Masaryk University
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Generalized planar curves and quaternionic geometries

Ph.D. Thesis

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Abstrakt / Abstract

Abstrakt: Se studiem zobecněných planárních křivek a planárních zobrazení jsme začali motivováni souvislostmi mezi projektivní a skoro kvaternionovou geometrií. Začínáme tím, že uvádíme přehled a obecné definice tak zvaných A–struktur, kde A je lineární obal daných afinorů, tedy zpracováváme a rozšiřujeme klasickou teorii planárních křivek a planárních zobrazení v její obecnosti. Dále využíváme dopad obecných výsledků na skoro kvaternionovou geometrii, kterou můžeme prezentovat z hlediska teorie parabolických geometrí a můžeme specifikovat koncept zobecněné planarity i v tomto případě. Konkrétně ukážeme že přirozeně definovaná třída $\mathbb{H}$–planárních křivek je shodná s třídou geodetik všech Weylových konexí a zachování této třídy se ukazuje být nutnou i postačující podmínkou pro morfismy skoro kvaternionové geometrie. Hlavní výsledky byly publikovány v [HS]

Abstract: Motivated by the analogies between the projective and the almost quaternionic geometries, we first study the generalized planar curves and mappings. We start to present an abstract and general definition of the so called A–structures, where A is the linear span of the given affinors, hence recover and extend the classical theory of planar curves in this general setup. Then we exploit the impact of the general results in the almost quaternionic geometry. We shall present the almost quaternionic structures from the viewpoint of the theory of parabolic geometries and we shall specify the classical generalizations of the concept of the planarity of curves to this case. In particular, we show, that the natural class of $\mathbb{H}$–planar curves coincides with the class of all geodesics of the so called Weyl connections and preserving this class turns out to be the necessary and sufficient condition on diffeomorphisms to become morphisms of almost quaternionic geometries. The main results have been published in [HS].

Contents

1. Introduction

One of the basic ideas in differential geometry is the idea of geodetics (for detailed exposition see e.g. [KMS], [KONO], [Mich]). On a smooth manifold M equipped with a linear connection ∇, geodetics is a parameterized curve $C \subset M$ whose tangent vector field is parallel along the curve itself. If we parameterize the curve arbitrarily, the condition on the parallel tangent vector field for another suitable parametrization reads $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c} \rangle$. Intuitively, we may say that a geodetic curve is a trajectory which is 'straight' and with 'constant velocity'. In the case of Riemanian geometry (M, g), this idea is very illustrative. There exists a distinguished connection without torsion which is called the Levi–Civita connection. Its geodetics are exactly the curves connecting two arbitrary points by the shortest curve (with respect to the Riemanian metric g). This connection is directly determined by the chosen metric g. Morphisms which preserve the geodetics (as unparameterized curves) are called geodesical mappings and they are not isometries in general. Let us remind that morphisms preserving geodetics of a linear connection ∇ are morphisms of the corresponding projective structure.

Our goal is to discuss similar relation between classes of curves and geometries for other structures. In fact, we want to find the class of curves which exactly determine morphisms of our structure. The fact that there is a connection without torsion is exceptional and it does not have to exist for our structures. An elegant solution exits for Cartan geometries (for detailed exposition see [ČS],[ČS-03]). There is a canonical class of distinguished connection for each Cartan geometry. Driven by the analogy to conformal Riemanian geometries, we call them Weyl connections. The general theory of parabolic geometries offers a simple description of the deformation of the covariant derivatives in the canonical class. It makes sense to talk about their geodesics. This text wants to resolve some of their properties for a concrete geometric structure, the almost quaternionic geometries.

Let us also note, that there is a narrower class of curves which are called the generalized geodetics and which are geodetics of the so called normal Weyl connections.

An almost quaternionic geometry is a real smooth manifold M of dimension $4n$ equipped with rank three subbundle $Q \subset TM \otimes T^{\star}M$ which is locally generated by tensors I, J, IJ of type $(1,1)$ (compare with [S], [J]). The tensors I, J, IJ replace the multiplication by imaginary quaternions i, j, k. Let us notice that an almost complex structure is a real manifold N of dimension $2n$ equipped with a tensor I of type $(1,1)$ representing the multiplication by the imaginary complex unit i. We shall present the almost quaternionic structures from the viewpoint of the theory of parabolic geometries and we shall specify the classical generalizations of the concept of the planarity of curves to this case. It turns out that the class of all such curves, the so called Q–planar curves, exhausts exactly the geodesics of all the Weyl connections. The main result of this work is the proof that isomorphisms of almost quaternionic geometries are just the diffeomorphisms which preserve the class of (unparameterized) Q–planar curves. This remarkable behavior corresponds nicely to the well known fact that the homogeneous model of the almost quaternionic geometries is the quaternionic projective space and so the similarity to the projective geometries is not that surprising.

On the way to this result, we follow the classical planarity concept with respect to several affinors $F_1, \dots, F_i$, i.e. tensors of type $(1,1)$. The idea is that the covariant derivative of the tangent vector $\dot{c}$ in the direction of the curve always belongs to the vector space $\langle F_1(\dot{c}), \dots, F_i(\dot{c})\rangle$. We present an abstract and general definition of the so called A–structures, where A is the linear span of the given affinors, and recover the classical theory of planar curves in this general setup. This is the contents of Chapter 4.

As we have seen, the almost quaternionic geometries represent a special example of such A–structures and the general results of Chapter 4 lead quickly to the proof of the main result mentioned above.

Generally, it is possible to read each chapter separately, following the references to other parts, if necessary.

The main line of our exposition starts in Chapter 3 which reviews the almost quaternionic geometry as an particular example of parabolic geometries. We first remind some basic notions and facts and then our main results are formulated.

The fourth chapter begins with a small introduction to the concept and history of planar morphism and continues with definition of A–structures. We recover and generalize the approach by Mikes and Sinjukov, see [MS] and the references therein. Our general results are applicable to a quite wide class of structures satisfying some generic rank conditions on the affinors.

The first chapter enhances our exposition by a gallery of geometries related to the almost quaternionic ones and it may serve as an introduction to the main topics of the dissertation for those coming from the classical theory of geometric structures rather than the parabolic geometry. In particular, we provide a quick link to the theory of G-structures and some related concepts.

The main results of this dissertation have been published in [HS].

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2. G–structures related to the quaternions

This chapter provides a quick introduction to G–structures, their connections and prolongations. A more detailed study of several basic examples of G–structures related to quaternions can be found also in this chapter. The concept of quaternionic–like geometries has been discussed by Dmitri Alekseevsky and S. Marchiafava (compare with paper [AM96]). The main observation is that it makes sense to talk about Q–planar curves for quaternionic–like geometries (compare with remark 2.42). The concept of Q–planar curves was first discussed by Shigeyoski Fujimura (see [Fujimura77]).

2.1. **Connections and prolongation of G–structures.** First order G–structures are the simplest examples of geometric structures. The definitions below appear in all standard textbooks on geometric structures, see for example the book [KO].

2.2**. Definition.** Let M be a smooth manifold of dimension n and let P^1M be the bundle of linear frames over M. The bundle P^1M is a principal bundle over M with the structure group $GL(n, \mathbb{R})$. Let G be a closed subgroup of $GL(n, \mathbb{R})$. The reduction $P \to P^1M$ of the bundle P^1M to the subgroup $G \subset GL(n, \mathbb{R})$ is called *G–structure*.

The following theorem shows a geometric way how such structures arise.

2.3**. Theorem** ([KO])**.** *Let M be a smooth manifold of dimension n. Let K be a tensor field over the vector space $\mathbb{R}^n$ and $G \subset GL(n, \mathbb{R})$ be a group of linear transformations $\mathbb{R}^n$ leaving K invariant. Let $P \to M$ be G–structure with structure group G and $\mathcal{K}$ the tensor filed on M defined by K and P. Then*

1. *A diffeomorphism $f : M \to M$ is an automorphism of the G–structure $P \to M$ if and only if f leaves $\mathcal{K}$ invariant.*
2. *A vector field X is an infinitesimal automorphism of the G–structure $P \to M$ if and only if $L_X\mathcal{K} = 0$, where L_X denotes the Lie derivation with respect to X.*

2.4**. Definition.** Let M be a smooth manifold of dimension n and let $\mathbb{R}^n$ be the vector space of the same dimension. The *soldering form* $\theta \in \Omega^1(P^1M, \mathbb{R}^n)$ is defined in the following way. For each $u \in P^1M$ (where u is viewed as a linear isomorphism $u : \mathbb{R}^n \to T_xM$),

$$\theta_u(\xi) := u^{-1}(T_u\pi \cdot \xi),$$

where $\pi : P^1M \to M$ $(T_u\pi \cdot \xi \in T_{\pi(u)}M)$, $x = \pi(u)$ and $\xi \in T_uP^1M$.

By construction, the soldering form θ is G–equivariant with respect to the standard action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ and it is strictly horizontal, i.e.:

$$\begin{aligned}(r^g)^\star\theta &= \ell_{g^{-1}}\circ\theta \text{ for } g\in GL(n,\mathbb{R})\\ \theta(\xi) &= 0 \text{ if and only if } \xi \text{ is vertical.}\end{aligned}$$

In fact, the principal bundle $\pi : P \to M$ with a structure group G equipped with a $\mathbb{R}^n$–valued G–equivariant strictly horizontal 1–form $\theta : TP \to \mathbb{R}^n$ is *a G–structure* on a manifold M.

The form θ induces an identification of P with a principal G–subbundle of the frame bundle P^1M in the following way. For any $p \in P$ there is an isomorphism

$$\bar{p} : T_{\pi(p)}M \to \mathbb{R}^n$$

induced by θ so that

$$\bar{p}\circ T\pi = \theta_p.$$

The map $p \mapsto \bar{p}$ defines this identification.

2.5. **Example.** Let M be a smooth manifold of dimension $2n$. Let $I : TM \to TM$ be a morphism satisfying $I^2 = -id$. Then $GL(n,\mathbb{C}) = \{A \in GL(2n,\mathbb{R}) | AI = IA\} \subset GL(2n,\mathbb{R})$ is a real Lie group. The G–structure (see definition 2.2) with structure group $GL(n,\mathbb{C})$ is called *almost complex structure.*

We also use the term G–structures with structure group G more generally. In this case the structure group G is not a closed subgroup of $GL(n,\mathbb{R})$ but a covering of a virtual subgroup, i.e. there is a homomorphism $j : G \to GL(n,\mathbb{R})$ such that the derivative $j' : \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{R})$ is injective. A well known example is a Rieamannian spin structure, corresponding to the universal covering $Spin(n) \to SO(n) \subset GL(n,\mathbb{R})$.

2.6. **Definition.** Let $\pi : P \to M$ be a principal bundle with the structure group G. The bundle $VP \to M$ is obtained by composing the bundle $\mathrm{Ker}(T\pi) : TP \to TM$ with the tangent bundle $\pi : TM \to M$. This bundle is called *vertical subbundle. A general connection* on P is a projection $\Psi : TP \to VP$, viewed as 1–form $\Psi \in \Omega(P, TP)$. The subbundle $\mathcal{H} := \ker\Psi$ is called *horizontal subbundle.* A general connection Ψ is called *principal connection* if this is G–equivariant for the principal right action $r : P \times G \to P$, i.e. $T(r^g)\cdot\Psi = \Psi\cdot T(r^g)$.

There are three equivalent ways to view this:

- We have $TP = VP \oplus \mathcal{H}$ and this decomposition can be equivalently described by the smooth vertical G–equivariant projection $\Psi : TP \to VP$ with kernel $\mathcal{H}$ (or horizontal projection $\chi = id_{TP} - \Psi$).
- We can consider the induced horizontal right invariant lift $\xi^{hor} \in \mathfrak{X}(P)$ of vector fields $\xi \in \mathfrak{X}(M)$, $(r^g)^\star\xi^{hor} = \xi^{hor}$, for all $g \in G$.

It is the unique projectable vector field lying over ξ whose value in each point is horizontal.

- We view the connection as the unique G–equivariant one jet of a section $s : M \to P$, such that the horizontal lift $T_xM \to T_yP$, where $\pi(y) = x$ is given by T_xs (i.e.: section of the first jet prolongation $J^1P \to P$).

There is the canonical isomorphism $i_p : V_pP \to \mathfrak{g}$ whose inverse is given by

$$\mathfrak{g} \ni X \mapsto \zeta_X(p) = \frac{d}{dt}(\exp(tX)\cdot p)|_{t=0} \in V_pP. \tag{1}$$

The vertical bundle VP is trivialized as a vector bundle over P by the principal action. So

$$\zeta_{\omega(X_p)}(u) = \Psi(X_p) \tag{2}$$

and in this way we get a G–equivariant $\mathfrak{g}$–valued 1–form $\omega \in \Omega(P, \mathfrak{g})$, which is called *connection form* of the principal connection Ψ.

2.7**. Definition.** Let $\mathbb{V}$ be a vector space of dimension n, let G be a subgroup of Lie group of linear transformations of $\mathbb{V}$ and let $\mathfrak{g}$ be the Lie algebra of G. The *first prolongation* $\mathfrak{g}^{(1)}$ *of a Lie algebra* $\mathfrak{g}$ is the space of all symmetric bilinear mappings $t : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ such that, for each fixed $v_1 \in \mathbb{V}$, the mapping $v \in \mathbb{V} \mapsto t(v, v_1) \in \mathbb{V}$ is contained in $\mathfrak{g}$.

For later use, the vector space $\mathbb{V}$ is equal to a vector space $\mathbb{R}^n$.

2.8**. Example.** *An almost complex structure* (M, I) (see the example 2.5) is a G–structure with structure group $GL(n, \mathbb{C})$. The Lie algebra $\mathfrak{gl}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n, \mathbb{R}) | AI = IA\}$ is the Lie algebra of Lie group $GL(n, \mathbb{C})$. The first prolongation $\mathfrak{gl}(n, \mathbb{C})^{(1)}$ is the space of symmetric bilinear mappings preserving I

$$\mathfrak{gl}(n, \mathbb{C})^{(1)} = \{t | t : \mathbb{V}\times\mathbb{V} \to \mathbb{V}, t(IX, Y) = It(X, Y), t(Y, X) = t(X, Y)\}.$$

2.9. **First prolongation of G–structure.** Our goal is to define a principal bundle whose sections are connections on a G–structure with a special torsion. The first step will be to construct a principal bundle whose sections are all connections of a G–structure. This bundle is called a *derivation of G–structure.* The second step will be to construct a *torsion tensor* and a subbundle of the derivation of G–structure which includes only connections with special torsion. This subbundle is called a *first prolongation of G–structure.* We follow the paper [AM96], which contains the results from our paragraphs 2.10, 2.13, and 2.15. The paper also contains several our definitions (includes in particular the definition of quaternionic–like structures) and the paragraph 2.38.

Every connection is described as a smooth horizontal subspace $\mathcal{H}(p)$, for any $p \in P$. The choice of a G–equivariant extension $\omega_p : T_pP \to \mathfrak{g}$ of the isomorphism i_p (see (1)) is the same as the choice of the kernel of ω ($\mathcal{H}(p) = \ker \omega(p)$), i.e. there is a one to one correspondence between connections on G–structure and G–equivariant forms $\omega : TP \to \mathfrak{g}$ which are equal on V_pP (the G–equivariant property cames from the definition of connection on a G–structure). On the other hand, restriction of the soldering form θ to a horizontal subspace $\mathcal{H}(p)$ defines an isomorphism

$$\theta_{\mathcal{H}} : \mathcal{H}(p) \to \mathbb{V}.$$

Together, we can associate horizontal space $\mathcal{H}(p)$ (i.e. connection) with an isomorphism

$$\bar{\mathcal{H}}(p) = \theta_{\mathcal{H}} \oplus i_p : T_pP = \mathcal{H}(p) \oplus V_pP \to \bar{\mathbb{V}} = \mathbb{V} + \mathfrak{g}.$$

The linear group $\bar{G} = \mathrm{Hom}(\mathbb{V}, \mathfrak{g}) \subset GL(\bar{\mathbb{V}})$ acts linearly on the vector space $\bar{\mathbb{V}}$ so that

$$\bar{G} \ni B : (v + A) \mapsto v + (A + B(v)), v \in \mathbb{V}, A \in \mathfrak{g}.$$

If $\bar{P}$ is the bundle of all linear connections on M then $\bar{P}$ is the bundle of all horizontal subspaces, i.e. manifold of all functions $\bar{\mathcal{H}}(p) : T_pP \to \mathbb{V} \oplus \mathfrak{g}$ with the properties and the action of $\bar{G}$ on $\bar{P}$ defined above. There is an orbit space $\bar{P}/\bar{G} = P$ and the natural projection $\bar{\pi} : \bar{P} \to P$ is a principal $\bar{G}$–bundle. The soldering form on $\bar{P}$ is

$$\bar{\theta}_{\mathcal{H}} = \bar{\mathcal{H}}(p) \circ \bar{\pi}_\star = (\theta_{\mathcal{H}} + i_p) \circ \bar{\pi}_\star,$$

where $\mathcal{H} \in \bar{P}, p = \bar{\pi}(\mathcal{H}) \in P$.

2.10. **Theorem.** *Let $\pi : P \to M$ be a G–structure with a canonical form $\theta : TP \to \mathbb{V}$. Then the bundle $\bar{\pi} : \bar{P} \to P$ of 1–jets of section of π is a $\bar{G}$–structure with soldering form $\bar{\theta} : T\bar{P} \to \bar{\mathbb{V}}$ defined above where $\bar{G} = Hom(\mathbb{V}, \mathfrak{g}) \subset GL(\bar{\mathbb{V}})$. The form $\bar{\theta} : T\bar{P} \to \bar{\mathbb{V}}$ is G–equivariant with respect to the natural action of $\bar{G}$ on the manifold $\bar{P}$ and on the vector space $\bar{\mathbb{V}} = \mathbb{V} + \mathfrak{g}$. A connection on the G–structure $\pi : P \to M$ may by identified with a G–equivariant section $s : P \ni p \mapsto \mathcal{H}(p) \in \bar{P}$ of the bundle $\bar{\pi} : \bar{P} \to P$.*

The principal bundle $\pi : \bar{P} \to P$ is equipped with a canonical form with values in $\bar{\mathbb{V}} = \mathbb{V} + \mathfrak{g}$.

On the G–structure $\bar{\pi} : \bar{P} \to P$ with structure group $\bar{G}$, there is the canonical $\mathbb{V} \otimes \wedge^2\mathbb{V}^\star$–valued *torsion function*:

$$T : \bar{P} \to \mathbb{V} \otimes \wedge^2\mathbb{V}^\star$$

which is given by

$$t_{\mathcal{H}}(u, v) = d\theta_p(\theta_{\mathcal{H}}^{-1}u, \theta_{\mathcal{H}}^{-1}v),$$

where $\mathcal{H} \in \bar{P},\ u, v \in \mathbb{V},\ p = \bar{\pi}(\mathcal{H}) \in P,\ \theta_{\mathcal{H}} = \theta_p|_{\mathcal{H}}$.

This torsion function is equivariant with respect to the natural action of the semidirect product $G \rtimes \bar{G}$, where the action of $\bar{G}$ on the vector space $\mathbb{V} \otimes \wedge^2\mathbb{V}^\star$ is given by

$$T \mapsto T + \partial B,$$

where $B \in \bar{G}$ and $T \in \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$. Here

$$\partial : \mathrm{Hom}(\mathbb{V}, \mathfrak{g}) \ni \bar{G} = \mathfrak{g} \otimes \mathbb{V}^\star \to \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star \tag{3}$$

is the Spencer operator of alternation.

Let $s_\omega : p \to \mathcal{H}(p)$ be a section of the bundle $\bar{\pi} : \bar{P} \to P$ that defines the connection ω. Torsion function of the connection ω is the function

$$t^\omega := t \circ s_\omega : P \to \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star.$$

Clearly, any choice of a complement $\mathcal{D}$ of the image $\partial(\mathfrak{g} \otimes \mathbb{V}^\star)$ in $\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$, i.e.

$$\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star = \partial(\mathfrak{g} \otimes \mathbb{V}^\star) \oplus \mathcal{D}, \tag{4}$$

as the required space for the torsion fixes the torsion completely. For good reasons, the G–invariant complements $\mathcal{D}$ are required. We will see that for any parabolic geometry there is such a complement, see (22).

Let us write

$$P^{(1)} := t^{-1}(\mathcal{D}) = \{\mathcal{H} \in \bar{P}, t^\omega \in \mathcal{D}\}.$$

Then $\pi^{(1)} = \bar{\pi}|_{P^{(1)}} : P^{(1)} \to P$ is a principal bundle with the structure group.

$$G^{(1)} = (\mathfrak{g} \otimes \mathbb{V}^\star) \cap (\mathbb{V} \otimes S^2 \mathbb{V}^\star) \subset \mathfrak{g} \otimes \mathbb{V}^\star = \bar{G} \subset GL(\bar{\mathbb{V}}).$$

2.11**. Definition.** Let $\pi^{(1)} : P^{(1)} \to P$ be the principal bundle with structure group $G^{(1)}$ constructed above. This bundle together with the $\bar{\mathbb{V}}$–valued 1–form $\theta^{(1)} = \bar{\theta}|_{P^{(1)}}$ is called the *first prolongation of the G–structure*.

2.12**. Definition.** Let $\pi : P \to M$ be a G–structure and let $\mathcal{D}$ be a G–invariant subspace introduced in (2.9). A connection ω on P is called a *$\mathcal{D}$–connection* if its torsion function

$$t^\omega = t \circ s_\omega : P \to \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star = \partial(\mathfrak{g} \otimes \mathbb{V}^\star) \oplus \mathcal{D}$$

takes values in $\mathcal{D}$.

2.13**. Theorem** ([AM96])**.**

(1) *Any G–structure $\pi : P \to M$ with a fixed G–equivariant subspace $\mathcal{D}$ as in (4) admits a $\mathcal{D}$–connection ω.*

(2) *Let $\omega, \bar{\omega}$, be two $\mathcal{D}$–connections on M. Then the corresponding covariant derivatives $\nabla, \bar{\nabla}$ are related by*

$$\bar{\nabla} = \nabla + S,$$

where S is a symmetric tensor field of type $(2,1)$, such that for any $x \in M$, the tensor S_x belongs to the first prolongation $\mathfrak{g}_x^{(1)}$ of the Lie algebra $\mathfrak{g}_x \subset \mathfrak{gl}(T_xM)$.

2.14. **Remark.** *A detailed study of the prolongation theory of higher order G–structures on manifolds can be found in* [K71],[K74], *or* [K75]. *Kolář defined generalized G–structures, as well as semi–holonomic and holonomic prolongations of generalized G–structures and related iterated differentials.*

2.15. **Corollary.** *Let $\pi : P \to M$ be a G–structure with structure group G, such that the first prolongation of its algebra vanishes, i.e. $\mathfrak{g}^{(1)} = 0$, and suppose that there is given a G–equivariant decomposition with respect to structure group G*

$$\mathbb{V} \otimes \wedge^2 \mathbb{V} = \partial(\mathfrak{g} \otimes \mathbb{V}^\star) \oplus \mathcal{D}.$$

Then there is an unique connection whose torsion tensor (calculated with respect to a frame $p \in P$) takes values in $\mathcal{D} \subset \mathbb{V} \otimes \wedge^2 \mathbb{V}^\star$.

Most of the structures related to quaternions satisfy the assumptions of the latter corollary, see below. The first prolongation of an almost complex structure from example 2.5 is not trivial, because $\mathfrak{g}^{(1)} \neq 0$ in this case (see example 2.8).

2.16. **Quaternionic numbers.** The quaternionic–like geometry is geometry based on quaternions. The descriptions of all these geometries are given in [AM96].

Quaternionic numbers $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ are extension of complex numbers $\mathbb{C}$, in the same way as complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ are a generalization of real numbers $\mathbb{R}$. Together it is described as $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$. Usually the number ij is denoted by k and

$$\mathbb{H} = \{x = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},$$

where $i^2 = j^2 = k^2 = -1, ij = k = -ji$.

In this notation we have $\bar{x} = a - bi - bj - ck$, $|x|^2 = a^2 + b^2 + c^2 + d^2$, and $|x| = x\bar{x}$, $Re(x) = \frac{x+\bar{x}}{2}$, $Im(x) = \frac{x-\bar{x}}{2}$.

Let us notice that in the previous section we introduced an almost complex structure as a G–structure (see example 2.5) and in the sense of theorem 2.3, there is tensor field I of type $(1, 1)$ on an almost complex structure.

2.17. **Definition.** Let $\mathbb{V}$ be a real vector space of dimension $4n$. A pair (I, J) of anti–commuting complex structures on $\mathbb{V}$ $(IJ + JI = 0)$ is called *hypercomplex structure.*

2.18. **Lemma.** *The following three definitions of hypercomplex structure are equivalent,*

1. *A pair (I, J) such that $IJ + JI = 0$, $I^2 = J^2 = -E$.*
2. *A triple (I, J, K) such that $K = IJ$, $IJ = -JI$, $IK = -KI$, $JK = -KJ$, $I^2 = J^2 = K^2 = -E$.*
3. *A triple (I, J, K) such that $K = IJ$ and $(IJK) = 0$, where $()$ is symmetrization.*

Proof. (1) $\Leftrightarrow$ (2). By definition of (1) and (2) it suffices to prove only the implication (1) $\Rightarrow$ (2). Consider complex structures I, J from (1) and denote $K = IJ$. A short computation shows:

$$KK = IJIJ = -IJJI = II = -E, IK = IIJ = -J$$

$$JK = JIJ = -IJJ = I, KI = IJI = -IIJ = JKJ = IJJ = -I.$$

(2) $\Rightarrow$ (3). Consider complex structures I, J, K from (2) and substitute $K = IJ$. We get

$$IJK + IKJ + JIK + JKI + KIJ + KJI = 0.$$

(3) $\Leftrightarrow$ (1). Consider complex structures I, J, K from (3). Thus $(IJK) = 0$, i.e.

$$KIJ + KJI + IJK + IKJ + JIK + JKI = 0$$

and since $K = IJ$, $J = -IK$, $KJ = -IJJ = I$ we get

$$0 = -E + KJI - E + E + E + JKI = KJI + JKI = (KJ + JK)I = 0,$$

i.e. $KJ = -JK$ and $IJ = KJJ = -JKJ = -JI$. $\square$

The second property from lemma 2.18 above is used as a definition of hypercomplex structure in many papers ([J], [MNP]) because it shows a similarity with quaternionic numbers. We prefer the first definition, because this is the simplest one.

2.19**. Definition.** Let $\mathbb{V}$ be a real vector space of dimension $4n$ and let a pair (I, J) be a hypercomplex structure. We define the subset $Q(I, J) \subset \mathrm{Aut}(\mathbb{V})$

$$Q(I, J) := \{aI + bJ + cIJ \mid a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\}. \tag{5}$$

The next lemma shows a geometrical meaning of our notation.

2.20**. Lemma.** *Let $\mathbb{V}$ be a real vector space of dimension $4n$ and let us choose a hypercomplex structure (I, J), $K := IJ$. If we consider a new hypercomplex structures $\bar{I}, \bar{J} \in \langle E, I, J, K\rangle$, $\bar{K} = \bar{I}\bar{J}$ then $\bar{I}, \bar{J}, \bar{K} \in Q(I, J)$.*

Proof. Consider hypercomplex structure

$$\bar{I} = a_0 + a_1 I + a_2 J + a_3 K,$$

$$\bar{J} = b_0 + b_1 I + b_2 J + b_3 K,$$

$$\bar{K} = c_0 + c_1 I + c_2 J + c_3 K.$$

If we compute the coefficient of I, J, K in $\bar{I}\bar{J} = -\bar{J}\bar{I}$, than we get

$$a_0 b_1 = -b_0 a_1, \tag{6}$$

$$a_0 b_2 = -b_0 a_2, \tag{7}$$

$$a_0 b_3 = -b_0 a_3, \tag{8}$$

and if we compose coefficients of the constant than we get

$$a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 = 0. \tag{9}$$

Supposing that $b_0 \neq 0$ we can express the coefficients a_1, a_2, a_3 from (6), (7), and (8) and thus after evaluation in (9) and computation we get

$$a_0b_0^2 + a_0b_1^2 + a_0b_2^2 + a_0b_3^2 = 0.$$

If $a_0 \neq 0$ we will get the equations

$$b_0^2 + b_1^2 + b_2^2 + b_3^2 = 0$$

and this is contradiction with $\bar{J} \neq 0$.

If $a_0 = 0$ we will get the equations $b_0a_1 = 0$, $b_0a_2 = 0$, $b_0a_3 = 0$, and $a_1 = a_2 = a_3 = 0$ because of $b_0 \neq 0$. This is contradiction with $\bar{I} \neq 0$

We obtain $b_0 = 0$ and if we use the relation $\bar{J}^2 = -b_1^2 - b_2^2 - b_3^2 = -id$ we will get, that the coefficients b_1, b_2, b_3 belong to the unit sphere in $\mathbb{R}^3$. $\square$

For better understanding, we describe the hypercomplex structures with the help of real matrices. In suitable real coordinates, the affinors I, J, K can be nicely described as real $4n \times 4n$ matrices:

$$I = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix},$$

where E denotes the identity real (n, n)–matrix.

The structure group of a hypercomplex structure $GL(n, \mathbb{H}) \subset GL(4n, \mathbb{R})$ consists of matrices:

$$GL(n, \mathbb{H}) := \left\{ \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \middle| \, a, b, c, d \in \mathrm{Mat}_n(\mathbb{R}) \right\}.$$

2.21. **Remarks.** *1.* Let I be a complex structure. Any nonzero vector $X \in \mathbb{V}$ and its image $I(X)$ are linearly independent. Let us assume $X = b \cdot I(X)$. If we apply the operator I then we get $IX = -bX$ and direct computations gives

$$X = bIX = -b^2X$$

$$b^2 = -1.$$

This is contradiction with the fact that $b \in \mathbb{R}$.

2. Let (I, J) be a hypercomplex structure, $K = IJ$. We shall observe that any nonzero $X \in \mathbb{V}$ the dimension of the real vector space $\langle X, IX, JX, KX\rangle$ equals to four. Indeed, let us consider $X \in \mathbb{V}$ such that $\dim\langle X, IX, JX, KX\rangle < 4$. There are $a, b, c, d \in \mathbb{R}$ such that:

$$aX + bIX + cJX + dKX = 0$$

and multiplying by E, I, J, K we get 4 equations:

$$aX + bIX + cJX + dKX = 0$$

$$-bX + aIX - dJX + cKX = 0$$

$$-cX + dIX + aJX - bKX = 0$$

$$-dX - cIX + bJX + aKX = 0$$

The determinant of the following matrix has to be zero and short computation shows that there is only one possibility $a = b = c = d = 0$ and for any $X \in TM$, $\dim\langle X, IX, JX, KX\rangle = 4$.

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} =$$

$$= a^4 + a^2c^2 + a^2b^2 + a^2d^2 + b^2a^2 + b^2c^2 + b^2a^2 + b^4 + c^4 +$$
$$+ c^2d^2 + c^2a^2 + c^2b^2 + b^2d^2 + d^2c^2 + d^2c^2 + d^4.$$

2.22. **Definition.** Let $\mathbb{V}$ be a vector space of dimension $4n$. A 3–dimension subspace $Q \subset \mathrm{Aut}(\mathbb{V})$ generated by some hypercomplex structure $Q(I, J) = Q$ is called a *quaternionic structure.*

The difference between hypercomplex and quaternionic structure is described as follows. Let a pair (I, J) be a hypercomplex structure. The subset $Q(I, J)$ (see (5)) is the quaternionic structure, but a quaternionic structure Q is not hypercomplex, because there is not the unique basis of Q. This is similar to the relation between a Rieamannian and a conformal structure.

2.23. **Lemma.** *Let Q be a quaternionic structure. Let $I, J \in Q \subset \mathrm{Aut}(\mathbb{V})$ be a hypercomplex structure and let $H \simeq GL(n, \mathbb{H})$ be the group preserving I and J. Then H preserves all hypercomplex structures in Q.*

Proof. The group $GL(n, \mathbb{H})$ is group preserving a hypercomplex structure I, J such that $Q = Q(I, J)$. Consider a morphism $A \in GL(n, \mathbb{H})$ and assume that $\tilde{I} = a_0 + a_1I + a_2J + a_3K$. A simple computation shows that $A\tilde{I} = A(a_0 + a_1I + a_2J + a_3K) = a_0A + a_1AI + a_2AJ + a_3AK = a_0A + a_1IA + a_2JA + a_3KA = (a_0 + a_1I + a_2J + a_3K)A = \tilde{I}A$ and morphism preserving I, J preserves $\tilde{I}, \tilde{J}$. $\square$

The class of morphisms of a quaternionic structure is bigger than $GL(n, \mathbb{H})$ because Q is invariant with respect to the right multiplication by unit quaternions.

2.24. **Definition.** A bilinear form $F \in \mathrm{Bil}(\mathbb{V})$ is called *Hermitian bilinear form* with respect to a hypercomplex structure (I, J) or with respect to a quaternionic structure Q if

$$F(IX, IY) = F(X, Y) \tag{10}$$

$$F(JX, JY) = F(X, Y) \tag{11}$$

or

$$F(AX, AY) = F(X, Y), \forall A \in Q, \tag{12}$$

respectively.

The formula (12) is only reformulation of (10) and (11) for a structure without fixed (I, J).

2.25. **Definition.** Let $\mathbb{V}$ be a vector space equipped with a quaternionic structure Q (or hypercomplex structure (I, J)) and let g be an Euclidean metric on $\mathbb{V}$. The triple $(\mathbb{V}, g, Q)$ (or $(\mathbb{V}, g, (I, J))$) is called *Hermitian structure* if and only if the bilinear form g is Hermitian with respect to Q (or with respect to (I, J)).

2.26. **Definition.** Let $\mathbb{V}$ be a real vector space of dimension $4n$, Q be a quaternionic structure, (I, J) be a hypercomplex structure, g be a bilinear Hermitian form with respect Q or (I, J) and (vol) be a volume form invariant with respect to Q or (I, J).

- The pair (Q, vol), where (vol) is a volume form is called *unimodular quaternionic structure*.
- The pair $((I, J), \mathrm{vol}))$ is called *unimodular hypercomplex structure*.
- The pair (Q, g) is called a *quaternionic Hermitian structure*.
- The pair $((I, J), g)$ is called a *hypercomplex Hermitian structure*.

2.27. **Quaternionic–like structures.** All structures considered above will be called *quaternionic–like structures*, again for more details see the paper [AM96].

If S is a quaternionic–like structure we denote by $\mathrm{Aut}(S)$ the group of all automorphisms of $\mathbb{V}$ which preserve S, where S is a quaternionic–like structure. The algebra $\mathfrak{aut}(S)$ is Lie algebra of $\mathrm{Aut}(S)$.

We have the following isomorphisms of quaternionic–like Lie groups (for more details see [AM96]):

$$
\begin{aligned}
\mathrm{Aut}(Q) &= Sp(1)GL(n,\mathbb{H})\\
\mathrm{Aut}(Q,\mathrm{vol}) &= Sp(1)SL(n,\mathbb{H})\\
\mathrm{Aut}(Q,g) &= Sp(1)Sp(n)\\
\mathrm{Aut}((I,J,K)) &= GL(n,\mathbb{H})\\
\mathrm{Aut}((I,J,K),\mathrm{vol}) &= SL(n,\mathbb{H})\\
\mathrm{Aut}((I,J,K),g) &= Sp(n)
\end{aligned}
$$

Collecting the information we obtain the following diagram which describes the relations between the group $Sp(1)GL(n,\mathbb{H})$ and its subgroups.

$$
\begin{tikzcd}
 & Sp(1)GL(n,\mathbb{H}) \arrow[dl] \arrow[dr] & \\
Sp(1)SL(n,\mathbb{H}) \arrow[d] \arrow[drr] & & GL(n,\mathbb{H}) \arrow[d] \\
Sp(1)Sp(n) \arrow[dr] & & SL(n,\mathbb{H}) \arrow[dl] \\
 & Sp(n) &
\end{tikzcd}
$$

2.28. **Lemma.** *The first prolongation of Lie algebras of quaternionic–like structures is zero except of the quaternionic structure. The first prolongation of a quaternionic structure is* $\mathbb{V}^\star$.

Proof. (1) The Lie algebra of the Lie group $GL(n,\mathbb{H})Sp(1)$ is $\mathfrak{gl}(n,\mathbb{H})\oplus\mathfrak{sp}(1)$ and the first prolongation of $\mathfrak{gl}(n,\mathbb{H})\oplus\mathfrak{sp}(1)$ is $\mathfrak{g}^{(1)}\cong\mathbb{V}^\star$. The identification $\mathbb{V}^\star\ni\xi\mapsto S^\xi\in\mathfrak{g}^{(1)}$ is described as

$$S^\xi = 2\mathrm{Sym}\left[\xi\otimes 1-(\xi\circ I)\otimes I-(\xi\circ J)\otimes J-(\xi\circ K)\otimes K\right], \tag{13}$$

where Sym is the operator of symmetrization. Let us note that the contraction defines an isomorphism $Tr:\mathfrak{g}^{(1)}\to\mathbb{V}^\star$,

$$\mathrm{Tr}(S^\xi)=4(n+1)\xi.$$

For complete description and for more details see [AM96].

(2) The Lie algebra of the hypercomplex structure group $GL(n,\mathbb{H})$ is $\mathfrak{gl}(n,\mathbb{H})=\{A\in\mathfrak{gl}(n,\mathbb{R})|AI=IA,AJ=JA\}$ and straightforward computation $IJt(X,Y)=It(JX,Y)=t(JX,IY)=Jt(X,IY)=JIt(X,Y)=-IJt(X,Y)$ shows that the first prolongation of this Lie algebra is zero.

(3) The vanishing condition for trace defines the reduction of the Lie algebra of quaternionic structure to the lie algebra of unimodular quaternionic structure. The first prolongation of $\mathfrak{sl}(n,\mathbb{H})\oplus$

$\mathfrak{sp}(1)$ is zero because there is no $S^\xi \in \mathfrak{g}^{(1)}$, such that the trace is zero because the property $\mathrm{Tr}(S^\xi) = 4(n+1)\xi = 0$ implies $\xi = 0$.

(4) For any subset of $\mathfrak{h} \subset \mathfrak{g}$ it holds that $\mathfrak{h}^{(1)} \subset \mathfrak{g}^{(1)}$.

□

2.29. **Quaternionic–like structures on manifolds.** Now, we introduce the G–structures corresponding to the structures groups of the structures in 2.27.

Choosing the necessary data in the individual tangent spaces of a manifold M leads to reductions of the structures groups, as described above. In particular, choosing a three–dimensional smooth subbundle $Q \subset TM \otimes T^*M$ yields an almost quaternionic manifold with the structure group $GL(n, \mathbb{H})Sp(1)$.

2.30. **Examples.** *1. A quaternionic Hermitian structure.* The group $G := Sp(n+1)$ acts transitively on the $\mathbb{H}^{n+1}$ and preserves quaternionic lines (the group G acts transitively on $\mathbb{PH}^n$). Let e be the first vector of standard basis:

$$e = \begin{bmatrix} 1 \\ 0_n \end{bmatrix} \in \mathbb{PH}^n.$$

Let us compute the stabilizer of e. Consider a matrix satisfying:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} 1 \\ 0_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0_n \end{bmatrix},$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}^n$, $c \in (\mathbb{R}^n)^\star$, $d \in Mat_n(\mathbb{R})$. From this identity it follows $c = 0$ and the matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ acts on the projective space. The matrix above is element of $Sp(n+1)$, i.e.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a^\intercal & 0 \\ b^\intercal & d^\intercal \end{pmatrix} = \begin{pmatrix} aa^\intercal + bb^\intercal & bd^\intercal \\ db^\intercal & dd^\intercal \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and this computation show that $d \in Sp(n)$, $a \in Sp(1)$, $b = 0$. The stabilizer of a ray represented by e is

$$H := G_e = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| d \in Sp(n), a \in Sp(1) \right\}$$

and the homogeneous model is

$$\begin{array}{c} Sp(n+1) \\ \big\downarrow{\scriptstyle \pi} \\ \mathbb{PH}^n \cong Sp(n+1)/(Sp(n)Sp(1)) \end{array}$$

2. An unimodular hypercomplex structure. The affine group

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \,\middle|\, c \in \mathbb{H}^n,\ d \in Sp(n) \right\}$$

acts transitively on $\mathbb{H}^n$ and stabilizer of

$$e = \begin{pmatrix} 1 \\ 0_n \end{pmatrix} \in \mathbb{H}^n$$

consists of matrices

$$H := G_e = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \,\middle|\, d \in Sp(n) \right\} \cong Sp(n)$$

and the homogeneous model is

$$\begin{array}{c} G \\ \big\downarrow{\scriptstyle \pi} \\ \mathbb{H}^n \cong G/Sp(n) \end{array}$$

We have seen that there is a homogeneous model for hypercomplex Hermitian structure where $\mathbb{H}^n$ is the base. Similarly, we may construct homogeneous models of all other quaternionic–like structures, except the case of almost quaternionic manifolds which behaves completely different. One of the reasons is that the latter geometry is of second order whereas all the remaining one are first order geometries. The homogeneous model for the almost quaternionic geometries is the quaternionic projective space $\mathbb{PH}^n = SL(n+1, \mathbb{H})/P$, see the beginning of the next chapter.

3. The quaternionic Iwasawa manifold. The numbers $a, b, c \in \mathbb{H}$ with coefficients in $\mathbb{Z}$, $\mathcal{H} = \mathbb{Z} \oplus I\mathbb{Z} \oplus J\mathbb{Z} \oplus K\mathbb{Z} \subset \mathbb{H}$ are called *Hamiltonian integers. The quaternionic Iwasawa manifold* is the homogeneous space $\mathbb{H}^{1,2}/\Pi$, where

$$\mathbb{H}^{1,2} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{H} \right\}$$

$$\Pi = \left\{ \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, d, e, f \in \mathcal{H} \right\}.$$

Because open neighborhood of any point of Iwasawa manifold is diffeomorphic to $\mathbb{H}^3$ there are any quaternionic–like structures on Iwasawa manifold.

2.31**. Remark.** For all quaternionic–like structures, there are distinguished G–invariant decompositions $\mathbb{V} \otimes \wedge^2 \mathbb{V}^\star = \partial(\mathfrak{g} \otimes \mathbb{V}^\star) \oplus \mathcal{D}$ with respect to their structure groups G. This was proved in [AM96]. Quaternionic–like structures are all of first order, except of the almost quaternionic one, i.e. there is a unique $\mathcal{D}$–connection for all almost quaternionic–like structures except of an almost quaternionic one (see 2.15). We will get nice description of these connections in theorem 2.38.

2.32. **The Obata and Oproiu connection.** It is a very well known fact that on a Riemanian manifold there is the unique torsion free connection preserving the metric (for more details see [KMS], [Mich]). This connection is called the *Levi–Civita connection.* On an almost hypercomplex structure and on an almost quaternionic structure, there are special connections too.

There are results about affine connections on almost hypercomplex manifolds given by Morio Obata in the 50th of the twentieth century. The original results on almost hypercomplex structures are in [Obata56], [Obata57], and [Obata58] with slightly different terminology used. Obata was using the term an 'almost quaternionic geometry' for what we call almost hypercomplex and the term almost quaternionic geometry was reserved for general structures. In his papers, Obata introduced affine connection $\nabla^{(I,J)}$ with special torsion on almost hypercomplex geometry. This connection was later called the $\nabla^{(I,J)}$ connection (Obata connection).

For two given tensor field I and J of type $(1,1)$ on a differentiable manifold the expression

$$[\![I,J]\!](X,Y) = [IX,JY] - I[X,JY] - J[IX,Y] + [JX,IY]-$$

$$-J[X,IY] - I[JX,Y] + (IJ+JI)[X,Y]$$

defines a tensor field $[\![I,J]\!]$ of type $(1,2)$ and this tensor plays very important role in the discussion of the integrability of an almost complex structure (I,J). We call $[\![I,J]\!]$ the *Nijenhuis bracket* of I and J. If I and J satisfy $IJ+JI=0$, then the expression takes the form

$$[\![I,J]\!](X,Y) = [IX,JY] - I[X,JY] - J[IX,Y] + [JX,IY]-$$

$$-J[X,IY] - I[JX,Y].$$

This tensor field of type $(1,2)$ depends I, J and partial derivatives of I, J of the first order.

2.33. **Lemma** ([YA]). *Let (I,J) be a pair of almost complex structures on a smooth manifold M and $K=IJ$. All following Nijenhus brackets vanish if and only if any two of them vanish:*

$$[\![I,I]\!], [\![J,J]\!], [\![K,K]\!], [\![I,J]\!], [\![J,K]\!], [\![K,I]\!].$$

The Nijenhuis bracket of an almost complex structure I is important, because $[\![I,I]\!]=0$ is a necessary and sufficient condition for the integrability of I.

Next, we are going to discuss on the distinguished connections on quaternionic–like structures.

2.34. **Definition.** Let ∇ be an affine connection on a manifold M and let (I, J) be an almost hypercomplex structure on M. If a covariant derivative satisfies

$$\nabla I = 0, \tag{14}$$

$$\nabla J = 0, \tag{15}$$

and consequently $\nabla K = 0$, then the connection ∇ is called (I, J)–*connection.*

2.35. **Definition.** Let (I, J) be an almost hypercomplex structure. The tensor

$$T^{(I,J)} = \frac{1}{12}([\![I, I]\!] + [\![J, J]\!] + [\![IJ, IJ]\!]) \tag{16}$$

is called the *structure tensor* of an almost hypercomplex structure.

2.36. **Theorem** ([Obata56]). *For an almost hypercomplex structure (I, J) there is a unique linear (I, J)–connection $\nabla^{(I,J)}$ whose torsion tensor equals $T^{(I,J)}$.*

The connection $\nabla^{(I,J)}$ is called *Obata connection.*

2.37. **Theorem** ([YA]). *On a differentiable manifold with an almost hypercomplex structure (I, J) there is the unique affine connection ∇ with torsion $T^{(I,J)}$ such that*

$$\nabla_X Y = \frac{1}{2} IJ([IX, JY] - I[X, JY] - J[IX, Y]) + \frac{1}{2}[X, Y]. \tag{17}$$

All invariants of the Obata connection $\nabla^{(I,J)}$ are invariants of the almost hypercomplex structure (I, J).

2.38. **Remark.** The $\mathcal{D}$–connections ∇^S of quaternionic–like structure

$$S = (H, \mathrm{vol}), (Q, \mathrm{vol}), (H, g), (Q, q)$$

are uniquely given by

(1) $\nabla_X^{((I,J,K),\mathrm{vol})} = \nabla_X^{(I,J,K)} + (\frac{1}{4})\omega(X)Id$, where $\nabla_X^{(I,J,K)}\mathrm{vol} = \omega(X)\mathrm{vol}$.
(2) $\nabla_X^{(Q,\mathrm{vol})} = \nabla_X^{(I,J,K)} + \sum_{L\in\{I,J,K\}} \tau_L^{(I,J,K)}(X)L + [\frac{1}{4}(n+1)](S^{\omega(I,J,K)})_X$, where $H = (I, J, IJ)$ is a admissible basis of Q and the local 1–forms τ_L^H are defined by (19).
(3) $\nabla^{((I,J,K),g)} = \nabla^{(I,J,K)} + A$, where $A = (\frac{1}{2})g^{-1}\nabla_X^{(I,J,K)} g$.
(4) $\nabla_X^{(Q,g)} = \nabla_X^{((I,J,K),g)} + \sum_{L\in\{I,J,K\}} \tau_L^{(I,J,K)}(X)L + [\frac{1}{8}(n+1)][S_X^\omega - S_{g^{-1}\omega}^{g\circ X}]$.

2.39. **Definition.** Let Q be an almost quaternionic geometry on a manifold M. The tensor

$$T^Q = T^{(I,J)} + \partial(\tau_I^{(I,J)} \otimes I) + \partial(\tau_J^{(I,J)} \otimes J) + \partial(\tau_{(IJ)}^{(I,J)} \otimes IJ) \tag{18}$$

is called a *structure tensor* of an almost quaternionic structure Q where ∂ is the operator of alternation (2.9), (I, J, IJ) is a local basis of Q and

$$\tau_Z^{(I,J)} = \frac{1}{4n-2} tr(ZT^{(I,J)}) \tag{19}$$

is a *structure 1–form on* (I, J) .

The tensor T^Q does not depend on the base of Q.

2.40**. Definition.** Let M be real smooth manifold of dimension $4n$ and let $Q \subset \mathrm{End}(TM \otimes T^\star M)$ be an almost quaternionic structure. A linear connection ∇ in M is called a *Q–connection* if for each section ψ of Q, the covariant derivative $\nabla_X \psi$ is also a section of Q, where X is an arbitrary vector field on M.

On a differentiable manifold with an almost quaternionic structure Q there always exists a Q–connection ∇ with torsion T^Q.

Now we will show that using $\mathcal{D}$–connections one gets a nice description of connection on almost quaternionic–like geometries. We still follow the paper [AM96].

For an almost quaternionic structure there is uniquely defined complementary $GL(n, \mathbb{H})Sp(1)$–module $\mathcal{D}$, such that

$$V \otimes \wedge^2 V^\star = \partial(\mathfrak{gl}(n, \mathbb{H}) + \mathfrak{sp}) \oplus \mathcal{D}. \tag{20}$$

Note that $T^Q \in \mathcal{D}$. Hence the concept of $\mathcal{D}$–connections is well defined, but since

$$(\mathfrak{gl}(n, \mathbb{H}) + \mathfrak{sp}_1)^{(1)} = V^\star,$$

the $\mathcal{D}$–connection is not unique (compare with [AM96]).

Generally, any two $\mathcal{D}$–connections $\nabla, \bar{\nabla}$ are related by symmetric tensor S^ξ such that $\bar{\nabla} = \nabla + S^\xi$, where $\xi \in T_x^\star M$ and $S^\xi \in \mathfrak{g}^{(1)}$. The symmetric tensor S^ξ is defined in (13).

2.41**. Definition.** *Let M be real smooth manifold of dimension $4n$, let Q be and almost quaternionic structure and let ∇ be any linear connection on M. A curve C on (M, Q, ∇) is called Q–*planar *if there is trajectory $c = c(t)$, such that the property $\nabla_{\dot{c}} \dot{c} \in \langle \dot{c}, I(\dot{c}), J(\dot{c}), K(\dot{c}) \rangle$ holds.*

2.42. **Remarks.** *1.* We should like also to remark at this point that the term Q–planar curves makes sense for any quaternionic–like geometry, because the property of Q–planarity does not depend of basis (I, J, K) of Q.

2. In the chapter three we will see that morphisms preserving Q–planar curves are exactly morphisms of the almost quaternionic geometries.

3. On each almost quaternionic–like structures except an almost quaternionic structure, there is the unique $\mathcal{D}$–connection. At the same time, each such structure induces the almost quaternionic structure by the obvious extension of the structure group. We will see later that any morphism preserving Q–planar curves of the unique $\mathcal{D}$–connection of any quaternionic–like structure is a morphism of the induced almost quaternionic structure.

4. The point of view we taken in this chapter was that the $\mathcal{D}$–connections are useful for a description of the geometry of an almost quaternionic structure. The important fact that on an almost quaternionic structure there is the G–invariant decomposition $\mathbb{V} \otimes \wedge^2\mathbb{V} = \partial(\mathfrak{g} \otimes \mathbb{V}^\star) \oplus \mathcal{D}$ is not obvious. A straightforward computation is given in [AM96]. We shall use a different conceptual approach, namely the general theory of parabolic geometries and the so called Weyl connections, see below.

3. Almost quaternionic geometries

Throughout this chapter, we describe an almost quaternionic geometry as a parabolic geometry and we prove all facts about Q–planar curves by technique of Weyl connections. The parabolic geometry is a Cartan geometry of type (G, P), where $P \subset G$ is a parabolic subgroup. Cartan generalized spaces are curved analogs of the homogenous spaces G/H defined by means of an absolute parallelism on a principal H–bundle. We follow the book [ČS].

3.1. **Definition.** Let $H \subset G$ be a Lie subgroup in a Lie group G, and $\mathfrak{g}$ be the Lie algebra of G. A *Cartan geometry* of type (G, H) on a manifold M is a principal fiber bundle $p : \mathcal{G} \to M$ with structure group H which is endowed with a $\mathfrak{g}$–valued one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the *Cartan connection* such that:

(1) ω is H–equivariant, i.e. $(r^h)^*\omega = \mathrm{Ad}_{h^{-1}} \circ \omega$ for all $h \in H$.
(2) ω reproduces the fundamental vector fields, i.e. $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}$.
(3) ω is an absolute parallelism, i.e. $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$.

The *homogenous model* for Cartan geometries of type (G, H) is the canonical bundle $p : G \to G/H$ endowed with the left Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$. The Cartan geometry is called split if and only if there is a Lie subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ which is complementary to $\mathfrak{h}$ as a vector space, i.e. such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$ as a vector space. The Cartan geometry is called reductive if and only if there is a H–invariant subspace $\mathfrak{n} \subset \mathfrak{g}$ which is complementary to $\mathfrak{h}$, i.e. such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as a H–module. On a reductive homogenous space, the Maurer–Cartan form is a sum of a principal connection form and soldering form. In particular, the linear connections on manifolds appear in this setting as the curved version of the affine space.

Given a Cartan geometry $(\mathcal{G} \to M, \omega)$ there are the *constant vector fields* $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ defined for all $X \in \mathfrak{g}$ by $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in G$. By equivariancy of ω we get

$$\omega^{-1}(X)(u \cdot g) = Tr^g \cdot \omega^{-1}(Ad(g) \cdot X)(u).$$

In the case of the homogenous model, the constant vector field $\omega^{-1}(X)$ is the left invariant field L_X by definition of the Maurer–Cartan form.

3.2. **Example.** *Affine almost hypercomplex n–dimensional space $A^n_{\mathbb{H}}$.* Let $A(n, \mathbb{H})$ be the group of affine motions

$$x \mapsto Ax + b \text{ for } A \in Gl(n, \mathbb{H}), b \in \mathbb{H}^n.$$

Viewing $A^n_{\mathbb{H}}$ as the affine hyperplane $x_1 = 1$ in $\mathbb{H}^{n+1}$ the affine motions are exactly the subgroup of $GL(n+1, \mathbb{H})$ which map this affine hyperplane to itself, i.e.

$$A(n, \mathbb{H}) = \left\{ \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix} \Big| A \in GL(n, \mathbb{H}), b \in \mathbb{H}^n \right\} \subset GL(n+1, \mathbb{H}).$$

On the Lie algebra level, we get

$$\mathfrak{a}(n, \mathbb{H}) = \left\{ \begin{pmatrix} 0 & 0 \\ X & B \end{pmatrix} \Big| B \in \mathfrak{gl}(n, \mathbb{H}), X \in \mathbb{H}^n \right\}.$$

Natural projection

$$\begin{tikzcd} A(n, \mathbb{H}) \arrow[d, "\pi"] \\ \mathbb{H}^n \cong A(n, \mathbb{H})/GL(n, \mathbb{H}) \end{tikzcd}$$

is a principal bundle with structure group $GL(n, \mathbb{H})$ and with induced Maurer–Cartan form $\omega \in \Omega^1(A(n, \mathbb{H}), \mathfrak{a}(n, \mathbb{H}))$. Now we may split $\omega = \theta + \gamma$ according to the splitting $\mathfrak{a}(n, \mathbb{H}) = \mathbb{H}^n \oplus \mathfrak{gl}(n, \mathbb{H})$ and since this splitting is $GL(n, \mathbb{H})$–equivariant, both θ and γ are $GL(n, \mathbb{H})$–equivariant.

Now we can view affine hypercomplex space as smooth manifold equipped with a Cartan connection.

Further, we want to understand the almost quaternionic structure as a homogeneous space. The almost quaternionic structure is a G–structure with the structure group

$$G_0 := GL(n, \mathbb{H})Sp(1) := GL(n, \mathbb{H}) \times_{\mathbb{Z}_2} Sp(1),$$

see e.g. [S].

The group $G = SL(n+1, \mathbb{H})$ acts transitively on the $\mathbb{H}^{n+1}$ and of course, this actions descends to the action on the points in quaternionic projective space. Hence the group $G = SL(n+1, \mathbb{H})$ acts transitively on $\mathbb{PH}^n$. The stabilizer of

$$e = \begin{bmatrix} 1 \\ 0_n \end{bmatrix} \in \mathbb{PH}^n$$

is a $4(n+1) \times 4(n+1)$ matrix which preserves e, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} 1 \\ 0_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0_n \end{bmatrix}.$$

From this identity it follows $c = 0$ and the matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ acts on the projective space. The stabilizer of a ray represented by e is

$$H := G_e = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in GL(n,\mathbb{H})Sp(1), b \in (\mathbb{H}^n)^\star \right\}$$

and the homogeneous model is

$$\begin{array}{c} SL(n+1,\mathbb{H}) \\ \Big\downarrow \pi \\ \mathbb{PH}^n \cong SL(n+1,\mathbb{H})/(GL(n,\mathbb{H})Sp(1) \rtimes ((\mathbb{H}^n)^\star)) \end{array}$$

Of course, G_0 is exactly the subgroup fixing the origin and mapping infinite points to infinite points.

The Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} a - \mathcal{R}e(\mathrm{tr}(A)) & Z \\ X & A \end{pmatrix} \middle| A \in \mathfrak{gl}(n,\mathbb{H}), X, Z^T \in \mathbb{H}^n, a \in \mathcal{I}m(\mathbb{H}) \right\}$$

is naturally split into the sum $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \middle| X \in \mathbb{H}^n \right\},$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} a - \mathcal{R}e(\mathrm{tr}(A)) & Z \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{gl}(n,\mathbb{H}), Z \in (\mathbb{H}^n)^\star, a \in \mathcal{I}m(\mathbb{H}) \right\},$$

$$\mathfrak{g} \cong \mathfrak{sp}(1) + \mathfrak{gl}(n,\mathbb{H}).$$

The commutative subalgebra $\mathfrak{n}$ is not H–invariant and there is no H–invariant complementary subalgebra to $\mathfrak{h}$. Hence an almost quaternionic geometry is naturally split but not reductive.

3.3. **Parabolic geometries.** Parabolic geometries are defined as Cartan geometries of type (G,P) for semisimple Lie group G and parabolic subgroup P. The general idea of Cartan geometries is to model the individual tangent spaces by the Lie algebra $\mathfrak{g}/\mathfrak{p}$, i.e. the tangent space to the homogeneous model in the origin inclusive its algebraic structure. In the special case of the parabolic geometries, this amounts to special understanding of the corresponding $|k|$–gradings of semisimple Lie algebras. This in turn shows up as filtrations of the tangent bundles, as we shall see below.

3.4. **Definition.** *Filtered vector space* is a vector space $\mathbb{V}$ together with a sequence $\{\mathbb{V}^i \mid i \in \mathbb{Z}\}$ of subspaces $\mathbb{V}^{i+1} \subset \mathbb{V}^i \subset \mathbb{V}$ for $i \in \mathbb{Z}$, $\cup_{i\in\mathbb{Z}}\mathbb{V}^i = \mathbb{V}$, and $\cap_{i\in\mathbb{Z}}\mathbb{V}^i = \{0\}$. A filtration is called finite if $0 = \mathbb{V}^{k+1} \subset \mathbb{V}^k \subset \mathbb{V}^{k-1} \subset \cdots \subset \mathbb{V}^{j+1} \subset \mathbb{V}^j = \mathbb{V}$ and all other V^i are trivial (i.e the whole space $\mathbb{V}$ or the zero subspace). All nontrivial subspaces are assumed to be different.

From a filtration $\{\mathbb{V}^i \mid i \in \mathbb{Z}\}$ of a vector space $\mathbb{V}$, we construct graded vector space $gr(\mathbb{V}) = \oplus_{i\in\mathbb{Z}} gr_i(\mathbb{V})$ by putting $gr_i(\mathbb{V}) := \mathbb{V}^i/\mathbb{V}^{i+1}$ for all $i \in \mathbb{Z}$ which is called the *associated graded vector space* of a filtered vector space $\mathbb{V}$. In the case of finite filtration we obtain a finite grading

$$gr(\mathbb{V}) := gr_j(\mathbb{V}) \oplus \cdots \oplus gr_k(\mathbb{V}).$$

In general, there is no natural extension of the canonical projections $\mathbb{V}^i \to gr_i(\mathbb{V}) = \mathbb{V}^i/\mathbb{V}^{i+1}$ to a linear isomorphism on $\mathbb{V}$. Of course one may construct such a linear isomorphism $\mathbb{V} \to gr(\mathbb{V})$ by making choices. Choosing some subspace $\mathbb{V}_i \subset \mathbb{V}^i$ complementary to V^{i+1}, for each $i \in \mathbb{Z}$ the restriction of the canonical projection induces the linear isomorphism $\mathbb{V}_i \to gr_i(\mathbb{V})$.

3.5**. Definition.** A *filtered vector bundle* over a smooth manifold M is a smooth vector bundle $p : E \to M$, together with a sequence $\{E^i | i \in \mathbb{Z}\}$ of smooth subbundles such that there is $i_0 < j_0 \in \mathbb{Z}$ satisfying $E^i = E$, for $i \leq i_0$ and $E^i = M$ (i.e. the zero subbundle in E) for $i > j_0$.

Given such a filtration of a bundle, we get the quotient bundles $gr_i(E) := E^i/E^{i+1}$ and *associated graded vector bundle*

$$gr(E) = \bigoplus_{i\in\mathbb{Z}} gr_i(E).$$

3.6**. Definition.** A *filtered Lie algebra* is a Lie algebra $(\mathfrak{g}, [,])$ together with filtration $\{\mathfrak{g}^i, i \in \mathbb{Z}\}$ on the vector space $\mathfrak{g}$ such that for all $i, j \in \mathbb{Z}$ we have $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. *The associated graded Lie algebra* $gr(\mathfrak{g})$ is $gr(\mathfrak{g}) = \bigoplus_{i\in\mathbb{Z}} gr_i(\mathfrak{g})$, where $gr_i(\mathfrak{g}) := \mathfrak{g}^i/\mathfrak{g}^{i+1}$.

3.7**. Definition.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and let* $k > 0$ *be an integer.* A $|k|$–grading on $\mathfrak{g}$ *is a decomposition* $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ *of* $\mathfrak{g}$ *into a direct sum of subspaces, which defines a grading on* $\mathfrak{g}$, *i.e.* $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, *where we agree that* $\mathfrak{g}_i = \{0\}$ *for* $|i| > k$, *such that the subalgebra* $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ *is generated by* fg_{-1}

By definition, if $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ *is a* $|k|$*–grading, then* $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ *is a subalgebra of* $\mathfrak{g}$, *and* $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{\mathfrak{k}}$ *is a nilpotent ideal in* $\mathfrak{p}$.

3.8. **The group level.** Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$–graded semisimple Lie algebra, and let G be a Lie group with Lie algebra $\mathfrak{g}$. We want to associate to the $|k|$–grading subgroups $G_0 \subset P \subset G$ corresponding to the Lie algebras $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$.

We define subgroups $G_0 \subset P \subset G$ by

$$G_0 := \{g \in G | \operatorname{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \; \forall i = -k \ldots k\}$$
$$P := \{g \in G | \operatorname{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \; \forall i = -k \ldots k\}$$

i.e. G_0 (respectively P) consists of all elements of G whose adjoint action preserves that grading (respectively the filtration) of $\mathfrak{g}$.

Note that the choice of the structure group P is not canonical. There is a possibility that the group preserving filtration is not connected. The choice of P is in between this group and its connected component containing the identity. The choice has not got effect on the local properties but the choice has got a big effect on the global properties. In our case, we chose the biggest one.

3.9. **Definition.** Let $\mathfrak{g}$ be a $|k|$–graded semisimple Lie algebra. A *parabolic geometry* is a Cartan geometry of type (G, P), where G is a semisimple Lie group and $P \subset G$ is the subgroup of all elements of G whose adjoint action preservers the filtration associated to a $|k|$–grading of the Lie algebra $\mathfrak{g}$ of G.

Note that a parabolic geometry is canonically a split Cartan geometry, since we always have the subalgebras $\mathfrak{g}_-$, which is complementary to the subalgebra $\mathfrak{p} \subset \mathfrak{g}$. This complement is however very far from being $\mathfrak{p}$–invariant

Let us discuss the special choice of G and P relevant to the quaternionic like geometries. Thus, consider the algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$:

$$\left\{ \begin{pmatrix} a & Z \\ X & A \end{pmatrix} \Big| X, Z^\top \in \mathbb{H}^n, a \in \mathbb{H}, A \in Mat_n(\mathbb{H}), Re(a) + Re(tr(A)) = 0 \right\}$$

The corresponding gradation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{21}$$

looks like:

$$\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{g}_0, \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_1.$$

The group $SL(n+1, \mathbb{H})$ consists of all invertible quaternionic linear endomorphisms of $\mathbb{H}^{n+1}$, which we can consider as endomorphisms of $\mathbb{R}^{4n+4}$ with determinant equal one. We define $G := PSL(n+1, \mathbb{H})$ as the quotient of $SL(n+1, \mathbb{H})$ by its center $\{\pm \mathrm{id}\}$. As a parabolic subgroup we obtain quotient of the stabilizer of the quaternionic line generated by the first vector from the standard basis, i.e.

$$P := \left\{ \begin{pmatrix} \gamma & \rho \\ 0 & \psi \end{pmatrix} \Big| \gamma \in \mathbb{H}, \rho \in \mathbb{H}^n, \psi \in GL(n, \mathbb{H}), |\gamma|^4 \cdot \det(\psi) = 1 \right\} / \{\pm \mathrm{id}\}$$

and

$$G_0 := \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & \psi \end{pmatrix} \Big| \gamma \in \mathbb{H}, \psi \in GL(n, \mathbb{H}), |\gamma|^4 \cdot \det(\psi) = 1 \right\} / \{\pm \mathrm{id}\}$$

3.10. **The general theory.** Let us note that the structure theory of graded semisimple Lie algebras leads to a classification which is nicely formulated in terms of the so called Satake diagrams with crosses. We shall not go into any details, but let us remark that the almost quaternionic geometry is a parabolic geometry of type (G, P), where the Lie algebra of P is the parabolic subalgebra in the real form $\mathfrak{sl}(n+1, \mathbb{H})$ of the complex algebra $\mathfrak{gl}(2n+2, \mathbb{C})$ corresponding to the Satake diagram with cross over the second node:

$$\bullet\!-\!\!-\times-\!\!-\bullet-\!\!-\cdots-\!\!-\bullet-\!\!-\circ-\!\!-\bullet$$

Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$ be the Lie algebra of quaternionic $(n+1)\times(n+1)$ matrices with zero real part of the trace. The parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ obtained by crossing out the second simple root is the stabilizer of the quaternionic line generated by the first vector from the standard basis (for detailed exposition see [Y]).

Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P). The filtration of $\mathfrak{g}$ by the $\mathfrak{p}$–submodules $\mathfrak{g}^j$ is transferred to the right invariant filtration $T^j\mathcal{G}$ on the tangent space $T\mathcal{G}$ by the parallelism ω. The filtration $TM = T^{-k}M \supset T^{-k+1} \supset \cdots \supset T^{-1}M$ of the tangent space of the manifold M is defined from this filtration by the projection $T\mathcal{G} \to TM$. Let us note that structure group of the associated graded tangent space $gr(TM)$ is the group G_0. Hence $\mathcal{G}_0 := \mathcal{G}/P_+ \to M$ is a principal bundle with structure group G_0. We call the filtration of TM with the fixed reduction of $gr(TM)$ to G_0 the *infinitesimal flag structure*.

If $\mathfrak{g}$ is a $|1|$–graded Lie algebra than our parabolic geometry is called *irreducible parabolic geometry* and several examples of such structures have been studied intensively. The classification of all such simple real Lie algebras is well known. There are several obvious examples of irreducible parabolic geometries, in particular almost Grasmanian (a projective geometry is a special case of this), almost quaternionic, (pseudo) conformal and Lagrangian geometries. In the case of irreducible parabolic geometries, the situation becomes very simple, since the filtration degenerates to $TM = T^{-1}M$. Hence from above we conclude that infinitesimal flag structures of type (G, P) are simply reductions of structure group of TM to the group G_0. Consider the $|1|$–grading on the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$, thus $\mathfrak{g}_{-1} \cong \mathbb{H}^n$, $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{H}) + \mathfrak{sp}(1, \mathbb{H})$, and $\mathfrak{g}_1 \cong \mathbb{H}^{n\star}$ and the standard action of $(A, q) \in \mathfrak{g}_0$ on $X \in \mathfrak{g}_{-1}$ is AXq^{-1}. Consequently, an infinitesimal flag structures of type (G, P) in this case is exactly a first order G–structure with structure group $GL(n, \mathbb{H})Sp(1)$.

3.11. **The adjoint tractor bundle.** Let $\lambda : G \to GL(\mathbb{W})$ be a linear representation on a vector space $\mathbb{W}$ and $(\mathcal{G} \to M, \omega)$ a Cartan connection of type (G, P). The corresponding natural vector bundle $\mathcal{G} \otimes_G \mathbb{W}$ is called *tractor bundle*.

The *adjoint tractor bundle* is the natural vector bundle $\mathcal{A}$ corresponding to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$, so we have $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$, where P acts on $\mathfrak{g}$ by the restriction of the adjoint action.

Because the gradation of $\mathfrak{g}$ is G_0–equivariant, we can compute the action of G_0 on $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, for irreducible geometries. For an almost quaternionic geometry, the adjoint actions G_0 on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are:

$$\mathrm{Ad}\begin{pmatrix}\gamma & 0\\ 0 & \psi\end{pmatrix}\begin{pmatrix}0 & 0\\ X & 0\end{pmatrix} = \begin{pmatrix}\gamma & 0\\ 0 & \psi\end{pmatrix}\begin{pmatrix}0 & 0\\ X & 0\end{pmatrix}\begin{pmatrix}\gamma^{-1} & 0\\ 0 & \psi^{-1}\end{pmatrix} = \begin{pmatrix}0 & 0\\ \psi X\gamma^{-1} & 0\end{pmatrix}$$

$$\mathrm{Ad}\begin{pmatrix}\gamma & 0\\ 0 & \psi\end{pmatrix}\begin{pmatrix}0 & Z\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\gamma & 0\\ 0 & \psi\end{pmatrix}\begin{pmatrix}0 & Z\\ 0 & 0\end{pmatrix}\begin{pmatrix}\gamma^{-1} & 0\\ 0 & \psi^{-1}\end{pmatrix} = \begin{pmatrix}0 & \psi^{-1}Z\gamma\\ 0 & 0\end{pmatrix}$$

respectively, i.e. $\mathrm{Ad}(\gamma,\psi)(X) = \psi X\gamma^{-1}$ for $X \in \mathfrak{g}_{-1}$ and $\mathrm{Ad}(\gamma,\psi)(Z) = \psi^{-1}Z\gamma$ for $Z \in \mathfrak{g}_1$, where $\gamma \in \mathfrak{sp}(1)$ and $\psi \in \mathfrak{gl}(n,\mathbb{H})$.

The P–submodules $\mathfrak{g}^j \subset \mathfrak{g}$ give rise to the filtration

$$\mathcal{A} = \mathcal{A}^{-1} \supset \mathcal{A}^0 \supset \mathcal{A}^1,$$

where the natural subbundles are $\mathcal{A}^j = \mathcal{G} \times_P \mathfrak{g}^j$. Graded adjoint tractor bundle is

$$Gr(\mathcal{A}) = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1,$$

where $\mathcal{A}_j = \mathcal{A}^j/\mathcal{A}^{j+1}$. By the definition, there is the algebraic bracket on $\mathcal{A}$ defined by means of the graded Lie bracket in $\mathfrak{g}$, such that

$$\{\mathcal{A}_i, \mathcal{A}_j\} \mapsto \mathcal{A}_{i+j}.$$

Let us note that $TM = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$, $T^\star M = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$,

$$\mathcal{A}_{-1} = \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \cong TM, \quad \mathcal{G} \times \mathfrak{g}_- \ni (u,X) \mapsto Tp(\omega^{-1}(X)(u))$$

and we obtain on the level of vector bundles

$$\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 \subset TM \oplus (TM \otimes T^*M) \oplus T^\star M$$

where $\mathcal{A}_0 = \mathcal{G}_0 \times_H \mathfrak{g}_0$ is the adjoint bundle of the Lie algebra $\mathfrak{g}_0$.

The key feature of $\mathcal{A}$ is that all further G_0–invariant object on $\mathfrak{g}$ are carried over to the adjoint tractors, too. In particular, the Lie bracket on $\mathfrak{g}$ defines a bundle map $\{,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. By definition, this means that for $u \in \mathcal{G}$, X, $Y \in \mathfrak{g}$, and $[\![u,X]\!] \in \mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$ one has $\{[\![u,X]\!],[\![u,Y]\!]\} = [\![u,[X,Y]]\!]$ and this is well defined since for any $g \in G$ the map $\mathrm{Ad}(g)$ is a Lie algebra homomorphism. In particular, applying the this bracket pointwise, we obtain a Lie bracket on the space $\Gamma(\mathcal{A}M)$ of adjoint tractors on M.

3.12**. Definition.** The mapping φ between two principal fiber bundles $(\mathcal{G}_1,\omega_1)$ and $(\mathcal{G}_2,\omega_2)$ is a *morphism of parabolic geometry* if and only if φ is a morphism of principal bundles $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\varphi^*\omega_2 = \omega_1$.

3.13. **The Kostant codifferential.** The G–equivariant Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ induces an isomorphism $\mathfrak{g} \cong \mathfrak{g}^\star$ on the Lie algebra. We define n–chain space $C^n(\mathfrak{g}_-, \mathfrak{g}) := L(\wedge^n \mathfrak{g}_-, \mathfrak{g})$ and codifferential

$$\begin{aligned}\partial^\star(Z_0 \wedge \cdots \wedge Z_n \otimes A) = &\sum_{i=0}^{n} (-1)^{i+1} Z_0 \wedge \ldots \widehat{i} \cdots \wedge Z_n \otimes [Z_i, A] \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \ldots \widehat{i} \ldots \widehat{j} \cdots \wedge Z_n \otimes A.\end{aligned}$$

The codifferential computes the same cohomology as the standard Lie algebra differential $\partial : C^n \to C^{n+1}$ on the cochains. This partly follows from the next lemma and it serves as the main ingredient of the well known Kostant's computation of the Lie algebra cohomologies.

3.14. **Lemma** ([ČS], [Y]). *For any $n \geq 0$, the chain space $C^n(\mathfrak{g}_-, \mathfrak{g})$ naturally splits into a direct sum of G_0–submodules as*

$$C^n(\mathfrak{g}_-, \mathfrak{g}) = \mathrm{im}(\partial^\star) \oplus \ker(\Box) \oplus \mathrm{im}(\partial), \tag{22}$$

where the sum of the first two summands is $\ker(\partial^\star)$ *while the last two summands add up to* $\ker(\partial)$*, where* $\Box$ *is an* Kostant Laplacian $\Box := \partial\partial^\star + \partial^\star\partial$.

In the $|1|$–graded case, the Lie algebra differential coincides with the Spencer operator. The conclusion in this case is that there is an G_0–invariant complement $\mathcal{D}$ for any infinitesimal flag structures (compare with remark 2.31).

The *curvature form* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan geometry $(\mathcal{P} \to M, \omega)$ is defined by the structure equation

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

where $\xi, \eta \in T\mathcal{G}$.

The curvature function $\kappa : \mathcal{G} \to \wedge^2 \mathfrak{g}_-^\star \otimes \mathfrak{g} \cong C^2(\mathfrak{g}_-, \mathfrak{g})$ defined by

$$\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)),$$

may be decomposed according to the values of the target components κ_i in $\mathfrak{g}_i$. The whole $\mathfrak{g}_-$–component κ_- is called the *torsion* of the Cartan connection ω.

3.15. **Definition.** The irreducible parabolic geometry $(\mathcal{G}, \omega)$ with the curvature function κ is called *flat* if $\kappa = 0$, *torsion–free* if $\kappa_- = 0$, *normal parabolic geometry* if $\partial^\star \circ \kappa = 0$.

Since the almost quaternionic geometry is irreducible we shall continue to discuss this case only.

3.16. **Theorem** ([ČS], [Y]). *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $|1|$–graded semi–simple Lie algebra without summands isomorphic to* $\mathfrak{sl}(n+1, \mathbb{R}) =$

$\mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R}) \oplus \mathbb{R}^{n*}$ *(i.e. a graded subalgebra corresponding to the projective geometry). Let G be a Lie group with Lie algebra $\mathfrak{g}$, $G_0 \subset P \subset G$ the subgroups determined by the $|1|$–grading. There is the bijective correspondence between the isomorphism classes of normal parabolic geometries of type (G,P) and infinitesimal flag structures of type $\mathfrak{g}/\mathfrak{p}$ on M.*

3.17. **Weyl structures.** Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $|1|$–graded semisimple Lie algebra. G is a Lie group with Lie algebra $\mathfrak{g}$ and let $G_0 \subset P \subset G$ be the subgroups determined by the $|1|$–grading. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G,P), and consider underlying principal G_0–bundle $p_0 : \mathcal{G}_0 \to M$. By definition $\mathcal{G}_0 = \mathcal{G}/P_+$ and there is the natural projection $\pi : \mathcal{G} \to \mathcal{G}_0$, which is a principal bundle with structure group P_+.

3.18. **Definition.** Let $(p : \mathcal{G} \to M, \omega)$ be an irreducible parabolic geometry on a smooth manifold M and consider the underlying principal G_0–bundle $p_0 : \mathcal{G}_0 \to M$ and the canonical projection $\pi : \mathcal{G} \to \mathcal{G}_0$. A *Weyl structure* for $(\mathcal{G}, \omega)$ is a global G_0–equivariant smooth section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of $\pi : \mathcal{G} \to \mathcal{G}_0$.

3.19. **Theorem** ([ČS],[ČS-03]). *For any irreducible parabolic geometry $(p : \mathcal{G} \to M, \omega)$, there exists a Weyl structure. Moreover, if σ and $\bar{\sigma}$ are two Weyl–structures, then there is a unique smooth section Υ of $\mathcal{A}_1$ such that*

$$\bar{\sigma}(u) = \sigma(u) \exp(\Upsilon(u)).$$

Finally, each Weyl–structure σ and section Υ define another Weyl–structure $\bar{\sigma}$ by the above formula.

3.20. **Underlying structures.** Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure on an irreducible parabolic geometry $(\pi : \mathcal{G} \to M, \omega)$ of type (G,P). The pullback $\sigma^\star\omega : T\mathcal{G}_0 \to \mathfrak{g}$ is G_0–equivariant and the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as G_0–module. Hence we may decompose $\sigma^\star\omega$ as $\sigma^\star\omega_{-1} \oplus \sigma^\star\omega_0 \oplus \sigma^\star\omega_1$.

Let $\omega = (\omega_{-1} \oplus \omega_0 \oplus \omega_1) \in \Omega^1(\mathcal{G}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1)$ be the above mentioned decomposition of Cartan connection ω. The component $\sigma^\star\omega_{-1}$ is an element of $\Omega^1(\mathcal{G}_0, \mathbb{R}^n)$, which enjoys the properties of the *soldering form.* An important observation is that this soldering form is independent of the chosen Weyl structure σ. In fact, the bijective correspondence between the normal Cartan connections and the G_0–structures in the theorem above is provided by this soldering form.

By the definition of Cartan connection ω, the component $\sigma^\star\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ represents a principal connection on the bundle $\mathcal{G}_0 \to M$, which is called the *Weyl connection* associated to the Weyl structure σ. The component $\sigma^\star\omega_1$ clearly determines a one–form $P \in \Omega^1(M, T^\star M)$, which is called the *Rho tensor* associated to the Weyl structure σ. The latter tensor measures the difference between the horizontal vectors with respect to the Weyl connection and the original Cartan connection.

Now we may write down easily the transformation formula. Let $\bar\nabla$ and ∇ be two Weyl connections, and let Υ be the appropriate unique smooth section of $\mathcal{A}_1$, from theorem 3.19. For all ξ, $\eta \in \Gamma(TM)$, the connections transform as

$$\bar\nabla_\xi \eta = \nabla_\xi \eta + \{\{\xi, \Upsilon\}, \eta\}, \tag{23}$$

where the vector fields are understand as adjoint tractors in $\mathcal{A}_{-1}$ and the bracket is the natural Lie bracket, see 3.11. Further, the Rho–tensor transform as

$$\bar{\mathsf{P}}(\xi) = \mathsf{P}(\xi) + \nabla_\xi \Upsilon + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}.$$

Notice that the internal bracket of $\xi \in TM$ and $\Upsilon \in T^*M$ is in $\mathcal{A}_0$ (i.e. an endomorphism of TM) while the external bracket is exactly the evaluation of this endomorphism on η (all this is read off the brackets in the Lie algebra easily). For more details see [ČS-03], [GS], or [BE].

3.21**. Definition.** *A (unparameterized) curve $C \subset M$ is called $\mathbb{H}$–*planar *if it is Q–planar with respect to each Weyl connection ∇ on M (compare with definition 2.41).*

The next theorem explains the link between the two concepts:

3.22**. Theorem** ([H])**.** *A curve C is Q–planar with respect to at least one Weyl connection ∇ on M if and only if C is $\mathbb{H}$–planar.*

Proof. For a Weyl connection ∇ and a trajectory $c : \mathbb{R} \to M$, the defining equation for Q–planarity reads $\nabla_{\dot c}\dot c \in Q(\dot c)$. If we choose some hypercomplex structure within Q, we may rephrase this condition as: $\nabla_{\dot c}\dot c = \dot c \cdot q$ where $\dot c(t)$ is a trajectory in the tangent bundle TM while $q(t)$ is a suitable trajectory in quaternions $\mathbb{H}$. Now the formula (23) for the change of the Weyl connections implies

$$\hat\nabla_{\dot c}\dot c = \nabla_{\dot c}\dot c + \{\{\dot c, \Upsilon\}, \dot c\} = \nabla_{\dot c}\dot c + 2\dot c \cdot \Upsilon(\dot c).$$

Indeed, this is the consequence of the computation of the Lie bracket in $\mathfrak{g}$ of the corresponding elements $\dot c \in \mathfrak{g}_{-1}$, $\Upsilon(u) \in \mathfrak{g}_1$:

$$\begin{aligned}[][\dot c, \Upsilon], \dot c] &\simeq \left[\left[\begin{pmatrix} 0 & 0 \\ \dot c & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Upsilon \\ 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 \\ \dot c & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 0 & 0 \\ 2\dot c \cdot \Upsilon(\dot c) & 0 \end{pmatrix} \simeq 2\dot c \cdot \Upsilon(\dot c),\end{aligned}$$

where $\Upsilon(\dot c)$ is the standard evaluation of the linear form $\Upsilon \in \mathfrak{g}_1 = (\mathbb{H}^n)^*$ on the vector $\dot c \in \mathfrak{g}_{-1} = \mathbb{H}^n$. Thus we see that if there is such a quaternion q for one Weyl connection, then it exists also for all of them. $\square$

Next, we will see that Q–planar curves are exactly geodetics of all Weyl connections.

3.23. **Theorem** ([H]). *Let M be a manifold with an almost quaternionic structure. Then, a curve $C \subset M$ is $\mathbb{H}$–planar if and only if there is parametrization $c : \mathbb{R} \to C \subset M$ which is a geodesic trajectory of some Weyl connection.*

Proof. Let C be a geodesics for ∇. Let us remark that $c : \mathbb{R} \to M$ is a trajectory of C if and only if $\nabla_{\dot{c}}\dot{c} = 0$. Thus, the statement follows immediately from the computation in the proof of lemma 3.22. Indeed, if C is Q–planar, then choose any Weyl connection ∇ and pick up Υ so that $\hat{\nabla}_{\dot{c}}\dot{c}$ vanishes. □

Now, we are able to formulate the main result about diffeomorphisms between two almost quaternionic manifolds.

3.24. **Theorem.** [HS] *Let $f : M \to M'$ be a diffeomorphism between two almost quaternionic manifolds of dimension at least eight. Then f is a morphism of the geometries if and only if it preserves the class of unparameterized geodesics of all Weyl connections on M and M'.*

The proof is based on our theory of A–structures which will be discussed in detail in the next section. We shall come back to it later.

3.25. **Unimodular quaternionic geometry.** We shall conclude this section by short discussion on unimodular quaternionic geometries, i.e. almost quaternionic geometries equipped with volume forms. An unimodular quaternionic geometry (compare with definition 2.26) is not a parabolic geometry but we shall indicate that the choice of a volume form plays a similar role as the choice of a metric in conformal Riemanian geometry. Of course the volume forms on an almost quaternionic geometry are sections of a natural line bundle. Let us observe that this is a principal bundle associated to $\mathcal{G}_0$ at the same time:

For an almost quaternionic geometry, the center of $\mathfrak{g}_0$ is described in matrices as couple $(b, B) \in \mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)$, which satisfies

$$\left[\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & B \end{pmatrix}\right] = 0,$$

for any $(a, A) \in \mathfrak{g}_0$. This is equivalent of fact that $[a, b] = 0$ in $\mathfrak{sp}_1$ and $[A, B] = 0$ in $\mathfrak{gl}(n, \mathbb{H})$. The solution is only $\{(b, cE) | b \in \mathbb{R}, c \in \mathbb{R}\}$ and the center of $\mathfrak{g}_0$ is

$$\mathfrak{z}(\mathfrak{g}_0) = \left\{ \begin{pmatrix} -nk & 0 \\ 0 & kE \end{pmatrix} \right\}.$$

We define $Z(G_0) \subset G_0$ to be the image of $\mathfrak{z}(\mathfrak{g}_0)$ under the exponential mapping

$$Z(G_0) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{\sqrt[n]{a}}E \end{pmatrix} \middle| a \in \mathbb{R}^+ \right\}$$

and the factor group $G_0/Z(G_0) \cong Sp(1) \times_{\mathbb{Z}_2} SL(n, \mathbb{H})$ is the semisimple part of G_0.

A bundle of scales for irreducible parabolic geometries of type (G, P) is a natural principal $\mathbb{R}_+$–bundle $\mathcal{L}^\lambda$ associated to a homomorphism $\lambda : G_0 \to \mathbb{R}_+$, where the kernel λ is exactly the semisimple part of G_0. For the homomorphism $\lambda : Sp(1)GL(n, \mathbb{H}) \to \mathbb{R}^+$ such that

$$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \cdot x \mapsto \left(\mathcal{R}e \det \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \right)^n x$$

the sections of the line–bundle $\mathcal{L}^\lambda$ correspond to volume forms.

Now, each Weyl connection obviously induces a connection on $\mathcal{L}^\lambda$ and surprisingly enough, this yields a bijection between Weyl structures and connections on the bundle of scales, [ČS-03]. Of course, the sections of $\mathcal{L}^\lambda$ correspond to trivial connections on the line bundle and thus the corresponding Weyl connections belong to the unimodular quaternionic geometry.

Such a construction works for all parabolic geometries and, similarly to the conformal case, the Weyl connections coming from scales are called exact.

4. Planar curves and planar morphisms

Various concepts generalizing geodetics have been studied for almost quaternionic and similar geometries. Also various structures on manifolds are defined as smooth distribution in the vector bundle $T^\star M \otimes TM$ of all endomorphisms of the tangent bundle. We have seen the two examples of almost complex and almost quaternionic structures above. Let as extract some formal properties from these examples. Unless otherwise stated, all manifolds are smooth and they have the dimension n. Let ∇ be a linear connection and let $c : \mathbb{R} \to M$ be a trajectory. Then there is the trajectory $\dot{c} := \frac{\partial c(t)}{\partial t} : \mathbb{R} \to TM$.

4.1. **Definition.** Let M be a smooth manifold and let ∇ be a linear connection. Then a map $c : \mathbb{R} \to M$ is called *geodesic trajectory* if and only if $\nabla_{\dot{c}} \dot{c} = 0$.

In other words, a curve c is a geodesic trajectory if and only if its tangent vectors $\dot{c}(t)$ are parallely transported along trajectory $c(t)$.

In local coordinates on M, the condition is expressed as the system of 2nd order differential equations for geodesics (for more details see [KMS])

$$\ddot{c}^i + \sum_{j,k=1}^{n} \Gamma^i_{jk} \dot{c}^j \dot{c}^k = 0. \tag{24}$$

This immediately shows the unique local existence of geodesics in each tangent direction.

4.2. **Lemma.** *Let M be a smooth manifold and let ∇ be a linear connection. If a trajectory $c(t)$ is a geodesic trajectory on M then the trajectory $c(at + b)$ is also a geodesic trajectory on M, for every $a, b \in \mathbb{R}$.*

Proof. We call $\bar{c}^i = c(at+b)^i$. The first and second derivation of $\bar{c}$ is $\dot{\bar{c}} = \dot{c}a$ and $\ddot{\bar{c}} = \ddot{c}a^2$. The equation (24) is

$$a^2\ddot{\bar{c}}^i + \sum_{j,k=1}^{n} \Gamma^i_{jk} a\dot{\bar{c}}^j a\dot{\bar{c}}^k = a^2(\ddot{\bar{c}}^i + \sum_{j,k=1}^{n} \Gamma^i_{jk}\dot{\bar{c}}^j\dot{\bar{c}}^k) = 0$$

for the trajectory $\bar{c}$. The trajectory $\bar{c}$ is a geodesics. □

4.3**. Definition.** Let M be a smooth manifold and ∇ be a linear connection. A curve $C \subset M$ is called *geodesic curve* of the linear connection ∇ if there is some its parametrization $c(t)$ such that $c(t)$ is a geodesic trajectory. We shortly call it *geodesic.*

4.4**. Lemma.** *Let ∇ be a linear connection on smooth manifold M. A curve C is geodesic if and only if one parametrization $c(t)$ (and then all parameterizations) satisfies:*

$$\nabla_{\dot{c}}\dot{c} = \dot{c}\cdot k,\ k \in C^\infty(M,\mathbb{R}).$$

Proof. We will work in an arbitrary coordinate system.
($\Leftarrow$) $(\nabla_{\dot{c}}\dot{c})^h = (\ddot{c})^h + \Gamma^h_{ij}\dot{c}^i\dot{c}^j$ where $c(t) = (c^h(t))$. We shall find another parametrization for which the right hand side will vanish. For a reparametrization $\bar{c}(t) = c\circ\varphi(t)$ we have $\bar{c}^h = c^h(\varphi(t))$ and after derivation by t we have $\dot{\bar{c}}^h = \dot{c}^h\dot{\varphi}$ a $\ddot{\bar{c}}^h = \ddot{c}^h\dot{\varphi}^2 + \dot{c}^h\ddot{\varphi}$. We can substitute $(\nabla_{\dot{\bar{c}}}\dot{\bar{c}})^h = (\ddot{\bar{c}})^h + \Gamma^h_{ij}\dot{\bar{c}}^i\dot{\bar{c}}^j = \ddot{c}^h\dot{\varphi}^2 + \dot{c}^h\ddot{\varphi} + \dot{\varphi}^2\Gamma^h_{ij}\dot{c}^i\dot{c}^j = \dot{\varphi}^2(\ddot{c}^h + \Gamma^h_{ij}\dot{c}^i\dot{c}^j) + \dot{c}^h\ddot{\varphi} = \dot{\varphi}^2(\dot{c}^h\cdot k) + \dot{c}^h\ddot{\varphi} = \dot{c}^h(\dot{\varphi}^2\cdot k + \ddot{\varphi})$. Since we require this to by zero, it is necessary that $\ddot{\varphi} + k\cdot\dot{\varphi}^2 = 0$. The general solution of this equation for small δ is $\varphi = c_1\int \exp^{-\int k dt} dt + c_2$ where $c_1 \neq 0$, $-\delta \le t \le \delta$ and $c_1, c_2, \delta \in \mathbb{R}$, $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$.
($\Rightarrow$) If $(\nabla_{\dot{c}}\dot{c})^h = 0$ then any reparametrization leads to $(\nabla_{\dot{\bar{c}}}\dot{\bar{c}})^h = (\ddot{\bar{c}})^h + \Gamma^h_{ij}\dot{\bar{c}}^i\dot{\bar{c}}^j = \ddot{c}^h\dot{\varphi}^2 + \dot{c}^h\ddot{\varphi} + \dot{\varphi}^2\Gamma^h_{ij}\dot{c}^i\dot{c}^j = \dot{\varphi}^2(\ddot{c}^h + \Gamma^h_{ij}\dot{c}^i\dot{c}^j) + \dot{c}^h(\ddot{\varphi}) = \dot{c}^h\ddot{\varphi} = \dot{\bar{c}}^h\frac{\ddot{\varphi}}{\dot{\varphi}}$, where $\dot{\varphi} \neq 0$. Finally, the choice $k = \frac{\ddot{\varphi}}{\dot{\varphi}} \in C^\infty(\mathbb{R},\mathbb{R})$ leads to the required formula. □

We will write $\nabla_{\dot{c}}\dot{c} \in \langle\dot{c}\rangle$ (i.e. $\nabla_{\dot{c}}\dot{c}$ is in the real vector space spaned by $\dot{c}$).

4.5**. Definition.** Let (M,∇), $(\bar{M},\bar{\nabla})$ be smooth manifolds equipped with linear connections. A diffeomorphism $f : M \to \bar{M}$ is called a *geodesic morphism* if it maps geodesics on M with respect to ∇ to geodesics on $\bar{M}$ with respect to $\bar{\nabla}$.

Two connections ∇ and $\bar{\nabla}$ on a manifold M are called projectively equivalent if they share the geodesics. A classical computation in local coordinates reveals that this is equivalent to the existence of a one form ψ, such that their Christofell symbols $\bar{\Gamma}^i_{jk}$ and Γ^i_{jk} are related by the equation

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\psi_j(x) + \delta^h_j\psi_i(x) = \Gamma^h_{ij}(x) + \psi_{(i}\delta^h_{j)}. \tag{25}$$

More generally, a morphism $f : M \to \bar{M}$ mapping geodesics of ∇ onto geodesics of $\bar{\nabla}$ has to satisfy

$$f^\star\bar{\nabla} - \nabla = (\psi \odot E), \tag{26}$$

where ψ is a suitable one form, $\odot$ is the symmetric tensor product, and E is the identity affinor. Such an f is called a morphism of the projective structures represented by the connections ∇ and $\bar{\nabla}$, see [M2] for more details.

4.6. **F–planar curves and morphisms.** Let M be a smooth manifold equipped with a linear connection ∇ and let F be an affinor on M. A curve C is called F*–planar curve* if there is its parametrization $c : \mathbb{R} \to M$ satisfying the condition

$$\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}, F(\dot{c})\rangle.$$

It is easy to see that geodesics are F–planar curves for all affinors F, because of $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}\rangle \subset \langle \dot{c}, F(\dot{c})\rangle$.

The best known example is an almost complex structure (see example 2.5). We have to be careful about the dimension of M. Let M be a manifold of dimension two and let I be a complex structure. A curve C is F–planar for $F = I$ if and only if there is trajectory c satisfying the identity $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}, I\dot{c}\rangle \cong \mathbb{R}^2$, and any trajectory c satisfy the identity $\nabla_{\dot{c}}\dot{c} \in \mathbb{R}^2$. In other words any curve C is a F–planar on the manifold of dimension two. The concept of I–planar curves makes sense for dimension at least four.

A diffeomorphism $f : (M, F) \to (\bar{M}, \bar{F})$, where couples (M, F) and $(\bar{M}, \bar{F})$ are structure equipped with one affinor is called $(F, \bar{F})$–planar mappings if it maps F–planar curves on M to $\bar{F}$–planar curves on $\bar{M}$.

Now, let (M, ∇, F), $(\bar{M}, \bar{\nabla}, \bar{F})$ be smooth manifolds equipped with an affinor and a linear connection. Let us assume that the connections $\nabla, \bar{\nabla}$ have the same torsion. It is a well known fact (see [MS]) that diffeomorphism $f : M \to \bar{M}$ is F–planar if and only if satisfy following:

$$f^\star\bar{\nabla} - \nabla = (\psi_1 \odot E) + (\psi_2 \odot F), \tag{27}$$

$$f^*\bar{F} \subset \langle F\rangle, \tag{28}$$

where $\odot$ is symmetric tensor product, E is the identity affinor, and ψ_i are one forms. The F–planar curves and morphisms were introduced and studied in detail in [MS].

4.7. **4–planar curves and mappings.** Consider almost hypercomplex structure (I, J, IJ) (definition 2.17). The curve C with a trajectory $c : \mathbb{R} \to M$ such that $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}, I(\dot{c}), J(\dot{c}), IJ(\dot{c})\rangle$ is called 4–*planar*. It is easy to see that all geodesics are 4–planar curve, because of $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}\rangle \subset \langle \dot{c}, I(\dot{c}), J(\dot{c}), IJ(\dot{c})\rangle$ and also all F–planar curves are 4–planar, where $F \in \langle I, J, IJ, E\rangle$. A diffeomorphism of an almost hypercomplex structures with an linear connection $f : (M, (I, J, IJ), \nabla) \to$

$(\bar{M}, (F_1, F_2, F_1F_2), \bar{\nabla})$ is called 4–*planar morphism* if and only if this maps 4–planar curves on M with respect to ∇ and I, J, IJ to 4–planar curves on $\bar{M}$ with respect to $\bar{\nabla}$ and F_1, F_2, F_1F_2.

Let (M, I, J, ∇), $(\bar{M}, F_1, F_2, \bar{\nabla})$ be a smooth manifolds equipped with almost hypercomplex structures and connections with the same torsion. A diffeomorphism $f : M \to \bar{M}$ is 4–planar if and only if satisfies followings:

$$f^{\star}\bar{\nabla} - \nabla = (\psi_1 \odot E) + (\psi_2 \odot I) + (\psi_3 \odot J) + (\psi_4 \odot IJ), \tag{29}$$

$$f^{\star}\bar{F}_i = a_i I + b_i J + c_i IJ, \tag{30}$$

where $\odot$ is symmetric tensor product, E is identity affinor, ψ_i are on forms and $a_i, b_i, c_i \in \mathbb{R}$ (compare with [MNP]).

Finally, the 4–planar curves and morhpisms have brought us to the main topic of this Thesis, the Q–planar curves and morphisms. As defined in definition 2.22 and subsection 2.29, an *almost quaternionic geometry* is a rank four subbundle $Q \subset T^*M \otimes TM$ locally generated by the identity E and an almost hypercomplex structure.

4.8**. Definition.** *Let (M, Q) be a manifold with an almost quaternionic structure. A curve $C \subset M$ is called* Q–planar *if there is its trajectory $c : \mathbb{R} \to M$ such that:*

$$\nabla_{\dot{c}}\dot{c} \in Q(\dot{c}) = \{A(\dot{c}) \mid A \in Q\}.$$

4.9**. Lemma.** *Let $(M, \mathcal{Q}, \nabla)$ be a manifold equipped with linear connection and almost quaternionic structure. For every chosen bases I, J, IJ of Q the Q–planar curves are exactly 4–planar curves with respect to I, J, IJ.*

Proof. Let I, J, $IJ \in Q$ be a chosen basis of Q. The curve C is a Q–planar curve if and only if there is its trajectory $c : \mathbb{R} \to M$ such that

$$\nabla_{\dot{c}}\dot{c} \in Q(\dot{c}).$$

This property implies 4–planarity:

$$\nabla_{\dot{c}}\dot{c} \in (aE + bI + cJ + dK)\dot{c}$$

because $aE + bI + cI + dK \in Q$. The other implication is obvious. $\square$

Let $(M, Q), \nabla)$, $(\bar{M}, \bar{Q}, \bar{\nabla})$ be manifolds equipped with linear connections and almost quaternionic structures. A diffeomorphism $f : M \to \bar{M}$ is called *Q–planar morphism* if it maps Q–planar curves on M to Q–planar curves on $\bar{M}$.

The Q–planar curves and related morphisms were discussed by Fujimora, see [Fujimura79, Fujimura80].

4.10. **A–planar curves and morphisms.** In the lemma 4.9 we made an important observation: the 4–planar curves are fully determined by the almost quaternionic structure induced by the chosen hypercomplex structure.

This simple consequence of standard behavior of the generators of a vector subspace suggests the generalization of the planarity concept below.

4.11. **Definition.** Let M be a smooth manifold of dimension m. Let A be a smooth ℓ–rank ($\ell < m$) vector subbundle in $T^*M \otimes TM$, such that the identity affinor $E = id_{TM}$ restricted to T_xM belongs to $A_x \subset T_x^*M \otimes T_xM$ at each point $x \in M$. We say that M is equipped by ℓ–rank A*–structure.*

In definition 4.11, the dimension of M is higher than the rank of A. This is not a restriction, because there are no A–structures of rank ℓ higher then m. The possibility $\ell = m$ is not interesting, because in this event every curve is A–planar (see remark 4.6).

4.12. **Definition.** For any tangent vector $X \in T_xM$ we shall write $A(X)$ for the vector subspace

$$A(X) = \{F(X)|F \in A_xM\} \subset T_xM$$

and we call $A(X)$ the A*–hull of the vector* X. Similarly, the A*–hull of vector field* will be subbundle in TM obtained pointwise.

For every smooth parameterized curve $c : \mathbb{R} \to M$ we write $\dot{c}$ and $A(\dot{c})$ for the tangent vector field and its A–hull along the curve c.

4.13. **Definition.** Let (M, A) be a smooth manifold M equipped with an ℓ–rank A–structure. We say that the A–structure has

(1) *generic rank* ℓ if for each $x \in M$ the subset of vectors $(X, Y) \in T_xM \oplus T_xM$, such that the A–hulls $A(X)$ and $A(Y)$ generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ is open and dense.

(2) *weak generic rank* ℓ if for each $x \in M$ the subset of vectors

$$\mathcal{V} := \{X \in T_xM|\dim A(X) = \ell\}$$

is open and dense in T_xM.

One immediately checks that any A–structure which has generic rank ℓ has week generic rank ℓ. Indeed, if $U \subset T_xM$ is an open subset of vectors X with $A(X)$ of dimension lower than ℓ, then $U \times U$ is an open subset with to low dimension, too.

4.14. **Theorem.** *Let (M, A) be a smooth manifold of dimension n equipped with A–structure of rank ℓ, such that $2\ell \leq n$. If A_x is an algebra (i.e. for all $f, g \in A_x$, $fg := f \circ g \in A_x$) for all $x \in M$, and A has weak generic rank ℓ then the structure has generic rank ℓ.*

Proof. Since the A–structure has a weak generic rank ℓ, there is the open and dense subset $\mathcal{V} \subset TM$ such that $\dim A(X) = \ell$ for all $X \in \mathcal{V}$.

Because A is an algebra, for any $X, Z \in TM$, $Z \in A(X)$ implies also $A(Z) \subset A(X)$, and moreover $A(Z) = A(X)$ for all $X, Z \in \mathcal{V}$ because of the dimension. Thus, whenever there is a non–trivial vector $0 \neq Z \in A(X) \cap A(Y)$, the entire subspaces coincide, i.e. $A(X) = A(Y)$.

In particular, whenever $X, Y \in \mathcal{V}$ and the dimension of $A(X)+A(Y)$ is less then 2ℓ, we know $A(X) = A(Y)$.

Let us consider a couple of vectors $(Y, Z) \in A(X) \oplus A(X)$ for some $X \in \mathcal{V}$. Consider a vector $W \notin A(X)$. The open neighborhood $\mathcal{U}$ of Y has to include $(Y + aW, Y)$ for all sufficiently small $a \in \mathbb{R}$. But if $Y + aW \in A(X)$ for some $a \neq 0$ then $W \in A(X)$ and this is not true. Thus, for every couple of vectors in $A(X) \oplus A(X)$ and for every its open neighborhood, we have found another couple $(Y' = Y + aW, Z)$ for which the dimension of $A(Y') + A(Z)$ is 2ℓ. This proves the density of the set of couples of vectors generating the maximal dimensions 2ℓ.

Of course, the requirement on the maximal dimension is an open condition and the theorem is proved. □

4.15. **Corollary.** *Let (M, A) be a smooth manifold with A–structure of rank ℓ, such that $\ell \geq \dim M$. If $A_x \subset T^\star_x M \otimes T_x M$ is an algebra with inversion then A has weak generic rank. Moreover, if $\dim M \geq 2\ell$ than A has generic rank ℓ.*

Proof. If $\dim A(X) < \ell$ for some non–zero X, then there is $F \in A$ such that $F(X) = 0$ for some non–zero. But this is not possible because of the existence of the inverses.

The remaining claim follows from the theorem above. □

4.16. **Lemma.** *Let M be a smooth manifold of dimension at least two and F be an affinor such that $F \neq q \cdot E$. Then the $\langle E, F\rangle$–structure has weak generic rank 2.*

Proof. Consider A–structure $A = \langle E, F\rangle$. The complement of $\mathcal{V}$ consists vectors $X \in T_x M$ such that:

$$X + aF(X) = 0,\ a \in \mathbb{R},$$

i.e. eigenspace of F. Dimension of A is two and F is not multiple of the identity. Thus, the union of eigenspaces of F is closed or trivial vector subspace of $T_x M$. Thus, the complement $\mathcal{V}$ is open and nontrivial, i.e. open and dense. □

There is only one possibility for the A–structures in the lowest dimension one $A = \langle E\rangle$. The algebra $\langle E\rangle$ is an algebra with inversion, such that $E \cdot E = E$. For every $X \in T_x M$, $A(X)$ is the strait line containing A.

4.17. **Example.** (1) *An almost complex geometry.* The pair (M, F) is a called a complex structure on M if and only if $F^2 = -E$. An almost complex structure has generic rank two on all manifolds of dimension at least four, because of theorem 4.14.

(2) *An almost product geometry.* The pair (M, F) is a called a product structure on M if and only if $F^2 = E$ and $f \neq E$. An almost product structure has a weak generic rank ℓ because of lemma 4.16 and an almost product structure has generic rank two because of theorem 4.14. We proved that an almost quaternionic geometry has a weak generic rank four in lemma 2.21. The algebra $\langle E, I, J, K\rangle = Q$ is an algebra with inversion, i.e. an almost quaternionic geometry has an generic rank four on all manifolds of dimensions at least eight, because of the theorem 4.14.

4.18**. Definition.** Let M be a smooth manifold equipped with an A–structure and a linear connection ∇. A smooth curve C is told to be *A–planar* if there is its parametrization $c : \mathbb{R} \to M$ such that

$$\nabla_{\dot c}\dot c \in A(\dot c).$$

Clearly, A planarity means that the parallel transport of any tangent vector to c has to stay within the A–hull $\mathrm{A}(\dot c)$ of the tangent vector field $\dot c$ along the curve. Moreover, this concept does not depend on the parametrization of the curve c.

4.19**. Definition.** Let M be a manifold with a linear connection ∇ and an A–structure, while $\bar M$ be another manifold with a linear connection $\bar\nabla$ and B–structure. A diffeomorphism $f : M \to \bar M$ is called *(A, B)–planar* if each A–planar curve C on M is mapped onto the B–planar curve $f(C)$ on M.

4.20**. Remark.** The 1–dimensional $A = \langle E\rangle$ structure must be given just as the linear hull of the identity affinor E, by the definition. Obviously, the $\langle E\rangle$–planar curves on a manifold M with a linear connection ∇ are exactly the unparameterized geodesics. Moreover, two connections ∇ and $\bar\nabla$ without torsion are projectively equivalent (i.e. they share the same unparameterized geodesics) if and only if their difference satisfies $\bar\nabla_X Y - \nabla_X Y = \alpha(X)Y + \alpha(Y)X$ for some one–form α on M. The latter condition can be rewritten as

$$\bar\nabla - \nabla \in \Gamma(T^\star M \odot \langle E\rangle) \subset \Gamma(S^2T^\star M \otimes TM)$$

where the symbol $\odot$ stays for the symmetrized tensor product. Compare to (25).

The latter condition on projective structures may be also rephrased in the terms of morphisms. A diffeomorphism $f : M \to M$ is called geodesical (or an automorphism of the projective structure) if $f \circ c$ is an trajectory of geodesic C for each trajectory c of geodesic C and this happens if and only if the symmetrization of the difference $f^\star\nabla - \nabla$ is a section of $T^\star M \odot \langle E\rangle$. We are going to generalize the above example in the rest of the section.

In the case $A = \langle E\rangle$, the $(\langle E\rangle, B)$–planar mappings are called simply *B–planar.* They map each geodesic curve on (M, ∇) onto a B–planar curve on $(\bar{M}, \bar{\nabla}, B)$

Each ℓ dimensional A structure $A \subset T^*M \otimes TM$ determines the distribution

$$A^{(1)}_x M := \langle \alpha_1 \odot F_1 + \dots + \alpha_l \odot F_l | \alpha_i \in T^*_x M, F_i \in A_x M\rangle.$$

Let us remind that there is not direct coherence of the term $A^{(1)}$ and a first prolongation of a Lie algebra. The term $A^{(1)}$ is only label for our expression.

4.21. **Theorem.** *Let M be a manifold with a linear connection ∇, let N be a manifold of the same dimension with a linear connection $\bar{\nabla}$ and with A–structure of generic rank ℓ, and suppose $dim M \geqq 2\ell$. Then a diffeomorphism $f : M \to N$ is A–planar if and only if*

$$\mathrm{Sym}(f^*\bar{\nabla} - \nabla) \in f^*(A^{(1)}) \tag{31}$$

where Sym *denotes the symmetrization of the difference of the two connections.*

This theorem will be proved later in this chapter.

4.22. **Remark.**

1. The theorem is of local character. We may assume that $M = N$ and $f = id_M$ without any loss of generality.
2. A–planarity of $f : M \to N$ does not at all depend on the possible torsions of the connections. Indeed, we always test expressions of the type $\nabla_{\dot{c}}\dot{c}$ for a trajectory c and thus a deformation of ∇ into $\bar{\nabla} = \nabla + T$ by adding same torsion will not effect the results. Thus, without any loss of generality we may assume that the connection ∇ and $\bar{\nabla}$ live at the same manifold and share the same torsion. Then we may omit the symmetrization from equation (31).
3. We may fix same (local) basis $E = F_0$, F_i, $i = 1, \dots \ell - 1$ of A, i.e. $A = \langle F_0, \dots, F_{l-1}\rangle$. Then the condition in the theorem says

$$\bar{\nabla} = \nabla + \sum_{i=0}^{l-1} \alpha_i \odot F_i$$

for some suitable one–forms α_i on M. Of course, the existence of such forms does not depend on our choice of the basis of A.

With respect to the above remarks, the theorem 4.21 is equivalent to alternative statement below.

4.23. **Equivalent theorem to 4.21.** *Let M be a manifold of dimension at least 2ℓ, ∇ and $\bar{\nabla}$ two connections on M with the same torsion, and consider an A–structure of generic rank ℓ on M. Then each geodesic curve with respect to ∇ is A–planar with respect to $\bar{\nabla}$ if and only if there are one–forms α_i satisfying equation*

$$\bar{\nabla} = \nabla + \sum_{i=0}^{\ell-1} \alpha_i \odot F_i.$$

The proof will require several steps. Assume first we have such forms α_i, and let c be a geodesic trajectory for ∇. Then equation (31) implies $\bar{\nabla}_{\dot{c}}\dot{c} \in A(\dot{c})$ so that c is an A–planar trajectory, by definition. The other implication is the more difficult one. Assume that each geodesic C is A–planar. This implies that the symmetric difference tensor $P = \bar{\nabla} - \nabla \in \Gamma(S^2T^\star M \otimes TM)$ satisfies

$$P(\dot{c}, \dot{c}) = \langle \dot{c}, F_1(\dot{c}), \ldots, F_{\ell-1}(\dot{c}) \rangle.$$

In fact, the main argument of the entire proof boils down to a purely algebraic claim:

4.24. **Lemma.** *Let $A \subset V^\star \otimes V$ be a vector subspace of generic rank ℓ, and assume that $P(X, X) \in A(X)$ for some fixed symmetric tensor $P \in V^\star \otimes V^\star \otimes V$ and each vector $X \in V$. Then the induced mapping $P \in V \to V^\star \otimes V$ has values in A.*

Proof. Let us fix a basis $F_0 = id_{\mathbb{V}}, F_1, \ldots, F_{\ell-1}$ of A. Since A is of generic rank ℓ, there is the open and dense subset $\mathcal{V} \subset V$ of all vectors $X \in TM$ for which $\{X, F_1(X), \ldots, F_{\ell-1}(X)\}$ are linearly independent. Now, for each $X \in \mathcal{V}$ there are unique coefficient $\alpha_i(X) \in \mathbb{R}$ such that

$$P(X, X) = \sum_{i=0}^{\ell-1} \alpha_i(X) F_i(X). \tag{32}$$

The essential technical step in the proof is to show that all functions α_i are in fact restrictions of smooth one–forms on M. Let us notice, that P is symmetric bilinear tensor and thus it is determined by the restriction of $P(X, X)$ to arbitrary small open non–empty subset of the arguments X in V. □

4.25. **Lemma.** *If a smooth symmetric tensor*

$$P(X, X) = \sum_{i=0}^{l-1} \alpha_i(X) F(X)$$

is determined over the above defined subspace $\mathcal{V}$, then the function $\alpha : \mathcal{V} \to \mathbb{R}$ are smooth and their restriction to the individual rays (half–lines) generated by vectors in $\mathcal{V}$ are linear.

Proof. Let us fix a local smooth basis $e_i \in TM$, the dual basis e^i, and consider the induced dual bases e_I and e^I on the multivectors and exterior forms. Let us consider the smooth mapping

$$\chi : \Lambda^l TM \setminus \{0\} \to \Lambda^l T^\star M$$
$$\chi\left(\sum a_I e^I\right) = \sum \frac{a_I}{\sum a_I^2} e_I.$$

Now, for all non–zero tensors

$$\Xi = \sum a_I e^I,$$

the evaluation $\langle \Xi, \chi(\Xi)\rangle$ is the constant function 1, while $\chi(k \cdot \Xi) = k^{-1}\chi(\Xi)$. Next we define for each $X \in \mathcal{V}$

$$\tau(X) = \chi(X \wedge F_1(X) \wedge \cdots \wedge F_{l-1}(X))$$

and we may compute the unique coefficient α_i :

$$\alpha_0(X) = \langle P(X,X) \wedge F_1(X) \wedge F_2(X) \wedge \cdots \wedge F_{\ell-1}(X)), \tau(X)\rangle$$
$$\alpha_1(X) = \langle X \wedge P(X,X) \wedge F_2(X) \wedge \cdots \wedge F_{\ell-1}(X)), \tau(X)\rangle$$
$$\alpha_{\ell-1}(X) = \langle X \wedge F_1(X) \wedge F_2(X) \wedge \cdots \wedge P(X,X)), \tau(X)\rangle$$

In particular, this proves the first part.

Let us now consider a fixed vector $X \in \mathcal{V}$. The defining formula formula for α_i implies $\alpha_i(kX) = k\alpha_i(X)$, for each real number $k \neq 0$. Passing to zero with positive k shows that α does have limit 0 in the origin and so we may extend the definition of the forms α_i to the entire cone $\mathcal{V} \cup \{0\}$ by setting $\alpha_i(0) = 0$ for all i.

Finally, along the ray $\{tX | t < 0\} \subset \mathcal{V}$, the derivative $\frac{d}{dt}\alpha(tX)$ has the constant value $\alpha(X)$. This proves the rest of the lemma. □

4.26. **Lemma.** *If a smooth symmetric tensor P is determined over the above defined subspace $\mathcal{V} \cup \{0\}$, then the coefficients α_i, are smooth one–forms on M and the tensor P is given by*

$$P(X,Y) = \frac{1}{2}\sum_{i=0}^{\ell-1}(\alpha_i(Y)F_i(X) + \alpha_i(X)F_i(Y)).$$

Proof. The entire tensor P is obtained through polarization from its evaluation $P(X,X), X \in TM$,

$$P(X,X) = \frac{1}{2}(P(X+Y,X+Y) - P(X,X) - P(Y,Y)), \tag{33}$$

and again, the entire tensor is determined by its values on arbitrarily small non–empty open subset of X and Y in each fiber. The summands on the right hand side have values in the following subspaces:

$$P(X+Y,X+Y) \in \langle X+Y, F_1(X+Y), \ldots, F_{\ell-1}(X+Y)\rangle \subset$$
$$\langle X, F_1(X), \ldots, F_{\ell-1}(X), Y, F_1(Y), \ldots, F_{\ell-1}(Y)\rangle,$$
$$P(X,X) \in \langle X, F_1(X), \ldots, F_{\ell-1}(X)\rangle$$

$$P(Y,Y) \in \langle Y, F_1(Y), \dots, F_{\ell-1}(Y)\rangle.$$

Since we have assumed that A has generic rank ℓ, the subspace $\mathcal{W} \in TM \times_M TM$ of vectors (X, Y) such that all the values

$$\{X, F_1(X), \dots, F_{\ell-1}(X), Y, F_1(Y), \dots, F_{\ell-1}(Y)\}$$

are linearly independent is open and dense. Clearly $\mathcal{W} \subset \mathcal{V} \times_M \mathcal{V}$. Moreover, if $(X, Y) \in \mathcal{W}$ than $F_0(X + Y), \dots, F_{\ell-1}(X + Y)$ are independent, i.e. $X + Y \in \mathcal{V}$. Inserting (32) into (33), we obtain

$$P(X,Y) = \sum_{i=0}^{\ell-1} (d_i(X,Y)F_i(X) + e_i(X,Y)F_i(Y)).$$

For all $(X, Y) \in \mathcal{W}$, the coefficients $d_i(X,Y) = \frac{1}{2}(\alpha_1(X+Y) - \alpha_i(X))$ at $F_i(X)$, and $e_i(X,Y) = \frac{1}{2}(\alpha_i(X+Y) - \alpha_i(Y))$ at $F_i(Y)$ in the latter expression are uniquely determined. The symmetry of P implies $d_i(X, Y) = e_i(Y, X)$. If $(X, Y) \in \mathcal{W}$ then also $(sX, tY) \in \mathcal{W}$ for all non–zero reals s, t and the linearity of P in the individual arguments yields for all real parameters s, t

$$std_i(X,Y) = sd_i(sX, tY).$$

Thus the functions α_i satisfy

$$\alpha_i(sX + tY) - \alpha_i(sX) = t(\alpha_i(X+Y) - \alpha_i(X)).$$

Since $\alpha_i(tX) = t\alpha_i(X)$, in the limit $a \to 0$ this means

$$\alpha_i(Y) = \alpha_i(X+Y) - \alpha_i(X).$$

Thus α_i are additive over the open and dense set $(X, Y) \in \mathcal{W}$. Choosing a basis of V such that each couple of basis elements is in $\mathcal{W}$, this shows that α_i are restrictions of linear forms, as required. □

Now the completion of the proof of theorem 4.21 is straightforward. Following the equivalent local claim in the alternative theorem 4.23 and the pointwise algebraic description of P achieved in lemma 4.25, we just have to apply the latter lemma to individual fibers over the points $x \in M$ and verify, that the linear forms α_i may be chosen in a smooth way. But this is obvious from the explicit expression for the coefficients α_i in the proof of lemma 4.26. This concludes the complete proof.

4.27. **Theorem.** *Let M be a manifold with linear connection ∇ and an A–structure, N be a manifold of the same dimension with a linear connection $\bar{\nabla}$ and B–structure with generic rank ℓ. Then a diffeomorphism $f : M \to N$ is (A, B)–planar if and only if f is B–planar and $A(X) \subset (f^\star(B))(X)$ for all $X \in TM$.*

As before, we may restrict ourselves to same open submanifolds, fix generators F_i for B, assume that $f = id_M$ and booth connection ∇ and

$\bar{\nabla}$ share the same torsion, and restrict ourselves to prove the equivalent local assertion to our theorem:

4.28. **Equivalent theorem.** *Let M be a manifold with linear connections ∇ and $\bar{\nabla}$, together with an A–structure and B–structure with generic rank ℓ. Each A–planar curve c with respect to ∇ is B–planar with respect to $\bar{\nabla}$, if and only if the symmetric difference tensor $P = \bar{\nabla} - \nabla$ is of the form (32) with smooth one–forms $\alpha_i, i = 0, \ldots, \ell - 1$, and $A(X) \subset B(X)$ for each $X \in TM$.*

All geodesics with respect to ∇ on M are in particular A–planar and thus also B–planar. Therefore, we may use the result of the theorem 4.21 to deduce that

$$P(X,X) = \sum_{j=0}^{l} \alpha_i(X)F_i(X)$$

for uniquely given smooth one–forms α_i.

Now, consider a fixed $F \in A$ and suppose $F(X) \notin B(X)$. Since we assume that all $\langle E, F\rangle$–planar curves c in M are B–planar, we may proceed exactly as in beginning of the proof of theorem 4.21 to deduce that

$$P(X,X) = \sum_{j=0}^{l} \alpha_i(X)F_i(X) + \beta_i(X)F_i(X)$$

on a neighborhood of X, with some unique functions α_i and β.

The comparison of the latter two unique expressions for $P(X,X)$ shows that $\beta(X)$ vanishes. But since $F(X) \neq X$, there definitely are curves which are $\langle E, F\rangle$–planar and tangent to X, but not $\langle E\rangle$–planar. Thus, the assumption in the theorem would lead to $\beta(X) \neq 0$. Consequently, our choice $F(X) \notin B(X)$ cannot be achieved and we have proved $A(X) \subset B(X)$ for all $X \in TM$.

4.29**. Example.** Let us summarize consequences of the latter theorem for the three examples of projective, complex and quaternionic structures. We obtain the well knowned results mentioned earlier. Consider connections with the same torsion on a manifold M.

(1) *Projective structures.* In this case $A = \langle E\rangle$ and we have
$$Sym(f^*\bar{\nabla} - \nabla) = f^*\bar{\nabla} - \nabla \in f^*(A^{(1)})$$
$$f^*\bar{\nabla} = \nabla + \alpha \odot E$$
for some one–form α. The second property $f^*\langle E\rangle \subset \langle E\rangle$ is trivial and this case and so we have got the classical result on geodesical mappings.

(2) *Almost complex structure.* i.e $A = \langle E, I\rangle$, with $I^2 = -E$. For this choice,
$$f^*\bar{\nabla} = \nabla + \alpha \odot E + \beta \odot I$$

The second property $f^*\langle E, I\rangle \subset \langle I\rangle$ imply $f^*\langle I\rangle \subset \langle I\rangle$ becomes $f^*I = \pm I$ and so we have recovered the fact that mappings respecting the A–planar curves are either holomorphic or anti–holomorphic.

(3) *Almost quaternionic space.* With the choice $A = \langle E, I, J, K\rangle$ the first condition becomes

$$f^*\bar{\nabla} = \nabla + \alpha \odot E + \beta \odot I + \gamma \odot J + \delta \odot K$$

which suitable one–forms α, β, γ, δ, while the second property reads

$$f^*\langle E, I, J, K\rangle \subset \langle E, I, J, K\rangle$$

and equivalently

$$f^*\langle I, J, K\rangle \subset \langle I, J, K\rangle.$$

The almost quaternionic geometries enjoy the following property, which is not necessarily true for all A–structures:

$$\forall X \in T_xM,\ \forall F \in A, \exists c_X \mid \dot{c}_X = X,\ \nabla_{\dot{c}_X}\dot{c}_X = \beta(X)F(X), \tag{34}$$

where $\beta(X) \neq 0$.

For example, this property is not true for conformal geometries of all signatures except the positive and negative definite ones, or the almost product structures.

4.30. **Theorem.** *Let (M, A), (M', A') be smooth manifolds of dimension m equipped with A–structure and A'–structure of the same generic rank $\ell \leq 2m$ and assume that the A–structure satisfies the property (34). If $f : M \to M'$ is an (A, A')–planar mapping, then f is a morphism of the A–structures, i.e $f^*A' = A$.*

Proof. Assume we have got two manifolds with A–structures (M, A), (M', A') and a diffeomorphism $f : M \to M'$ which is (A, A')–planar. Then theorem 4.27 implies that $A(X) = (f^*A')(X)$ for each $X \in TM$ (since they both have the same dimension). In order to conclude the theorem, we have to verify $A = f^*A'$ instead, since this is exactly the requirement that f preserves the defining subbundles A and A'.

Let us look at the subsets of all second jets of A–planar curves. According to the property (34), the accelerations fill just the complete A–hulls of the velocities at each point. Thus, for a given point $x \in M$ and a fixed $F \in A$, we may locally choose a smoothly parameterized system c_X of A–planar curves with parameter $X \in T_xM$ such that $\dot{c}_X = X$ and $\nabla_{\dot{c}_X}\dot{c}_X = \beta(X)F(X)$ where F is one of the generators of A and $\beta(X) \neq$. Then

$$(\nabla_{\dot{c}_X} - \hat{\nabla}_{\dot{c}_X})\dot{c}_X = \beta(X)F(X) + \sum_k \alpha_k(X)F_k(X)$$

where F_k are the generators of A' and α_k are smooth 1–forms, cf. the proof of theorem 4.21. Moreover, the affinors F_k as well as the one–forms α_k are independent of X. But this shows that $F(X) = \sum_k \gamma_k(X)F_k(X)$ for some smooth functions γ_k of X. Since F is linear in X, γ_k have to be constants and we are done. $\square$

4.31. **Proof of theorem 3.24.** By example 4.17 an almost quaternionic structure has generic rank 4. Further, we saw in 3.22 that the almost quaternionic structures have the property (34). Furthermore, theorem 3.23 asserts that the Q–planar curves are just the geodesics of the Weyl connections. Thus, the theorem 4.30 concludes the proof.

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