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CHAPTER 1

Introduction

In this thesis we investigate oscillatory and asymptotic properties of solutions of the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{1.1}$$

where $\Phi(x) := |x|^{p-1} \operatorname{sgn} x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$. This equation can be regarded as a generalization of the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0 \tag{1.2}$$

and it was shown that in the concept of oscillation theory their solutions behave very similarly. Some results like Sturmian separation and comparison theorems extend from the linear theory to the half-linear theory almost verbatim. Note, that the term "half-linear equations" is motivated by the fact that the solution space of (1.1) has just one of the properties of linearity, concretely homogenity, but not aditivity. This fact brings a new phenomena to the oscillation theory because the lack of aditivity does not permit us to use the same proving techniques as in the linear case.

The first appearence of half-linear equations in the form of (1.1) can be found in the papers of Bihary [**2, 3**]. Oscillation theory of (1.1) was developed in the second half of the last century when many authors devoted attention to half-linear equations, namely Elbert and Mirzov established oscillation theory of (1.1) in their papers [**15, 29**]. The investigation of solutions of (1.1) continues even in the new millennium and the summary of already known results for half-linar equations was made in 2005 in the monograph [**11**] by Došlý and Řehák.

The central idea which goes through the entire thesis is the concept of perturbations. We regard equation (1.1) as a perturbation of another half-linear equation in the same form to get some results for a class of half-linear equations for which the classical approach fails.

This thesis is organized as follows. In Chapter 2 we recall basic results of the theory of half-linear equation (1.1). We start with the oscillatory properties of this equation, in particular, we present the half-linear Prüfer transformation, which plays an important role in the existence and uniqueness theory of (1.1). Next, we introduce the half-linear Riccati type equation associated to (1.1). We formulate the so called Picone's identity in the form as needed in our computations. In the next part of this chapter we present the Roundabout theorem which plays the fundamental role in proofs of the majority of results of the half-linear oscillation theory. We also formulate Sturmian separation and comparison theorems. Further, we recall the concept of the principal solution of (1.1) and its basic properties. We conclude Chapter 2 with the introduction to the approach dealing with perturbations and we derive the modified Riccati equation which plays the crucial role in the proofs of our results.

Chapter 3 considers Hille-Wintner type comparison criteria. First we formulate the known results for linear and half-linear differential equations in the classical form and then we introduce the generalization of these kind of statements for (1.1) regarded as a perturbation of another nonoscillatory half-linear equation (as it was published in [**9**]). Some immediate consequences are included too.

In Chapter 4 we present some results based on the Hartman-Wintner theorem for perturbed equation (1.1). Again, we summarize the Harman-Wintner type theorems and then we introduce a generalized criterion (published in [**32**]), which gives a sufficient qualitative condition of oscillation. We conclude this chapter with some Q-type criteria.

The last chapter of this thesis deals with asymptotics of some solutions of equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) + g(t)\Phi(x) = 0,$$

seen again as a perturbation. We derive also an improved asymptotic formula for the solution of the so called Euler-Weber type half-linear equation (5.5). These results were formulated in the paper [**31**].

CHAPTER 2

Elements of half-linear oscillation theory

2.1. Sturmian theory

This section contains an introduction to the oscillation theory for half-linear differential equations. The statements as well as its proofs can be found in [**11**] and [**6**].

2.1.1. Existence and uniqueness

In proving the existence and uniqueness result for (1.1), the fundamental role is played by the generalized half-linear Prüfer transformation. Consider a special half-linear equation

$$(\Phi(x'))' + (p-1)\Phi(x) = 0 \tag{2.1}$$

and denote by $S = S(t)$ its solution given by the initial conditions $S(0) = 0, S'(0) = 1$. Multiplying (2.1) (with x replaced by S) and using the fact that $(\Phi(S'))' = (p-1)|S'|^{p-2}S''$, we get the identity $(|S'|^p + |S|^p)' = 0$. Substituing here $t = 0$ and using the initial condition for S we have the generalized Pythagorian identity

$$|S'|^p + |S|^p \equiv 1. \tag{2.2}$$

The function S is positive in some right neighbourhood of $t = 0$ and using (2.2) $S' = \sqrt[p]{1-S^p}$ i.e., $\frac{dS}{\sqrt[p]{1-S^p}} = dt$, hence

$$t = \int_0^{S(t)} (1-s^p)^{-\frac{1}{p}}\, ds. \tag{2.3}$$

The formula (2.3) defines uniquely the function $S = S(t)$ on $[0, \pi_p/2]$ with $S(\pi_p/2) = 1$ and hence by (2.2) $S'(\pi_p/2) = 0$, where

$$\pi_p = \frac{2\pi}{p \sin\frac{\pi}{p}}.$$

Let us define the generalized sine function $\sin_p$ on the whole real line as the odd $2\pi_p$ periodic continuation of the function

$$\sin_p(t) = \begin{cases} S(t) & 0 \le t \le \frac{\pi_p}{2} \\ S(\pi_p - t) & \frac{\pi_p}{2} \le t \le \pi_p. \end{cases}$$

and denote the derivation of $\sin_p t$ by $\cos_p t$. The behavior of these functions is similar to that of the classical sine and cosine function. Half-linear tangent and cotangent functions are then defined by

$$\tan_p t = \frac{\sin_p t}{\cos_p t}, \quad \cot_p t = \frac{\cos_p t}{\sin_p t}.$$

Half-linear Prüfer transformation uses these trigonometric functions and its iverses in the following manner. Let x be a solution of (1.1) and let q be a conjugate number of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Put

$$\rho(t)=\sqrt[p]{|x(t)|^p+r^q(t)|x'(t)|^p}$$

and let $\varphi(t)$ be a continuous function defined at all points where $x(t)\neq 0$ by the formula

$$\varphi(t)=\operatorname{arccot}_p\frac{r^{1-q}(t)x'(t)}{x(t)}.$$

The inverse transformation has then the form

$$x(t)=\rho(t)\sin_p\varphi(t),\quad x'(t)=r^{1-q}\rho(t)\cos_p\varphi(t) \tag{2.4}$$

and the equation (1.1) is equivalent to the first order system

$$\begin{aligned}\varphi' =&\frac{c(t)}{p-1}|\sin_p\varphi|^p+r^{1-q}(t)|\cos_p\varphi|^p,\\ \rho' =&\Phi(\sin_p\varphi)\cos_p\varphi\left[r^{1-q}(t)-\frac{c(t)}{p-1}\right]\rho.\end{aligned}$$

Since the right-hand side of the system is Lipschitzian in ρ,φ, the initial value problem for this system is uniquely solvable and its solution exists on the whole interval, where r,c are continuous and $r(t)>0$. Hence the same holds for (1.1):

Theorem 2.1. *Suppose that the functions r,c are continuous in an interval $I\subseteq\mathbb{R}$ and $r(t)>0$ for $t\in I$. Given $t_0\in I$ and $A,B\in\mathbb{R}$, there exists a unique solution of (1.1) satisfying $x(t_0)=A, x'(t_0)=B$ which is extensible over the whole interval I. This solution depends continuously on the initial values A,B.*

2.1.2. Riccati equation

Let x be a solution of (1.1). Then the function $w=r\Phi(x'/x)$ solves the Riccati-type differential equation

$$w'+c(t)+(p-1)r^{1-q}(t)|w|^q=0. \tag{2.5}$$

Indeed, in view of (1.1) we have

$$\begin{aligned}w'&=\frac{(r\Phi(x'))'\Phi(x)-(p-1)r\Phi(x')|x|^{p-2}x'}{\Phi^2(x)}=-c-(p-1)\frac{r|x'|^p}{|x|^p}\\ &=-c-(p-1)r^{1-q}|w|^q.\end{aligned}$$

2.1.3. Picone's identity

The original Picone's identity for the linear second order equation (1.2) was established in 1910. Since then, it has been extended and generalized in many ways. We introduce the half-linear version of this identity proved in [**20**] and already simplified as needed in our purpose.

Lemma 2.1. *Suppose that w is a solution of (2.5) defined on the interval $I=[a,b]$. Then for any $y\in C^1(I)$ the following identity holds:*

$$r(t)|x'|^p-c(t)|x|^p=(w(t)|x|^p)'+pr^{1-q}(t)P(r^{q-1}(t)x',\Phi(x)w(t)), \tag{2.6}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0 \tag{2.7}$$

with the equality $P(u,v) = 0$ if and only if $v = \Phi(u)$.

We shall frequently use the following lemma (proved for exapmle in [**12**]) to estimate some terms containing the function $P(u,v)$ from below and from above with quadratic functions.

LEMMA 2.2. *The function $P(u,v)$, defined in (2.7), satisfies the following inequalities*

$$\begin{aligned} P(u,v) &\geq \frac{1}{2}|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \leq 2, \\ P(u,v) &\leq \frac{1}{2}|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \geq 2,\ u \neq 0. \end{aligned}$$

Futhermore, let $T > 0$ be arbitrary. There exists a constant $K = K(T) > 0$ such that

$$\begin{aligned} P(u,v) &\geq K|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \geq 2 \\ P(u,v) &\leq K|u|^{2-p}(v-\Phi(u))^2 \quad \text{for} \quad p \leq 2, \end{aligned}$$

and every $u, v \in \mathbb{R}$ satisfying $\left|\frac{v}{\Phi(u)}\right| \leq T$.

2.1.4. Roundabout theorem

Equation (1.1) is said to be *disconjugate* on the closed interval [a,b] if the solution x given by the initial condition $x(a) = 0, r(a)\Phi(x'(a)) = 1$ has no zero in $(a,b]$, in the opposite case (1.1) is said to be *conjugate* on [a,b].

THEOREM 2.2. *The following statements are equivalent:*

(i) *Equation* (1.1) *is disconjugate on the interval $I = [a,b]$.*
(ii) *There exists a solution of* (1.1) *having no zero in $[a,b]$.*
(iii) *There exists a solution w of the generalized Riccati equation* (2.5) *which is defined on the whole interval $[a,b]$.*
(iv) *The p-degree functional $F(y;a,b) = \int_a^b [r(t)|y'|^p - c(t)|y|^p]\, dt$ is positive for every $0 \not\equiv y \in W_0^{1,p}(a,b)$.*

Equation (1.1) is said to be *nonoscillatory* if there exists $T_0 \in \mathbb{R}$ such that (1.1) is disconjugate on $[T_0, T]$ for every $T > T_0$, in the opposite case equation (1.1) is said to be *oscillatory.*

The Roundabout theorem is one of fundamental statements of oscillation theory. According to this theorem we can prove the fact that equation (1.1) is oscillatory or nonoscillatory, i.e., whether it has oscillatory or nonoscillatory solutions, by following one of the two main concepts. The first proving tool (the so-called variational principle) uses the equivalence between nonoscillation of (1.1) and positivity of the functional $F(y; T_0, \infty)$. The second one relates the nonoscillation (1.1) to the solvability of the Riccati equation (2.5). We use the so-called Riccati technique in proving all of our (non)oscillation criteria.

2.1.5. Sturmian separation and comparison theorems

Sturmian separation and comparison theorems extend almost verbatim from the linear oscillation theory to the half-linear theory (see [**1**]).

THEOREM 2.3. *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1). *Then any other solution of this equation which is not proportional to x has exactly one zero on (t_1, t_2).*

Consider the equation

$$(R(t)\Phi(y'))' + C(t)\Phi(y) = 0, \tag{2.8}$$

where the functions R, C satisfy the same assumptions as r, c, respectively.

THEOREM 2.4. *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1) *and suppose that*

$$C(t) \geq c(t), \quad r(t) \geq R(t) > 0 \tag{2.9}$$

for $t \in [t_1, t_2]$. Then any solution of (2.8) *has a zero in (t_1, t_2) or it is a multiple of the solution x. The last possibility is excluded if one of the inequalities in* (2.9) *is strict on a set of positive measure.*

2.1.6. Leighton-Wintner oscillation criterion and Riccati integral equation

Supposing that integral $\int^\infty r^{1-q}(t)\,dt$ diverges, the investigation of oscillation properties of (1.1) divides into two subcases. The first one, when $\int^\infty c(t)\,dt$ also diverges, is considered by the Leighton-Wintner oscillation criterion.

THEOREM 2.5. *Equation* (1.1) *is oscillatory provided*

$$\int^\infty r^{1-q}(t)\,dt = \infty \quad \text{and} \quad \int^\infty c(t)\,dt = \lim_{b\to\infty} \int^b c(t)\,dt = \infty.$$

For the second case, when $\int^\infty c(t)\,dt$ is convergent, there are many oscillation criteria in the literature comparing the functions r and c pointwise or in an integral form. As an important tool, Riccati integral equation is often used.

LEMMA 2.3. *Suppose that*

$$\int^\infty r^{1-q}(t)\,dt = \infty$$

and the integral $\int^\infty c(t)\,dt$ is convergent. Then (1.1) *is nonoscillatory if and only if there exists a solution of the Riccati integral equation*

$$w(t) = \int_t^\infty c(s)\,ds + (p-1)\int_t^\infty r^{1-q}(s)|w(s)|^q\,ds. \tag{2.10}$$

2.2. Principal solution

Further, let us recall the concept of the principal solution of the nonoscillatory equation (1.1) as introduced by Mirzov and later independently by Elbert and Kusano.

2.2.1. Mirzov's definition of principal solution

Mirzov [**30**] extended the concept of the principal solution to half-linear equation (1.1) and defined this solution via the eventually minimal solution of the associated Riccati equation (2.5). The main idea of Mirzov's definition of the principal solution reads as follows. Suppose that (1.1) is nonoscillatory and let $\bar{x}$ be its solution for which $\bar{x}(t) \neq 0$ for $t > T$. Further, for $b_1 > T$ let x_{b_1} be the solution of (1.1) given by the initial condition $x_{b_1}(b_1) = 0, r(b_1)\Phi(x'_{b_1}(b_1)) = -1$. Let $\bar{w} = \frac{r\Phi(\bar{x}')}{\Phi(\bar{x})}, w_{b_1} = \frac{r\Phi(x'_{b_1})}{\Phi(x_{b_1})}$ be the corresponding solutions of (2.5). Then $w(b_1-) = -\infty$ and $w_{b_1} < \bar{w}(t)$ on $[T, b_1)$. Moreover, if $T < b_1 < b_2$ then $w_{b_1}(t) < w_{b_2}(t) < \bar{w}(t)$ on $[T, b_1)$. As $b \to \infty$, the functions w_b converge uniformly on every compact interval $[T, T_1] \subset [T, \infty)$ to a function $\tilde{w} = \lim_{b\to\infty} w_b(t)$ which is also a solution of (2.5). This solution has the property that any other solution w of (2.5) which is defined on the whole interval $[T, \infty)$ satisfies $w(t) > \tilde{w}(t)$ in this interval. Now, if

$$\tilde{x}(t) = \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\, ds\right\},$$

then $\tilde{x}$ is a solution of (1.1) which is called the *principal solution* of this equation. I.e., the principal solution $\tilde{x}$ of (1.1) is a solution which "produces" the minimal solution of (2.5).

2.2.2. Elbert and Kusano's definition of principal solution

The notion of the principal solution by Elbert and Kusano [**16**] was introduced using the Prüfer transformation and reads as follows.

Suppose that (1.1) is nonoscillatory and consider a solution $x(t)$ of this equation. Then $x(t)$ has at most finitely many zeros for large t, i.e., $x(t)$ is either positive or negative on an interval $[T, \infty)$. Let us suppose that $x(t) > 0$ on $[T, \infty)$. Now, using the half-linear Prüfer transformation, we can express this solution $x(t)$ in the form (2.4). From (2.4) we have that $\sin_p \varphi(t) > 0$ on $[T, \infty)$. Define the function $\varphi_\tau(t)$ as a solution of

$$\varphi' = \frac{c(t)}{p-1} \left|\sin_p \varphi(t)\right|^p + r^{1-q}(t) \left|\cos_p \varphi(t)\right|^p, \tag{2.11}$$

and satisfying the initial condition

$$\varphi_\tau(\tau) = 0, \qquad \text{for } T < \tau < \infty.$$

From (2.11) we have $\varphi'_\tau(\tau) > 0$, hence $\varphi_\tau(t) < 0$ for $t < \tau$ and $\varphi_\tau(t) > 0$ for $t > \tau$. Consequently,

$$\varphi(t) < \varphi_{\tau_2}(t) < \varphi_{\tau_1}(t) \quad \text{ for } t \geq T \quad \text{ provided } T < \tau_1 < \tau_2.$$

Hence, the function φ_τ is a strictly monotonic function with respect to τ, therefore there exists the limit

$$\lim_{\tau\to\infty} \varphi_\tau(T) = \varphi^*(T).$$

Let $x^*(t)$ be the solution of (1.1) with the initial conditions

$$x^*(T) = \sin_p \varphi^*(T), \quad x^{*\prime}(T) = r^{1-q}(T) \cos_p \varphi^*(T), \tag{2.12}$$

i.e., $\rho(T) = 1$ in (2.4). By (2.12) the solution $x^*(t)$ is uniquely defined and we can introduce the notion of the *principal solution* as follows.

THEOREM 2.6. *Suppose that* (1.1) *is nonoscillatory. Then there exists* $T \in \mathbb{R}$ *and a solution* x^* *of* (1.1) *such that* $x^*(t) \not\equiv 0$ *for* $t > T$ *and for any solution* x *of* (1.1) *satisfying* $x(t) \not\equiv 0$ *for* $t \geq T$ *either*

$$\frac{x^{*\prime}(t)}{x^*(t)} < \frac{x'(t)}{x(t)} \quad \text{for } t \in [T, \infty),$$

or $x(t) \equiv \operatorname{const} x^*(t)$ *for* $t \in [T, \infty)$.
The solution $x^*(t)$ *is called the principal solution of* (1.1) *at* ∞.

2.2.3. Some properties of the half-linear principal solution

We shall make use of the following comparison theorem for principal solutions.

THEOREM 2.7. *Consider a pair of half-linear equations* (1.1), (2.8), *and suppose that* (2.8) *is a Sturmian majorant of* (1.1) *for large* t, *i.e., there exists* $T \in \mathbb{R}$ *such that* $0 < R(t) \leq r(t), c(t) \leq C(t)$ *for* $t \in [T, \infty)$. *Suppose that the majorant equation* (2.8) *is nonoscillatory and denote by* $\tilde{w}, \tilde{v}$ *eventually minimal solutions of* (2.5) *and of*

$$v' + C(t) + (p-1)R^{1-q}(t)|v|^q = 0,$$

respectively. Then $\tilde{w}(t) \leq \tilde{v}(t)$ *for large* t.

EXAMPLE 2.1. Consider the one-term half-linear equation

$$(r(t)\Phi(x'))' = 0. \tag{2.13}$$

The solution space is a two-dimensional linear space with the basis $x_1 \equiv 1$, $x_2 = \int^t r^{1-q}(s)\, ds$. The Riccati equation associated with (2.13) is

$$w' + (p-1)r^{1-q}(t)|w|^q = 0 \tag{2.14}$$

and the general solution of this equation is

$$w(t) = \frac{1}{\Phi(C + \int_T^t r^{1-q}(s)\, ds)}, \quad w(t) \equiv 0. \tag{2.15}$$

If $\int^\infty r^{1-q}(t)\, dt = \infty$, then $\tilde{w}(t) \equiv 0$ is the eventually minimal solution of this equation and hence $\tilde{x}(t) = 1$ is the principal solution of (2.13).

If $\int^\infty r^{1-q}(t)\, dt < \infty$, the eventually minimal solution of (2.14) is

$$w(t) = -\frac{1}{\Phi(\int_t^\infty r^{1-q}(s)\, ds)}$$

(we take $C = -\int_T^\infty r^{1-q}(s)\, ds$ in formula (2.15)) and $\tilde{x}(t) = \int_t^\infty r^{1-q}(s)\, ds$ is the principal solution of (2.13).

EXAMPLE 2.2. The Euler-type equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0 \tag{2.16}$$

is nonoscillatory if and only if $\gamma \le \gamma_p = \left(\frac{p-1}{p}\right)^p$. If $\gamma = \gamma_p$, then (2.16) has a solution $x(t) = t^{\frac{p-1}{p}}$ and all linearly independent solutions are asymptotically equivalent to $t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$. Consequently, $\tilde{x}(t) = t^{\frac{p-1}{p}}$ is the principal solution of (2.16).

Regarding equation (1.1) as a majorant of (2.13), we have the next statement.

Corollary 2.1. *Let* $\int^\infty r^{1-q}(t)\,dt = \infty$, $c(t) \ge 0$ *for large* t *and suppose that* (1.1) *is nonoscillatory. Then the eventually minimal solution of the associated Riccati equation* (2.5) *satisfies* $\tilde{w} \ge 0$ *for large* t.

2.3. Perturbations

In the all above mentioned criteria, equation (1.1) is regarded as a perturbation of the one-term differential equation (2.13). We use a modified approach and consider equation (1.1) as a perturbation of a general (nonoscillatory on $[T,\infty)$) two-term equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{2.17}$$

i.e., (1.1) can be seen in the form

$$(r(t)\Phi(x')' + \tilde{c}(t)\Phi(x) + (c(t) - \tilde{c}(t))\Phi(x) = 0.$$

This idea is motivated by Elbert and Schneider, who used it first in the paper [**17**]. Since then this approach has been extended and used in many other papers (for example [**5**], [**8**], [**7**], [**10**], [**13**], [**14**], [**36**]).

Now we derive the so-called modified Riccati equation which plays the crucial role in the proofs of our main results. Let $x \in C^1$ be any function such that $x(t) \ne 0$ for $t \in [T,\infty)$ and w be a solution of the Riccati equation (2.5). Then from Picone's identity (2.6) we get

$$(w|x|^p)' = r|x'|^p - c|x|^p - pr^{1-q}|x|^p P(\Phi^{-1}(w_x), w), \tag{2.18}$$

where $w_x = r\Phi(x'/x)$ and Φ^{-1} is the inverse function of Φ. At the same time, let h be a (positive) solution of (2.17) and $w_h = r\Phi(h'/h)$ be the solution of the Riccati equation associated with (2.17), then

$$(w_h|x|^p)' = r|x'|^p - \tilde{c}|x|^p - pr^{1-q}|x|^p P(\Phi^{-1}(w_x), w_h). \tag{2.19}$$

Substituting $x = h$ into (2.18), (2.19) and subtracting these equalities we get the following equation (in view of the identity $P(\Phi^{-1}(w_h), w_h) = 0$)

$$((w - w_h)h^p)' + (c - \tilde{c})h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0. \tag{2.20}$$

Observe that if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (2.20) reduces to (2.5) and this is also the reason why we call this equation the *modified Riccati equation*.

CHAPTER 3

Hille-Wintner type comparison criteria

The classical Sturm comparison theorem (see Theorem 2.4) compares the pair of equations with coefficients c, r and C, R *pointwise*, while Hille-Wintner type criteria compare *integrals*. More precisely, together with (1.1) consider the equation

$$(r(t)\Phi(x'))' + C(t)\Phi(x) = 0. \tag{3.1}$$

The following theorem was proved in [**24**].

THEOREM 3.1. *Suppose that* $\int^\infty r^{1-q}(t)\,dt = \infty$ *and the integral* $\int^\infty c(t)\,dt$ *converges, then if*

$$0 \le \int_t^\infty c(s)\,ds \le \int_t^\infty C(s)\,ds \quad \textit{for large } t \tag{3.2}$$

and (3.1) *is nonoscillatory, equation* (1.1) *is nonoscillatory as well.*

Concerning the complementary case $\int^\infty r^{1-q}(t)\,dt < \infty$ (which is treated in [**25**]), denote $\rho(t) := \int_t^\infty r^{1-q}(s)\,ds$.

THEOREM 3.2. *Suppose that* $\int^\infty r^{1-q}(t)\,dt < \infty$ *and* $c(t) \ge 0$, $C(t) \ge 0$ *for large* t. *If*

$$\int_t^\infty c(s)\rho^p(s)\,ds \le \int_t^\infty C(s)\rho^p(s)\,ds < \infty \tag{3.3}$$

for large t, *then nonoscillation of* (3.1) *implies that of* (1.1).

We generalize these theorems to the case when equation (1.1) is seen as a perturbation of (2.17). This statement is a main result of the paper [**9**].

THEOREM 3.3. *Let* $\int^\infty r^{1-q}(t)\,dt = \infty$. *Suppose that equation* (2.17) *is nonoscillatory and possesses a positive principal solution* h *such that there exist a finite limit*

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) =: L > 0 \tag{3.4}$$

and

$$\int^\infty \frac{dt}{r(t)h^2(t)(h'(t))^{p-2}} = \infty. \tag{3.5}$$

Further suppose that $0 \le \int_t^\infty C(s)\,ds < \infty$ *and*

$$0 \le \int_t^\infty (c(s) - \tilde c(s))h^p(s)\,ds \le \int_t^\infty (C(s) - \tilde c(s))h^p(s)\,ds < \infty, \tag{3.6}$$

all for large t. *If equation* (3.1) *is nonoscillatory, then* (1.1) *is also nonoscillatory.*

PROOF. As we have already mentioned before, to prove that (1.1) is nonoscillatory, it is sufficient to find a solution of associated Riccati equation (2.5) which is defined on some interval $[T, \infty)$. This solution we will find (using the Schauder-Tychonov theorem) as a fixed point of a suitably constructed integral operator.

By our assumption, equation (3.1) is nonoscillatory, i.e., there exists an eventually positive principal solution x of this equation. Denote by $w := r\Phi(x'/x)$ the solution of the associated Riccati equation

$$w' + C(t) + (p-1)r^{1-q}(t)|w|^q = 0.$$

From the previous section, with (1.1) replaced by (3.1), i.e., with c replaced by C, we know that the modified Riccati equation

$$((w - w_h)h^p)' + (C - \tilde{c})h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0$$

holds, where h is the principal solution of (2.17) and $w_h = r\Phi(h'/h)$ is the minimal solution of the Riccati equation corresponding to equation (2.17). By integrating we get

$$h^p(w_h - w)|_T^t = \int_T^t (C(s) - \tilde{c}(s))h^p(s)\,ds + p\int_T^t r^{1-q}(s)P(r^{q-1}h', w\Phi(h))\,ds. \tag{3.7}$$

Since $\int^\infty r^{1-q}(t)\,dt = \infty$ and $0 \le \int_t^\infty C(s)\,ds < \infty$, w solves also the integral Riccati equation (see Lemma 2.3)

$$w(t) = \int_t^\infty C(s)\,ds + (p-1)\int_t^\infty r^{1-q}(s)|w(s)|^q\,ds,$$

and therefore $w(t) \ge 0$ for large t. Hence

$$h^p(w_h - w)|_T^t \le h^p w_h(t) + h^p(w(T) - w_h(T))$$

and letting $t \to \infty$ in (3.7) we have (with L given by (3.4))

$$L + h^p(w(T) - w_h(T)) \ge \int_T^\infty (C(s) - \tilde{c}(s))h^p(s)\,ds + p\int_T^\infty r^{1-q}(s)P(r^{q-1}h', w\Phi(h))\,ds.$$

Since $P(u, v) \ge 0$ and (3.6) holds, this means that

$$\int^\infty r^{1-q}(t)P(r^{q-1}(t)h'(t), w(t)\Phi(h(t)))\,dt < \infty. \tag{3.8}$$

Now, since (3.4), (3.6), (3.8) hold, from (3.7) follows that there exist finite limits

$$\lim_{t\to\infty} h^p(t)(w(t) - w_h(t)) =: \beta$$

and

$$\lim_{t\to\infty} \frac{w(t)}{w_h(t)} = \lim_{t\to\infty} \frac{h^p(t)w(t)}{h^p(t)w_h(t)} = \frac{L+\beta}{L}. \tag{3.9}$$

Therefore, letting $t \to \infty$ in (3.7) and then replacing T by t, we get the equation

$$\begin{aligned} h^p(t)(w(t) - w_h(t)) - \beta = &\int_t^\infty (C(s) - \tilde{c}(s))h^p(s)\,ds \\ &+p\int_t^\infty r^{1-q}(s)P(r^{q-1}h', w\Phi(h))\,ds. \end{aligned} \tag{3.10}$$

Since (3.9) holds, according to Lemma 2.2 there exists a positive constant K such that

$$K|\Phi^{-1}(w_h)|^{2-p}(w-w_h)^2 \le P(\Phi^{-1}(w_h), w),$$

and hence

$$Kr^{1-q}h^p w_h^{q-2}(w-w_h)^2 \le r^{1-q}h^p P(\Phi^{-1}(w_h), w) = r^{1-q}P(r^{q-1}h', w\Phi(h)).$$

Now, using the fact that $w_h^{q-2} = r^{q-2}(h')^{2-p}h^{p-2}$, we get the inequality

$$\frac{K}{r(t)h^2(t)(h'(t))^{p-2}}[(w(t)-w_h(t))h^p(t)]^2 \le r^{1-q}(t)P(r^{q-1}(t)h'(t), w(t)\Phi(t))). \tag{3.11}$$

Denote $G(t) = r^{-1}(t)h^{-2}(t)(h'(t))^{2-p}$, then the last inequality after integrating over $[T,\infty)$ reads

$$K\int_T^\infty G(t)[(w(t)-w_h(t))h^p(t)]^2\,dt \le \int_T^\infty r^{1-q}(t)P(r^{q-1}(t)h'(t), w(t)\Phi(h(t)))\,dt.$$

By (3.5) we have $\int^t G(s)\,ds \to \infty$ as $t\to\infty$. This implies that $\beta = \lim_{t\to\infty} h^p(t)(w(t)-w_h(t)) = 0$ since if $\beta \ne 0$, we have

$$\int^\infty G(t)[(w(t)-w_h(t))h^p(t)]^2\,dt = \infty,$$

which, in view of (3.11), implies that $\int^\infty r^{1-q}P(r^{q-1}h', w\Phi(h))\,dt = \infty$ and this contradicts (3.8). Consequently from (3.10), we get the integral equation

$$\begin{aligned} h^p(t)(w(t)-w_h(t)) &= \int_t^\infty (C(s)-\tilde{c}(s)h^p(s)\,ds \\ &\quad + p\int_t^\infty r^{1-q}(s)P(r^{q-1}h', w\Phi(h))\,ds. \end{aligned} \tag{3.12}$$

This equation we use in constructing the integral operator whose fixed point is a solution of (2.5) which we are looking for.

Define the function set U and the mapping F by

$$U = \{u \in C[T,\infty) : w_h(t) \le u(t) \le w(t) \text{ for } t \in [T,\infty)\},$$

where T is sufficiently large,

$$\begin{aligned} F(u)(t) &= w_h(t) + h^{-p}(t)\left\{\int_t^\infty (c(s)-\tilde{c}(s))h^p(s)\,ds\right. \\ &\quad \left. + p\int_t^\infty r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), u)\,ds\right\} \end{aligned}$$

Observe that the set U is well defined since $w(t) \ge w_h(t)$ for large t by (3.6) and (3.12). Obviously, U is a convex and closed subset of the Frechet space $C[T,\infty)$ with the topology of the uniform convergence on compact subintervals of $[T,\infty)$. Denote $H(s) := \frac{|s|^q}{q} - \Phi^{-1}(w_h)s$. Then $H'(s) = \Phi^{-1}(s) - \Phi^{-1}(w_h) \ge 0$ for $s \ge w_h$. This means that $P(\Phi^{-1}(w_h), u)$ is nondecreasing in the second variable and hence if $w_h(t) \le u_1(t) \le u_2(t) \le w(t)$, $t \in [T,\infty)$, we have $F(u_1)(t) \le F(u_2)(t)$ for $t \in [T,\infty)$.

Next we show that F maps U into itself. To this end, it is sufficient to show that $w_h(t) \le F(w_h)(t) \le F(u)(t) \le F(w)(t) \le w(t)$ for large t. We have

$$F(w_h)(t) = w_h(t) + h^{-p}(t)\left\{\int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds\right\} \ge w_h(t)$$

and, at the same time, using (3.6) and (3.12) (suppressing the argument t)

$$\begin{aligned} F(w) &= w_h + h^{-p}\left\{\int_t^\infty (c - \tilde{c})h^p + p\int_t^\infty r^{1-q}h^p P(\Phi^{-1}(w_h), w)\right\} \\ &\le w_h + h^{-p}\left\{\int_t^\infty (C - \tilde{c})h^p + p\int_t^\infty r^{1-q}h^p P(\Phi^{-1}(w_h), w)\right\} \\ &= w. \end{aligned}$$

Let $T_1 > T$ be arbitrary. As $w_h(t) \le F(u)(t) \le w(t)$ for $u \in U$ and w_h, w exist on the whole interval $[T, \infty)$, the set $F(U)|_{[T,T_1]}$ is bounded. Next we show that this set is also uniformly continuous. Let $u \in U$ be arbitrary, $\varepsilon > 0$, and $t_1, t_2 \in [T, T_1]$, without a loss of generality we may suppose that $t_1 < t_2$. Denote

$$f(t) := (c(t) - \tilde{c}(t))h^p(t) + pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t)), u(t)),$$

then by the monotonicity of P in the second argument

$$\int_T^\infty f(s)\,ds \le \int_T^\infty \left[c(t) - \tilde{c}(t))h^p(t) + pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t)), w(t))\right]\,dt =: R$$

and hence

$$\begin{aligned} |F(u)(t_2) - F(u)(t_1)| &\le |w_h(t_2) - w_h(t_1)| + \left|h^{-p}(t_2)\int_{t_2}^\infty f(s)\,ds - h^{-p}(t_1)\int_{t_1}^\infty f(s)\,ds\right| \\ &= |w_h(t_2) - w_h(t_1)| + \left|h^{-p}(t_2)\int_{t_2}^\infty f(s)\,ds - h^{-p}(t_1)\int_{t_2}^\infty f(s)\,ds\right. \\ &\qquad \left. + h^{-p}(t_1)\int_{t_2}^\infty f(s)\,ds - h^{-p}(t_1)\int_{t_1}^\infty f(s)\,ds\right| \\ &\le |w_h(t_2) - w_h(t_1)| + |h^{-p}(t_2) - h^{-p}(t_1)|\int_{t_2}^\infty f(s)\,ds + |h^{-p}(t_1)|\int_{t_1}^{t_2} f(s)\,ds \\ &\le |w_h(t_2) - w_h(t_1)| + |h^{-p}(t_2) - h^{-p}(t_1)|\int_T^\infty f(s)\,ds + |h^{-p}(t_1)|\int_{t_1}^{t_2} f(s)\,ds. \end{aligned}$$

Since w_h is continuous, there exists δ_1 such that $|w_h(t_2) - w_h(t_1)| < \frac{\varepsilon}{3}$ provided $|t_2 - t_1| < \delta_1$. Similarly, as h^{-p} is continuous, there exists δ_2 such that $|h^{-p}(t_2) - h^{-p}(t_1)| < \frac{\varepsilon}{3R}$ if $|t_2 - t_1| < \delta_2$. Finally, for $\tilde{R} := \sup_{t\in[T,T_1]} h^{-p}(t)$ there exists δ_3 such that $\int_{t_1}^{t_2} f(s)\,ds < \frac{\varepsilon}{3\tilde{R}}$ provided $|t_2 - t_1| < \delta_3$.

Altogether,

$$|F(u)(t_2) - F(u)(t_1)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3R}R + \tilde{R}\frac{\varepsilon}{3\tilde{R}} = \varepsilon$$

if $|t_2 - t_1| < \min\{\delta_1, \delta_2, \delta_3\}$. Hence $F(U)|_{[T,T_1]}$ is uniformly continuous.

It is obvious that F is a continuous mapping and using the Arzela-Ascoli theorem $F(U)$ is relatively compact subset of $C[T, \infty)$. Now, from the Schauder-Tychonov fixed point theorem follows that there exists $v \in U$ such that $v = F(v)$. Hence v satisfies the modified Riccati integral equation

$$h^p(t)(v(t) - w_h(t)) = \int_t^\infty (c(s) - \tilde{c}(s)h^p(s)\, ds + p \int_t^\infty r^{1-q}(s) P(r^{q-1}h', v\Phi(h))\, ds.$$

By differentiating, one can see that v satisfies the modified Riccati equation (2.20) and hence v solves also (2.5). This implies that equation (1.1) is nonoscillatory and the proof is complete. □

As an immediate consequence of the previous theorem we have the following statement.

COROLLARY 3.1. *Let the assumptions of Theorem 3.3 be satisfied. Then the oscillation of equation* (1.1) *implies that of* (2.17).

COROLLARY 3.2. *Let* $r(t) \equiv 1$, $\tilde{c} = \frac{\gamma_p}{t^p}$, *where* $\gamma_p = \left(\frac{p-1}{p}\right)^p$, *i.e.,* (2.17) *is the generalized Euler equation* (2.16) *with the critical coefficient. If equation* (3.1) *is nonoscillatory,* $\int_t^\infty C(s)\, ds \geq 0$ *for large* t, *and*

$$0 \leq \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds \ \leq \int_t^\infty \left(C(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds < \infty \tag{3.13}$$

for large t, *then* (1.1) *is also nonoscillatory.*

PROOF. The function $h(t) = t^{\frac{p-1}{p}}$ is the principal solution of (2.16) (see Example 2.2 and [**17**]),

$$\lim_{t\to\infty} h(t)\Phi(h'(t)) = \lim_{t\to\infty} t^{\frac{p-1}{p}} \left(\frac{p-1}{p} t^{-\frac{1}{p}}\right)^{p-1} = \left(\frac{p-1}{p}\right)^{p-1},$$

and

$$\int^\infty \frac{dt}{h^2(t)(h'(t))^{p-2}} = \left(\frac{p}{p-1}\right)^{p-2} \int^\infty \frac{dt}{t} = \infty.$$

Since all remaining assumptions of Theorem 3.3 are obviously satisfied, the statement follows from this theorem. □

COROLLARY 3.3. *Suppose that*

$$0 \leq \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds < \infty$$

for large t. *If*

$$\log t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds \leq \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

for large t, *then equation* (1.1) *is nonoscillatory.*

PROOF. Let us replace the nonoscillatory equation (2.17) by the so-called half-linear Euler-Weber equation with the critical coefficients

$$(\Phi(y'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right] \Phi(y) = 0,$$

which is nonoscillatory too (see [**17**]). Then the statement follows from the previous corollary. □

Now let us introduce the version of the Theorem 3.3 in the case when the limit (3.4) is infinite (see also [**34**]).

THEOREM 3.4. *Let $\int^\infty r^{1-q}(t)\,dt = \infty$. Suppose that h is the positive principal solution of* (2.17), (3.5) *holds, and*

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) = \infty. \tag{3.14}$$

Further suppose that

$$C(t) \geq \tilde{c}(t) \quad \textit{for large } t \tag{3.15}$$

and (3.6) *holds, all for large t. If equation* (3.1) *is nonoscillatory, then* (1.1) *is also nonoscillatory.*

PROOF. As (3.1) is nonoscillatory, w satisfies the modified Riccati equation

$$((w-w_h)h^p)' + (C-\tilde{c})h^p + pr^{1-q}h^pP(\Phi^{-1}(w_h), w) = 0. \tag{3.16}$$

By a direct computation for $G(t) := r(t)h(t)\Phi(h'(t))$ and $v(t) = h^p(t)(w(t)-w_h(t))$ we have

$$pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h(t)), w(t)) = (p-1)r^{1-q}(t)h^{-q}(t)H(t,v),$$

where $H(t,v) = |v+G(t)|^q - qG^{q-1}(t)v - G^q(t) \geq 0$ with the equality $H(t,v)=0$ if and only if $v(t)=0$. Indeed, (suppressing the argument t) we have

$$pr^{1-q}h^pP(\Phi^{-1}(w_h), w) = pr^{1-q}h^p\left(\frac{|\Phi^{-1}(w_h)|^p}{p} - \Phi^{-1}(w_h)w + \frac{|w|^q}{q}\right)$$

$$= pr^{1-q}h^p\left(\frac{r^q|\frac{h'}{h}|^p}{p} - r^{q-1}\frac{h'}{h}w + \frac{|w|^q}{q}\right) = r|h'|^p - ph'h^{p-1}w + pr^{1-q}h^p\frac{|w|^q}{q}$$

$$= (p-1)\left((q-1)r|h'|^p - qh'h^{p-1}w + r^{1-q}h^p|w|^q\right)$$

and

$$\begin{aligned}
&r^{1-q}h^{-q}H(t,v)\\
&\quad= r^{1-q}h^{-q}\left(|h^p(w-w_h)+rh\Phi(h')|^q - q(rh\Phi(h'))^{q-1}v - r^qh^q(h')^{q(p-1)}\right)\\
&\quad= r^{1-q}h^{-q+pq}|w|^q - qh^{p-1}h'w + qh^{p-1}h'\left(\frac{h'}{h}\right)^{p-1} - r(h')^p\\
&\quad= r^{1-q}h^p|w|^q - qh^{p-1}h'w + (q-1)r(h')^p.
\end{aligned}$$

Hence $v(t)$ satisfies the equality

$$v' + (C-\tilde{c})h^p + (p-1)r^{1-q}h^{-q}H(t,v) = 0. \tag{3.17}$$

By (3.16), in view of (3.15), $v(t)$ is nondecreasing. Using Theorem 2.7 we have $w(t) \geq w_h(t)$ for large t and hence $v(t) \geq 0$ for large t and there exists a nonnegative limit $v(\infty) = \lim_{t\to\infty} v(t)$. Following the idea introduced in [**14**], we can find out that $v(\infty)=0$. In fact, integrating (3.17) from T to t, $t>T$, T sufficiently large, we have

$$v(T) - v(t) = \int_T^t (C(s)-\tilde{c}(s))h^p(s)\,ds + (p-1)\int_T^t r^{1-q}(s)h^{-q}(s)H(s,v(s))\,ds$$

and hence

$$v(T) \geq \int_T^t (C(s) - \tilde{c}(s))h^p(s)\,ds + (p-1)\int_T^t r^{1-q}(s)h^{-q}(s)H(s, v(s))\,ds.$$

As the integral $\int_T^t (C(s) - \tilde{c}(s))h^p(s)\,ds$ is convergent, the integral

$$\int_T^t r^{1-q}(s)h^{-q}(s)H(s, v(s))\,ds$$

is convergent too.

We have (suppressing the argument t)

$$r^{1-q}h^{-q}[|v + G|^q - qG^{q-1}v - G^q] = r(h')^p(|z+1|^q - qz - 1),$$

where $z(t) = v(t)/G(t) \to 0$ as $t \to \infty$ since $0 \leq v(\infty) < \infty$ and (3.14) holds. By the second degree Taylor formula

$$|z+1|^q - qz - 1 = \frac{q(q-1)}{2}z^2 + o(z^2) \quad \text{as} \quad z \to 0,$$

hence, for every $\varepsilon > 0$ there exists T_0 such that

$$\frac{(q-\varepsilon)(q-1)}{2}z^2(t) \leq |z(t)+1|^q - qz(t) - 1 \leq \frac{(q+\varepsilon)(q-1)}{2}z^2(t)$$

for $t \geq T_0$. Consequently

$$\infty > (p-1)\int^\infty r^{1-q}(t)h^{-q}(t)H(t, v(t))\,dt > \frac{q-\varepsilon}{2}\int^\infty r(t)(h'(t))^p \left|\frac{v(t)}{G(t)}\right|^2 dt$$

$$= \frac{q-\varepsilon}{2}\int^\infty \frac{v^2(t)}{r(t)h^2(t)(h'(t))^{p-2}}\,dt.$$

Since (3.5) holds, we have $v(\infty) = 0$, otherwise we get a contradiction with the last inequality. Hence v satisfies integral equation (3.12) and the rest of the proof is the same as in the previous theorem. $\square$

CHAPTER 4

Hartman-Wintner type theorems

The classical Hartman-Wintner theorem for nonoscillatory equation (1.2) (see [**18**, p. 365] for $r \equiv 1$) relates the existence of the limit

$$\lim_{t\to\infty} \frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,d\tau\right)\,ds}{\int^t r^{-1}(s)\,ds} \tag{4.1}$$

to the convergence of the integral $\int^\infty r^{-1}(t)w^2(t)\,dt$, where w is any solution of the Riccati equation

$$w'(t) + c(t) + \frac{w^2(t)}{r(t)} = 0. \tag{4.2}$$

THEOREM 4.1. *Let* $\int^\infty r^{-1}(t)\,dt = \infty$. *If equation* (1.2) *is nonoscillatory, then a necessary and suffitient condition that*

$$\int^\infty r^{-1}(t)w^2(t)\,dt < \infty$$

for a solution w *of* (4.2) *is that*

$$\lim_{t\to\infty} \frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,d\tau\right)\,ds}{\int^t r^{-1}(s)\,ds}$$

exists as a finite number.

As a consequence of the Harman-Wintner theorem we have the statement that if $\int^\infty r^{-1}(t)\,dt = \infty$, equation (1.2) is oscillatory provided

$$-\infty < \liminf_{t\to\infty} \frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,d\tau\right)\,ds}{\int^t r^{-1}(s)\,ds} < \limsup_{t\to\infty} \frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,d\tau\right)\,ds}{\int^t r^{-1}(s)\,ds}$$

or

$$\lim_{t\to\infty} \frac{\int^t r^{-1}(s)\left(\int^s c(\tau)\,d\tau\right)\,ds}{\int^t r^{-1}(s)\,ds} = \infty.$$

The authors of [**4**] studied (for $r(t) \equiv 1$) the case when the limit (4.1) exists finite and proved the next oscillation criterion.

THEOREM 4.2. *Let* $r(t) \equiv 1$ *in* (1.2), *suppose that the limit* (4.1) *exists finite and*

$$\limsup_{t\to\infty} \frac{t}{\log t}\left(c(\infty) - \frac{1}{t}\int_1^t \int_1^s c(\tau)\,d\tau\,ds\right) > \frac{1}{4},$$

where $c(\infty) = \lim_{t\to\infty} \frac{1}{t}\int_1^t \int_1^s c(\tau)\,d\tau\,ds$. *Then equation* (1.2) *is oscillatory.*

The results for Sturm-Liouville linear equation can be naturally generalized to half-linear equations (see [**26**], [**27**], [**28**]). The classical half-linear extension of the Hartman-Wintner theorem is presented for example in [**11**, Sec. 2.2.4] or in [**23**] and reads as follows.

THEOREM 4.3. *Suppose that* $\int^\infty r^{1-q}(t)\,dt = \infty$ *and* (1.1) *is nonoscillatory. Then the following statements are equivalent.*

(i) *It holds*

$$\int^\infty r^{1-q}(t)|w(t)|^q\,dt < \infty$$

for every solution w *of* (2.5).

(ii) *There exists a finite limit*

$$\lim_{t\to\infty} \frac{\int^t r^{1-q}(s)(\int^s c(\tau)\,d\tau)\,ds}{\int^t r^{1-q}(s)\,ds}.$$

(iii) *For the lower limit we have*

$$\liminf_{t\to\infty} \frac{\int^t r^{1-q}(s)(\int^s c(\tau)\,d\tau)\,ds}{\int^t r^{1-q}(s)\,ds} > -\infty.$$

Similarly as for linear equations, it was shown in [**23**] that equation (1.1) with $r \equiv 1$ is oscillatory provided

$$\lim_{t\to\infty} c_p(t) = \infty \quad \text{or} \quad -\infty < \liminf_{t\to\infty} c_p(t) < \limsup_{t\to\infty} c_p(t),$$

where

$$c_p(t) = \frac{(p-1)}{t^{p-1}} \int_1^t s^{p-2} \int_1^s c(\tau)\,d\tau\,ds.$$

Moreover, if $\lim_{t\to\infty} c_p(t) = c_p(\infty)$ exists finite the following statement can be formulated.

PROPOSITION 4.1. *Suppose that* $c_p(\infty)$ *exists as a finite number and*

$$\limsup_{t\to\infty} \frac{t^{p-1}}{\log t}(c_p(\infty) - c_p(t)) > \left(\frac{p-1}{p}\right)^p.$$

Then equation (1.1) *with* $r(t) \equiv 1$ *is also oscillatory.*

It is quite natural to look for some consequences of this statement in the form of the so called Q, H-type criteria (see [**23**] and also [**11**, Sec. 3.3.1]). These criteria are formulated using the functions and quantities

$$Q_p(t) := t^{p-1}\left(c_p(\infty) - \int_1^t c(s)\,ds\right), \quad H_p(t) := \frac{1}{t}\int_1^t s^p c(s)\,ds,$$

$$Q_* := \liminf_{t\to\infty} Q_p(t), \quad Q^* := \limsup_{t\to\infty} Q_p(t),$$

$$H_* := \liminf_{t\to\infty} H_p(t), \quad H^* := \limsup_{t\to\infty} H_p(t).$$

THEOREM 4.4. *Let $r(t) \equiv 1$. Supposing that $c_p(\infty)$ exists as a finite number, each of the next two conditions is sufficient for oscillation of equation* (1.1)*:*

$$Q_* > -\infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{1}{\log t}\int_1^t s^{p-1}c(s)\,ds > \left(\frac{p-1}{p}\right)^p,$$

or

$$Q_* > \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}.$$

First we introduce the Hartman-Wintner type theorem, which is a completion of results published in [**35**]. The idea of our proof is similar to that used in [**35**], but for the sake of completeness and further references we include the proof too.

THEOREM 4.5. *Suppose that equations* (1.1) *and* (2.17) *are nonoscillatory and let h be a solution of* (2.17) *such that $h'(t) \neq 0$ for large t and*

$$\int^\infty H^{-1}(t)\,dt = \infty, \quad H(t) := r(t)h^2(t)|h'(t)|^{p-2}. \tag{4.3}$$

Let w be a solution of the Riccati equation (2.5) *corresponding to* (1.1) *and $w_h = r\frac{\Phi(h')}{\Phi(h)}$ a solution of the Riccati equation corresponding to* (2.17) *such that*

$$\limsup_{t\to\infty}\left|\frac{w(t)}{w_h(t)}\right| < \infty. \tag{4.4}$$

Then for $u(t) = h^p(t)(w(t) - w_h(t))$ and T sufficiently large the following statements are equivalent.

(1) *The inequality*

$$\int_T^\infty \frac{u^2(t)}{H(t)}\,dt < \infty \tag{4.5}$$

holds.

(2) *There exists a finite limit*

$$\lim_{t\to\infty}\frac{\int_T^t H^{-1}(s)\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\,d\tau\,ds}{\int_T^t H^{-1}(s)\,ds}. \tag{4.6}$$

(3) *For the lower limit we have*

$$\liminf_{t\to\infty}\frac{\int_T^t H^{-1}(s)\int_T^s (c(\tau)-\tilde c(\tau))h^p(\tau)\,d\tau\,ds}{\int_T^t H^{-1}(s)\,ds} > -\infty. \tag{4.7}$$

PROOF. $(1 \Rightarrow 2)$:
We can write (2.20) in the form

$$u'(t) + (c(t) - \tilde c(t))h^p(t) + pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) = 0.$$

Integrating from T to t we get

$$u(t) = u(T) - \int_T^t (c(s)-\tilde c(s))h^p(s)\,ds - p\int_T^t r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,ds$$

and multiplying by H^{-1} and applying the same integration we obtain

$$\int_T^t H^{-1}(s)u(s)\,ds = u(T)\int_T^t H^{-1}(s)\,ds - \int_T^t H^{-1}(s)\left(\int_T^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau\right) ds$$
$$-p\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h),w)\,d\tau\right) ds$$

and hence

$$\frac{\int_T^t H^{-1}(s)u(s)\,ds}{\int_T^t H^{-1}(s)\,ds} = u(T) - \frac{\int_T^t H^{-1}(s)\left(\int_T^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau\right) ds}{\int_T^t H^{-1}(s)\,ds}$$
$$-p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h),w)\,d\tau\right) ds}{\int_T^t H^{-1}(s)\,ds}.$$

Using the Cauchy-Schwartz inequality (suppressing the argument s in the integrated functions) we arrive at

$$0 \le \frac{\left|\int_T^t H^{-1}u\,ds\right|}{\int_T^t H^{-1}\,ds} \le \frac{\left[\int_T^t H^{-1}\,ds\right]^{\frac{1}{2}}\left[\int_T^t \frac{u^2}{H}\,ds\right]^{\frac{1}{2}}}{\int_T^t H^{-1}\,ds} = \left(\frac{\int_T^t \frac{u^2}{H}\,ds}{\int_T^t H^{-1}\,ds}\right)^{\frac{1}{2}} \to 0, \quad t\to\infty.$$

From Lemma 2.2 we know that provided (4.4) holds, there exist constants K_1, K_2 such that

$$K_1\frac{u^2}{H} \le r^{1-q}h^p P(\Phi^{-1}(w_h),w) \le K_2\frac{u^2}{H}. \tag{4.8}$$

As $\int_T^\infty \frac{u^2}{H}\,dt < \infty$, the integral $\int_T^\infty r^{1-q}h^pP(\Phi^{-1}(w_h),w)\,dt$ converges too and using L'Hospital's rule we have

$$\lim_{t\to\infty} p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h),w)\,d\tau\right) ds}{\int_T^t H^{-1}(s)\,ds} < \infty.$$

Therefore,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\int_T^t H^{-1}(s)(\int_T^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau)\,ds}{\int_T^t H^{-1}(s)\,ds}\\
&= u(T) - \lim_{t\to\infty} p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h),w)\,d\tau\right) ds}{\int_T^t H^{-1}(s)\,ds}\\
&= u(T) - p\int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h),w)\,dt < \infty.
\end{aligned} \tag{4.9}$$

$(2 \Rightarrow 3)$: This implication is trivial.
$(3 \Rightarrow 1)$:

From the first part of this proof we have

$$\frac{\int_T^t H^{-1}(s)u(s)\,ds}{\int_T^t H^{-1}(s)\,ds} = u(T) - \frac{\int_T^t H^{-1}(s)\left(\int_T^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds}$$
$$-p\frac{\int_T^t H^{-1}(s)\left(\int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h),w)\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds}.$$

The Cauchy-Schwartz inequality together with (4.7) and (4.8) implies that there exists a constant $M \in \mathbb{R}$ such that

$$-\left(\frac{\int_T^t \frac{u^2(s)}{H(s)}\,ds}{\int_T^t H^{-1}(s)\,ds}\right)^{\frac{1}{2}} \le M - pK_1\frac{\int_T^t H^{-1}(s)\left(\int_T^s \frac{u^2(\tau)}{H(\tau)}\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds}.$$

Suppose, by contradiction, that $\int^\infty \frac{u^2(t)}{H(t)}\,dt = \infty$. Then by L'Hospital's rule

$$\lim_{t\to\infty}\frac{\int_T^t H^{-1}(s)\left(\int_T^s \frac{u^2(\tau)}{H(\tau)}\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds} = \infty$$

and

$$pK_1\frac{\int_T^t H^{-1}(s)\left(\int_T^s \frac{u^2(\tau)}{H(\tau)}\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds} - M \ge \frac{1}{2}pK_1\frac{\int_T^t H^{-1}(s)\left(\int_T^s \frac{u^2(\tau)}{H(\tau)}\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds}$$

for t sufficiently large, i.e.,

$$\left(\frac{\int_T^t \frac{u^2(s)}{H(s)}\,ds}{\int_T^t H^{-1}(s)\,ds}\right)^{\frac{1}{2}} \ge \frac{1}{2}pK_1\frac{\int_T^t H^{-1}(s)\left(\int_T^s \frac{u^2(\tau)}{H(\tau)}\,d\tau\right)\,ds}{\int_T^t H^{-1}(s)\,ds}.$$

Denote $S(t) := \int_T^t H^{-1}(s)\left(\int_T^s \frac{u^2(\tau)}{H(\tau)}\,d\tau\right)\,ds$. Then

$$\left(\frac{S'(t)H(t)}{\int_T^t H^{-1}(s)\,ds}\right)^{\frac{1}{2}} \ge \frac{1}{2}pK_1\frac{S(t)}{\int_T^t H^{-1}(s)\,ds}.$$

By a simple calculation we obtain

$$\frac{S'(t)}{S^2(t)} \ge \frac{1}{4}p^2K_1^2\frac{H^{-1}(t)}{\int_T^t H^{-1}(s)\,ds}.$$

Integrating from $T_1 > T$ to t we get

$$\frac{1}{S(T_1)} > \frac{1}{S(T_1)} - \frac{1}{S(t)} \ge \frac{1}{4}p^2K_1^2\log\left(\int_{T_1}^t H^{-1}(s)\,ds\right) \to \infty$$

for $t \to \infty$, and this is a contradiction with the convergence of $\int^\infty \frac{u^2}{H}\,dt$.

$\square$

For an easier manipulation with certain terms in the subsequent parts of this paper, let us denote

$$L(t) := \frac{\int_T^t H^{-1}(s)(\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,d\tau)\,ds}{\int_T^t H^{-1}(s)\,ds}, \quad L(\infty) := \lim_{t\to\infty} L(t).$$

Corollary 4.1. *Assume that the assumptions of Theorem 4.5 hold. Let either*

$$L(\infty) = \infty \quad \textit{or} \quad -\infty < \liminf_{t\to\infty} L(t) < \limsup_{t\to\infty} L(t). \tag{4.10}$$

Then (1.1) *is oscillatory.*

Proof. Let $L(\infty) = \infty$ and suppose that (1.1) is nonoscillatory. Then (4.7) holds and by Theorem 1 the integral (4.5) converges for every solution u of (2.20) and hence the limit (4.6) exists as a finite number, which is a contradiction. The proof of sufficiency of the second condition in (4.10) is similar. □

The next theorem is the main result of our paper [**32**]. It can be seen as a kind of generalization of Hartman-Wintner type criteria.

Theorem 4.6. *Let $\int^\infty r^{1-q}(t)\,dt = \infty$. Suppose that equation* (2.17) *is nonoscillatory and let h be a principal solution of* (2.17) *such that*

$$\int^\infty H^{-1}(t)\,dt = \infty, \quad \lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) := M > 0,$$

where the funcion H is defined by (4.3).

Further, let $0 \le \int^\infty c(t)\,dt < \infty$ and

$$0 \le \int^\infty (c(t) - \tilde{c}(t))h^p(t)\,dt < \infty.$$

If the limit $L(\infty) < \infty$ exists and

$$\limsup_{t\to\infty} \frac{\int_T^t H^{-1}(s)\,ds}{\log \int_T^t H^{-1}(s)\,ds}(L(\infty) - L(t)) > \frac{1}{2q}, \tag{4.11}$$

then (1.1) *is oscillatory.*

Proof. Suppose, by contradiction, that (1.1) is nonoscillatory. Similarly as in the proof of Theorem 3.3 our assumptions ensure the existence of the finite limit

$$\lim_{t\to\infty} \frac{w(t)}{w_h(t)} = 1, \tag{4.12}$$

where w is a solution of the Riccati equation (2.5) corresponding to (1.1) and $w_h = r\frac{\Phi(h')}{\Phi(h)}$ the solution of the Riccati equation corresponding to (2.17). Let us investigate the behavior of the function $P(u,v)$,

$$P(u,v) = \frac{u^p}{p} - uv + \frac{v^q}{q} = u^p\left(\frac{1}{q}\frac{v^q}{u^p} - vu^{1-p} + \frac{1}{p}\right) = u^p Q(vu^{1-p}),$$

where $Q(x) = \frac{1}{q}x^q - x + \frac{1}{p} \geq 0$ and $Q(1) = 0$. By L'Hospital's rule (used twice) we have

$$\lim_{x \to 1} \frac{Q(x)}{(x-1)^2} = \frac{q-1}{2}.$$

Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$-\varepsilon \leq \frac{Q(x)}{(x-1)^2} - \frac{q-1}{2} \leq \varepsilon \tag{4.13}$$

for x satisfying $|x-1| < \delta$, and inequality (4.13) can be rewritten as

$$\left(\frac{q-1}{2} - \varepsilon\right)(x-1)^2 \leq Q(x) \leq \left(\frac{q-1}{2} + \varepsilon\right)(x-1)^2.$$

For $x = vu^{1-p}$ we have

$$\left(\frac{q-1}{2} - \varepsilon\right)(vu^{1-p}-1)^2 \leq Q(vu^{1-p}) \leq \left(\frac{q-1}{2} + \varepsilon\right)(vu^{1-p}-1)^2,$$

which is for $u \neq 0$ equivalent to

$$u^p\left(\frac{q-1}{2} - \varepsilon\right)(vu^{1-p}-1)^2 \leq P(u,v) \leq u^p\left(\frac{q-1}{2} + \varepsilon\right)(vu^{1-p}-1)^2.$$

By virtue of (4.12) there exists T_1 such that $\left|\frac{w}{w_h} - 1\right| < \delta$ for $t \geq T_1$ and hence for $u = \Phi^{-1}(w_h(t)), v = w(t)$ we have

$$w_h^q\left(\frac{q-1}{2} - \varepsilon\right)\left(\frac{w}{w_h} - 1\right)^2 \leq P(\Phi^{-1}(w_h), w) \leq w_h^q\left(\frac{q-1}{2} + \varepsilon\right)\left(\frac{w}{w_h} - 1\right)^2.$$

From the definition of w_h we get

$$\begin{aligned} h^{2p-2}(t)r^{-1}(t)(h'(t))^{2-p}\left(\frac{q-1}{2} - \varepsilon\right)(w(t)-w_h(t))^2 &\leq r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) \\ &\leq h^{2p-2}(t)r^{-1}(t)(h'(t))^{2-p}\left(\frac{q-1}{2} + \varepsilon\right)(w(t)-w_h(t))^2, \end{aligned}$$

which, in terms of $u = (w - w_h)h^p$ and $H = rh^2|h'|^{p-2}$, yields

$$\left(\frac{q-1}{2} - \varepsilon\right)\frac{u^2(t)}{H(t)} \leq r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) \leq \left(\frac{q-1}{2} + \varepsilon\right)\frac{u^2(t)}{H(t)}. \tag{4.14}$$

As (1.1) and (2.17) are nonoscillatory, the modified Riccati equation (2.20) holds and by its integration and using the fact that $\int^\infty r^{1-q}h^pP(\Phi^{-1}(w_h), w) < \infty$ (which follows from the proof of Theorem 3.3), we get

$$u(t) = u(T) - \int_T^t (c(s) - \tilde{c}(s))h^p(s)\,ds - p\int_t^T r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,ds,$$

hence

$$u(t) = u(T) - p\int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w)\,dt + p\int_t^\infty r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w)\,ds$$
$$- \int_T^t (c(s) - \tilde{c}(s))h^p(s)\,ds.$$

Using (4.9), we get in view of the definition of $L(\infty)$ and (4.14)

$$u(t) \geq L(\infty) + \Big(\frac{q}{2} - p\varepsilon\Big)\int_t^\infty \frac{u^2(s)}{H(s)}\,ds - \int_T^t (c(s) - \tilde{c}(s))h^p(s)\,ds,$$

which implies (suppressing the integration variable)

$$\int_T^t H^{-1}u \geq \int_T^t L(\infty)H^{-1} + \frac{q}{2}\int_T^t H^{-1}\int_s^\infty \frac{u^2}{H} - \int_T^t H^{-1}\int_T^s (c - \tilde{c})h^p$$
$$-p\varepsilon\int_T^t H^{-1}\int_s^\infty \frac{u^2}{H},$$

and hence

$$\int_T^t L(\infty)H^{-1}(s)\,ds - \int_T^t H^{-1}(s)\int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,d\tau\,ds \leq \int_T^t H^{-1}(s)u(s)\,ds$$
$$-\frac{q}{2}\int_T^t H^{-1}(s)\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,d\tau\,ds + p\varepsilon\int_T^t H^{-1}(s)\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,d\tau\,ds.$$

Using the definition of $L(t)$ on the left-hand side and integrating by parts on the right-hand side of the last inequality, we have

$$(L(\infty) - L(t))\int_T^t H^{-1}(s)\,ds \leq \int_T^t H^{-1}(s)u(s)\,ds - \frac{q}{2}\left[\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,d\tau\cdot\int_T^s H^{-1}(\tau)\,d\tau\right]_T^t$$
$$-\frac{q}{2}\int_T^t\left(\frac{u^2(s)}{H(s)}\int_T^s H^{-1}(\tau)\,d\tau\,ds\right) + p\varepsilon\int_T^t H^{-1}(s)\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,d\tau\,ds$$

and

$$(L(\infty) - L(t))\int_T^t H^{-1}(s)\,ds$$
$$\leq \int_T^t \frac{H^{-1}(s)}{\int_T^s H^{-1}(\tau)\,d\tau}\left(u(s)\int_T^s H^{-1}(\tau)\,d\tau - \frac{q}{2}\left(u(s)\int_T^s H^{-1}(\tau)\,d\tau\right)^2\right)ds$$
$$-\frac{q}{2}\int_t^\infty \frac{u^2(s)}{H(s)}\,ds\int_T^t H^{-1}(s)\,ds + p\varepsilon\int_T^t H^{-1}(s)\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,d\tau\,ds$$

and by virtue of the inequality $\alpha - \frac{q}{2}\alpha^2 \leq \frac{1}{2q}$ for $\alpha = u\int^s H^{-1}$ we get

$$(L(\infty) - L(t)) \leq \frac{1}{2q}\frac{\log\int_T^t H^{-1}(s)\,ds}{\int_T^t H^{-1}(s)\,ds} - \frac{q}{2}\int_t^\infty \frac{u^2(s)}{H(s)}\,ds + p\varepsilon\frac{\int_T^t H^{-1}(s)\int_s^\infty \frac{u^2(\tau)}{H(\tau)}\,d\tau\,ds}{\int_T^t H^{-1}(s)\,ds}.$$

From Theorem 4.5 we obtain that $\int_t^\infty \frac{u^2}{H} < \infty$ and thus

$$\limsup_{t\to\infty} \frac{\int_T^t H^{-1}(s)\,ds}{\log \int_T^t H^{-1}(s)\,ds}(L(\infty) - L(t)) \le \frac{1}{2q} + p\varepsilon \int_t^\infty \frac{u^2(s)}{H(s)}\,ds.$$

As $\lim_{t\to\infty} \frac{w}{w_h} = 1$, ε and also the last term of the above inequality are arbitrarily small and we have a contradiction with our assumption. □

Corollary 4.2. *Let* $r(t) \equiv 1$, $\tilde{c} = \frac{\gamma_p}{t^p}$ *where* $\gamma_p = \left(\frac{p-1}{p}\right)^p$, *i.e.,* (2.17) *is the generalized Euler equation* (2.16) *with the critical coefficient. Let* $\int_t^\infty c(s)\,ds \ge 0$ *for large* t *and*

$$0 \le \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}(s)\,ds < \infty.$$

If, for T *sufficiently large, the limit*

$$L(\infty) = \lim_{t\to\infty} \frac{\int_T^t s^{-1} \int_T^s \left(c - \frac{\gamma_p}{\tau^p}\right) \tau^{p-1}\,d\tau\,ds}{\log\left|\frac{t}{T}\right|} < \infty$$

exists and

$$\limsup_{t\to\infty} \frac{\log\left|\frac{t}{T}\right|}{\log\log\left|\frac{t}{T}\right|} \left(L(\infty) - \frac{\int_T^t s^{-1} \int_T^s \left(c - \frac{\gamma_p}{\tau^p}\right) \tau^{p-1}\,d\tau\,ds}{\log\left|\frac{t}{T}\right|}\right) > \frac{1}{2q},$$

then (1.1) *is oscillatory.*

Proof. The proof can be made in the same manner as the proof of Corollary 3.2 and the statement follows from Theorem 4.6. □

To introduce the next statement, denote

$$Q(t) = \int^t H^{-1}(s)\,ds \cdot \left(L(\infty) - \int^t (c(s) - \tilde{c}(s))h^p(s)\,ds\right), \quad Q_* := \liminf_{t\to\infty} Q(t).$$

Similarly as in the classical approach to half-linear equations (see [**23**]) we can formulate the following corollaries.

Corollary 4.3. *Let condition* (4.11) *in Theorem 4.6 be replaced by the assumptions that* $Q_* > -\infty$ *and*

$$\limsup_{t\to\infty} \frac{1}{\log \int^t H^{-1}(s)\,ds} \int^t \left(\int^s H^{-1}(\tau)\,d\tau \cdot (c(s) - \tilde{c}(s))h^p(s)\right) ds > \frac{1}{2q}.$$

Then equation (1.1) *is oscillatory.*

Proof. In order to prove our statement, we have to show that inequality (4.11) holds. Integrating per partes (and surppressing the arguments), we have

$$\frac{\int^t H^{-1}(s)\,ds}{\log \int^t H^{-1}(s)\,ds} \left(L(\infty) - \frac{\int^t H^{-1}(s) \int^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,d\tau\,ds}{\int^t H^{-1}(s)\,ds}\right)$$

$$= \frac{L(\infty) \int^t H^{-1}(s)\,ds}{\log \int^t H^{-1}(s)\,ds} - \frac{1}{\log \int^t H^{-1}(s)\,ds} \cdot \int^t \left[H^{-1}(s) \int^s (c(\tau) - \tilde{c}(\tau))h^p(\tau)\,d\tau\right] ds$$

$$= \frac{L(\infty)\int^t H^{-1}(s)\,ds}{\log\int^t H^{-1}(s)\,ds} - \frac{1}{\log\int^t H^{-1}(s)\,ds}\left[\int^t H^{-1}(s)\,ds\cdot\int^t (c(s)-\tilde{c}(s))h^p(s)\,ds \right.$$
$$\left. - \int^t \left(\int^s H^{-1}(\tau)\,d\tau\cdot(c(s)-\tilde{c}(s))h^p(s)\right)ds\right]$$

$$= \frac{\int^t H^{-1}(s)\,ds}{\log\int^t H^{-1}(s)\,ds}\left(L(\infty) - \int^t (c(s)-\tilde{c}(s))h^p(s)\,ds\right)$$
$$+\frac{1}{\log\int^t H^{-1}(s)\,ds}\int^t\left(\int^s H^{-1}(\tau)\,d\tau\cdot(c(s)-\tilde{c}(s))h^p(s)\right)ds$$

$$= \frac{Q}{\log\int^t H^{-1}(s)\,ds} + \frac{1}{\log\int^t H^{-1}(s)\,ds}\int^t\left(\int^s H^{-1}(\tau)\,d\tau\cdot(c(s)-\tilde{c}(s))h^p(s)\right)ds.$$

This shows that the conditions of the previous theorem are satisfied. □

COROLLARY 4.4. *Condition* (4.11) *in Theorem 4.6 can be replaced by the assumption*

$$Q_* > \frac{1}{2q}.$$

PROOF. By a direct computation we have

$$\frac{\int^t H^{-1}(s)\,ds}{\log\int^t H^{-1}(s)\,ds}\left(L(\infty) - \frac{\int^t H^{-1}(s)\int^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau\,ds}{\int^t H^{-1}(s)\,ds}\right)$$

$$= \frac{1}{\log\int^t H^{-1}(s)\,ds}\int^t\left(H^{-1}(s)\cdot\left[L(\infty) - \int^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau\right]\right)ds$$

$$= \frac{1}{\log\int^t H^{-1}(s)\,ds}\int^t\left(\frac{H^{-1}(s)}{\int^s H^{-1}(\tau)\,d\tau}\int^s H^{-1}(\tau)\,d\tau\cdot[L(\infty)\right.$$
$$\left. - \int^s (c(\tau)-\tilde{c}(\tau))h^p(\tau)\,d\tau]\right)ds$$

$$= \frac{1}{\log\int^t H^{-1}(s)\,ds}\left(\int^t \frac{H^{-1}(s)}{\int^s H^{-1}(\tau)\,d\tau}\cdot Q\,ds\right).$$

Therefore,

$$\limsup_{t\to\infty}\frac{\int^t H^{-1}(s)\,ds}{\log\int^t H^{-1}(s)\,ds}(L(\infty)-L(t))$$
$$\geq \limsup_{t\to\infty} Q_*\frac{1}{\log\int^t H^{-1}(s)\,ds}\int^t \frac{H^{-1}(s)}{\int^s H^{-1}(\tau)\,d\tau}\,ds > \frac{1}{2q}.$$

Hence condition (4.11) in Theorem 4.6 holds and therefore equation (1.1) is oscillatory. □

Corollary 4.5. *Let condition* (4.11) *in Theorem 4.6 be replaced by the assumption*

$$\liminf_{t\to\infty} \frac{1}{\log \int^t H^{-1}(s)\,ds} \int^t \left(\int^s H^{-1}(\tau)\,d\tau \cdot (c(s)-\tilde{c}(s))h^p(s) \right) ds > \frac{1}{2q}.$$

Then equation (1.1) *is oscillatory.*

Proof. With the notation $C(t) := (c(t)-\tilde{c}(t))h^p(t))$ we observe that

$$L'(t) = \frac{H^{-1}(t) \int^t \int^s H^{-1}(\tau)\,d\tau \cdot C(s)\,ds}{\left(\int^t H^{-1}(s)\,ds\right)^2}.$$

Integrating from t to T ($T > t > 1$) we have

$$L(T) = L(t) + \int_t^T \left[\log\left(\int^s H^{-1}(\tau)\,d\tau\right) \frac{H^{-1}(s)}{\left(\int^s H^{-1}(\tau)\,d\tau\right)^2} \cdot \left(\frac{1}{\log \int^s H^{-1}(\tau)\,d\tau} \cdot \int^s \int^\tau H^{-1}(\zeta)\,d\zeta \cdot C(\tau)\,d\tau \right) \right] ds$$

Hence using the assumption

$$\begin{aligned} L(T) &> L(t) + \left(\frac{1}{2q}+\varepsilon\right) \cdot \int_t^T \left[\log\left(\int^s H^{-1}(\tau)\,d\tau\right) \frac{H^{-1}(s)}{\left(\int^s H^{-1}(\tau)\,d\tau\right)^2} \right] ds \\ &= L(t) + \left(\frac{1}{2q}+\varepsilon\right) \left[\frac{-\log\left(\int^s H^{-1}(\tau)\,d\tau\right)}{\int^s H^{-1}(\tau)\,d\tau} - \frac{1}{\int^s H^{-1}(\tau)\,d\tau} \right]_t^T \\ &= L(t) + \left(\frac{1}{2q}+\varepsilon\right) \left[\frac{\log\left(\int^t H^{-1}(s)\,ds\right)}{\int^t H^{-1}(s)\,ds} + \frac{1}{\int^t H^{-1}(s)\,ds} - \frac{\log\left(\int^T H^{-1}(s)\,ds\right)}{\int^T H^{-1}(s)\,ds} - \frac{1}{\int^T H^{-1}(s)\,ds} \right] \end{aligned}$$

If $T \to \infty$ then

$$\frac{\int^t H^{-1}(s)\,ds}{\log\left(\int^t H^{-1}(s)\,ds\right)} (L(\infty) - L(t)) > \left(\frac{1}{2q}+\varepsilon\right) \left(1 + \frac{1}{\log\left(\int^t H^{-1}(s)\,ds\right)}\right)$$

for t large enough. Hence condition (4.11) in Theorem 4.6 is satisfied and equation (1.1) is oscillatory. $\square$

CHAPTER 5

Asymptotic formulas for nonoscillatory solutions of perturbed half-linear Euler equation

In this chapter we present asymptotic formulas for some solutions of the half-linear differential equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) + g(t)\Phi(x) = 0, \tag{5.1}$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$. The main results of this chapter were formulated in the paper [**31**].

The studied equation (5.1) can be seen as a perturbation of the half-linear Euler equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0. \tag{5.2}$$

As Euler equation (5.2) has a solution $x(t) = t^{\frac{p-1}{p}}$, it is nonoscillatory. The coeficient γ_p is a critical constant, critical in that sence that it lies between the oscillation and nonoscillation of (5.2), more precisely, (5.2) is oscillatory if γ_p is replaced by a bigger constant and remains nonoscillatory for less constants (see e.g. [**11**, Sec. 1.4.2]). (Non)oscillation of equation (5.1) depends on the asymptotic behavior of the perturbed term $g(t)\Phi(x)$ as $t \to \infty$.

The asymptotics of solutions of equation (1.1) with $r \equiv 1$ were studied in [**21**]. It was proved in that paper that under the assumption

$$\lim_{t\to\infty} t^{p-1} \int_t^\infty c(s)\, ds = c \in (-\infty, \gamma_p)$$

the equation

$$(\Phi(x'))' + c(t)\Phi(x) = 0$$

is nonoscillatory and has two solutions $x_{1,2}(t)$ of the form

$$x_i(t) = t^{\lambda_i^{q-1}} \exp\left\{\int_{t_0}^t \frac{\varepsilon(s)}{s}\, ds\right\} \quad \text{for } t \geq t_0, \tag{5.3}$$

where q is the conjugate number of p, $\varepsilon(t) \to 0$ as $t \to \infty$ and λ_i are roots of the equation

$$|\lambda|^q - \lambda + c = 0.$$

We consider equation (5.1) under the assumption

$$\lim_{t\to\infty} \log t \int_t^\infty g(s)s^{p-1}\, ds \in (-\infty, \mu_p)\,, \tag{5.4}$$

where $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. In the case (5.4) we establish asymptotic formulas for two linearly independent solutions of (5.1). These formulas can be viewed as an extension of (5.3) to the situation when (5.1) is regarded as a perturbation of (5.2).

We also consider the case $g(t) = \frac{\mu_p}{t^{p-1}\log^2 t}$. Equation (5.1) then reduces to the so-called half-linear Euler-Weber equation with the critical coefficients

$$(\Phi(y'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right]\Phi(y) = 0. \tag{5.5}$$

This nonoscillatory equation was studied in [**17**] and the asymptotic formula for its principal solution (see [**11**]) derived there is

$$\tilde{x}(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t\,(1+o(1)) \quad \text{as} \quad t \to \infty.$$

Our second main result shows that the term $(1+o(1))$ is a special slowly varying function.

Let us recall that a positive measurable function $L(t)$ defined on $(0,\infty)$ is said to be a *slowly varying function* in the sence of Karamata (see e.g. [**21**], [**22**]) if it satisfies

$$\lim_{t\to\infty}\frac{L(\lambda t)}{L(t)} = 1 \quad \text{for any} \quad \lambda > 0.$$

From the representation theorem for slowly varying functions (see [**19**]) we know that they are in the form

$$L(t) = l(t)\exp\left\{\int_{t_0}^{t}\frac{\varepsilon(s)}{s}\,ds\right\}, \quad t \ge t_0,$$

for some $t_0 > 0$, where $l(t)$ and $\varepsilon(t)$ are measurable functions such that

$$\lim_{t\to\infty} l(t) = l \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty}\varepsilon(t) = 0.$$

If $l(t)$ is identically a positive constant, we say that $L(t)$ is a *normalized slowly varying function*.

Now we introduce the main result. We show that equation (5.1) has a pair of solutions of a special asymptotic form.

THEOREM 5.1. *Suppose that*

$$c(t) := \frac{\gamma_p}{t^p} + g(t) \ge 0 \quad \textit{for large } t, \tag{5.6}$$

the integral $\int^\infty g(t)t^{p-1}\,dt$ *converges, and let*

$$c := \lim_{t\to\infty}\log t\int_t^\infty g(s)s^{p-1}\,ds < \mu_p$$

holds. Then (5.1) *possesses a pair of solutions*

$$x_i(t) = t^{\frac{p-1}{p}}(\log t)^{\nu_i} L_i(t),$$

where $\lambda_i := \left(\frac{p-1}{p}\right)^{p-1}\nu_i$ *are roots of the equation*

$$\frac{\lambda^2}{4\mu_p} - \lambda + c = 0 \tag{5.7}$$

and $L_i(t)$ are normalized slowly varying functions of the form $L_i(t) = \exp\left\{\int^t \frac{\varepsilon_i(s)}{s \log s}\, ds\right\}$ with $\varepsilon_i(t) \to 0$ for $t \to \infty$, $i = 1, 2$.

PROOF. In order to express the solutions x_i of (5.1), we will find the solutions of modified Riccati equations corresponding to them as fixed points of suitably constructed operators.

First we formulate the modified Riccati equation associated with (5.1). Let w be a solution of the Riccati equation

$$w' + \frac{\gamma_p}{t^p} + g(t) + (p-1)|w|^q = 0. \tag{5.8}$$

Since (5.6) holds, from [**11**, Cor. 4.2.1] we have $w(t) \geq 0$ for large t. Let

$$w_h(t) = \Phi\left(\frac{h'}{h}\right) = \left(\frac{p-1}{p}\right)^{p-1} t^{1-p}$$

be the solution of Riccati equation associated with (5.2) generated by the solution $h(t) = t^{\frac{p-1}{p}}$, and denote

$$v(t) = (w(t) - w_h(t))h^p(t) = t^{p-1}\left(w - \left(\frac{p-1}{p}\right)^{p-1} t^{1-p}\right). \tag{5.9}$$

Modified Riccati equation (2.20), where $\tilde{c}(t) = \frac{\gamma_p}{t^p}$, $c(t) = \frac{\gamma_p}{t^p} + g(t)$, has then the form

$$v' + g(t)t^{p-1} + pt^{p-1}P\left(\left(\frac{p-1}{p}\right)\frac{1}{t}, w\right) = 0,$$

which, by an easy calculation, arrives at

$$v' + g(t)t^{p-1} + \frac{p-1}{t}G(v) = 0, \tag{5.10}$$

where

$$G(v) = \left|v + \left(\frac{p-1}{p}\right)^{p-1}\right|^q - v - \left(\frac{p-1}{p}\right)^p,$$

with the equality $G(v) = 0$ if and only if $v = 0$.

Now we show that $v(t) \to 0$ for $t \to \infty$. Integrating (5.10) from T to t, $T \leq t$, we have

$$\left[\left(\frac{p-1}{p}\right)^{p-1} - t^{p-1}w\right]_T^t = \int_T^t g(s)s^{p-1}\, ds + (p-1)\int_T^t \frac{G(v)}{s}\, ds$$

Letting $t \to \infty$ and taking into account that $w(t) \geq 0$ for large t,

$$\left[\left(\frac{p-1}{p}\right)^{p-1} - t^{p-1}w\right]_T^\infty \leq T^{p-1}w(T).$$

Hence

$$\int_T^\infty g(s)s^{p-1}\, ds + (p-1)\int_T^\infty \frac{G(v)}{s}\, ds \leq T^{p-1}w(T)$$

and since the integral $\int_T^\infty g(s)s^{p-1}\,ds$ converges, the integral $\int_T^\infty \frac{G(v)}{s}\,ds$ converges too. This means that $\lim_{t\to\infty} v(t)$ exists and $v(t)\to 0$, since if $v(t)\to v_0\neq 0$, then $G(v(t))\to G(v_0)>0$ which contradicts the convergence of $\int_T^\infty \frac{G(v)}{s}\,ds$.

Let us investigate the behavior of the function $G(v)$. By L'Hospital's rule (used twice) we have

$$\lim_{v\to 0}\frac{G(v)}{v^2}=\frac{q-1}{2}\left(\frac{p}{p-1}\right)^{p-1}=\frac{q-1}{4\mu_p}.$$

Hence, for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\left(\frac{q-1}{4\mu_p}-\varepsilon\right)v^2\le G(v)\le\left(\frac{q-1}{4\mu_p}+\varepsilon\right)v^2. \tag{5.11}$$

for v satisfying $|v|<\delta$. Similarly for $\frac{\partial G}{\partial v}$, as

$$\lim_{v\to 0}\frac{\frac{\partial G}{\partial v}}{v}=(q-1)\left(\frac{p}{p-1}\right)^{p-1}=\frac{q-1}{2\mu_p},$$

to every $\varepsilon>0$ one can find $\delta>0$ such that

$$\left(\frac{q-1}{2\mu_p}-\varepsilon\right)v\le\frac{\partial G}{\partial v}\le\left(\frac{q-1}{2\mu_p}+\varepsilon\right)v \tag{5.12}$$

as $|v|<\delta$.

Denote

$$\psi(t):=\log t\int_t^\infty g(s)s^{p-1}\,ds-c.$$

The equation

$$\frac{\lambda^2}{4\mu_p}-\lambda+c=0$$

has for $c<\mu_p$ two real zeros $\lambda_{1,2}$ satisfying the inequalities

$$\lambda_1=2\mu_p\left(1-\sqrt{1-\frac{c}{\mu_p}}\right)<2\mu_p<\lambda_2=2\mu_p\left(1+\sqrt{1-\frac{c}{\mu_p}}\right). \tag{5.13}$$

In the rest of the proof we take $i=1,2$. We assume that solutions of modified Riccati equation (5.10) are in the form

$$v_i(t)=\frac{\lambda_i+\psi(t)+z(t)}{\log t}.$$

Then for its derivative we have

$$v_i'(t)=\frac{\left(\frac{1}{t}\int_t^\infty g(s)s^{p-1}ds-g(t)t^{p-1}\log t+z'(t)\right)\log t-(\lambda_i+\psi(t)+z(t))\frac{1}{t}}{\log^2 t}$$

and substituing into the modified Riccati equation we get the equation

$$z'(t)-\frac{z(t)}{t\log t}+\frac{1}{t\log t}(c-\lambda_i)+\frac{(p-1)\log t}{t}G(v_i)=0,$$

which can be rewritten as

$$z'(t) + \frac{(\lambda_i - 2\mu_p)z(t)}{2\mu_p t\log t} + \left[\frac{1}{t\log t}(c-\lambda_i) - \frac{\lambda_i z(t)}{2\mu_p t\log t} + \frac{(p-1)\log t}{t}G(v_i)\right] = 0.$$

If we denote

$$r_i(t) = \exp\left\{\int^t \frac{\lambda_i - 2\mu_p}{2\mu_p s\log s}\,ds\right\},$$

then the previous equation is equivalent to

$$(r_i(t)z(t))' + r_i(t)\frac{1}{t\log t}H_i(z,t) = 0, \tag{5.14}$$

where $H_i(z,t) = c - \lambda_i - \frac{\lambda_i z(t)}{2\mu_p} + (p-1)\log^2 tG(v_i)$.

Let $C_0[t_i,\infty)$ denote the set of all continuous functions on $[t_i,\infty)$ tending to zero as $t\to\infty$; concrete t_i will be specified later. $C_0[t_i,\infty)$ is a Banach space with the norm $\|z\| = \sup\{|z(t)| : t\ge t_i\}$. For $i=1$ we consider the integral operator

$$F_1z(t) = \frac{1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{s\log s}H_1(z,s)\,ds$$

on the set

$$V_1 = \{z\in C_0[t_1,\infty) : |z(t)| < \varepsilon_1, t\ge t_1\},$$

where ε_1, t_1 are suitably chosen (will be specified later). Now our aim is to show that the operator F_1 is a contraction on the set V_1 and maps V_1 to itself.

First we show that $\int_t^\infty \frac{r_1(s)}{s\log s}\,ds$ converges. Denote $a_1 = \frac{\lambda_1 - 2\mu_p}{2\mu_p} < 0$, then we have $r_1(t)\to 0$ for $t\to\infty$,

$$\int_t^\infty \frac{r_1(s)}{s\log s}\,ds = \int_t^\infty \frac{\exp\{\int^s \frac{a_1}{\tau\log\tau}\,d\tau\}}{s\log s}\,ds = \int_t^\infty \frac{(\log s)^{a_1}}{s\log s}\,ds = \left[\frac{(\log s)^{a_1}}{a_1}\right]_t^\infty < \infty.$$

Furthermore,

$$r_i'(t) = \frac{\lambda_i - 2\mu_p}{2\mu_p}\frac{r_i(t)}{t\log t}$$

and by L'Hospital's rule we have

$$\lim_{t\to\infty}\frac{1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{s\log s}\,ds = \frac{-\frac{r_1(t)}{t\log t}}{r_1'(t)} = \frac{2\mu_p}{2\mu_p - \lambda_1} > 0.$$

Let T_1 be large enough such that

$$\frac{1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{s\log s}\,ds < \frac{4\mu_p}{2\mu_p - \lambda_1} \tag{5.15}$$

for $t\ge T_1$.

Let $\varepsilon_1 > 0$, such that

$$\frac{4\mu_p}{2\mu_p - \lambda_1}\left(\frac{|\lambda_1|\varepsilon_1}{2\mu_p} + \frac{\varepsilon_1(\varepsilon_1+1)^2}{4\mu_p} + \varepsilon_1\right) \le 1 \tag{5.16}$$

and

$$\frac{4\mu_p}{2\mu_p - \lambda_1}\left(\frac{\varepsilon_1^2 + \varepsilon_1}{2\mu_p} + \varepsilon_1\right) < \frac{1}{2}. \tag{5.17}$$

Let T_2 be such that $|\psi(t)| < \varepsilon_1^2$ for $t \geq T_2$.

In order to show that $H_1(z,t) \to 0$ for $t \to \infty$, the estimates for $H_1(z,t)$ are:

$$\begin{aligned}
|H_1(z,t)| &= \left| c - \lambda_1 - \frac{\lambda_1 z(t)}{2\mu_p} + (p-1)\log^2 t\ G(v_1) \right| \\
&\leq \left| c - \lambda_1 - \frac{\lambda_1 z(t)}{2\mu_p} + \frac{v_1^2 \log^2 t}{4\mu_p} \right| + \left| (p-1)\log^2 t\ G(v_1) - \frac{v_1^2 \log^2 t}{4\mu_p} \right| \\
&= \left| \frac{\lambda_1 \psi}{2\mu_p} + \frac{(\psi + z)^2}{4\mu_p} \right| + \left| (p-1)\log^2 t\ G(v_1) - \frac{v_1^2 \log^2 t}{4\mu_p} \right|,
\end{aligned}$$

where (5.7) and the definition of v have been used in the step between the second and the third line of the above computation. Now, according to (5.11), the second term in the last expression is arbitrarily small for small v_1, i.e., as $v_1(t) \to 0$ for $t \to \infty$, there exists T_3 large enough such that for $t \geq T_3$

$$\begin{aligned}
&\left| \frac{\lambda_1 \psi}{2\mu_p} + \frac{(\psi + z)^2}{4\mu_p} \right| + \left| (p-1)\log^2 t\ G(v_1) - \frac{v_1^2 \log^2 t}{4\mu_p} \right| \\
&\leq \left| \frac{\lambda_1 \psi}{2\mu_p} + \frac{(\psi + z)^2}{4\mu_p} \right| + \varepsilon_1^2 \leq \frac{|\lambda_1| \varepsilon_1^2}{2\mu_p} + \frac{(\varepsilon_1^2 + \varepsilon_1)^2}{4\mu_p} + \varepsilon_1^2
\end{aligned}$$

for $t \geq \max\{T_2, T_3\}$.

Similarly, we will need an estimate for the difference $|H_1(z_1,t) - H_1(z_2,t)|$. Using the mean value theorem (with $z \in V_1$ such that $\min\{z_1(t), z_2(t)\} \leq z(t) \leq \max\{z_1(t), z_2(t)\}$) we have

$$\begin{aligned}
|H_1(z_1,s) - H_1(z_2,s)| &= \left| -\frac{\lambda_1 (z_1 - z_2)}{2\mu_p} + (p-1)\log^2 t\, \frac{\partial G(v_1, z)}{\partial z}(z_1 - z_2) \right| \\
&\leq \|z_1 - z_2\| \left(\left| -\frac{\lambda_1}{2\mu_p} + \frac{v_1 \log t}{2\mu_p} \right| + \left| (p-1)\log^2 t\, \frac{\partial G(v_1, z)}{\partial z} - \frac{v_1 \log t}{2\mu_p} \right| \right) \\
&= \|z_1 - z_2\| \left(\left| \frac{\psi + z}{2\mu_p} \right| + \left| (p-1)\log^2 t\, \frac{\partial G(v_1, z)}{\partial z} - \frac{v_1 \log t}{2\mu_p} \right| \right) \\
&\leq \|z_1 - z_2\| \left(\left| \frac{\psi + z}{2\mu_p} \right| + \varepsilon_1 \right) \leq \|z_1 - z_2\| \left(\frac{\varepsilon_1^2 + \varepsilon_1}{2\mu_p} + \varepsilon_1 \right)
\end{aligned}$$

for $t \geq \max\{T_2, T_4\}$, where T_4 is such that $|v(t)| < \delta$ for $t \geq T_4$ (such T_4 exists because of (5.12)).

We take $t_1 = \max\{T_1, T_2, T_3, T_4\}$. Then

$$\begin{aligned}
|F_1 z(t)| &\leq \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s \log s} |H_1(z,s)|\, ds \leq \left(\frac{|\lambda_1| \varepsilon_1^2}{2\mu_p} + \frac{(\varepsilon_1^2 + \varepsilon_1)^2}{4\mu_p} + \varepsilon_1^2 \right) \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s \log s}\, ds \\
&< \frac{4\mu_p}{2\mu_p - \lambda_1} \left(\frac{|\lambda_1| \varepsilon_1^2}{2\mu_p} + \frac{(\varepsilon_1^2 + \varepsilon_1)^2}{4\mu_p} + \varepsilon_1^2 \right) \leq \varepsilon_1
\end{aligned}$$

using (5.16) and hence F_1 maps V_1 to itself.

Next we show that F_1 is a contraction. We have (using the definition of F_i)

$$|F_1 z_1(t) - F_1 z_2(t)| = \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s \log s} |H_1(z_1, s) - H_1(z_2, s)|\, ds$$

$$\leq \|z_1 - z_2\| \left| \frac{\varepsilon_1^2 + \varepsilon}{2\mu_p} + \varepsilon_1 \right| \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s \log s}\, ds,$$

which is, according to (5.15) and (5.17), less than $\frac{1}{2}\|z_1 - z_2\|$ and hence F_1 is a contraction.

By the Banach fixed point theorem, F_1 has a fixed point z_1 that satisfies $z_1 = F_1 z_1$, i.e.

$$z_1(t) = \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s \log s} H_1(z_1, s)\, ds.$$

Differentiating the last equality we can see that $z_1(t)$ is a solution of (5.14) and hence $v_1(t) = \frac{\lambda_1 + \psi(t) + z_1(t)}{\log t}$ is a solution of modified Riccati equation (5.10).

Now we turn our attention to the case $i = 2$. We consider the operator

$$F_2 z(t) = -\frac{1}{r_2(t)} \int^t \frac{r_2(s)}{s \log s} H_2(z, s)\, ds$$

on the set

$$V_2 = \{z \in C_0[t_2, \infty) : |z(t)| < \varepsilon_2, t \geq t_2\},$$

where ε_2, t_2 are suitably chosen. It is easy to see that

$$\lim_{t\to\infty} r_2(t) = \infty, \quad \lim_{t\to\infty} \int^t \frac{r_2(s)}{s \log s}\, ds = \infty$$

and

$$\lim_{t\to\infty} \frac{1}{r_2(t)} \int^t \frac{r_2(s)}{s \log s}\, ds = \frac{2\mu_p}{\lambda_2 - 2\mu_p} > 0.$$

We can choose t_2 and ε_2 similarly as in the case $i = 1$. Let $\varepsilon_2 > 0$, such that

$$\frac{4\mu_p}{\lambda_2 - 2\mu_p} \left(\frac{|\lambda_2|\varepsilon_2}{2\mu_p} + \frac{\varepsilon_2(\varepsilon_2 + 1)^2}{4\mu_p} + \varepsilon_2 \right) \leq 1$$

and

$$\frac{4\mu_p}{\lambda_2 - 2\mu_p} \left(\frac{\varepsilon_2^2 + \varepsilon_2}{2\mu_p} + \varepsilon_1 \right) < \frac{1}{2}.$$

We take t_2 large enough such that $|\psi(t)| < \varepsilon_2^2$,

$$\frac{1}{r_2(t)} \int_t^\infty \frac{r_2(s)}{s \log s}\, ds < \frac{4\mu_p}{\lambda_2 - 2\mu_p}$$

and the estimates for $|H_2(z, t)|$ and $|H_2(z_1, t) - H_2(z_2, t)|$ hold similarly as in the previous case, all for $t \geq t_2$. Since the estimates for $|H_2(z, t)|$ and $|H_2(z_1, t) - H_2(z_2, t)|$ are essentially the same as for $H_1(z, t)$, it is a matter of an almost verbatim repetition of the calculations from the previous part, to show that F_2 maps V_2 to itself and that it is a contraction. Therefore, there exists a fixed point z_2 of F_2 such that

$$z_2(t) = -\frac{1}{r_2(t)} \int^t \frac{r_2(s)}{s \log s} H_2(z_2, s)\, ds.$$

Differentiating we can see that $z_2(t)$ is a solution of (5.14) and $v_2(t) = \frac{\lambda_2+\psi(t)+z_2(t)}{\log t}$ is the second solution of modified Riccati equation (5.10).

Now, for the solutions of Riccati equation (5.8) $w_i(t)$, we have

$$w_i(t) = t^{1-p}\left(v_i + \left(\frac{p-1}{p}\right)^{p-1}\right) = t^{1-p}\left(\frac{\lambda_i+\psi(t)+z_i(t)}{\log t} + \left(\frac{p-1}{p}\right)^{p-1}\right)$$

and the solutions x_i of equation (5.1) are

$$x_i(t) = \exp\int^t \Phi^{-1}(w_i(s))\,ds.$$

Furthermore,

$$\Phi^{-1}(w_i) = \Phi^{-1}\left(t^{1-p}\left(\frac{p-1}{p}\right)^{p-1}\left[\left(\frac{p-1}{p}\right)^{1-p} v_i + 1\right]\right)$$

$$= \frac{1}{t}\frac{p-1}{p}\left[\left(\frac{p-1}{p}\right)^{1-p} v_i + 1\right]^{q-1} = \frac{1}{t}\frac{p-1}{p}\left[1 + (q-1)\left(\frac{p-1}{p}\right)^{1-p} v_i + o(v_i)\right]$$

$$= \frac{1}{t}\frac{p-1}{p} + \frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p}\frac{(\lambda_i+\psi(t)+z(t))}{t\log t} + o(\frac{v_i}{t})$$

$$= \frac{p-1}{p}\frac{1}{t} + \frac{\lambda_i\frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p}}{t\log t} + \frac{\frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p}(\psi(t)+z(t)) + o(\lambda_i+\psi(t)+z(t))}{t\log t}.$$

Denote

$$\lambda_i\frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} = \nu_i, \quad \frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p}(\psi(t)+z(t)) + o(\lambda_i+\psi(t)+z(t)) = \varepsilon_i(t),$$

then the solutions of (5.1) are in the form

$$x_i(t) = \exp\int^t \Phi^{-1}(w_i(s))\,ds = t^{\frac{p-1}{p}}(\log t)^{\nu_i}\exp\left\{\int^t \frac{\varepsilon_i(s)}{s\log s}\,ds\right\}$$

and the theorem is proved. □

Next we introduce an asymptotic formula for a solution of half-linear Euler-Weber equation (5.5), which is the special perturbation of half-linear Euler equation (5.2) with $\lim_{t\to\infty}\log t\int_t^\infty g(s)s^{p-1}\,ds = \mu_p$.

THEOREM 5.2. *Equation* (5.5) *has a solution satisfying the asymptotic formula*

$$x(t) = t^{\frac{p-1}{p}}(\log t)^{\frac{1}{p}}L(t), \tag{5.18}$$

where $L(t)$ is a normalized slowly varying function in the form $L(t) = \exp\left\{\int^t \frac{\varepsilon_i(s)}{s\log s}\right\}$.

PROOF. The perturbed term $g(t)$ is equal to $\frac{\mu_p}{t^p \log^2 t}$ and hence

$$\lim_{t\to\infty} \log t \int_t^\infty g(s)s^{p-1}\, ds = \mu_p.$$

The equation

$$\frac{\lambda^2}{4\mu_p} - \lambda + \mu_p = 0$$

has one (double) real root $\lambda = 2\mu_p$. The modified Riccati equation is of the form

$$z'(t) - \frac{z(t)}{t\log t} + \frac{1}{t\log t}(c-\lambda) + \frac{(p-1)\log t}{t} G(v) = 0 \tag{5.19}$$

and if we take

$$r(t) = \exp\left\{-\int^t \frac{ds}{s\log s}\right\},$$

then equation (5.19) is equivalent to

$$(r(t)z(t))' + r(t)\frac{1}{t\log t}\left(c - \lambda + (p-1)\log^2 t\, G(v)\right) = 0.$$

We again denote $H(z,t) = c-\lambda+(p-1)\log^2 t\, G(v)$ and we consider the integral operator

$$Fz(t) = \frac{1}{r(t)}\int_t^\infty \frac{r(s)}{s\log s} H(z,s)\, ds$$

on the set

$$V = \{z \in C_0[t_0,\infty) : |z(t)| < \varepsilon_0, t \ge t_0\},$$

where ε_0, t_0 are suitably chosen. The rest of the proof can be made in the same manner as in the previous theorem. □

REMARK 5.1. It is known (see [**17**]) that in addition to the solution x given by (5.18), any solution of (5.5) linearly independent of x is given by the asymptotic formula

$$\tilde{x}(t) = t^{\frac{p-1}{p}} \log t^{\frac{1}{p}} (\log(\log t))^{\frac{2}{p}} (1 + o(1)). \tag{5.20}$$

Observe that

$$\lim_{t\to\infty} \frac{L(t)}{[\log(\log(t)]^{\frac{2}{p}}} = 0$$

and it is an open problem whether the function $(1+o(1))$ in (5.20) is also a slowly varying function.

CHAPTER 6

Conclusion

The central idea of this thesis is to make use of the concept of perturbations in the oscilation theory of half-linear differential equations. It turns out that if we regard the studied half-linear differential equation (1.1) as a perturbation of another half-linear equation in the same form, we can arrive at new results and generalizations of the classical statements of the oscillation theory that deals with the qualitative properties of solutions of (1.1). Compared to the classical results, we obtain new concrete (non)oscillatory criteria primarily for equations that are in some sence close to half-linear Euler-type equation (2.16), for which the classical citeria could not have been used.

After the first preliminary part, that contains the introduction to the studied problem, we formulated the Hille-Wintner type comparison theorem for half-linear equation (1.1) seen as a perturbation with its consequences for Euler-type half-linear equation (2.16). In Chapter 4 we utilized the Hartman-Wintner type theorem for "perturbed" half-linear equations to prove an oscillation criterion and its so-called Q-type corollaries. In the last part we presented asymptotic formulas for some solutions of equations which are seen as a perturbation of the half-linear Euler equation (2.16), as well as an improvement of the asymtotic formula of the principal solution of the half-linear Euler-Weber equation (5.5).

The method of perturbations offers quite wide field for further applications and can be used to solve some other open problems which arise.

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