,\beta}"] & \Pi(f_{0,\beta}) \arrow[d, "\hat{f}_{0,\beta}"] \\
X_\beta \arrow[urr, "c_\beta"'] \arrow[rr, equal] & & X_\beta
\end{tikzcd} \tag{3.3.2}$$

commutes for all $\alpha \leq \beta < \lambda$.

(i) At $\beta = 0$ set $c_0 = \tau_X \colon X \to \Gamma X$.

(ii) At a successor step from β to $\beta + 1$, first observe that diagram (3.3.2) with $\alpha = 0$ already establishes $f_{0,\beta} \in \mathcal{Z}$ by Lemma 3.3.8. Because of $f_{0,\beta} \in \mathcal{C}$, we have $f_{0,\beta} \in \mathcal{C} \cap \mathcal{W}$ and hence $f_{0,\beta+1} = f_{0,\beta} f_{\beta,\beta+1} \in \mathcal{C} \cap \mathcal{W}$. In particular $\hat{f}_{0,\beta+1} \in \mathcal{C}^\square$ and therefore the square

$$\begin{tikzcd}
X_\beta \arrow[r, "c_\beta"] \arrow[d, "f_{\beta,\beta+1}"'] & \Pi(f_{0,\beta}) \arrow[r, "u_{\beta,\beta+1}"] & \Pi(f_{0,\beta+1}) \arrow[d, "\hat{f}_{0,\beta+1}"] \\
X_{\beta+1} \arrow[urr, "c_{\beta+1}"'] \arrow[rr, equal] & & X_{\beta+1}
\end{tikzcd}$$

has a diagonal which we can take as $c_{\beta+1}$.

(iii) At a limit ordinal $\beta < \lambda$ define $c_\beta \colon X_\beta \to \Pi(f_{0,\beta})$ as the map induced by the cocone $c_\alpha u_{\alpha,\beta} \colon X_\alpha \to \Pi(f_{0,\beta})$ for $\alpha < \beta$.

Now set $d_\alpha = c_\alpha u_\alpha \colon X_\alpha \to \Pi(f)$ for all $\alpha < \lambda$. By diagram (3.3.2) these maps form a cocone and therefore induce a map $d\colon Y \to \Pi(f)$ from the colimit which satisfies $fd = \tilde{f}$ and $d\hat{f} = Y$. $\square$

Proof of Theorem 3.3.1. The conditions of Theorem 1.3.9 and Theorem 1.3.13 are given by the Lemmas 3.3.5, 3.3.6, 3.3.3, 3.3.10, 3.3.11 and 3.3.4.

By Corollary 3.3.9, every map in $\mathcal{C} \cap \mathcal{W}$ is an immersion and hence in particular a split monomorphism. Therefore every object is fibrant.

It remains to check that $\mathcal{W}$ is contained in the smallest localizer $\mathcal{W}_\mathcal{C}(\emptyset)$. Suppose $f\colon X \to Y$ lies in $\mathcal{W}$ and consider the glueing factorization $f = \tilde{f}\hat{f}$ from Definition 3.2.5. In diagram (3.2.1) we have $\tilde{f}\pi_f = X$ and $\pi_f \in \mathcal{C}^\square \subseteq \mathcal{W}(\emptyset)$. Therefore $\tilde{f} \in \mathcal{W}_\mathcal{C}(\emptyset)$. By Lemma 3.2.6 we also have $\hat{f} \in \mathcal{W}_\mathcal{C}(\emptyset)$. Hence $f \in \mathcal{W}_\mathcal{C}(\emptyset)$. $\square$

3.4 The case of adjoint cylinders

In the previous sections we initially used (relative) homotopy only between parallel cells and introduced homotopy between general parallel maps through cocylinders. Now we consider the situation where already a cylinder $(\mathrm{C}, \gamma, \sigma)$ for $({}^\square(I^\square), I^\square)$ is available and C has a right adjoint Γ.

We first describe how to extend Γ to a cocylinder (Γ, π, τ) and rephrase properties of (Γ, π, τ) in terms of $(\mathrm{C}, \gamma, \sigma)$. This works in general and does not require any particular assumptions about the weak factorization system for which $(\mathrm{C}, \gamma, \sigma)$ is a cylinder.

3.4.1 Lemma. *Let $(\mathrm{C}, \gamma, \sigma)$ be a cylinder for a weak factorization system and suppose that C has a right adjoint Γ. Let π and τ be the natural maps conjugate to γ and σ respectively.*

Then (Γ, π, τ) is a cocylinder and the following holds:

(a) $f \sim g \pmod{\mathcal{C}} \iff f \sim g \pmod{\Gamma}$ *for any maps $f, g\colon X \to Y$. In this equivalence, homotopies $h\colon \mathrm{C}X \to Y$ correspond by adjointness to homotopies $\hat{h}\colon X \to \Gamma Y$.*

(b) In the situation of (a), the map $\hat{h}\colon X \to \Gamma Y$ is trivial on $u\colon A \to X$ iff $(\mathrm{C}u)h$ factors through $\sigma_A\colon \mathrm{C}A \to A$.

(c) Let $h, k\colon \mathrm{C}X \to Y$ be two maps with adjoints $\hat{h}, \hat{k}\colon X \to \Gamma Y$ and consider a map $u\colon A \to X$. Then $\hat{h}$ and $\hat{k}$ are u-parallel (i.e. $u\hat{h} = u\hat{k}$) iff $(\mathrm{C}u)h = (\mathrm{C}u)k$.

(d) A map $f\colon X \to Y$ is an immersion iff there exist maps $g\colon Y \to X$ and $h\colon \mathrm{C}Y \to Y$ with

(i) $fg = X$.

(ii) $\iota^0_Y h = gf$ *and* $\iota^1_Y h = Y$.

(iii) $(\mathrm{C}f)h = \sigma_X f$.

Proof. Write $(-)$ for the identity functor, $2(-)$ for the copower functor that takes f to $f+f$, and $(-)^2$ for the squaring functor that takes f to $f \times f$. Then $2(-)$ is left adjoint to $(-)^2$ and under this adjointness, maps $(f|g)\colon X+X \to Y$ correspond to maps $(f,g)\colon X \to Y \times Y$. Therefore the natural coproduct inclusions $\iota^0, \iota^1\colon (-) \to 2(-)$ have the natural product projections $p^0, p^1\colon (-)^2 \to (-)$ as conjugates. We then have the two corresponding diagrams

$$\begin{tikzcd} & 2(-) \arrow[d, "\gamma"] & \\ (-) \arrow[ur, bend left, "\iota^0"] \arrow[r, "\gamma^0"] \arrow[dr, equal] & \mathrm{C} \arrow[d, "\sigma"] & (-) \arrow[ul, bend right, "\iota^1"'] \arrow[l, "\gamma^1"'] \arrow[dl, equal] \\ & (-) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & (-)^2 \arrow[dl, bend right, "p^0"'] \arrow[dr, bend left, "p^1"] & \\ (-) \arrow[dr, equal] & \Gamma \arrow[u, "\pi"'] \arrow[l, "\pi^0"'] \arrow[r, "\pi^1"] & (-) \arrow[dl, equal] \\ & (-) \arrow[u, "\tau"'] & \end{tikzcd}$$

where the left one states that $(\mathrm{C}, \gamma, \sigma)$ is a cylinder and the right one is obtained from conjugation. Hence (Γ, π, τ) is a cocylinder.

(a) We have the correspondence between

$$\begin{tikzcd} X+X \arrow[r, "(f|g)"] \arrow[d, "\gamma_X"'] & Y \\ \mathrm{C}X \arrow[ur, "h"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "(f,g)"] \arrow[d, "\hat{h}"'] & Y \times Y \\ \Gamma Y \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

where the left diagram means $f \sim g \pmod{\mathrm{C}}$ and the right diagram means $f \sim g \pmod{\Gamma}$.

(b) This follows from the correspondence between

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "u"'] & Y \arrow[d, "\tau_Y"] \\ X \arrow[r, "\hat{h}"'] & \Gamma Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{C}A \arrow[r, "\sigma_A"] \arrow[d, "\mathrm{C}u"'] & A \arrow[d, "d"] \\ \mathrm{C}X \arrow[r, "h"'] & Y \end{tikzcd}$$

(c) This already follows from adjointness.

(d) To check that conditions (i)–(iii) correspond to conditions (i)–(iii) in Definition 3.3.7, use the corresponding diagrams

$$\begin{tikzcd} Y \arrow[r, "\iota_Y^0"] \arrow[d, "g"'] & \mathrm{C}Y \arrow[d, "h"] & Y \arrow[l, "\iota_Y^1"'] \arrow[dl, equal] \\ X \arrow[r, "f"'] & Y & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "g"'] \arrow[d, "\hat{h}"] \arrow[dr, equal] & \\ Y & \Gamma Y \arrow[l, "\pi_Y^0"] \arrow[r, "\pi_Y^1"'] & Y \end{tikzcd}$$

for condition (ii) and use the corresponding diagrams

$$\begin{tikzcd} \mathrm{C}X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{C}f"'] & X \arrow[d, "f"] \\ \mathrm{C}Y \arrow[r, "h"'] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Xf \arrow[r, "r"] \arrow[d, "f"'] & Y \arrow[d, "\tau_Y"] \\ Y \arrow[r, "\hat{h}"'] & \Gamma Y \end{tikzcd}$$

for condition (iii). ☐

3.4.2 Remark. The conditions in 3.4.1(d) are those for a strong deformation retract in [4, Définition 2.15], the dual of which we already used in Definition 2.2.8 (for the last equation observe that $\sigma_X f = (\mathrm{C}f)\sigma_Y$).

We now return to the special case where $I = \{s_n \colon S^{n-1} \to B^n \mid n \geq 0\}$ and a system of final cylinder objects $(P^n, \gamma_n, \sigma_n)$ for B^n relative to s_n is given. The following definition is a translation of Definition 3.2.2 via Lemma 3.4.1.

3.4.3 Definition. A cylinder $(\mathrm{C}, \gamma, \sigma)$ has the **homotopy exchange property** at X if it satisfies the following condition:

Let $\{i, j\} = \{0, 1\}$ and suppose that there are maps $h \colon \mathrm{C}B^n \to X$ and $x \colon B^n \to X$ with $\iota^j_{\mathrm{C}B^n} h \sim x$. Then there exists a $h' \colon \mathrm{C}B^n \to X$ with $(\mathrm{C}s_n)h = (\mathrm{C}s_n)h$ that satisfies $\iota^i_{\mathrm{C}B^n} h = \iota^i_{\mathrm{C}B^n} h'$ and $\iota^j_{\mathrm{C}B^n} h' = x$.

A cylinder has **homotopy exchange property** if it satisfies the above condition at each object X.

3.4.4 Remark. Remark 3.2.3 applies here as well: in terms of the homotopy relations given by C and the P^n, the homotopy exchange property (at X) means that for any three parallel n-cells $x, y, z \colon B^n \to X$ the implications

$$(x \sim y \bmod \mathrm{C}) \wedge (y \overset{s}{\sim} z \bmod P^n) \implies (x \sim z \bmod \mathrm{C})$$
$$(y \sim x \bmod \mathrm{C}) \wedge (z \overset{s}{\sim} y \bmod P^n) \implies (z \sim x \bmod \mathrm{C})$$

hold, with the additional requirement that the homotopy between x and z can be taken to be parallel to the one between x and y.

If C has a right adjoint Γ and (Γ, π, τ) is constructed as in Lemma 3.4.1, then $(\mathrm{C}, \gamma, \sigma)$ has the homotopy exchange property iff (Γ, π, τ) has it.

Now we have a translation of Theorem 3.3.1 for cylinders.

3.4.5 Theorem (3.3.1 for cylinders)**.** *Let $\{B^n \mid n \geq 0\}$ be a system of small balls in a complete and cocomplete category $\mathcal{K}$ with sphere maps $I = \{s_n \colon S^{n-1} \to B^n \mid n \geq 0\}$. Suppose that $({}^{\Box}(I^{\Box}), I^{\Box})$ has a cylinder $(\mathrm{C}, \gamma, \sigma)$ with the homotopy exchange property such that* C *has a right adjoint.*

If the final relative homotopy relation between parallel cells is transitive and every object X satisfies $(X \to 1) \in \Lambda^0(\mathrm{C}, \emptyset, I)^{\Box}$, then there is a cofibrantly generated model *structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ where $\mathcal{C} = {}^{\square}(I^{\square})$ and $\mathcal{W}$ is the class of those maps that have the relative homotopy lifting property with respect to all maps in I. This model structure has the following properties:*

(a) every object is fibrant.

(b) the trivial cofibrations are generated by the set $J = \{\gamma_n^0 \colon B^n \to P^n \mid n \geq 0\}$. obtained from final cylinder objects $(P^n, \gamma_n, \sigma_n)$ for B^n relative to s_n.

(c) $\mathcal{W} = \mathcal{W}_{\mathcal{C}}(\emptyset)$. In particular, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is left determined.

Proof. By Lemma 3.4.1 the right adjoint can be made into a cocylinder (Γ, π, τ) which also has the homotopy exchange property. It remains to verify that the cocylinder is final, i.e. that the components of π^0 and π^1 lie in $I^{\square}$. First observe that, up to isomorphism, $\pi^0 \star (X \to 1)$ is equal to $\pi_X^0 \colon \Gamma X \to X$. Therefore Lemma 2.1.3 gives

$$(s_n \star \gamma^0) \,\square\, (X \to 1) \iff s_n \,\square\, (\pi^0 \star (X \to 1)) \iff s_n \,\square\, \pi_X^0$$

for all $n \geq 0$. By assumption $(X \to 1) \in (I \star \gamma^0)^{\square}$ and therefore $\pi_X^0 \in I^{\square}$. The same calculation gives $\pi_X^1 \in I^{\square}$. $\square$

3.4.6 Corollary. *Let I and $(\mathrm{C}, \gamma, \sigma)$ be as in Theorem 3.4.5 and set $\mathcal{C} = {}^{\square}(I^{\square})$. Suppose that $(\mathcal{C}, \mathcal{C}^{\square})$ is cofibrant and that $(\mathrm{C}, \gamma, \sigma)$ is cartesian and has the homotopy exchange property. Then the following conditions are equivalent:*

(1) All maps in $\mathcal{W}(\mathrm{C}, \emptyset, I)$ have the relative homotopy lifting property with respect to all maps in $\mathcal{C}$.

(2) Every object X is fibrant in the model structure given by $\mathcal{C}$ and $\mathcal{W}(\mathrm{C}, \emptyset, I)$, i.e. $(X \to 1) \in \Lambda(\mathrm{C}, \emptyset, I)^{\square}$.

Under these conditions the models structures given by Theorem 3.4.5 and Theorem 2.2.5 coincide and $\mathcal{W}(\mathrm{C}, \emptyset, I) = \mathcal{W}_{\mathcal{C}}(\emptyset)$. In particular Cisinski's construction produces a left determined model structure.

Proof. If condition (1) holds, then every map in $\mathcal{C} \cap \mathcal{W}(\mathrm{C}, \emptyset, I)$ must have the relative homotopy lifting property with respect to itself, and hence must be a split monomorphism. Therefore every object is fibrant. This shows (1) $\Rightarrow$ (2).

Conversely assume that condition (2) holds. We already observed in Remark 3.3.2 that two parallel cells $x, y \in \mathcal{K}(B^n, X)$ can be viewed as maps in $(S^{n-1} {\downarrow} \mathcal{K})(s_n, s_n x)$ and that the relative homotopy relation must be transitive whenever X is fibrant. Therefore condition (2) ensures that Theorem 3.4.5 is applicable and provides a model structure with $\mathcal{W}_{\mathcal{C}}(\emptyset)$ as the class of weak equivalences. By Corollary 3.2.7, every map in $\mathcal{W}_{\mathcal{C}}(\emptyset)$ has the relative homotopy lifting property with respect to all maps in $\mathcal{C}$, so it is enough to show $\mathcal{W}(\mathrm{C}, \emptyset, I) \subseteq \mathcal{W}_{\mathcal{C}}(\emptyset)$.

Let $f\colon X \to Y$ be in $\mathcal{W}(\mathrm{C}, \emptyset, I)$. If f is not in $\mathcal{C}$, factor f as $f = cr$ with $c \in \mathcal{C}$ and $r \in \mathcal{C}^{\square}$. Then $c \in \mathcal{C} \cap \mathcal{W}(\mathrm{C}, \emptyset, I)$ and $r \in \mathcal{W}_{\mathcal{C}}(\emptyset)$, so that it suffices to show $c \in \mathcal{W}_{\mathcal{C}}(\emptyset)$.

We may therefore assume $f \in \mathcal{C}$. Then there is a $g\colon Y \to X$ with $fg = X$ because X is fibrant. The equation

$$(f+f)(gf|Y) = (f|f) = (X|X)f$$

gives the solid arrows in the diagram

$$\begin{tikzcd}
X+X \arrow[r, "\gamma_X"] \arrow[d, "f+f"'] & \mathrm{C}X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{C}f"] & X \arrow[dd, "f"] \\
Y+Y \arrow[r, "\gamma_Y"] \arrow[drr, bend right, "(gf|Y)"'] & \mathrm{C}Y \arrow[dr, dotted, "h"] & \\
 & & Y
\end{tikzcd}$$

and $h\colon \mathrm{C}Y \to Y$ exists because Y is fibrant. Together g and h exhibit f as an immersion by Lemma 3.4.1(d). By Corollary 3.3.9, f is in $\mathcal{W}_{\mathcal{C}}(\emptyset)$. $\square$

4 Examples

4.1 New examples from old

There are quite a few category theoretic constructions by which new model categories can be obtained from old ones. One can form products of model categories ([8, Proposition 7.1.7]) and given a model category $\mathcal{K}$, one can put model structures on its dual $\mathcal{K}^{op}$ ([8, Proposition 7.1.9]), on slices $(\mathcal{K}{\downarrow}K)$ and coslices $(K{\downarrow}\mathcal{K})$ ([8, Theorem 7.6.5]), which we already used. If the original $\mathcal{K}$ is locally presentable, then all the above constructions, except dualizing, will also give locally presentable categories again.

If $\mathcal{K}$ is cofibrantly generated and $\mathcal{A}$ is a small category, then $\mathcal{K}^{\mathcal{A}}$ can also be given a (cofibrantly generated) model structure ([8, Theorem 11.6.1]). We describe this construction for the special case where $\mathcal{K}$ is locally presentable. Recall that cocompleteness of $\mathcal{K}$ ensures that for any objects $K\colon \mathbf{1} \to \mathcal{K}$ and $a\colon \mathbf{1} \to \mathcal{A}$ the left Kan extension $\mathrm{Lan}_a(K)\colon \mathcal{A} \to \mathcal{K}$ of K along a exists and can be computed via the formula $\mathrm{Lan}_a(K)_c = \mathrm{colim}((a{\downarrow}c) \to \mathcal{A} \xrightarrow{K} \mathcal{K})$ (see e.g. [17, X-3, Theorem 1]), where we use lower case letters for the objects of $\mathcal{A}$. Then $\mathrm{Lan}_a\colon \mathcal{K} \to \mathcal{K}^{\mathcal{A}}$ is left adjoint to the evaluation functor $ev_a\colon \mathcal{K}^{\mathcal{A}} \to \mathcal{K}$. Instead of $ev_a(\varphi)$ we will often write φ_a in the following.

4.1.1 Lemma. *Let $\mathcal{A}$ be small and $\mathcal{K}$ be a locally presentable category. Suppose that $\mathcal{K}$ has a set I and a class $\mathcal{W}$ of maps such that the conditions of Smith's Theorem 1.3.9 are satisfied. Then*

$$\tilde{I} = \{\mathrm{Lan}_a(i) \mid i \in I,\, a \in \mathcal{A}\} \quad \textit{and} \quad \widetilde{\mathcal{W}} = \bigcap_{a\in\mathcal{A}} ev_a^{-1}(\mathcal{W}) = \{\varphi \mid \forall a \in \mathcal{A}:\ \varphi_a \in \mathcal{W}\}$$

again satisfy the conditions of Smith's Theorem. In particular $\mathcal{K}^{\mathcal{A}}$ has a cofibrantly generated model structure with generating cofibrations $\tilde{I}$, where a map $\varphi\colon X \to Y$ is a weak equivalence iff $\varphi_a\colon X_a \to Y_a$ is a weak equivalence in $\mathcal{K}$ for every object $a \in \mathcal{A}$.

Proof. First observe that for objects $a, c \in \mathcal{A}$ the comma category $(a{\downarrow}c)$ is just the set $\mathcal{A}(a, c)$ and hence the above colimit formula gives

$$\mathrm{Lan}_a(K)_c = \coprod_{\mathcal{A}(a,c)} K \qquad \text{and} \qquad \mathrm{Lan}_a(f)_c = \coprod_{\mathcal{A}(a,c)} f \tag{4.1.1}$$

on objects and maps respectively. We now go through the conditions of Theorem 1.3.9:

(1) Each $ev_a^{-1}(\mathcal{W})$ has the 2-3 property and is closed under retracts. Therefore the same holds for their intersection $\widetilde{\mathcal{W}}$.

(2) Let $f\colon K \to L$ in $\mathcal{K}$ and $\varphi\colon X \to Y$ in $\mathcal{K}^{\mathcal{A}}$. For any $a \in \mathcal{A}$ we have the equivalence

$$\mathrm{Lan}_a(f) \,\Box\, \varphi \iff f \,\Box\, \varphi_a$$

which gives

$$\varphi \in \tilde{I}^{\Box} \iff \forall a \in \mathcal{A} : \varphi_a \in I^{\Box} \implies \forall a \in \mathcal{A} : \varphi_a \in \mathcal{W}$$

and hence $\tilde{I}^{\Box} \subseteq \widetilde{\mathcal{W}}$.

(3) From (4.1.1) we obtain in particular $\mathrm{Lan}_a(i)_c \in {}^{\Box}(I^{\Box})$ for all $i \in I$ and $a, c \in \mathcal{A}$. Therefore all maps in $\tilde{I}$ have their components in ${}^{\Box}(I^{\Box})$. Because colimits in $\mathcal{K}^{\mathcal{A}}$ are computed pointwise, the same holds for maps in $\mathrm{cell}(\tilde{I})$. Consequently, every map in ${}^{\Box}(\tilde{I}^{\Box})$ has its components in ${}^{\Box}(I^{\Box})$ because it is a retract of some map in $\mathrm{cell}(\tilde{I})$.

Therefore every map in ${}^{\Box}(\tilde{I}^{\Box}) \cap \widetilde{\mathcal{W}}$ has its components in ${}^{\Box}(I^{\Box}) \cap \mathcal{W}$ and, again because colimits are computed pointwise, the same holds for pushouts and transfinite compositions of such maps.

(4) We want to verify that $\widetilde{\mathcal{W}}$ satisfies the solution set condition at every $\mathrm{Lan}_a(i)$ (for $a \in \mathcal{A}$ and $i \in I$). Suppose $\varphi\colon X \to Y$ is in $\mathcal{W}$ and consider the following diagram:

$$\begin{tikzcd}
\mathrm{Lan}_a(K) \arrow[r, "\alpha"] \arrow[d, "\mathrm{Lan}_a(i)"'] & X \arrow[d, "\varphi"] \\
\mathrm{Lan}_a(L) \arrow[r, "\beta"] & Y
\end{tikzcd}$$

Switch via the adjunction and factor the resulting square

$$\begin{tikzcd}
K \arrow[r, "u"'] \arrow[d, "i"'] \arrow[rr, bend left, "\hat{\alpha}"] & U \arrow[r, "x"'] \arrow[d, "j"] & X_a \arrow[d, "\varphi_a"] \\
L \arrow[r, "v"] \arrow[rr, bend right, "\hat{\beta}"'] & V \arrow[r, "y"] & Y_a
\end{tikzcd}$$

with $j \in \mathcal{W}_i$ where $\mathcal{W}_i$ is the solution set for $\mathcal{W}$ at i. Switch back via the adjunction to obtain the following factorization:

$$\begin{tikzcd}
\mathrm{Lan}_a(K) \arrow[r, "\mathrm{Lan}_a(u)"'] \arrow[d, "\mathrm{Lan}_a(i)"'] \arrow[rr, bend left, "\alpha"] & \mathrm{Lan}_a(U) \arrow[r, "\hat{x}"'] \arrow[d, "\mathrm{Lan}_a(j)"] & X \arrow[d, "\varphi"] \\
\mathrm{Lan}_a(L) \arrow[r, "\mathrm{Lan}_a(v)"] \arrow[rr, bend right, "\beta"'] & \mathrm{Lan}_a(V) \arrow[r, "\hat{y}"] & Y
\end{tikzcd}$$

By Corollary 1.3.10 we may assume $\mathcal{W}_i \subseteq {}^{\Box}(I^{\Box}) \cap \mathcal{W}$. Therefore $\mathrm{Lan}_a(j)$ has its components in ${}^{\Box}(I^{\Box}) \cap \mathcal{W}$ and hence $\mathrm{Lan}_a(j) \in \widetilde{\mathcal{W}}$. Consequently, the set $\mathrm{Lan}_a(\mathcal{W}_i)$ is a solution set for $\widetilde{\mathcal{W}}$ at $\mathrm{Lan}_a(i)$. $\square$

At the end of the next section we will present a class of examples where the model structure on $\mathcal{K}$ is given by Cisinski's construction but the model structure on $\mathcal{K}^{\mathcal{A}}$ cannot be constructed in this way. This is again related to the cofibrancy condition in Theorem 2.2.5.

In contrast, the following Lemma provides a method where new model structures are build by transporting Cisinski's construction itself from $\mathcal{K}$ to a reflexive subcategory. We will usually assume that full reflective subcategories are isomorphism closed.

4.1.2 Lemma. *Let $\mathcal{K}$ be a locally presentable category with a cofibrant weak factorization system generated by a set I, a cylinder $(\mathrm{C}, \gamma, \sigma)$ and a reflection $R\colon \mathcal{K} \to \mathcal{A}$ onto a full subcategory $\mathcal{A}$ which is also locally presentable. Then the restriction of $R\mathrm{C}\colon \mathcal{K} \to \mathcal{A}$ to $\mathcal{A}$ provides a cylinder $(R\mathrm{C}, R\gamma, R\sigma)$ for the cofibrant weak factorization system generated by RI in $\mathcal{A}$. Moreover, the following holds:*

(a) The two cylinders $(\mathrm{C}, \gamma, \sigma)$ and $(R\mathrm{C}, R\gamma, R\sigma)$ determine the same homotopy relation on $\mathcal{A}$.

(b) For any $S \subseteq {}^{\Box}(I^{\Box})$ one has $\Lambda(R\mathrm{C}, RS, RI) = R\Lambda(\mathrm{C}, S, I)$. Therefore $\Lambda(R\mathrm{C}, RS, RI)$ and $\Lambda(\mathrm{C}, S, I)$ determine the same fibrant objects in $\mathcal{A}$.

(c) Suppose that $(\mathrm{C}, \gamma, \sigma)$ is cartesian and that the right adjoint of C leaves $\mathcal{A}$ invariant. Then the cylinder $(R\mathrm{C}, R\gamma, R\sigma)$ is also cartesian.

(d) Given $S \subseteq {}^{\Box}(I^{\Box})$, if in the situation of (c) every object of $\mathcal{A}$ is fibrant w.r.t. $\Lambda(\mathrm{C}, S, I)$ then $\mathcal{W}(R\mathrm{C}, RS, RI) = \mathcal{A} \cap \mathcal{W}(\mathrm{C}, S, I)$.

Proof. First observe that by part (a) of Theorem 1.3.5 the set RI indeed generates a weak factorization system in $\mathcal{A}$, which is cofibrant because $\mathcal{A}$ is full. We will repeatedly use the equivalence

$$Rf \,\Box\, g \iff f \,\Box\, g \qquad \text{for all } f \in \mathcal{K}, g \in \mathcal{A} \tag{$*$}$$

which holds by Remark 1.1.2(d). Given any object $A \in \mathcal{A}$, its coproduct with itself in $\mathcal{A}$ is $R(A+A)$ and also $RA \cong A$. Application of R to diagram (1.2.1) in Definition 1.2.1 therefore shows that RCA is indeed a cylinder object for A.

(a) Consider any two maps $f, g\colon A \to B$ in $\mathcal{A}$ and the induced map $(f|g)\colon A + A \to B$ from the coproduct in $\mathcal{K}$. Then $\widehat{(f|g)}\colon R(A+A) \to B$ is the induced map from the coproduct in $\mathcal{A}$. The equivalence $f \sim g \pmod{\mathrm{C}} \iff f \sim g \pmod{R\mathrm{C}}$ now follows with $(*)$.

(b) Because R preserves pushouts, we have $Rf\star R\gamma = R(f\star\gamma)$ and $Rf\star R\gamma^k = R(f\star\gamma^k)$ (for $k = 0, 1$), which gives the equality $\Lambda(R\mathrm{C}, RS, RI) = R\Lambda(\mathrm{C}, S, I)$. By $(*)$ we have

$$R\Lambda(\mathrm{C}, S, I) \,\Box\, (A \to 1) \iff \Lambda(\mathrm{C}, S, I) \,\Box\, (A \to 1)$$

and hence $\Lambda(\mathrm{C}, S, I)$ and $\Lambda(R\mathrm{C}, RS, RI)$ determine the same class of fibrant objects.

(c) Let $G\colon \mathcal{K} \to \mathcal{K}$ be a right adjoint of C with $G\mathcal{A} \subseteq \mathcal{A}$. The isomorphisms (natural in $A, B \in \mathcal{A}$)

$$\mathcal{A}(R\mathrm{C}A, B) \cong \mathcal{K}(\mathrm{C}A, B) \cong \mathcal{K}(A, GB) \cong \mathcal{A}(A, GB)$$

exhibit the cylinder functor as a left adjoint. The second condition in Definition 2.1.9 holds because of (b).

(d) By Corollary 2.2.14 and part (a) above, both $\mathcal{W}(R\mathrm{C}, RS, RI)$ and $\mathcal{A} \cap \mathcal{W}(\mathrm{C}, S, I)$ coincide with the class of homotopy equivalences in $\mathcal{A}$. $\square$

4.1.3 Question. Does Lemma 4.1.2 work for other adjunctions?

More specifically, can it be modified to include reflective subcategories that are not full? Of particular interest would be the case of suitable monads on $\mathcal{K}$. This question is also related to the condition on objects to be cofibrant, because fullness of the inclusion $\mathcal{A} \hookrightarrow \mathcal{K}$ ensured that the induced weak factorization system on $\mathcal{A}$ is cofibrant if the original one on $\mathcal{K}$ is.

4.1.4 Corollary. *Let $\mathcal{K}$ be a locally λ-presentable category with $\mathcal{A}$ the set of λ-presentable objects as in Definition 1.3.1, regarded as a full subcategory of $\mathcal{K}$. Suppose that $\mathcal{K}$ is cartesian closed. Then the cofibrant model structure on $\mathbf{Set}^{\mathcal{A}^{op}}$ of Example 2.3.5 induces a model structure on the full reflective subcategory $\mathcal{K}$.*

Proof. Let $E\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{op}}$ be the functor defined by $E(K)A = \mathcal{K}(A, K)$ (so in particular $E(A) = \mathcal{A}(-, A)$ for $A \in \mathcal{A}$). By [2, Proposition 1.26(i) and Proposition 1.27] this functor is full, faithful and has a left adjoint. Hence we can regard $\mathcal{K}$ as a full reflective subcategory of $\mathbf{Set}^{\mathcal{A}^{op}}$. In order to distinguish between the cartesian closed structures on $\mathbf{Set}^{\mathcal{A}^{op}}$ and $\mathcal{K}$ we use $[L, K]$ instead of K^L for the latter.

We already know that a final cartesian cartesian cylinder for $(\mathrm{Mono}, \mathrm{Mono}^{\Box})$ in $\mathbf{Set}^{\mathcal{A}^{op}}$ is given by $\mathrm{C} = (-) \times \Omega$. So it remains to verify $K^\Omega \in \mathcal{K}$ for any object $K \in \mathcal{K}$. We will do so by showing $K^\Omega \cong [R\Omega, K]$, where $R\colon \mathbf{Set}^{\mathcal{A}^{op}} \to \mathcal{K}$ is the reflection. Fix $K \in \mathcal{K}$.

(1) For all $L \in \mathcal{K}$ and $A \in \mathcal{A}$ we have

$$\mathbf{Set}^{\mathcal{A}^{op}}(L, K^A) \cong \mathbf{Set}^{\mathcal{A}^{op}}(L\times A, K) \cong \mathcal{K}(L\times A, K) \cong \mathcal{K}(L, [A, K]) \cong \mathbf{Set}^{\mathcal{A}^{op}}(L, [A, K])$$

where we use that limits are the same in $\mathcal{K}$ and $\mathbf{Set}^{\mathcal{A}^{op}}$.

(2) Write $\Omega \in \mathbf{Set}^{\mathcal{A}^{op}}$ as a colimit $\Omega = \operatorname{colim}_i A_i$ of representable functors. Then $R\Omega = R(\operatorname{colim}_i A_i)$ is the colimit of the A_i taken in $\mathcal{K}$.

Because exponentiation with a fixed base takes colimits into limits, we obtain $K^\Omega = \lim_i K^{A_i}$ and $[R\Omega, K] = \lim_i [A_i, K]$.

(3) Combining (1) with (2) we obtain

$$\begin{aligned}
\mathbf{Set}^{\mathcal{A}^{op}}(L, K^\Omega) &\cong \mathbf{Set}^{\mathcal{A}^{op}}(L, \lim_i K^{A_i}) \\
&\cong \lim_i \mathbf{Set}^{\mathcal{A}^{op}}(L, K^{A_i}) \\
&\cong \lim_i \mathbf{Set}^{\mathcal{A}^{op}}(L, [A_i, K]) \\
&\cong \mathbf{Set}^{\mathcal{A}^{op}}(L, \lim_i [A_i, K]) \\
&\cong \mathbf{Set}^{\mathcal{A}^{op}}(L, [R\Omega, K])
\end{aligned}$$

for all $L \in \mathcal{K}$. But because $\mathcal{A}$ is dense in $\mathbf{Set}^{\mathcal{A}^{op}}$, this gives also

$$\mathbf{Set}^{\mathcal{A}^{op}}(X, K^\Omega) \cong \mathbf{Set}^{\mathcal{A}^{op}}(X, [R\Omega, K])$$

for all $X \in \mathbf{Set}^{\mathcal{A}^{op}}$. Therefore $K^\Omega \cong [R\Omega, K]$. □

4.1.5 Remark. In the above proof we used that $\mathcal{K}$ contains the dense subcategory $\mathcal{A}$ of $\mathbf{Set}^{\mathcal{A}^{op}}$. In general a cartesian closed category may have a full reflective subcategory that is not invariant under exponentiation but still cartesian closed in its own right (see e.g. Wyler [27, 9.4-9.8]).

Moreover, the proof also shows that $\mathcal{K}$ is invariant under $(-)^X$ for all objects X in $\mathbf{Set}^{\mathcal{A}^{op}}$, not just invariant under $(-)^\Omega$. This suggest the following question.

4.1.6 Question. Does Corollary 4.1.4 still work for categories that are not cartesian closed?

4.1.7. In the situation of Lemma 4.1.2 one cannot expect in general that a final cylinder on $\mathcal{K}$ will induce a final cylinder on the subcategory $\mathcal{A}$ or that $\mathcal{W}(R\mathrm{C}, \emptyset, I)$ is a smallest localizer. Therefore the induced model structure may fail to be left determined even if the original one was. Whenever we use Lemma 4.1.2 in the following section, one can check directly that the induced cylinders are final and hence the induced model structures are left determined. The situation is nicer in the special case where the original model structure comes from a system of balls as in Theorem 3.3.1.

4.1.8 Lemma. *Let $\mathcal{K}$ be a locally presentable category with a weak factorization system generated by sphere maps $I = \{s_n : S^{n-1} \to B^n \mid n \geq 0\}$ and a reflection $R: \mathcal{K} \to \mathcal{A}$ onto a full subcategory which is also locally presentable.*

Suppose that $({}^\square(I^\square), I^\square)$ has a cocylinder (Γ, π, τ) such that Γ leaves $\mathcal{A}$ invariant. Then the restriction of (Γ, π, τ) to $\mathcal{A}$ gives a cocylinder $(\Gamma_{|\mathcal{A}}, \pi, \tau)$ for $({}^\square(RI^\square), RI^\square)$. Moreover the following holds:

(a) For any object X in $\mathcal{A}$ we have $\pi^0_X, \pi^1_X \in RI^{\square} \iff \pi^0_X, \pi^1_X \in I^{\square}$. In particular, if (Γ, π, τ) is fibrant at each object of $\mathcal{A}$ then $(\Gamma_{|A}, \pi, \tau)$ is a fibrant cocylinder.

(b) If (Γ, π, τ) has the homotopy exchange property at each object of $\mathcal{A}$, then the same holds for $(\Gamma_{|A}, \pi, \tau)$.

(c) Suppose that (Γ, π, τ) is fibrant at all objects of $\mathcal{A}$ and has the homotopy exchange property at all objects of $\mathcal{A}$, and that the relative final homotopy relation between parallel cells in $\mathcal{A}$ is transitive. Then $\mathcal{C} = {}^{\square}(RI^{\square})$ and $\mathcal{W}_{\mathcal{C}}(\emptyset)$ give a left determined model structure on $\mathcal{A}$.

Proof. From the system of balls $\{b^0_n, b^1_n \colon B^n \to B^{n+1} \mid n \geq 0\}$ in $\mathcal{K}$ we obtain a system of balls $\{R(b^0_n), R(b^1_n) \colon RB^n \to RB^{n+1} \mid n \geq 0\}$ in $\mathcal{A}$. Also $RI = \{R(s_n) \colon RS^{n-1} \to RB^n \mid n \geq 0\}$ is its system of sphere maps as in Definition 3.1.1 because the left adjoint $R \colon \mathcal{K} \to \mathcal{A}$ preserves colimits. For any two cells $x, y \colon RB^n \to X$ in $\mathcal{A}$ and corresponding cells $\hat{x}, \hat{y} \colon B^n \to X$ in $\mathcal{K}$, we have $(Rs_n)x = (Rs_n)y \iff s_n\hat{x} = s_n\hat{y}$ and also the following equivalence:

$$x \sim y \pmod{\Gamma_{|\mathcal{A}}} \iff \hat{x} \sim \hat{y} \pmod{\Gamma}$$

We will show (a) and (b). Part (c) then follows via Theorem 3.3.1.

(a) By Remark 1.1.2(d) we have $g \in RI^{\square} \iff g \in I^{\square}$ for all $g \in \mathcal{A}$.

(b) We will use the description in Remark 3.2.3. Let $x, y, z \colon RB^n \to X$ be three parallel cells in $\mathcal{A}$ and let $\hat{x}, \hat{y}, \hat{z} \colon B^n \to X$ be the corresponding cells in $\mathcal{K}$. To describe relative final homotopy between parallel n-cells in $\mathcal{A}$ we may take a final relative cylinder object $(P^n, \gamma_n, \sigma_n)$ for S^n in $\mathcal{K}$, apply R to obtain a good relative cylinder object $(RP^n, R\gamma_n, R\sigma_n)$ for RS^n in $\mathcal{A}$, and then use a $(\mathcal{C}, \mathcal{C}^{\square})$-factorization of σ_n into $\lambda_n \colon RP^n \to Q^n$ and $\rho_n \colon Q^n \to RB^n$ to obtain a final relative cylinder object $(Q^n, \gamma_n\lambda_n, \rho_n)$. Now suppose $x \sim y \pmod{\Gamma_{|\mathcal{A}}}$ and $y \overset{R(s_n)}{\sim} z \pmod{Q^n}$:

$$\begin{tikzcd}
 & RB^n \arrow[dl, "x"'] \arrow[d, "g"] \arrow[dr, "y"] & \\
X & \Gamma X \arrow[l, "\pi^0_X"] \arrow[r, "\pi^1_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
RS^n \arrow[d, "R\gamma_n"'] \arrow[drr, "(y|z)"] & & \\
RP^n \arrow[r, "\lambda_n"'] & Q^n \arrow[r] & X
\end{tikzcd}$$

Switching via adjointness gives the diagrams

$$\begin{tikzcd}
 & B^n \arrow[dl, "\hat{x}"'] \arrow[d, "\hat{g}"] \arrow[dr, "\hat{y}"] & \\
X & \Gamma X \arrow[l, "\pi^0_X"] \arrow[r, "\pi^1_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
S^n \arrow[d, "\gamma_n"'] \arrow[dr, "(\hat{y}|\hat{z})"] & \\
P^n \arrow[r] & X
\end{tikzcd}$$

and homotopy exchange for Γ gives $\hat{x} \sim \hat{z} \pmod{\Gamma}$ via a homotopy parallel to $\hat{g}$. This homotopy corresponds via adjointness to a homotopy from x to z which is parallel to g. $\square$

4.1.9 Corollary. *Suppose that in Lemma 4.1.2 the cofibrant weak factorization system is generated by sphere maps $I = \{s_n \colon S^{n-1} \to B^n \mid n \geq 0\}$ and that the cartesian cylinder $(\mathrm{C}, \gamma, \sigma)$ has the homotopy exchange property at all objects of $\mathcal{A}$.*

Suppose further that the right adjoint of C *leaves $\mathcal{A}$ invariant and that every object of $\mathcal{A}$ is fibrant with respect to $\Lambda(\mathrm{C}, \emptyset, I)$.*

Then $\mathcal{W}(R\mathrm{C}, \emptyset, RI) = \mathcal{W}_{\mathcal{C}}(\emptyset)$ where $\mathcal{C} = {}^{\square}((RI)^{\square}$. In particular the model structure induced from $\mathcal{K}$ on $\mathcal{A}$ is left determined.

Proof. First construct (Γ, π, τ) as in Lemma 3.4.1. By part (b) of Lemma 4.1.8, the cocylinder $(\Gamma_{|\mathcal{A}}, \pi, \tau)$ also has the homotopy exchange property. Consequently, its left adjoint $(R\mathrm{C}, R\gamma, R\sigma)$ then also has the homotopy exchange property, as we already noted in Remark 3.4.4. By part (b) of Lemma 4.1.2, every object of $\mathcal{A}$ is fibrant with respect to $\Lambda(R\mathrm{C}, \emptyset, RI)$.

Therefore $\mathcal{C} = {}^{\square}(RI^{\square})$ and $\mathcal{W}(R\mathrm{C}, \emptyset, RI)$ satisfy the conditions in Corollary 3.4.6. $\square$

4.2 Cartesian closed examples

In this section, our main focus is on examples, where the underlying categories are locally presentable, but not toposes. However, they are still cartesian closed and cylinders can be obtained from suitable factorizations of the codiagonals $2 \to 1$ as indicated in Remark 2.3.4.

Moreover, the homotopy relation is already determined by $\mathrm{C}(1)$ in the sense that two maps $f, g \colon X \to Y$ are homotopic if and only if their exponential adjoints $\hat{f}, \hat{g} \colon 1 \to Y^X$ are homotopic. This latter condition often has a direct description in terms of the structure of Y^X, so that it is sufficient to know when two elements $x, y \colon 1 \to X$ are homotopic.

The first example also provides an instance of the second line of generalization, in that the class of cofibrations is not the class of monomorphisms.

4.2.1 Example. Consider $\mathcal{K} = \mathbf{Cat}$, the category of small categories and functors. It has a model structure, the so called "folk model structure", where the cofibrations are those functors that are injective on objects, and the weak equivalences are the usual categorical equivalences. This model structure has been known for some time (hence the name), the first published source seems to be Joyal and Tierney [11]. It has also been later reproved and described in detail by Rezk [21]. We will show that this model structure is left determined by rebuilding it from a generating set of cofibrations and a final cartesian cylinder.

Recall that for any set S one has the discrete category on its elements (written also as S) and the indiscrete category (i.e. the connected groupoid with trivial object groups) on its elements, which we will write as $\overline{\underline{S}}$. These two constructions give functors in the obvious way to provide left and right adjoints for the underlying object functor $\mathrm{Ob} \colon \mathbf{Cat} \to \mathbf{Set}$. In particular we write 2 and $\overline{\underline{2}}$ for the discrete and the indiscrete category on two objects. Moreover, we write $\underline{2}$ for the linearly ordered set $\{0,1\}$ and S^1 for the "parallel pair", i.e. the pushout of the inclusion $2 \hookrightarrow \underline{2}$ with itself.

Consider $I = \{(0 \hookrightarrow 1), (2 \hookrightarrow \underline{2}), s\colon S^1 \to \underline{2}\}$, where the last functor maps both nontrivial arrows of S^1 to the nontrivial arrow of $\underline{2}$.

(1) We first check that I is a set of generating cofibrations. Clearly $I^\square$ consists of all those functors, which are full, faithful and surjective on objects. Moreover, for any map f one has

$$f \in {}^\square(I^\square) \iff \mathrm{Ob}(f) \text{ is a monomorphism}$$

For the direction "$\Rightarrow$", observe that the functor $(\overline{2} \to 1)$ is in $I^\square$ and that $f \,\square\, (\overline{2} \to 1)$ forces $\mathrm{Ob}(f) \,\square\, (2 \to 1)$ in **Set**.

Conversely, consider a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

where $p \in I^\square$ and i is injective on objects. Define $h\colon B \to X$ on objects by $h(i(a)) = f(a)$ and $h(b) \in p^{-1}(g(b))$ for $b \notin i(A)$. This can be done because $\mathrm{Ob}(i)$ is injective and $\mathrm{Ob}(p)$ is surjective. For a morphism $u\colon b \to b'$ in B, define $h(u)\colon h(b) \to h(b')$ to be the unique element of $X(h(b), h(b')) \cap p^{-1}(g(u))$. This works because p is full and faithful. Then h is the desired diagonal.

(2) The cylinder functor $\mathrm{C} = (-) \times \underline{2}$ is obtained from the factorization $2 \hookrightarrow \overline{2} \to 1$ and $\gamma_X\colon X \times 2 \to X \times \underline{2}$ is the usual inclusion. Because $(\overline{2} \to 1)$ is in $I^\square$, the resulting cylinder is final. Two objects $x, y\colon 1 \to X$ of a category X are homotopic iff they are isomorphic. Therefore two functors $f, g\colon X \to Y$ are homotopic iff they are naturally isomorphic.

(3) It remains to check condition (b) of Definition 2.1.9, i.e. stability of I under $(-) \star \gamma$ and $(-) \star \gamma^k$.

For the case of γ, consider a diagram

$$\begin{tikzcd} X + X \arrow[r, "\gamma_X"] \arrow[d, "f+f"'] & \mathrm{C}X \arrow[d] \arrow[dr, bend left, "\mathrm{C}f"] & \\ Y + Y \arrow[r, "q"] \arrow[rr, bend right, "\gamma_Y"'] & Q \arrow[r, "f\star\gamma"] & \mathrm{C}Y \end{tikzcd}$$

where Q is a pushout of $f+f$ and γ_X. The maps $\mathrm{Ob}(\gamma_X)$ and $\mathrm{Ob}(\gamma_Y)$ are bijective. Because the functor Ob preserves pushouts, the map $\mathrm{Ob}(q)$ is also bijective and hence $\mathrm{Ob}(f \star \gamma)$ is bijective.

For the case of γ^0 and γ^1 one can calculate directly that the following two diagrams

$$\begin{tikzcd}
2 \arrow[r, "\gamma_2^k"] \arrow[d, "\gamma_1"'] & 2 \times \overline{\underline{2}} \arrow[d, "\gamma_1 \times \overline{\underline{2}}"] \\
\underline{2} \arrow[r, "\gamma_{\underline{2}}^k"'] & \underline{2} \times \overline{\underline{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1 \arrow[r, "\gamma_{S^1}^k"] \arrow[d, "s"'] & S^1 \times \overline{\underline{2}} \arrow[d, "s \times \overline{\underline{2}}"] \\
\underline{2} \arrow[r, "\gamma_{\underline{2}}^k"'] & \underline{2} \times \overline{\underline{2}}
\end{tikzcd}$$

are pushout squares and hence $(2 \hookrightarrow \underline{2}) \star \gamma^k$ and $p \star \gamma^i$ are isomorphisms. Moreover, $(0 \to 1) \star \gamma^k = \gamma_1^k$.

(4) Now for the computation of $\Lambda(\emptyset, I)$. By (3) above, $\Lambda^0(\emptyset, I)$ consists of isomorphisms and the two inclusions $\gamma_1^0, \gamma_1^1 \colon 1 \to \overline{\underline{2}}$. A direct computation gives that

$$\begin{tikzcd}
1 + 1 \arrow[r, "\gamma_1"] \arrow[d, "\gamma_1^k + \gamma_1^k"'] & 1 \times \overline{\underline{2}} \arrow[d, "\gamma_1^k \times \overline{\underline{2}}"] \\
\overline{\underline{2}} + \overline{\underline{2}} \arrow[r, "\gamma_{\overline{\underline{2}}}"'] & \overline{\underline{2}} \times \overline{\underline{2}}
\end{tikzcd}$$

is a pushout square and hence $\gamma_1^k \star \gamma$ is an isomorphism. Therefore $\Lambda(\emptyset, I)^{\Box} = \Lambda^0(\emptyset, I)^{\Box} = \{\gamma_1^0, \gamma_1^1\}^{\Box}$ and every object of **Cat** is fibrant.

(5) From Corollary 2.2.14 we obtain that $\mathcal{W} = \mathcal{W}(\emptyset, I)$ consists of the categorical equivalences, which completes the construction.

4.2.2 Remark. The set I of generating cofibrations in Examples 4.2.1 can be obtained as sphere maps of a system of balls as in Definition 3.1.1.

(1) Set $B^0 = 1$ and $B^n = \underline{2}$ for $n \geq 1$. As $b_0^0, b_0^1 \colon 1 \to \underline{2}$ we take the two inclusions and let all other b_n^k be identity maps. Then $S^0 = 2$, S^1 is indeed the parallel pair as introduced in the example, $S^n = B^n$ for all $n \geq 2$, and we have $I = \{s_0, s_1, s_2\}$. All higher s_n are isomorphisms and can therefore be omitted. As to be expected, 0-cells are objects, 1-cells are arrows and two 1-cells are parallel if they have the same domain and codomain. The n-cells for $n \geq 2$ also correspond to arrows, but the parallel relation is equality.

(2) For the relative final cylinder objects we can use $P^0 = \overline{\underline{2}}$, $P^1 = \underline{2}$ and $P^n = B^n$ for $n \geq 2$. Two parallel 0-cells are homotopic iff they are isomorphic and for $n \geq 1$ homotopy between parallel n-cells is equality.

(3) Now we check that the cylinder $(\mathrm{C}, \gamma, \sigma)$ in Example 4.2.1 has the homotopy exchange property.

Let x, y, z be parallel n-cells and suppose $x \sim y \pmod{\mathrm{C}}$ via some homotopy h and $y \overset{s}{\sim} z \pmod{P^n}$. We want to show $x \sim z \pmod{\mathrm{C}}$ via some homotopy k

parallel to h For $n \geq 1$ we already noted that homotopy $(\mathrm{mod}\, P^n)$ is equality. Therefore $y = z$ and we can take $k = h$. For $n = 0$ relative homotopy coincides with ordinary homotopy because any two 0-cells (and also homotopies between them) are parallel. Moreover $\mathrm{C}(S^0) = P^0$. The above condition therefore reduces to the requirement that homotopy between 0-cells is transitive, which we verified in Example 4.2.1. The other condition in Remark 3.4.4 is checked in the same way.

Because every object of **Cat** is fibrant with respect to $\Lambda(\mathrm{C}, \emptyset, I)$, the conditions in Corollary 3.4.6 are satisfied. We obtain $J = \{\gamma^0 \colon 1 \to \overline{2}\}$ as a set of generating trivial cofibrations.

The next three examples are applications of Lemma 4.1.2. In view of the above Remark one can also regard them as applications of Lemma 4.1.8.

4.2.3 Example. Let $\mathcal{K} = \mathbf{Cat}$ and $\mathcal{A} = \mathbf{PrOrd}$, the category of preordered sets (i.e. sets with a reflexive and transitive relation) and monotone maps. **PrOrd** has a model structure where the cofibrations are the monomorphisms and the weak equivalences are the categorical equivalences. We will obtain it from the previous one on **Cat**.

The reflection $R \colon \mathbf{Cat} \to \mathbf{PrOrd}$ is bijective on objects and identifies parallel arrows. We will keep the notation from Example 4.2.1. Discarding the isomorphism Rs from RI, we obtain the generating set $I' = RI \setminus \{Rs\} = \{(0 \to 1), (2 \hookrightarrow \underline{2})\}$. One has ${}^{\square}(I'^{\square}) = \mathrm{Mono}$, which is obtained exactly as in Example 4.2.1, keeping in mind that functors between preorders are always faithful and that the monomorphisms in **PrOrd** are exactly the functors that are injective on objects. The right adjoint to $(-) \times \overline{2}$ is $(-)^{\overline{2}}$ which leaves **PrOrd** invariant. Every object is fibrant and therefore $\mathcal{W}' = \mathcal{W}(\emptyset, I')$ consists of the categorical equivalences.

4.2.4 Example. Let $\mathcal{K} = \mathbf{PrOrd}$ and $\mathcal{A} = \mathbf{Ord}$, the category of ordered sets (i.e. sets with a reflexive, transitive and antisymmetric relation) and monotone maps. **Ord** has a model structure where the cofibrations are all maps and the weak equivalences are the isomorphisms. We will obtain it from the previous one on **PrOrd**.

The reflection $R \colon \mathbf{PrOrd} \to \mathbf{Ord}$ assigns to every preordered set X the quotient $X/{\sim}$ obtained from identifying homotopic elements. The generating set $I' = \{(0 \to 1), (2 \hookrightarrow \underline{2})\}$ is already contained in **Ord** and hence $I' = RI'$. Because a full surjective functor between ordered sets must be an isomorphism, the class $I^{\square}$ consists of all isomorphisms and consequently ${}^{\square}(I^{\square}) = \mathbf{Ord}$. For any ordered set P one has $P^{\overline{2}} = P$. Therefore **Ord** is invariant under $(-)^{\overline{2}}$. Every object is fibrant and therefore $\mathcal{W}' = \mathcal{W}(\emptyset, I')$ is the class of isomorphisms.

4.2.5 Example. Let $\mathcal{K} = \mathbf{PrOrd}$ and $\mathcal{A} = \mathbf{Set}$. Here we identify **Set** with the full subcategory of indiscrete preordered sets. It has a model structure where the cofibrations are the monomorphisms and the weak equivalences are the maps between nonempty sets together with the identity map of the empty set. This (almost trivial) model structure is also mentioned in [4, Exemple 3.7] and [26, Section 3]. It can be constructed with the cylinder in Example 2.3.5, with the set of generating cofibrations given by the proof in [3, Proposition 1.12]. Instead we will obtain it from the one on **PrOrd** in Example 4.2.3.

The reflection R: **PrOrd** $\to$ **Set** assigns to every preordered set the indiscrete preorder on its elements. Let I' be as in Example 4.2.3. Discarding the identity map $\underline{\overline{2}}$ from RI', we obtain the generating set $I'' = \{(0 \to 1)\}$ in **Set**. Then $I''^{\square}$ is the class of surjective maps and ${}^{\square}(I''^{\square}) = \text{Mono}$. For any indiscrete preorder X, the preorder $X^{\underline{\overline{2}}}$ is again indiscrete. Therefore **Set** is invariant under $(-)^{\underline{\overline{2}}}$. Every object is fibrant and therefore $\mathcal{W}'' = \mathcal{W}(\emptyset, I'')$ consists of the identity map of the empty set and of all maps with nonempty domain.

In the previous examples, all objects were fibrant and consequently the homotopy relation already determined the weak equivalences via Corollary 2.2.14. Here is an example where this does not happen.

4.2.6 Example. Let $\mathcal{K} = \mathbf{rsRel}$, the category of plain undirected graphs (i.e. sets with a reflexive and symmetric relation together with maps preserving such relations). We will construct a left determined model structure on **rsRel** where the cofibrations are the monomorphisms and the weak equivalences are those maps that induce bijections between path components. It can be seen as the one-dimensional version of the left determined model structure on simplicial complexes as described in [26, Remark 3.7].

We will write n for the discrete graph on n vertices, K_n for the indiscrete (i.e. complete) graph on n vertices and K_n^- for the graph obtained from K_n by deleting one edge. Consider the set $I = \{(0 \to 1), (2 \hookrightarrow K_2)\}$, where the second map is the usual inclusion.

(1) We first check that I is a set of generating cofibrations. The class $I^{\square}$ consists of those maps $f\colon (X, \alpha) \to (Y, \beta)$ that are surjective and full (i.e. satisfy $f(x)\beta f(x') \implies x\alpha x'$).

Moreover one has ${}^{\square}(I^{\square}) = \text{Mono}$. This follows by the same argument as in the case of categories (step (1) in Example 4.2.1) with K_2 in place of $\underline{\overline{2}}$.

(2) The cylinder functor $\mathrm{C} = (-) \times K_2$ is obtained from the factorization $2 \hookrightarrow K_2 \to 1$ and $\gamma_X\colon X \times 2 \to X \times K_2$ is the usual inclusion. Because $(K_2 \to 1)$ is in $I^{\square}$, the resulting cylinder is final. Two vertices $x, y\colon 1 \to X$ of a graph are homotopic iff they are joined by an edge in X. Therefore, for two maps $f, g\colon (X, \alpha) \to (Y, \beta)$ one has

$$f \sim g \iff \forall x, x' \in X : (x\alpha x' \implies f(x)\beta g(x'))$$

because Y^X is $\mathbf{rsRel}(X, Y)$ equipped with the relation β^α defined by the condition on the right side of the above equivalence. In particular the homotopy relation is not transitive in general. The homotopy relation on $\mathbf{rsRel}(X, Y)$ is transitive whenever Y (i.e. its relation) is transitive. Moreover, if Y is discrete then homotopy coincides with equality.

(3) For a partial description of $\Lambda = \Lambda(\emptyset, I)$ first observe, that the forgetful functor **rsRel** $\to$ **Set** preserves pushouts. In particular, in a pushout diagram

$$\begin{tikzcd}
A \times 2 \arrow[r, "\gamma_A"] \arrow[d, "f\times 2"'] & A \times K_2 \arrow[d] \\
B \times 2 \arrow[r] & Q
\end{tikzcd}$$

one can assume that the underlying set of Q is $B\times 2$, that the horizontal underlying maps are identity maps and that the two vertical underlying maps coincide. Now suppose that A is nonempty and B is indiscrete.

Then Q is path connected: given any $b, b' \in B$ and $i, j \in 2$, take some $a \in A$ with b —— $f(a)$ —— b' . Then

(i) (b, i) —— $(f(a), i)$ in $B \times 2$

(ii) (a, i) —— (a, j) in $A \times K_2$

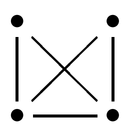
(iii) $(f(a), j)$ —— (b', j) in $B \times 2$

and passing to Q gives a path (b, i) —— $(f(a), i)$ —— $(f(a), j)$ —— (b', j) in Q. Hence, if $f\colon A \to B$ is an inclusion then $f \star \gamma$ is the inclusion of the (nonempty) path connected Q into the indiscrete $B \times K_2$.

As in Example 4.2.1 we have $(0 \to 1) \star \gamma^k = \gamma_1^k\colon 1 \to K_2$. From the inclusion $\gamma_1\colon 2 \to K_2$ we obtain the following diagram

$$\begin{tikzcd}
2 \arrow[r, "\gamma_2^0"] \arrow[d, "\gamma_1"'] & 2 \times K_2 \arrow[d] \arrow[ddr, bend left, "\gamma_1 \times K_2"] & \\
K_2 \arrow[r] \arrow[drr, bend right, "\gamma_{K_2}^0"'] & K_4^- \arrow[dr, "\gamma_1 \star \gamma^0"] & \\
 & & K_2 \times K_2
\end{tikzcd}$$

where (according to the notation introduced) K_4^- is the graph



and $\gamma_1 \star \gamma^0$ is the inclusion of K_4^- into $K_4 = K_2 \times K_2$. Up to a permutation of vertices, the same inclusion is obtained as $\gamma_1 \star \gamma^1$.

Hence each map in Λ^0 is the inclusion of a nonempty path connected subgraph of some suitable K_n Applying the above observation gives (via induction) that each

Λ^n consists only of maps of this type. Except for the two inclusions γ_1^0 and γ_1^1, the included subgraph of K_n is wide, i.e. it has the maximal number of vertices.

Consequently, every transitive graph T is fibrant: given some inclusion $P \hookrightarrow K_n$ with P path connected and $|P| = n$, any map $f\colon P \to T$ can be extended to $h\colon K_n \to T$ by $h(x) := f(x)$.

Conversely, assume that X is fibrant. Observe that $K_3^- \hookrightarrow K_3$ is in ${}^{\square}(\Lambda^{\square})$ because it can be obtained from $K_4^- \hookrightarrow K_4$ as a pushout

$$\begin{tikzcd} K_4^- \arrow[r, "p"] \arrow[d, hook] & K_3^- \arrow[d, hook] \\ K_4 \arrow[r] & K_3 \end{tikzcd}$$

where p is the surjection that collapses the two vertices of degree 3. Therefore, every map $f\colon K_3^- \to X$ can be extended to a map $f'\colon K_3 \to X$, which is precisely the definition of transitivity.

In summary, the fibrant graphs are exactly the transitive graphs.

(4) For a graph (X, α), a path component is an equivalence class of the transitive closure α^* of the relation α. We write $[x]$ for the equivalence class of any $x \in X$ and $\pi_0 X$ for the discrete graph on the set $\{[x] \mid x \in X\}$. Setting $\pi_0 f([x]) := [f(x)]$ for any $f\colon X \to Y$ makes π_0 into a functor and the canonical map $r_X\colon X \to \pi_0 X$ with $r(x) = [x]$ gives a reflection into the subcategory of discrete graphs. For two maps $f, g\colon (X, \alpha) \to (Y, \beta)$ one has:

$$\pi_0 f = \pi_0 g \iff \forall x, x' \in X : (x \alpha^* x' \implies f(x) \beta^* g(x'))$$

Comparing this with the homotopy condition

$$f \sim g \iff \forall x, x' \in X : (x \alpha x' \implies f(x) \beta g(x'))$$

one obtains that always $f \sim g \implies \pi_0 f = \pi_0 g$ and that the converse implication $\pi_0 f = \pi_0 g \implies f \sim g$ holds whenever β is already transitive. In the general case of a map $f\colon X \to Y$ one has:

$$f \in \mathcal{W} \iff \pi_0 f \text{ is an isomorphism}$$

For the direction "$\Rightarrow$" assume $f \in \mathcal{W}$. Remark 2.2.13 with $t = r_X$ and $T = \pi_0 X$ gives a map $u\colon X \to \pi_0 X$ such that in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r_X"'] & Y \arrow[dl, "u"] \arrow[d, "r_Y"] \\ \pi_0 X \arrow[r, "\pi_0 f"'] & \pi_0 Y \end{tikzcd}$$

we have $r_X \sim fu$. Then also $fr_Y = r_X(\pi_0 f) \sim fu(\pi_0 f)$ and by Remark 2.2.13 with $t = r_X(\pi_0 f)$ and $T = \pi_0 Y$ this forces $r_Y \sim u(\pi_0 f)$. But for discrete codomains, homotopy means equality and hence the above diagram strictly commutes. Applying the functor π_0 to that diagram exhibits $\pi_0 u$ as the two-sided inverse of $\pi_0 f$.

For the direction "$\Leftarrow$" assume that $\pi_0 f$ is an isomorphism and let $t\colon X \to T$ be a map to a transitive graph T. Uniqueness up to homotopy follows from the equivalence

$$\begin{aligned} fh \sim fh' &\iff (\pi_0 f)(\pi_0 h) = (\pi_0 f)(\pi_0 h') \\ &\iff \pi_0 h = \pi_0 h' \iff h \sim h' \end{aligned}$$

for any $h, h'\colon Y \to T$ because T is transitive.

For existence, let $s\colon \pi_0 T \to T$ be a section of r_T with $\pi_0 s = \pi_0 T$ (i.e. a choice of representatives of the path components) and define $h\colon Y \to T$ as the composite $h = r_Y(\pi_0 f)^{-1}(\pi_0 t)s$. Then $\pi_0(fh) = \pi_0 t$ and hence $fh \sim t$.

4.2.7 Remark. Example 4.2.6 illustrates how the constructions in Section 3.1 can be carried out without producing a class of weak equivalences. We first build the set of generating cofibrations from a system of balls: Set $B^0 = 1$ and $B^n = K_2$ for $n \geq 1$. As $b_0^0, b_0^1\colon 1 \to K_2$ we take the two inclusions and let all other b_n^k be identity maps. Then $S^0 = 2$ and $S^n = K_2 = B^n$ for all $n \geq 1$. We obtain $I = \{s_0, s_1\}$ because all higher s_n are isomorphisms. The 0-cells of a graph are its vertices. For $n \geq 1$, the n-cells correspond to pairs of related vertices and the parallel relation is equality.

For the relative final cylinder objects we can use $P^n = K_2$ for all $n \geq 0$. Two 0-cells are homotopic iff they are connected by an edge, and for $n \geq 1$ any two parallel n-cells are homotopic (because they are equal). In particular, homotopy between parallel cells of a graph X is transitive iff the graph X is transitive.

A map $f\colon (X, \alpha) \to (Y, \beta)$ has the relative homotopy lifting property w.r.t. I iff it is full and surjective up to homotopy. The latter means that it satisfies the following condition:

$$\forall y \in Y : \exists x \in X : f(x)\alpha y$$

Let $\mathcal{W}$ be the class of all these maps.

To see that $\mathcal{W}$ is not related to the class $\mathcal{W}(\mathrm{C}, \emptyset, I)$ of weak equivalences, consider for example the graph $\underline{3} = K_3^-\colon\ 0 \!-\!\!-\!\!-\! 1 \!-\!\!-\!\!-\! 2$. The inclusion of $\{0\}$ into $\underline{3}$ is a weak equivalence but is not in $\mathcal{W}$, and the inclusion of (the discrete graph) $\{0, 2\}$ into $\underline{3}$ is in $\mathcal{W}$ but is not a weak equivalence.

The cylinder $(\mathrm{C}, \gamma, \sigma)$ has the homotopy exchange property at a graph X iff X is transitive. Therefore Corollary 4.1.9 can also be applied to the next example to ensure that the induced model structure is left determined.

4.2.8 Example. Keep the notation of the Example 4.2.6 and consider the full reflective subcategory **eqRel** of transitive graphs, i.e. sets equipped with an equivalence relation.

It has a model structure where the cofibrations are the monomorphisms and the weak equivalences are those maps that induce bijections between equivalence classes. This model structure has been described in detail by Lárusson [15]. We will obtain it via Lemma 4.1.2 from the previous one on **rsRel**.

The reflection R: **rsRel** $\to$ **eqRel** assigns to every graph (X, α) its transitive closure (X, α^*). Because the graphs 0, 1, 2 and K_2 are already transitive, one obtains $RI = I$ and also ${}^{\square}(RI^{\square}) = \mathrm{Mono} \cap \mathbf{eqRel}$ as in step (1) above. Moreover, if X is transitive then so is X^{K_2} and we already noted in step (3) that all transitive graphs are fibrant. From Lemma 4.1.2 we now obtain that $\mathcal{W}' = \mathcal{W}(\emptyset, I)$ consists of those maps f where $\pi_0 f$ is an isomorphism, i.e. those maps that induce a bijection between equivalence classes. Finally observe, that R preserves full surjections. Therefore the induced cylinder is again final and the induced model structure is left determined. By the above remark one can also obtain this model structure via Lemma 4.1.8. In particular, this gives the set $\{1 \hookrightarrow K_2\}$ of generating trivial cofibrations from [15].

4.2.9 Question. Can fibrant objects be characterized by homotopy alone?

By Lemma 2.2.12, the homotopy relation on $\mathcal{K}(X, Y)$ is transitive whenever Y is fibrant. Can this property be used as a characterization of fibrant objects?

Of course this cannot work in general because, as introduced in 2.2.7, the class of fibrant objects depends on $\Lambda(\mathrm{C}, S, I)$ and this depends on the choice of the set S in Definition 2.1.6. So one should restrict attention to the special case $S = \emptyset$ which gives the largest possible class of fibrant objects. So far the only nontrivial example (i.e. where not all objects are fibrant) is Example 4.2.6, where indeed the fibrant objects are exactly the transitive graphs.

4.2.10 Remark. We already noted in Remark 2.2.6 that the condition that all objects are cofibrant cannot simply be dropped from Theorem 2.2.5. A closer look at its proof shows that the assumption that objects are cofibrant enters at three different points:

(a) Whenever we use that γ^0_X and γ^1_X and hence $\gamma^0_X + \gamma^1_X$ are in $\mathcal{C}$. This happens e.g. in Lemma 2.2.12 (the homotopy relation is transitive on $\mathcal{K}(X, Y)$ for fibrant Y).

(b) Whenever we use $(0 \to X) \star \gamma \in {}^{\square}(\Lambda^{\square})$. This happens in Lemma 2.2.4 (the natural maps γ^0 and γ^1 have their components in ${}^{\square}(\Lambda^{\square})$) and indirectly in Corollary 2.2.16 and Lemma 2.2.22.

(c) Whenever we use that every map in $\mathcal{C}^{\square}$ is a retraction. This happens in Lemma 2.2.9 (every map in $\mathcal{C}^{\square}$ is a dual strong deformation retract).

The first two points could be repaired: introduce the assumption of X cofibrant, whenever a usage of type (a) appears, and restrict $\Lambda(\mathrm{C}, S, I)$ to the case where all maps in I and S have cofibrant domains and codomains.

The serious obstacle is posed by (c) because, if every map in $\mathcal{C}^{\square}$ is a retraction then every map $(0 \to X)$ is in ${}^{\square}(\mathcal{C}^{\square}) = \mathcal{C}$, i.e. that every object is cofibrant. This is also exploited in the next example, where Cisinski's construction cannot be used even for presheaf toposes and when each generating cofibration is a monomorphism with cofibrant domain and codomain.

A dual situation occurs in the proof of Theorem 3.3.1. By Corollary 3.3.9, every map in $\mathcal{C} \cap \mathcal{W}$ is a split monomorphism and therefore every map $(X \to 1)$ is in $(\mathcal{C} \cap \mathcal{W})^{\square}$, i.e every object is fibrant.

4.2.11 Example. Let $\mathcal{K} = \mathbf{Set}$ and recall the model structure from Example 4.2.5 where $I = \{0 \to 1\}$ is the set of generating cofibrations and the weak equivalences are the maps with nonempty domain together with the identity on the empty set. Let $\mathcal{A}$ be a small category $\mathcal{A}$. Then Lemma 4.1.1 gives a cofibrantly generated model structure on $\mathbf{Set}^{\mathcal{A}^{op}}$ with generating cofibrations $\tilde{I} = \{\mathrm{Lan}_a(0 \to 1) \mid a \in \mathcal{A}\}$.

Because each Lan_a is a left adjoint, we must have $\mathrm{Lan}_a(0) = 0$ and the formula (4.1.1) with $\mathcal{A}^{op}$ in place of $\mathcal{A}$, gives $\mathrm{Lan}_a(1) = \mathcal{A}(-, a)$. Therefore $\tilde{I} = \{0 \to \mathcal{A}(-, a) \mid a \in \mathcal{A}\}$ and $\tilde{I}^{\square}$ consists of those (natural) maps with surjective components, i.e. the epimorphisms in $\mathbf{Set}^{\mathcal{A}^{op}}$.

Now, if ${}^{\square}(\tilde{I}^{\square})$ and $\mathcal{W}(\mathrm{C}, S, \tilde{I})$ are to form a model structure for a suitable choice of C and S, then in particular $\tilde{I}^{\square} \subseteq \mathcal{W}(\mathrm{C}, S, \tilde{I})$ must hold. This means in particular that whenever the map $X \to 1$ is an epimorphism (and hence X fibrant), $\mathbf{Set}^{\mathcal{A}^{op}}(1, X)$ must not be empty. We now give two popular choices for $\mathcal{A}$ so that this condition fails:

(a) Let $\mathcal{A} = S^1$, where S^1 is the 'parallel pair' from Example 4.2.1. Then $\mathbf{Set}^{S^{1\,op}}$ is the category of directed graphs (with multiple edges and loops allowed). Let X be any graph with an edge but no loop.

(b) Let $\mathcal{A} = G$ for some nontrivial group G. Then $\mathbf{Set}^{G^{op}}$ is the category of G-sets, i.e. sets equipped with an action of G. Let X be any nonempty G-set without fixpoints.

The model structure on G-sets can be constructed from balls as follows. The construction works also when G is only a monoid.

Set $B^0 = G$ and $B^n = G + G$ for $n \geq 1$, where G acts on itself via left-multiplication. For $b_0^0, b_0^1 \colon G \to G + G$ we take the two coproduct inclusions and let b_n^k be identity maps for $n \geq 1$. Then $S^n = G + G$ for $n \geq 0$ and $I = \{\emptyset \hookrightarrow G\}$ is the set of generating cofibrations because all other sphere maps are isomorphisms.

We can take $P^n = G + G$, and any two parallel cells are homotopic. The trivial cocylinder (Γ, π, τ) with $\Gamma X = X \times X$, $\pi = X \times X$ and $\tau_X = (X, X)$ is fibrant and has the homotopy exchange property. In the resulting model structure, a map $f \colon X \to Y$ is a weak equivalence iff it satisfies the condition $X = \emptyset \implies Y = \emptyset$. This is indeed the condition in the description of $\widetilde{\mathcal{W}}$ in Lemma 4.1.1.

4.3 Modules

We now turn from "space-like" to "linear" examples.

4.3.1 Lemma. *Let $\mathcal{K}$ be an abelian category.*

(a) For maps $m\colon A \to B$ and $f\colon X \to Y$ the following are equivalent

(i) $m \,\Box\, f$.

(ii) $m \,\Box\, (\ker(f) \to 0)$ and for any given square

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "m"'] & X \arrow[d, "f"] \\ B \arrow[r, "g"'] & Y \end{tikzcd} \tag{4.3.1}$$

the map g can be lifted through f by a map $h\colon B \to X$.

(b) $\mathrm{Mono}^{\Box}$ *consists of all epimorphisms with injective kernel.*

(c) ${}^{\Box}\mathrm{Epi}$ *consists of all monomorphisms with projective cokernel.*

Proof. Note that (c) is just (b) interpreted in the opposite category $\mathcal{K}^{op}$. It is therefore sufficient to prove (a) and (b). The calculations are as in [1, Example 1.8(i)]).

(a) Suppose $m \,\Box\, f$. Then any diagonal of the square (4.3.1) will in particular be a lifting of g through f. Let $k\colon \ker(f) \to X$ be the kernel inclusion and consider any map $u\colon A \to \ker(f)$. We obtain the following diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "m"'] & \ker(f) \arrow[r, "k"] & X \arrow[d, "f"] \\ B \arrow[ur, "t"] \arrow[urr, "d"'] \arrow[rr, "0"'] & & Y \end{tikzcd}$$

where d exists because $m \,\Box\, f$ and where t is the factorization of d through k induced by $df = 0$. The remaining equation $mt = u$ follows from $mtk = md = uk$ because k is monic.

Conversely suppose that the conditions in (ii) are satisfied and consider diagram (4.3.1). Let $h\colon B \to X$ be a map with $hf = g$. Then $(v - mh)f = vf - mg = 0$ and therefore $v - mh$ factors through $k\colon \ker(f) \to X$ via some $u\colon A \to \ker(f)$. Let $t\colon B \to \ker(f)$ be a map with $mt = u$ and set $d = h + tk$. Then $df = hf = g$ and $md = mh + uk = mh + (v - mh) = v$ show that d is the desired diagonal.

(b) Any epimorphism f with injective kernel is a split epimorphism and satisfies $\ker(f) \to 0 \in \mathrm{Mono}^\Box$. Therefore condition (ii) of part (a) holds with respect to every $m \in \mathrm{Mono}$.

Conversely, if $f\colon X \to Y$ is in $\mathrm{Mono}^\Box$ then $\ker(f)$ is injective by part (a) and f must be a (split) epimorphism because of $(0 \to Y) \,\Box\, f$. $\Box$

Let R be a ring and let $\mathcal{K} = {}_R\mathbf{Mod}$, the category of left R-modules. We also write $\mathbf{Mod}_R$ and ${}_R\mathbf{Mod}_R$ for the categories of right and two-sided R-modules respectively.

4.3.2 Corollary. *In $\mathcal{K} = {}_R\mathbf{Mod}$ the pairs $({}^\Box\mathrm{Epi}, \mathrm{Epi})$ and $(\mathrm{Mono}, \mathrm{Mono}^\Box)$ are cofibrantly generated weak factorization systems. The singleton set $\{0 \to R\}$ generates $({}^\Box\mathrm{Epi}, \mathrm{Epi})$, and the set $I = \{\mathfrak{a} \hookrightarrow R \mid \mathfrak{a} \trianglelefteq R\}$ of all inclusions of left ideals generates $(\mathrm{Mono}, \mathrm{Mono}^\Box)$.*

Proof. Clearly $\{0 \to R\}^\Box = \mathrm{Epi}$ and $\mathrm{Mono}^\Box \subseteq I^\Box$. It remains to verify $I^\Box \subseteq \mathrm{Mono}^\Box$. Suppose $f \in I^\Box$. With $m = (0 \to R)$ in condition (ii) of Lemma 4.3.1(a) we obtain that f is an epimorphism. Condition (ii) also gives $(\ker(f) \to 0) \in I^\Box$. By Baer's criterion this implies $(\ker(f) \to 0) \in \mathrm{Mono}^\Box$. Therefore $f \in \mathrm{Mono}^\Box$ by part (b) of the Lemma. $\Box$

We will only consider model structures on $\mathcal{K} = {}_R\mathbf{Mod}$ constructed from the weak factorization systems of the above Corollary. The following example concerns $({}^\Box\mathrm{Epi}, \mathrm{Epi})$ and illustrates that there is only the trivial result.

4.3.3 Example. The model structure $({}^\Box\mathrm{Epi}, {}_R\mathbf{Mod}, \mathrm{Epi})$ on ${}_R\mathbf{Mod}$ is left determined. We construct it via Theorem 3.3.1.

(1) Set $B^0 = R$ and $B^n = R + R$ for $n \geq 1$. Take the two coproduct inclusions $\iota^0, \iota^1\colon R \to R + R$ as b_0^0 and b_0^1, and let all other b_n^k be identity maps.

Then $S^n = R + R$ for all $n \geq 0$ and the $s_n\colon S^{n-1} \to B^n$ are isomorphisms for $n \geq 1$. This leaves $s_0 = (0 \to R)$ as the only relevant generating cofibration. The 0-cells of a module are its elements, its n-cells for $n \geq 1$ are pairs of elements and any two n-cells are parallel.

(2) For the relative final cylinder objects we can use $P^n = R + R = S^n$ for all $n \geq 0$. Any two parallel n-cells are homotopic.

(3) Let (Γ, π, τ) be the trivial cylinder with $\Gamma M = M + M = M \times M$, $\pi_M = M + M$ and $\tau_M = (M, M)$. Then (Γ, π, τ) is fibrant. For two (parallel) n-cells of M, the induced n-cell of the product $M \times M$ gives a homotopy between them. Therefore (Γ, π, τ) has the homotopy exchange property.

Therefore Theorem 3.3.1 can be applied. Because every map has the relative homotopy lifting property with respect to $(0 \to R)$, the resulting model structure has $\mathcal{W} = {}_R\mathbf{Mod}$.

We now consider the case of $(\mathrm{Mono}, \mathrm{Mono}^{\square})$. For this it remains to find cartesian cylinders.

In order to find possible examples, we first characterize cartesian cylinders for the weak factorization system $(\mathrm{Mono}, \mathrm{Mono}^{\square})$ in $\mathcal{K}$. Recall that a map $f\colon U \to V$ of right modules is **pure** (or equivalently that $f(U)$ is a pure submodule of V) if for every (finitely generated) left module M, the map $f \otimes_R M\colon U \otimes_R M \to V \otimes_R M$ is a monomorphism. We use another characterization of pure submodules: $U \subseteq V$ is pure iff every finite system of equations

$$u_j = \sum_i x_i r_{ij} \quad (u_j \in U,\ r_{ij} \in R)$$

which has a solution with $x_i \in V$ also has a solution with $x_i \in U$. For a direct proof, which can easily be adapted to the non-commutative setting, see e.g. Matsumura [18, Theorem 7.13].

4.3.4 Proposition. *Suppose V is a two-sided R-module together with a map $v\colon R \to V$ in ${}_R\mathbf{Mod}_R$ and let $\mathrm{C}_v\colon \mathcal{K} \to \mathcal{K}$ be the functor with $\mathrm{C}_v(M) = (R+V) \otimes_R M = M + V \otimes_R M$. Let $\gamma_R^0\colon R \to R+V$ be the coproduct injection, $\sigma_R\colon R+V \to R$ be the product projection and $\gamma_R^1 = (R, v)\colon R \to R+V$. Set $\sigma = \sigma_R \otimes_R (-)$ and $\gamma = (\gamma_R^0|\gamma_R^1) \otimes_R (-)$.*

Then $(\mathrm{C}_v, \gamma, \sigma)$ is a cylinder. For any two maps $f, g\colon M \to N$ we have

$$f \sim g \pmod{\mathrm{C}_v} \iff g - f\colon M \to N \textit{ factors through } v \otimes_R M\colon M \to V \otimes_R M$$

Moreover the following holds:

(a) Every left adjoint cylinder $(\mathrm{C}, \gamma, \sigma)$ arises as $(\mathrm{C}_v, \gamma, \sigma)$ for some $v\colon R \to \ker(\sigma_R)$.

(b) $(\mathrm{C}_v, \gamma, \sigma)$ is good $\iff$ $v\colon R \to V$ is a pure monomorphism (in $\mathbf{Mod}_R$).

(c) Suppose that $(\mathrm{C}_v, \gamma, \sigma)$ is a good cylinder. Then we have:

$$(\mathrm{C}_v, \gamma, \sigma) \textit{ is cartesian} \iff V \textit{ is a flat right module}$$

(d) Suppose that $(\mathrm{C}_v, \gamma, \sigma)$ is a good cylinder. Then we have:

$$(\mathrm{C}_v, \gamma, \sigma) \textit{ is final} \iff V \otimes_R M \textit{ is injective for every } M$$

Proof. We use familiar matrix notation for maps between (co)products and omit the object names for identities and zero maps. Then the maps introduced above can be written as $\gamma_R^0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\gamma_R^1 = \begin{pmatrix} 1 & v \end{pmatrix}$, $\gamma_R = \begin{pmatrix} 1 & 0 \\ 1 & v \end{pmatrix}$ and $\sigma_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Abbreviating $v \otimes_R M$ as v_M and $V \otimes f$ as f_V, we can also write $\gamma_M = \begin{pmatrix} 1 & 0 \\ 1 & v_M \end{pmatrix}$ and $\mathrm{C}f = \begin{pmatrix} f & 0 \\ 0 & f_V \end{pmatrix}$.

Because of $\begin{pmatrix} 1 & 0 \\ 1 & v_M \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ the maps γ_M and σ_M clearly factor the codiagonal and $(\mathrm{C}_v, \gamma, \sigma)$ is a cylinder.

Given two maps $f, g\colon M \to N$, the map $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\colon M + M \to N$ can be extended along $\gamma_M\colon M + M \to M + V \otimes_R M$ iff the equation

$$\begin{pmatrix} 1 & 0 \\ 1 & v_M \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

can be solved with some $h_1\colon M \to N$ and $h_2\colon V \otimes_R M \to N$. This is equivalent to the condition that $g - f\colon M \to N$ extends along $v_M\colon M \to V \otimes_R M$.

(a) Let $(\mathrm{C}, \gamma, \sigma)$ be a cylinder such that C has a right adjoint G.

Application of C to the right translations $\rho_r\colon R \to R$ for each $r \in R$ gives a right action of R on $\mathrm{C}R$ which makes $\mathrm{C}R$ into a two-sided module such that the isomorphisms

$$\mathcal{K}(\mathrm{C}R, M) \cong \mathcal{K}(R, G(M)) \cong G(M)$$

are isomorphisms of left modules and hence $\mathrm{C} \cong \mathrm{C}(R) \otimes_R (-)$. Moreover, the diagrams

$$\begin{tikzcd} R \arrow[r, "\gamma_R^k"] \arrow[d, "\rho_r"'] & \mathrm{C}R \arrow[r, "\sigma_R"] \arrow[d, "\mathrm{C}(\rho_r)"] & R \arrow[d, "\rho_r"] \\ R \arrow[r, "\gamma_R^k"] & \mathrm{C}R \arrow[r, "\sigma_R"] & R \end{tikzcd}$$

show that σ_R and the γ_R^k are maps of two-sided modules. Therefore the decomposition $\mathrm{C}R = \gamma_R^0(R) + \ker(\sigma_R)$ is indeed a decomposition as two-sided modules. With respect to this decomposition, we obtain $\gamma_R^0 = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$, and $\sigma_R = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Moreover, $\gamma_R^1 = \left(\begin{smallmatrix} 1 & v \end{smallmatrix}\right)$ for some $v\colon R \to \ker(\sigma_R)$. Application of naturality of γ and σ to an $m\colon R \to M$ then gives $\gamma_M = \gamma_R \otimes_R M$ and $\sigma_M = \sigma_R \otimes_R M$.

(b) From $\gamma_M = \left(\begin{smallmatrix} 1 & 0 \\ 1 & v_M \end{smallmatrix}\right)$ we obtain that γ_M is a monomorphism iff v_M is a monomorphism.

(c) Let $i\colon M \to N$ be a monomorphism.

The pushout of i and γ_M^0 is $N + V \otimes_R M$ and $i \star \gamma^0$ is the map $\left(\begin{smallmatrix} 1 & 0 \\ 0 & i_V \end{smallmatrix}\right)\colon N + V \otimes_R M \to N + V \otimes_R N$. Therefore $i \star \gamma^0$ is a monomorphism iff i_V is a monomorphism. In particular, flatness of V is necessary for $(\mathrm{C}_v, \gamma, \sigma)$ to be cartesian.

Now suppose V is flat. As seen above, $i \star \gamma^0$ is a monomorphism. Because of $\left(\begin{smallmatrix} 1 & v_M \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & v_M \\ 0 & 1 \end{smallmatrix}\right)$ the maps γ_M^0 and γ_M^1 differ only by an automorphism of their codomain. Moreover, for any $f\colon M \to N$ one has $v_M f_V = v \otimes_R f = f v_N$ and hence $\left(\begin{smallmatrix} 1 & v_M \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} f & 0 \\ 0 & f_V \end{smallmatrix}\right) = \left(\begin{smallmatrix} f & 0 \\ 0 & f_V \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & v_N \\ 0 & 1 \end{smallmatrix}\right)$. Therefore these automorphisms are part of a natural automorphism on the cylinder functor. Consequently $i \star \gamma^1$ is the pushout of $i \star \gamma^0$ along an isomorphism and hence $i \star \gamma^1$ is also a monomorphism.

For $i \star \gamma$, it is enough to consider the special case where i is the inclusion $\mathfrak{a} \hookrightarrow R$ of a left ideal. Let $j \colon V \otimes_R \mathfrak{a} \to V$ be the map with $j(w \otimes a) = wa$. The pushout Q of i and γ_M can be calculated as the cokernel in the exact row below

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a}+\mathfrak{a} \arrow[r, "k"] & R+R+\mathfrak{a}+V\otimes_R\mathfrak{a} \arrow[r] \arrow[d, "h"'] & Q \arrow[r] \arrow[dl, "i\star\gamma"] & 0 \\ & & R+V & & \end{tikzcd}$$

where

$$k = \begin{pmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 1 & v_{\mathfrak{a}} \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 1 & i & 0 \\ 0 & v & 0 & j \end{pmatrix}^{\top}$$

and $i \star \gamma$ is induced by h because $\mathrm{im}(k) \subseteq \ker(h)$. To show that $i \star \gamma$ is a monomorphism, it remains to verify $\ker(h) \subseteq \mathrm{im}(k)$.

Assume $(x, y, a, w) \in \ker(h)$ for some $x, y \in R$, $a \in \mathfrak{a}$ and $w = \sum_n w_n \otimes b_n \in V \otimes_R \mathfrak{a}$. This corresponds to equations $x + y + a = 0$ and $-vy = \sum_n w_n b_n$. Because vR is a pure submodule of V, there are $r_n \in R$ with $-vy = \sum_n v r_n b_n$. Since v is a monomorphism, we have $y = -\sum_n r_n b_n \in \mathfrak{a}$ and $x \in \mathfrak{a}$.

Therefore $(x, y, a, w) = (-x, -y)\left(\begin{smallmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 1 & v_{\mathfrak{a}} \end{smallmatrix}\right) \in \mathrm{im}(k)$.

(d) Tensoring the split exact sequence

$$0 \longrightarrow V \xrightarrow{(0\ 1)} R + V \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} R \longrightarrow 0$$

with M, we obtain $\ker(\sigma_M) = V \otimes_R M$ from which the equivalence follows. □

Observe that in the situation of 4.3.4(d), two maps $f, g \colon M \to N$ are homotopic iff $g - f \colon M \to N$ factors through some injective module. This relation is known as stable equivalence (see e.g. [13, Section 4] or [9, Definition 2.2.2]) and the homotopy equivalences are then also called stable equivalences.

4.3.5 Corollary. *Let $(\mathrm{C}, \gamma, \sigma)$ be a final cartesian cylinder in ${}_R\mathrm{Mod}$ and suppose that the ring R is injective. Then each map in $\Lambda = \Lambda(\mathrm{C}, \emptyset, I)$ has injective domain and codomain. In particular, every object is fibrant and $\mathcal{W} = \mathcal{W}(\mathrm{C}, \emptyset, I)$ is the class of stable equivalences.*

Proof. By part (a) of Proposition 4.3.4 one can assume $\mathrm{C} = \mathrm{C}_v$ for some $v \colon R \to V$. Moreover, C_v preserves injective objects by part (d). We prove by induction that each map in Λ^n has injective domain and codomain.

For the inclusion of a left ideal $i \colon \mathfrak{a} \to R$ we already noted in the proof of part (c), that the two maps $i \star \gamma^0$ and $i \star \gamma^1$ have isomorphic domains. We also calculated

$i \star \gamma^0 \colon R + V \otimes_R \mathfrak{a} \to R + V \otimes_R R$. Therefore every map in Λ^0 has injective domain and codomain.

Now assume that the claim holds for Λ^n and let $f\colon M \to N$ be a map in Λ^n. Then the codomain of $f \star \gamma$ is $N + V \otimes_R N$, which is injective. Its domain Q is the cokernel of a split exact sequence

$$0 \longrightarrow M + M \longrightarrow N + N + M + V \otimes_R M \longrightarrow Q \longrightarrow 0$$

and is therefore also injective. □

4.3.6 Example. Let H be a finite dimensional Hopf algebra over a field k, i.e. a (finite dimensional) k-algebra together with algebra maps $\Delta\colon H \to H \otimes_k H$ (comultiplication) and $\varepsilon\colon H \to k$ (counit), and an anti-algebra map $S\colon H \to H$ (antipode) satisfying certain conditions (for details see e.g. Montgomery [19]). ${}_H\mathbf{Mod}$ has a model structure where the weak equivalences are the stable equivalences [9, Theorem 2.2.12 and Proposition 4.2.15] We will show that this model structure is left determined by verifying the conditions of Proposition 4.3.4 and Corollary 4.3.5.

(1) Due to results of Larson and Sweedler [16, Theorem 2 (p79) and Proposition 2 (p83)] on finite dimensional Hopf algebras over a field, H satisfies the following conditions:

   (a) the antipode $S\colon H \to H$ is invertible.

   (b) there exists a nonzero $d \in H$ with $hd = \varepsilon(h)d$ for all $h \in H$. Giving k a left H-module structure via $\varepsilon\colon H \to k$, such a d corresponds to a (nonzero) H-linear map $d\colon k \to H$.

   (c) a left H-module is injective iff it is projective

(2) Let M and N be two H-modules. Then $M \otimes_k N$ has an $H \otimes_k H$-module structure with $(c \otimes c')(m \otimes n) = cm \otimes c'n$. Via the map $\Delta\colon H \to H \otimes_k H$ this induces an H-module structure on $M \otimes_k N$. Observe that with this definition $k \otimes_k M \cong M \cong M \otimes_k k$ and for a two sided module V also $M \otimes_k (V \otimes_H N) \cong (M \otimes_k V) \otimes_H N$ as H-modules.

   Let $\mathrm{Hom}(M, N)$ be the group of all k-linear maps from M to N. Then $\mathrm{Hom}(M, N)$ has a $H \otimes_k H^{op}$-module structure with $((c \otimes c')f)m = c(f(c'm))$. From this one obtains two different H-module structures on $\mathrm{Hom}(M, N)$:

   The first one is induced via $H \xrightarrow{\Delta} H \otimes_k H \xrightarrow{H \otimes S} H \otimes_k H^{op}$. We write $\mathrm{Hom}^r(M, N)$ for this module structure.

   The second one is induced via $H \xrightarrow{\Delta} H \otimes_k H \xrightarrow{tw} H \otimes_k H \xrightarrow{H \otimes S^{-1}} H \otimes_k H^{op}$, where tw is defined by $tw(c \otimes c') = c' \otimes c$. We write $\mathrm{Hom}^l(M, N)$ for this module structure.

Then one can verify that this gives bifunctors on ${}_H\mathbf{Mod}$ and that for any given M, the k-linear evaluation maps

$$e_N\colon \operatorname{Hom}^r(M,N)\otimes_k M \to N \quad \text{and} \quad e'_N\colon M\otimes_k \operatorname{Hom}^l(M,N)\to N$$

defined by $e_N(f,m) = fm = e'_N(m,f)$ are indeed H-linear and provide counits of two adjunctions $(-)\otimes_k M \dashv \operatorname{Hom}^r(M,-)$ and $M\otimes_k(-)\dashv \operatorname{Hom}^l(M,-)$.

(3) We fix some $d\colon k\to H$ as in (1b) above. Set $V = H\otimes_k H$. Then V is a two sided H-module. Define $v\colon H\to V$ by the composition $H\cong k\otimes_k H \xrightarrow{d\otimes H} H\otimes_k H$. Then this gives a map of two sided H-modules.

(4) Tensoring over the field k with a fixed module preserves monomorphisms. In particular the above $v\colon H\to V$ is a monomorphism. Moreover the natural isomorphisms $v\otimes_H(-)\cong d\otimes_k(-)$ and $V\otimes_H(-)\cong H\otimes_k(-)$ yield that $v\colon H\to V$ is pure and V is flat.

(5) For a fixed module M, both $\operatorname{Hom}^l(M,-)$ and $\operatorname{Hom}^r(M,-)$ preserve epimorphisms. Therefore their left adjoints $M\otimes_k(-)$ and $(-)\otimes_k M$ preserve projective H-modules. In particular, $V\otimes_H M\cong H\otimes_k M$ is projective and therefore injective.

We end with a question which, thanks to Proposition 4.3.4, is of purely ring-theoretical nature:

4.3.7 Question. For which rings R does ${}_RMod$ have a final cartesian cylinder for $(\mathrm{Mono}, \mathrm{Mono}^{\square})$?

Bibliography

[1] Jiří Adámek, Horst Herrlich, Jiří Rosický, and Walter Tholen, *Weak factorization systems and topological functors*, Appl. Categ. Structures **10** (2002), no. 3, 237–249. Papers in honour of the seventieth birthday of Professor Heinrich Kleisli (Fribourg, 2000). MR **1916156 (2003i:**18001)

[2] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. MR **1294136 (95j:**18001)

[3] Tibor Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 447–475. MR **1780498 (2001i:**18015)

[4] Denis-Charles Cisinski, *Théories homotopiques dans les topos*, J. Pure Appl. Algebra **174** (2002), no. 1, 43–82 (French, with English summary). MR **1924082 (2003i:**18021)

[5] Daniel Dugger and Daniel C. Isaksen, *Weak equivalences of simplicial presheaves*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 97–113. MR **2066498 (2005e:**18018)

[6] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR **1361887 (96h:**55014)

[7] Peter Gabriel and Friedrich Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, Vol. 221, Springer-Verlag, Berlin, 1971 (German). MR 0327863 (48 #6205)

[8] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR **1944041 (2003j:**18018)

[9] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR **1650134 (99h:**55031)

[10] P. T. Johnstone, *Topos theory*, Academic Press [Harcourt Brace Jovanovich Publishers], London, 1977. London Mathematical Society Monographs, Vol. 10. MR 0470019 (57 #9791)

[11] André Joyal and Myles Tierney, *Strong stacks and classifying spaces*, Category theory (Como, 1990), Lecture Notes in Math., vol. 1488, Springer, Berlin, 1991, pp. 213–236. MR **1173014 (93h:**18019)

[12] K. H. Kamps and T. Porter, *Abstract homotopy and simple homotopy theory*, World Scientific Publishing Co. Inc., River Edge, NJ, 1997. MR **1464944 (98k:**55021)

[13] Alexander Kurz and Jiří Rosický, *Weak factorizations, fractions and homotopies*, Appl. Categ. Structures **13** (2005), no. 2, 141–160. MR **2141595 (2006c:**18001)

[14] Yves Lafont, François Métayer, and Krzysztof Worytkiewicz, *A folk model structure on omega-cat* (June 2009), available at `arXiv:0712.0617v2[math.CT]`.

[15] Finnur Lárusson, *The homotopy theory of equivalence relations* (December 2006), available at `arXiv:math/0611344v1[math.AT]`.

[16] Richard Gustavus Larson and Moss Eisenberg Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. **91** (1969), 75–94. MR 0240169 (39 #1523)

[17] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR **1712872 (2001j:**18001)

[18] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR **1011461 (90i:**13001)

[19] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. MR **1243637 (94i:**16019)

[20] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR 0223432 (36 #6480)

[21] Charles Rezk, *A Model Category for Categories* (November 1996), available at `http://www.math.uiuc.edu/~rezk/cat-ho.dvi`.

[22] Claus Michael Ringel, *Diagonalisierungspaare. I*, Math. Z. **117** (1970), 249–266 (German). MR 0272864 (42 #7745)

[23] ———, *Faserungen und Homotopie in Kategorien*, Math. Ann. **190** (1970/71), 215–230 (German). MR 0301068 (46 #226)

[24] Jiří Rosický, *Generalized Brown representability in homotopy categories*, Theory Appl. Categ. **14** (2005), no. 19, 451–479 (electronic). MR **2211427 (2007c:**18009)

[25] J. Rosický, *On combinatorial model categories*, Appl. Categ. Structures **17** (2009), no. 3, 303–316. MR 2506258

[26] Jiří Rosický and Walter Tholen, *Left-determined model categories and universal homotopy theories*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3611–3623 (electronic). MR **1990164 (2004e:**55023)

[27] Oswald Wyler, *Lecture notes on topoi and quasitopoi*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1991. MR **1094373 (92c:**18004)

List of Symbols

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