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Half-linear Euler Differential Equation: Perturbations and Oscillatory Properties

Ph.D. Dissertation

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Advisor: prof. RNDr. Ondřej Došlý, DrSc. **Brno 2014**

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Abstract

In this dissertation, we investigate oscillatory properties of the perturbed half-linear Euler differential equation

$$\big(\Phi(x')\big)' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

First, we consider perturbations of the form

$$\left[\left(1 + \sum_{j=1}^{n} \frac{\alpha_j}{\operatorname{Log}_j^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p \operatorname{Log}_j^2 t}\right]\Phi(x) = 0,$$

where $\operatorname{Log}_k t = \prod_{j=1}^{k} \log_k t, \quad \log_k t = \log_{k-1}(\log t),\ \log_1 t = \log t$, and in the second part perturbations involving periodic functions $r(t)$, $c(t)$, $\alpha_j(t)$, $\beta_j(t) > 0$

$$\left[\left(r(t) + \sum_{j=1}^{n} \frac{\alpha_j(t)}{\operatorname{Log}_j^2 t}\right)^{1-p}\Phi(x')\right]' + \left[\frac{c(t)}{t^p} + \sum_{j=1}^{n} \frac{\beta_j(t)}{t^p \operatorname{Log}_j^2 t}\right]\Phi(x) = 0,$$

are studied.

In the both cases, the method of transformation of the so-called modified Riccati equation is used.

Abstrakt

V této disertační práci studujeme oscilační vlastnosti perturbované Eulerovy pololineární diferenciální rovnice

$$\left(\Phi(x')\right)' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

Nejdříve uvažujeme perturbace ve tvaru

$$\left[\left(1 + \sum_{j=1}^{n} \frac{\alpha_j}{\mathrm{Log}_j^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p \,\mathrm{Log}_j^2 t}\right]\Phi(x) = 0,$$

kde $\mathrm{Log}_k t = \prod_{j=1}^{k} \log_k t, \quad \log_k t = \log_{k-1}(\log t),\ \log_1 t = \log t$, a v druhé části studujeme perturbace obsahující periodické funkce $r(t)$, $c(t)$, $\alpha_j(t)$, $\beta_j(t) > 0$

$$\left[\left(r(t) + \sum_{j=1}^{n} \frac{\alpha_j(t)}{\mathrm{Log}_j^2 t}\right)^{1-p}\Phi(x')\right]' + \left[\frac{c(t)}{t^p} + \sum_{j=1}^{n} \frac{\beta_j(t)}{t^p \,\mathrm{Log}_j^2 t}\right]\Phi(x) = 0.$$

V obou případech je k vyšetřování rovnic využita metoda transformací tzv. modifikované Riccatiho rovnice.

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Declaration

I declare that my dissertation was developed independently, using the information sources that are cited in the work.

Brno 2nd October 2014 . . . .

Hana Funková

Contents

Introduction

The half-linear differential equation is a second order nonlinear differential equation of the form

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1, \tag{1}$$

with continuous functions r, c and $r(t) > 0$. The terminology "half-linear" differential equation is motivated by the fact that its solution space has just one half of the properties that characterize linearity, namely homogeneity but not generally aditivity. The terminology half-linear equation was introduced in the papers of Bihari [1, 2], but as pioneers of this theory are usually regarded Elbert and Mirzov with their papers [16, 17, 26].

There are several motivations for the investigation of qualitative properties of half-linear differential equations. One of them comes from the fact that the partial differential equation with the so-called p-Laplacian

$$\operatorname{div}\left(||\nabla u||^{p-2}\nabla u\right) + c(x)\Phi(u) = 0, \quad x \in \mathbb{R}^n$$

with a spherically symetric potential $c(x)$ can be transformed into half-linear equation (1) which is also sometimes called equation with one-dimensional p-Laplacian. Another motivation is that if $p = 2$ then (1) reduces to the linear Sturm-Liouville second order differential equation

$$\big(r(t)x'\big)' + c(t)x = 0, \tag{2}$$

whose qualitative theory is deeply developed and a natural question is which "linear" results for (2) can be extended to (1).

Even if the aditivity of the solution space of (1) is lost, the remaining homogeneity is sufficient to establish a Sturmian theory for (1). This means, among others, that equation

(1) does not admit coexistence of oscillatory and nonoscillatory solutions and hence it can be classified as oscillatory or nonoscillatory as linear equation (2). A comprihensive treatment of the qualitative theory of half-linear equations can be found in the books [4, 13].

The principal concern of the dissertation are oscillation criteria for (1) which are related, in a certain sense, to the so-called half-linear Euler differential equation

$$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1,\ \gamma \in \mathbb{R}.$$

This equation is one of a few half-linear differential equations which can be solved (at least partially) explicitly and when $\gamma = \left(\frac{p-1}{p}\right)^p$, this equation represents a natural ”borderline” between oscillatory and nonoscillatory half-linear differential equations and hence it plays an important role in many (non)oscillation criteria for (1).

The organization of the dissertation is following. In the first chapter, we deal with the principal methods used for studying the oscillation properties. The reader gets familiar with techniques using Prüfer angle, Riccati equation or modified Riccati equation, as well as the basic types of half-linear differential equations and their essential properties.

In the next chapter, we focus on the Euler half-linear differential equation having a two-term perturbation and we lead the reader step by step to the oscillation criterion for half-linear Euler differential equation with a two-term perturbation.

The third chapter gives a more general view on the problem of oscillation criteria for Euler half-linear differential equation. We investigate the Euler half-linear differential equation with perturbations of at maximum n terms also in the term involving derivative. This chapter shows that even if more complicated methods had to be used, the oscillation criteria are perfect generalization of oscillation criteria presented in the Chapter 2.

The fourth chapter includes the oscillatory properties of Euler half-linear differential equation with perturbations involving periodic functions.

The fifth and last chapter gives some ideas for the possible new research directions.

Chapter 1

Basic Theory

Let us consider the equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-1}\operatorname{sgn} x,\ p > 1, \tag{1.1}$$

where r and c are continuous functions and $r(t) > 0$. We will first deal with the existence and uniqueness of a solution of (1.1).

1.1 Half-linear trigonometric functions

We start with introducing the so-called half-linear trigonometric functions. Consider a special half-linear equation of the form (1.1)

$$\big(\Phi(x')\big)' + (p-1)\Phi(x) = 0 \tag{1.2}$$

and denote by $S = S(t)$ its solution given by the initial conditions $S(0) = 0$, $S'(0) = 1$. We will show that the behavior of this solution si very similar to that of the classical sine function. Multiplying (1.2) (with x replaced by S) by S' and using the fact that $(\Phi(S'))' = (p-1)|S'|^{p-2}S''$, we get the identity $[|S'|^p + |S|^p]' = 0$. Substituting here $t = 0$ and using the initial condition for S we have the generalized Pythagorian identity

$$|S(t)|^p + |S'(t)|^p = 1. \tag{1.3}$$

The function S is positive in some right neighbourhood of $t = 0$ and using (1.3) $S' = (1-S^p)^{\frac{1}{p}}$, i.e., $\frac{dS}{(1-S^p)^{\frac{1}{p}}} = dt$ in this neighbourhood, hence

$$t = \int_0^{S(t)} (1-s^p)^{-\frac{1}{p}} ds. \tag{1.4}$$

Following the analogy with the case $p = 2$, we denote

$$\frac{\pi_p}{2} = \int_0^1 (1-s^p)^{-\frac{1}{p}} ds = \frac{1}{p}\int_0^1 (1-u)^{-\frac{1}{p}} u^{-\frac{1}{q}} du = \frac{1}{p} B\left(\frac{1}{p}, \frac{1}{q}\right),$$

where $q = \frac{p}{p-1}$ and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

is the Euler beta function. Using the formulas

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

with the Euler gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, we have

$$\pi_p = \frac{2\pi}{p \sin \frac{\pi}{p}}.$$

The formula (1.4) defines uniquely the function $S = S(t)$ on $[0, \frac{\pi_p}{2}]$ with $S(\frac{\pi_p}{2}) = 1$ and hence by (1.3) $S'(\frac{\pi_p}{2}) = 0$. Now, we define the half-linear sine function $\sin_p t$ as the $2\pi_p$ odd continuation of the function

$$S_p(t) = \begin{cases} S(t), & 0 \le t \le \frac{\pi_p}{2}, \\ S(\pi_p - t), & \frac{\pi_p}{2} \le t \le \pi_p. \end{cases}$$

The function S_p reduces to the classical function sine in the case $p = 2$ and in some literature this function is denoted by $\sin_p t$ (in this dissertation we will be using this notation as well).

The remaining half-linear trigonometric functions are defined by the formulas

$$\cos_p t = (\sin_p t)', \quad \tan_p t = \frac{\sin_p t}{\cos_p t}, \quad \cot_p t = \frac{\cos_p t}{\sin_p t}.$$

The function $\tan_p t$ is periodic with the period π_p and has discontinuities at $\frac{\pi_p}{2} + k\pi_p$, $k \in \mathbb{Z}$. By (1.2) and (1.3) we have

$$(\tan_p t)' = \frac{1}{|\cos_p t|^p} = 1 + |\tan_p t|^p, \quad (\cot_p t)' = -|\cot_p t|^{2-p}(1 + |\cot_p t|^p). \qquad (1.5)$$

Hence $(\tan_p t)' > 0$, $(\cot_p t)' < 0$ on their definition domains and there exist the inverse functions $\operatorname{arctan}_p t$, $\operatorname{arccot}_p t$ which are defined as inverse functions of $\tan_p t$ and $\cot_p t$ in the domains $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ and $(0, \pi_p)$, respectively. From (1.5) we have

$$(\operatorname{arctan}_p t)' = \frac{1}{1 + |t|^p}.$$

1.2 Half-linear Prüfer transformation

Using the generalized trignometric functions and their inverse functions defined in the previous section, we can introduce the generalized Prüfer transformation as follows. Let x be a nontrivial solution of (1.1) and $q = \frac{p}{p-1}$ be the conjugate exponent of p. Put

$$\rho(t) = \left(|x(t)|^p + r^q(t)|x'(t)|^p\right)^{\frac{1}{p}}$$

and let φ be a continuous function defined at all points where $x(t) \neq 0$ by the formula

$$\varphi(t) = \operatorname{arccot}_p \frac{r^{q-1}(t)x'(t)}{x(t)}.$$

Hence

$$x(t) = \rho(t) \sin_p \varphi(t), \qquad (1.6)$$

$$r^{q-1}(t)x'(t) = \rho \cos_p \varphi(t). \qquad (1.7)$$

Differentiating equality (1.6) and comparing it with (1.7) we get

$$r^{1-q}(t)\rho(t)\cos_p \varphi(t) = \rho'(t)\sin_p \varphi(t) + \rho(t)\varphi'(t)\cos_p \varphi(t). \qquad (1.8)$$

Similarly, applying the function Φ to both sides of (1.7), differentiating the obtained identity and substituting from (1.1) we get

$$\begin{aligned}-c(t)\rho^{p-1}(t)\Phi(\sin_p\varphi(t)) = \\ (p-1)\left[\rho^{p-2}(t)\rho'(t)\Phi(\cos_p\varphi(t)) - \rho^{p-1}(t)\varphi'(t)\Phi(\sin_p\varphi(t))\right].\end{aligned} \tag{1.9}$$

Now, multiplying (1.8) by $\frac{\Phi(\cos_p\varphi)}{\rho}$, (1.9) by $\frac{\sin_p\varphi}{\rho^{p-1}}$ and combining the obtained equations we get the first order system for φ and ρ

$$\varphi' = \frac{c(t)}{p-1}|\sin_p\varphi|^p + r^{1-q}(t)|\cos_p\varphi|^p,$$

$$\rho' = \Phi(\sin_p\varphi)\cos_p\varphi\left[r^{1-q}(t) - \frac{c(t)}{p-1}\right]\rho.$$

The right hand-side of the last system for φ and ρ is Lipschitzian in these variables, hence its solution is uniquely determined by an initial condition. But this means that the solution of (1.1) is determined by an initial condition as well.

1.3 Riccati technique and variational principle

One of the basic methods of the half-linear oscillatory theory, the so-called Riccati technique, is based on the relationship between nonoscillation of equation (1.1) and solvability of the associated Riccati type equation.

Let x be a solution of equation (1.1). Then $w(t) = \frac{r(t)\Phi(x'(t))}{\Phi(x(t))}$ is a solution of the Riccati type differential equation

$$R[w](t) := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0. \tag{1.10}$$

We can easily prove that in view of equation (1.1) we have

$$\begin{aligned}w' &= \frac{(r\Phi(x'))'\Phi(x) - (p-1)r\Phi(x')|x|^{p-2}x'}{\Phi^2(x)} \\ &= -c-(p-1)\frac{r|x'|^p}{|x|^p} = -c-(p-1)r^{1-q}|w|^q.\end{aligned}$$

Recall that equation (1.1) is *disconjugate* on the closed interval $[a,b]$ if the solution x given by the initial condition $x(a)=0$, $r(a)\Phi(x'(a))=1$ has no zero in $(a,b]$, in the opposite case equation (1.1) is said to be *conjugate* on $[a,b]$.

Let us also remind that equation (1.1) is said to be *nonoscillatory* if there exists $T\in\mathbb{R}$ such that (1.1) is disconjugate on $[T,\infty)$, i.e., every nontrivial solution of this equation has at most one zero in this interval and this means that every nontrivial solution is eventually positive or negative. Equation (1.1) is said to be *oscillatory* in the opposite case, i.e., when there exists a nontrivial oscillatory solution and this is equivalent, as we show below, that every nontrivial solution has infinitely many zeros tending to ∞.

The following statement which describes two basic methods of the half-linear oscillation theory is usually referred to as the Roundabout theorem.

Theorem 1. *The following statements are equivalent.*

1. *Equation* (1.1) *is disconjugate on the interval* $[a,b]$.
2. *There exists a solution of equation* (1.1) *having no zero in* $[a,b]$.
3. *There exists a solution w of Riccati equation* (1.10) *which is defined on the whole interval* $[a,b]$.
4. *The energy functional* $\mathscr{F}(y;a,b)=\int\limits_a^b[r(t)|y'|^p-c(t)|y|^p]\,\mathrm{d}t$ *is positive for every* $0\not\equiv y\in W_0^{1,p}(a,b)$.

For the proof of the above Roundabout Theorem 1 see for instance [4, p. 175].

The relationship between (1.1) and Riccati equation (1.10) shows that the classification of (1.1) as oscillatory or nonoscillatory is correct, in the sense that if one solution of (1.1) (non)oscillates, then any other solution also (non)oscillates. Indeed, let x be a nontrivial solution of (1.1) with consecutive zeros at $t_1<t_2$ and $w=\frac{r\Phi(x')}{\Phi(x)}$. Then $w(t_1+)=\infty$ and $w(t_2-)=-\infty$. Now, if $\tilde{x}$ is a solution for which $\tilde{x}(t)\neq 0$ for $t\in[t_1,t_2]$, then the graph of $\tilde{w}=\frac{r\Phi(\tilde{x}')}{\Phi(\tilde{x})}$ has to intersect the graph of w and this contradicts the unique solvability of (1.10) (since (1.10) is Lipschitzian in w). Therefore, $\tilde{x}$ has to vanish somewhere in $[t_1,t_2]$ and hence, the zeros of linearly independent solutions of (1.1) interchange.

Theorem 1 shows us the way the properties of Riccati equation (1.10) and second order half-linear differential equation (1.1) are linked. For the investigation of oscillatory properties of equation (1.1), the Riccati technique uses the equivalence of disconjugacy of equation (1.1) and solvability of the associated Riccati equation (1.10).

More precisely, the following statement holds (see [13, Theorem 2.2.1]).

Proposition 1. *Equation* (1.1) *is nonoscillatory if and only if there exists a differentiable function w such that* (1.10) *holds for large t.*

1.4 Modifed Riccati equation

The *modified Riccati equation* associated with (1.1) is introduced explicitly in [14], but it can be found implicitly already in some earlier papers, e.g. [21, 22, 31]. Suppose that (1.1) is nonoscillatory (i.e., every its nontrivial solution is eventually positive or negative) and let h be a positive differentiable function. Consider the substitution

$$v = h^p(t)w - G(t), \quad G(t) := r(t)h(t)\Phi(h'(t)), \tag{1.11}$$

where w is a solution of (1.10). Then v is a solution of the *modified Riccati equation*

$$v' + \tilde{c}(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v, G(t)) = 0, \tag{1.12}$$

with

$$H(v, G) := |v+G|^q - q\Phi^{-1}(G)v - |G|^q, \tag{1.13}$$

$\Phi^{-1}(s) = |s|^{q-2}s$ being the inverse function of Φ, and

$$\tilde{c}(t) = h(t)\left[\left(r(t)\Phi(h'(t))\right)' + c(t)\Phi(h(t))\right]. \tag{1.14}$$

This can be verified by a direct computation, see also computation below formula (2.5) in the next chapter. Note that the function $H(v, G)$ satisfies $H(v, G) \geq 0$ for every $v, G \in \mathbb{R}$ and $H(v, G) = 0 = H_v(v, G)$ if and only if $v = 0$. Observe also that Riccati equation (1.10) is a special case of (1.12) with $h(t) \equiv 1$, i.e., $G(t) \equiv 0$.

1.5 Half-linear Euler differential equation

The half-linear Euler differential equation

$$\big(\Phi(x')\big)' + \frac{\gamma}{t^p}\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1,\ \gamma \in \mathbb{R}, \tag{1.15}$$

is one of a few half-linear second order differential equations which can be solved explicitly. Similarly to the linear case $p = 2$, if we look for a solution of (1.15) in the form $x(t) = t^\lambda$, we get

$$\begin{aligned} x' &= \lambda t^{\lambda-1}, \\ \Phi(x') &= \Phi(\lambda)t^{(p-1)(\lambda-1)} = \Phi(\lambda)t^{p\lambda-\lambda-p+1}, \\ \big(\Phi(x')\big)' &= \Phi(\lambda)(p-1)(\lambda-1)t^{p\lambda-\lambda-p}. \end{aligned}$$

Substituting into (1.15) we obtain

$$\begin{aligned} \Phi(\lambda)(p-1)(\lambda-1)t^{p\lambda-\lambda-p} + \gamma t^{(p-1)\lambda-p} &= 0, \\ t^{p\lambda-\lambda-p}(p-1)\left[\Phi(\lambda)(\lambda-1) + \frac{\gamma}{p-1}\right] &= 0. \end{aligned}$$

Hence, we find that λ has to be a solution of the algebraic equation

$$\Phi(\lambda)(\lambda-1) + \frac{\gamma}{p-1} = 0,$$

$$(p-1)\Phi(\lambda)(\lambda-1) + \gamma = 0,$$

see also [17].

We denote $F(\lambda) := (p-1)\Phi(\lambda)(\lambda-1)$. By a direct computation of the derivative of the function $F(\lambda)$ we get the extrema of this function

$$\begin{aligned} p\Phi(\lambda) - |\lambda|^{p-2}(p-1) &= 0, \\ |\lambda|^{p-2}\left[\lambda p - (p-1)\right] &= 0. \end{aligned}$$

Since $F(\pm\infty) = \infty$, we see that the function $F(\lambda) := (p-1)\Phi(\lambda)(\lambda-1)$ has a global minimum at $\lambda^* = \frac{p-1}{p}$ and the value of this minimum is

$$\begin{aligned} F(\lambda^*) &= (p-1)\Phi\left(\frac{p-1}{p}\right)\left(\frac{p-1}{p}-1\right) = (p-1)\left(\frac{p-1}{p}\right)^{p-1}\frac{p-1-p}{p} \\ &= \left(\frac{p-1}{p}\right)^{p-1}\frac{(p-1)(-1)}{p} = -\left(\frac{p-1}{p}\right)^{p} =: -\gamma_p. \end{aligned}$$

Consequently, the equation $F(\lambda)+\gamma = 0$ has two real roots if $\gamma < \gamma_p$, one double real root if $\gamma = \gamma_p$, and no real root if $\gamma > \gamma_p$.

Equation (1.15) is a particular case of the general half-linear second order differential equation (1.1). This means, in particular, that (1.15) is nonoscillatory if and only if $\gamma \le \gamma_p$. Also, equation (1.15) with the critical coefficient $\gamma = \gamma_p$ serves as a comparison equation for the Kneser-type (non)oscillation test which states that (1.1) with $r(t) = 1$ is oscillatory provided

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p \tag{1.16}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^p c(t) < \gamma_p \tag{1.17}$$

The potential $c(t) = \gamma_p/t^p$ "separates" potentials c in (1.1) with $r(t) \equiv 1$ for which this equation is oscillatory or nonoscillatory. Criteria (1.16), (1.17) can be extended to the general case $r(t) \not\equiv 1$. In this general setting, the Kneser type criterion is fomulated in terms of the lower and upper limit of the expression

$$r^{q-1}(t)\left(\int^t r^{1-q}(s)\,ds\right)^p c(t) \tag{1.18}$$

if $\int^\infty r^{1-q}(t)\,dt = \infty$, and of the expression

$$r^{q-1}(t)\left(\int_t^\infty r^{1-q}(s)\,ds\right)^p c(t)$$

if $\int^\infty r^{1-q}(t)\,dt < \infty$. The constant γ_p in this criterion remains the same. In the linear case $p = 2$, (1.16) and (1.17) are the classical Kneser (non)oscillation criteria.

The Kneser test does not apply when $\lim_{t\to\infty} t^p c(t) = \gamma_p$ and this situation is one of the concerns of this dissertation.

1.6 Euler equation - nonoscillatory case

Even if the first possibility, when the equation $F(\lambda)+\gamma = 0$ has two real roots happens, since the additivity of the solution space is lost in the half-linear case, we are not able to compute other solutions explicitly. To get a more detailed information about their asymptotic behavior, we use the procedure which is also typical in the linear case, namely the transformation of (1.15) into an equation with constant coefficients.

The change of independent variable $s = \log t$ converts (1.15) into the equation (where the dependent variable will be denoted again by x and $' = \frac{d}{ds}$)

$$\big(\Phi(x')\big)' - (p-1)\Phi(x') + \gamma\Phi(x) = 0. \tag{1.19}$$

The Riccati equations corresponding to (1.15) and (1.19) are

$$w' = -\gamma t^{-p} - (p-1)|w|^p \tag{1.20}$$

and

$$v' = -\gamma + (p-1)v - (p-1)|v|^q := E(v). \tag{1.21}$$

The solutions w and v are related by the formula $w(t) = t^{1-p}v(\log t)$ and, moreover, we have $F(\Phi^{-1}(v)) + \gamma = -E(v)$ with F defined in the previous section.

In the case where $\gamma < \gamma_p$ the function $E(\lambda) = 0$ has two real roots $\lambda_1 < \tilde{\lambda} < \lambda_2$, $\tilde{\lambda} = \left(\frac{p-1}{p}\right)^{p-1}$. The constant functions $v(s) \equiv \lambda_1$, $v(s) \equiv \lambda_2$ are solutions of (1.21). Clearly, if v is a solution of (1.21) such that $v(s) < \lambda_1$, for some $s \in \mathbb{R}$, then $v'(s) < 0$, if $v(s) \in (\lambda_1, \lambda_2)$, then $v'(s) > 0$, and $v'(s) < 0$ for $v(s) > \lambda_2$, a picture of the direction field of (1.21) helps to visualize the situation. Any solution of (1.21) different from $v(s) = \lambda_{1,2}$ can be expressed (implicitly) in the form ($S \in \mathbb{R}$ being fixed)

$$\int_{v(S)}^{v(s)} \frac{d\lambda}{E(\lambda)} = s - S. \tag{1.22}$$

Observe that the integral $\int_{s_1}^{s_2} \frac{ds}{E(s)}$ is convergent whenever the integration interval does not contain zeros $\lambda_{1,2}$ of E, in particular, for any $\varepsilon > 0$

$$\int_{-\infty}^{\lambda_1-\varepsilon} \frac{d\lambda}{E(\lambda)} > -\infty \quad \text{and} \quad \int_{\lambda_2+\varepsilon}^{\infty} \frac{d\lambda}{E(\lambda)} > -\infty.$$

In the case where $\gamma = \gamma_p$ the function $E(\lambda) = 0$ has the double root $\tilde{\lambda} = \left(\frac{p-1}{p}\right)^{p-1}$ and equation (1.15) has a solution $x(t) = t^{\Phi^{-1}(\tilde{\lambda})} = t^{\frac{p-1}{p}}$. That implies nonoscillation of (1.15).

Since $E(\tilde{\lambda}) = 0 = E'(\tilde{\lambda})$,

$$E(\lambda) = \frac{1}{2}E''(\tilde{\lambda})(\lambda - \tilde{\lambda})^2 + O\left((\lambda - \tilde{\lambda})^3\right) \quad \text{as} \quad \lambda \to \tilde{\lambda},$$

hence, taking into account that $E''(\tilde{\lambda}) = -\frac{1}{\tilde{\lambda}}$,

$$\frac{1}{E(\lambda)} = \frac{1}{\frac{1}{2}E''(\tilde{\lambda})(\lambda - \tilde{\lambda})^2\left[1 + O(\lambda - \tilde{\lambda})\right]} = -\frac{2\tilde{\lambda}}{(\lambda - \tilde{\lambda})^2} + O\left((\lambda - \tilde{\lambda})^{-1}\right) \quad \text{as} \quad \lambda \to \tilde{\lambda}.$$

On the other hand, we see form (1.22) that any solution v which starts with the initial value $v(S) < \tilde{\lambda}$ fails to be extensible up to ∞ and solutions with $v(S) > \tilde{\lambda}$ tend to $\tilde{\lambda}$ as $S \to \infty$. Substituting for $E(\lambda)$ in (1.22) we have

$$\frac{2\tilde{\lambda}}{\lambda - \tilde{\lambda}} + O(\log|\lambda - \tilde{\lambda}|) = s - S,$$

hence

$$2\tilde{\lambda} + (\lambda - \tilde{\lambda})O(\log|\lambda - \tilde{\lambda}|) = (\lambda - \tilde{\lambda})(s - S).$$

Since $\lim_{\lambda \to \tilde{\lambda}}(\lambda - \tilde{\lambda})O(\log|\lambda - \tilde{\lambda}|) = 0$, we have

$$\lim_{s\to\infty}(s - S)(v(s) - \tilde{\lambda}) = \lim_{s\to\infty} s(v(s) - \tilde{\lambda}) = 2\tilde{\lambda}.$$

Consequently,

$$O(\log|v(s) - \tilde{\lambda}|) = O(\log s^{-1}) = O(\log s) \quad \text{as} \quad s \to \infty,$$

and thus $(v(s)-\tilde{\lambda})^{-1}=\frac{s}{2\lambda}+O(\log s)$, which means

$$v(s)-\tilde{\lambda}=\frac{2\lambda}{s}\frac{1}{1+O(\frac{\log s}{s})}=\frac{2\tilde{\lambda}}{s}\left(1+O\left(\frac{\log s}{s}\right)\right).$$

Now, taking into account that solutions of (1.20) and (1.21) are related by $w(t)=t^{1-p}v(\log t)$, we have

$$t^{p-1}w(t)-\tilde{\lambda}=\frac{2\tilde{\lambda}}{\log t}+O\left(\frac{\log(\log t)}{\log^2 t}\right),$$

which means that the solution x of (1.15) which determines the solution w of (1.20) satisfies

$$\frac{x'(t)}{x(t)}\sim\frac{\Phi^{-1}(\tilde{\lambda})}{t}\left(1+\frac{2}{\log t}\right)^{\frac{1}{p-1}}\sim\frac{p-1}{pt}+\frac{2}{pt\log t}$$

and thus

$$x(t)\sim t^{\frac{p-1}{p}}\log^{\frac{2}{p}}t\quad\text{as}\quad t\to\infty.$$

1.7 Riemann-Weber half-linear differential equation

An important role in our treatment is played by the so-called half-linear Riemann-Weber equation which is the equation

$$\left(\Phi(x')\right)'+\left(\frac{\gamma_p}{t^p}+\frac{\mu}{t^p\log^2 t}\right)\Phi(x)=0. \tag{1.23}$$

It is shown in [18] that equation (1.23) is oscillatory if and only if

$$\mu>\mu_p:=\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

In the linear case $p=2$, equation (1.23) reduces to the classical Riemann-Weber equation

$$x''+\left(\frac{1}{4t^2}+\frac{\mu}{t^2\log^2 t}\right)x=0. \tag{1.24}$$

In the critical case $\mu = \frac{1}{4}$, equation (1.24) has linearly independent solutions

$$x_1(t) = \sqrt{t \log t}, \qquad x_2(t) = \sqrt{t \log t} \log(\log t).$$

In the half-linear case, solutions in the critical case $\mu = \mu_p$ cannot be computed exactly, but using a method similar to that from the preceeding Section 1.6, one can prove that (1.23) with $\mu = \mu_p$ has a solution

$$\tilde{x}(t) \sim t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \quad \text{as} \quad t \to \infty$$

and any other solution which is not proportional to $\tilde{x}$ behaves asymptotically for $t \to \infty$ as

$$x(t) = ct^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \log^{\frac{2}{p}}(\log t), \quad 0 \neq c \in \mathbb{R},$$

see [18].

Here $f(t) \sim g(t)$ for a pair of functions f, g means $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

Chapter 2

Two-terms perturbation

2.1 Auxiliary results

In a general framework, we suppose that equation (1.1) is nonoscillatory and we study oscillatory properties of its perturbation

$$\big[\big(r(t)+\tilde{r}(t)\big)\Phi(x')\big]' + \big(c(t)+\tilde{c}(t)\big)\Phi(x) = 0 \tag{2.1}$$

with continuous functions $\tilde{r}, \tilde{c}$ such that $r(t)+\tilde{r}(t) > 0$ for large t.

An important role is played by the concept of conditionally oscillatory half-linear equation. Following [13], equation (1.1) with $\lambda c(t)$ instead of $c(t)$ is said to be *conditionally oscillatory* if there exists a constant λ_0 such that this equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The constant λ_0 is called the *oscillation constant* of (1.1). A typical example of a conditionally oscillatory equation is just Euler equation (1.15) and its oscillation constant is $\lambda_0 = \gamma_p$. Concerning a more detailed treatment of conditional oscillation of half-linear differential equations we refer to [25].

Here we will deal with conditionally oscillatory half-linear equations in a more general sense. We will consider the equation of the form

$$[(r(t)+\lambda\tilde{r}(t))\Phi(x')]' + (c(t)+\mu\tilde{c}(t))\Phi(x) = 0 \tag{2.2}$$

and we say that (2.2) is conditionally oscillatory if there exist constants $\alpha, \beta, \omega \in \mathbb{R}$, $\alpha \neq 0$,

$\beta \neq 0$, such that (2.2) is oscillatory for $\alpha\lambda + \beta\mu > \omega$ and nonoscillatory for $\alpha\lambda + \beta\mu < \omega$. A typical example of conditionally oscillatory equation with two parameters is perturbed Euler equation (1.15) with the critical coefficient γ_p

$$\left[\left(1+\frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right]\Phi(x) = 0. \tag{2.3}$$

It is proved in [6] that (2.3) is oscillatory if $\mu - \lambda\gamma_p > \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$ and nonoscillatory if $\mu - \lambda\gamma_p < \mu_p$. It was conjectured in [6] that (2.3) is nonoscillatory also in the limiting case

$$\mu - \lambda\gamma_p = \mu_p. \tag{2.4}$$

In this chapter we prove that the conjecture is true.

Next we derive the modified Riccati equation in a more general setting than in the Section 1.4. Together with (1.1) we consider equation (2.1) which we regard as a perturbation of (1.1). Let $h(t) \neq 0$ be a differentiable function, denote

$$\Omega(t) = (r+\tilde{r})h\Phi(h')$$

and put

$$z := h^p w - \Omega,$$

where w is a solution of the Riccati equation associated with (2.1)

$$w' + c(t) + \tilde{c}(t) + (p-1)(r(t)+\tilde{r}(t))^{1-q}|w|^q = 0.$$

Then z is a solution of the *modified Riccati equation* of the form

$$z' + C(t) + (p-1)(r(t)+\tilde{r}(t))^{1-q}h^{-q}(t)H(z,\Omega) = 0,$$

where

$$C(t) = h(t)\left[\left((r(t)+\tilde{r}(t))\Phi(h'(t))\right)' + (c(t)+\tilde{c}(t))\Phi(h(t))\right]. \tag{2.5}$$

Indeed, by a direct computation we have

$$
\begin{aligned}
z' &= \big[h^p w - (r+\tilde r) h \Phi(h')\big]' \\
&= p\Phi(h) h' w + h^p\big[-(c+\tilde c) - (p-1)(r+\tilde r)^{1-q}|w|^q\big] \\
&\quad - \big((r+\tilde r)\Phi(h')\big)' h - (r+\tilde r)|h'|^p \\
&= p\Phi(h) h' h^{-p}(z+\Omega) - h^p(c+\tilde c) - (p-1)(r+\tilde r)^{1-q} h^p |h^{-p}(z+\Omega)|^q \\
&\quad - \big((r+\tilde r)\Phi(h')\big)' h - (r+\tilde r)|h'|^p \\
&= p\frac{h'}{h} z + p\frac{h'}{h}\Omega - hL(h) - (r+\tilde r)|h'|^p - (p-1)(r+\tilde r)^{1-q} h^{p-pq}|z+\Omega|^q \\
&= -hL(h) - (p-1)h^{-q}(r+\tilde r)^{1-q}\big[|z+\Omega|^q - q\Phi^{-1}(\Omega) z - |\Omega|^q\big].
\end{aligned}
$$

Note that in contrast to [6], here we do not suppose that h is a solution of (1.1), so the extra term $h[(r\Phi(h'))' + c\Phi(h)]$ appears in the definition of the function C in (2.5).

In the investigation of perturbations of Euler equation (1.15), we will need the following results. The first one is a slight modification of [6, Theorem 3] (here, in contrast to [6], we do not require that the function h is a solution of (1.1) but this only means that the above mentioned extra terms appear in the definition of the function C, otherwise everything is the same), so we omit its proof.

Theorem 2. *Let h be a positive differentiable function such that $h'(t) \neq 0$ for large t. Denote*

$$P(t) := \big(r(t) + \tilde r(t)\big) h^2(t) |h'(t)|^{p-2},$$

and suppose that

$$\int^\infty \frac{dt}{P(t)} = \infty, \qquad \int^\infty C(t)\, dt \text{ is convergent},$$

where C is given by (2.5)*, and that*

$$\liminf_{t\to\infty} |\Omega(t)| > 0.$$

If

$$\limsup_{t\to\infty}\int^t \frac{ds}{P(s)}\int_t^\infty C(s)\,ds < \frac{1}{2q}$$

and

$$\liminf_{t\to\infty}\int^t \frac{ds}{P(s)}\int_t^\infty C(s)\,ds > -\frac{3}{2q},$$

then equation (2.1) *is nonoscillatory.*

We will also need the next two statements concerning the existence of proper solutions (i.e., solutions which can be extended up to ∞) of modified Riccati equation (1.12) which we, for the sake of the later application, rewrite into the form

$$v' + C(t) + (p-1)R^{-1}(t)H(v, G(t)) = 0 \tag{2.6}$$

with continuous functions C, R and $R(t) > 0$.

The proofs of the below given theorems can be found (in a modified form) in [6, 7].

Theorem 3. (i) *If* $C(t) \le 0$ *for large* t*, then* (2.6) *possesses a (nonnegative) proper solution. In the remaining part of the theorem suppose that*

$$\liminf_{t\to\infty}|G(t)| > 0 \quad \textit{and} \quad C(t) \ge 0 \quad \textit{for large } t.$$

Denote

$$\mathscr{R}(t) = R^{-1}(t)|G(t)|^{q-2},$$

and suppose that

$$\int^\infty \mathscr{R}(t)\,dt = \infty, \quad \int^\infty C(t)\,dt < \infty.$$

(ii) *If*

$$\limsup_{t\to\infty}\left(\int^t \mathscr{R}(s)\,ds\right)\left(\int_t^\infty C(s)\,ds\right) < \frac{1}{2q},$$

then (2.6) *has a proper solution.*

(iii) *If*

$$\liminf_{t\to\infty}\left(\int^t \mathscr{R}(s)\,ds\right)\left(\int_t^\infty C(s)\,ds\right) > \frac{1}{2q},$$

then (2.6) *possesses no proper solution.*

Theorem 4. *Together with* (2.6) *consider an equation of the same form*

$$v' + D(t) + (p-1)R^{-1}(t)H(v, G(t)) = 0 \tag{2.7}$$

with the function D satisfying $D(t) \geq C(t)$ for large t. If the (majorant) equation (2.7) *has a proper solution, then* (2.6) *has a proper solution as well.*

2.2 Equation (2.3) in the limiting case (2.4)

In this section we deal with equation (2.3) in the limiting case (2.4). The result of the below given Theorem 5 is a special case of Theorem 6 from the Chapter 4, but because the idea of the proof of Theorem 5 is completely different from that of Theorem 6, we present it here with details. This result has been published in [11].

Theorem 5. *Suppose that* (2.4) *holds. Then the perturbed Euler equation with the critical coefficient* (2.3) *is nonoscillatory.*

Proof. We rewrite (2.3) into the form

$$\left[\left(1 + \frac{\lambda}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + \frac{\mu - \mu_p}{t^p \log^2 t}\right]\Phi(x) = 0$$

and we use the computation below (2.5) with $r(t) = 1$, $\tilde{r}(t) = \frac{\lambda}{\log^2 t}$, $c(t) = \frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}$, $\tilde{c}(t) = \frac{\mu - \mu_p}{t^p \log^2 t}$, $h(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t$.

We have

$$h' = \frac{p-1}{p} t^{-\frac{1}{p}} \log^{\frac{1}{p}} t + \frac{1}{p} t^{-\frac{1}{p}} \log^{\frac{1}{p}-1} t = \frac{p-1}{p} t^{-\frac{1}{p}} \log^{\frac{1}{p}} t \left(1 + \frac{1}{(p-1)\log t}\right),$$

$$\Phi(h') = \left(\frac{p-1}{p}\right)^{p-1} t^{-\frac{p-1}{p}} \log^{\frac{p-1}{p}} t \left(1 + \frac{1}{(p-1)\log t}\right)^{p-1},$$

$$
\begin{aligned}
\big(\Phi(h')\big)' = {} & \left(\frac{p-1}{p}\right)^{p-1}\Bigg[-\frac{p-1}{p}t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left(1+\frac{1}{(p-1)\log t}\right)^{p-1} \\
& +\frac{p-1}{p}t^{-2+\frac{1}{p}}\log^{-\frac{1}{p}}t\left(1+\frac{1}{(p-1)\log t}\right)^{p-1} \\
& -t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\log^{-2}t\left(1+\frac{1}{(p-1)\log t}\right)^{p-2}\Bigg] \\
= {} & \left(\frac{p-1}{p}\right)^{p-1}\frac{p-1}{p}t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left(1+\frac{1}{(p-1)\log t}\right)^{p-2} \\
& \times\left[-1-\frac{1}{(p-1)\log t}+\frac{1}{\log t}\left(1+\frac{1}{(p-1)\log t}\right)-\frac{p}{p-1}\frac{1}{\log^2 t}\right] \\
= {} & \gamma_p t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left[1+\frac{p-2}{(p-1)\log t}+\binom{p-2}{2}\frac{1}{(p-2)^2\log^2 t}\right. \\
& \left.+\binom{p-2}{3}\frac{1}{(p-1)^3\log^3 t}+o(\log^{-3}t)\right]\left[-1+\frac{1}{\log t}\left(1-\frac{1}{p-1}\right)\right. \\
& \left.+\frac{1}{\log^2 t}\left(\frac{1}{p-1}-\frac{p}{p-1}\right)\right] \\
= {} & \gamma_p t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left[-1+\frac{1}{\log t}\left(-\frac{p-2}{p-1}+\frac{p-2}{p-1}\right)\right. \\
& \left.+\frac{1}{\log^2 t}\left(-\frac{(p-2)(p-3)}{2(p-1)^2}+\frac{(p-2)^2}{(p-1)^2}-1\right)+o(\log^{-2}t)\right] \\
= {} & \gamma_p t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left[-1-\frac{p}{2(p-1)}\frac{1}{\log^2 t}+O(\log^{-3}t)\right] \\
= {} & t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left[-\gamma_p-\frac{\mu_p}{\log^2 t}+O\big(\log^{-3}t\big)\right], \\
\big(\Phi(h')\big)'+{} & \frac{1}{t^p}\left[\gamma_p+\frac{\mu_p}{\log^2 t}\right]\Phi(h) \\
= {} & (\frac{p-1}{p})^{p-1}t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left[-\frac{p-1}{p}-\frac{1}{2\log^2 t}-\frac{(p-2)^2}{3(p-1)^2\log^3 t}+o(\log^{-3}t)\right] \\
& +t^{-2+\frac{1}{p}}\log^{\frac{p-1}{p}}t\left[\left(\frac{p-1}{p}\right)^p+\frac{1}{2}(\frac{p-1}{p})^{p-1}\frac{1}{\log^2}t\right] \\
= {} & -(\frac{p-1}{p})^{p-1}t^{-2+\frac{1}{p}}\log^{-2-\frac{1}{p}}t\frac{(p-2)^2}{3(p-1)^2}\big(1+o(\log^{-1}t)\big)\le 0
\end{aligned}
$$

and hence for $t\to\infty$

$$
h\left[\big(\Phi(h')\big)'+\frac{1}{t^p}\left(\gamma_p+\frac{\mu_p}{\log^2 t}\right)\Phi(h)\right]\sim\frac{1}{t\log^2 t}\to 0.
$$

Similarly,

$$\begin{aligned}
\left(\frac{\lambda}{\log^2 t}\Phi(h')\right)' =& \lambda\left(\frac{p-1}{p}\right)^{p-1}\left[t^{-\frac{p-1}{p}}\log^{-1-\frac{1}{p}}\left(1+\frac{1}{(p-1)\log t}\right)^{p-1}\right]' \\
=& \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[1+\frac{p-2}{(p-1)\log t}+o\big(\log^{-2} t\big)\right] \\
&\times\left[-\frac{p-1}{p}\left(1+\frac{1}{(p-1)\log t}\right)\right. \\
&\left.-(1+\frac{1}{p})\frac{1}{\log t}\left(1+\frac{1}{(p-1)\log t}\right)-\frac{1}{\log^2 t}\right] \\
=& \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[1+\frac{p-2}{(p-1)\log t}+o\big(\log^{-1} t\big)\right] \\
&\times\left[-\frac{p-1}{p}-\frac{p+2}{p\log t}+o\big(\log^{-1}\big)\right] \\
=& \lambda\left(\frac{p-1}{p}\right)^{p-1} t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\left[-\frac{p-1}{p}-\frac{2}{\log t}+o\big(\log^{-1} t\big)\right].
\end{aligned}$$

Hence, in the limiting case (2.4) it holds

$$\begin{aligned}
[(\tilde{r}\Phi(h'))'+\tilde{c}\Phi(h)] =& t^{-2+\frac{1}{p}}\log^{-1-\frac{1}{p}} t\,[-\lambda\gamma_p+\mu-\mu_p \\
&\left.-\frac{2}{p}\left(\frac{p-1}{p}\right)^{p-1}\frac{1}{\log t}+o(\log^{-2} t)\right] = -\frac{2\gamma_p}{p-1}t^{-2+\frac{1}{p}}\log^{-2-\frac{1}{p}} t(1+o(1))
\end{aligned}$$

as $t\to\infty$. Consequently,

$$h[(\tilde{r}\Phi(h'))'+\tilde{c}\Phi(h)] = -t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t\,\frac{2\gamma_p}{p-1}t^{-2+\frac{1}{p}}\log^{-2-\frac{1}{p}} t(1+o(1)) = O\big(t^{-1}\log^{-2} t\big)$$

as $t\to\infty$.

Now we use Theorem 2. In this theorem

$$P=(r+\tilde{r})h^2|h'|^{p-2}=t\log t(1+o(1))\sim t\log t,$$

$$G=rh\Phi(h')=\left(\frac{p-1}{p}\right)^{p-1}\log t\left(1+\frac{1}{(p-1)\log t}\right)^{p-1}$$

and using the previous computations

$$C = h\big[\big((r+\tilde{r})\Phi(h')\big)' + (c+\tilde{c})\Phi(h)\big] = O\big(t^{-1}\log^{-2} t\big)$$

as $t \to \infty$, i.e., there exists a constant $M > 0$ such that $|C(t)| \leq M\frac{1}{t\log^2 t}$. Now, by a direct computation

$$\lim_{t\to\infty}\left|\int^t P^{-1}(s)\,ds \int_t^\infty C(s)\,ds\right| \leq M \lim_{t\to\infty} \frac{\log(\log t)}{\log t} = 0,$$

so by Theorem 2 equation (2.3) with λ and μ satisfying (2.4) is nonoscillatory. $\square$

Chapter 3

General perturbation

3.1 Perturbations of linear Euler equation

The results of this chapter are taken from [9]. Our investigation is motivated by the papers [11, 18], and [23]. In [18], perturbations of (1.15) of the form

$$\big(\Phi(x')\big)' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p \operatorname{Log}_j^2 t}\right]\Phi(x) = 0, \tag{3.1}$$

were investigated. Here, the notation

$$\operatorname{Log}_k t = \prod_{j=1}^{k} \log_k t, \quad \log_k t = \log_{k-1}(\log t),\ \log_1 t = \log t$$

is again used. It was shown that the crucial role in (3.1) plays the constant $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. In particular, if $n = 1$ in (3.1), i.e., this equation reduces, as we have mentioned in Section 1.7, to the so-called Riemann-Weber half-linear differential equation, then this equation is oscillatory if $\beta_1 > \mu_p$ and nonoscillatory in the opposite case. In general, if $\beta_j = \mu_p$ for $j = 1, \dots, n-1$, then (3.1) is oscillatory if and only if $\beta_n > \mu_p$, see [18].

In [23], perturbations of the *linear* Euler differential equation were investigated and a perturbation was also allowed in the term involving derivative. More precisely, the differential equation

$$\left[\left(1+\sum_{j=1}^{n}\frac{\alpha_j}{\operatorname{Log}_j^2 t}\right)x'\right]'+\left[\frac{1}{4t^2}+\sum_{j=1}^{n}\frac{\beta_j}{t^2\operatorname{Log}_j^2 t}\right]x=0 \tag{3.2}$$

was considered. It was shown that if there exists $k\in\{1,\dots,n\}$ such that $\beta_j-\alpha_j/4=1/4$ for $j=1,\dots,k-1$, and $\beta_k-\alpha_k/4\neq\frac{1}{4}$, then (3.2) is oscillatory if and only if $\beta_k-\alpha_k/4>1/4$. If $\beta_j-\alpha_j/4=\frac{1}{4}$ for all $j=1,\dots,n$, then (3.2) is nonoscillatory. This result was first partially extended to the half-linear equation

$$\left[\left(1+\frac{\alpha}{\log^2 t}\right)\Phi(x')\right]'+\left[\frac{\gamma_p}{t^p}+\frac{\beta}{t^p\log^2 t}\right]\Phi(x)=0 \tag{3.3}$$

and it was shown that (3.3) is oscillatory if and only $\beta-\alpha\gamma_p>\mu_p$, see Section 2.2. For some related results see also [7].

In this chapter we deal with perturbations of the Euler half-linear differential equation in full generality. We consider the equation

$$\left[\left(1+\sum_{j=1}^{n}\frac{\alpha_j}{\operatorname{Log}_j^2 t}\right)\Phi(x')\right]'+\left[\frac{\gamma_p}{t^p}+\sum_{j=1}^{n}\frac{\beta_j}{t^p\operatorname{Log}_j^2 t}\right]\Phi(x)=0 \tag{3.4}$$

and we find an explicit formula for the relationship between the constants α_j,β_j in (3.4) which implies (non)oscillation of this equation. Our result is based on a new method which consists in transformations of the modified Riccati equations associated with (1.1). The main result along this line is established in the next section, while its application to the perturbed Euler equation is presented in the last section of this chapter.

Next, we recall the transformation method of the investigation of (3.2) which we extend in a modified form to half-linear equations. Recall that the Sturm-Liouville differential equation

$$(r(t)x')'+c(t)x=0 \tag{3.5}$$

is the special case $p=2$ in (1.1). The transformation $x=f(t)y$ gives the identity (suppressing the argument t)

$$f[(rx')'+cx]=(rf^2y')'+f[(rf')'+cf]y. \tag{3.6}$$

In particular, if $f(t) \neq 0$, then x is a solution of (3.5) if and only if y is a solution of the equation

$$\big(r(t)f^2(t)y'\big)' + f(t)[(r(t)f'(t))' + c(t)f(t)]y = 0. \tag{3.7}$$

Let us emphasize at this moment that we have in disposal *no half-linear version* of transformation identity (3.6).

Let us denote

$$r(t) = 1 + \sum_{j=1}^{n} \frac{\alpha_j}{\operatorname{Log}_j^2 t}, \quad c(t) = \frac{1}{4t^2} + \sum_{j=1}^{n} \frac{\beta_j}{t^2 \operatorname{Log}_j^2 t}.$$

First we apply the transformation $x = \sqrt{t}y$ to (3.2). Using (3.6) and the fact that $f(t) = \sqrt{t}$ is a solution of the critical Euler linear equation $x'' + \frac{1}{4t^2}x = 0$, we find that y is a solution of the equation

$$\big(tr(t)y'\big)' + \left[\sum_{j=1}^{n} \frac{\beta_j - \alpha_j/4}{t \operatorname{Log}_j^2 t}\right] y = 0.$$

Now, we change the independent variable $t \longmapsto e^t$, the resulting equation is

$$\big(r(e^t)y'\big)' + \left[\sum_{j=1}^{n} \frac{\beta_j - \alpha_j/4}{t^2 \operatorname{Log}_{j-1}^2 t}\right] y = 0. \tag{3.8}$$

Here we take $\operatorname{Log}_0 t = 1$. Equation (3.8) is oscillatory by Kneser oscillation criterion if $\beta_1 - \alpha_1/4 > 1/4$ and nonoscillatory if $\beta_1 - \alpha_1/4 < 1/4$. Indeed, since $r(e^t) \sim 1$ as $t \to \infty$, we have in (1.18) with $p = 2$

$$r(e^t)\left(\int^t r^{-1}(e^s)\,ds\right)^2 \sim t^2$$

as $t \to \infty$, and hence

$$\lim_{t\to\infty} r(e^t)\left(\int^t r^{-1}(e^s)\,ds\right)^2 \left[\sum_{j=1}^{\infty} \frac{\beta_j - \alpha_j/4}{t^2 \operatorname{Log}_{j-1}^2 t}\right] = \beta_1 - \alpha_1/4.$$

If $\beta_1 - \alpha_1/4 = 1/4$, we can repeat the previous transformations and we obtain the equation

$$\big(r(e_2(t))y'\big)' + \left[\frac{\beta_2 - \alpha_2/4}{t^2} + \cdots + \frac{\beta_n - \alpha_n/4}{t^2 \operatorname{Log}^2_{n-2} t}\right] y = 0,$$

here $e_2(t) := e^{e^t}$. Now it should be clear how one can obtain the result of [23] concerning oscillation of (3.2). We repeat the transformation of dependent variable $y \longmapsto \sqrt{t}y$ followed by the change of independent variable $t \longmapsto e^t$ as long as the condition $\beta_j - \alpha_j/4 = 1/4$ is satisfied.

As we have emphasized above, we have no half-linear version of the linear transformation identity (3.6). Consequently, the above procedure cannot be applied directly to (1.1). However, as observed e.g. in [6, 7], the modified Riccati equation in the linear case $p = 2$ is the equation

$$v' + h[(rh')' + ch] + \frac{v^2}{rh^2} = 0$$

which is just the Riccati equation associated with differential equation (3.7) (with $h = f$). Hence, modified Riccati equation can be regarded, in a certain sense, as a half-linear substitution for the linear transformation identity (3.6). This is just the idea which we develop in the next section and apply it in the investigation of the perturbed Euler equation.

3.2 Transformation of modified Riccati equation

As a starting point of this section we consider the modified Riccati equation in the form

$$v' + C(t) + (p-1)R^{-1}(t)H(v, G(t)) = 0, \tag{3.9}$$

where the function H is given by (1.13), the functions R, C are supposed to be continuous and $R(t) > 0$. In this equation, we call the function C the *absolute term* (since this term does not contain the unknown function v).

We consider the transformation

$$z = f^p(t)v - U(t) \tag{3.10}$$

with a positive differentiable function f and with a function U which we determine as follows. We have (again suppressing the argument t, this argument we will suppress also now and then in the next parts of the dissertation)

$$\begin{aligned} z' =& p\frac{f'}{f}(z+U)+f^p\Big\{-C-(p-1)R^{-1}\left[f^{-pq}|z+U+f^pG|^q\right. \\ & \left.-q\Phi^{-1}(G)f^{-p}(z+U)-|G|^q\right]\Big\}-U' \\ =& -(p-1)R^{-1}f^{-q}\big|z+U+f^pG\big|^q+p\left[\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right]z \\ & +p\left[\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right]U-(p-1)f^pR^{-1}|G|^q-U'-f^pC. \end{aligned}$$

Next we determine the function U in such a way that the differential equation for z is again an equation of the form (3.9) (in which $H(0,G)=0=H_v(0,G)$). Denote $\Omega := U+f^pG$. The terms on the third line of the previous computation

$$-(p-1)R^{-1}f^{-q}|z+\Omega|^q+p\left[\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right]z \tag{3.11}$$

we will take as the first two terms in the function of the same form as H in (3.9). Differentiating (3.11) with respect to z, substituting $z=0$, and setting the obtained expression equal to zero, we obtain

$$R^{-1}f^{-q}\Phi^{-1}(\Omega)=\frac{f'}{f}+R^{-1}\Phi^{-1}(G),$$

hence

$$\Omega = f\Phi(Rf'+f\Phi^{-1}(G)).$$

Consequently, we obtain the transformed modified Riccati equation

$$z'+\tilde{C}+(p-1)R^{-1}f^{-q}\left[|z+\Omega|^q-q\Phi^{-1}(\Omega)z-|\Omega|^q\right]=0,$$

where

$$\tilde{C} = -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}(G)\right)U + f^p C - (p-1)R^{-1} f^p |G|^q \qquad (3.12)$$
$$+ (p-1)R^{-1} f^{-q} |\Omega|^q + U'$$

and

$$U = -f^p G + f\Phi(Rf' + f\Phi^{-1}(G)). \qquad (3.13)$$

Note that formula (3.12) can be simplyfied as

$$\tilde{C} = R^{-1} f^p |G|^q - R^{-1} f^{-q} |\Omega|^q - f^p G' + \Omega' + f^p C.$$

3.3 General perturbations of Euler differential equation

Now we apply the results of the previous section to the perturbed Euler half-linear differential equation

$$\left[\left(\sum_{j=0}^{n} \frac{\alpha_j}{\operatorname{Log}_j^2 t}\right)\Phi(x')\right]' + \left(\sum_{j=0}^{n} \frac{\beta_j}{t^p \operatorname{Log}_j^2 t}\right)\Phi(x) = 0, \qquad (3.14)$$

where $\alpha_0 = 1$, $\beta_0 = \gamma_p := \left(\frac{p-1}{p}\right)^p$.

To simplify the next computations, we denote

$$r(t) = 1 + \sum_{j=1}^{n} \frac{\alpha_j}{\operatorname{Log}_j^2 t}, \quad c(t) = \frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p \operatorname{Log}_j^2 t}. \qquad (3.15)$$

The Riccati equation associated with (3.14) is

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0. \qquad (3.16)$$

In order to understand better the next transformation procedure, we recommend the reader to compare it with the linear transformation idea presented in Section 3.1. The transformation

$$v_1 = t^{p-1}w - U_1,$$

with U_1 specified later, transforms (3.16) into the equation

$$v_1' + \frac{\tilde{c}_1(t)}{t} + \frac{p-1}{t} r^{1-q}(t) H(v_1, \Omega_1(t)) = 0 \tag{3.17}$$

with $\tilde{c}_1/t$ given by (3.12), i.e. $\frac{\tilde{c}_1(t)}{t} = \tilde{C}(t)$ with $f(t) = t^{\frac{p-1}{p}}$, $R = r^{q-1}$, and $G = 0$. This means that $\frac{\tilde{c}_1}{t} = X_1 + Y_1 + Z_1 + U_1' + f^p c$, where

$$\begin{aligned}
U_1 &= -f^p G + f\Phi(Rf' + f\Phi^{-1}(G)) = r\Gamma_p, \quad \Gamma_p := \left(\frac{p-1}{p}\right)^{p-1}, \\
X_1 &= -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}(G)\right) U_1 = -(p-1)t^{-1} r\Gamma_p, \\
Y_1 &= -(p-1)R^{-1} f^p |G|^q = 0, \\
\Omega_1 &= f\Phi(Rf' + f\Phi^{-1}(G)) = r\Gamma_p, \\
Z_1 &= (p-1)R^{-1} f^{-q} |\Omega_1|^q = (p-1)r^{1-q} t^{-1} r^q \gamma_p = (p-1) r t^{-1} \gamma_p.
\end{aligned}$$

Hence, by a direct computation we obtain

$$\frac{\tilde{c}_1(t)}{t} = \sum_{j=1}^{n} \frac{B_j}{t \operatorname{Log}_j^2 t} + O(t^{-1} \log^{-3} t),$$

where

$$B_j = \beta_j - \alpha_j \gamma_p.$$

In equation (3.17), with the above given $\frac{\tilde{c}_1(t)}{t}$, we change the independent variable $t \longmapsto e^t$ and the resulting equation is

$$v_1' + c_1(t) + (p-1) r^{1-q}(e^t) \left[|v_1 + \Omega_1(e^t)|^q - q\Phi^{-1}(\Omega_1(e^t)) v - \Omega_1^q(e^t)) \right] = 0$$

with

$$c_1(t) := \tilde{c}_1(e^t) = \frac{B_1}{t^2} + \frac{B_2}{t^2 \log^2 t} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-1}^2 t} + O(t^{-3}).$$

As the next step, we consider the modified Riccati equation

$$v_1' + \frac{B_1}{t^2} + \frac{B_2}{t^2 \log^2 t} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-1}^2 t} + O(t^{-3}) + (p-1)\tilde{r}_1^{1-q}(t)H(v, \tilde{\Omega}_1(t)) = 0,$$

where now

$$\begin{aligned}
\tilde{r}_1(t) &:= r(e^t) = 1 + \frac{\alpha_1}{t^2} + \frac{\alpha_2}{t^2 \log^2 t} + \cdots + \frac{\alpha_n}{t^2 \operatorname{Log}_{n-1}^2 t}, \\
\tilde{\Omega}_1(t) &:= \Omega_1(e^t) = r(e^t)\Gamma_p = \tilde{r}_1(t)\Gamma_p.
\end{aligned}$$

We apply the transformation $v_2 = tv_1 - U_2$, the quantity U_2 is again determined in such a way that we obtain a modified Riccati equation containing H type function for v_2. Hence, using the results from formula (3.12), with $f(t) = t^{1/p}$, $G(t) = \tilde{\Omega}_1(t) = \tilde{r}_1(t)\Gamma_p$, and $R^{-1}(t) = \tilde{r}_1^{1-q}(t)$, we have

$$\begin{aligned}
\Omega_2 &= f\Phi(Rf' + f\Phi^{-1}(\tilde{\Omega}_1)) = t^{\frac{1}{p}}\Phi\big(\frac{1}{p}\tilde{r}_1^{q-1}t^{\frac{1}{p}-1} + t^{\frac{1}{p}}\Phi^{-1}(\tilde{r}_1\Gamma_p)\big) \\
&= \tilde{r}_1 t\Gamma_p \left(1 + \frac{1}{(p-1)t}\right)^{p-1}
\end{aligned}$$

and using the binomial expansion

$$\begin{aligned}
U_2 &= -f^p\tilde{\Omega}_1 + \Omega_2 = -t\tilde{r}_1\Gamma_p + \tilde{r}_1 t\Gamma_p \left(1 + \frac{1}{(p-1)t}\right)^{p-1} \\
&= \tilde{r}_1\Gamma_p \left[1 + \frac{p-2}{2(p-1)t} + O\big(t^{-2}\big)\right].
\end{aligned}$$

Further,

$$\begin{aligned}
X_2 &= -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}(\tilde{\Omega}_1)\right)U_2 \\
&= -p\left(\frac{1}{p}t^{\frac{1}{p}-1}t^{\frac{1}{p}} + \tilde{r}_1^{1-q}\Phi^{-1}(\tilde{r}_1\Gamma_p)\right)\tilde{r}_1\Gamma_p\left(1 + \frac{p-2}{2(p-1)t} + O(t^{-2})\right) \\
&= \tilde{r}_1\Gamma_p\left(-\frac{1}{t} - \frac{p-2}{2(p-1)t^2} - (p-1) - \frac{p-2}{2t} + O(t^{-2})\right),
\end{aligned}$$

$$
\begin{aligned}
Y_2 &= -(p-1)R^{-1}f^p|\tilde{\Omega}_1|^q = -(p-1)\tilde{r}_1^{1-q}t\tilde{r}_1^q\gamma_p = -t(p-1)\gamma_p\tilde{r}_1,\\
Z_2 &= (p-1)R^{-1}f^{-q}|\Omega_2|^q = (p-1)\tilde{r}_1^{1-q}t^{-\frac{q}{p}}\tilde{r}_1^q\gamma_p t^q\left(1+\frac{1}{(p-1)t}\right)^p\\
&= (p-1)\gamma_p\tilde{r}_1 t + p\gamma_p\tilde{r}_1 + \frac{p}{2t}\gamma_p\tilde{r}_1 + O(t^{-2}).
\end{aligned}
$$

Hence, the absolute term in the resulting modified Riccati equation is

$$
\begin{aligned}
\frac{\tilde{c}_2(t)}{t} &:= X_2+Y_2+Z_2+U_2'+t\left(\sum_{j=1}^{n}\frac{B_j}{t^2\operatorname{Log}_{j-1}^2}\right)\\
&= \tilde{r}_1\left\{t\left[-(p-1)\gamma_p+(p-1)\gamma_p\right]+\left[-(p-1)\Gamma_p+p\gamma_p\right]\right.\\
&\quad \left.+\frac{1}{t}\left[-\Gamma_p-\frac{p-2}{2}\Gamma+\frac{p}{2}\gamma_p\right]+O(t^{-2})\right\}+\sum_{j=1}^{n}\frac{B_j}{t\operatorname{Log}_{j-1}^2 t}\\
&= \tilde{r}_1\left[-\frac{\mu_p}{t}+O(t^{-2})\right]+\frac{B_1}{t}+\ldots+\frac{B_n}{t\operatorname{Log}_{n-1}^2 t}\\
&= \frac{1}{t}(-\mu_p+B_1)+\frac{B_2}{t\log^2 t}+\ldots+\frac{B_n}{t\operatorname{Log}_{n-1}^2 t}+O(t^{-2}).
\end{aligned}
$$

Observe that the O term in U_2 and later in other U_j can be differentiated because of its special form. Hence, if $B_1=\mu_p$, we obtain the equation

$$
v_2'+\frac{B_2}{t\log^2 t}+\cdots+\frac{B_n}{t\operatorname{Log}_{n-1}^2 t}+O(t^{-2})+(p-1)\tilde{r}_1^{q-1}(t)t^{1-q}H(v_2,\Omega_2)=0.
$$

In this equation we apply again the change of independent variable $t\longmapsto e^t$ and the resulting equation is

$$
v_2'+c_2(t)+(p-1)\tilde{r}_1^{1-q}(e^t)e^{(2-q)t}H(v_2,\tilde{\Omega}_2)=0
$$

with $c_2(t)=\tilde{c}_1(e^t)$, $\tilde{\Omega}_2(t)=\Omega_2(e^t)$, and

$$
c_2(t)=\frac{B_2}{t^2}+\ldots+\frac{B_n}{t^2\operatorname{Log}_{n-2}^2 t}+O(e^{-t}).
$$

We use the notation

$$
\tilde{r}_2(t):=\tilde{r}_1(e^t)e^t,\ldots,\tilde{r}_k(t):=\tilde{r}_{k-1}(e^t)e^t \tag{3.18}
$$

in the next computations. With this notation, we have

$$v_2' + \frac{B_2}{t^2} + \dots \frac{B_n}{t^2 \operatorname{Log}_{n-2}^2 t} + O(e^{-t}) + (p-1)\tilde{r}_2^{1-q} e^t H(v_2, \tilde{\Omega}_2) = 0. \tag{3.19}$$

We apply the transformation $v_3 = t v_2 - U_3$ to (3.19). We obtain the equation

$$v_3' + \frac{\tilde{c}_3(t)}{t} + (p-1)\tilde{r}_2^{1-q} e^t t^{1-q} H(v_3, \Omega_3) = 0,$$

where, with $f(t) = t^{\frac{1}{p}}$ and $R^{-1}(t) = \tilde{r}_2^{1-q} e^t$,

$$\Omega_3 = f\Phi\big(Rf' + \Phi^{-1}(\tilde{\Omega}_2)\big) = \tilde{r}_2 t \Gamma_p \left(1 + \frac{1}{(p-1)e^t} + \frac{1}{(p-1)te^t}\right)^{p-1}$$

and

$$\frac{\tilde{c}_3(t)}{t} := X_3 + Y_3 + Z_3 + U_3' + t c_2(t) \tag{3.20}$$

with

$$\begin{aligned}
U_3 = -t\tilde{\Omega}_2 + \Omega_3 = t\tilde{r}_2 \Gamma_p &\left\{ -\left(1 + \frac{1}{(p-1)e^t}\right)^{p-1} \right.\\
&\left. + \left(1 + \frac{1}{(p-1)e^t} + \frac{1}{(p-1)te^t}\right)^{p-1} \right\}\\
= t\tilde{r}_2 \Gamma_p &\left\{ -\left(1 + \frac{1}{e^t} + \frac{p-2}{2(p-1)e^{2t}} + O\left(e^{-3t}\right)\right) \right.\\
&\left. + 1 + \frac{1}{e^t} + \frac{1}{te^t} + \frac{p-2}{2(p-1)e^{2t}}\left(1 + \frac{1}{t}\right)^2 + O\left(e^{-3t}\right) \right\}\\
= \tilde{r}_2 \Gamma_p &\left\{ \frac{1}{e^t} + \frac{p-2}{(p-1)e^{2t}} + \frac{p-2}{2(p-1)te^{2t}} + O\left(te^{-3t}\right) \right\},
\end{aligned}$$

$$
\begin{aligned}
X_3 = & -p\left(\frac{f'}{f}+R^{-1}\Phi^{-1}(\tilde{\Omega}_2)\right)U_3 \\
= & -\left[\frac{1}{t}+(p-1)e^t\left(1+\frac{1}{(p-1)e^t}\right)\right]\tilde{r}_2\Gamma_p \\
& \times\left[\frac{1}{e^t}+\frac{p-2}{(p-1)e^{2t}}+\frac{p-2}{2(p-1)te^{2t}}+O\left(te^{-3t}\right)\right] \\
= & -\tilde{r}_2\Gamma_p\left[\frac{1}{te^t}+\frac{p-2}{(p-1)te^{2t}}+\frac{p-2}{2(p-1)t^2e^{2t}}+(p-1)+\frac{p-2}{e^t}\right. \\
& \left.+\frac{p-2}{2te^t}+\frac{1}{e^t}+\frac{p-2}{(p-1)e^{2t}}+\frac{p-2}{2(p-1)te^{2t}}+O\left(te^{-2t}\right)\right] \\
= & -\tilde{r}_2\Gamma_p\left[(p-1)+\frac{p-1}{e^t}+\frac{p}{2te^t}+O\left(te^{-2t}\right)\right], \\
Y_3 = & -(p-1)\tilde{r}_2^{1-q}e^tt^{1-q}\tilde{\Omega}_2^q \\
= & -(p-1)\tilde{r}_2^{1-q}e^tt^{1-q}\tilde{r}_2^q\Gamma_p^qt^q\left[1+\frac{1}{(p-1)e^t}\right]^p \\
= & -(p-1)\gamma_p\tilde{r}_2e^tt\left[1+\frac{p}{(p-1)e^t}+\frac{p}{2(p-1)e^{2t}}+O\left(e^{-2t}\right)\right] \\
= & \tilde{r}_2[-(p-1)\gamma_pte^t-p\gamma_pt-\frac{p}{2}\gamma_p\frac{t}{e^t}+O\left(te^{-2t}\right)], \\
Z_3 = & (p-1)\tilde{r}_2^{1-q}e^tt^{1-q}\Omega_3^q \\
= & (p-1)\tilde{r}_2^{1-q}e^tt^{1-q}t^q\tilde{r}_2^q\gamma_p\left[1+\frac{1}{(p-1)e^t}+\frac{1}{(p-1)te^t}\right]^p \\
= & (p-1)t\tilde{r}_2\gamma_pe^t\left[1+\frac{p}{(p-1)e^t}+\frac{p}{(p-1)te^t}+\frac{p}{2(p-1)e^{2t}}\left(1+\frac{1}{t}\right)^2+O\left(e^{-3t}\right)\right] \\
= & \tilde{r}_2\left[(p-1)\gamma_pe^tt+p\gamma_pt+p\gamma_p+\frac{pt}{2e^t}\gamma_p+p\gamma_p\frac{1}{e^t}+\frac{p}{2}\gamma_p\frac{1}{te^t}+O\left(te^{-2t}\right)\right].
\end{aligned}
$$

Substituting into (3.20) the above computed quantities, we have

$$
\frac{\tilde{c}_3(t)}{t}=\frac{B_2-\mu_p}{t}+\frac{B_3}{t\log^2 t}+\cdots+\frac{B_n}{t\,\mathrm{Log}_{n-2}^2 t}+O(te^{-t}).
$$

Consequently, if $B_2=\mu_p$, we obtain the equation

$$
v_3'+\sum_{j=3}^{n}\frac{B_j}{t\,\mathrm{Log}_{j-2}^2 t}+O(te^{-t})+(p-1)\tilde{r}_2^{1-q}(t)e^tt^{1-q}H(v_3,\Omega_3(t))=0.
$$

In this equation, the change of independent variable $t \longmapsto e^t$ results the equation

$$v_3' + c_3(t) + (p-1)\tilde{r}_3^{1-q}(t)E_2(t)H(v_3, \tilde{\Omega}_3(t)) = 0, \quad c_3(t) := \tilde{c}_3(e^t).$$

Here, and also in the sequel, we use the notation

$$e_1(t) := e^t, \dots, e_n(t) := e_{n-1}(e^t), \quad E_n(t) := e_n(t) \cdots e_1(t)$$

where n is the integer in (3.14).

Now we are already in a position to make the induction step in transformations of modified Riccati equations. We suppose that $B_j = \mu_p$ for $j = 1, \dots, k-2$ for some $k \in \{3, \dots, n\}$, so we have the equation

$$v_{k-1}' + c_{k-1}(t) + (p-1)\tilde{r}_{k-1}^{1-q}(t)E_{k-2}(t)H(v_{k-1}, \tilde{\Omega}_{k-1}(t)) = 0$$

with

$$\tilde{\Omega}_{k-1}(t) = \Gamma_p \tilde{r}_{k-1}(t) \left(1 + \frac{1 + E_1(t) + \cdots + E_{k-3}(t)}{E_{k-2}(t)}\right)^{p-1}, \tag{3.21}$$

$$c_{k-1}(t) = \frac{B_{k-1}}{t^2} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-k+1}^2 t} + O\big(tE_{k-3}^3(t)/E_{k-2}(t)\big),$$

where $\tilde{r}_k$ is given by (3.18). We will also use the notation

$$r_k(t) := r_{k-1}(e^t), \quad r_1(t) := \tilde{r}_1(t) = r(e^t).$$

Then $\tilde{r}_k(t) = r_k(t)E_{k-1}(t)$ and $r_k(t) = r(e_k(t))$ with r given by (3.15).

We put $v_k = tv_{k-1} - U_k$. We have

$$\begin{aligned}
U_k =& -t\tilde{\Omega}_{k-1} + \Omega_k \\
=& t\tilde{r}_{k-1}\Gamma_p \left\{ -\left[1 + \frac{1+\cdots+E_{k-3}}{(p-1)E_{k-2}}\right]^{p-1} \right. \\
&+ \left. \left[1 + \frac{1+\cdots+E_{k-3}}{(p-1)E_{k-2}} + \frac{1}{(p-1)tE_{k-2}}\right]^{p-1} \right\} \\
=& r_{k-1}\Gamma_p \left[1 + \frac{(p-2)(1+\cdots+E_{k-3})}{(p-1)E_{k-2}} + \frac{p-2}{2(p-1)tE_{k-2}} + O\big(tE_{k-3}^3/E_{k-2}^2\big)\right]
\end{aligned}$$

and with $f(t) = t^{\frac{1}{p}}$, $R^{-1} = \tilde{r}_{k-1}^{1-q}E_{k-2}$ and $\tilde{\Omega}_{k-1}$ given by (3.21)

$$\begin{aligned}
X_k =& -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}(\tilde{\Omega}_{k-1})\right)U_k \\
=& -\tilde{r}_{k-1}\Gamma_p \left[\frac{1}{t} + (p-1)E_{k-2} + (1+\cdots+E_{k-3})\right] \\
&\times \left[\frac{1}{E_{k-2}} + \frac{(p-2)(1+\cdots+E_{k-3})}{(p-1)E_{k-2}^2} + \frac{p-2}{2(p-1)tE_{k-2}^2} + O\big(tE_{k-3}^3/E_{k-2}^2\big)\right], \\
=& -r_{k-1}\Gamma_p \left[(p-1)E_{k-2} + (p-1)(1+\cdots+E_{k-3}) + \frac{p}{2t} + \frac{(p-2)(1+\cdots+E_{k-3})^2}{(p-1)E_{k-2}} \right. \\
&+ \left. O\big(tE_{k-3}^3/E_{k-2}\big)\right], \\
Y_k =& -(p-1)R^{-1}f^p\tilde{\Omega}_{k-1}^q = -(p-1)\tilde{r}_{k-1}^{1-q}E_{k-2}t\tilde{r}_{k-1}^q\gamma_p\left(1 + \frac{1+\cdots+E_{k-3}}{(p-1)E_{k-2}}\right)^p \\
=& -r_{k-1}\left[(p-1)\gamma_p tE_{k-2}^2 - p\gamma_p tE_{k-2}(1+\cdots+E_{k-3}) \right. \\
&+ \left. \frac{p\gamma_p}{2}t(1+\cdots+E_{k-3})^2 + O\big(tE_{k-3}^3/E_{k-2}\big)\right], \\
Z_k =& (p-1)R^{-1}f^{-q}\tilde{\Omega}_{k-1}^q = (p-1)\tilde{r}_{k-1}E_{k-2}t\gamma_p \\
&\times \left[1 + \frac{(1+\cdots+E_{k-3})}{(p-1)E_{k-2}} + \frac{1}{(p-1)tE_{k-2}}\right]^p \\
=& r_{k-1}\left[(p-1)\gamma_p tE_{k-2}^2 + p\gamma_p tE_{k-2}(1+\cdots+E_{k-3}) + p\gamma_p E_{k-2} \right. \\
&+ \left. \frac{p\gamma_p}{2}t(1+\cdots+E_{k-3})^2 + p\gamma_p(1+\cdots+E_{k-3}) + \frac{p\gamma_p}{2t} + O\big(tE_{k-3}^3/E_{k-2}\big)\right].
\end{aligned}$$

Then, using that $(p-1)\Gamma_p = p\gamma_p$ and

$$\frac{p}{2}(\gamma_p - \Gamma_p) = -\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} = -\mu_p,$$

we have

$$X_k + Y_k + Z_k = -\frac{\mu_p}{t} + O\big(tE_{k-3}^3/E_{k-2}\big).$$

The last formula is the result of a direct computation where one needs to show that all terms with the faster growth than t^{-1} vanish. Further, again by a direct computation

$$U_k' = \left\{ r_{k-1}\Gamma_p \left[1 + \frac{(p-2)(1+\cdots+E_{k-3})}{(p-1)E_{k-2}} + \frac{p-2}{2(p-1)tE_{k-2}} + O\big(tE_{k-3}^3/E_{k-2}^2\big) \right] \right\}' \\ = O\big(tE_{k-3}^2/E_{k-2}\big).$$

Consequently, in the resulting modified Riccati equation for v_k

$$v_k' + \frac{\tilde{c}_k(t)}{t} + (p-1)R^{-1}(t)H(v_k, \Omega_k(t)) = 0 \tag{3.22}$$

with $R^{-1}(t) = \tilde{r}_{k-1}^{-1}(t)E_{k-2}(t)t^{1-q}$ we have

$$\frac{\tilde{c}_k(t)}{t} := X_k + Y_k + Z_k + U_k' + tc_{k-1} = \frac{B_{k-1} - \mu_p}{t} + \sum_{j=k}^{n} \frac{B_j}{t\,\mathrm{Log}_{j+1-k}^{\,2}} + O\big(tE_{k-3}^3/E_{k-2}\big)$$

as $t \to \infty$.

Now we can summarize the previous computations as follows.

Theorem 6. *Suppose that there exists $k \in \{2, \ldots, n\}$ such that*

$$\beta_j - \gamma_p \alpha_j = \mu_p, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}, \; j = 1, \ldots, k-1, \tag{3.23}$$

and $\beta_k - \gamma_p\alpha_k \neq 0$. Then (3.4) *is oscillatory if $\beta_k - \gamma_p\alpha_k > \mu_p$ and nonoscillatory if $\beta_k - \gamma_p\alpha_k < \mu_p$. If* (3.23) *holds for all $j = 1, \ldots, n$, equation* (3.4) *is nonoscillatory.*

Proof. We apply Theorem 3 to the modified Riccati equation (3.22) for v_k. In this equation, with the notation from Theorem 3,

$$\begin{aligned}\mathscr{R} &= R^{-1}|\Omega_k|^{q-2} \sim t^{q-2}\tilde{r}_{k-1}^{q-2}\Gamma_p^{q-2}\tilde{r}_{k-1}^{1-q}E_{k-2}t^{1-q}\\ &= r_{k-1}t^{-1}q^{p-2} \sim q^{p-2}t^{-1},\end{aligned}$$

hence $\int^t \mathscr{R}(s)\,ds \sim q^{p-2}\log t$ and

$$\int_t^\infty c_k(s)\,ds \sim \frac{B_k}{\log t}, \quad B_k = \beta_k - \alpha_k\gamma_p.$$

Recall that $f \sim g$ for a pair of functions f,g means $\lim_{t\to\infty}\frac{f(t)}{g(t)} = 1$. Consequently, if $B_k q^{p-2} > \frac{1}{2q}$, what happens if and only if $B_k > \mu_p$, modified Riccati equation (3.22) for v_k has no proper solution in view of Theorem 3 (iii). Now, via the "back" transformations

$$v_{j-1} = -U_j + \frac{v_j}{t}, \quad j = 2,\ldots,k, \quad w = -U_1 + t^{\frac{1-p}{p}}v_1,$$

the same holds for the Riccati equation associated with (3.4) and hence this equation is oscillatory by Proposition 1.

If $B_k < \mu_p$, nonoscillation of (3.4) follows from parts (i) (when $B_k < 0$) and (ii) (when $0 \le B_k < \mu_p$) of Theorem 3 since the existence of a proper solution of the modified Riccati equation for v_{k+1} implies the existence of a proper solution for the Riccati equation (3.16) associated with (3.4), hence this equation is nonoscillatory by Proposition 1.

Finally, if (3.23) holds for all $j = 1,\ldots,n$, then the absolute term in the modified Riccati equation for v_{n+1} is $d(t) := \frac{\tilde{c}_{n+1}(t)}{t} = O\big(tE_{n-2}^3(t)/E_{n-1}(t)\big)$ and replacing d by its nonnegative part $d^+ = \max\{0,d\}$, we get a majorant of the modified Riccati equation for v_{n+1} (in the sense of Theorem 4). The function d^+ satisfies the same asymptotic estimate as d. To estimate the integral $\int_t^\infty d^+(s)\,ds$ we proceed as follows. We have, via the substitution $e_{n-2}(s) = u$, $E_{n-2}(s)\,ds = du$, using the inequality $\log_j u \le u$, and followed by integration by parts,

$$\begin{aligned}\int_t^\infty \frac{sE_{n-2}^3(s)}{E_{n-1}(s)}\,ds &= \int_t^\infty \frac{s(e_1(s)\cdots e_{n-2}(s))^3}{e_{n-1}(s)\cdots e_1(t)}\,ds \\ &= \int_{e_{n-2}(t)}^\infty \frac{\log_{n-2} u \log_{n-3} u \ldots \log u \cdot u}{e^u}\,du \\ &\le \int_{e_{n-2}(u)}^\infty \frac{u^{n-1}}{e^u}\,du \\ &\sim -u^{n-1}e^{-u}\big|_{e_{n-2}(u)}^\infty = \frac{e_{n-2}^{n-1}(u)}{e_{n-1}(u)}.\end{aligned}$$

Consequently,

$$\lim_{t\to\infty} \log t \int_t^\infty d^+(s)\,ds = 0,$$

hence the modified Riccati equation for v_{n+1} with d^+ instead of $\frac{\tilde{c}_{n+1}(t)}{t}$ possesses a proper solution by Theorem 3 and, by Theorem 4, the Riccati equation for v_{n+1} has the same property. This implies that (3.4) is nonoscillatory using the same argument as in the previous part of the proof. □

Chapter 4

Perturbation with periodic functions

4.1 Formulation of the problem

One of the typical problems in the qualitative theory of various differential equations is to study what happens when constants in an equation are replaced by periodic functions. Our investigation follows this line and it is mainly motivated by the papers [6, 7, 9, 11, 12, 19, 23, 29, 30]. In [29, 30], linear second order differential equations with periodic coefficients were considered which using a transformation of dependent variable can be transformed into the equation of the form

$$(r(t)x')' + \frac{1}{t^2}\left[c(t) + \frac{d(t)}{\log^2 t}\right]x = 0 \tag{4.1}$$

with α-periodic functions r, c, d. It was shown that (4.1) behaves essentially in the same way as the classical Riemann-Weber equation where the functions r^{-1}, c, d are replaced by their mean values

$$\bar{r} = \frac{1}{\alpha}\int_0^\alpha r^{-1}(t)\,dt, \quad \bar{c} = \frac{1}{\alpha}\int_0^\alpha c(t)\,dt, \quad \bar{d} = \frac{1}{\alpha}\int_0^\alpha d(t)\,dt.$$

More precisely, equation (4.1) is nonoscillatory if $\bar{c}\bar{r} < 1/4$ and oscillatory if $\bar{c}\bar{r} > 1/4$. In the limiting case $\bar{c}\bar{r} = 1/4$, equation (4.1) is nonoscillatory if $\bar{d}\bar{r} < 1/4$ and oscillatory if $\bar{d}\bar{r} > 1/4$.

This result was extended in [23], where the second order linear differential equation

$$\left[r(t)+\sum_{j=1}^{n}\frac{\alpha_j(t)}{\operatorname{Log}_j^2(t)}x'\right]'+\left[\frac{c(t)}{t^2}+\sum_{j=1}^{n}\frac{\beta_j(t)}{t^2\operatorname{Log}_j^2(t)}\right]x=0$$

with periodic functions r, c, α_j, β_j, $j=1,\dots,n$, was considered. It was shown, using an averaging argument, that the oscillation result for (3.2) remains essentially to hold when the constants α_j,β_j, $j=1,\dots,n$, in (3.2) are replaced by periodic functions. The role of constants is taken in this result by the mean values of periodic functions $\alpha_j(t),\beta_j(t)$.

As a next step, the effort was concentrated to extend the previous linear results to half-linear equations. In [19], the equation

$$\big(r(t)\Phi(x')\big)'+\frac{c(t)}{t^p}\Phi(x)=0 \tag{4.2}$$

with α-periodic r,c was considered. Similarly to the linear case, it was shown that (4.2) is oscillatory provided $\bar{c}\bar{r}^{p-1}>\gamma_p$ and nonoscillatory when $\bar{c}\bar{r}^{p-1}<\gamma_p$,

$$\bar{r}=\frac{1}{\alpha}\int_0^\alpha r^{1-q}(t)\,dt,\quad \bar{c}=\frac{1}{\alpha}\int_0^\alpha c(t)\,dt,$$

the limiting case $\bar{c}\bar{r}^{p-1}=\gamma_p$ remained undecided in [19]. This problem was resolved in the later paper [12], we will mention this result later.

In this chapter we consider the equation

$$\left[\left(r(t)+\sum_{j=1}^{n}\frac{\alpha_j(t)}{\operatorname{Log}_j^2(t)}\right)^{1-p}\Phi(x')\right]'+\left[\frac{c(t)}{t^p}+\sum_{j=1}^{n}\frac{\beta_j(t)}{t^p\operatorname{Log}_j^2(t)}\right]\Phi(x)=0. \tag{4.3}$$

with T-periodic functions r,c,α_j,β_j, $r(t)>0$. One of the reasons why we consider the coefficient of $\Phi(x')$ in the power $1-p$ is that then this equation can be written as the first order system

$$x'=\left(r(t)+\sum_{j=1}^{n}\frac{\alpha_j(t)}{\operatorname{Log}_j^2(t)}\right)\Phi^{-1}(u),\quad u'=-\frac{1}{t^p}\left(c(t)+\sum_{j=1}^{n}\frac{\beta_j(t)}{\operatorname{Log}_j^2(t)}\right)\Phi(x)$$

and perturbation terms in both equations of this system have essentially the same form.

The main statement of this chapter is based on the result of the previous chapter presented in Theorem 6. We show, similarly to the above mentioned papers, that constants α_j, β_j in (3.4) can be replaced by periodic functions and the resulting oscillation formula is essentially the same as that one in Theorem 6, only constants are replaced by mean values of periodic functions appearing in perturbation terms.

We start with a slight modification of Theorem 6. We reformulate it in such a way that it is applicable to equation (4.3) with constants α_j, β_j instead of T-periodic functions.

Proposition 2. *Suppose that there exists $k \in \{1, \dots, n\}$ such that*

$$\beta_j + (p-1)\gamma_p \alpha_j = \mu_p, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}, \ j = 1, \dots, k-1$$

(drop this condition for $k = 1$) and $\beta_k + (p-1)\gamma_p \alpha_k \neq \mu_p$. Then the equation

$$\left[\left(1 + \sum_{j=1}^{n} \frac{\alpha_j}{Log_j^2 t}\right)^{1-p} \Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p \, Log_j^2 t}\right] \Phi(x) = 0$$

is oscillatory if $\beta_k + (p-1)\gamma_p \alpha_k > \mu_p$ and nonoscillatory if $\beta_k + (p-1)\gamma_p \alpha_k < \mu_p$.

Using the binomial expansion

$$\left(1 + \sum_{j=1}^{n} \frac{\alpha_j}{\mathrm{Log}_j^2(t)}\right)^{1-p} = 1 + (1-p)\left(\sum_{j=1}^{n} \frac{\alpha_j}{\mathrm{Log}_j^2(t)}\right) + O(\log^{-4} t) \quad \text{as } t \to \infty$$

it is not difficult to see that Theorem 6 can be really reformulated as stated in Proposition 2.

We will need in the proof of the main result of this chapter the following modification of the Prüfer transformation from Section 1.2.

Let x be a nontrivial solution of (1.1) and consider the modified half-linear Prüfer transformation

$$x(t) = \rho(t) \sin_p \varphi(t), \quad r^{q-1}(t) x'(t) = \frac{\rho(t)}{t} \cos_p \varphi(t).$$

Then the angular variable φ satisfies the differential equation

$$\varphi' = \frac{1}{t}\left[r^{1-q}(t)|\cos_p \varphi|^p - \Phi(\cos_p \varphi)\sin_p \varphi + \frac{t^p c(t)}{p-1}|\sin_p \varphi|^p\right],$$

see [12].

The proof of our main result relies on the following averaging lemma, which can be found in [12], see also [23, Sec. 5], [30, Proposition 2 in the linear case $p = 2$].

Lemma 1. *Let φ be a solution of the equation*

$$\varphi' = \frac{1}{t}\left[a(t)|\cos_p \varphi|^p - \Phi(\cos_p \varphi)\sin_p \varphi + b(t)|\sin_p \varphi|^p\right]$$

with bounded functions $a(t)$ and $b(t)$, $a(t) > 0$, and let $T > 0$. Denote

$$\theta(t) := \frac{1}{T}\int_t^{t+T} \varphi(s)\, ds.$$

Then θ is a solution of the equation

$$\theta' = \frac{1}{t}\left[A(t)|\cos_p \theta|^p - \Phi(\cos_p \theta)\sin_p \theta + B(t)|\sin_p \theta|^p\right] + O\left(\frac{1}{t^2}\right) \tag{4.4}$$

with

$$A(t) = \frac{1}{T}\int_t^{t+T} a(\tau)\, d\tau, \quad B(t) = \frac{1}{T}\int_t^{t+T} b(\tau)\, d\tau, \tag{4.5}$$

and $\varphi(t) - \theta(t) = o(1)$ as $t \to \infty$.

The term $O(t^{-2})$ in (4.4) can also be written as (compare (1.3))

$$(|\cos_p \theta|^p + |\sin_p \theta|^p)O(t^{-2}),$$

hence equation (4.4) can be rewritten into the form considered later on

$$\theta' = \frac{1}{t}\left[\big(A(t) + O(t^{-1})\big)|\cos_p \theta|^p - \Phi(\cos_p \theta)\sin_p \theta + \big(B(t) + O(t^{-1})\big)|\sin_p \theta|^p\right]. \tag{4.6}$$

4.2 Main result

The results of this section are taken from [10].

The formulation of Lemma 1 from the previous section shows another reason why we consider the perturbations of Euler equation in the form as appears in (4.3), in particular, why we consider the term by $\Phi(x')$ with the power $1-p$. With this power (since $(1-p)(1-q)=1$), the function A in (4.5) is just the mean value over the interval $[t,t+T]$ of the function $r(t)+\sum_{j=1}^{n}\frac{\alpha_j(t)}{\mathrm{Log}_j^2(t)}$. Our main result reads as follows.

Theorem 7. *Let r,c and α_j,β_j, $j=1,\dots,n$, be T-periodic continuous functions, $r(t)>0$, and denote by $\bar r,\bar c$, $\bar\alpha_j,\bar\beta_j$, $j=1,\dots,n$, their mean values over the period T.*

(i) *If $\bar c\bar r^{p-1}>\gamma_p$, then* (4.3) *is oscillatory and if $\bar c\bar r^{p-1}<\gamma_p$, then it is nonoscillatory.*

(ii) *Let $\bar c\bar r^{p-1}=\gamma_p$. If there exists $k\in\{1,\dots,n\}$ such that*

$$\bar\beta_j\bar r^{p-1}+(p-1)\gamma_p\bar\alpha_j\bar r^{-1}=\mu_p,\quad j=1,\dots,k-1$$

(drop this condition if $k=1$) and $\bar\beta_k\bar r^{p-1}+(p-1)\gamma_p\bar\alpha_k\bar r^{-1}\neq\mu_p$, then (4.3) *is oscillatory if*

$$\bar\beta_k\bar r^{p-1}+(p-1)\gamma_p\bar\alpha_k\bar r^{-1}>\mu_p \tag{4.7}$$

and nonoscillatory if

$$\bar\beta_k\bar r^{p-1}+(p-1)\gamma_p\bar\alpha_k\bar r^{-1}<\mu_p. \tag{4.8}$$

Proof. First of all, let us note that the statement (i) is given for completeness, it is proved in [19], see also the text below (4.2). The statement (ii) for $n=1$, $\alpha_1=0$ is the main result of [12]. It remains to prove the statement (ii) in full generality.

Let x be a nontrivial solution of (4.3) and let φ be its Prüfer angle, i.e., the solution x of (4.3) and its quasiderivative are given by the formulas

$$x(t)=\rho(t)\sin_p\varphi(t),\qquad \left(r(t)+\sum_{j=1}^{n}\frac{\alpha_j(t)}{\mathrm{Log}_j^2(t)}\right)^{-1}x'=\frac{\rho(t)}{t}\cos_p\varphi(t).$$

Then $\varphi(t) = 0 \pmod{\pi_p}$ if and only if $x(t) = 0$ and for large t

$$\varphi'(t) = \frac{1}{t}\left(r(t) + \sum_{j=1}^{n} \frac{\alpha_j(t)}{\operatorname{Log}_j^2(t)}\right)|\cos_p \varphi|^p = \frac{1}{t}\left(r(t) + \sum_{j=1}^{n} \frac{\alpha_j(t)}{\operatorname{Log}_j^2(t)}\right) > 0$$

at these points (see the below equation (4.9)). Hence, (4.3) is oscillatory if and only if $\varphi(t)$ is unbounded as $t \to \infty$ and this happens, by Lemma 1, if and only its mean value over the interval $[t, t+T]$ $\theta(t) = \frac{1}{T}\int_t^{t+T} \varphi(s)\,ds$ is unbounded.

The function φ is a solution of the differential equation

$$\begin{aligned} \varphi' = \frac{1}{t}\Bigg[&\left(r(t) + \sum_{j=1}^{n} \frac{\alpha_j(t)}{\operatorname{Log}_j^2(t)}\right)|\cos_p \varphi|^p - \Phi(\cos_p \varphi)\sin_p \varphi \\ &+ \frac{1}{p-1}\left(c(t) + \sum_{j=1}^{n} \frac{\beta_j(t)}{\operatorname{Log}_j^2(t)}\right)|\sin_p \varphi|^p \Bigg], \end{aligned} \tag{4.9}$$

i.e., in differential equation (4.4) we have (compare (4.5))

$$A(t) = \frac{1}{T}\int_t^{t+T} \left(r(s) + \sum_{j=1}^{n} \frac{\alpha_j(s)}{\operatorname{Log}_j^2(s)}\right) ds, \tag{4.10}$$

$$B(t) = \frac{1}{(p-1)T}\int_t^{t+T} \left(c(s) + \sum_{j=1}^{n} \frac{\beta(s)}{\operatorname{Log}_j^2(s)}\right) ds. \tag{4.11}$$

Let f be a continuous T periodic function and $\bar{f} = \frac{1}{T}\int_0^T f(s)\,ds$ be its mean value over the period, then integration by parts yields

$$\begin{aligned} \frac{1}{T}\int_t^{t+T} \frac{f(s)}{\operatorname{Log}_j^2(s)}\,ds &= \frac{1}{T\operatorname{Log}_j^2(s)}\int_t^s f(u)\,du\Bigg|_t^{t+T} \\ &\quad - \frac{1}{T}\int_t^{t+T}\left[\left(\frac{1}{\operatorname{Log}_j^2(s)}\right)'\int_t^s f(u)\,du\right] ds \\ &= \frac{\bar{f}}{\operatorname{Log}_j^2(t)} + \bar{f}\left[\frac{1}{\operatorname{Log}_j^2(t+T)} - \frac{1}{\operatorname{Log}_j^2(t)}\right] \\ &\quad - \frac{1}{T}\int_t^{t+T}\left[\left(\frac{1}{\operatorname{Log}_j^2(s)}\right)'\int_t^s f(u)\,du\right] ds. \end{aligned} \tag{4.12}$$

Since the function f is bounded, there exists a constant $K > 0$ such that

$$\left| \int_t^s f(s)\, ds \right| \leq K, \quad \text{for } t \leq s \leq t+T,$$

and hence we can estimate the last term in the previous computation as follows

$$\begin{aligned} &\left| \int_t^{t+T} \left[\left(\frac{1}{\operatorname{Log}_j^2(s)} \right)' \int_t^s f(u)\, du \right] ds \right| \leq K \left[\frac{1}{\operatorname{Log}_j^2(t+T)} - \frac{1}{\operatorname{Log}_j^2(t)} \right] \\ =&KT \left(\frac{1}{\operatorname{Log}_j^2(t)} \right)' \bigg|_{t=\xi} = \frac{-KT}{\xi \log \xi \operatorname{Log}_j^2(\xi)} [1 + o(1)] = O\left(\frac{1}{t \log t \operatorname{Log}_j^2(t)} \right) \end{aligned}$$

as $t \to \infty$, $\xi \in (t, T+t)$. Here we have used that

$$\left(\frac{1}{\operatorname{Log}_j^2(t)} \right)' = -\frac{1}{t \log t \operatorname{Log}_j^2(t)} (1 + o(1)), \quad j = 1, \dots, n,$$

as can be verified by a direct computation. The same argument shows that also the term in brackets in (4.12) has the same asymptotic behavior as $t \to \infty$. Altogether, we have

$$\frac{1}{T} \int_t^{t+T} \frac{f(s)}{\operatorname{Log}_j^2(s)} ds = \frac{\bar{f}}{\operatorname{Log}_j^2(t)} + O\left(\frac{1}{t \log t \operatorname{Log}_j^2(t)} \right) = \frac{\bar{f}}{\operatorname{Log}_j^2(t)} \left(1 + O\left(\frac{1}{t \log t} \right) \right).$$

This implies that the functions A and B in (4.10), (4.11) are

$$\begin{aligned} A(t) &= \bar{r} + \left[1 + O\left(t^{-1} \log^{-1} t \right) \right] \sum_{j=1}^n \frac{\bar{\alpha}_j}{\operatorname{Log}_j^2(t)}, \\ B(t) &= \frac{1}{p-1} \left\{ \bar{c} + \left[1 + O\left(t^{-1} \log^{-1} t \right) \right] \sum_{j=1}^n \frac{\bar{\beta}_j}{\operatorname{Log}_j^2(t)} \right\}. \end{aligned}$$

Hence, substituting into (4.6), we obtain

$$\theta' = \frac{1}{t}\Bigg\{\left[\bar{r} + \left(1 + O(t^{-1}\log^{-1}t)\right)\sum_{j=1}^{n}\frac{\bar{\alpha}_j}{\operatorname{Log}_j^2(t)} + O(t^{-1})\right]|\cos_p\theta|^p$$
$$-\Phi(\cos_p\theta)\sin_p\theta$$
$$+\frac{1}{p-1}\left[\bar{c} + \left(1 + O(t^{-1}\log^{-1}t)\right)\sum_{j=1}^{k-1}\frac{\bar{\beta}_j}{\operatorname{Log}_j^2(t)} + O(t^{-1})\right]|\sin_p\theta|^p\Bigg\}.$$

Now, since all terms

$$\frac{O\left(\frac{1}{t\log t}\right)}{\operatorname{Log}_j^2(t)},\ j = 1,\dots,n, \quad \text{and} \quad O(t^{-1}) \quad \text{as } t\to\infty$$

are asymptotically less than $\frac{o(1)}{\operatorname{Log}_n^2(t)}$, we obtain the differential equation for θ which can be written in the form

$$\theta' = \frac{1}{t}\left[\left(\bar{r} + \sum_{j=1}^{n}\frac{\bar{\alpha}_j}{\operatorname{Log}_j^2(t)} + \frac{o(1)}{\operatorname{Log}_n^2(t)}\right)|\cos_p\theta|^p - \Phi(\cos_p\theta)\sin_p\theta\right.$$
$$\left.+\frac{1}{p-1}\left(\bar{c} + \sum_{j=1}^{n}\frac{\bar{\beta}_j}{\operatorname{Log}_j^2(t)} + \frac{o(1)}{\operatorname{Log}_n^2(t)}\right)|\sin_p\theta|^p\right].$$

This equation is a “Prüfer angle” equation for the second order half-linear differential equation

$$\left[\left(\bar{r}+\sum_{j=1}^{n}\frac{\bar{\alpha}_j}{\operatorname{Log}_j^2(t)} + \frac{o(1)}{\operatorname{Log}_n^2(t)}\right)^{1-p}\Phi(x')\right]'$$
$$+\frac{1}{t^p}\left(\bar{c}+\sum_{j=1}^{n}\frac{\bar{\beta}_j}{\operatorname{Log}_j^2(t)}+\frac{o(1)}{\operatorname{Log}_n^2(t)}\right)\Phi(x) = 0. \tag{4.13}$$

which is the same as the equation

$$\left[\left(1+\sum_{j=1}^{n}\frac{\bar\alpha_j/\bar r}{\operatorname{Log}_j^2(t)}+\frac{o(1)}{\operatorname{Log}_n^2(t)}\right)^{1-p}\Phi(x')\right]' \\ +\frac{1}{t^p}\left(\bar c\bar r^{p-1}+\sum_{j=1}^{n}\frac{\bar\beta\bar r^{p-1}}{\operatorname{Log}_j^2(t)}+\frac{o(1)}{\operatorname{Log}_n^2(t)}\right)\Phi(x)=0. \tag{4.14}$$

Suppose that assumptions (ii) of Theorem 7 are satisfied and that (4.7) holds for $k\in\{1,\dots,n-1\}$. Then equation (4.14) is oscillatory as a direct consequence of Proposition 2. If (4.7) holds for $k=n$, let $\varepsilon>0$ be so small that still

$$\bar r^{p-1}\bar\beta_n-\varepsilon+(p-1)\gamma_p\left(\bar r^{-1}\bar\alpha_n-\varepsilon\right)>\mu_p. \tag{4.15}$$

and consider the equation

$$\left[\left(1+\sum_{j=1}^{n-1}\frac{\bar\alpha_j/\bar r}{\operatorname{Log}_j^2(t)}+\frac{\bar\alpha_n/\bar r-\varepsilon}{\operatorname{Log}_n^2(t)}\right)^{1-p}\Phi(x')\right]' \\ +\frac{1}{t^p}\left(\bar c\bar r^{p-1}+\sum_{j=1}^{n-1}\frac{\bar\beta_j\bar r^{p-1}}{\operatorname{Log}_j^2(t)}+\frac{\bar\beta_n\bar r^{p-1}-\varepsilon}{\operatorname{Log}_j^2(t)}\right)\Phi(x)=0.$$

This equation is a Sturmian minorant for t sufficiently large (when the $o(1)$ term in (4.14) is less than ε) of (4.14) and (4.15) implies by Proposition 2 that this minorant equation is oscillatory and hence (4.14) (which is the same as (4.13)) is oscillatory as well. This means that the Prüfer angle θ of a solution of (4.13) is unbounded and by Lemma 1 the Prüfer angle φ of a solution of (4.3) is unbounded as well. This means that (4.3) is oscillatory. A slightly modified argument implies that (4.3) is nonoscillatory provided (4.8) holds. □

Chapter 5

Concluding remarks

(i) In equation (2.3), the functions $\tilde{r}(t) = \frac{1}{\log^2 t}$, $\tilde{c}(t) = \frac{1}{t^p \log^2 t}$ "match together", i.e., for $r(t) = 1$ and $c(t) = \gamma_p t^{-p}$ they have such asymptotic growth for $t \to \infty$ that equation (2.3) is conditionally oscillatory. This fact is a special case of the general situation which is a subject of a possible next investigation. More precisely, given the functions r, c, we look for functions $\tilde{r}, \tilde{c}$ with such asymptotic growth that equation (2.1) is conditionally oscillatory. For $\tilde{r} = 0$, this problem has been studied in [14, 15], where conditions on unperturbed equation (1.1) is found under which its perturbation

$$\big(r(t)\Phi(x')\big)' + \left[c(t) + \frac{\mu}{h^p(t)R(t)\big(\int^t R^{-1}(s)\,ds\big)^2}\right]\Phi(x) = 0$$

is conditionally oscillatory (and its oscillation constant is $\mu_0 = \frac{1}{2q}$, where q is the conjugate exponent to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Here h is the so-called principal solution of (1.1) and $R = rh^2|h'|^{p-2}$. In [7], the explicit formula for the function $\tilde{r}$ is found in such a way that together with the function

$$\tilde{c}(t) = \frac{1}{h^p(t)R(t)\big(\int^t R^{-1}(s)\,ds\big)^2}$$

equation (2.1) is conditionally oscillatory. This formula is

$$\tilde{r}(t) = \frac{1}{|h'(t)|^p \tilde{R}(t)\big(\int^t \tilde{R}^{-1}(s)\,ds\big)^2},$$

where

$$\tilde{G} := c^{1-q} g \Phi^{-1}(g') = -rh\Phi(h'), \quad \tilde{R} := c^{1-q} g^2 |g'|^{q-2} = \frac{r^2 |h'|^{2p-2}}{ch^{p-2}}.$$

However, the results of that paper are proved under rather restrictive assumptions and it would be interesting to know how to relax these restrictions.

(ii) In [31], the authors establish a "power comparison theorem" for the Riemann-Weber half-linear equation

$$\left(\Phi(x')\right)' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right] \Phi(x) = 0. \tag{5.1}$$

They proved a (non)oscillation criterion for this equation where this equation is compared with an equation of the same form, but with a different power in the function Φ and other functions and constants appearing in (5.1). It suggests to investigate a similar problem for the more general equation (2.3). Partial results along this line can be found in [3, 8].

(iii) In Chapter 3, we applied successively the transformation $v_k = tv_{k-1} - U_k$ to the modified Riccati equation, followed by the change of independent variable $t \longmapsto e^t$. This change of the independent variable was motivated by the linear case and also by the fact that upon this transformation the modified Riccati equation simplifies. Without this change of independent variable, the transformation procedure can be "reformulated" as follows. As showed at the beginning of Chapter 3, the transformation (1.11), i.e., $v = h^p(t)w - G(t)$, $G(t) = r(t)h(t)\Phi(h'(t))$, transforms Riccati equation (1.10) associated with (1.1) into the modified Riccati equation (1.12). The transformation (3.10), i.e., $z = f^p(t)v - U(t)$, transforms (1.12) into an equation of the same form, with the function $\tilde{C}$ given by a relatively complicated formula (3.12). The composition of these transformations gives

$$z = (f(t)h(t))^p w - (f^p(t)G(t) + U(t))$$

and by a direct computation, using (3.13), we have $f^p G + U = rfh\Phi((fh)')$. So, the resulting modified Riccati equation for z is just the modified Riccati equation resulting from (1.10) via (1.11) with h replaced by fh. In this equation, the function $\tilde{c}$ is given by (1.14) with h replaced by fh, i.e. $\tilde{c} = fh[(r\Phi((fh)')' + c\Phi(fh)]$.

Now, consider the function

$$h(t) = t^{\frac{p-1}{p}} (\operatorname{Log}_n t)^{\frac{1}{p}} = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \cdots \log_n^{\frac{1}{p}} t. \tag{5.2}$$

In view of the previous consideration, the application of transformation (1.11) with this h can be decomposed into the successive transformations $v_1 = t^{p-1}w - G$, $v_j = \log_{j-1} t v_j - U_j$, $j = 2, \dots, k$. Hence, the successive transformations treated in the previous chapter can be replaced by just one transformation, with the transformation function (5.2).

This idea has been used in Chapter 2 in the case $n = 1$ in (3.4) and in (5.2). However, as shows the computations in Section 2.2 (where also substantially the results of [5] have been used), this method is technically complicated even in this relatively simple case. This is also the reason why we developed the method of successive transformations of modified Riccati equation presented in Section 3.2.

(iv) The reason why the perturbation terms in (3.4) are just $\frac{\alpha_j}{\operatorname{Log}_j^2 t}$ in the differential term and $\frac{\beta_j}{t^p \operatorname{Log}_j^2 t}$ by $\Phi(x)$ is motivated by the fact that in this form they "match together", similarly as in the simple case mentioned in the part (i) of this chapter. More precisely, if we replace some of them by a term with a faster asymptotic growth, then this term "overrules" remaining terms and equation becomes (non)oscillatory for any positive value of the corresponding parameter α_j or β_j. On the other hand, functions with slower asymptotic growth have no influence on the oscillatory behavior. These considerations are closely related to concepts of strong (non)oscillation of half-linear equations as treated for example in [25].

(v) In [7], and partially also in [6], the equation

$$\left[(r(t) + \lambda \tilde{r}(t))\Phi(x')\right]' + \left[c(t) + \mu \tilde{c}(t)\right]\Phi(x) = 0 \tag{5.3}$$

is considered as a perturbation of (1.1). Assumptions on the functions $r, \tilde{r}, c, \tilde{c}$ (which are satisfied in case of the perturbed Euler equation) were found, which guarantee that there exists a constant γ such that (5.3) is oscillatory if $\mu - \lambda > \gamma$ and nonoscillatory if $\mu - \lambda < \gamma$. The limiting case $\mu - \lambda = \gamma$ remained undecided, mainly because of technical computational problems. In view of perturbations of Euler equation with $n = 1$, $r(t) = 1$,

$\tilde{r}(t) = \gamma_p \log^{-2} t$, $c(t) = t^{-p}$, $\tilde{c}(t) = t^{-p} \log^{-2} t$ (then $\gamma = \mu_p$) we conjecture that (5.3) is nonoscillatory also in the limiting case $\mu - \lambda = \gamma$. We also hope that the method of transformations of modified Riccati equation elaborated in Section 3.2 can be applied to treat the "multiparametric" general case, not only for perturbations of Euler equation.

(vi) A possible new research direction is to investigate oscillatory properties of (4.3) for a larger class of functions r, c, α_j, β_j than are periodic ones. A first step in this direction has been made in [20], where oscillatory properties of (4.3) with asymptotically almost periodic functions r, c are investigated. It is an open problem whether results of [20] can be extended to the setting treated in Chapter 4.

(vii) Another open problem is the investigation of asymptotic properties of the perturbed Euler equation in the framework of regularly varying functions. It would be interesting to see how far the results of the papers [22, 24, 27, 28] can be extended to (4.3) with regularly varying functions r, c, α_j, β_j.

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