MASARYK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

Ph.D. Dissertation

BRNO 2014 JANA KREJČOVÁ

MASARYK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

Oscillation of a class of the fourth-order nonlinear difference equations

Ph.D. Dissertation

Jana Krejčová

Advisor: prof. RNDr. Zuzana Došlá, DSc. **Brno 2014**

Bibliographic Entry

Author:	RNDr. Jana Krejčová Faculty of Science, Masaryk University Department of Mathematics and Statistics
Title of Dissertation:	Oscillation of a class of the fourth-order nonlinear difference equations
Degree Programme:	Mathematics
Field of Study:	Mathematical Analysis
Supervisor:	prof. RNDr. Zuzana Došlá, DSc. Faculty of Science, Masaryk University Department of Mathematics and Statistics
Academic Year:	2013/2014
Number of Pages:	74
Keywords:	Difference equation; Difference; Difference system; Oscillatory solution; Nonoscillatory solution; Maximal solution; Minimal solution; Oscillation; Cyclic permutation

Bibliografický záznam

Autor:	RNDr. Jana Krejčová Přírodovědecká fakulta, Masarykova univerzita Ústav matematiky a statistiky
Název práce:	Oscilace třídy nelineárních diferenčních rovnic čtvrtého řádu
Studijní program:	Matematika
Studijní obor:	Matematická analýza
Školitel:	prof. RNDr. Zuzana Došlá, DSc. Přírodovědecká fakulta, Masarykova univerzita Ústav matematiky a statistiky
Akademický rok:	2013/2014
Počet stran:	74
Klíčová slova:	Diferenční rovnice; Diference; Soustava diferenčních rovnic; Oscilatorické řešení; Neoscilatorické řešení; Maximální řešení; Minimální řešení; Oscilace; Cyklická permutace

Abstract

In this doctoral dissertation we deal with nonoscillatory solutions of fourth-order difference equations and their asymptotic properties. The dissertation is organized into seven chapters. The study of the difference equation of our type is motivated in Chapter 1, where the historical overview of different types of fourth-order difference equations and difference systems studied in the recent years is also given. The second chapter presents the basic properties of two-terms difference equations such as the existence of quickly oscillatory solutions, the cyclic permutation of the coefficients of the difference equation and the classification of possible types of nonoscillatory solutions of the difference equation. In Chapter 3 we study the asymptotic behavior of nonoscillatory solutions of type (a) and we present sufficient conditions for the nonexistence of this type of solution depending on the type of equation (sub-linear, half-linear and super-linear case). Analogously, the same type of problems for solutions of type (b) is studied in Chapter 4. The main body of the text is represented by the following two chapters. Using a combination of conditions for the nonexistence of the solutions of type (a) and type (b) we obtain the oscillation criteria which are illustrated by examples and applications in Chapter 5. In Chapter 6 we deal with the asymptotic behavior of nonoscillatory solutions and we define the maximal and the minimal solution of difference equations. We state the necessary condition for the difference equation to have the maximal or the minimal solution and we present theorems that provide a connection between maximal, resp. minimal solutions and solutions of type (a), resp. (b). Examples are provided to illustrate most of the theorems. For completeness, the dissertation is finished with a sketch of a further research in the presented theory. The main methods used in this dissertation are the asymptotic integration and the cyclic permutation of the coefficients of the equation.

Abstrakt

V této disertační práci se zabýváme neoscilatorickými řešeními diferenčních rovnic čtvrtého řádu a jejich asymptotickými vlastnostmi. Práce je rozdělena do sedmi kapitol. V první kapitole je uveden historický přehled různých typů diferenčních rovnic čtvrtého řádu a diferenčních systémů studovaných v posledních letech. Druhá kapitola uvádí základní vlastnosti dvoučlenné diferenční rovnice, jako je existence rychle oscilatorického řešení, cyklická permutace koeficientů a klasifikace možných typů neoscilatorických řešení dané rovnice. V Kapitole 3 studujeme asymptotické vlastnosti neoscilatorických řešení typu (a) a uvádíme postačující podmínky pro neexistenci tohoto typu řešení v závislosti na typu rovnice (sublineární, pololineární a superlineární případ). Analogicky je stejná problematika pro řešení typu (b) studována v Kapitole 4. Nosnou část práce tvoří následující dvě kapitoly. Kombinací podmínek neexistence řešení typu (a) a typu (b) získáme oscilační kritéria, která jsou ilustrována příklady a aplikacemi v páté kapitole. V šesté kapitole se zabýváme asymptotickým chováním neoscilatorických řešení, definujeme zde maximální a minimální řešení diferenční rovnice. Udáváme nutnou podmínku pro to, aby daná rovnice měla maximální či minimální řešení, a věty, které uvádí spojitost mezi řešením maximálním, resp. minimálním, a řešením typu (a), resp. (b). Většina uvedených výsledků je ilustrována příklady. Disertační práce je uzavřena nástinem možného směřování dalšího výzkumu řešené problematiky. Hlavní metody použité v této práci jsou asymptotická integrace a cyklická záměna koeficientů v diferenční rovnici.

MASARYKOVA UNIVERZITA
Přírodovědecká fakulta

ZADÁNÍ PRÁCE

Akademický rok: 2013/2014

Ústav: Ústav matematiky a statistiky

Studentka: RNDr. Jana Krejčová, DiS.

Program: Matematika (čtyřleté)

Obor: Matematická analýza

Ředitel *Ústavu matematiky a statistiky* PřF MU Vám ve smyslu Studijního a zkušebního řádu MU určuje práci s tématem:

Téma práce: Limitní vlastnosti řešení diferenciálních a diferenčních rovnic vyšších řádů

Téma práce anglicky: Limit properties of solutions of higher order differential and difference equations

Oficiální zadání: Vyšetřete asymptotické vlastnosti neoscilatorických řešení diferenciálních rovnic čtvrtého a vyšších řádů. Srovnejte získané výsledky s odpovídajícími diskrétními modely.

Literatura:

AGARWAL, Ravi P. *Difference equations and inequalities :theory, methods, and applications*. 2nd ed., rev. and expanded. New York: Marcel Dekker, 2000. xiii, 971. ISBN 0-8247-9007-3.

CHANTURIA, T. A. a I. T. KIGURADZE. *Asymptotic properties of solutions of nonautonomous ordinary differential equations*. Dordrecht: Kluwer Academic Publishers, 1993. 331 s. ISBN 0-7923-2059-X.

Jazyk závěrečné práce:

Vedoucí práce: prof. RNDr. Zuzana Došlá, DSc.

Datum zadání práce: 24. 9. 2009

V Brně dne: 9. 1. 2014

Souhlasím se zadáním (podpis, datum):

RNDr. Jana Krejčová, DiS.
studentka

prof. RNDr. Zuzana Došlá, DSc.
vedoucí práce

prof. RNDr. Jiří Rosický, DrSc.
ředitel Ústavu matematiky a statistiky

Acknowledgement

I would like to express my heartiest thanks to my respected supervisor, **prof. RNDr. Zuzana Došlá, DSc.**, for her valuable suggestions, sincere encouragement, immeasurable support and constant help during the period of my research and also in the preparation of the doctoral dissertation.
I sincerely thank my parents and my sister for their encouragement and support during my study.

Pronouncement

I pronounce that I formulated the dissertation independently using the information sources properly cited.

Brno 28 February 2014 . . . .

Jana Krejčová

Contents

Preface

Many interesting dynamic problems in applied science can be modelled by difference equations (for example the vibration of particles and lattices in physics, problem of elasticity, deformation of structures or soil settlement, phenomena in crystals, electric circuit analysis, dynamic systems, molecular chains, control theory). The theory of difference equations, the methods used, and their wide applications occupy a central position in the broad area of mathematical analysis. Difference equations are used as mathematical models describing real life situations in probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, mechanics, geometry, electrical networks, etc. In general, we expect difference equations to occur whenever the system under study depends on one or more variables that can only assume a discrete set of possible values.

In the last few years, an increasing attention has been paid to the study of oscillatory and asymptotic behavior of solutions of difference equations. Determination of oscillatory behavior for solutions of second-order difference equations has occupied a great part of researchers' interest. Compared to this, however, the study of fourth-order difference equations receives considerably less attention in the literature even though such equations often arise in the study of economics, statistics, mathematical biology and many other areas of mathematics whose discrete models are used. In this dissertation we present new contributions to the theory of a fourth-order difference equation.

This dissertation consists of seven chapters which are organized as follows: In the first chapter we introduce the most frequently occurring forms of fourth-order difference equations and four-dimensional difference systems. In Chapter 2 we recall fundamental definitions and necessary basic properties of solutions of difference equations. We classify nonoscillatory solutions of a fourth-order difference equation according to the sign of their quasi-differences. In Chapter 3 and Chapter 4 we give sufficient conditions that the difference equation does not have any of these types of solutions. Finally, in Chapter 5, we establish oscillation criteria for the difference equation and we present some applications. We illustrate our criteria by examples. Furthermore, we deal with the asymptotic behavior of nonoscillatory solutions and we introduce a definition of a maximal and a minimal solution in Chapter 6. In the last chapter we conclude with some remarks and open problems.

This doctoral dissertation comprises of results which the author achieved as the PhD student in the years 2009-2014. Some results in this dissertation have not been published yet. Some reported results were published by the author jointly with prof. RNDr. Zuzana Došlá, DSc. The exact list of the published results is presented in the appendix.

List of author's publications

The publications are completed with the corresponding impact factors (IF) and the publication cited in the Web of Science (WOS) database is denoted by these letters.

- Došlá, Z., Krejčová, J.: *Nonoscillatory solutions of the four-dimensional difference system*, Electron. J. Qual. Theory Differ. Equ., Proc. 9'th Coll. Qualitative Theory of Diff. Equ., No. 4 (2011), 1-11, IF 0.74, WOS.

- Došlá, Z., Krejčová, J.: *Oscillation of a class of the fourth-order nonlinear difference equations*, Adv. Difference Equ., 2012, 2012:99 (2 July 2012), IF 0.76, WOS.

- Došlá, Z., Krejčová, J.: *Asymptotic and oscillatory properties of the fourth-order nonlinear difference equations*, Appl. Math. Comput. (submitted, November 2013).

- Došlá, Z., Krejčová, J.: *Minimal and maximal solutions of the fourth-order nonlinear difference equations* , (in preparation, February 2014).

- Krejčová, J., Matucci, S.: *A nonlocal boundary value problem to functional difference equations*, (in preparation, February 2014).

The results were also presented at the international conferences.

Conferences with active participation

- *Conference on Differential and Difference Equations and Applications*,
  Rajecké Teplice, Slovakia, June 21 – 25, 2010,
  Poster: ”Oscillation of the fourth-order nonlinear difference equations”.

- *9th Colloquium on the Qualitative Theory of Differential Equations*,
  Szeged, Hungary, June 28 – July 1, 2011,
  Talk: ”Oscillation of the fourth-order nonlinear difference equations”.

- *International Student Conference on Applied Mathematics and Informatics*
  Malenovice, Czech Republic, May 10 - 13, 2012,
  Talk: ”Oscillation of the fourth-order nonlinear difference equations”.

- *Conference on Differential and Difference Equations and Applications*,
  Těrchová, Slovakia, June 25 – 29, 2012,
  Talk: ”Oscillatory and nonoscillatory solutions of the fourth-order nonlinear difference equations”.

- *18th International Conference on Difference Equations and Applications*
  Barcelona, Spain, July 22 – 27, 2012,
  Talk: ”Emden-Fowler type difference equations of the fourth-order”.

Scientific stay abroad

- Erasmus: University of Florence, February - July, 2013, Florence, Italy.

Chapter 1

Introduction

Consider a class of fourth-order nonlinear difference equations of the form

$$\Delta a_n \left(\Delta b_n \left(\Delta c_n \left(\Delta x_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha} + d_n x_{n+\tau}^{\lambda} = 0 \tag{E}$$

where $\alpha, \beta, \gamma, \lambda$ are the ratios of odd positive integers, $\tau \in \mathbb{Z}$ is a deviating argument and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are positive real sequences defined for $n \in \mathbb{N}_0 = \{n_0, n_0+1, \dots\}$, n_0 is a positive integer, and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

In the oscillation problem we assume that sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{c_n^{1/\gamma}} = \infty, \tag{H1}$$

or

$$\begin{cases} \displaystyle\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} < \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{c_n^{1/\gamma}} = \infty, \\ \displaystyle\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n_0}^{n-1} \frac{1}{a_k^{1/\alpha}}\right)^{1/\beta} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{c_n^{1/\gamma}} \left(\sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}}\right)^{1/\gamma} = \infty. \end{cases} \tag{H2}$$

We say that the equation (E) is in the *sub-linear case* when $\lambda < \alpha\beta\gamma$, in the *half-linear case* when $\lambda = \alpha\beta\gamma$ and in the *super-linear case* when $\lambda > \alpha\beta\gamma$.

In recent years, great attention has been paid to the study of oscillatory and asymptotic behavior of solutions of difference equations. Compared to second-order difference equations the study of higher-order equations and, in particular, fourth-order difference equations has received considerably less attention. Practical use of difference equations is evident in [10, 11, 12, 35].

Fourth-order difference equations were investigated in different forms, but widely considered in the literature have been special cases of (E). The most frequently occurring forms of fourth-order difference equations are summarized as follows.

The simplest form of equation (E) when $\alpha = \beta = \gamma = 1$ and $a_n = b_n = c_n = 1$ is presented by an equation of the form

$$\Delta^4 x_n = f(n, x_{n+2}),$$

where the function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ satisfies the condition $xf(n,x) < 0$ for all $n \in \mathbb{N}$, $x \in \mathbb{R} \setminus \{0\}$. This equation was investigated by Popenda and Schmeidel [24] in 1995. They studied the oscillatory behavior of solutions of this equation.

In 2003, Schmeidel [27] studied the similar equation with a different shift of indexes

$$\Delta^4 x_n = f(n, x_n).$$

Thandapani and Arockiasamy [29] studied necessary and sufficient conditions for the existence of nonoscillatory solutions with a specified asymptotic behavior for the equation in more general form

$$\Delta^2 \left(r_n \left(\Delta^2 x_n \right) \right) + f\left(n, x_n\right) = 0,$$

where $\{r_n\}$ is a positive real sequence and the continuous function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ satisfies $uf(n,u) > 0$ for all $u \neq 0$ and $n \in \mathbb{N}$. The oscillatory and asymptotic behavior of solutions of this equation was discussed by Yan and Liu [36].

If $\alpha = \gamma = 1$, $a_n = c_n = 1$ and $\tau = 3$, then equation (E) reduces to the difference equation

$$\Delta^2 \left(b_n \left(\Delta^2 x_n \right)^{\beta} \right) + d_n x_{n+3}^{\lambda} = 0. \tag{1.1}$$

If

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} < \infty, \tag{1.2}$$

then the assumption (H2) applies the condition

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n_0}^{n-1} 1 \right)^{1/\beta} = \sum_{n=n_0}^{\infty} \frac{(n-n_0)^{1/\beta}}{b_n^{1/\beta}} = \infty,$$

which is equivalent with

$$\sum_{n=n_0}^{\infty} \left(\frac{n}{b_n} \right)^{1/\beta} = \infty, \tag{1.3}$$

and the condition

$$\sum_{n=n_0}^{\infty} 1 \left(\sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} \right)^{1/\gamma} = \sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \sum_{k=n_0}^{n} 1 = \sum_{n=n_0}^{\infty} \frac{n-n_0+1}{b_n^{1/\beta}} = \infty,$$

which is equivalent with

$$\sum_{n=n_0}^{\infty} \frac{n}{b_n^{1/\beta}} = \infty. \tag{1.4}$$

Hence, conditions (1.3) and (1.4) are special cases of condition (H2). The oscillatory and asymptotic properties of solutions of equation (1.1) have been investigated with these special assumptions (1.3) and (1.4) by Agarwal and Manojlović in [5] and Thandapani et al. in [31, 32, 33]. While Thandapani and Vijaya [34] deal with a case where these series are convergent (see also the references therein).

If $\alpha = \beta = \gamma = \lambda = 1$, $a_n = b_n = 1$ and $\tau = 1$, then equation (E) reduces to the difference equation

$$\Delta^3 \left(c_n \left(\Delta x_n \right) \right) + d_n x_{n+1} = 0.$$

The oscillatory behavior of solutions of this difference equation was investigated by Selvaraj and Jaffer in [26].

Later, the results in [36] were extended by Graef and Thandapani [30] to the more general equation

$$\Delta a_n \left(\Delta b_n \left(\Delta c_n \left(\Delta x_n \right) \right) \right) + f \left(n, x_n \right) = 0, \tag{1.5}$$

where $n \in \mathbb{N}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences of positive real numbers, f is a function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. The oscillation criteria of (1.5) was investigated by Schmeidel, Migda, Musielak [28]. Migda and Schmeidel [23] studied nonoscillatory solutions with special asymptotic properties of equation (1.5).

The following equation

$$\Delta\frac{1}{a_3(n)}\left(\Delta\frac{1}{a_2(n)}\left(\Delta\frac{1}{a_1(n)}\left(\Delta x(n)\right)^{\alpha_1}\right)^{\alpha_2}\right)^{\alpha_3}+\delta q(n)f\left(x[g(n)]\right)=0, \tag{1.6}$$

where $\delta=\pm 1$, $\{a_i(n)\}$, $\{q(n)\}$ are sequences of positive real numbers, $g(n):\mathbb{N}\to\mathbb{R}$, $\Delta g(n)\geq 0$ for $n\geq n_0$ and $lim_{n\to\infty}g(n)=\infty$, f is a function such that $xf(x)>0$, $f'(x)\geq 0$ for $x\neq 0$, and α_i for $i=1,2,3$ are the ratios of positive odd integers, was considered in the recent papers by Agarwal, Grace, Wong and Manojlović [2, 3]. In [2], necessary and sufficient conditions for the oscillation of all bounded solutions of (1.6) (the so called B-oscillation) have been given. In [3], oscillation criteria for (1.6) have been established using the analysis of nonoscillatory solutions and by comparison with certain first and second-order difference equations.

In addition to the above, other special types of equation (E) have been widely investigated in the literature for a particular deviating argument τ. In the case when $\tau=0$, see e.g. [22, 23, 27, 28, 29, 31, 33, 36], in the case when $\tau=1$, see e.g. [26], in the case when $\tau=2$, see e.g. [24], in the case when $\tau=3$, see e.g. [4, 5, 32, 34], and references therein.

Equation (E) with $\tau=2$ can be seen as a coupled system of two second-order difference equations of the form

$$\begin{cases}\Delta\left(r_n\left(\Delta x_n\right)^{\alpha}\right)=-\varphi_n z_{n+1}^{\eta}\\ \Delta\left(q_n\left(\Delta z_n\right)^{\beta}\right)=\psi_n x_{n+1}^{\lambda},\end{cases} \tag{1.7}$$

where $\alpha,\beta,\eta,\lambda$ are the ratios of odd positive integers and $\{r_n\}$, $\{q_n\}$, $\{\varphi_n\}$, $\{\psi_n\}$ are positive real sequences defined for $n\in\mathbb{N}_0$. Indeed, eliminating z from the first equation, this system can be rewritten as

$$\Delta q_{n+1}\left(\Delta\varphi_n^{-1/\eta}\left(\Delta r_n\left(\Delta x_n\right)^{\alpha}\right)^{1/\eta}\right)^{\beta}+\psi_{n+1}x_{n+2}^{\lambda}=0, \tag{1.8}$$

System (1.7) is a special case of more general coupled systems of the form

$$\begin{cases} \Delta\left(r_n \Phi_\alpha\left(\Delta x_n\right)\right) = -f\left(n, y_{n+1}\right) \\ \Delta\left(q_n \Phi_\beta\left(\Delta y_n\right)\right) = g\left(n, x_{n+1}\right), \end{cases} \tag{1.9}$$

where $\Phi_\lambda(u) = |u|^{\lambda-1} \operatorname{sgn} u$ with $\lambda > 1$ and f, $g : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, nondecreasing with respect to the second variable, such that $uf(k,u) > 0$, $ug(k,u) > 0$ for every $u \neq 0$ and $k \in \mathbb{N}$. Oscillatory properties of system (1.9) have been investigated by Marini, Matucci and Řehák in [21].

Motivated by these papers, we study the asymptotic and oscillatory properties of solutions of equation (E) and we state new oscillation theorems. Our results unify, improve and extend many well-known oscillation criteria that have appeared in the literature for some special cases of equation (E). Oscillation criteria established in the above papers are based on a different approach than that applied here. Namely in [3], they used comparing (1.6) with certain first and second-order difference equations whose oscillatory properties are known.

The approach here is based on considering equation (E) as a four-dimensional system. By using the notation

$$y_n = c_n\left(\Delta x_n\right)^\gamma, \; z_n = b_n\left(\Delta y_n\right)^\beta, \; w_n = a_n\left(\Delta z_n\right)^\alpha \tag{1.10}$$

equation (E) can be written as the four-dimensional nonlinear difference system

$$\begin{cases} \Delta x_n = C_n y_n^{1/\gamma} \\ \Delta y_n = B_n z_n^{1/\beta} \\ \Delta z_n = A_n w_n^{1/\alpha} \\ \Delta w_n = -D_n x_{n+\tau}^\lambda, \end{cases} \tag{S}$$

where

$$A_n = a_n^{-1/\alpha}, \quad B_n = b_n^{-1/\beta}, \quad C_n = c_n^{-1/\gamma}, \quad D_n = d_n.$$

Thus, if x is a solution of (E) and

$$x_n^{[1]} = c_n\,(\Delta x_n)^{\gamma}, \quad x_n^{[2]} = b_n\left(\Delta x_n^{[1]}\right)^{\beta}, \quad x_n^{[3]} = a_n\left(\Delta x_n^{[2]}\right)^{\alpha}$$

are the so called *quasi-differences* of x, then the vector

$$(x,y,z,w) = (x, x^{[1]}, x^{[2]}, x^{[3]})$$

is a solution of (S). Therefore, we can use system (S) instead of equation (E). It was more appropriate in proofs of our theorems.

System (S) is a prototype of even-order k-dimensional difference systems

$$\Delta x_i(n) = a_i(n) f_i(x_{i+1}(n)), \quad x_{k+1} = x_1, \quad i = 1,\ldots k, \quad k \geq 2, \tag{1.11}$$

where a_i are functions and f_i are continuous functions on $\mathbb{R}$ such that

$$u f_i(u) > 0 \text{ for } u \neq 0.$$

Let us note that, system (1.11) can be viewed as a discrete analogue of the four-dimensional differential system investigated by Kusano et al. [20], and by Chanturia [13]. In these papers the oscillation of the n-dimensional differential systems was investigated in terms of *Property A* (which reads for equations of even-order as the oscillation of all solutions) and *Property B* (which means that any nonoscillatory solution is either unbounded or vanishing at infinity in all their components). The terminology *Property A* and *Property B* is due to [13] and [18].

The properties of system (S) with the assumption $D_n < 0$ and A_n, B_n, C_n positive are described in [14] and in author's rigorous thesis [19]. In [19], we study the asymptotic properties of nonoscillatory solutions of difference systems and we give sufficient conditions that any bounded nonoscillatory solution tends to zero and any unbounded nonoscillatory solution tends to infinity in all its components.

In this dissertation, we study (E) via system (S) with the assumption $D_n > 0$ and A_n, B_n, C_n positive. First, we show the influence of the deviating argument τ on the existence of quickly oscillatory solutions and we describe the so called cyclic permutation for (E). Our main goal is to state new oscillation theorems for equation (E) and to extend the existing oscillation results in the literature, in the case where the difference operator in (E) is in the canonical form, i.e. when (H1) holds, as well as in the case when (H2) holds. We give oscillation theorems in the sub-linear, in the half-linear and in the super-linear case. We state *a-priori* bounds for nonoscillatory solutions that lead to conditions for the oscillation theorems. Our results are based on the conditions for the nonexistence of nonoscillatory solutions and on the change of summation for double series. Due to our approach considering (E) as a four-dimensional system, we extend for any $\tau \in \mathbb{Z}$ some results of [3] stated for a delay $\tau \leq 0$. Using the cyclic permutation we show how it is possible to extend oscillation criteria to the case when one of the series in (H1) is convergent.

Thereafter, we deal with the asymptotic behavior of nonoscillatory solutions. We define the maximal and the minimal solution of the difference equation and we state the necessary condition for the difference equation to have the maximal, resp. the minimal solution. Finally, we find a connection between maximal, resp. minimal solutions and solutions of type (a), resp. (b).

Our main tools are an *asymptotic integration* and the *cyclic permutation* of the coefficients of a difference equation described in Chapter 2.2. The asymptotic integration means that we use the summation of an equation from n_0 to n (or to ∞). This term was introduced by William Trench. This enables us to establish precise lower and upper bounds for both types of nonoscillatory solutions, see Chapter 3 and Chapter 4.

Now we present definitions that we use below.

By a solution of equation (E) we mean a real sequence $\{x_n\}$ defined for all $n \in \mathbb{N}_0$ and satisfying equation (E) for all $n \in \mathbb{N}_0$. A solution of (E) is called a *nontrivial* if for any $n_0 \geq 1$ there exists $n > n_0$ such that $x_n \neq 0$. Otherwise, the solution is called a *trivial*. By a solution of system (S) we mean a vector sequence (x, y, z, w) which satisfies the system

(S) for $n \in \mathbb{N}_0$. We consider only such solutions that are nontrivial for large n.

Observe that if (x, y, z, w) is a solution of system (S) and if there exists $n_0 \in \mathbb{N}$ such that $x_n \neq 0$ for $n \geq n_0$, then $y_n \neq 0$, $z_n \neq 0$ and $w_n \neq 0$ for $n \geq n_0$. Obviously, if (x, y, z, w) is a solution of system (S) and one of its components is of one sign, then all its components are of one sign.

A nontrivial solution $\{x_n\}$ of equation (E) is said to be *nonoscillatory* if it is either eventually positive or eventually negative. Otherwise, the nontrivial solution is said to be *oscillatory*. Equation (E) is said to be *oscillatory* if all its solutions are oscillatory. Oscillatory types of solutions occur in many physical phenomena, such as vibrating mechanical systems and electric circuits.

If (H1) holds, we say that the difference operator in equation (E) (resp. system (S)) is in the *canonical form*.

The important role plays the following definition.

Definition 1. *A solution x of* (E) *is of* ***type (a)*** *if*

$$x_n > 0, \quad x_n^{[1]} > 0, \quad x_n^{[2]} > 0, \quad x_n^{[3]} > 0 \quad \textit{for large } n.$$

A solution x of (E) *is of* ***type (b)*** *if*

$$x_n > 0, \quad x_n^{[1]} > 0, \quad x_n^{[2]} < 0, \quad x_n^{[3]} > 0 \quad \textit{for large } n.$$

Chapter 2

Basic properties of two-terms difference equations

First, we point out some basic properties of solutions of equation (E). Equation (E) is called a two-term difference equation, because it can be written as

$$L_4 x_n + d_n x_{n+\tau}^{\lambda} = 0,$$

where

$$L_4 x_n = \Delta a_n \left(\Delta b_n \left(\Delta c_n \left(\Delta x_n \right)^{\gamma} \right)^{\beta} \right)^{\alpha}.$$

The terminology of two-terms equations comes from Uri Elias [8], who has introduced it for n-order differential equations.

The results of this chapter hold without assumptions (H1), (H2).
We begin with the necessary condition for the existence of quickly oscillatory solutions.

2.1 Quickly oscillatory solutions

Prototypes of oscillatory solutions of (E) are solutions of the form

$$x_n = (-1)^n p_n, \quad p_n > 0 \ \text{ for } n \in \mathbb{N}_0.$$

Such solutions are called quickly oscillatory and the following result can be seen as a necessary condition for their existence.

Theorem 1. *Equation* (E) *with τ even has no quickly oscillatory solutions.*

Proof. Let $x_n = (-1)^n p_n$ be a quickly oscillatory solution of (E). Then

$$\Delta x_n = (-1)^{n+1}\left(p_{n+1} + p_n\right).$$

From the first equation of system (S) we have

$$y_n = \left(\frac{\Delta x_n}{C_n}\right)^{\gamma} = (-1)^{n+1} q_n,$$

where $q_n = \left(\frac{p_{n+1}}{C_n} + \frac{p_n}{C_n}\right)^{\gamma} > 0$. From the second equation of (S) we obtain

$$z_n = \left(\frac{\Delta y_n}{B_n}\right)^{\beta} = (-1)^n r_n,$$

where $r_n = \left(\frac{q_{n+1}}{B_n} + \frac{q_n}{B_n}\right)^{\beta} > 0$. Repeating the argument, we get from the third equation of (S)

$$w_n = \left(\frac{\Delta z_n}{A_n}\right)^{\alpha} = (-1)^{n+1} s_n,$$

where $s_n = \left(\frac{r_{n+1}}{A_n} + \frac{r_n}{A_n}\right)^{\alpha} > 0$. Consequently, from here and from the fourth equation of system (S) we have

$$\Delta w_n = (-1)^n\left(s_{n+1} + s_n\right) = -D_n(-1)^{(n+\tau)\lambda} p_{n+\tau}^{\lambda} = (-1)^{n+1+\tau} D_n p_{n+\tau}^{\lambda},$$

which gives a conclusion. □

Remark 1. *Theorem 1 explains why equation* (E) *is often considered with τ odd.*

By the method used in the proof of Theorem 1 we can easily construct equations possessing a quickly oscillatory solution.

Example 1. *Consider the equation*

$$\Delta^2\left(\Delta^2 x_n\right)^\beta + \frac{3^{2\beta}\left(2^\beta+1\right)^2}{2^{\tau\lambda}} 2^{n(\beta-\lambda)} x_{n+\tau}^\lambda = 0, \tag{2.1}$$

where τ is an odd positive integer. This equation has a quickly oscillatory solution

$$x_n = (-1)^n 2^n.$$

Indeed, $p_n = 2^n$, $q_n = 2^n 3$, $r_n = 2^{n\beta}3^{2\beta}$, $s_n = 2^{n\beta}3^{2\beta}\left(2^\beta+1\right)$ and the value of d_n follows from the relation $d_n = \left(s_{n+1}+s_n\right)/p_{n+\tau}^\lambda$.

Example 2. *Consider the equation*

$$\Delta^3 n\left(\Delta x_n\right) + 8\left(2n+3\right)x_{n+\tau} = 0. \tag{2.2}$$

If τ is an even positive integer, then (2.2) *has no quickly oscillatory solution. If τ is an odd positive integer, then* (2.2) *has a quickly oscillatory solution*

$$x_n = (-1)^n.$$

2.2 Cyclic permutation

The left-ordered cyclic permutation of coefficients in system (S) is described in author's rigorous thesis [19].

Lemma 1. [19, Lemma 7]
The following statements are equivalent:

(i) (x,y,z,w) *is a solution of system (S),*

(ii) $(\tilde{x},\tilde{y},\tilde{z},\tilde{w})$, *where* $(\tilde{x},\tilde{y},\tilde{z},\tilde{w}) = (w,x,y,z)$, *is a solution of system*

$$\begin{cases} \Delta\tilde{x}_n = D_n\,\tilde{y}^{\delta}_{n+\tau} \\ \Delta\tilde{y}_n = A_n\,\tilde{z}^{\alpha}_n \\ \Delta\tilde{z}_n = B_n\,\tilde{w}^{\beta}_n \\ \Delta\tilde{w}_n = C_n\,\tilde{x}^{\gamma}_n, \end{cases} \tag{S1}$$

(iii) $(\tilde{x},\tilde{y},\tilde{z},\tilde{w})$, *where* $(\tilde{x},\tilde{y},\tilde{z},\tilde{w}) = (z,w,x,y)$, *is a solution of system*

$$\begin{cases} \Delta\tilde{x}_n = C_n\,\tilde{y}^{\gamma}_n \\ \Delta\tilde{y}_n = D_n\,\tilde{z}^{\delta}_{n+\tau} \\ \Delta\tilde{z}_n = A_n\,\tilde{w}^{\alpha}_n \\ \Delta\tilde{w}_n = B_n\,\tilde{x}^{\beta}_n, \end{cases} \tag{S2}$$

(iv) $(\tilde{x},\tilde{y},\tilde{z},\tilde{w})$, *where* $(\tilde{x},\tilde{y},\tilde{z},\tilde{w}) = (y,z,w,x)$, *is a solution of system*

$$\begin{cases} \Delta\tilde{x}_n = B_n\,\tilde{y}^{\beta}_n \\ \Delta\tilde{y}_n = C_n\,\tilde{z}^{\gamma}_n \\ \Delta\tilde{z}_n = D_n\,\tilde{w}^{\delta}_{n+\tau} \\ \Delta\tilde{w}_n = A_n\,\tilde{x}^{\alpha}_n. \end{cases} \tag{S3}$$

Now, we describe the left-ordered cyclic permutation of coefficients in equation (E).

Lemma 2. *The following statements are equivalent:*

(i) x *is a solution of* (E).

(ii) $y = \{y_n\}$, *where* $y_n = c_n\,(\Delta x_n)^{\gamma}$, *is a solution of*

$$\Delta \frac{1}{d_n^{1/\lambda}} \left(\Delta a_n \left(\Delta b_n\,(\Delta y_n)^{\beta}\right)^{\alpha}\right)^{1/\lambda} + \frac{1}{c_{n+\tau}^{1/\gamma}}\, y_{n+\tau}^{1/\gamma} = 0. \tag{R1}$$

(iii) $z = \{z_n\}$, *where* $z_n = b_n\,(\Delta y_n)^{\beta}$, *is a solution of*

$$\Delta c_{n+\tau} \left(\Delta \frac{1}{d_n^{1/\lambda}} \left(\Delta a_n\,(\Delta z_n)^{\alpha}\right)^{1/\lambda}\right)^{\gamma} + \frac{1}{b_{n+\tau}^{1/\beta}}\, z_{n+\tau}^{1/\beta} = 0. \tag{R2}$$

(iv) $w = \{w_n\}$, *where* $w_n = a_n (\Delta z_n)^{\alpha}$ *is a solution of*

$$\Delta b_{n+\tau} \left(\Delta c_{n+\tau} \left(\Delta \frac{1}{d_n} (\Delta w_n)^{1/\lambda} \right)^{\gamma} \right)^{\beta} + \frac{1}{a_{n+\tau}^{1/\alpha}} w_{n+\tau}^{1/\alpha} = 0. \tag{R3}$$

Proof. First, we prove that (i) is equivalent to (ii). If we express x from the last equation in (S) we obtain

$$x_{n+\tau} = -\frac{1}{d_n^{1/\lambda}} (\Delta w_n)^{1/\lambda} = -\frac{1}{d_n^{1/\lambda}} \left(\Delta a_n \left(\Delta b_n (\Delta y_n)^{\beta} \right)^{\alpha} \right)^{1/\lambda}. \tag{2.3}$$

Thus, from here and the first equation in (S) we have

$$\Delta x_{n+\tau} = -\Delta \frac{1}{d_n^{1/\lambda}} \left(\Delta a_n \left(\Delta b_n (\Delta y_n)^{\beta} \right)^{\alpha} \right)^{1/\lambda} = \frac{1}{c_{n+\tau}^{1/\gamma}} y_{n+\tau}^{1/\gamma},$$

which yields equation (R1). To prove that (i) is equivalent to (iii) we use the same process. Using (1.10) and (2.3) we have

$$\Delta x_n = -\Delta \frac{1}{d_{n-\tau}^{1/\lambda}} \left(\Delta a_{n-\tau} (\Delta z_{n-\tau})^{\alpha} \right)^{1/\lambda}.$$

Substituing this into

$$\Delta y_n = \Delta c_n (\Delta x_n)^{\gamma}$$

and using the second equation of (S) we get equation (R2).

To prove that (i) is equivalent to (iv) we proceed as before, expressing Δz in terms of w from the third equation of (S) and from (1.10) and comparing both expressions. □

Theorem 2. *Equation* (E) *is oscillatory if and only if any of equations (R_1), (R_2), (R_3) is oscillatory.*

Proof. The validity implies from Lemma 2. □

Remark 2. *By Theorem 2 equation* (1.1) *is oscillatory if and only if the equation*

$$\Delta^2 \left(\frac{1}{d_n^{1/\lambda}} \left(\Delta^2 z_n \right)^{1/\lambda} \right) + \frac{1}{b_{n+3}^{1/\beta}} z_{n+3}^{1/\beta} = 0$$

is oscillatory. Observe that the difference operator in this equation is in the canonical form if

$$\sum_{n=n_0}^{\infty} d_n = \infty.$$

If we apply Lemma 2 to equation (1.8) we get that the cyclic permutation for the coupled system (1.7) means that equations in (1.7) are considered in the opposite order. From here and Theorem 2 we get the following corollary.

Corollary 1. *Vector* (x,z) *is a solution of* (1.7) *if and only if the vector* $(u,v) = (-z,x)$ *is a solution of the coupled system*

$$\begin{cases} \Delta\left(q_n\left(\Delta u_n\right)^{\beta}\right) = -\psi_n v_{n+1}^{\lambda} \\ \Delta\left(r_n\left(\Delta v_n\right)^{\alpha}\right) = \varphi_n u_{n+1}^{\eta}, \end{cases} \tag{2.4}$$

which is again system of the form (1.7).
The coupled system (1.7) *is oscillatory if and only if the coupled system* (2.4) *is oscillatory.*

Oscillation results of Marini, Matucci, Řehák in [21] for (1.7) assume

$$\sum \frac{1}{r_n^{1/\alpha}} = \infty, \quad \sum \frac{1}{q_n^{1/\beta}} = \infty, \quad \sum \varphi_n < \infty, \quad \sum \psi_n = \infty,$$

which means that the difference operator in (1.8) is not in the canonical form. Hence, to compare results of [21] and our oscillation criteria for the equation with the difference operator in the canonical form we have to apply results of [21] to the coupled system (2.4). Observe that the coupled system is oscillatory if all solutions are oscillatory, i.e. both components are neither eventually positive nor negative.

The aim of the following section is to describe the possible types of nonoscillatory solutions of equation (E). Throughout the next sections we use the convention

$$\sum_{i=n_1}^{n_2} u_i = 0 \text{ if } n_1 > n_2.$$

2.3 Nonoscillatory solutions

We assume system (S) instead of equation (E). If (S) has a solution (x,y,z,w), then $(-x,-y,-z,-w)$ is a solution of (S), too. Hence, when studying the nonexistence conditions for nonoscillatory solutions, for the sake of simplicity, we restrict our attention to solutions such that $x_n > 0$ for large n.

The component x of the solution (x,y,z,w) of system (S) is said to be *oscillatory* if for any $n_0 \geq 1$ there exists $n > n_0$ such that $x_{n+1}x_n \leq 0$. The oscillation of the components y, z, w is defined in the same way. A solution of system (S) is said to be *oscillatory* if all of its components x,y,z,w are oscillatory. Otherwise, a solution is said to be *nonoscillatory*.

A solution of the system (S) is said to be *bounded* if all of its components x,y,z,w are bounded. Otherwise, a solution is said to be *unbounded*.

The following Lemma 3 has been presented for system (S) with the assumption $D_n < 0$ in author's rigorous thesis [19].

Lemma 3. *Let* (x,y,z,w) *be a solution of system* (S)*. The solution* (x,y,z,w) *is nonoscillatory if and only if any of its components* x*,* y*,* z*,* w *is either positive or negative for large* n*.*

Proof. It is sufficient to prove that if (x,y,z,w) is an oscillatory solution of (S), then all components are either positive or negative for large n. First, we assume that $x_n > 0$ for $n \geq n_0$ and $n_0 \in \mathbb{N}$. From the fourth equation of the system (S) we have that w_n is strictly decreasing for $n \geq n_0$. Hence, it is of one sign for large n. Proceeding by the same argument we get that z and y are monotone and of one sign for large n, too. The remaining cases when any of the components y, z, w are eventually positive or negative can be treated in the same way. $\square$

We start with the following lemma which provides the classification of nonoscillatory solutions of (S).

Lemma 4. *Assume* (H1) *or* (H2)*. Then any solution* (x,y,z,w) *of system* (S) *such that* $x_n > 0$ *for large* n *is of type (a) or of type (b).*

Proof. Assume (H1). Let (x,y,z,w) be a nonoscillatory solution of (S). Assume that there exists a solution such that $y_n > 0$, $z_n < 0$, $w_n < 0$ for large n. Since $\Delta z_n < 0$, there exists $k > 0$ such that $z_n \le -k$ for large n. Using the summation of the second equation of system (S) we get

$$y_n - y_{n_0} = \sum_{i=n_0}^{n-1} B_i z_i^{1/\beta} \le -k^{1/\beta} \sum_{i=n_0}^{n-1} B_i.$$

Passing $n \to \infty$ we get $\lim y_n = -\infty$, which is a contradiction.

Let there exist a solution so that $y_n < 0$, $z_n > 0$, $w_n > 0$ for large n. Since z is positive increasing there exists $k > 0$ so that $z_n \ge k$ for large n. Summation of the second equation of system (S) leads to $\lim y_n = +\infty$, which is a contradiction with the fact $y_n < 0$.

Let there exist a solution so that $y_n < 0$, $z_n < 0$ for large n. Since y is negative decreasing there exists $k > 0$ so that $y_n \le -k$ for large n. By summation of the first equation of system (S) and passing $n \to \infty$, we arrive at a contradiction.

The case when $z_n > 0$ and $w_n < 0$ for large n can be treated in a similar way by summation of the third equation of (S).

Assume (H2). First, assume that there exists a nonoscillatory solution (x,y,z,w) such that $x_n > 0$ and $z_n > 0$ for large n. Assume $y_n < 0$. Since y_n is increasing we can assume that there exists $k \le 0$ such that $y_n \le k$ for large n. By summation of the first and the second equation of system (S) we get

$$y_n \le -\sum_{i=n}^{\infty} B_i z_i^{1/\beta},$$

$$x_n = \sum_{j=n_0}^{n-1} C_j y_j^{1/\gamma} + x_{n_0}. \tag{2.5}$$

Thus,

$$x_n \le -\sum_{j=n_0}^{n-1} C_j \left(\sum_{i=j}^{\infty} B_i z_i^{1/\beta} \right)^{1/\gamma} + x_{n_0}$$

and passing $n \to \infty$ we obtain a contradiction with positivity of x. Therefore $y_n > 0$. Now assume $w_n < 0$. Since w_n is decreasing there exists $k < 0$ such that $w_n \le k$ for large n. By summation of the third equation of system (S) we obtain

$$z_n = \sum_{i=n_0}^{n-1} A_i w_i^{1/\alpha} + z_{n_0} \le k^{1/\alpha} \sum_{i=n_0}^{n-1} A_i + z_{n_0}, \tag{2.6}$$

passing $n \to \infty$ we have a contradiction with positivity of z. Therefore $w_n > 0$ and this is a type (a) of a nonoscillatory solution.

Assume that there exists a nonoscillatory solution (x,y,z,w) such that $x_n > 0$ and $z_n < 0$ for large n. Assume $y_n < 0$. Since y_n is decreasing, we can assume that there exists $k < 0$ such that $y_n \leq k$ for large n. Then using (2.5) we get

$$x_n \leq k^{1/\gamma} \sum_{i=n_0}^{n-1} C_i + x_{n_0},$$

which is a contradiction with positivity of x. Therefore, $y_n > 0$. Now assume that $w_n < 0$. Since w_n is decreasing there exists $k < 0$ such that $w_n \leq k$ for large n. Then using substitution of (2.6) into the second equation of system (S) we obtain

$$y_n \leq \sum_{j=n_0}^{n-1} B_j \left(k^{1/\alpha} \sum_{i=n_0}^{j-1} A_i + z_{n_0} \right)^{1/\beta} + y_{n_0} \leq L \sum_{j=n_0}^{n-1} B_j \left(\sum_{i=n_0}^{j-1} A_i \right)^{1/\beta},$$

where L is a suitable constant. Passing $n \to \infty$ we obtain a contradiction with positivity of y. Therefore, $w_n > 0$ and it is a type (b) of a nonoscillatory solution. □

Remark 3. *A solution x of equation* (E) *is of type (a) [type (b)] if the corresponding solution (x,y,z,w) of system* (S) *is of type (a) [type (b)].*

Theorem 3. *Assume* (H1) *or* (H2). *If*

$$\sum_{n=n_0}^{\infty} d_n = \infty, \tag{2.7}$$

then equation (E) *is oscillatory.*

Proof. In view of Lemma 4 we can assume without loss of generality that $x_n > 0$, $y_n > 0$ and $w_n > 0$. Hence, there exist $k > 0$ and $n_0 > 1$ such that $x_n \geq k$ for $n \geq n_0$. By summation of the fourth equation of system (S), we find that (2.7) leads to a contradiction with the positiveness of w_n. □

Example 3. *Consider equation* (2.2) *from Example 2. By Theorem 3, this equation has all solutions oscillatory for any $\tau \in \mathbb{Z}$. However, by Theorem 1 no oscillatory solution is quickly oscillatory for τ even.*

Example 4. *Consider the difference equation*

$$\Delta^4 x_n + x_{n+\tau} = 0. \tag{2.8}$$

We see that $a_n = b_n = c_n = d_n = 1$. Therefore, it satisfies assumption (H1) *and* (2.7). *In virtue of Theorem 3 equation* (2.8) *is oscillatory for any $\tau \in \mathbb{Z}$.*
In addition, one can check (see [1]) that (2.8) *with $\tau = 0$ has these solutions*

$$x_n^{(1)} = \alpha^n cos\beta n,$$

$$x_n^{(2)} = \alpha^n sin\beta n,$$

$$x_n^{(3)} = \gamma^n cos\delta n,$$

$$x_n^{(4)} = \gamma^n sin\delta n,$$

where

$$\alpha = |1-(1-i)/\sqrt{2}|,$$
$$\beta = tan^{-1} 1/(\sqrt{2}-1),$$
$$\gamma = |1+(1+i)/\sqrt{2}|,$$
$$\delta = tan^{-1} 1/(\sqrt{2}+1).$$

Example 5. *Consider equation* (2.1) *from Example 1. We have $a_n = b_n = c_n = 1$ and*

$$d_n = \frac{3^{2\beta}\left(2^\beta+1\right)^2}{2^{\tau\lambda}} 2^{n(\beta-\lambda)}.$$

If $\beta \geq \lambda$, then by Theorem 3 equation (2.1) *has all solutions oscillatory.*
However, if $\beta < \lambda$, then by [5, Theorems 3.5,3.6] equation (2.1) *has also nonoscillatory solutions.*

Hence, under assumptions (H1) or (H2), if (E) has a nonoscillatory solution, then

$$\sum_{n=n_0}^{\infty} d_n < \infty. \tag{2.9}$$

2.4 Lemmas on integration and summation

In the following chapters we give sufficient conditions for the nonexistence of both types of nonoscillatory solutions of (E). To this goal the following lemmas will be used.

Lemma 5. *(i) Let $k \in (0;1)$ and $\{w_n\}$ be a sequence such that $w_n > 0$ and $\Delta w_n < 0$. Then*

$$\sum_{n=1}^{\infty} \frac{-\Delta w_n}{w_n^k} < \infty.$$

(ii) Let $k > 1$ and $\{w_n\}$ be a sequence such that $w_n > 0$ and $\Delta w_n > 0$. Then

$$\sum_{n=1}^{\infty} \frac{\Delta w_n}{w_{n+1}^k} < \infty.$$

Proof. Claim (i). We suppose that $k < 1$ and $w_n > 0$, $\Delta w_n < 0$. This implies

$$\frac{-\Delta w_n}{w_n^k} \leq \int_{w_{n+1}}^{w_n} \frac{1}{t^k} dt.$$

Summing from N to ∞ we obtain

$$\sum_{n=N}^{\infty} \frac{-\Delta w_n}{w_n^k} \leq \sum_{n=N}^{\infty} \int_{w_{n+1}}^{w_n} \frac{1}{t^k} dt \leq \int_0^{w_N} \frac{1}{t^k} dt < \infty.$$

Claim (ii). If $k > 1$ and $w_n > 0$, $\Delta w_n > 0$, then we get

$$\frac{\Delta w_n}{w_{n+1}^k} \leq \int_{w_n}^{w_{n+1}} \frac{1}{t^k} dt.$$

Using summation from N to ∞ we obtain

$$\sum_{n=N}^{\infty} \frac{\Delta w_n}{w_{n+1}^k} \leq \sum_{n=N}^{\infty} \int_{w_n}^{w_{n+1}} \frac{1}{t^k} dt \leq \int_{w_N}^{\infty} \frac{1}{t^k} dt < \infty.$$

□

The important tool in our investigation is the following change of summation, see [6, 7].

Lemma 6. *Let $\{a_n\}$ and $\{d_n\}$ be positive real sequences defined for $n \in \mathbb{N}_0$. Assume case* *(i)* $\alpha > \lambda$ *or* $\alpha = \lambda \geq 1$.

$$\text{If} \quad \sum_{n=n_0}^{\infty} d_n \left(\sum_{k=n_0}^{n} \frac{1}{a_k^{1/\alpha}} \right)^{\lambda} = \infty, \text{ then} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} \left(\sum_{k=n}^{\infty} d_k \right)^{1/\alpha} = \infty.$$

(ii) $\alpha < \lambda$ *or* $\alpha = \lambda \leq 1$.

$$\text{If} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} \left(\sum_{k=n}^{\infty} d_k \right)^{1/\alpha} = \infty, \text{ then} \quad \sum_{n=n_0}^{\infty} d_n \left(\sum_{k=n_0}^{n} \frac{1}{a_k^{1/\alpha}} \right)^{\lambda} = \infty.$$

Proof. Conditions with $\alpha = \lambda$ have been proved in [7], conditions $\alpha \neq \lambda$ in [6]. □

Remark 4. *Observe that the opposite implications in Lemma 6 in general need not hold. For example, choosing*

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n-1)} \left(\sum_{k=1}^{n} 1 \right)^{\lambda} \quad \text{and} \quad T = \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{1}{k(k-1)} \right)^{1/\alpha},$$

we have $S = \infty$ *and* $T < \infty$ *for* $\lambda \geq 1$ *and* $\alpha < 1$*; the opposite case holds for* $\lambda < 1$ *and* $\alpha \geq 1$.

In order to construct illustrative examples we use the following connection between power and generalized power.
Define for $k \in \mathbb{N}$

$$k^{(\alpha)} := \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)},$$

where Γ is the Gamma function

$$\Gamma(t) := \int_0^{\infty} e^{-s} s^{t-1} ds.$$

Lemma 7. *We have*

$$\lim_{k \to \infty} \frac{k^{\alpha}}{k^{(\alpha)}} = 1 \quad (\alpha \in \mathbb{R}).$$

Proof. The proof of this result was suggested by M. Bohner by personal communication and published in [9, Lemma 5.1].

Stirling's formula [25, Chapter 8] says that

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1.$$

Thus,

$$\alpha_k := \frac{\Gamma(k+1-\alpha)\left(\frac{k}{e}\right)^k \sqrt{2\pi k}}{\Gamma(k+1)\left(\frac{k-\alpha}{e}\right)^{k-\alpha} \sqrt{2\pi(k-\alpha)}} \to 1 \quad \text{as} \quad k\to\infty.$$

Hence, we can conclude that

$$\begin{aligned}
\frac{k^\alpha}{k^{(\alpha)}} &= k^\alpha \alpha_k \frac{\left(\frac{k-\alpha}{e}\right)^{k-\alpha} \sqrt{2\pi(k-\alpha)}}{\left(\frac{k}{e}\right)^k \sqrt{2\pi k}} \\
&= \alpha_k \frac{\left(\frac{k-\alpha}{k}\right)^{k-\alpha} \sqrt{k-\alpha}}{e^{-\alpha}\sqrt{k}} \\
&= \alpha_k e^\alpha \sqrt{1-\frac{\alpha}{k}} \left(1-\frac{\alpha}{k}\right)^k \\
&\to 1\cdot e^\alpha \cdot 1 \cdot e^{-\alpha} = 1 \quad \text{as} \quad k\to\infty.
\end{aligned}$$

$\square$

Chapter 3

Nonoscillatory solutions of type (a)

3.1 Asymptotic properties of solutions of type (a)

To establish oscillation theorems, conditions for the nonexistence of solutions of type (a) and of type (b) are crucial. In the sequel, we give a lower bound for solutions of type (a) and we describe asymptotic properties of these solutions.

Recall that a solution x of (E) is of type (a) if

$$x_n > 0, \quad x_n^{[1]} > 0, \quad x_n^{[2]} > 0, \quad x_n^{[3]} > 0 \quad \text{for large n},$$

where $x^{[1]}$, $x^{[2]}$, $x^{[3]}$ are quasi-differences of x. If x is of type (a), then

$$(x,y,z,w) = \left(x, x^{[1]}, x^{[2]}, x^{[3]}\right)$$

is a type (a) solution of (S).

Lemma 8. *If equation* (E) *has a solution of type (a), then*

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \right)^{1/\gamma} \right)^{\lambda} < \infty \tag{3.1}$$

and every solution x of type (a) satisfies for $n \geq n_0$

$$\frac{x_n}{x_{n+\tau-3}^{\lambda/(\alpha\beta\gamma)}} \geq I_n \left(\sum_{i=n-3}^{\infty} d_i \right)^{1/(\alpha\beta\gamma)}, \tag{3.2}$$

where n_0 is sufficiently large and

$$I_n = \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma}. \tag{3.3}$$

Proof. Let (x,y,z,w) be a type (a) solution of system (S), i.e. all components of the solution are positive. First we prove (3.1). Since z is positive increasing, there exists $k > 0$ such that $z_n^{1/\beta} \geq k$ for large n, say $n \geq n_0$. From the first and the second equation of system (S) we get

$$x_j \geq \sum_{i=n_0}^{j-1} C_i y_i^{1/\gamma}, \quad y_j \geq \sum_{i=n_0}^{j-1} B_i z_i^{1/\beta} \geq k \sum_{i=n_0}^{j-1} B_i,$$

so

$$x_j \geq \sum_{n=n_0}^{j-1} C_n \left(\sum_{k=n_0}^{n-1} B_k z_k^{1/\beta} \right)^{1/\gamma} \geq k^{1/\gamma} \sum_{n=n_0}^{j-1} C_n \left(\sum_{k=n_0}^{n-1} B_k \right)^{1/\gamma}. \tag{3.4}$$

By summation of the fourth equation of system (S) and using (3.4)

$$-w_n + w_{n_0} = \sum_{i=n_0}^{n-1} -\Delta w_i \geq k^{\lambda/\gamma} \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{i+\tau-1} C_j \left(\sum_{k=n_0}^{j-1} B_k \right)^{1/\gamma} \right)^{\lambda}$$

and from the boundedness of w we have (3.1). Since w is non-increasing, we get from the second and the third equation of system (S)

$$y_j \geq w_{j-2}^{1/(\alpha\beta)} \sum_{i=n_0}^{j-1} B_i \left(\sum_{k=n_0}^{i-1} A_k \right)^{1/\beta},$$

so

$$x_n \geq w_{n-3}^{1/(\alpha\beta\gamma)} \sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}} \left(\sum_{i=n_0}^{j-1} \frac{1}{b_i^{1/\beta}} \left(\sum_{k=n_0}^{i-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma}. \tag{3.5}$$

Using summation of the fourth equation of system (S) we get

$$w_n \geq \sum_{i=n}^{\infty} D_i x_{i+\tau}^{\lambda} \geq x_{n+\tau}^{\lambda} \sum_{i=n}^{\infty} D_i. \tag{3.6}$$

Therefore,

$$w_{n-3}^{1/(\alpha\beta\gamma)} \geq x_{n+\tau-3}^{\lambda/(\alpha\beta\gamma)} \left(\sum_{i=n-3}^{\infty} D_i \right)^{1/(\alpha\beta\gamma)}. \tag{3.7}$$

From (3.5) and (3.7) follows the validity of (3.2). □

Theorem 4. *Every solution x of type (a) satisfies for $n \geq n_0$*

$$k_1 \left(x_{n-3}^{[3]} \right)^{1/(\alpha\beta\gamma)} I_n \leq x_n \leq k_2 I_n, \tag{3.8}$$

where k_1, k_2 are suitable positive constants, n_0 is sufficiently large and I_n is defined by (3.3).

In addition, if (H1) *or* (H2) *holds, then*

$$\lim_{n\to\infty} x_n = \infty, \tag{3.9}$$

if

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma} \right)^{\lambda} = \infty, \tag{3.10}$$

then

$$\lim_{n\to\infty} x_n^{[3]} = 0. \tag{3.11}$$

Proof. Let $(x,y,z,w) = \left(x, x^{[1]}, x^{[2]}, x^{[3]}\right)$ be a type (a) solution of system (S), i.e. all components of the solution are positive.

First, we prove (3.8). Since $w = x^{[3]}$ is non-increasing, we get from the second and the third equation of system (S)

$$y_j \geq w_{j-2}^{1/(\alpha\beta)} \sum_{i=n_0}^{j-1} B_i \left(\sum_{k=n_0}^{i-1} A_k \right)^{1/\beta},$$

so

$$x_n \ge w_{n-3}^{1/(\alpha\beta\gamma)} \sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}} \left(\sum_{i=n_0}^{j-1} \frac{1}{b_i^{1/\beta}} \left(\sum_{k=n_0}^{i-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma}. \tag{3.12}$$

Since w is positive decreasing, there exists $l > 0$ such that $w_n \le l$ for $n \ge n_0$. From the third and the second equation of system (S) we obtain

$$z_n \le z_{n_0} + l^{1/\alpha} \sum_{i=n_0}^{n-1} A_i, \quad y_n \le y_{n_0} + \sum_{k=n_0}^{n-1} B_k \left(z_{n_0} + l^{1/\alpha} \sum_{i=n_0}^{k-1} A_i \right)^{1/\beta},$$

then using the first equation of system (S) we get the upper bound

$$x_n \le x_{n_0} + \sum_{j=n_0}^{n-1} C_j \left(y_{n_0} + \sum_{k=n_0}^{j-1} B_k \left(z_{n_0} + l^{1/\alpha} \sum_{i=n_0}^{k-1} A_i \right)^{1/\beta} \right)^{1/\gamma}.$$

Therefore, there exists $k_2 > 0$ such that

$$x_n \le k_2 \sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}} \left(\sum_{k=n_0}^{j-1} \frac{1}{b_k^{1/\beta}} \left(\sum_{i=n_0}^{k-1} \frac{1}{a_i^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma} = k_2 I_n.$$

From (3.2) in Lemma 8 we get the lower bound. From (E) we obtain

$$\Delta x_n^{[3]} = -d_n x_{n+\tau}^{\lambda}. \tag{3.13}$$

By using (3.13) in inequality (3.2) we get the lower bound in (3.8).

Now, we prove the asymptotic properties of solutions of type (a). If (H1) or (H2) holds, then we get from (3.4) that

$$\lim_{j\to\infty} x_j \ge k^{1/\gamma} \lim_{j\to\infty} \sum_{n=n_0}^{j-1} C_n \left(\sum_{k=n_0}^{n-1} B_k \right)^{1/\gamma} = \infty,$$

which implies the validity of (3.9).

As claimed above, $x^{[3]}$ is positive and non-increasing. Assume that

$$\lim_{n\to\infty} x_n^{[3]} = m,$$

where m is a positive constant. From (3.8) we get that $x_n \geq k \cdot I_n$, where k is a positive constant. Using this fact and summation (3.13) from n_0 to $n-1$ we obtain

$$x_n^{[3]} = x_{n_0}^{[3]} - \sum_{i=n_0}^{n-1} d_i x_{i+\tau}^{\lambda} \leq x_{n_0}^{[3]} - k^{\lambda} \sum_{i=n_0}^{n-1} d_i I_{i+\tau}^{\lambda}. \tag{3.14}$$

Passing $n \to \infty$ and assuming (3.10), we have from (3.14) that $\lim_{n\to\infty} x_n^{[3]} < 0$, which gives a contradiction. Thus:

$$\lim_{n\to\infty} x_n^{[3]} = 0.$$

This completes the proof. □

3.2 Sufficient conditions for the nonexistence solutions of type (a)

The nonexistence of solutions of type (a) is ensured by the following conditions.

Theorem 5. *Assume* (H1) *or* (H2). *Then equation* (E) *has no solution of type (a) if any of the following conditions hold:*

(i)
$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \right)^{1/\gamma} \right)^{\lambda} = \infty; \tag{3.15}$$

(ii) $\lambda < \alpha\beta\gamma$ *and*

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma} \right)^{\lambda} = \infty; \tag{3.16}$$

(iii) $\lambda \geq \alpha\beta\gamma$, $\tau \geq 3$ *and*

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma} \left(\sum_{m=n-3}^{\infty} d_m \right)^{1/(\alpha\beta\gamma)} > 1; \tag{3.17}$$

(iv) $\lambda > \alpha\beta\gamma$, $\tau \geq 3$ *and*

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma} \left(\sum_{m=n-3}^{\infty} d_m \right)^{1/(\alpha\beta\gamma)} > 0. \tag{3.18}$$

Proof. Let (x,y,z,w) be a type (a) solution of system (S), i.e. all components of the solution are positive. Since z is positive increasing, there exists $k>0$ such that $z_n^{1/\beta} \geq k$ for large n, say $n \geq n_0$. From the first and the second equation of system (S) we get

$$x_j \geq \sum_{i=n_0}^{j-1} C_i y_i^{1/\gamma}, \quad y_j \geq \sum_{i=n_0}^{j-1} B_i z_i^{1/\beta} \geq k \sum_{i=n_0}^{j-1} B_i,$$

so

$$x_j \geq \sum_{n=n_0}^{j-1} C_n \left(\sum_{k=n_0}^{n-1} B_k z_k^{1/\beta} \right)^{1/\gamma} \geq k^{1/\gamma} \sum_{n=n_0}^{j-1} C_n \left(\sum_{k=n_0}^{n-1} B_k \right)^{1/\gamma}. \tag{3.19}$$

Let condition (i) hold. By summation of the fourth equation of system (S) and using (3.19) we get

$$-w_n + w_{n_0} = \sum_{i=n_0}^{n-1} -\Delta w_i \geq k^{\lambda/\beta\gamma} k^{\lambda/\gamma} \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{i+\tau-1} C_j \left(\sum_{k=n_0}^{j-1} B_k \right)^{1/\gamma} \right)^{\lambda}.$$

Passing $n \to \infty$ we get the contradiction with the boundedness of w.

Let condition (ii) hold. Taking into account that w is positive and decreasing, we get by summation of the third equation of system (S)

$$z_j \geq \sum_{i=n_0}^{j-1} A_i w_i^{1/\alpha} \geq w_{j-1}^{1/\alpha} \sum_{i=n_0}^{j-1} A_i.$$

Thus,

$$-\Delta w_n = D_n x_{n+\tau}^{\lambda} \geq D_n \left(\sum_{m=n_0}^{n+\tau-1} C_m \left(\sum_{k=n_0}^{m-1} B_k \left(w_{k-1}^{1/\alpha} \sum_{i=n_0}^{k-1} A_i \right)^{1/\beta} \right)^{1/\gamma} \right)^{\lambda}.$$

Hence,

$$\frac{-\Delta w_n}{w_{n-1}{}^{\lambda/(\alpha\beta\gamma)}} \geq D_n \left(\sum_{m=n_0}^{n+\tau-1} C_m \left(\sum_{k=n_0}^{m-1} B_k \left(\sum_{i=n_0}^{k-1} A_i \right)^{1/\beta} \right)^{1/\gamma} \right)^{\lambda}.$$

Summing this inequality from n_0 to ∞ we have

$$\sum_{n=n_0}^{\infty} \frac{-\Delta w_n}{w_{n-1}^{\lambda/(\alpha\beta\gamma)}} \geq \sum_{n=n_0}^{\infty} D_n \left(\sum_{i=n_0}^{n+\tau-1} C_i \left(\sum_{j=n_0}^{i-1} B_j \left(\sum_{k=n_0}^{j-1} A_k \right)^{1/\beta} \right)^{1/\gamma} \right)^{\lambda}.$$

By Lemma 5 the expression on the left side is finite, which is a contradiction with (3.16).

Assume (iii). Using Lemma 8 we obtain from (3.2)

$$I_n \left(\sum_{i=n-3}^{\infty} d_i \right)^{1/(\alpha\beta\gamma)} \leq \frac{x_n}{x_{n+\tau-3}^{\lambda/(\alpha\beta\gamma)}} \leq 1. \tag{3.20}$$

Passing $n \to \infty$, we get a contradiction with (3.17).

Assume (iv). Because (H1) or (H2) holds, then by Theorem 4 we have that (3.9) holds. Thus, since $\lambda > \alpha\beta\gamma$, then

$$\lim_{n\to\infty} \frac{x_n}{x_{n+\tau-3}^{\lambda/(\alpha\beta\gamma)}} = 0.$$

By (3.20) we have

$$I_n \left(\sum_{i=n-3}^{\infty} d_i \right)^{1/(\alpha\beta\gamma)} \leq \frac{x_n}{x_{n+\tau-3}^{\lambda/(\alpha\beta\gamma)}}.$$

Passing $n \to \infty$, we get that

$$\lim_{n\to\infty} I_n \left(\sum_{i=n-3}^{\infty} d_i \right)^{1/(\alpha\beta\gamma)} = 0,$$

which is a contradiction with (3.18). Thus, the solution of type (a) cannot occur. □

Remark 5. *Theorem 5 extends Theorem 2.6 and Corollary 2.2 in [3] for equation* (E). *We extend these results also for the super-linear and the half-linear cases.*

3.3 Applications

The assumption (H1) or (H2) is important only in Theorem 5-(iv). Theorem 5 claims (i), (ii), (iii) hold without this assumption.
In the super-linear case $\lambda > \alpha\beta\gamma$ condition (3.18) is better than (3.17). It is clear that condition (3.17) implies the validity of (3.18). This fact is illustrated by the following example.

Example 6. *Consider equation* (E) *in the form*

$$\Delta^3\left(n^{(3)}(\Delta x_n)\right) + \Delta\left(-\frac{1}{\ln n}\right)x_{n+\tau}^{\lambda} = 0, \quad \tau \geq 3 \text{ and } \lambda > 1.$$

Thus, $\alpha = \beta = \gamma = 1$ *and* $a_n = b_n = 1$, $c_n = n^{(3)}$, $d_n = \Delta\left(-\frac{1}{\ln n}\right)$.

Therefore,

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}}\right)^{1/\beta}\right)^{1/\gamma} \left(\sum_{m=n-3}^{\infty} d_m\right)^{1/(\alpha\beta\gamma)}$$

$$= \limsup_{n\to\infty} \sum_{i=n_0}^{n-1} \frac{1}{i^{(3)}} \left(\sum_{j=n_0}^{i-1}\left(\sum_{k=n_0}^{j-1} 1\right)\right) \left(\sum_{m=n-3}^{\infty} \Delta\left(-\frac{1}{\ln m}\right)\right)$$

$$= \limsup_{n\to\infty} \frac{\frac{1}{2}\ln(n-2)}{\ln(n-3)} = \frac{1}{2}.$$

We can see that condition (3.18) *is satisfied, while* (3.17) *is not applicable. By Theorem 5 such equation has no solution of type (a).*

Theorem 5 together with the change of summation given in Lemma 6 enables us to show the role of the nonlinearity $f(n) = n^{\lambda}$ to the nonexistence of a solution of type (a). The following holds.

Corollary 2. *Let there exist* $\lambda_0 < \alpha\beta\gamma$ *such that* (3.16) *with* $\lambda = \lambda_0$ *holds. Then for any* $\lambda \geq \lambda_0$ *equation* (E) *has no solution of type (a).*

Proof. First, assume $\lambda_0 \leq \lambda < \alpha\beta\gamma$. Using notation

$$X_i = \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma},$$

we have

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} X_i \right)^{\lambda_0} \leq \sum_{n=n_0}^{\infty} d_n \left(\sum_{i=n_0}^{n+\tau-1} X_i \right)^{\lambda},$$

and by Theorem 5 equation (E) does not have any solution of type (a).

Now, assume $\lambda \geq \alpha\beta\gamma$. Then using the change of summation from Lemma 6 part (i) we get that conditions (3.18) and (3.17) hold. In virtue of Theorem 5, equation (E) does not have any solution of type (a) in this case as well. □

Roughly speaking, condition (3.16) is the ”universal” sufficient condition for the nonexistence of solutions of type (a) for any $\lambda > 0$.

Chapter 4

Nonoscillatory solutions of type (b)

Recall that a solution x of (E) is of type (b) if

$$x_n > 0, \quad x_n^{[1]} > 0, \quad x_n^{[2]} < 0, \quad x_n^{[3]} > 0 \quad \text{for large n.}$$

Similarly as in Chapter 3, we state the lower bound for solutions of type (b) and we describe asymptotic properties of these solutions.

4.1 Asymptotic properties of solutions of type (b)

First, we give a lower bound for solutions of type (b).

Lemma 9. *If equation* (E) *has a solution of type (b), then*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} \left(\sum_{k=n}^{\infty} d_k \right)^{1/\alpha} < \infty \tag{4.1}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n}^{\infty} \frac{1}{a_k^{1/\alpha}} \left(\sum_{i=k}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} < \infty. \tag{4.2}$$

Moreover, every solution x of type (b) satisfies for $n \geq n_0$

$$\frac{x_n}{x_{n+\tau-1}^{\lambda/\alpha\beta\gamma}} \geq J_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}}, \tag{4.3}$$

where n_0 is sufficiently large and

$$J_n = \sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta}. \tag{4.4}$$

Proof. Let x be a solution of type (b). Then (x,y,z,w) is a solution of (S) satisfying $x_n > 0$, $y_n > 0$, $z_n < 0$, $w_n > 0$ for large n. Since the components y, w and $-z$ are positive and decreasing, we have

$$\lim_{n\to\infty} y_n = y_\infty, \quad y_\infty \geq 0, \qquad \lim_{n\to\infty} w_n = w_\infty, \quad w_\infty \geq 0, \qquad \lim_{n\to\infty} z_n = z_\infty, \quad z_\infty \leq 0.$$

STEP 1. By summation of the fourth equation of (S) we have

$$w_n = w_\infty + \sum_{k=n}^{\infty} D_k x_{k+\tau}^{\lambda} \geq x_{n+\tau}^{\lambda} \sum_{k=n}^{\infty} D_k. \tag{4.5}$$

By summation of the third equation of (S) and substituting (4.5) we obtain

$$z_m \geq z_{n_0} + x_{n_0+\tau}^{\lambda/\alpha} \sum_{n=n_0}^{m-1} A_n \left(\sum_{k=n}^{\infty} D_k \right)^{1/\alpha}.$$

Since z is bounded, we get (4.1).

STEP 2. By summation of the third equation of (S) and substituting (4.5) we get

$$-z_n = -z_\infty + \sum_{k=n}^{\infty} A_k w_k^{1/\alpha} \geq x_{n+\tau}^{\lambda/\alpha} \sum_{k=n}^{\infty} A_k \left(\sum_{i=k}^{\infty} D_i \right)^{1/\alpha}. \tag{4.6}$$

By summation of the second equation of (S) we obtain

$$y_m - y_{n_0} = \sum_{n=n_0}^{m-1} B_n z_n^{1/\beta}.$$

Thus,

$$y_{n_0} = y_m + \sum_{n=n_0}^{m-1} B_n (-z_n)^{1/\beta} \geq L^{1/\beta} \sum_{n=n_0}^{m-1} B_n \left(\sum_{k=n}^{\infty} A_k \left(\sum_{i=k}^{\infty} D_i \right)^{1/\alpha} \right)^{1/\beta},$$

where $L > 0$ such that $x_{n+\tau}^{\lambda/\alpha} \ge L$ for $n \ge n_0$. From here we get (4.2).

STEP 3. We prove the inequality (4.3). Using summation of the second equation of system (S) we get

$$y_\infty - y_n = \sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} z_k^{1/\beta},$$

so

$$y_n \ge \sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} \left(-z_k^{1/\beta}\right).$$

Using (4.6) we obtain

$$y_n \ge \sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} x_{k+\tau}^{\lambda/(\alpha\beta)} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta},$$

$$y_n \ge x_{n+\tau}^{\lambda/(\alpha\beta)} \sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta},$$

$$y_n \ge x_{n+\tau}^{\lambda/(\alpha\beta)} J_n.$$

Using summation of the first equation of system (S) we get

$$x_n \ge x_{n_0} + \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}} y_i^{1/\gamma} \ge y_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}}, \tag{4.7}$$

thus,

$$x_n \ge x_{n+\tau-1}^{\lambda/(\alpha\beta\gamma)} J_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}},$$

which implies (4.3). □

4.2 Sufficient conditions for the nonexistence solutions of type (b)

The nonexistence of solutions of type (b) is ensured by the following conditions.

Theorem 6. *Equation* (E) *has no solution of type (b) if any of the following conditions* *hold:*

(i)

$$T := \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} \left(\sum_{k=n}^{\infty} d_k \right)^{1/\alpha} = \infty; \tag{4.8}$$

(ii) $T < \infty$ *and*

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n}^{\infty} \frac{1}{a_k^{1/\alpha}} \left(\sum_{i=k}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} = \infty; \tag{4.9}$$

(iii) $\lambda < \alpha\beta\gamma$, $T < \infty$ *and*

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} \left(\sum_{k=n_0}^{n+\tau-1} \frac{1}{c_k^{1/\gamma}} \right)^{\lambda/(\alpha\beta)} \left(\sum_{j=n}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} = \infty; \tag{4.10}$$

(iv) $\lambda > \alpha\beta\gamma$, $\tau \geq 1$ *and*

$$\sum_{n=n_0}^{\infty} \frac{1}{c_n^{1/\gamma}} \left(\sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} \right)^{1/\gamma} = \infty; \tag{4.11}$$

(v) $\lambda \geq \alpha\beta\gamma$, $\tau \geq 1$ *and*

$$\limsup_{n\to\infty} \left(\sum_{k=n-1}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} \right)^{1/\gamma} \left(\sum_{m=n_0}^{n-1} \frac{1}{c_m^{1/\gamma}} \right) > 1. \tag{4.12}$$

Proof. Claims (i) and (ii) follow from Lemma 9. To prove claims (iii) - (v), we assume that there exists a solution x of (E) which is of type (b) and we use estimations stated in the proof of Lemma 9.

Let (x,y,z,w) be a solution of (S) satisfying $x_n > 0$, $y_n > 0$, $z_n < 0$, $w_n > 0$ for large n. Then z satisfies (4.6) and x satisfies (4.7). Using (4.6) and (4.7) we get

$$-z_n \geq x_{n+\tau}^{\lambda/\alpha} \sum_{j=n}^{\infty} A_j \left(\sum_{i=j}^{\infty} D_i \right)^{1/\alpha} \geq y_{n+\tau-1}^{\lambda/(\alpha\gamma)} \left(\sum_{k=n_0}^{n+\tau-1} C_k \right)^{\lambda/\alpha} \sum_{j=n}^{\infty} A_j \left(\sum_{i=j}^{\infty} D_i \right)^{1/\alpha}.$$

Thus, using the second equation of system (S) we obtain

$$-\Delta y_n = B_n\left(-z_n\right)^{1/\beta} \geq B_n y_{n+\tau-1}^{\lambda/(\alpha\beta\gamma)} \left(\sum_{k=n_0}^{n+\tau-1} C_k\right)^{\lambda/(\alpha\beta)} \left(\sum_{j=n}^{\infty} A_j \left(\sum_{i=j}^{\infty} D_i\right)^{1/\alpha}\right)^{1/\beta},$$

so

$$\frac{-\Delta y_n}{y_{n+\tau-1}^{\lambda/(\alpha\beta\gamma)}} \geq B_n \left(\sum_{k=n_0}^{n+\tau-1} C_k\right)^{\lambda/(\alpha\beta)} \left(\sum_{j=n}^{\infty} A_j \left(\sum_{i=j}^{\infty} D_i\right)^{1/\alpha}\right)^{1/\beta}.$$

Since $\alpha\beta\gamma > \lambda$ we get by Lemma 5-(i)

$$\infty > \sum_{n=n_0}^{\infty} \frac{-\Delta y_n}{y_{n+\tau-1}^{\lambda/(\alpha\beta\gamma)}} \geq \sum_{n=n_0}^{\infty} B_n \left(\sum_{k=n_0}^{n+\tau-1} C_k\right)^{\lambda/(\alpha\beta)} \left(\sum_{j=n}^{\infty} A_j \left(\sum_{i=j}^{\infty} D_i\right)^{1/\alpha}\right)^{1/\beta},$$

which gives a contradiction with (4.10).

Assume (iv). From the third and the second equation of system (S) and (4.6) we obtain

$$y_n \geq x_{n+\tau}^{\lambda/(\alpha\beta)} \sum_{k=n}^{\infty} B_k \left(\sum_{j=k}^{\infty} A_j \left(\sum_{i=j}^{\infty} D_i\right)^{1/\alpha}\right)^{1/\beta} = x_{n+\tau}^{\lambda/(\alpha\beta)} J_n,$$

where J_n is defined by (4.4). Thus,

$$c_n\left(\Delta x_n\right)^{\gamma} = y_n \geq x_{n+\tau}^{\lambda/(\alpha\beta)} J_n,$$

$$\frac{\Delta x_n}{x_{n+\tau}^{\lambda/(\alpha\beta\gamma)}} \geq \frac{1}{c_n^{1/\gamma}} J_n^{1/\gamma}.$$

Since x is positive increasing, $\tau \geq 1$ and $\alpha\beta\gamma < \lambda$ we have by Lemma 5-(ii)

$$\infty > \sum_{n=n_0}^{\infty} \frac{\Delta x_n}{x_{n+1}^{\lambda/(\alpha\beta\gamma)}} \geq \sum_{n=n_0}^{\infty} \frac{\Delta x_n}{x_{n+\tau}^{\lambda/(\alpha\beta\gamma)}} \geq \sum_{n=n_0}^{\infty} \frac{1}{c_n^{1/\gamma}} J_n^{1/\gamma},$$

which leads to a contradiction with (4.11).

Assume (v). Since $x_n \leq x_{n+\tau-1}$, we get by (4.3)

$$1 \geq \frac{x_n}{x_{n+\tau-1}^{\lambda/\alpha\beta\gamma}} \geq J_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}}$$

a contradiction with (4.12). □

Remark 6. *The conditions* (H1) *and* (H2) *are not needed in Theorem 6.*

4.3 Discussion of conditions

1) In the super-linear case $\lambda > \alpha\beta\gamma$ we can apply conditions (4.11) or (4.12). We show that they are independent. Let the sequence J_n be defined by (4.4) and put

$$X_n = \frac{1}{c_n^{1/\gamma}}.$$

Then conditions (4.11) and (4.12) can be rewritten as

$$\sum_{n=n_0}^{\infty} X_n J_n^{1/\gamma} = \infty \quad \text{and} \quad \limsup_{n\to\infty} J_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} X_i > 1,$$

respectively. Since J_n is decreasing, we have

$$J_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} X_i \le \sum_{i=n_0}^{n-1} X_i J_i^{1/\gamma},$$

so

$$\limsup_{n\to\infty} J_{n-1}^{1/\gamma} \sum_{i=n_0}^{n-1} X_i \le \sum_{i=n_0}^{\infty} X_i J_i^{1/\gamma}.$$

Thus in general, if condition (4.11) holds, then (4.12) need not to hold and vice versa.

Example 7. *Consider equation* (E), *where* $c_n = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\lambda > 1$ *and* a_n, b_n *be such that* $J_n = \frac{1}{n}$.
Then (4.11) *reads as*

$$\sum_{n=n_0}^{\infty} J_n = \sum_{n=n_0}^{\infty} \frac{1}{n} = \infty,$$

so (4.11) *is satisfied. Condition* (4.12) *reads as*

$$\limsup_{n\to\infty} J_{n-1} \sum_{i=n_0}^{n-1} 1 = \limsup_{n\to\infty} \frac{1}{n-1} \sum_{i=n_0}^{n-1} 1 = \limsup_{n\to\infty} \frac{n-n_0}{n-1} = 1,$$

so (4.12) *is not valid.*

2) Now, we discuss conditions for the nonexistence of solutions of type (a) and type (b) stated in Theorem 5 and Theorem 6.

Parts (i) and (ii) of Theorem 5 and parts (i)-(iii) of Theorem 6 can be viewed as a discrete counterpart of similar results for differential systems of the n-th order, see [20, Propositions 4.1 and 4.5].

Comparing conditions for the nonexistence of solutions of type (a) and (b) in the sublinear case, part (ii) of Theorem 5 and part (iii) of Theorem 6 extend Corollary 2.2 and Corollary 2.1 in [3], respectively, where it is assumed that $\tau \leq 0$ and (H1). Moreover, assuming (H1), part (i) of Theorem 5 and part (ii) of Theorem 6 can be obtained from Theorem 2.6 and Theorem 2.4 in [3], respectively, but our proofs are completely different.

Chapter 5

Oscillation criteria and applications

In this section we establish oscillation criteria for equation (E) under assumptions (H1) or (H2) and (2.9). Oscillation criteria are based on conditions for the nonexistence of the nonoscillatory solutions given in the previous sections.

5.1 Oscillation criteria

Theorem 3 from Section 2.3 ensures the oscillation of (E) for any $\tau \in \mathbb{Z}$. Now we apply results of Chapter 3 and Chapter 4 and we state oscillation theorems in which the role of deviating argument τ is important.
Consider the double series

$$P = \sum_{n=n_0}^{\infty} d_n \left(\sum_{k=n_0}^{n} \frac{1}{c_k^{1/\gamma}} \right)^{\lambda}, \qquad T = \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} \left(\sum_{k=n}^{\infty} d_k \right)^{1/\alpha}.$$

Theorem 7. *Assume* (H1) *or* (H2), $\tau \geq 1$. *If*

$$P = \infty \quad \text{and} \quad T = \infty,$$

then (E) *is oscillatory.*

Proof. Lemma 4 implies that equation (E) has two possible types of solutions, type (a) or type (b). If $P = \infty$, then (3.15) holds for $\tau \geq 1$ and by Theorem 5 equation (E) has no solution of type (a). If $T = \infty$, then by Theorem 6 equation (E) has no solution of type (b). Thus, (E) is oscillatory. □

In a special case when $\alpha = \gamma = \lambda = 1$ and $a_n = c_n$ we have

$$P = \infty \Leftrightarrow T = \infty.$$

The interesting case occurs when $\alpha = \lambda \neq 1$ or $\alpha \neq \lambda$. The problem of comparison of conditions (3.15) and (4.8) leads to the problem of a change of summation for double series described in Lemma 6.

Theorem 8. *Assume* (H1) *or* (H2). *Equation* (E) *with* $\tau \geq 1$ *is oscillatory if any of the following conditions holds:*

(i) $\alpha > \lambda$ *or* $\alpha = \lambda \geq 1$, $P = \infty$ *and*

$$\liminf \frac{c_n^{1/\gamma}}{a_n^{1/\alpha}} > 0; \tag{5.1}$$

(ii) $\alpha < \lambda$ *or* $\alpha = \lambda \leq 1$, $T = \infty$ *and*

$$\limsup \frac{c_n^{1/\gamma}}{a_n^{1/\alpha}} < \infty.$$

Proof. Claim (i). Clearly, condition $P = \infty$ implies the validity of (3.15) for any $\tau \geq 1$. Hence, by Theorem 5, equation (E) with $\tau \geq 1$ has no type (a) solution. By comparison theorem for series and in view of (5.1), we have

$$\sum_{n=n_0}^{\infty} d_n \left(\sum_{k=n_0}^{n} \frac{1}{a_k^{1/\alpha}} \right)^{\lambda} = \infty.$$

Using Lemma 6 we get $T = \infty$. By Theorem 6 equation (E) has no type (b) solutions. Now, the conclusion follows from Lemma 4. Claim (ii) can be proved by a similar way. □

In general, when Theorem 8 cannot be applied, then we can apply Theorem 5, part (i) and Theorem 6, parts (i), (ii) and we obtain the following result.

Theorem 9. *Assume* (H1) *or* (H2). *If* (3.15) *and either* (4.8) *or* (4.9) *hold, then equation* (E) *is oscillatory.*

In the sub-linear case this result can be improved using part (ii) of Theorem 5 and part (iii) of Theorem 6 as follows.

Theorem 10. *Assume* $\lambda < \alpha\beta\gamma$ *and either* (H1) *or* (H2). *If* (3.16) *and either* (4.8) *or* (4.10) *hold, then equation* (E) *is oscillatory.*

In general, oscillation of (E) depends on the type of nonlinearity (whether the sub-linear, the half-linear or the super-linear case occurs) and on the deviating argument τ. The following holds.

Theorem 11. *Let* $\tau \geq 3$ *and either* (H1) *or* (H2) *hold. Equation* (E) *is oscillatory if any of the following conditions hold:*

(i) $\lambda = \alpha\beta\gamma$, (3.17) *and one of conditions* (4.8), (4.9) *or* (4.12);

(ii) $\lambda > \alpha\beta\gamma$, (3.18) *and one of conditions* (4.8), (4.9), (4.11) *or* (4.12);

(iii) $\lambda < \alpha\beta\gamma$, (3.16) *and one of conditions* (4.8), (4.9) *or* (4.10).

Proof. By Lemma 4 any nonoscillatory solution is of type (a) or (b). By Theorem 5 and 6 the conditions ensure that equation (E) has no solutions of type (a) and of type (b). □

Remark 7. *Theorem 9 generalizes [3, Theorem 2.10], where they assume only the case when* (H1) *holds.*

Theorems 8, 9, 10 can be compared with results in [21] using coupled system (2.4). *Application of Theorem 1 or Theorem 2' of [21] to system* (2.4) *leads to conditions* (3.15), (4.8) *or* (3.16), (4.8), *respectively. Observe that Theorem 4' of [21] ensures oscillation of* (2.4) *provided* $\lambda < 1$, (3.16) *and certain additional assumptions on* α, β, γ.

Theorem 11 case (ii) extends Corollary 2 in [33] and case (iii) extends Corollary 1 in [33], where equation (1.1) *was studied, the special kind of our more general equation* (E).

Theorem 11 extends Theorem 2.10 in [3], where the super-linear case was not treated at all.

Concluding remark

We discuss the role of the integer-valued argument τ in (E) to the behavior of nonoscillatory solutions. It is well-known that the deviating argument τ plays an important role in the oscillation.

We can notice that conditions (3.15) and (3.16) for the nonexistence of solutions of type (a) depend on τ but hold for $\tau \in \mathbb{Z}$. On the contrary, conditions (3.17) and (3.18) do not depend on τ and hold only for $\tau \geq 3$.

If we consider conditions for the nonexistence of solutions of type (b), the argument τ appears only in condition (4.10), the others do not depend on τ. Conditions (4.8), (4.9) and (4.10) hold for $\tau \in \mathbb{Z}$ and conditions (4.11) and (4.12) hold only for $\tau \geq 1$.

In example 12 of the following section we can see how the argument τ can influence the nonexistence of a solution of type (a). Thus, it is a question whether we can generalize the effect of τ to the nonexistence of both solutions of type (a) and (b).

5.2 Applications and examples

In this section there are examples which illustrate our results which were presented in the previous chapter.

First example shows that conditions in Theorem 10 are optimal.

Example 8. *Consider the equation*

$$\Delta\left(\Delta^3 x_n\right)^{\alpha} + d_n x_{n+\tau}^{\lambda} = 0 \tag{5.2}$$

where $\tau \geq 1$ *and* (2.9) *holds. Then*

$$P = \sum_{n=n_0}^{\infty} n^{\lambda} d_n, \quad T = \sum_{n=n_0}^{\infty} \left(\sum_{k=n}^{\infty} d_k\right)^{1/\alpha}$$

and by Theorems 8 and 10 we get that equation (5.2) *is oscillatory if any of the following conditions is satisfied*

(i) $\lambda < \alpha$ *or* $\alpha = \lambda \geq 1$, $P = \infty$;

(ii) $\lambda > \alpha$ *or* $\alpha = \lambda \le 1$, $T = \infty$;

(iii) $\lambda < \alpha$, $\sum_{n=n_0}^{\infty} n^{3\lambda} d_n = \infty$, $T < \infty$ *and*

$$\sum_{n=n_0}^{\infty} n^{\lambda/\alpha} \sum_{j=n}^{\infty} \left(\sum_{k=j}^{\infty} d_k \right)^{1/\alpha} = \infty.$$

The claim (iii) of Example 8 is not true for $\alpha = \lambda = 1$ as the next example shows.

Example 9. *Consider the Euler-type difference equation*

$$\Delta^4 x_n + \frac{15}{16} \frac{n^{(-3/2)}}{(n+3)^{(5/2)}} x_{n+3} = 0, \qquad (n \ge 2), \tag{5.3}$$

where $n^{(\mu)} = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}$ *is the factorial function,* Γ *is the Gamma function and* $\mu \in \mathbb{R}$ *for which* $\Gamma(n-\mu+1)$ *is defined. One can check that* $x_n = n^{(5/2)}$ *is a positive solution of* (5.3)*. Using the fact* $n^{(\mu)} \sim n^{\mu}$ *(see Lemma 7) we have* $\sum_{n=n_0}^{\infty} n^3 d_n = \infty$ *and*

$$\sum_{n=n_0}^{\infty} n \sum_{j=n}^{\infty} \left(\sum_{k=j}^{\infty} d_k \right) = \infty.$$

Another oscillation criteria can be obtained using the cyclic permutation described in Lemma 2 and Theorem 2. For instance, in the case when

$$\sum_{n=n_0}^{\infty} a_n^{-1/\alpha} = \infty, \quad \sum_{n=n_0}^{\infty} b_n^{-1/\beta} < \infty, \quad \sum_{n=n_0}^{\infty} c_n^{-1/\gamma} = \sum_{n=n_0}^{\infty} d_n = \infty,$$

we can apply Theorems 8–11 to the equation (R2).

We show the application of Theorem 2 and Theorem 10.

Consider equation

$$\Delta^2 \left(b_n \left(\Delta^2 x_n \right)^{\beta} \right) + d_n x_{n+\tau}^{\lambda} = 0, \tag{5.4}$$

where $\tau \in \mathbb{Z}$ and

$$\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\beta}} < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} d_n = \infty. \tag{5.5}$$

Then the cyclic permutated equation (R_2) to (5.4) is

$$\Delta^2\left(\frac{1}{d_n^{1/\lambda}}\left(\Delta^2 z_n\right)^{1/\lambda}\right)+\frac{1}{b_{n+\tau}^{1/\beta}}z_{n+\tau}^{1/\beta}=0, \tag{5.6}$$

whose difference operator is in the canonical form, i.e. (H1) holds. In equation (5.6) we have $\alpha=1$, $\beta=1/\lambda$, $\gamma=1$, $\lambda=1/\beta$. Hence, the condition $\lambda<\alpha\beta\gamma$ reads $\lambda<\beta$ and the series P and T for (5.6) as

$$\bar{P}=\sum_{n=n_0}^{\infty}\left(\frac{n}{b_{n+\tau}}\right)^{1/\beta}, \qquad \bar{T}=\sum_{n=n_0}^{\infty}\sum_{k=n}^{\infty}\frac{1}{b_{k+\tau}^{1/\beta}}=\sum_{n=n_0}^{\infty}\frac{n-n_0+1}{b_{n+\tau}^{1/\beta}}.$$

Since $\lim_{n\to\infty}\frac{n+\tau}{n}=1$, we have $\bar{P}=\infty$ if and only if

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{b_n}\right)^{1/\beta}=\infty. \tag{5.7}$$

Similarly, since $\lim_{n\to\infty}\frac{n+\tau}{n-n_0+1}=1$, we get $\bar{T}=\infty$ if and only if

$$\sum_{n=n_0}^{\infty}\frac{n}{b_n^{1/\beta}}=\infty. \tag{5.8}$$

Observe that if $\beta\geq 1$ and (5.7) holds, then (5.8) is satisfied, while if $\beta\leq 1$ and (5.8) holds, then (5.7) is satisfied.

It is worth noting that if (5.7) and (5.8) hold, then (H2) is satisfied for (5.6) and we can apply Theorems 7 – 11 to (5.6).

If (5.7) and (5.8) hold, then by Theorem 7 equation (5.6) is oscillatory.

By Theorem 7 we get the following corollary.

Corollary 3. *Assume* (5.5)*,* $\tau\geq 1$*,* $\beta>0$ *arbitrary and* (5.7)*,* (5.8) *hold. Then* (5.4) *is oscillatory.*

By Theorem 2 and Theorem 10, we get the following result.

Corollary 4. *Assume* (5.5) *and* $\lambda < \beta$, $\tau \in \mathbb{Z}$. *If*

$$\sum_{n=n_0}^{\infty} \frac{1}{b_{n+\tau}^{1/\beta}} \left(\sum_{j=n_0}^{n+\tau-1} j^{\lambda} d_j \right)^{1/\beta} = \infty$$

and either (5.8) *or*

$$\sum_{n=n_0}^{\infty} n^{\lambda/\beta} d_n \left(\sum_{k=n}^{\infty} \frac{k}{b_k^{1/\beta}} \right)^{\lambda} = \infty,$$

then equation (5.4) *is oscillatory.*

Remark 8. *Corollary 4 completes the oscillation criteria for equation* (5.4) *with* $\tau = 3$ *given in [33] and [34], where instead of the condition* $\sum d_n = \infty$, *it is assumed that both series in conditions* (5.7) *and* (5.8) *are divergent or convergent respectively.*

The following examples illustrate Theorem 11.

Example 10. *Consider the equation*

$$\Delta^2 \left(\frac{1}{n-1} (\Delta(n-1)\Delta x_n) \right) + \frac{\mu}{(n+2)(n+3)} x_{n+3}^{\lambda} = 0, \tag{5.9}$$

where $\mu > 1$ *and* $\lambda \geq 1$ *are real constants.*

Thus, $a_n = 1$, $b_n = \frac{1}{n-1}$, $c_n = n-1$, *and* $\alpha = \beta = \gamma = 1$. *We have*

$$X_n = \frac{1}{c_n} \sum_{i=n_0}^{n-1} \frac{1}{b_i} \left(\sum_{j=1}^{i-1} \frac{1}{a_j} \right) = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{i-1} \left(\sum_{j=1}^{i-1} 1 \right) = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{i-1}{i-1} = 1,$$

$$\sum_{k=n-3}^{\infty} d_k = \sum_{k=n-3}^{\infty} \frac{\mu}{(k+2)(k+3)} = \mu \sum_{k=n-3}^{\infty} -\Delta \frac{1}{k+2} = \frac{\mu}{n-1},$$

and so (3.17) *reads as*

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} X_i \sum_{k=n-3}^{\infty} d_k = \limsup_{n\to\infty} (n-n_0) \frac{\mu}{n-1} = \mu.$$

Therefore, if $\lambda > 1$, *then the condition* (3.18) *is satisfied and by Theorem 5-(iv) equation* (5.9) *has no solution of type (a).*

If $\lambda = 1$ *, then we apply Theorem 5-(iii). Thus,* (5.9) *has no solution of type (a).*

Applying condition (4.8) *we get*

$$\sum_{n=n_0}^{\infty}\left(\sum_{k=n}^{\infty}\frac{\mu}{(k+2)(k+3)}\right)=\infty.$$

Hence, by Theorem 6 - (i) equation (5.9) *has no solution of type (b) for any* $\lambda \geq 1$.

Summarizing, (5.9) *is oscillatory for* $\lambda \geq 1$.

Example 11. *Consider the equation*

$$\Delta^4 x_n + \mu n^{(-4)} x_{n+3} = 0, \tag{5.10}$$

where $\mu > 0$ *is a real constant.*

If $\mu > 6$, *then*

$$\limsup_{n\to\infty}\left(\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\sum_{s=j}^{\infty}\mu n^{(-4)}\right)\left(\sum_{k=n_0}^{n}1\right)$$
$$=\limsup_{n\to\infty}\frac{\mu n^{(-1)}}{6}\cdot(n+1-n_0)=\limsup_{n\to\infty}\frac{\mu(n+1-n_0)}{6(n+1)}>1.$$

Thus, the condition (4.12) *is satisfied and by Theorem 6 equation* (5.10) *has no solution of type (b).*

Similarly, if $\mu > 18$, *then*

$$\limsup_{n\to\infty}\left(\sum_{i=n_0}^{n-1}\sum_{j=n_0}^{i-1}\sum_{k=n_0}^{j-1}1\right)\left(\sum_{k=n-3}^{\infty}\mu k^{(-4)}\right)=\limsup_{n\to\infty}\frac{n^3}{6}\cdot\frac{\mu\,(n-3)^{(-3)}}{3}>1,$$

and the condition (3.17) *is satisfied. Then by Theorem 5 equation* (5.10) *has no solution of type (a).*

Summarizing, (5.10) *is oscillatory for* $\mu > 18$.

Next example illustrates the role of the deviating argument τ.

Example 12. *Assume equation* (E) *with* $\lambda = 2$, $d_n = e^{-n^2}$, $a_n = 1$, $\alpha = 1$ *and* b_n, c_n, β, γ *satisfy*

$$\frac{1}{c_i^{1/\gamma}}\left(\sum_{j=n_0}^{i-1}\frac{1}{b_j^{1/\beta}}\right)^{1/\gamma}=\Delta e^{\frac{(i-2)^2}{2}}.$$

First, assume equation (E) *with* $\tau = 1$. *Rewriting the condition* (3.15) *we obtain*

$$\sum_{n=n_0}^{\infty} e^{-n^2} \left(\sum_{i=n_0}^{n} \Delta e^{\frac{(i-2)^2}{2}} \right)^2 = \sum_{n=n_0}^{\infty} e^{-n^2} \left(e^{\frac{(n-1)^2}{2}} - K \right)^2$$

$$< \sum_{n=n_0}^{\infty} e^{-n^2} e^{(n-1)^2} = \sum_{n=n_0}^{\infty} e^{-2n+1} < \infty,$$

where $K = e^{\frac{(n_0-2)^2}{2}}$.

Thus, Theorem 5-(i) is not aplicable and we can not decide if (E) *has a solution of type (a).*

However, for (E) *with* $\tau = 2$ *the condition* (3.15) *is satisfied because*

$$\sum_{n=n_0}^{\infty} e^{-n^2} \left(\sum_{i=n_0}^{n+1} \Delta e^{\frac{(i-2)^2}{2}} \right)^2 = \sum_{n=n_0}^{\infty} e^{-n^2} \left(e^{\frac{n^2}{2}} - K \right)^2$$

$$= \sum_{n=n_0}^{\infty} e^{-n^2} \left(e^{n^2} - 2Ke^{\frac{n^2}{2}} + K^2 \right) = \sum_{n=n_0}^{\infty} 1 - 2Ke^{-\frac{n^2}{2}} + K^2 e^{-n^2} = \infty$$

Therefore (E) *does not have any solution of type (a) for* $\tau \geq 2$.

Chapter 6

Maximal and minimal solutions

In this section we study maximal and minimal solutions of (E) under assumption (H1) or (H2) and their relationship to the type (a) and (b) solutions.
According to Lemma 4, any eventually positive solution of (E) falls into one of the two types (a) or (b).
Recall

$$x_n^{[1]} = c_n\left(\Delta x_n\right)^{\gamma}, \quad x_n^{[2]} = b_n\left(\Delta x_n^{[1]}\right)^{\beta}, \quad x_n^{[3]} = a_n\left(\Delta x_n^{[2]}\right)^{\alpha}.$$

If $x_n > 0$, then there exists $k > 0$ such that $x_n \geq k$ for large n and furthermore $x^{[3]}$ is positive and decreasing.

Therefore, we can state that for any eventually positive solution x of (E) there exist positive constants r, R such that

$$r \leq x_n \leq R I_n \quad \text{for large n},$$

where I_n is defined by (3.3), i.e.

$$I_n = \sum_{i=n_0}^{n-1} \frac{1}{c_i^{1/\gamma}} \left(\sum_{j=n_0}^{i-1} \frac{1}{b_j^{1/\beta}} \left(\sum_{k=n_0}^{j-1} \frac{1}{a_k^{1/\alpha}} \right)^{1/\beta} \right)^{1/\gamma}.$$

That leads to the following definition of a minimal and a maximal solution.

Definition 2. *In the set of all eventually positive solutions of equation* (E)*, a solution x* *which satisfies*

$$\lim_{n\to\infty} x_n = r \tag{6.1}$$

is called a minimal solution, and a solution x of equation (E) *satisfying*

$$\lim_{n\to\infty} \frac{x_n}{I_n} = R \tag{6.2}$$

is called a maximal solution.

The problem of the existence of minimal and maximal solutions has been studying by Agarwal and Manojlović [4], Migda and Schmeidel [23], Thandapani and Arockiasamy [29], Thandapani and Selvaraj [32] for special types of the fourth-order difference equations.

6.1 Maximal solutions

Lemma 10. *If x is a maximal solution of* (E)*, then*

$$\lim_{n\to\infty} x_n = \infty. \tag{6.3}$$

Proof. Assumptions (H1) and (H2) imply that

$$\lim_{n\to\infty} I_n = \infty. \tag{6.4}$$

If x is a maximal solution, then x satisfies (6.2) and from this and (6.4) we get the validity of (6.3). □

The following result shows the relation between a maximal solution and a solution of type (a).

Theorem 12. *Assume* (H1) *or* (H2)*. If x is a maximal solution of* (E)*, then x is of type (a).*

Proof. By Lemma 4, any eventually positive solution of (E) is of type (a) or type (b). Let x be a solution of (E) of type (b). Hence, $x^{[1]}$ is positive and decreasing. Thus, there exists

$k > 0$ such that $x_n^{[1]} \leq k$ for large n. From this we obtain

$$c_n (\Delta x_n)^\gamma \leq k,$$

$$x_n \leq k^{1/\gamma} \sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}}.$$

From (6.2) we get that there exists n_0 such that

$$x_n \geq \frac{R}{2} \cdot I_n$$

for all $n \geq n_0$. Hence,

$$\frac{R}{2} \cdot I_n \leq x_n \leq k^{1/\gamma} \sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}},$$

and therefore,

$$\limsup_{n\to\infty} \frac{I_n}{\sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}}} < \infty.$$

However, by discrete l'Hospital's rule (Stolz theorem), see Agarwal [1, Theorem 1.8.9], we get

$$\limsup_{n\to\infty} \frac{I_n}{\sum_{j=n_0}^{n-1} \frac{1}{c_j^{1/\gamma}}} = \infty,$$

which gives a contradiction. Therefore a solution x of type (b) can not be a maximal solution. □

Theorem 13. *A necessary condition for equation* (E) *to have a maximal solution x is that*

$$\sum_{n=n_0}^{\infty} d_n I_{n+\tau}^{\lambda} < \infty. \tag{6.5}$$

Proof. Let x be a maximal solution of equation (E). Then there exists an integer n_0 such that

$$\frac{R}{2} \cdot I_n \leq x_n \leq 2R \cdot I_n,$$

for $n \geq n_0$. Summing equation (E) from n_0 to $n-1$ we have

$$x_n^{[3]} = a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta x_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha} \geq \sum_{i=n_0}^{n-1} d_i x_{i+3}^{\lambda}$$

$$\geq \sum_{i=n_0}^{n-1} d_i\left(\frac{R}{2}\cdot I_{i+3}\right)^{\lambda} \geq \left(\frac{R}{2}\right)^{\lambda}\cdot\sum_{i=n_0}^{n-1} d_i\cdot I_{i+3}^{\lambda}$$

for all $n \geq n_0$. Since $x^{[3]}$ is bounded, passing $n \to \infty$ we arrive at a contradiction. □

The problem whether the condition (6.5) is also sufficient for the existence of a maximal solution is a subject of our study in [17]. Observe that this problem has been studied for equation

$$\Delta^2\left(b_n\left(\Delta^2 x_n\right)^{\beta}\right) + d_n x_{n+3}^{\lambda} = 0$$

by Thandapani and Selvaraj [32, Theorem 1] and Agarwal and Manojlović [4, Theorem 5.1]. The proof of Theorem 1 in [32] is based on Schauder Fixed-Point Theorem. However, the continuity of the operator is not given there and the proof of the relatively compactness is not clear. In [4] Theorem 5.1 is given without proof with the argument that it is the same as that of Theorem 1 in [32].

6.2 Minimal solutions

We continue with the relation between a minimal solution and a solution of type (b).

Theorem 14. *Assume* (H1) *or* (H2). *If x is a minimal solution of* (E)*, then x is of type (b).*

Proof. By Lemma 4, any eventually positive solution of (E) is of type (a) or type (b). Let x be a type (a) solution of (E). Hence, by Theorem 4, we have

$$\lim_{n\to\infty} x_n = \infty,$$

which gives a contradiction with the definition of the minimal solution. Therefore, the minimal solution must be of type (b). □

Theorem 15. *A necessary condition for equation* (E) *to have a minimal solution x is that*

$$\sum_{n=n_0}^{\infty} \frac{1}{c_n^{1/\gamma}} \left(\sum_{k=n}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} \right)^{1/\gamma} < \infty. \tag{6.6}$$

Proof. Let x be a positive minimal solution of equation (E). There exists an integer n_0 such that

$$\frac{r}{2} \le x_n \le 2r,$$

for all $n \ge n_0$. In view of Theorem 14 the solution is of type (b). Thus, from (4.3) we get

$$x_n \ge \sum_{m=n_0}^{n-1} \frac{1}{c_m^{1/\gamma}} \left(\sum_{k=m}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i x_{i+\tau}^{\lambda} \right)^{1/\alpha} \right)^{1/\beta} \right)^{1/\gamma}.$$

We suppose that $2r \ge x_n$, letting $n \to \infty$, we get

$$\infty > 2r \ge x_n \ge \sum_{m=n_0}^{n-1} \frac{1}{c_m^{1/\gamma}} \left(\sum_{k=m}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i x_{i+\tau}^{\lambda} \right)^{1/\alpha} \right)^{1/\beta} \right)^{1/\gamma}.$$

Because x_n has a positive finite limit as $n \to \infty$, we obtain

$$\infty > \sum_{m=n_0}^{\infty} \frac{1}{c_m^{1/\gamma}} \left(\sum_{k=m}^{\infty} \frac{1}{b_k^{1/\beta}} \left(\sum_{j=k}^{\infty} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{\infty} d_i \right)^{1/\alpha} \right)^{1/\beta} \right)^{1/\gamma}.$$

$\square$

In [17] we show that (6.6) is also the sufficient condition.

Similarly as maximal solutions, minimal solutions were studied by Thandapani and Selvaraj in [32], see Theorem 2. The proof is based on Schauder Fixed-Point Theorem but the operator in their proof is defined incorrectly. In addition, the proof lacks the proof of the continuity of the operator and the proof of the relatively compactness.

Chapter 7

Concluding remarks and open problems

We present new oscillation results and we indicate future directions which may be pursued in the context of our research. Due to the fact that studying fourth-order difference equations has received considerably less attention, there is a great number of open problems in this direction. Thus, the topics presented in this dissertation can be extended in various ways. We sketch some of the related problems in this section.

- The first possible extension of our results could be generalization of our theorems for a two-term difference equation of the form

$$L_4 x_n + d_n f(x_n) = 0,$$

  where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies

$$u f(u) > 0 \quad \text{for} \quad u \neq 0,$$

  and for $\lambda > 0$

$$\lim_{u\to\infty} \frac{f(u)}{u^{\lambda}} > 0.$$

- The other possibility is to study equation (E) without assumptions (H1) and (H2). In this case, there are eight possible types of nonoscillatory positive solutions, therefore it will be a more difficult problem to find some conditions for oscillation.

- The results in Chapter 5 can be generalized and applied to more general even-order difference equations and $2n$-dimensional systems.

- In this dissertation there are sufficient conditions given for the oscillation of equation (E). Thus, another possible direction of our research can be effort to find such conditions that all solutions of equation (E) are nonoscillatory. This is an open problem.

Appendix

In this dissertation we present new results in the theory of fourth-order difference equations obtained and published by the author jointly with Zuzana Došlá. The thesis consists of the results from articles [15, 16, 17]. For completeness, this dissertation is finished with a list of results presented in this text that have been published.

Theorem 1 [15, Theorem 1]

Lemma 1 [19, Lemma 7]

Lemma 2 [15, Lemma 1]

Theorem 2 [15, Theorem 2]

Lemma 3 - This lemma was proved in [14, Lemma 1] for system (S) with the assumption $D_n < 0$.

Lemma 4 - This lemma was proved in [16, Lemma 1] with assumption (H2) and in [15, Lemma 2] with assumption (H1).

Theorem 3 - This theorem was proved in [16, Proposition 1] with assumption (H2) and in [15, Proposition 1] with assumption (H1).

Lemma 5 claim (i) [15, Lemma 3]

Lemma 5 claim (ii) [16, Lemma 4]

Lemma 8 [16, Proposition 2]

Theorem 5 claims (i), (ii) [15, Lemma 4 (ii), (iii)]

Theorem 5 claim (iii) [16, Theorem 5 (ii)]

Lemma 9 [16, Proposition 3]

Theorem 6 claims (i), (ii), (iii) [15, Lemma 5 (i), (ii), (iii)]

Theorem 6 claims (iv), (v) [15, Theorem 6 (i), (ii)]

Theorem 7 [15]

Theorem 8 - This theorem was proved in [15, Theorem 3] only with assumption (H1).

Theorem 9 - This theorem was proved in [16, Theorem 2 (i)] with assumption (H2) and in [15, Theorem 4] with assumption (H1).

Theorem 10 - This theorem was proved in [16, Theorem 2 (ii)] with assumption (H2) and in [15, Theorem 5] with assumption (H1).

Theorem 11 claim (i) [16, Corollary 2 (ii)]

Theorem 11 claim (iii) [16, Theorem 2 (ii)]

Corollary 4 [15, Corollary 1]

The following results are contained in [17]:

Theorem 4, Theorem 5 claim (iv), Lemma 10, Theorems 11 – 15

Bibliography

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, 2nd ed., rev. and expanded. New York: Marcel Dekker, 2000. xiii, 971. ISBN 0-8247-9007-3.

[2] R. P. Agarwal, S.R. Grace, P.J.Y. Wong, *Oscillatory behavior of fourth order nonlinear difference equations*, New Zealand J. Math., 36 (2007), 101-111.

[3] R. P. Agarwal, S. R. Grace, J. V. Manojlović, *On the oscillatory properties of certain fourth order nonlinear difference equations*, J. Math. Anal. Appl., 322 (2006), 930-956.

[4] R. P. Agarwal, J. V. Manojlović, *Asymptotic behavior of positive solutions of fourth order nonlinear difference equations*, Ukrainian Math. J., 60 (2008), 6-28.

[5] R. P. Agarwal, J. V. Manojlović, *Asymptotic behavior of nonoscillatory solutions of fourth order nonlinear difference equations*, Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal., 16 (2009), 155-174.

[6] M. Cecchi, Z. Došlá, M. Marini, I. Vrkoč, *Summation inequalities and half-linear difference equations*, J. Math. Anal. Appl., 302 (2005), 1-13.

[7] M. Cecchi, Z. Došlá, M. Marini, I. Vrkoč, *Asymptotic properties for half-linear difference equation*, Math. Bohem., 131 (4) (2006), 347-363.

[8] U. Elias, *Oscillation Theory of Two-Term Differential Equations*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1997.

[9] S. Fišnarová, *Oscillation of two-term Sturm-Liouville difference equations*, International J. Difference Equ., (1) (2006), 81-99.

[10] T. Fort, *Finite differences and difference equations in the real domain*, Oxford University Press, London 1948.

[11] H.I. Freedman, *Deterministic mathematical models in population ecology*, Marcel Dekker, New York, 1980.

[12] D. Greenspan, *Discrete models*, Addison-Wesley, Reading, Massachusetts, 1973.

[13] T. A. Chanturia, *On oscillatory properties of systems of nonlinear ordinary differential equations* (in Russian), Proc. of I. N. Vekua Inst. of Appl. Math., Tbilisi, 14 (1983), 163-204.

[14] Z. Došlá, J. Krejčová, *Nonoscillatory solutions of the four-dimensional difference system*, Electron. J. Qual. Theory Differ. Equ., Proc. 9'th Coll. Qualitative Theory of Diff. Equ., No. 4 (2011), 1-11.

[15] Z. Došlá, J. Krejčová, *Oscillation of a class of the fourth-order nonlinear difference equations*, Adv. Difference Equ., 2012, 2012:99 (2 July 2012).

[16] Z. Došlá, J. Krejčová, *Asymptotic and oscillatory properties of the fourth-order nonlinear difference equations*, Appl. Math. Comput., (submitted, November 2013).

[17] Z. Došlá, J. Krejčová, *Minimal and maximal solutions of the fourth-order nonlinear difference equations*, (in preparation, February 2014).

[18] I. T. Kiguradze, A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[19] J. Krejčová, *Asymptotic properties of solutions of difference systems*, Rigorous thesis, Faculty of Science, Masaryk University, Brno, 2012.

[20] T. Kusano, M. Naito, F. Wu, *On the oscillation of solutions of 4-dimensional Emden-Fowler differential systems*, Adv. Math. Sci. Appl., 11 (2) (2001), 685-719.

[21] M. Marini, S. Matucci, P. Řehák, *Oscillation of coupled nonlinear discrete systems*, J. Math. Anal. Appl., 295 (2004), 459-472.

[22] M. Migda, A. Musielak, E. Schmeidel, *On a class of fourth order nonlinear difference equations*, Adv. Difference Equ., 1 (2004), 23-36.

[23] M. Migda, E. Schmeidel, *Asymptotic properties of fourth order nonlinear difference equations*, Math. Comput. Modelling, 39 (2004), 1203-1211.

[24] J. Popenda, E. Schmeidel, *On the solution of fourth order difference equations*, Rocky Mountain J.Math., 25 (4) (1995), 1485-1499.

[25] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Book Co., New York, third edition, 1976.

[26] B. Selvaraj, I. Mohammed Ali Jaffer, *Oscillation behavior of certain fourth order difference equations*, Int. J. Nonlinear Sci., (2011), 11.2: 131-136.

[27] E. Schmeidel, *Oscillation and nonoscillation theorems for fourth order difference equations*, Rocky Mountain J.Math., 33 (3) (2003), 1083-1094.

[28] E.Schmeidel, M.Migda, A.Musielak, *Oscillatory properties of fourth order nonlinear difference equations with quasidifferences*, Opuscula Math. 26, 2 (2006), 371-380.

[29] E. Thandapani, I. M. Arockiasamy, *Fourth-order nonlinear oscillations of difference equations*, Comput. Math. Appl., 42 (2001), 357-368.

[30] E. Thandapani, J. Graef, *Oscillatory and asymptotic behavior of fourth order nonlinear delay difference equations*, Fasc. Math., 31 (2001), 23-36.

[31] E. Thandapani, S. Pandian, R. Dhanasekaran, J. Graef, *Asymptotic results for a class of fourth order quasilinear difference equations*, J. Difference Equ. Appl., Vol.13, 12 (2007), 1085-1103.

[32] E. Thandapani, B. Selvaraj, *Oscillatory and nonoscillatory behavior of fourth order quasilinear difference equations*, Far East J. Appl. Math., 17 (3) (2004), 287-307.

[33] E. Thandapani, B. Selvaraj, *Oscillations of fourth order quasilinear difference equations*, Fasc. Math., 37 (2007), 109-119.

[34] E. Thandapani, M. Vijaya, *Oscillatory and asymptotic behavior of fourth order quasilinear difference equations*, Electron. J. Qual. Theory Differ. Equ., 64 (2009), 1-15.

[35] J. H. Van Lint, *Introduction to coding theory*, Springer-Verlag, New York, 1982.

[36] J. Yan, B. Liu, *Oscillatory and asymptotic behavior of fourth-order difference equations*, Acta Math. Sinica, 13 (1997), 105-115.