Masaryk University Brno, Faculty of Science
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Dynamic Equations with Mixed Derivatives

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CHAPTER 1

INTRODUCTION

The principal concern of this thesis is to investigate some aspects of the qualitative theory of dynamic equations on time scales. The main attention is focused to half-linear second order dynamic equations with mixed derivatives and to even order linear dynamic equations with mixed derivatives and their relationship to symplectic dynamic systems.

The time scales theory was introduced by Stefan Hilger in his PhD dissertation [**32**]. In this thesis, a "tool" which enabled unifying of discrete and continuous calculus has been developed. Before that, there were "parallel" branches of the differential equations and difference equations theory. In some aspects, they were very similar, in other aspects they seemed to be completely different. Introducing the notion of time scale was an elegant way how to unify these two theories into *dynamic equations* theory. The main idea of the papers concerning dynamic equations on time scales is to prove certain result for general time scale. If only the set of the real numbers, resp. integers, is taken into account (as a special time scale), the general result leads to a result applicable to an ordinary differential equation, resp. difference equation.

This thesis is divided into three parts. In the first part (Chapter 2) we define all notions and state all basic statements, that we will need later on. Further, a brief overview of the theory of half-linear equations and symplectic dynamic systems, which precedes the results of this thesis, can be found here. The main two parts of the thesis are Chapter 3, where we deal with half-linear dynamic equations, and Chapter 4, where the theory of certain type of the even order linear dynamic equation is discussed. Chapter 3 is based on the paper [**25**] and Chapter 4 contains results of the papers [**26**] and [**37**].

The half-linear second order *differential* equation is an equation of the form

$$\text{(HLD)} \qquad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn}(x), p > 1,$$

where the functions r, c are continuous and $r(t) > 0$. The space of all solutions of (HLD) is just homogenous, but generally not additive. So this space has only one half of the properties of a linear space, and this is the reason why equation (HLD) is called *half-linear*. The basic qualitative theory of equation (HLD) was developed by Elbert and Mirzov within their papers [**28**], [**39**] and a comprehensive treatment of this topic can be found in the book [**27**]. Equation (HLD) has similar properties as the Sturm-Liouville differential equation

$$\text{(SLD)} \qquad (r(t)x')' + c(t)x = 0,$$

which is a special case of equation (HLD) when $p = 2$. In particular, the Sturmian theory extends verbatim to (HLD).

A discrete counterpart of equation (HLD) is the difference equation

$$\text{(HL}\Delta\text{)} \qquad \Delta(r_k\Phi(\Delta x_k)) + c_k\Phi(x_{k+1}) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn}(x), p > 1,$$

where $\Delta x_k = x_{k+1} - x_k$ is the forward difference operator, r, c are real-valued sequences and $r_k \neq 0$. Properties of equation (HLΔ) are similar to properties of the equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \tag{SLΔ}$$

i.e., the Sturm-Liouville difference equation. The basic qualitative theory of half-linear difference equations has been established in a series of papers of P. Řehák [**40, 41, 42, 43, 44**] and the results of these papers are summarized in the book [**1**].

Natural unifications of (HLD) and (HLΔ) within the theory of dynamic equations on time scales are the equation

$$(r(t)\Phi(x^\Delta))^\Delta + c(t)\Phi(x^\sigma) = 0, \tag{HL}$$

and

$$(r(t)\Phi(x^\Delta))^\nabla + c(t)\Phi(x) = 0, \tag{HLMD}$$

where Δ, ∇ and σ are the so-called delta derivative, nabla derivative and forward jump operator, respectively. The basic qualitative theory of (HL) has been established in the papers of P. Řehák [**45, 46**] and it is summarized in [**2**]. The main concern of Chapter 3 is equation (HLMD). Motivated by the results given in [**2**], we prove the so-called Roundabout theorem for (HLMD) and we present oscillation and nonoscillation criteria for this equation.

As mentioned before, Chapter 4 deals with even order dynamic equations with mixed derivatives. A typical example of such equations are the fourth order dynamic equations

$$(r(t)y^{\Delta\nabla})^{\Delta\nabla} - (p(t)y^\Delta)^\nabla + q(t)y = 0$$

or

$$(r(t)y^{\nabla\Delta})^{\nabla\Delta} - (p(t)y^\nabla)^\Delta + q(t)y = 0.$$

Equations of this kind appeared only very recently in [**7, 9, 30**] and the basic qualitative theory of these equations has not been elaborated yet. The main tool we use is the relationship of even order dynamic equations with mixed derivatives to the so-called symplectic dynamic systems.

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Brno, September 2007 Daniel Marek

CHAPTER 2

PRELIMINARIES

In this chapter we recall, for reader's convenience, all basic facts about the topics treated in this thesis, i.e., essentials of the time scales calculus and basic theory of half-linear equations and of symplectic systems. More specific results will be stated in those subsections, where they are immediately used.

2.1. Time scales

A time scale $\mathbb{T}$ is any nonempty closed subset of the set of real numbers $\mathbb{R}$. The main examples of time scales (which will be mentioned several times) are the sets of integers $\mathbb{Z}$ and the real numbers $\mathbb{R}$. We define operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}.$$

Operator σ is called *forward jump operator* and operator ρ is called *backward jump operator*. We put $\sigma(M) = M$, if $\mathbb{T}$ has a maximum M, and $\rho(m) = m$, if $\mathbb{T}$ has a minimum m. The functions $\mu, \nu : \mathbb{T} \to [0, \infty)$, where

$$\mu(t) = \sigma(t) - t,$$
$$\nu(t) = t - \rho(t),$$

are called *graininess function* and *backward graininess function*, respectively. Depending on whether the graininess functions for $t \in \mathbb{T}$ are positive or equal to zero, we distinguish several types of time scales points. A point $t \in \mathbb{T}$ is said to be

right dense, if $\mu(t) = 0$,
left dense, if $\nu(t) = 0$,
right scattered, if $\mu(t) > 0$,
left scattered, if $\nu(t) > 0$ and
dense, if t is left dense or right dense.

We will use the abbreviations rd, ld, rs, ls-point, respectively. If a time scale $\mathbb{T}$ has a left scattered maximum M (right scattered minimum m), then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ ($\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$), otherwise $\mathbb{T}^\kappa = \mathbb{T}$ ($\mathbb{T}_\kappa = \mathbb{T}$).

We define the delta and nabla derivatives as follows

$$f^\Delta(t) = \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t}, & \text{if } \mu(t) = 0, \\ \frac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t) > 0, \end{cases} \tag{2.1}$$

$$f^\nabla(t) = \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t} & \text{if } \nu(t) = 0, \\ \frac{f(t)-f(\rho(t))}{\nu(t)} & \text{if } \nu(t) > 0. \end{cases} \tag{2.2}$$

It is obvious that $f^{\Delta}(t) = f'(t) = f^{\nabla}(t)$ if $\mathbb{T} = \mathbb{R}$, and $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, $f^{\nabla}(t) = f(t) - f(t-1)$ if $\mathbb{T} = \mathbb{Z}$. By f^{σ} and f^{ρ} we denote the composition $f \circ \sigma$ and $f \circ \rho$, respectively. Provided that f is a Δ-differentiable function, resp. ∇-differentiable function, i.e., f^{Δ}, resp. f^{∇} exists, then

$$\begin{aligned} f^{\sigma}(t) &= f(t) + \mu f^{\Delta}(t), \quad \text{resp.} \\ f^{\rho}(t) &= f(t) - \nu f^{\nabla}(t) \end{aligned} \tag{2.3}$$

holds. All basic differential formulas can be generalized also for the case of time scales, e.g., the product of two Δ-differentiable (∇-differentiable) functions satisfies

$$\begin{aligned} (fg)^{\Delta} &= f^{\Delta}g^{\sigma} + fg^{\Delta} = f^{\sigma}g^{\Delta} + f^{\Delta}g, \\ (fg)^{\nabla} &= f^{\nabla}g^{\rho} + fg^{\nabla} = f^{\rho}g^{\nabla} + f^{\nabla}g, \end{aligned} \tag{2.4}$$

and the ratio of two differentiable functions is given by

$$\begin{aligned} \left(\frac{f}{g}\right)^{\Delta} &= \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}, \\ \left(\frac{f}{g}\right)^{\nabla} &= \frac{f^{\nabla}g - fg^{\nabla}}{gg^{\rho}}. \end{aligned} \tag{2.5}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* (*ld-continuous*) if it is right continuous (left continuous) at all rd-points (ld-points) and the left limit (right limit) at ld-points (rd-points) exists (finite). Provided f is rd-continuous (ld-continuous) then there exists a Δ-differentiable function F (a ∇-differentiable function G) such that $F^{\Delta}(t) = f(t)$ ($G^{\nabla}(t) = f(t)$). Using these functions we define the integrals

$$\int_a^b f(t)\,\Delta t = F(b) - F(a),$$

$$\int_a^b f(t)\,\nabla t = G(b) - G(a).$$

In some proofs we will also need the nabla version of integration by parts

$$\int_a^b f^{\nabla}(t)g^{\rho}(t)\,\nabla t = [f(t)g(t)]_a^b - \int_a^b f(t)g^{\nabla}(t)\,\nabla t. \tag{2.6}$$

Further we recall the relationship between the delta and nabla derivatives. The proof of this statement can be found in [**16**, Chap. 4].

Lemma 2.1. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable function on $\mathbb{T}^{\kappa}$ and f^{Δ} is rd-continuous on $\mathbb{T}^{\kappa}$, then f is ∇-differentiable on $\mathbb{T}_{\kappa}$, and*

$$f^{\nabla}(t) = \begin{cases} \lim_{s \to t-} f^{\Delta}(s) & \textit{if } t \textit{ is ld and rs}, \\ f^{\Delta}(\rho(t)) & \textit{otherwise.} \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable function on $\mathbb{T}_{\kappa}$ and g^{∇} is ld-continuous on $\mathbb{T}_{\kappa}$, then g is Δ-differentiable on $\mathbb{T}^{\kappa}$, and

$$g^{\Delta}(t) = \begin{cases} \lim_{s \to t+} g^{\nabla}(s) & \textit{if } t \textit{ is ls and rd}, \\ g^{\nabla}(\sigma(t)) & \textit{otherwise.} \end{cases}$$

Especially, if f^Δ is continuous on $\mathbb{T}^\kappa$, resp. g^∇ is continuous on $\mathbb{T}_\kappa$, then

$$\begin{aligned} f^\nabla(t) &= f^\Delta(\rho(t)), \quad \text{resp.} \\ g^\Delta(t) &= g^\nabla(\sigma(t)) \end{aligned} \tag{2.7}$$

holds for any $t \in \mathbb{T}_\kappa$, resp. $t \in \mathbb{T}^\kappa$.

2.2. Half-linear equations

We will mention explicitly the associated Riccati equation, Picone's identity and Roundabout theorem.

LEMMA 2.2. *Let x be a solution of* (HLD) *such that $x(t) \neq 0$ in an interval I. Then $w(t) = \frac{r(t)\Phi(x'(t))}{\Phi(x(t))}$ is a solution of the Riccati-type differential equation*

$$w' + c(t) + (p-1)\frac{|w|^q}{\Phi^{-1}(r)} = 0, \quad \Phi^{-1}(r) = |r|^{q-1}\,\mathrm{sgn}\,(r) \tag{2.8}$$

on I, where q is the conjugate number of p, i.e., $q = \frac{p}{p-1}$.

PROPOSITION 2.1 (Picone's identity). *Suppose that w is a solution of the Riccati equation* (2.8) *on $[a,b]$ and let $y \in C^1[a,b]$. Then for $t \in [a,b]$*

$$(w|y|^p)' = r|y'|^p - c|y|^p - G(y,w),$$

holds, where

$$G(y,w) = \frac{p}{\Phi^{-1}(r)}\left[\frac{|\Phi^{-1}(r)y'|^p}{p} - w\Phi(y)\Phi^{-1}(r)y' + \frac{|w\Phi(y)|^q}{q}\right].$$

Equation (HLD) is said to be *disconjugate* on the closed interval $[a,b]$ if the solution x given by the initial condition $x(a) = 0$, $r(a)\Phi(x'(a)) = 1$ has no zero in the interval $(a,b]$. Otherwise (HLD) is said to be *conjugate* on $[a,b]$. The so-called Roundabout theorem relates the Riccati equation, the energy functional and the basic oscillatory properties of the solutions of equation (HLD).

PROPOSITION 2.2 (Roundabout theorem). *The following statements are equivalent:*

(i) *Equation* (HLD) *is disconjugate on the interval $[a,b]$.*
(ii) *Equation* (HLD) *has a positive solution on $[a,b]$.*
(iii) *There exists a solution w of the Riccati equation* (2.8) *which is defined on whole interval $[a,b]$.*
(iv) *The energy functional*

$$\mathcal{F}(y;a,b) = \int_a^b \left[r(t)|y'|^p - c(t)|y|^p\right] \mathrm{d}t,$$

is positive for every nontrivial function y, such that $y(a) = 0 = y(b)$ and y' is piecewise continuous on $[a,b]$.

Next, we will state the Roundabout theorem for equation (HLΔ). For that we need the following notion. We say that a solution x of equation (HLΔ) contains a *generalized zero* on an interval $(m, m+1]$ if $x_m \neq 0$ and $r_m x_m x_{m+1} \leq 0$.

PROPOSITION 2.3 (Roundabout theorem (difference version)). *The following statements are equivalent:*

(i) *Equation* (HLΔ) *is disconjugate on the interval* $[0, N]$, *i.e., the solution* $\tilde{x}$ *given by the initial conditions* $\tilde{x}_0 = 0, r_0\Phi(\tilde{x}_1) = 1$ *has no generalized zero in* $(0, N+1]$.
(ii) *There exists a solution of* (HLΔ) *having no generalized zero in* $[0, N+1]$.
(iii) *There exists a solution* w *of the Riccati-type difference equation (related to* (HLΔ) *by substitution* $w_k = r_k\Phi(\frac{\Delta x_k}{x_k})$*)*

$$\Delta w_k + c_k + w_k\left(1 - \frac{r_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))}\right) = 0$$

which is defined on whole interval $[0, N+1]$ *and satisfies* $\Phi^{-1}(r_k) + \Phi^{-1}(w_k) > 0$ *on interval* $[0, N]$.
(iv) *The discrete* p*-degree functional*

$$\mathcal{F}_d(y; 0, N) = \sum_{k=0}^{N} \left[r_k|\Delta y_k|^p - c_k|y_{k+1}|^p\right],$$

is positive for every nontrivial sequence $y = \{y_k\}_{k=0}^{N+1}$, *such that* $y_0 = 0 = y_{N+1}$.

More details about equations (HLD), (HLΔ), including proofs of the statements mentioned above, can be found in [**20**].

The next step in development of the basic theory of half-linear dynamic equations is the dynamic equation

$$(\mathrm{HL}^\Delta) \qquad (r(t)\Phi(x^\Delta))^\Delta + c(t)\Phi(x^\sigma) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn}(x), p > 1,$$

investigated by Řehák, see [**45**], [**46**]. This equation involves both of previously mentioned equations (HLD), (HLΔ) as special cases, it is sufficient to choose as a time scale the reals $\mathbb{R}$ or the integers $\mathbb{Z}$. The main results for (HL$^\Delta$) important for our thesis can be found in Chapter 3.

2.3. Symplectic dynamic systems

There exists well developed theory of linear Hamiltonian systems (further denoted LHS), i.e., systems of the form

$$\begin{aligned} x' &= A(t)x + B(t)u, \\ u' &= C(t)x - A^T(t)u, \end{aligned} \tag{2.9}$$

where $x, u \in \mathbb{R}^n$, A, B, C being $n \times n$ matrices with B, C symmetric. Overview of results concerning (2.9) achieved until 1995 can be found in monographs of Reid [**48**] and of Kratz [**36**].

Before passing to the main theorem of this section which summarizes oscillatory properties of system (2.9), we need to recall some basic definitions. We say that two points t_1, t_2 are *conjugate* relative to (2.9) if there exists a solution x such that $x(t_1) = 0 = x(t_2)$ and $x(t) \not\equiv 0$ in $[t_1, t_2]$. System (2.9) is said to be *conjugate* in an interval $[a, b]$ if there exist points $t_1, t_2 \in [a, b]$ which are conjugate relative to (2.9). In the opposite case system (2.9) is said to be *disconjugate*. We say that system (2.9) is *oscillatory* if for every $c \in \mathbb{R}$

this system is conjugate in $[c, \infty)$, otherwise system (2.9) is said to be *nonoscillatory*. The system (2.9) is said to be *identically normal* (*controllable*) on an interval $I \subset \mathbb{R}$, if the trivial solution $(x, u) \equiv (0, 0)$, is the only solution for which $x(t) \equiv 0$ on a nondegenerate subinterval of I.

The matrix analogy of (2.9) is the system

$$\begin{aligned} X' &= A(t)X + B(t)U, \\ U' &= C(t)X - A^T(t)U, \end{aligned} \tag{2.10}$$

where X, U are $n \times n$ matrices. A solution (X, U) of system (2.10) is said to be *conjoined* if $X^T U$ is symmetric matrix and it is said to be *conjoined basis* if, moreover, rank $\left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right) = n$.

In oscillation theory of (2.9), an important role is played by the associated quadratic functional

$$\mathcal{F}(x, u) = \int_a^b \left[u^T(t)B(t)u(t) + x^T(t)C(t)x(t) \right] \mathrm{d}t, \tag{2.11}$$

and the Riccati matrix equation

$$Q' - C(t) + A^T(t)Q + QA(t) + QB(t)Q = 0, \tag{2.12}$$

where the matrix Q is related to (2.10) by the substitution $Q = UX^{-1}$.

THEOREM 2.1. *Assume that* (2.9) *is identically normal on interval* $[a, b]$ *and that the matrix* B *is nonnegative definite in this interval. Then the following statements are equivalent:*

(i) *System* (2.9) *is disconjugate on the interval* $[a, b]$.
(ii) *The quadratic functional* (2.11) *is positive for every nontrivial* (x, u) *satisfying conditions* $x'(t) = A(t)x + B(t)u$, $x(a) = 0 = x(b)$ *and* $x(t) \not\equiv 0$ *in* $[a, b]$.
(iii) *The solution* (X, U) *of* (2.10) *given by the initial condition* $X(a) = 0, U(a) = I$ *satisfies* $\det X(t) \neq 0$ *for* $t \in [a, b]$.
(iv) *There exists a conjoined basis* (X, U) *of* (2.10) *such that* $X(t)$ *is nonsingular for* $t \in [a, b]$.
(v) *There exists a symmetric matrix* Q *which for* $t \in [a, b]$ *solves the Riccati matrix differential equation* (2.12).

System (2.9) can be rewritten as the first order system

$$z' = \mathcal{H}(t)z, \quad z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \tag{2.13}$$

where the matrix $\mathcal{H}$ satisfies the identity

$$\mathcal{H}^T(t)\mathcal{J} + \mathcal{J}\mathcal{H}(t) = 0, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.14}$$

I being the $n \times n$ identity matrix.

One of the fundamental properties of LHS is that its fundamental matrix Z is *symplectic*, i.e., $Z^T(t)\mathcal{J}Z(t) = \mathcal{J}$, whenever it is symplectic in the initial condition. Indeed,

$$(Z^T \mathcal{J} Z)' = Z^T \mathcal{H}^T \mathcal{J} Z + Z^T \mathcal{J} \mathcal{H} Z = Z^T (\mathcal{H}^T \mathcal{J} + \mathcal{J}\mathcal{H}) Z = 0.$$

The discrete counterpart of (2.13) is the so-called *symplectic difference system* (further referred to as SDS)

$$z_{k+1} = \mathcal{S}_k z_k, \tag{2.15}$$

where $z_k \in \mathbb{R}^{2n}, \mathcal{S}_k \in \mathbb{R}^{2n\times 2n}$, $\mathcal{S}_k$ being symplectic matrix, i.e.,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The symplecticity of the fundamental matrix of system (2.15) is caused by the fact that the set of all symplectic $(2n \times 2n)$-matrices forms a group with respect to the matrix multiplication. The basic properties of (2.15) are described in [**12**].

A (delta) symplectic dynamic system

$$z^\Delta = \mathcal{S}(t) z \tag{2.16}$$

with $z \in \mathbb{R}^{2n}$, $\mathcal{S} : \mathbb{T} \to \mathbb{R}^{2n\times 2n}$, then represents a unification of the previous two cases. Similarly as before, the desired property of the fundamental matrix of this system is that the fundamental matrix should be symplectic, i.e., $Z^T(t)\mathcal{J}Z(t) = \mathcal{J}$ ($\mathcal{J}$ being the same matrix as in (2.14)), whenever it has this property at one point of $\mathbb{T}$. In the case of symplectic dynamic system, this condition is of the form

$$\begin{aligned}(Z^T\mathcal{J}Z)^\Delta &= (Z^T)^\Delta \mathcal{J} Z^\sigma + Z^T\mathcal{J}Z^\Delta = Z^T\mathcal{S}^T\mathcal{J}(Z + \mu Z^\Delta) + Z^T\mathcal{J}\mathcal{S}Z = \\ &= Z^T\mathcal{S}^T\mathcal{J}Z + \mu Z^T\mathcal{S}^T\mathcal{J}\mathcal{S}Z + Z^T\mathcal{J}\mathcal{S}Z = Z^T(\mathcal{S}^T\mathcal{J} + \mathcal{J}\mathcal{S} + \mu\mathcal{S}^T\mathcal{J}\mathcal{S})Z.\end{aligned}$$

Therefore, the symplecticity condition of system (2.16) reads as

$$\mathcal{S}^T(t)\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\mathcal{S}^T(t)\mathcal{J}\mathcal{S}(t) = 0, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.17}$$

I being the $n \times n$ identity matrix. If we write the matrix $\mathcal{S}$ in the form $\mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ with $n \times n$ matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, then (2.17) translates as

$$\begin{aligned} \mathcal{C} - \mathcal{C}^T + \mu(\mathcal{A}^T\mathcal{C} - \mathcal{C}^T\mathcal{A}) &= 0, \\ \mathcal{B}^T - \mathcal{B} + \mu(\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B}) &= 0, \\ \mathcal{A}^T + \mathcal{D} + \mu(\mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B}) &= 0. \end{aligned} \tag{2.18}$$

The matrix symplecticity condition $Z^T(t)\mathcal{J}Z(t) = \mathcal{J}$ can be equivalently written as $Z(t)\mathcal{J}Z^T(t) = \mathcal{J}$ and using this equation one can easily derive a complementary set of conditions to (2.18), i.e.

$$\begin{aligned} \mathcal{C} - \mathcal{C}^T + \mu(\mathcal{C}\mathcal{D}^T - \mathcal{D}\mathcal{C}^T) &= 0, \\ \mathcal{B}^T - \mathcal{B} + \mu(\mathcal{A}\mathcal{B}^T - \mathcal{B}\mathcal{A}^T) &= 0, \\ \mathcal{D} + \mathcal{A}^T + \mu(\mathcal{D}\mathcal{A}^T - \mathcal{C}\mathcal{B}^T) &= 0. \end{aligned} \tag{2.19}$$

Basic qualitative properties of delta symplectic systems have been established in the papers [**14, 22, 24, 33**] and are summarized in [**16**, Chap. IX]. The main tool in the investigation of qualitative properties of (2.16) is the so-called Roundabout theorem which relates oscillatory properties of this system to the positivity of the associated quadratic functional and the solvability of the Riccati matrix equation. In this statement, system (2.16) is considered on a time scale interval $[a,b] \subset \mathbb{T}$.

PROPOSITION 2.4. ([**33**]). *Suppose that* (2.16) *is dense-normal on every interval* $[a,s]$, *where* $s \in [a,b]$ *is a dense point, i.e., the trivial solution* $z = \binom{x}{u} \equiv 0$ *is the the only solution for which* $x(t) \equiv 0$ *on* $[a,s]$. *Then the following statements are equivalent:*

(i) *The quadratic functional*

$$\mathcal{F}(z) = \int_a^b \left\{ z^T(\mathcal{S}^T\mathcal{K} + \mathcal{K}\mathcal{S} + \mu\mathcal{S}^T\mathcal{K}\mathcal{S})z \right\}(t)\,\Delta t, \quad \mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

is positive definite, i.e., $F(z) > 0$ *for every* $z = \binom{x}{u} : [a,b] \to \mathbb{R}^{2n}$ *for which* $x(a) = 0 = x(b)$ *and* $x \not\equiv 0$ *on* $[a,b]$.

(ii) *The* $2n \times n$ *solution* $Z = \binom{X}{U}$ *given by the initial condition* $X(a) = 0, U(a) = I$ *is such that* $X(t)$ *is invertible in all dense points in* $(a,b]$, $Ker\, X(\sigma(t)) \subseteq Ker\, X(t)$, *and* $X(t)X^\dagger(\sigma(t))\mathcal{B}(t) \geq 0$ *for* $t \in [a,\rho(b)]$. *Here Ker,* †, *and* $\geq$ *denote the kernel, Moore-Penrose generalized inverse, and nonnegative definiteness of the matrix indicated.*

(iii) *There exists a symmetric solution* Q *on* $[a,b]$ *of the Riccati matrix equation*

$$Q^\Delta = \mathcal{C}(t) + \mathcal{D}(t)Q - Q^\sigma(\mathcal{A}(t) + \mathcal{B}(t)Q)$$

such that $I + \mu(\mathcal{A} + \mathcal{B}Q)$ *is nonsingular and* $[I + \mu(\mathcal{A} + \mathcal{B}Q)]^{-1}\mathcal{B} \geq 0$ *on* $[a,\rho(b)]$.

A nabla symplectic system is the first order system

$$z^\nabla = \mathcal{S}(t)z \tag{2.20}$$

with the $2n \times 2n$ matrix $\mathcal{S}$ satisfying

$$\mathcal{S}^T(t)\mathcal{J} + \mathcal{J}\mathcal{S}(t) - \nu(t)\mathcal{S}^T(t)\mathcal{J}\mathcal{S}(t) = 0$$

and in terms of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ this identity reads as

$$\begin{aligned} \mathcal{C} - \mathcal{C}^T - \nu(\mathcal{A}^T\mathcal{C} - \mathcal{C}^T\mathcal{A}) &= 0, \\ \mathcal{B}^T - \mathcal{B} - \nu(\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B}) &= 0, \\ \mathcal{A}^T + \mathcal{D} - \nu(\mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B}) &= 0. \end{aligned} \tag{2.21}$$

The concept of the nabla symplectic system is quite new and these systems have not been studied in the literature yet (at least, as far as we know), but it can be shown that basic properties of solutions of these systems are the same as those of (2.16). In particular, the fundamental matrix of this system is symplectic whenever it is symplectic at one point of $\mathbb{T}$.

CHAPTER 3

HALF-LINEAR DYNAMIC EQUATIONS WITH MIXED DERIVATIVES

3.1. Introduction

In this chapter we investigate oscillatory properties of solutions of the half-linear second order dynamic equation with mixed derivatives

$$(r(t)\Phi(x^{\Delta}))^{\nabla} + c(t)\Phi(x) = 0. \tag{3.1}$$

Recently, several papers dealing with the Sturm-Liouville second order dynamic equation of the form (which is the special case $p = 2$ in (3.1))

$$(r(t)x^{\Delta})^{\nabla} + c(t)x = 0 \tag{3.2}$$

appeared, see [**10, 38**] and also [**16**, Chap. IV], where the basic qualitative theory of (3.2) has been established. It was shown that qualitative properties of solutions of this equation are very similar to those of the "normal" Sturm-Liouville dynamic equation

$$(r(t)x^{\Delta})^{\Delta} + c(t)x^{\sigma} = 0, \tag{3.3}$$

the theory of which is now relatively deeply developed, see [**15**] and the references given therein.

Another motivation for our research is a series of papers [**2, 45, 46**], where the half-linear dynamic equation

$$(r(t)\Phi(x^{\Delta}))^{\Delta} + c(t)\Phi(x^{\sigma}) = 0, \quad \Phi(x) := |x|^{p-1}\operatorname{sgn}(x),\ p > 1, \tag{3.4}$$

is investigated and a theory unifying the theory of half-linear differential and difference equations is established.

3.2. Basic facts

Here we start with several lemmas, that are used later on in this chapter. In the theory of half-linear equations, the frequently used tool is the Young inequality, see [**31**].

LEMMA 3.1. *If $p > 1$ and $q > 1$ are mutually conjugate numbers, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, then for any $u, v \in \mathbb{R}$*

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \geq |uv|, \tag{3.5}$$

and equality holds if and only if $u = |v|^{q-2}v$.

The next lemma can be considered as a time scale version of the second mean value theorem of integral calculus. Its proof can be found in [**45**].

LEMMA 3.2. *Let f be a function such that its Δ-derivative f^Δ is rd-continuous and f^Δ does not change its sign for $t \in [a,b]$. Then for any rd-continuous function g there exist points $c,d \in [a,b]^\kappa$ such that*

$$\int_a^b f^\sigma(t)g(t)\,\Delta t \le f(a)\int_a^c g(t)\,\Delta t + f(b)\int_c^b g(t)\,\Delta t$$

and

$$\int_a^b f^\sigma(t)g(t)\,\Delta t \ge f(a)\int_a^d g(t)\,\Delta t + f(b)\int_d^b g(t)\,\Delta t.$$

Lemma 2.1, applied to the Δ-integral and ∇-integral, gives the following result.

LEMMA 3.3. *Let f be a ld-continuous function and let*

$$\hat{f}(t) = \begin{cases} \lim_{s\to t+} f(s) & \textit{if } t \textit{ is ls and rd point},\\ f^\sigma(t) & \textit{otherwise.}\end{cases}$$

Then

$$\int_a^b f(t)\,\nabla t = \int_a^b \hat{f}(t)\,\Delta t.$$

PROOF. Let F be the ∇-antiderivative of f, i.e., $F^\nabla = f$. Then by Lemma 2.1 we have

$$F^\Delta(t) = \begin{cases} \lim_{s\to t+} F^\nabla(s) = \lim_{s\to t+} f(s) & \text{if } t \text{ is ls and rd},\\ F^\nabla(\sigma(t)) = f^\sigma(t) & \text{otherwise.}\end{cases}$$

Hence, $F^\Delta(t) = \hat{f}(t)$, and thus

$$\int_a^b \hat{f}(t)\,\Delta t = F(t)|_a^b = \int_a^b f(t)\,\nabla t.$$

□

Further we present a formula for the ∇-derivative of a composite function, the proof of this statement is the same as for Δ-derivative and it is based on the Lagrange Mean Value Theorem.

LEMMA 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $g : \mathbb{T} \to \mathbb{R}$ be ∇-differentiable. Then we have*

$$[f(g(t))]^\nabla = f'(\xi)g^\nabla(t),$$

where ξ is between $g^\rho(t)$ and $g(t)$.

Now we recall some results of the above mentioned papers [**2**] and [**38**] that deal with equations (3.2) and (3.4). These results are summarized in statements which are usually referred to as the Reid Roundabout theorem. Recall that by a solution of (3.2) it is understood a function x which is Δ-differentiable, rx^Δ is ∇-differentiable and (3.2) is satisfied. A solution of (3.4) is defined in a similar way. We use the standard notation for time scale intervals. An interval $[a,b]$ actually means $\{t \in \mathbb{T} : a \le t \le b\}$, open and half-open intervals have the same meaning.

PROPOSITION 3.1 ([**38**], [**16**, Chap. 4]). *Suppose that the function c is ld-continuous, r is continuous and $r(t) > 0$. Then the following statements are equivalent:*

(i) *Equation* (3.2) *is disconjugate on an interval $[\rho(a), \sigma(b)]$, i.e., the solution x of* (3.2) *given by the initial condition $x^\rho(a) = 0$, $(rx^\Delta)^\rho(a) = 1$ has no generalized zero in $(\rho(a), \sigma(b)]$, i.e., it satisfies $x^\rho(t)x(t) > 0$ for $t \in (\rho(a), \sigma(b)]$.*
(ii) *There exists a solution of* (3.2) *having no generalized zero in $[\rho(a), \sigma(b)]$.*
(iii) *The quadratic functional*

$$\mathcal{F}(y) = \int_{\rho(a)}^{\sigma(b)} \left[r^\rho(t)(y^\nabla)^2 - c(t)y^2\right] \nabla t > 0$$

over nontrivial $y : [\rho(a), \sigma(b)] \to \mathbb{R}$ for which y^∇ exists, it is ld-continuous, and $y^\rho(a) = 0 = y^\sigma(b)$.
(iv) *There exists a solution of the Riccati equation*

$$w^\nabla + c(t) + \frac{(w^\rho)^2}{r^\rho(t) + \nu(t)w^\rho} = 0,$$

related to (3.2) *by the substitution $w = \frac{r(t)x^\Delta}{x}$, which is defined on $[\rho(a), \sigma(b)]$ and satisfies there $r^\rho(t) + \nu(t)w^\rho > 0$.*

Note that it is supposed in [**10**] that *both* functions c, r in (3.2) are continuous. However, under this assumption the ∇-derivative $(r(t)x^\Delta)^\nabla$ is continuous, in particular, ld-continuous, hence applying the forward jump operator to (3.2), using (2.7) we get the equation

$$(r(t)x^\Delta)^\Delta + c^\sigma(t)x^\sigma = 0$$

which is just the equation of the form (3.3) and the above formulated Proposition 3.1 can be essentially deduced from a corresponding statement for (3.3), see [**15**]. Also, a statement analogous to Proposition 3.1 can be formulated without positivity assumption on the function r, however, as showed, e.g., in [**24**] where (3.3) is investigated, "reasonable" oscillation criteria can be derived only under some sign restrictions on the function r, we refer to [**24**] for details. Finally, note that our presentation of Proposition 3.1 follows exactly the presentation of [**16**] and [**38**]. Later, in Section 3.3, we give a similar result for half-linear equation (3.1), but instead of the interval $[\rho(a), \sigma(b)]$ considered in Proposition 3.1, we formulate our results for $t \in [a, b]$.

Now we turn our attention to the Roundabout theorem for (3.4), see [**45**].

PROPOSITION 3.2. *Suppose that the functions r, c are rd-continuous and $r(t) \neq 0$. Then the following statements are equivalent.*

(i) *Equation* (3.4) *is disconjugate on a time scale interval $[a, b]$, i.e., the solution x given by the initial condition $x(a) = 0$, $r(a)\Phi(x^\Delta(a)) = 1$ has no generalized zero in $(a, b]$, i.e., $r(t)\Phi(x(t))\Phi(x^\sigma(t)) > 0$ for $t \in (a, b]$.*
(ii) *There exists a solution of* (3.4) *having no generalized zero in $[a, b]$.*
(iii) *The energy functional*

$$\mathcal{F}(y) = \int_a^b \left[r(t)|y^\Delta|^p - c(t)|y^\sigma|^p\right] \Delta t > 0$$

for every nontrivial y whose Δ-derivative is piecewise rd-continuous and at endpoins $y(a) = 0 = y(b)$ holds.

(iv) *There exists a solution of the Riccati-type equation (related to (3.4) by the substitution $w = r\Phi(x^\Delta/x)$)*

$$w^\Delta + c(t) = \begin{cases} -(p-1)r^{1-q}(t)|w|^q & \text{if } \sigma(t) = t, \\ -\frac{w}{\mu(t)}\left(1 - \frac{r(t)}{\Phi(\Phi^{-1}(r(t))+\mu(t)\Phi^{-1}(w))}\right) & \text{if } \sigma(t) > t \end{cases}$$

which is defined for $t \in [a,b]$ and satisfies $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t)) > 0$ in this interval.

3.3. Picone's identity and Roundabout theorem

Before passing to the main subjects of this section which are basic statements for the elaboration of the oscillation theory of (3.1), let us note that we are not concerned with the *existence and uniqueness* problem for (3.1) in this thesis. This result can be proved using the time scales induction essentially in the same way as in [**10**, Theorem 3.1] and [**45**, Section 3].

Throughout what follows we suppose that

(H) $r(t)$ is continuous, $c(t)$ is ld-continuous, and $r(t) \neq 0$

on a time scale interval under consideration. Under this assumption, system (3.1) can be written as a 2×2 system

$$x^\nabla = \Phi^{-1}(u^\rho/r^\rho(t)), \quad u^\nabla = -c(t)\Phi(x^\rho + \nu(t)\Phi^{-1}(u^\rho/r^\rho(t))),$$

and the existence and uniqueness problem for (3.1) is investigated via this first order system. We have the same statement as [**10**, Theorem 3.1], namely that a solution of (3.1) is uniquely determined by the initial condition $x(t_0) = x_0$, $x^\nabla(t_0) = x_1$, $t_0 \in \mathbb{T}$, $x_0, x_1 \in \mathbb{R}$, it exists on any interval where the hypotheses (H) are satisfied and depends continuously on the initial condition. We conjucture, that the results of this section remain to hold under the weaker assumption that r is only ld-continuous, but under this weaker assumption we have till now some difficulties with the existence problem for (3.1).

We start with the Riccati substitution for (3.1).

LEMMA 3.5. *Suppose that x is a solution of (3.1) such that $x(t) \neq 0$ on a time scale interval $I = [a,b]$. Then $w = r\Phi(x^\Delta/x)$ is a solution of the Riccati-type equation*

$$w^\nabla + c(t) = \begin{cases} -(p-1)\frac{|w|^q}{\Phi^{-1}(r(t))} & \text{if } t = \rho(t), \\ -\frac{w^\rho}{\nu(t)}\left(1 - \frac{r^\rho(t)}{\Phi(\Phi^{-1}(r^\rho(t))+\nu(t)\Phi^{-1}(w^\rho))}\right) & \text{if } \rho(t) < t. \end{cases} \tag{3.6}$$

Moreover, if

$$r^\rho(t)x(t)x^\rho(t) > 0 \quad \text{for} \quad t \in [a,b]_\kappa, \tag{3.7}$$

holds, then

$$\Phi^{-1}(r^\rho(t)) + \nu(t)\Phi^{-1}(w^\rho(t)) > 0 \tag{3.8}$$

for $t \in [a,b]_\kappa$.

PROOF. Let $w = r\Phi(\frac{x^\Delta}{x})$. Then using (2.5) and (2.3) we have (suppressing the argument t)

$$\begin{aligned} w^\nabla &= \frac{(r\Phi(x^\Delta))^\nabla \Phi(x) - r\Phi(x^\Delta)\Phi^\nabla(x)}{\Phi(x^\rho)\Phi(x)} \\ &= \frac{(r\Phi(x^\Delta))^\nabla (\Phi(x^\rho) + \nu\Phi^\nabla(x)) - r\Phi(x^\Delta)\Phi^\nabla(x)}{\Phi(x^\rho)\Phi(x)} \\ &= -c + \frac{[\nu(r\Phi(x^\Delta))^\nabla - r\Phi(x^\Delta)]\Phi^\nabla(x)}{\Phi(x^\rho)\Phi(x)} \\ &= -c - \frac{(r\Phi(x^\Delta))^\rho \Phi^\nabla(x)}{\Phi(x^\rho)\Phi(x)} = -c - \frac{w^\rho \Phi^\nabla(x)}{\Phi(x)}. \end{aligned}$$

Now we have to distinguish two cases:
(i) Suppose that t is left dense. Then the nabla derivative reduces to the "normal derivative" ($\Phi^\nabla(x) = \Phi'(x)$) and the ρ-operator has no effect, so that

$$\begin{aligned} w^\nabla &= -c - w\frac{\Phi'(x)}{\Phi(x)} = -c - w\frac{(p-1)|x|^{p-2}x'}{|x|^{p-1}} = -c - (p-1)w\frac{x'}{x}\frac{\Phi^{-1}(r)}{\Phi^{-1}(r)} \\ &= -c - (p-1)w\frac{\Phi^{-1}(w)}{\Phi^{-1}(r)} = -c - (p-1)\frac{|w|^q}{\Phi^{-1}(r)} \end{aligned}$$

which is equation (3.6).
(ii) Suppose that t is left scattered. Then because of (2.3) and (2.7)

$$\begin{aligned} \frac{\Phi^\nabla(x)}{\Phi(x)} &= \frac{\Phi(x) - \Phi(x^\rho)}{\nu\Phi(x)} = \frac{1}{\nu}\left(1 - \frac{\Phi(x^\rho)}{\Phi(x)}\right) = \frac{1}{\nu}\left(1 - \Phi\left(\frac{x^\rho}{x^\rho + \nu(x^\Delta)^\rho}\right)\right) \\ &= \frac{1}{\nu}\left(1 - \frac{1}{\Phi(1 + \nu(\frac{x^\Delta}{x})^\rho)}\right) = \frac{1}{\nu}\left(1 - \frac{r^\rho}{\Phi(\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho))}\right) \end{aligned}$$

which implies the second case of relation (3.6). The last fact we need to prove is that the inequality $\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho) > 0$ is valid for $t \in [a,b]_\kappa$. But

$$\begin{aligned} \Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho) &= \Phi^{-1}(r^\rho) + \nu\Phi^{-1}(r^\rho)\frac{x^{\Delta\rho}}{x^\rho} = \Phi^{-1}(r^\rho)\left(1 + \nu\frac{x^{\Delta\rho}}{x^\rho}\right) \\ &= \Phi^{-1}(r^\rho)\frac{x^\rho + \nu x^{\Delta\rho}}{x^\rho} = \Phi^{-1}(r^\rho)\frac{x^\rho + \nu x^\nabla}{x^\rho} = \Phi^{-1}(r^\rho)\frac{x}{x^\rho} \end{aligned}$$

and the last expression is positive if and only if (3.7) holds. □

In the next statement and also later we will denote by C^1_{ld} the class of functions $y : [a,b] \subset \mathbb{T} \to \mathbb{R}$ such that y^∇ exists and it is ld-continuous.

THEOREM 3.1 (Picone's Identity). *Assume that w is a solution of Riccati equation (3.6) on $[a,b]$. Let $y \in C^1_{ld}[a,b]$. Then for $t \in [a,b]$ (suppressing the argument)*

$$(w|y|^p)^\nabla = r^\rho|y^\nabla|^p - c|y|^p - G(y,w), \tag{3.9}$$

holds, where

$$G(y,w)=\begin{cases}\frac{p}{\Phi^{-1}(r)}\left[\frac{|\Phi^{-1}(r)y^{\nabla}|^p}{p}-w\Phi(y)\Phi^{-1}(r)y^{\nabla}+\frac{|w\Phi(y)|^q}{q}\right] & \text{if } \rho(t)=t,\\ r^{\rho}|y^{\nabla}|^p-\frac{w^{\rho}r^{\rho}}{\nu\Phi(\Phi^{-1}(r^{\rho})+\nu\Phi^{-1}(w^{\rho}))}|y^{\rho}+\nu y^{\nabla}|^p+\frac{w^{\rho}}{\nu}|y^{\rho}|^p & \text{if } \rho(t)<t.\end{cases}\tag{3.10}$$

PROOF. First suppose that t is left dense, i.e. $\rho(t)=t$. Then

$$\begin{aligned}(w|y|^p)^{\nabla}&=w^{\nabla}|y|^p+w(|y|^p)^{\nabla}=\Big(-c-(p-1)\frac{|w|^q}{\Phi^{-1}(r)}\Big)|y|^p+pw\Phi(y)y^{\nabla}\\&=r|y^{\nabla}|^p-c|y|^p-p\{\frac{r|y^{\nabla}|^p}{p}-w\Phi(y)y^{\nabla}+\frac{1}{q}\frac{|w|^q|y|^p}{\Phi^{-1}(r)}\}\\&=r|y^{\nabla}|^p-c|y|^p-\frac{p}{\Phi^{-1}(r)}\{\frac{|\Phi^{-1}(r)y^{\nabla}|^p}{p}-w\Phi(y)\Phi^{-1}(r)y^{\nabla}+\frac{|w\Phi(y)|^q}{q}\}.\end{aligned}$$

For ls-point t we have (using (2.4) and (3.6))

$$\begin{aligned}&(w|y|^p)^{\nabla}\\&=w^{\nabla}|y|^p+w^{\rho}(|y|^p)^{\nabla}=\big[-c-\frac{w^{\rho}}{\nu}\big(1-\frac{r^{\rho}}{\Phi(\Phi^{-1}(r^{\rho})+\nu\Phi^{-1}(w^{\rho}))}\big)\big]|y|^p\\&\quad+w^{\rho}\frac{|y|^p-|y^{\rho}|^p}{\nu}\\&=r^{\rho}|y^{\nabla}|^p-c|y|^p+\frac{w^{\rho}r^{\rho}}{\nu\Phi(\Phi^{-1}(r^{\rho})+\nu\Phi^{-1}(w^{\rho}))}|y|^p-\frac{w^{\rho}}{\nu}|y^{\rho}|^p-r^{\rho}|y^{\nabla}|^p\\&=r^{\rho}|y^{\nabla}|^p-c|y|^p-\big\{r^{\rho}|y^{\nabla}|^p-\frac{w^{\rho}r^{\rho}}{\nu\Phi(\Phi^{-1}(r^{\rho})+\nu\Phi^{-1}(w^{\rho}))}|y|^p+\frac{w^{\rho}}{\nu}|y^{\rho}|^p\big\},\end{aligned}$$

i.e., (3.10) holds since $y=y^{\rho}+\nu y^{\nabla}$. □

THEOREM 3.2. *Let the assumptions of the previous theorem be satisfied and, in addition, suppose that*

$$\Phi^{-1}(r^{\rho}(t))+\nu(t)\Phi^{-1}(w^{\rho}(t))>0\tag{3.11}$$

for $t\in\mathbb{T}_{\kappa}$. Then $G(y,w)(t)\ge 0$ for $t\in[a,b]_{\kappa}$, where the equality holds if and only if $w\Phi(y)=r\Phi(y^{\Delta})$.

PROOF. Again, suppose first that t is left dense. Then because $\nu(t)=0$ holds, condition (3.11) implies $\Phi^{-1}(r(t))>0$. We have

$$G(y,w)=\frac{p}{\Phi^{-1}(r)}\{\frac{|\Phi^{-1}(r)y^{\nabla}|^p}{p}-w\Phi(y)\Phi^{-1}(r)y^{\nabla}+\frac{|w\Phi(y)|^q}{q}\}.$$

This case is very easy to prove, because the expression in brackets is nonnegative according to Young's inequality (Lemma 3.1 with $u=\Phi^{-1}(r)y^{\nabla}, v=w\Phi(y)$). Equality occurs if and only if $v=\Phi(u)$, i.e., if and only if $w\Phi(y)=r\Phi(y^{\Delta})$. Note that this equality holds iff w is related to y by the Riccati substitution.

Now suppose that t is ls-point. If we set $\alpha = \nu y^\nabla, \beta = y^\rho$, we can write the function G in variables α, β as

$$G(\alpha,\beta) = \frac{1}{\nu}\Big\{\frac{r^\rho}{\nu^{p-1}}|\alpha|^p - \frac{w^\rho r^\rho}{\Phi(\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho))}|\alpha+\beta|^p + w^\rho|\beta|^p\Big\}.$$

If the case $\alpha+\beta=0$ occurs, then $|\alpha|=|\beta|$ and the function $G(\alpha,\beta)$ is of following form

$$G(\alpha,\beta) = G(\alpha) = \frac{|\alpha|^p}{\nu}\Big\{\frac{r^\rho}{\nu^{p-1}} + w^\rho\Big\}$$

and the expression in brackets is positive according to (3.11).

If $\alpha+\beta \neq 0$ then our aim is to prove that

$$\frac{r^\rho\nu^{1-p}|\alpha|^p + w^\rho|\beta|^p}{|\alpha+\beta|^p} \geq \frac{w^\rho r^\rho}{\Phi(\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho))}. \tag{3.12}$$

The left-hand side of the last inequality is homogeneous in variables α, β, i.e., it is not changed by the transformation $\alpha \mapsto k\alpha, \beta \mapsto k\beta$ for any $k \in \mathbb{R}\setminus\{0\}$. For this reason, we can assume that $\alpha+\beta=\pm1$, for example $\alpha+\beta=1$. We will show that the minimum of the function $\widetilde{G}(\alpha,\beta) := \frac{r^\rho}{\nu^{p-1}}|\alpha|^p + w^\rho|\beta|^p$, provided $\alpha+\beta=1$, is equal to the right-hand side of the inequality (3.12).

First we will express $\widetilde{G}$ as a function of only one variable using the condition $\alpha+\beta=1$. So we have

$$\widetilde{G}(\alpha) = \frac{r^\rho}{\nu^{p-1}}|\alpha|^p + w^\rho|1-\alpha|^p.$$

The derivative of this function is

$$\widetilde{G}'(\alpha) = p\Big\{\frac{r^\rho}{\nu^{p-1}}\Phi(\alpha) - w^\rho\Phi(1-\alpha)\Big\}$$

with the only stationary point

$$\alpha^* = \frac{\nu\Phi^{-1}(w^\rho)}{\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho)}.$$

The second derivative is given by

$$\widetilde{G}''(\alpha) = p(p-1)\left\{\frac{r^\rho}{\nu^{p-1}}|\alpha|^{p-2} + w^\rho|1-\alpha|^{p-2}\right\} \tag{3.13}$$

and at the stationary point α^* satisfies

$$\begin{aligned}
&\widetilde{G}''(\alpha^*) = \\
&= p(p-1)\frac{1}{|\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho)|^{p-2}}\Big\{\frac{r^\rho}{\nu^{p-1}}|\nu\Phi^{-1}(w^\rho)|^{p-2} + w^\rho|\Phi^{-1}(r^\rho)|^{p-2}\Big\} \\
&= p(p-1)\frac{1}{\nu|\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho)|^{p-2}}\left\{r^\rho|w^\rho|^{2-q} + \nu w^\rho|r^\rho|^{2-q}\right\} \\
&= p(p-1)\frac{|r^\rho w^\rho|^{2-q}}{\nu|\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho)|^{p-2}}\left\{\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho)\right\}
\end{aligned}$$

so that the sign of $\widetilde{G}''(\alpha^*)$ depends only on the last bracket, which is positive due to our assumption (3.11). This implies that α^* is a local minimum point of the function $\widetilde{G}$ and one can directly verify that the value $\widetilde{G}(\alpha^*)$ just equals the expression on the right-hand side of inequality (3.12). Finally, using again condition (3.11) and (3.13), by a similar computation as above one can verify that $\widetilde{G}''(\alpha) \geq 0$, i.e., $\widetilde{G}$ is convex and hence α^* is also the global minimum of $\widetilde{G}$. $\square$

Oscillatory properties of (3.1) are defined via the concept of a generalized zero of a solution of this equation. We say that a solution x of equation (3.1) has a *generalized zero* at t if $x(t) = 0$ or, if t is ls-point and $(r^\rho x x^\rho)(t) < 0$. We say that equation (3.1) is *disconjugate* on an interval $[a, b]$ if the nontrivial solution y satisfying $y(a) = 0$ has no generalized zero in $(a, b]$ and any other nontrivial solution of (3.1) has at most one generalized zero in $[a, b]$.

Now, let us define $\mathbb{A}$ to be the set of functions

$$\mathbb{A} := \{y \in C^1_{ld}([a, b], \mathbb{R}) : y(a) = y(b) = 0\}$$

and the p-degree functional $\mathcal{F}$ on $\mathbb{A}$ by

$$\mathcal{F}(y; a, b) = \int_a^b \{r^\rho(t)|y^\nabla|^p - c(t)|y|^p\} \nabla t. \tag{3.14}$$

We say $\mathcal{F}$ is *positive definite* (and write $\mathcal{F} > 0$) on $\mathbb{A}$ provided $\mathcal{F}(y) \geq 0$ for all $y \in \mathbb{A}$ and $\mathcal{F}(y) = 0$ if and only if $y \equiv 0$.

The next theorem establishes basic methods of the oscillation theory of (3.1) and relates disconjugacy of this equation to the solvability of the Riccati equation (3.6) and positivity of the energy functional (3.14).

THEOREM 3.3 (Roundabout theorem). *The following statements are equivalent:*

(i) *Equation* (3.1) *is disconjugate on* $[a, b]$.
(ii) *There exists a solution of* (3.1) *having no generalized zero in* $[a, b]$.
(iii) *The Riccati equation* (3.6) *has a solution* w *satisfying for all* $t \in [a, b]_\kappa$ *the inequality* $\{\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)\}(t) > 0$.
(iv) *The* p*-degree functional* $\mathcal{F}$ *is positive definite on* $\mathbb{A}$.

PROOF. We prove that the following four implications are valid:
(i)⇒(ii): Let $\widetilde{y}$ be the solution of (3.1) satisfying the initial conditions $\widetilde{y}(a) = 0, \widetilde{y}^\nabla(a) = 1$. From (i) we get that $(r^\rho \widetilde{y}\widetilde{y}^\rho)(t) > 0$ for $t \in (a, b]$. Consider a solution y_ε given by the initial conditions (with $\varepsilon > 0$)

$$y_\varepsilon(a) = \varepsilon, \quad y_\varepsilon^\nabla(a) = \widetilde{\nu}(a)\Big(\frac{\varepsilon r^\rho(a) - 1}{r^\rho(a)} - \nu(a)\Big) + 1,$$

where $\widetilde{\nu} = 0$ if $\nu = 0$ and $\widetilde{\nu} = \frac{1}{\nu}$ if $\nu > 0$. Then $y_\varepsilon \to \widetilde{y}$ for $\varepsilon \to 0$. Hence, if we choose $\varepsilon > 0$ sufficiently small, then $y \equiv y_\varepsilon$ fulfills $(r^\rho y y^\rho)(t) > 0$ for $t \in (a, b]$. Moreover, for ls-point a we get

$$(r^\rho y y^\rho)(a) = r^\rho(a)\frac{\varepsilon}{r^\rho(a)} = \varepsilon > 0$$

because

$$y^\nabla(a) = \big(\frac{y - y^\rho}{\nu}\big)(a) = \frac{\varepsilon r^\rho(a) - 1}{\nu(a) r^\rho(a)}$$

by (2.2). In the case when a is ld-point we have

$$(r^\rho yy^\rho)(a) = (ry^2)(a) = r(a)\varepsilon^2,$$

which is positive if and only if $r(a) > 0$. Suppose conversely that $r(a) < 0$. Consider a solution $\widehat{y}$ that satisfies the initial conditions $\widehat{y}(d) = 0, \widehat{y}^\Delta(d) = 1$, where $d \in (a, b]$. The disconjugacy of the equation (3.1) implies $(r^\rho \widehat{y}\widehat{y}^\rho)(a) > 0$. Since a is left dense, we get $r(a) > 0$ which is contradiction. Altogether, y is the solution of (3.1) with $(r^\rho yy^\rho)(t) > 0$ for $t \in [a, b]$, so that (ii) holds.
(ii)⇒(iii): This implication is the Riccati substitution already proved in Lemma 3.5.
(iii)⇒(iv): Suppose that w is a solution of Riccati equation (3.6) satisfying the inequality $\{\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)\}(t) > 0$ for $t \in [a, b]_\kappa$. Let $y \in \mathbb{A}$, i.e., $y(a) = y(b) = 0$. From the Picone identity we have

$$r^\rho(t)|y^\nabla|^p - c(t)|y|^p = (w(t)|y|^p)^\nabla + G(y, w)$$

and by integrating from a to b we obtain

$$\begin{aligned}\mathcal{F}(y; a, b) &= \int_a^b \{r^\rho(t)|y^\nabla|^p - c(t)|y|^p\}\,\nabla t\\ &= [w(t)|y|^p]_a^b + \int_a^b G(y, w)\,\nabla t = \int_a^b G(y, w)\,\nabla t.\end{aligned}$$

Hence $\mathcal{F}(y; a, b) \geq 0$ because of Theorem 3.2 and, moreover, the case $\mathcal{F}(y; a, b) = 0$ can occur if and only if $w\Phi(y) = r\Phi(y^\Delta)$, i.e., $y^\Delta = \Phi^{-1}(w/r)y$. But since $y(a) = 0$, the initial value problem admits only the trivial solution. Consequently, $\mathcal{F}(y; a, b) > 0$ for all nontrivial $y \in \mathbb{A}$.
(iv)⇒(i): Suppose, by contradiction, that $\mathcal{F}(y; a, b) > 0$ and (3.1) is not disconjugate on $[a, b]$. Then either the nontrivial solution $\tilde{y}$ of (3.1) given by the initial condition $y(a) = 0$ has a generalized zero in $(a, b]$ or there is a nontrivial solution y of (3.1) such that y has at least two generalized zeros in $(a, b]$. Consider the latter possibility, the former one can be treated in a similar way. Let $\alpha, \beta \in (a, b]$, where $\alpha < \rho(\beta)$, be two smallest generalized zeros of y in $(a, b]$. There are four possibilities according to whether α, β are ld- or ls-points. We consider here the case when β ld-point (i.e., $\rho(\beta) = \beta$) and we construct a nontrivial piecewise continuous function $y \in C^1$ with $y(a) = 0 = y(b)$, such that $\mathcal{F}(y; a, b) \leq 0$. If the remaining two possibilities happen ($\rho(\beta) < \beta$), we proceed in a similar way as in the remaining part of the proof.

First suppose that α is ls-point and define

$$u(t) = \begin{cases} 0 & \text{for } t \in [a, \alpha),\\ y(t) & \text{for } t \in [\alpha, \beta],\\ 0 & \text{for } t \in (\beta, b],\end{cases}$$

which implies $u \in \mathbb{A}$ and $u(t) \neq 0$ for $t \in (\alpha, \beta)$. In the next computation we use integration by parts (2.6), the definition of function u, the fact that $\int_{\rho(t)}^t f(s)\nabla s = f(t)\nu(t)$, and that $(r\Phi(y^\Delta))(\alpha) = (r\Phi(y^\Delta))^\rho(\alpha) + \nu(\alpha)(r\Phi(y^\Delta))^\nabla(\alpha)$.

We have

$$
\begin{aligned}
&\mathcal{F}(u;a,b)\\
&= \int_a^b \left[r^\rho(t)|u^\nabla|^p - c(t)|u|^p\right] \nabla t\\
&= \int_{\rho(\alpha)}^{\alpha} \left[r^\rho(t)|u^\nabla|^p - c(t)|u|^p\right] \nabla t + \int_\alpha^\beta \left[r^\rho(t)|u^\nabla|^p - c(t)|u|^p\right] \nabla t\\
&= \left\{(r^\rho|u^\nabla|^p - c|u|^p)\nu\right\}(\alpha) + ur\Phi(u^\Delta)\Big|_\alpha^\beta - \int_\alpha^\beta u\left[(r(t)\Phi(u^\Delta))^\nabla + c(t)\Phi(u)\right] \nabla t\\
&= \left\{\nu r^\rho |u^\nabla|^p\right\}(\alpha) - u\left\{c\Phi(u)\nu + (r\Phi(u^\Delta))^\rho + \nu(r\Phi(u^\Delta))^\nabla\right\}(\alpha)\\
&= r^\rho(\alpha)\left|\frac{u(\alpha)-u^\rho(\alpha)}{\nu(\alpha)}\right|^p \nu(\alpha) - y(\alpha)r^\rho(\alpha)\Phi(y^\nabla(\alpha))\\
&= \frac{y(\alpha)r^\rho(\alpha)}{\Phi(\nu(\alpha))}\left[\Phi(y(\alpha)) - \Phi(\nu(\alpha)y^\nabla(\alpha))\right].
\end{aligned}
$$

Hence, it suffices to show that

$$
\left\{yr^\rho\Phi(y) - yr^\rho\Phi(\nu y^\nabla)\right\}(\alpha) \le 0. \tag{3.15}
$$

This inequality is equivalent to the inequality

$$
\left\{\Phi^{-1}(yr^\rho)(y-\nu y^\nabla)\right\}(\alpha) = \left\{\Phi^{-1}(yr^\rho\Phi(y^\rho))\right\}(\alpha) \le 0,
$$

but this inequality holds because according to our assumption α is generalized zero of solution y, so (3.15) holds and hence $\mathcal{F}(u;a,b) \le 0$, a contradiction.

Now suppose that α is an ld-point, i.e., $\rho(\alpha) = \alpha$. Since $r(t) \ne 0$, the inequality $\{r^\rho y^\rho y\}(\alpha) \le 0$ means that either $y(\alpha) = 0$ or $r(\alpha) < 0$. If $y(\alpha) = 0$, the same function u as in the previous part of the proof gives $\mathcal{F}(u;a,b) = 0$, a contradiction, so we suppose that $y(\alpha) \ne 0$ and $r^\rho(\alpha) < 0$. In this case we proceed in the same way as in the continuous case (see, e.g., [**47**]). Let $t_m \to \alpha-$, as $m \to \infty$, be the left-sequence for α and put

$$
u_m(t) = \begin{cases} \frac{t-t_m}{(\alpha-t_m)^{1/p}} & \text{for } t \in [t_m,\alpha] \cap \mathbb{T},\\ 0 & \text{otherwise.} \end{cases}
$$

Now, the same computation as in [**45, 46**] yields

$$
\mathcal{F}(u_m;a,b) \to r^\rho(\alpha) < 0 \quad \text{as } m \to \infty,
$$

a contradiction. □

Remark 3.1. (i) The previous theorem implies that the Sturm Comparison theorem extends verbatim to (3.1). In particular, let the equation

$$
(R(t)\Phi(x^\Delta))^\nabla + C(t)\Phi(x) = 0 \tag{3.16}
$$

be a Sturmian majorant of (3.1) on $[a,b]$, i.e.,

$$
0 < R(t) \le r(t), \quad C(t) \ge c(t), \quad t \in [a,b].
$$

If (3.1) is not disconjugate on $[a,b]$, i.e., there exists a nontrivial function $y \in \mathbb{A}$ such that

$$\mathcal{F}_{rc}(y;a,b) = \int_a^b \left[r^\rho(t)|y^\nabla|^p - c(t)|y|^p\right] \nabla t \le 0,$$

then also

$$\mathcal{F}_{RC}(y;a,b) = \int_a^b \left[R^\rho(t)|y^\nabla|^p - C(t)|y|^p\right] \nabla t \le 0,$$

and hence (3.16) is not disconjugate as well. Conversely, if (3.16) is disconjugate on $[a,b]$, i.e., $\mathcal{F}_{RC}(y;a,b) > 0$ for every $0 \not\equiv y \in \mathbb{A}$, then $\mathcal{F}_{rc}(y;a,b) > 0$ and (3.1) is also disconjugate on $[a,b]$.
(ii) Theorem 3.3 also shows that (3.1) does not admit coexistence of a solution without generalized zero in $[a,b]$ and a solution having two or more generalized zeros in this interval. Indeed, the existence of a solution of (3.1) without a generalized zero in $[a,b]$ implies $\mathcal{F}_{rc}(y;a,b) > 0$ for $0 \not\equiv y \in \mathbb{A}$, while the existence of a solution with two or more generalized zeros enables to construct a function $0 \not\equiv \tilde{y} \in \mathbb{A}$ for which $\mathcal{F}_{RC}(\tilde{y};a,b) \le 0$.
(iii) The previous remark also justifies the classification of (3.1) on time scales unbounded above as *oscillatory* and *nonoscillatory* in the same way as for the classical linear Sturm-Liouville differential equations.

3.4. Oscillation criteria

Throughout this section we suppose that a time scale under consideration is *unbounded above*; i.e., there exists a sequence $t_n \in \mathbb{T}$ such that $t_n \to \infty$.

Equation (3.1) is said to be *nonoscillatory* if there exists $\alpha \in \mathbb{T}$ such that (3.1) is disconjugate on $[\alpha,\beta]$ for every $\beta > \alpha$. In the opposite case, (3.1) is said to be *oscillatory.*

As a direct consequence of the equivalence (i) and (iv) in the Roundabout theorem, we have the following statement.

LEMMA 3.6. *Equation* (3.1) *is nonoscillatory if and only if there exists* $a \in \mathbb{T}$ *such that*

$$\mathcal{F}(y;a,\infty) = \int_a^\infty \{r^\rho|y^\nabla|^p - c|y|^p\}(t)\,\nabla t > 0$$

for every nontrivial $y : [a,\infty) \to \mathbb{R}$ *with* y^∇ *piecewise ld-continuous, satisfying* $y(a) = 0$, *and for which there exists* $d > a$ *with* $y(t) \equiv 0$ *for* $t > d$.

THEOREM 3.4 (Leighton-Wintner criterion). *Suppose that* $r(t) > 0$ *for large* t

$$\int^\infty (r^\rho(t))^{1-q}\,\nabla t = \infty \quad \text{and} \quad \int^\infty c(t)\,\nabla t = \infty. \tag{3.17}$$

Then equation (3.1) *is oscillatory.*

PROOF. Let $a \in \mathbb{T}$ be arbitrary and $t_1, t_2, t_3, t_4 \in [a, \infty)$ be such that $a \le t_1 < t_2 < t_3 < t_4$. Define function y by

$$y(t) = \begin{cases} 0 & \text{for } t \in [a, t_1), \\ f(t) & \text{for } t \in [t_1, t_2), \\ 1 & \text{for } t \in [t_2, t_3), \\ g(t) & \text{for } t \in [t_3, t_4), \\ 0 & \text{for } t \in [t_4, \infty), \end{cases}$$

where f, g are given by the formulas

$$f(t) = \frac{\int_{t_1}^{t} (r^\rho(s))^{1-q} \, \nabla s}{\int_{t_1}^{t_2} (r^\rho(s))^{1-q} \, \nabla s}, \quad g(t) = \frac{\int_{t}^{t_4} (r^\rho(s))^{1-q} \, \nabla s}{\int_{t_3}^{t_4} (r^\rho(s))^{1-q} \, \nabla s},$$

i.e., they satisfy the boundary conditions $f(t_1) = 0, f(t_2) = 1, g(t_3) = 1, g(t_4) = 0$. This yields $y(t_1) = y(t_4) = 0, y(t) > 0$ for $t \in (t_1, t_4)$ and y^∇ is piecewise ld-continuous. It holds

$$f^\nabla(t) = \frac{(r^\rho(t))^{1-q}}{\int_{t_1}^{t_2} (r^\rho(s))^{1-q} \, \nabla s}, \quad g^\nabla(t) = -\frac{(r^\rho(t))^{1-q}}{\int_{t_3}^{t_4} (r^\rho(s))^{1-q} \, \nabla s},$$

and consequently, using integration by parts,

$$\begin{aligned}
\int_a^\infty r^\rho(t) |y^\nabla(t)|^p \, \nabla t &= \int_{t_1}^{t_4} r^\rho(t) |y^\nabla(t)|^p \, \nabla t \\
&= \int_{t_1}^{t_2} r^\rho(t) \Phi(f^\nabla(t)) f^\nabla(t) \, \nabla t + \int_{t_3}^{t_4} r^\rho(t) \Phi(g^\nabla(t)) g^\nabla(t) \, \nabla t \\
&= [r^\rho(t) \Phi(f^\nabla(t)) f(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} (r^\rho(t) \Phi(f^\nabla(t)))^\nabla f^\rho \, \nabla t \\
&\quad + [r^\rho(t) \Phi(g^\nabla(t)) g(t)]_{t_3}^{t_4} - \int_{t_3}^{t_4} (r^\rho(t) \Phi(g^\nabla(t)))^\nabla g^\rho \, \nabla t \\
&= r^\rho(t_2) \Phi(f^\nabla(t_2)) f(t_2) - r^\rho(t_3) \Phi(g^\nabla(t_3)) g(t_3) \\
&= \left(\int_{t_1}^{t_2} (r^\rho(t))^{1-q} \, \nabla t \right)^{1-p} + \left(\int_{t_3}^{t_4} (r^\rho(t))^{1-q} \, \nabla t \right)^{1-p}.
\end{aligned}$$

Now we compute the second term in $\mathcal{F}(y; a, \infty)$ by Lemma 3.3 (with $\hat{c}, \hat{g}$ defined in the same way as $\hat{f}$ in Lemma 3.3. We obtain

$$\int_{t_3}^{t_4} c(t) g^p(t) \, \nabla t = \int_{t_3}^{t_4} \hat{c}(t) \hat{g}^p(t) \, \Delta t = \int_{t_3}^{t_4} \hat{c}(t) g^p(\sigma(t)) \, \Delta t$$

since the function g is continuous. Using the second mean value theorem of integral calculus (Lemma 3.2) there exists $s_2 > t_3$ such that

$$\int_{t_3}^{t_4} \hat{c}(t) g^p(\sigma(t)) \, \Delta t \ge \int_{t_3}^{s_2} \hat{c}(t) \, \Delta t = \int_{t_3}^{s_2} c(t) \, \nabla t.$$

By the same argument, there exists $s_1 \in (t_1, t_2)$ such that

$$\int_{t_1}^{t_2} c(t) f^p(t)\,\nabla t \geq \int_{s_1}^{t_2} c(t)\,\nabla t.$$

Summarizing the previous computations, we get

$$\mathcal{F}(y; a, \infty) \leq \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} + \Big(\int_{t_3}^{t_4} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} - \int_{s_1}^{s_2} c(t)\,\nabla t.$$

Now, if t_1, t_2 are fixed, for sufficiently large t_3, t_4 the assumptions (3.17) of this theorem imply that $\mathcal{F}(y; a, \infty) < 0$. □

When the assumption of the previous theorem concerning the divergence of the integral $\int^\infty c(t)\,\nabla t$ is violated, the next criterion applies.

THEOREM 3.5. *Suppose that $r(t) > 0$ for large t,*

$$\int^\infty (r^\rho(t))^{1-q}\,\nabla t = \infty,$$

and

$$\liminf_{t\to\infty} \Big(\int_a^t (r^\rho(s))^{1-q}\,\nabla s\Big)^{p-1} \Big(\int_t^\infty c(s)\,\nabla s\Big) > 1. \tag{3.18}$$

Then equation (3.1) *is oscillatory.*

PROOF. Define the function y in the same way as in the previous proof. Then $\mathcal{F}(y; a, \infty)$ satisfies

$$\begin{aligned}
\mathcal{F}(y; a, \infty) &\leq \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} + \Big(\int_{t_3}^{t_4} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} - \int_{s_1}^{s_2} c(t)\,\nabla t \\
&= \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} \\
&\quad \times \Big[1 - \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{p-1} \int_{s_1}^{s_2} c(t)\,\nabla t + \Big(\frac{\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t}{\int_{t_3}^{t_4} (r^\rho(t))^{1-q}\,\nabla t}\Big)^{p-1}\Big].
\end{aligned}$$

It is not so difficult to show that if (3.18) holds, then the expression in brackets is negative for sufficiently large $t_2 < t_3 < t_4$. This proof is exactly the same as for differential equation, i.e. $\mathbb{T} = \mathbb{R}$, see [**18**]. □

3.5. Nonoscillation criteria

In the proof of the next nonoscillation criterion for (3.1) we will need the following refinement of the Riccati equivalence of (i) and (iii) in Theorem 3.3. We will denote by $\mathcal{R}[w]$ the so-called Riccati operator (compare (3.6)), i.e.,

$$\mathcal{R}[w] := \begin{cases} w^\nabla + c(t) + (p-1)\frac{|w|^q}{\Phi^{-1}(r(t))} & \text{if } t = \rho(t), \\ w^\nabla + c(t) + \frac{w^\rho}{\nu(t)}\big(1 - \frac{r^\rho(t)}{\Phi(\Phi^{-1}(r^\rho(t)) + \nu(t)\Phi^{-1}(w^\rho))}\big) & \text{if } \rho(t) < t, \end{cases} \tag{3.19}$$

and by $\mathcal{L}(x)$ the left-hand side of (3.1), i.e.,

$$\mathcal{L}(x) := (r(t)\Phi(x^\Delta))^\nabla + c(t)\Phi(x).$$

The proof of the next lemma follows the same idea as in the continuous case, but for the reader's convenience we present here the main ideas of this proof.

Lemma 3.7. *Equation* (3.1) *is nonoscillatory if and only if there exists a* ∇*-differentiable function* w *satisfying* (3.8) *such that* $\mathcal{R}[w] \le 0$ *for large* t.

Proof. The implication "$\Rightarrow$" is trivial since it is only a restatement of the Riccati equivalence (i) $\iff$ (iii) for large t. To prove the opposite implication, suppose that there exists a function w satisfying assumptions of the lemma on an interval $[T, \infty)$. To prove that (3.1) is nonoscillatory, we will construct a nonoscillatory majorant of this equation in such a way that w is a solution of the Riccati equation associated to this majorant equation.

Let y be the solution of the initial value problem

$$y^\Delta = r^{1-q}(t)\Phi^{-1}(w(t))y, \quad y(T) = 1,$$

where T is sufficiently large. Using the computation at the beginning of Lemma 3.5 we have

$$\mathcal{R}[w] = w^\nabla + c(t) + \frac{r^\rho \Phi^\rho(y^\Delta)(\Phi(y))^\nabla}{\Phi(y^\rho)\Phi(y)}.$$

Then we have, again following the computation in the proof of Lemma 3.5, in particular, splitting the cases $\rho(t) < t$ and $\rho(t) = t$,

$$0 \ge |y|^p \mathcal{R}[w] = |y|^p \Big[w^\nabla + c(t) + \frac{r^\rho \Phi^\rho(y^\Delta)(\Phi(y))^\nabla}{\Phi(y^\rho)\Phi(y)}\Big] = y\mathcal{L}(y).$$

Now, let $\tilde{c}(t) := c(t) - \frac{y(t)\mathcal{L}[y](t)}{y^p(t)}$. Then $\tilde{c}(t) \ge c(t)$ and y is a solution of the equation (which is a Sturmian majorant of (3.1))

$$(r(t)\Phi(y^\Delta))^\nabla + \tilde{c}(t)\Phi(y) = 0 \tag{3.20}$$

for which $r^\rho(t)y^\rho(t)y(t) > 0$ for large t, i.e., (3.20) is nonoscillatory and hence (3.1) is nonoscillatory as well. $\square$

Now we apply Lemma 3.7 to prove the Hille-Nehari-type nonoscillation criterion for (3.1). The idea of the proof is the same as in the continuous case $\mathbb{T} = \mathbb{R}$, but the particularities of time scale calculus require some additional assumptions (which are automatically satisfied for $\mathbb{T} = \mathbb{R}$) and also some technical modifications, compare the proof of [**18**, Theorem 2.1].

THEOREM 3.6. *Suppose that $r(t) > 0$ for large t, $\int^\infty (r^\rho(t))^{1-q}\,\nabla t = \infty$, the integral $\int^\infty c(t)\,\nabla t$ is convergent,*

$$\lim_{t\to\infty} \frac{\nu(t)\,[r^\rho(t)]^{1-q}}{\int_a^{\rho(t)} [r^\rho(s)]^{1-q}\,\nabla s} = 0, \tag{3.21}$$

$$\liminf_{t\to\infty} \Big(\int_{t_0}^{\rho(t)} (r^\rho(s))^{1-q}\,\nabla s\Big)^{p-1} \Big(\int_{\rho(t)}^{\infty} c(s)\,\nabla s\Big) > -\frac{2p-1}{p}\big(\frac{p-1}{p}\big)^{p-1}, \tag{3.22}$$

$$\limsup_{t\to\infty} \Big(\int_{t_0}^{\rho(t)} (r^\rho(s))^{1-q}\,\nabla s\Big)^{p-1} \Big(\int_{\rho(t)}^{\infty} c(s)\,\nabla s\Big) < \frac{1}{p}\big(\frac{p-1}{p}\big)^{p-1}, \tag{3.23}$$

then (3.1) *is nonoscillatory.*

PROOF. By the previous lemma, we will construct a function w such that $\mathcal{R}[w](t) \le 0$ and (3.8) holds for large t. To this end, we denote (for the notational convenience)

$$\tilde r := r^\rho, \quad \tilde w = w^\rho,$$

we also denote

$$\mathcal{A}(t) := \Big(\int_{t_0}^{t} \tilde r^{1-q}(s)\,\nabla s\Big)^{p-1} \Big(\int_{t}^{\infty} c(s)\,\nabla s\Big).$$

Let

$$w(t) = \big(\frac{p-1}{p}\big)^{p-1} \Big(\int_{t_0}^{t} \tilde r^{1-q}(s)\,\nabla s\Big)^{1-p} + \int_t^\infty c(s)\,\nabla s. \tag{3.24}$$

Using Lemma 3.4 (a ∇-chain rule for differentiation) we have

$$\Big[\Big(\int_{t_0}^{t} \tilde r^{1-q}(s)\,\nabla s\Big)^{1-p}\Big]^\nabla = (1-p)\tilde r^{1-q}(t)\theta^{-p}(t),$$

where

$$\int_{t_0}^{\rho(t)} \tilde r^{1-q}(s)\,\nabla s \le \theta(t) \le \int_{t_0}^{t} \tilde r^{1-q}(s)\,\nabla s.$$

Also, using the Lagrange mean value theorem we have

$$\begin{aligned}\frac{\tilde w}{\nu}\Big(1 - \frac{\tilde r}{\Phi\left(\Phi^{-1}(\tilde r) + \nu\Phi^{-1}(\tilde w)\right)}\Big) &= \frac{\tilde w}{\nu}\frac{\Phi(\Phi^{-1}(\tilde r) + \nu\Phi^{-1}(\tilde w)) - \Phi(\Phi^{-1}(\tilde r))}{\Phi(\Phi^{-1}(\tilde r) + \nu\Phi^{-1}(\tilde w))} \\ &= (p-1)\frac{|\eta|^{p-2}|\tilde w|^q}{\Phi\left(\Phi^{-1}(\tilde r) + \nu\Phi^{-1}(\tilde w)\right)},\end{aligned}$$

where η is between $\Phi^{-1}(\tilde r)$ and $\Phi^{-1}(\tilde r) + \nu\Phi^{-1}(\tilde w)$. By adding $(\frac{p-1}{p})^p$ to the pair of inequalities

$$-\frac{2p-1}{p}\big(\frac{p-1}{p}\big)^{p-1} < \mathcal{A}^\rho(t) < \frac{1}{p}\big(\frac{p-1}{p}\big)^{p-1},$$

we obtain

$$-\big(\frac{p-1}{p}\big)^{p-1} < \mathcal{A}^\rho(t) + \Big(\frac{p-1}{p}\Big)^p < \big(\frac{p-1}{p}\big)^{p-1}.$$

More precisely, (3.22) implies the existence of $\varepsilon > 0$ such that

$$\big|\mathcal{A}^\rho(t) + \big(\frac{p-1}{p}\big)^p\big|^q(1+\varepsilon) < \big(\frac{p-1}{p}\big)^p$$

for large t. Now we will estimate the quantity

$$|\tilde{w}|^q = \Big(\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s\Big)^{-p}\big|\mathcal{A}^\rho(t) + \big(\frac{p-1}{p}\big)^p\big|^q.$$

(I) First consider the case that t is ld-point, i.e., $\rho(t) < t$. Using the previous computations, we obtain

$$\begin{aligned}
\mathcal{R}[w] &= w^\nabla + c + \frac{\tilde{w}}{\nu}\left(1 - \frac{\tilde{r}}{\Phi\left(\Phi^{-1}(\tilde{r}) + \nu\Phi^{-1}(\tilde{w})\right)}\right) \\
&= -(p-1)\big(\frac{p-1}{p}\big)^p|\theta|^{-p}\tilde{r}^{1-q} - c + c + (p-1)\frac{|\eta|^{p-2}|\tilde{w}|^q}{\Phi\left(\Phi^{-1}(\tilde{r}) + \nu\Phi^{-1}(\tilde{w})\right)} \\
&\le (p-1)\tilde{r}^{1-q}\Big[-\big(\frac{p-1}{p}\big)^p\Big(\int_{t_0}^{t}\tilde{r}^{1-q}(s)\,\nabla s\Big)^{-p} \\
&\quad + \Big(\int_{t_0}^{\rho(t)}\tilde{r}^{1-q}(s)\,\nabla s\Big)^{-p}\frac{|\eta|^{p-2}\tilde{r}^{q-1}\left|\big(\frac{p-1}{p}\big)^p + \mathcal{A}^\rho(t)\right|^q}{\Phi\left(\Phi^{-1}(\tilde{r}) + \nu\Phi^{-1}(\tilde{w})\right)}\Big] \\
&= \frac{(p-1)\tilde{r}^{1-q}}{\Big(\int_{t_0}^{t}\tilde{r}^{1-q}(s)\,\nabla s\Big)^p}\Big[-\big(\frac{p-1}{p}\big)^p + \mathcal{B}(t)\left|\mathcal{A}^\rho(t) + \big(\frac{p-1}{p}\big)^p\right|^q\Big],
\end{aligned}$$

where

$$\mathcal{B}(t) := \Big(\frac{\int_{t_0}^{t}\tilde{r}^{1-q}(s)\,\nabla s}{\int_{t_0}^{\rho(t)}\tilde{r}^{1-q}(s)\,\nabla s}\Big)^p\frac{|\eta|^{p-2}\tilde{r}^{q-1}}{\Phi\left(\Phi^{-1}(\tilde{r}) + \nu\Phi^{-1}(\tilde{w})\right)} \to 1$$

as $t \to \infty$, in particular, for any $\varepsilon > 0$, $\mathcal{B}(t) < (1+\varepsilon)$ for large t. Indeed, consider the case $p > 2$, the case $p \in (1,2)$ can be treated analogously. Using the fact that

$$\Phi^{-1}(\tilde{r}) - \nu|\Phi^{-1}(\tilde{w})| \le \eta \le \Phi^{-1}(\tilde{r}) + \nu|\Phi^{-1}(\tilde{w})|,$$

and that

$$\begin{aligned}
\nu\Big|\frac{\tilde{w}}{\tilde{r}}\Big|^{q-1} &= \\
&= \nu\frac{\left|\big(\frac{p-1}{p}\big)^p\Big[\int_{t_0}^{\rho(t)}\tilde{r}^{1-q}(s)\,\nabla s\Big]^{1-p} + \int_{\rho(t)}^{\infty}c(s)\,\nabla s\right|^{q-1}}{\tilde{r}^{q-1}} \\
&= \frac{\nu(t)\tilde{r}^{1-q}}{\int_{t_0}^{\rho(t)}\tilde{r}^{1-q}(s)\,\nabla s}\Big|\big(\frac{p-1}{p}\big)^p + \Big(\int_{t_0}^{\rho(t)}\tilde{r}^{1-q}(s)\,\nabla s\Big)^{p-1}\Big(\int_{\rho(t)}^{\infty}c(s)\,\nabla s\Big)\Big|^{q-1} \to 0,
\end{aligned}$$

as $t \to \infty$, since the second term in the last expression is bounded (see (3.22)) and the first one goes to zero by (3.21). The last calculation also implies that

$$\Phi^{-1}(\tilde{r}) + \nu\Phi^{-1}(\tilde{w}) > 0$$

for large t.

Hence

$$\begin{aligned}|\mathcal{B}(t)| &\le \left(\frac{\int_{t_0}^{\rho(t)}\tilde r^{1-q}(s)\,\nabla s+\nu\tilde r^{1-q}}{\int_{t_0}^{\rho(t)}\tilde r^{1-q}(s)\,\nabla s}\right)^p\frac{|\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)|^{p-2}\tilde r^{q-1}}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)}\\ &= \left(1+\frac{\nu\tilde r^{1-q}}{\int_{t_0}^{\rho(t)}\tilde r^{1-q}(s)\,\nabla s}\right)^p\frac{\tilde r^{q-1+(p-2)(q-1)}\,|1+\nu\Phi^{-1}(\tilde w/\tilde r)|^{p-2}}{\tilde r\Phi\left(1+\nu\Phi^{-1}(\tilde w/\tilde r)\right)}\\ &= \left(1+\frac{\nu\tilde r^{1-q}}{\int_{t_0}^{\rho(t)}\tilde r^{1-q}(s)\,\nabla s}\right)^p\frac{1}{1+\nu\Phi^{-1}(\tilde w/\tilde r)}\to 1,\quad \text{as } t\to\infty.\end{aligned}$$

Summarizing all estimates, if t_0 is so large that all statements claimed to hold for large t hold for $t \ge t_0$, we have

$$\mathcal{R}[w] \le \frac{(p-1)\tilde r^{1-q}}{(\int_{t_0}^{\rho(t)}\tilde r^{1-q}(s)\,\nabla s)^p}\Big[-(\frac{p-1}{p})^p+\Big|(\frac{p-1}{p})^p+\mathcal{A}^\rho(t)\Big|^q(1+\varepsilon)\Big]<0$$

and

$$\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)=\Phi^{-1}(\tilde r)(1+\nu\Phi^{-1}(\tilde w/\tilde r))>0$$

for large t.
(II) Case $\rho(t)=t$. This case is now easy to treat, since then $\tilde w = w$,

$$w^\nabla=(1-p)\left(\frac{p-1}{p}\right)^{p-1}\Big(\int_{t_0}^t\tilde r^{1-q}(s)\,\nabla s\Big)^{-p}\tilde r^{1-q}(t)-c(t)$$

and an easy modification of the previous computation shows that

$$\mathcal{R}[w]=w^\nabla+c(t)+(p-1)r^{1-q}(t)|w|^q\le 0$$

for large t. □

The following theorem complements the previous statement and deals with the case when $\int^\infty (r^\rho(t))^{1-q}\,\nabla t<\infty$.

Theorem 3.7. *Suppose that $r(t)>0$ for large t, $\int^\infty (r^\rho(t))^{1-q}\,\nabla t<\infty$, the integral $\int^\infty c(t)\,\nabla t=\infty$,*

$$\lim_{t\to\infty}\frac{\nu(t)\,[r^\rho(t)]^{1-q}}{\int_t^\infty [r^\rho(s)]^{1-q}\,\nabla s}=0,$$

$$\liminf_{t\to\infty}\Big(\int_{\rho(t)}^\infty (r^\rho(s))^{1-q}\,\nabla s\Big)^{p-1}\Big(\int_{t_0}^{\rho(t)}c(s)\,\nabla s\Big)>-\frac{2p-1}{p}(\frac{p-1}{p})^{p-1},$$

$$\limsup_{t\to\infty}\Big(\int_{\rho(t)}^\infty (r^\rho(s))^{1-q}\,\nabla s\Big)^{p-1}\Big(\int_{t_0}^{\rho(t)}c(s)\,\nabla s\Big)<\frac{1}{p}(\frac{p-1}{p})^{p-1},$$

then (3.1) *is nonoscillatory.*

Proof. The proof is similar to that of the previous theorem, we only take

$$w(t)=(\frac{p-1}{p})^{p-1}\Big(\int_t^\infty\tilde r^{1-q}(s)\,\nabla s\Big)^{1-p}+\int_{t_0}^t c(s)\,\nabla s,$$

instead of w defined by (3.24). $\square$

In this chapter we have formulated only basic results concerning qualitative theory of (3.1). A natural motivation for the continuation of the research are the results presented in [**2**] which concern equation (3.4).

CHAPTER 4

EVEN ORDER DYNAMIC EQUATIONS

4.1. Introduction

As a motivation for our research, let us start first with the case $\mathbb{T} = \mathbb{R}$ and consider the even order (formally) self-adjoint differential equation

$$\sum_{\nu=0}^{n} (-1)^\nu \left(r_\nu(t) y^{(\nu)}\right)^{(\nu)} = 0, \quad r_n(t) > 0. \tag{4.1}$$

The substitution

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} \sum_{\nu=1}^{n} (-1)^\nu (r_\nu y^{(\nu)})^{(\nu-1)} \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix}.$$

converts (4.1) into the linear Hamiltonian differential system (2.9)

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u,$$

with the matrices

$$B = \operatorname{diag}\left\{0, \ldots, 0, \frac{1}{r_n}\right\}, \quad C = \operatorname{diag}\{r_0, \ldots, r_{n-1}\},$$

$$A = A_{i,j} = \begin{cases} 1 & \text{if} \quad j = i+1,\ i = 1, \ldots, n-1, \\ 0 & \quad\quad \text{elsewhere}, \end{cases}$$

see [**17, 47**].

The discrete counterpart of (4.1) is the difference equation

$$\sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu}\right) = 0, \quad r_k^{[n]}(t) \neq 0, \tag{4.2}$$

and the substitution

$$x_k = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \quad u_k = \begin{pmatrix} \sum_{\nu=1}^{n} (-1)^\nu \Delta^{\nu-1} (r_k^{[\nu]} \Delta^\nu y_{k+n-\nu}) \\ \vdots \\ -\Delta(r_k^{[n]} \Delta^n y_k) + r_k^{[n-1]} \Delta^{n-1} y_{k+1} \\ r_k^{[n]} \Delta^n y_k \end{pmatrix}.$$

converts this equation into the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{4.3}$$

with the matrices

$$B = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{r_k^{[n]}}\right\}, \quad C = \operatorname{diag}\{r_k^{[0]}, \dots, r_k^{[n-1]}\},$$
$$A = A_{i,j} = \begin{cases} 1 & \text{if} \quad j = i+1,\ i = 1, \dots, n-1, \\ 0 & \quad \text{elsewhere.} \end{cases}$$

The comprehensive treatment of the qualitative theory of discrete Hamiltonian systems and higher order equations can be found in [**5**]. We also refer to the fundamental paper of Bohner [**11**], where the theory of (4.3) with the matrix B possibly singular is established.

Concerning a time scale unification of the results for continuous and discrete equations (4.1) and (4.2), except for some partial results (see, e.g., [**6, 7, 8, 9**]), no systematic theory has been developed yet. The possible reason is the following fact. Motivated by the discrete case, consider the fourth order dynamic equation (we consider this special case just to explain the main idea without technical details)

$$(r(t)y^{\Delta\Delta})^{\Delta\Delta} + q(t)y^{\sigma\sigma} = 0. \tag{4.4}$$

If we try the substitution (again motivated by the discrete case)

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y^\sigma \\ y^\Delta \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -(ry^{\Delta\Delta})^\Delta \\ ry^{\Delta\Delta} \end{pmatrix}.$$

with the aim to rewrite (4.4) as the Hamiltonian system (with the matrices given by analogous formulas as in the continuous and discrete case)

$$x^\Delta = A(t)x^\sigma + B(t)u, \quad u^\Delta = C(t)x^\sigma - A^T(t)u, \tag{4.5}$$

we easily find that we need the identity $(y^\Delta)^\sigma = (y^\sigma)^\Delta$, but this identity holds generally only for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, h being a positive real constant. Consequently, even-order equations of the form (4.4) *cannot* be written in the form (4.5) and this is likely the reason for the missing qualitative theory of even-order equations on time scales since the theory of Hamiltonian systems both in the continuous and discrete case is a natural background for the investigation of (4.1) and (4.2).

The aim of this chapter is to overcome this problem using the concept of *dynamic equations with mixed derivatives*. Second order equations of this type have been investigated in the recent papers [**10, 38**], see also [**16**, Chap. 3,4], and the principal role is played there by the concept of *nabla derivative* on time scales. Let us define the differential operators

$$D_k^\Delta y := \begin{cases} y^{\Delta\nabla\dots\Delta\nabla} & k \text{ even}, \\ y^{\Delta\nabla\dots\nabla\Delta} & k \text{ odd}, \end{cases} \qquad \widetilde{D}_k^\Delta y := \begin{cases} y^{\nabla\Delta\dots\nabla\Delta} & k \text{ even}, \\ y^{\Delta\nabla\dots\nabla\Delta} & k \text{ odd}, \end{cases} \tag{4.6}$$

$$D_k^\nabla y := \begin{cases} y^{\nabla\Delta\dots\nabla\Delta} & k \text{ even}, \\ y^{\nabla\Delta\dots\Delta\nabla} & k \text{ odd}, \end{cases} \qquad \widetilde{D}_k^\nabla y := \begin{cases} y^{\Delta\nabla\dots\Delta\nabla} & k \text{ even}, \\ y^{\nabla\Delta\dots\Delta\nabla} & k \text{ odd}, \end{cases} \tag{4.7}$$

(the operators D^Δ, D^∇ *start* with the nabla and delta derivative, respectively, while $\widetilde{D}^\Delta, \widetilde{D}^\nabla$ *end* with the corresponding derivative) and consider the higher order dynamic equations

$$L(y) := \sum_{\nu=0}^{n}(-1)^\nu \widetilde{D}_\nu^\nabla\left(r_\nu(t)D_\nu^\Delta y\right) = 0, \tag{4.8}$$

$$M(y) := \sum_{\nu=0}^{n}(-1)^\nu \widetilde{D}_\nu^\Delta\left(r_\nu(t)D_\nu^\nabla y\right) = 0. \tag{4.9}$$

We will show that these equations can be written in the form of (delta or nabla) symplectic dynamic systems and this enables to study (4.8), (4.9) using the (relatively deeply developed) theory of symplectic dynamic systems.

The next section contains the computations showing that (4.8), (4.9) can be written as a (delta or nabla) symplectic dynamic system. In the last section we discuss some open problems and the perspectives of the research in the area of higher order dynamic equations with mixed derivatives.

4.2. Conversion to symplectic systems

In this section we show that higher order equations (4.8) and (4.9) can be rewritten as a delta symplectic system or as a nabla symplectic system. The vector variables x, u and the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in resulting symplectic systems have slightly different form for the operators L or M, for n even or odd, and for transformation to delta or nabla symplectic system (altogether we have eight cases), but the approach is similar in all cases. For this reason we present detailed calculations only for two particular possibilities.

As a first representative case let us consider equation (4.9) with n odd and ld-continuous functions r_i.

THEOREM 4.1. *Suppose that n is odd and the functions r_i, $i = 0, \dots, n$, are ld-continuous. Then equation* (4.9) *can be transformed into nabla symplectic system*

$$\begin{pmatrix} x \\ u \end{pmatrix}^\nabla = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix},$$

where the blocks $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $n \times n$ matrices of the following form

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & 0 & 0 \\ 0 & 0 & 1 & -\nu & \cdots & & & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & & 0 & 1 & -\nu & 0 \\ & & & & & 0 & 0 & 1 & 0 \\ & & & & & 0 & 0 & 0 & 1 \\ 0 & \cdots & & & & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{-\nu}{r_n} \\ 0 & \cdots & 0 & \frac{1}{r_n} \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} r_0^\rho & -\nu r_0^\rho & 0 & 0 & \cdots & & 0 & 0 \\ 0 & r_1 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & \nu r_1 & r_2^\rho & -\nu r_2^\rho & & & 0 & 0 \\ 0 & 0 & 0 & r_3 & & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots & \vdots \\ & & & & & r_{n-3}^\rho & -\nu r_{n-3}^\rho & 0 \\ & & & & & 0 & r_{n-2} & 0 \\ 0 & \cdots & & & & 0 & \nu r_{n-2} & r_{n-1}^\rho \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ -1 & 0 & 0 & 0 & & & 0 & 0 \\ -\nu & -1 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & -1 & 0 & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & -1 & 0 & 0 \\ 0 & \cdots & & & & -\nu & -1 & -\frac{\nu r_{n-1}^\rho}{r_n} \end{bmatrix}.$$

PROOF. Using the usual type of substitution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y^\nabla \\ y^{\nabla\Delta} \\ \vdots \\ D_{n-2}^\nabla y \\ D_{n-1}^\nabla y \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} \sum_{\nu=1}^n (-1)^{\nu-1} \widetilde{D}_{\nu-1}^\nabla (r_\nu(t) D_\nu^\nabla y) \\ \sum_{\nu=2}^n (-1)^{\nu-2} \widetilde{D}_{\nu-2}^\Delta (r_\nu(t) D_\nu^\nabla y) \\ \vdots \\ -(r_n(t) D_n^\nabla y)^\Delta + r_{n-1}(t) D_{n-1}^\nabla y \\ r_n(t) D_n^\nabla y \end{bmatrix},$$

we get a system of $2n$ equations (suppressing the argument t)

$$\begin{aligned} x_1^\nabla &= x_2, & u_1^\Delta &= r_0 x_1, \\ x_2^\Delta &= x_3, & u_2^\nabla &= -u_1 + r_1 x_2, \\ &\vdots & &\vdots \\ x_{n-1}^\Delta &= x_n, & u_{n-1}^\nabla &= -u_{n-2} + r_{n-2} x_{n-1}, \\ x_n^\nabla &= \frac{1}{r_n} u_n, & u_n^\Delta &= -u_{n-1} + r_{n-1} x_n. \end{aligned}$$

The obtained system contains both nabla and delta derivatives. Because we want to get nabla symplectic system, we need to replace all delta derivatives by nabla derivatives.

The functions r_i, $i = 0, \dots, n$, are ld-continuous, hence according to (2.7) one can directly verify that

$$\begin{aligned} x_2^\nabla &= x_3 - \nu x_4, \\ x_4^\nabla &= x_5 - \nu x_6, \\ &\vdots \\ x_{n-1}^\nabla &= x_n - \nu x_n^\nabla = x_n - \frac{\nu}{r_n} u_n, \end{aligned}$$

and

$$\begin{aligned} u_1^\nabla &= r_0^\rho x_1 - \nu r_0^\rho x_2, \\ u_3^\nabla &= -u_2 - \nu u_1 + \nu r_1 x_2 + r_2^\rho x_3 - \nu r_2^\rho x_4, \\ &\vdots \\ u_{n-2}^\nabla &= -u_{n-3} - \nu u_{n-4} + \nu r_{n-4} x_{n-3} + r_{n-3}^\rho x_{n-2} - \nu r_{n-3}^\rho x_{n-1}, \\ u_n^\nabla &= -u_{n-1} - \nu u_{n-2} + \nu r_{n-2} x_{n-1} + r_{n-1}^\rho x_n - \frac{\nu r_{n-1}^\rho}{r_n} u_n. \end{aligned}$$

Thus we get a matrix nabla system

$$\begin{pmatrix} x \\ u \end{pmatrix}^\nabla = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{4.10}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $n \times n$ matrices of desired form. It remains to prove that (4.10) is really a nabla symplectic system, i.e., that the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ verify equalities (2.21). It holds

$$\mathcal{A}^T\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 & 0 \\ r_0^\rho & -\nu r_0^\rho & 0 & & & 0 & 0 \\ 0 & r_1 & 0 & & & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ & & & & 0 & 0 & 0 \\ & & & & r_{n-3}^\rho & -\nu r_{n-3}^\rho & 0 \\ 0 & \cdots & & & 0 & r_{n-2} & 0 \end{bmatrix},$$

$$\mathcal{B}^T\mathcal{D} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & -\frac{1}{r_n} & \frac{-\nu r_{n-1}^\rho}{r_n^2} \end{bmatrix},$$

$$\mathcal{A}^T\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & 0 & & & 0 & 0 \\ -1 & 0 & 0 & & & 0 & 0 \\ 0 & -1 & 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ 0 & \cdots & & & -1 & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}^T\mathcal{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \frac{r_{n-1}^{\rho}}{r_n} \end{bmatrix}.$$

Using these calculations one can easily deduce that the needed equations (2.21) are really satisfied. $\square$

As a second representative case we choose equation (4.8) for n even. We suppose that the functions r_i are rd-continuous and we transform this equation to a delta symplectic system.

THEOREM 4.2. *Suppose that n is even and that the functions r_i, $i = 0, \ldots, n$, are rd-continuous. Then equation (4.8) can be transformed to the delta symplectic system*

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\Delta} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix},$$

where the blocks $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $n \times n$ matrices of the following form

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & 0 & 0 \\ 0 & 0 & 1 & \mu & \cdots & & & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & \mu \\ & & & & & & 0 & 0 & 1 \\ 0 & \cdots & & & & & 0 & 0 & \frac{\mu r_{n-1}}{r_n^{\sigma}} \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & -\frac{\mu}{r_n^{\sigma}} & \frac{1}{r_n^{\sigma}} \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} r_0^\rho & \mu r_0^\sigma & 0 & 0 & \cdots & & 0 & 0 \\ 0 & r_1 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & -\mu r_1 & r_2^\sigma & \mu r_2^\sigma & & & 0 & 0 \\ 0 & 0 & 0 & r_3 & \ddots & & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots & \vdots \\ & & & & & r_{n-3} & 0 & 0 \\ & & & & & -\mu r_{n-3} & r_{n-2}^\sigma & \mu r_{n-2}^\sigma \\ 0 & \cdots & & & & 0 & 0 & r_{n-1} \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ -1 & 0 & 0 & 0 & & & 0 & 0 \\ \mu & -1 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & -1 & 0 & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ & & & & -1 & 0 & 0 & 0 \\ & & & & \mu & -1 & 0 & 0 \\ 0 & \cdots & & & 0 & 0 & -1 & 0 \end{bmatrix}.$$

PROOF. A substitution of the similar form as that in the proof of Theorem 4.1, i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y^\Delta \\ y^{\Delta\nabla} \\ \vdots \\ D_{n-2}^\Delta y \\ D_{n-1}^\Delta y \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} \sum_{\nu=1}^n (-1)^{\nu-1} \widetilde{D}_{\nu-1}^\Delta (r_\nu(t) D_\nu^\nabla y) \\ \sum_{\nu=2}^n (-1)^{\nu-2} \widetilde{D}_{\nu-2}^\nabla (r_\nu(t) D_\nu^\nabla y) \\ \vdots \\ -(r_n(t) D_n^\Delta y)^\Delta + r_{n-1}(t) D_{n-1}^\Delta y \\ r_n(t) D_n^\Delta y \end{bmatrix},$$

leads to a system of $2n$ equations (again suppressing the argument t)

$$\begin{aligned} x_1^\Delta &= x_2, & u_1^\nabla &= r_0 x_1, \\ x_2^\nabla &= x_3, & u_2^\Delta &= -u_1 + r_1 x_2, \\ &\vdots & &\vdots \\ x_{n-1}^\Delta &= x_n, & u_{n-1}^\nabla &= -u_{n-2} + r_{n-2} x_{n-1}, \\ x_n^\nabla &= \frac{1}{r_n} u_n & u_n^\Delta &= -u_{n-1} + r_{n-1} x_n. \end{aligned}$$

Using (2.7) we replace nabla derivatives by delta derivatives (we can do it because of our assumption of rd-continuity of functions r_i) to get the equations

$$\begin{aligned}
x_2^\Delta &= x_3 + \mu x_4,\\
x_4^\Delta &= x_5 + \mu x_6,\\
&\vdots\\
x_n^\Delta &= \frac{1}{r_n^\sigma}u_n - \frac{\mu}{r_n^\sigma}u_{n-1} + \frac{\mu r_{n-1}}{r_n^\sigma}x_n,
\end{aligned}$$

and

$$\begin{aligned}
u_1^\Delta &= r_0^\sigma x_1 + \mu r_0^\sigma x_2,\\
u_3^\Delta &= -u_2 + \mu u_1 - \mu r_1 x_2 + r_2^\sigma x_3 + \mu r_2^\sigma x_4,\\
&\vdots\\
u_{n-1}^\Delta &= -u_{n-2} + \mu u_{n-3} - \mu r_{n-3}x_{n-2} + r_{n-2}^\sigma x_{n-1} + \mu r_{n-2}^\sigma x_n.
\end{aligned}$$

So we have the matrix delta system

$$\begin{pmatrix} x \\ u \end{pmatrix}^\Delta = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{4.11}$$

where the $n \times n$ matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the same as in the statement of this theorem.

It remains to prove that (4.11) is a symplectic system, i.e., to verify equations (2.18). It holds

$$\mathcal{A}^T\mathcal{C} = \begin{bmatrix}
0 & 0 & 0 & \cdots & & & 0 \\
r_0^\sigma & \mu r_0^\sigma & 0 & & & & \vdots \\
0 & r_1 & 0 & & & & \\
\vdots & & \ddots & \ddots & & & \\
 & & & & \mu r_{n-4}^\sigma & 0 & 0 \\
 & & & & r_{n-3} & 0 & 0 \\
0 & \cdots & & & 0 & r_{n-2}^\sigma & \mu r_{n-2}^\sigma + \frac{\mu r_{n-1}^2}{r_n^\sigma}
\end{bmatrix},$$

$$\mathcal{B}^T\mathcal{D} = \begin{bmatrix}
0 & \cdots & 0 & 0 & 0 \\
\vdots & & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & \frac{\mu}{r_n^\sigma} & 0 \\
0 & \cdots & 0 & \frac{-1}{r_n^\sigma} & 0
\end{bmatrix},$$

$$\mathcal{A}^T\mathcal{D} = \begin{bmatrix}
0 & 0 & 0 & \cdots & & 0 & 0 \\
0 & 0 & 0 & & & 0 & 0 \\
-1 & 0 & 0 & & & 0 & 0 \\
0 & -1 & 0 & \ddots & & \vdots & \vdots \\
\vdots & & \ddots & & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 \\
0 & \cdots & & & -1 & -\frac{\mu r_{n-1}}{r_n^\sigma} & 0
\end{bmatrix},$$

$$\mathcal{C}^T\mathcal{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & -\frac{\mu r_{n-1}}{r_n^\sigma} & \frac{r_{n-1}}{r_n^\sigma} \end{bmatrix}.$$

These computations directly imply that equations (2.18) are satisfied. □

REMARK 4.1. Observe that in cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ equations (4.8) and (4.9) really reduce to (4.1) and (4.2), respectively. In the continuous case $\mathbb{T} = \mathbb{R}$ it is clear since both nabla and delta derivatives are the usual derivative. Concerning the discrete case $\mathbb{T} = \mathbb{Z}$, using the identity $\Delta y_{k-1} = \nabla y_k$ and after suitable relabeling the sequences $r^{[\nu]}$, $\nu = 0, \dots, n$, to show that (4.2) can be written either in the form (4.8) or (4.9) is a matter of direct computation.

4.3. Self-adjointness of dynamic equations

In this section we prove that equations (4.8), (4.9) are formally self-adjoint. As it was shown in the previous section, the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in symplectic dynamic system corresponding to equation (4.8) or (4.9) are of slightly different form for the operators L, M as well as for n even or odd.

The terminology "formally" self-adjoint is used in this thesis in the sense which is usual in the theory of differential equations. A given higher order dynamic equation is converted to a *first* order vector dynamic system, the first entry of the vector solution of this system complies with the solution of the original higher order equation. Then, the *last* entry of the vector solution of the adjoint system satisfies the adjoint equation, and if this equation is the same as the original one, this equation is said to be formally self-adjoint. The adjective "formally" is skipped, if the problem is regarded from the differential operators theory point of view. In this setting, together with a given differential expression, the domain of a differential operator is determined by some boundary conditions and this operator is called self-adjoint if the differential expression and also the domain of the adjoint operator are the same as far as the original one. The "differential operators" approach to self-adjointness of even order dynamic equations with mixed derivatives has been used in [**8**].

The adjoint system to system (2.16) is the system

$$y^\Delta = -\mathcal{S}^T(t)y^\sigma. \tag{4.12}$$

Indeed, let W be a fundamental matrix of (2.16) and let $V = (W^T)^{-1}$. Then

$$V^\Delta = -(W^T)^{-1}(W^T)^\Delta((W^\sigma)^T)^{-1} = -(W^T)^{-1}W^T\mathcal{S}^T((W^\sigma)^T)^{-1} = -\mathcal{S}^T V^\sigma.$$

Equation (4.12) is equivalent to the equation

$$y^\Delta = -(I + \mu\mathcal{S}^T(t))^{-1}\mathcal{S}^T(t)y^\sigma. \tag{4.13}$$

Note that the matrix $I + \mu\mathcal{S}^T(t)$ is really invertible because (2.17) implies that the matrix $I + \mu\mathcal{S}^T(t)$ is symplectic and hence invertible. Observe also that (4.13) is again a symplectic dynamic system as can be verified by a direct computation.

THEOREM 4.3. *Suppose that the functions r_ν, $\nu = 0, \dots, n$, are rd-continuous. Then equation* (4.8) *is formally self-adjoint.*

PROOF. Let us suppose that n is even. Then the assumptions of Theorem 4.2 are satisfied (this theorem is formulated for n even, but for n odd the same statement holds only with slightly different block matrices) and therefore equation (4.8) is equivalent to the system

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\Delta} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{4.14}$$

where the blocks $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $n \times n$ matrices of the following form

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & 0 & 0 \\ 0 & 0 & 1 & \mu & \cdots & & & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & & 0 & 1 & 0 & 0 \\ & & & & & 0 & 0 & 1 & \mu \\ & & & & & 0 & 0 & 0 & 1 \\ 0 & \cdots & & & & 0 & 0 & 0 & \frac{\mu r_{n-1}}{r_n^\sigma} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & -\frac{\mu}{r_n^\sigma} & \frac{1}{r_n^\sigma} \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} r_0^\rho & \mu r_0^\sigma & 0 & 0 & \cdots & & 0 & 0 \\ 0 & r_1 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & -\mu r_1 & r_2^\sigma & \mu r_2^\sigma & & & 0 & 0 \\ 0 & 0 & 0 & r_3 & & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots & \vdots \\ & & & & & r_{n-3} & 0 & 0 \\ & & & & & \mu r_{n-3} & r_{n-2}^\sigma & \mu r_{n-2}^\sigma \\ 0 & \cdots & & & & 0 & 0 & r_{n-1} \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ -1 & 0 & 0 & 0 & & & 0 & 0 \\ \mu & -1 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & -1 & 0 & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ & & & & -1 & 0 & 0 & 0 \\ & & & & \mu & -1 & 0 & 0 \\ 0 & \cdots & & & 0 & 0 & -1 & 0 \end{bmatrix}.$$

According to (4.12), the corresponding adjoint system to system (4.14) is

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\Delta} = -\begin{pmatrix} \mathcal{A}^T & \mathcal{C}^T \\ \mathcal{B}^T & \mathcal{D}^T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}^{\sigma}. \tag{4.15}$$

This matrix system is equivalent to the system of $2n$ dynamic equations

$$\begin{array}{rcl}
y_1^\Delta &=& -r_0^\sigma z_1^\sigma, \\
y_2^\Delta &=& -y_1^\sigma - \mu r_0^\sigma z_1^\sigma - r_1 z_2^\sigma + \mu r_1 z_3^\sigma, \\
y_3^\Delta &=& -y_2^\sigma - r_2^\sigma z_3^\sigma, \\
y_4^\Delta &=& -\mu y_2^\sigma - y_3^\sigma - \mu r_2^\sigma z_3^\sigma - r_3 z_4^\sigma + \mu r_3 z_5^\sigma, \\
&\vdots& \\
y_{n-2}^\Delta &=& -\mu y_{n-4}^\sigma - y_{n-3}^\sigma - \mu r_{n-4}^\sigma z_{n-3}^\sigma - r_{n-3} z_{n-2}^\sigma + \mu r_{n-3} z_{n-1}^\sigma, \\
y_{n-1}^\Delta &=& -y_{n-2}^\sigma - r_{n-2}^\sigma z_{n-1}^\sigma, \\
y_n^\Delta &=& -\mu y_{n-2}^\sigma - y_{n-1}^\sigma - \dfrac{\mu r_{n-1}}{r_n^\sigma} y_n^\sigma - \mu r_{n-2}^\sigma z_{n-1}^\sigma - r_{n-1} z_n^\sigma,
\end{array}$$

$$\begin{array}{rcl}
z_1^\Delta &=& z_2^\sigma - \mu z_3^\sigma, \\
z_2^\Delta &=& z_3^\sigma, \\
z_3^\Delta &=& z_4^\sigma - \mu z_5^\sigma, \\
&\vdots& \\
z_{n-2}^\Delta &=& z_{n-1}^\sigma, \\
z_{n-1}^\Delta &=& \dfrac{\mu}{r_n^\sigma} y_n^\sigma + z_n^\sigma, \\
z_n^\Delta &=& -\dfrac{1}{r_n^\sigma} y_n^\sigma,
\end{array}$$

where $y_1, \dots, y_n$ and $z_1, \dots, z_n$ are entries of the vectors y and z, respectively.

Next we show that the first entry z_1 of the vector z in (4.15) satisfies equation (4.8), which proves that this equation is formally self-adjoint. Using (2.3) and (2.7) we have

$$\begin{array}{rcl}
z_1^\Delta &=& z_2^\sigma - \mu z_3^\sigma = z_2 + \mu z_2^\Delta - \mu z_3^\sigma = z_2 + \mu z_3^\sigma - \mu z_3^\sigma = z_2 \\
z_1^{\Delta\nabla} &=& z_2^\nabla = (z_2^\Delta)^\rho = z_3 \\
&\vdots& \\
D_{n-1}^\Delta z_1 &=& z_{n-1}^\Delta = \dfrac{\mu}{r_n^\sigma} y_n^\sigma + z_n^\sigma = \dfrac{\mu}{r_n^\sigma}(-r_n^\sigma z_n^\Delta) + z_n + \mu z_n^\Delta = z_n \\
D_n^\Delta z_1 &=& z_n^\nabla = (z_n^\Delta)^\rho = -\dfrac{1}{r_n} y_n.
\end{array}$$

The last equation implies

$$y_n = -r_n D_n^\Delta z_1$$

and therefore the next identity holds

$$\begin{aligned} y_n^{\Delta} &= -(r_n D_n^{\Delta} z_1)^{\Delta} = -\mu y_{n-2}^{\sigma} - y_{n-1}^{\sigma} - \frac{\mu r_{n-1}}{r_n^{\sigma}} y_n^{\sigma} - \mu r_{n-2}^{\sigma} z_{n-1}^{\sigma} - r_{n-1} z_n^{\sigma} \\ &= -\mu y_{n-2}^{\sigma} - y_{n-1} - \mu y_{n-1}^{\Delta} - \frac{\mu r_{n-1}}{r_n^{\sigma}}(-r_n^{\sigma} z_n^{\Delta}) - \mu r_{n-2}^{\sigma} z_{n-1}^{\sigma} - r_{n-1}(z_n + \mu z_n^{\Delta}) \\ &= -\mu y_{n-2}^{\sigma} - y_{n-1} - \mu(-y_{n-2}^{\sigma} - r_{n-2}^{\sigma} z_{n-1}^{\sigma}) - \mu r_{n-2}^{\sigma} z_{n-1}^{\sigma} - r_{n-1} z_n \\ &= -y_{n-1} - r_{n-1} z_n = -y_{n-1} - r_{n-1} D_{n-1}^{\Delta} z_1. \end{aligned}$$

So that

$$y_{n-1} = (r_n D_n^{\Delta} z_1)^{\Delta} - r_{n-1} D_{n-1}^{\Delta} z_1$$

and the ∇-derivative of y_{n-1} fulfills

$$y_{n-1}^{\nabla} = (r_n D_n^{\Delta} z_1)^{\Delta\nabla} - (r_{n-1} D_{n-1}^{\Delta} z_1)^{\nabla} = -y_{n-2} - r_{n-2} z_{n-1} = -y_{n-2} - r_{n-2} D_{n-2}^{\Delta} z_1,$$

hence

$$\begin{aligned} y_{n-2} &= -(r_n D_n^{\Delta} z_1)^{\Delta\nabla} + (r_{n-1} D_{n-1}^{\Delta} z_1)^{\nabla} - r_{n-2} D_{n-2}^{\Delta} z_1, \\ &= \sum_{\nu=n-2}^{n} (-1)^{\nu-(n-3)} \widetilde{D}_{\nu-(n-2)}^{\nabla} \left(r_\nu(t) D_\nu^{\Delta} z_1 \right). \end{aligned}$$

By similar computations, using only equations of the system of $2n$ dynamic equations and (2.3) and (2.7) we get after $(n-3)$ steps

$$y_1 = \sum_{\nu=1}^{n} (-1)^{\nu} \widetilde{D}_{\nu-1}^{\Delta} \left(r_\nu(t) D_\nu^{\Delta} z_1 \right).$$

The ∇-derivative of the first entry y_1 satisfies

$$y_1^{\nabla} = -r_0 z_1 = \sum_{\nu=1}^{n} (-1)^{\nu} \widetilde{D}_{\nu}^{\nabla} \left(r_\nu(t) D_\nu^{\Delta} z_1 \right).$$

Altogether we get the original equation for the entry z_1

$$\sum_{\nu=0}^{n} (-1)^{\nu} \widetilde{D}_{\nu}^{\nabla} \left(r_\nu(t) D_\nu^{\Delta} z_1 \right) = 0.$$

If n is odd, then the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in (4.14) are slightly different, but the same computation as above shows that (4.8) is formally self-adjoint also in this case. □

THEOREM 4.4. *Suppose that the functions r_ν, $\nu = 0, \dots, n$, are rd-continuous. Then the equation* (4.9) *is formally self-adjoint.*

PROOF. Equation (4.9) can be again rewritten as a delta symplectic system, see [**26**]. Therefore, one can directly verify that the same procedure as in the proof of the previous theorem proves the statement. □

4.4. Remarks and research perspectives

What we have done so far is just the starting point of the qualitative theory of even order dynamic equations with mixed derivatives – a "hint" that equations of the form (4.8) and (4.9) can be written as symplectic dynamic systems. This facts opens a relatively large area for the further investigation of these equations, one can follow the discrete and continuous methods and try to find their time scale unification. In this concluding section we outline some perspectives of the research along this line.

(i) The main research direction is oscillation theory of (4.8) and (4.9). Following the discrete and continuous case, these properties can be defined via (non)oscillation of the associated symplectic system. Let us consider the case that this associated system is a delta symplectic system (2.16). It is not difficult to see that the assumption of dense normality is satisfied and hence one can apply the Roundabout theorem (Proposition 2.4), in particular, the equivalence of disconjugacy and positivity of the corresponding quadratic functional. By a direct computation one can verify that the integrand of the functional $\mathcal{F}$ in Proposition 2.4 is

$$\begin{aligned} F(z) &:= z^T\{\mathcal{S}^T\mathcal{K}+\mathcal{K}\mathcal{S}+\mu\mathcal{S}^T\mathcal{K}\mathcal{S}\}z \\ &= \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} \mathcal{C}^T+\mu\mathcal{C}^T\mathcal{A} & \mu\mathcal{C}^T\mathcal{B} \\ \mathcal{D}^T+\mathcal{A}+\mu\mathcal{D}^T\mathcal{A} & \mathcal{B}+\mu\mathcal{D}^T\mathcal{B} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \end{aligned}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are block entries of $\mathcal{S}$. Consider the case n even (the case n odd is analogical), then substituting for the matrices $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ and for

$$x=\begin{pmatrix} y \\ y^{\Delta} \\ \vdots \\ D^{\Delta}_{n-1} \end{pmatrix}, \quad u=\begin{pmatrix} \sum_{\nu=1}^{n}(-1)^{\nu-1}\widetilde{D}^{\Delta}_{\nu-1}(r_\nu(t)D^{\nabla}_{\nu}y) \\ \sum_{\nu=2}^{n}(-1)^{\nu-2}\widetilde{D}^{\nabla}_{\nu-2}(r_\nu(t)D^{\nabla}_{\nu}y) \\ \vdots \\ -(r_n(t)D^{\Delta}_n y)^{\Delta}+r_{n-1}(t)D^{\Delta}_{n-1}y \\ r_n(t)D^{\Delta}_n y \end{pmatrix}$$

we have using a direct computation similar to that of the previous section

$$\begin{aligned} F(z) &= x^T(\mathcal{C}^T+\mu\mathcal{C}^T\mathcal{A})x+2\mu x^T\mathcal{C}^T\mathcal{B}u+u^T(\mathcal{B}+\mu\mathcal{D}^T\mathcal{B})u \\ &= r_0^{\sigma}(y+\mu y^{\Delta})^2+r_1(y^{\Delta})^2+r_2^{\sigma}(y^{\Delta\nabla}+\mu(y^{\Delta\nabla})^{\Delta})^2+\dots \\ &\quad +r_{n-2}^{\sigma}(D^{\Delta}_{n-2}y+\mu(D^{\Delta}_{n-2}y)^{\Delta})^2+r_{n-1}(D^{\Delta}_{n-1}y)^2+\frac{\mu^2r_{n-1}^2}{r_n^{\sigma}}(D^{\Delta}_{n-1}y)^2 \\ &\quad +\frac{1}{r_n^{\sigma}}\left[(u_n+\mu u_n^{\Delta})^2-\mu^2r_{n-1}^2(D^{\Delta}_{n-1}y)^2\right] \\ &= \sum_{i=0}^{n/2-1}\left\{r_{2i}^{\sigma}[(D^{\Delta}_{2i}y)^{\sigma}]^2+r_{2i+1}(D^{\Delta}_{2i+1}y)^2\right\}+r_n^{\sigma}[(D^{\Delta}_n y)^{\sigma}]^2. \end{aligned}$$

Here we have used the convention that $D^{\Delta}_0 y=y$.

Now suppose that the time scale under consideration is unbounded from above. The equivalence of disconjugacy of (2.16) and the positivity of the associated quadratic functional $\mathcal{F}$ (see Proposition 2.4) implies that (4.8) is *eventually disconjugate* (another terminology is *nonoscillatory*) if and only if for every $T \in \mathbb{T}$ the symplectic system (we consider here the *delta* symplectic system since its oscillation theory is relatively deeply developed) is disconjugate on $[T, T_1]$ for every $\mathbb{T} \ni T_1 > T$, and this is equivalent to (with the relationship between y and $z = \binom{x}{u}$)

$$\mathcal{F}(y) = \int_T^\infty F(z)\,\Delta t > 0$$

for every nontrivial y for which $D_n^\Delta y$ exists, it is piecewise rd-continuous, $D_i^\Delta y(T) = 0$, $i = 0, \dots, n-1$, and there exists $\tilde{T} \in \mathbb{T}$ such $y(t) \equiv 0$ for $t > \tilde{T}$. There exist various oscillation and nonoscillation criteria for (4.1) and (4.2) based on this variational principle, see, e.g., [**23, 29**]. The results of the previous section, coupled with the oscillation criteria given in [**13**] suggest to look for time scale unification of these criteria. An important role in this investigation may play the time scale version of the Wirtinger inequality proved in [**34**].

(ii) To explain another research possibility, consider the two-term differential equation

$$(-1)^n \left(r(t)y^{(n)}\right)^{(n)} = q(t)y, \tag{4.16}$$

where r, q are positive functions. It is known (see, e.g., [**3**]) that this equation is nonoscillatory if and only if the so-called reciprocal equation (related to (4.16) by the substitution $z = ry^{(n)}$)

$$(-1)^n \left(\frac{1}{q(t)} z^{(n)}\right)^{(n)} = \frac{1}{r(t)} z$$

is also nonoscillatory. A discrete version of this statement is established in [**12, 21**]. A natural question is whether a unifying time scale approach can be developed on the basis of the results of this paper. We refer also to the paper [**35**], where this problem is treated in the scope of time scale Hamiltonian systems.

(iii) Another problem closely related to the oscillation theory of (formally) self-adjoint higher order equations is the *factorization* of the corresponding differential operator. Denote by $L(y)$ the $2n$-th order differential operator defined by the left-hand side of (4.1). If this equation is disconjugate on an interval I, the classical result of the theory of differential operators states that in this case there exists an n-th differential operator

$$N(y) = y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y$$

with continuous functions $a_0, \dots, a_{n-1}$, such that the operator L admits in I the factorization

$$L(y) = N^*(r_n(t)N(y)),$$

where N^* is the adjoint operator of M. A discrete version of this statement can be found in [**19**] and suggests again to look for a time scale unification.

(iv) The last research problem which we point out here is the transformation theory of even order self-adjoint equations in the framework of transformations of Hamiltonian or symplectic systems. It is shown in [**4**] (continuous case) and in [**12**] (discrete case) that the transformation of dependent variable $y = hz$, where h is a transformation function (sequence), can be investigated as a special case of the general transformation of Hamiltonian or symplectic systems. The results of the previous section suggest to look for a time scale unifying approach to this problem.

(v) In the paper [**8**], a similar problem as in this chapter is investigated. In the main part of that paper the authors deal with another $2n$-order dynamic equations with mixed derivatives

$$L(y) := \sum_{i=0}^{n} (-1)^i \left(r_i(t) y^{\Delta^{i-1}\nabla} \right)^{\nabla^{i-1}\Delta} \tag{4.17}$$

and its "nabla" counterpart

$$M(y) := \sum_{i=0}^{n} (-1)^i \left(r_i(t) y^{\nabla^{i-1}\Delta} \right)^{\Delta^{i-1}\nabla} \tag{4.18}$$

(with the convention that for $i = 0$ and $i = 1$ the corresponding terms in L are $r_0(t)y$ and $\left(r_1(t)y^{\nabla}\right)^{\Delta}$, a similar convention is used in the operator M). It is shown that these equations can be written in the form of the time scale linear Hamiltonian system (4.5) and hence also in the form (2.16). Further it is shown that these equations are formally self-adjoint with respect to a certain inner product, provided some boundary conditions are satisfied. At the final part of [**8**], equations of the form (4.8) and (4.9) are briefly discussed and their transformation into Hamiltonian systems is suggested. However, the approach used there is different from ours. Finally, note that all research problems mentioned in this section "apply" also to equations (4.17), (4.18).

(vi) If the functions r_ν are ld-continuous, equations (4.8), (4.9) can be written in the form (2.20) and the adjoint system to this system is $y^{\nabla} = -\mathcal{S}^T(t)y^{\rho}$. Using the same idea as in the previous section, it can be shown, that (4.8), (4.9) are formally self-adjoint in this case as well. Only the block matrices in these nabla systems are slightly different from those in the delta symplectic systems, the technical computations are very similar.

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Abstract

The thesis consists essentially of two parts. The first one deals with oscillation theory of the half-linear second order dynamic equation with mixed derivatives

$$(r(t)\Phi(x^{\Delta}))^{\nabla}+c(t)\Phi(x)=0, \quad \Phi(x)=|x|^{p-1}\operatorname{sgn}(x),\ p>1.$$

It is established the so-called Roundabout theorem for this equation and this theorem is used to prove several oscillation and nonoscillation criteria for this equation. The second part is devoted to the investigation of even order dynamic equations

$$\begin{aligned} L(y) &:= \sum_{\nu=0}^{n}(-1)^{\nu}\widetilde{D}_{\nu}^{\nabla}\left(r_{\nu}(t)D_{\nu}^{\Delta}y\right)=0, \\ M(y) &:= \sum_{\nu=0}^{n}(-1)^{\nu}\widetilde{D}_{\nu}^{\Delta}\left(r_{\nu}(t)D_{\nu}^{\nabla}y\right)=0, \end{aligned}$$

where $D_{\nu}^{\Delta}, \widetilde{D}_{\nu}^{\Delta}, D_{\nu}^{\nabla}, \widetilde{D}_{\nu}^{\nabla}$ are certain ν-th order differential operators with mixed derivatives. It is shown that equations $L(y)=0, M(y)=0$ are formally self-adjoint and that they can be written in the form of the so-called delta and nabla symplectic systems.

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