MASARYK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

# Ph.D. Dissertation

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PETER ŠEPITKA

**MASARYK UNIVERSITY**
**FACULTY OF SCIENCE**
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# Theory of Principal Solutions at Infinity for Linear Hamiltonian Systems

Ph.D. Dissertation

**Peter Šepitka**

**Advisor: Prof. RNDr. Roman Šimon Hilscher, DSc.** **Brno 2014**

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# Bibliografický záznam

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# Abstract

In this dissertation we develop the theory of principal and antiprincipal solutions at infinity for linear Hamiltonian systems without any controllability assumption. We prove the existence and basic properties of principal and antiprincipal solutions for such nonoscillatory systems. Moreover, we show that principal and antiprincipal solutions can be classified according to the rank of their first component and that they exist for any rank in the range between explicitly given minimal and maximal values. The minimal rank then corresponds to the minimal principal and antiprincipal solutions at infinity, which generalize the classical principal and antiprincipal solutions at infinity developed by W. T. Reid, P. Hartman, W. A. Coppel, and C. D. Ahlbrandt for completely controllable systems. On the other hand, the maximal principal solution (corresponding to the maximal rank) coincides with the principal solution at infinity introduced by Reid for general, possibly abnormal linear Hamiltonian systems. By using a new concept of genera of conjoined bases, we also derive a classification of all principal and antiprincipal solutions, which have eventually the same image of their first component, as well as we establish a limit characterization of principal solutions. The proofs and methods are based on a detailed analysis of conjoined bases with a given rank and their construction from the minimal conjoined bases. Finally, we illustrate our new theory by several examples. This research was supported by grant MUNI/A/0821/2013 of Masaryk University.

**Mathematics Subject Classification (2010):** 34C10.

# Abstrakt

V této disertační práci budujeme teorii hlavních a antihlavních řešení v nekonečnu pro lineární hamiltonovské systémy bez předpokladu kontrolovatelnosti. Dokážeme existenci a základní vlastnosti hlavních a antihlavních řešení takových neoscilatorických systémů. Dále ukazujeme, že hlavní a antihlavní řešení lze klasifikovat podle hodnosti jejich první komponenty a že tyto řešení existují pro každou hodnost mezi explicitně danou minimální a maximální hodností. Minimální hodnost pak odpovídá minimálnímu hlavnímu a antihlavnímu řešení, které zobecňuje klasické hlavní a antihlavní řešení definované W. T. Reidem, P. Hartmanem, W. A. Coppelem a C. D. Ahlbrandtem pro úplně kontrolovatelné systémy. Naproti tomu maximální hlavní řešení (odpovídající maximální hodnosti) se shoduje s hlavním řešením v nekonečnu, které již dříve představil Reid pro obecné nekontrolovatelné lineární hamiltonovské systémy. Zavedením nového pojmu genu (nebo také rodu) izotropických bází jsme v práci také odvodili klasifikaci všech hlavních a antihlavních řešení, které mají eventuálně stejný obraz, a limitní charakterizaci hlavních řešení. Důkazy a metody jsou založeny na detailní analýze izotropických bází s danou hodností a na jejich konstrukci z minimálních izotropických bází. Naši novou teorii jsme také doplnili několika ilustrujícími příklady. Práce na disertaci byla podpořena projektem specifického výzkumu MUNI/A/0821/2013 Masarykovy univerzity.

MASARYKOVA UNIVERZITA
Přírodovědecká fakulta

# ZADÁNÍ PRÁCE

Akademický rok: 2014/2015

**Ústav:** Ústav matematiky a statistiky

**Student:** Mgr. Peter Šepitka

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Ředitel *Ústavu matematiky a statistiky* PřF MU Vám ve smyslu Studijního a zkušebního řádu MU určuje práci s tématem:

**Téma práce:** Teorie hlavních řešení v nekonečnu pro lineární hamiltonovské systémy

**Téma práce anglicky:** Theory of principal solutions at infinity for linear Hamiltonian systems

**Oficiální zadání:**
Student se bude věnovat teorii lineárních Hamiltonovských systémů a jejich diskrétní analogii - diskrétním symplektickým systémům. Tyto systémy vznikají v teorii variačního počtu a optimálního řízení v podmínkách optimality druhého řádu. Student bude zkoumat oscilační a spektrální vlastnosti řešení těchto systémů, zejména v souvislosti s odstraněním předpokladu kontrolovatelnosti (normality) v této teorii.

**Jazyk závěrečné práce:** angličtina

**Vedoucí práce:** prof. RNDr. Roman Šimon Hilscher, DSc.

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# Contents

# Preface

The principal solution at infinity represents a traditional object in the theory of differential equations with applications e.g. in the oscillation and spectral theory. In this thesis we open a new direction in the study of principal and antiprincipal solutions for nonoscillatory linear Hamiltonian systems. In the absence of any controllability assumption, we introduce concepts of principal and antiprincipal solutions at infinity, which are generalizations of the classical principal/antiprincipal solutions (sometimes called the recessive/dominant solutions) for completely controllable linear Hamiltonian systems. These new results were obtained and published by the author (jointly with his advisor prof. Roman Šimon Hilscher) during his Ph.D. study between years 2010 and 2014, see [35–37].

The dissertation consists of eight chapters. The first introductory chapter includes the motivation for the main subject of this work, preliminaries from the matrix analysis, and an overview of some classical parts of the theory of linear Hamiltonian systems needed in the subsequent chapters. The main results of the dissertation are contained in Chapters 2–6. In Chapters 2–4 we develop the theory of the representation and the construction of conjoined bases of linear Hamiltonian systems, as well as we present the asymptotic properties of their corresponding $S$-matrices. These results are then essentially utilized in the remaining parts of this work. The theory of principal and antiprincipal solutions at infinity is introduced in Chapters 5 and 6. In particular, we establish a precise classification and limit properties of principal and antiprincipal solutions. Some of these results are new even for controllable linear Hamiltonian systems. In Chapter 7 we present several examples which illustrate the topics in this work. The final chapter contains some notes about further research in the presented theory and open problems. This thesis is completed by two appendices, which include some auxiliary results from the matrix analysis, author's current list of publications, and his curriculum vitae.

The highlights of this dissertation are, in author's opinion, the following results and methods:

- the construction of conjoined bases via the relation "being contained" (Definition 3.2.1 and Theorems 3.2.4, 3.2.5, 3.2.7, 3.2.8, and 3.2.10),

- the concept of a minimal conjoined basis and its properties (Definition 3.3.1, Remark 3.3.5, and Theorem 3.3.6),

- a fundamental connection between the asymptotic behavior of $S$-matrices associated with minimal conjoined bases and the maximal order of abnormality (Theorems 4.1.12 and 4.3.2),

- the concepts of a principal solution at infinity (Definition 5.1.1) and an antiprincipal solution at infinity (Definition 5.2.1) general abnormal linear Hamiltonian systems,
- the existence and classification of principal and antiprincipal solutions at infinity according to the rank of their first component (Theorems 5.1.5, 5.1.6, and 5.2.7),
- the concept of a genus of conjoined bases and a classification of all principal and antiprincipal solutions at infinity, as well as of all conjoined bases, according to the image of their first component (Theorems 6.3.7 and 6.3.13 and Corollary 6.3.15),
- the characterization of all antiprincipal solutions in terms of their Wronskian with a given principal solution within one genus (Theorem 6.3.11 and Corollary 6.3.12),
- the limit characterization of all principal solutions belonging to a given genus (Theorems 6.4.1 and 6.4.5).

I would like to express gratitude and appreciation to my advisor, prof. Roman Šimon Hilscher, for his guidance, patience, and extraordinary care. He has taught me not only most what I know about the problems of this work, but also how to present the new results in the most optimal way. I am also grateful to prof. Ondřej Došlý for his continuous support throughout my Ph.D. study. Many thanks belong to prof. Werner Kratz for very fruitful and inspiring discussions, for a kind welcome and for the opportunity to speak about my research on their seminar during my stay at the University of Ulm. Last but not least, I wish to express my thanks to my family, and to all my friends for the permanent support during the years of my study.

Brno, October 2014 Peter Šepitka

# Chapter 1

# Introduction

In this chapter we introduce the central objects studied in this thesis. We also comment the main results of this work in the framework of the current literature. In Section 1.1 we provide some facts and reasons which motivated our research. In Sections 1.2–1.5 we review some important notions and results from the matrix analysis and the theory of controllable linear Hamiltonian systems.

## 1.1 Introduction and motivation

Let $n \in \mathbb{N}$ be a given dimension and $a \in \mathbb{R}$ be a fixed number. In this work we study the *linear Hamiltonian system*

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \qquad t \in [a,\infty), \tag{H}$$

where $A, B, C : [a,\infty) \to \mathbb{R}^{n\times n}$ are given piecewise continuous matrix-valued functions such that $B(t)$ and $C(t)$ are symmetric on $[a,\infty)$ and satisfying the Legendre condition

$$B(t) \geq 0 \quad \text{on } [a,\infty). \tag{1.1}$$

Such a study is motivated by the general interest in the differential equations of the form (H), which have their origin in the nonlinear variational theory [25, 44, 45].

The main purpose of this work is to develop a theory of *principal and antiprincipal solutions* for system (H) under no controllability assumption. In the literature one usually studies a completely controllable system (H). This means, roughly speaking, that the only solution $(x,u)$ of (H), whose first component $x$ vanishes on some nondegenerate subinterval, is the trivial solution $(x,u) \equiv (0,0)$. As we comment in Section 1.4 below, if (H) is completely controllable and nonoscillatory, then certain matrix solutions $(X,U)$ of (H), called conjoined bases, have their first component eventually invertible. In this case W. T. Reid defined in [29] the *principal solution* $(\hat{X},\hat{U})$ of (H) at infinity as a conjoined basis, for which

$$\lim_{t\to\infty} \hat{S}^{-1}(t) = 0, \qquad \text{where} \quad \hat{S}(t) := \int^t \hat{X}^{-1}(s)\,B(s)\,\hat{X}^{T-1}(s)\,\mathrm{d}s, \tag{1.2}$$

see also the monographs by P. Hartman [16, Section XI.10] or W. T. Reid [31, pg. 316] or W. A. Coppel [6, Setion 2.2]. Since then, the principal solution was used in many applications, which include for example the theory of Riccati matrix differential equations [8, 29, 31, 32], oscillation theory [1, 2, 10–12, 14, 20, 28], Sturmian theory [4, 15], and spectral theory (property BD and Friedrichs extension) [9, 13, 17, 42]. On the other hand, the principal solution $(\hat{X},\hat{U})$ in (1.2) is the smallest solution of (H) at infinity in the sense that

$$\lim_{t\to\infty} X^{-1}(t)\hat{X}(t) = 0 \tag{1.3}$$

for any conjoined basis $(X,U)$ of (H) which is linearly independent on $(\hat{X},\hat{U})$ and $X(t)$ is eventually invertible, see [1, Theorem 2.2], [6, Proposition 4, pg 43], [31, Theorem VII.3.2], [16, Theorem XI.10.5]. In this context the conjoined basis $(X,U)$ is called an *antiprincipal solution* (or *nonprincipal solution*) of (H) at infinity and similarly to (1.2) it is characterized by the property, see [1, Theorem 3.1(ii)],

$$\lim_{t\to\infty} \left( \int^t X^{-1}(s)\,B(s)\,X^{T^{-1}}(s)\,\mathrm{d}s \right)^{-1} = T, \qquad \text{with } T \text{ nonsingular.} \tag{1.4}$$

In the paper [30], Reid succeeded to remove the controllability assumption in the definition of the principal solution. More precisely, if (H) is completely controllable, then the function $\hat{S}(t)$ is strictly increasing for large $t$ and thus, the matrix $\hat{S}(t)$ is necessarily eventually invertible, see Section 1.3 below. Without controllability of (H), the function $\hat{S}(t)$ is only nondecreasing and Reid replaced the inverse of $\hat{S}(t)$ in (1.2) by its Moore–Penrose pseudoinverse. That is, according to [30, Section 4], a conjoined basis $(\hat{X},\hat{U})$ of (H) is a principal solution at infinity when

$$\lim_{t\to\infty} \hat{S}^{\dagger}(t) = 0, \qquad \text{where} \quad \hat{S}(t) := \int^t \hat{X}^{-1}(s)\,B(s)\,\hat{X}^{T^{-1}}(s)\,\mathrm{d}s. \tag{1.5}$$

Note that the definition of the matrix $\hat{S}(t)$ in (1.2) and (1.5) is the same, namely $\hat{S}(t)$ is in both cases constructed from an invertible $\hat{X}(t)$ near infinity, see Section 1.5.

It is the primary aim of this work to develop the most general concept of the principal solution for system (H) without assuming its controllability. The only assumptions we impose are the Legendre condition (1.1) and the nonoscillation of system (H), defined in appropriate way, see Section 1.3 below. We do not require that the principal solution $(\hat{X},\hat{U})$ has $\hat{X}(t)$ invertible for large $t$, but only that $\hat{X}(t)$ has constant kernel for large $t$. Our definition of the principal solution then involves the Moore–Penrose pseudoinverses of both $\hat{S}(t)$ and $\hat{X}(t)$,

$$\lim_{t\to\infty} \hat{S}^{\dagger}(t) = 0, \qquad \text{where} \quad \hat{S}(t) := \int^t \hat{X}^{\dagger}(s)\,B(s)\,\hat{X}^{\dagger T}(s)\,\mathrm{d}s. \tag{1.6}$$

Secondly, following the above study we introduce the corresponding concept of antiprincipal solutions at infinity for possibly abnormal linear Hamiltonian system (H). Assuming only the Legendre condition (1.1) and the nonoscillation of system (H), we define the antiprincipal solution $(X,U)$ of (H) as a conjoined basis with eventually constant kernel of $X(t)$ and, similarly to (1.4),

$$\lim_{t\to\infty} \left( \int^t X^{\dagger}(s)\,B(s)\,X^{\dagger T}(s)\,\mathrm{d}s \right)^{\dagger} = T, \qquad \text{with maximal rank of } T. \tag{1.7}$$

The change from (1.5) to (1.6) and from (1.4) to (1.7) are by no means trivial. As we shall see, it requires a whole new theory describing the properties of conjoined bases of system (H) with constant kernel and their corresponding $S$-matrices, as well as a precise analysis of the abnormality of system (H). Our research reveals the existence of a whole scale of new (anti)principal solutions with the rank of their first component equal to *any given value* between an explicitly given minimal rank (corresponding to the minimal (anti)principal solutions) and the maximal rank $n$ (corresponding to the maximal (anti)principal solutions). In this respect, this minimal principal solution is then the abnormal analogy of the classical Reid's principal solution in (1.2). On the other hand, Reid's principal solution for general system (H) defined in (1.5) corresponds to our maximal principal solution. Other important goals of this work are to describe the exact relationship between the minimal (anti)principal solutions and the (anti)principal solutions with any higher rank up to $n$ and to classify the (anti)principal solutions which have eventually the same image. This gives rise to a new concept called a *genus* of conjoined bases of (H). The classification of principal solutions within one genus completes the work by Reid in [30] on the invertible principal solutions (1.5). This topic opens a new field in the study of linear Hamiltonian systems and their solutions. Finally, we also provide a limit comparison of the principal and antiprincipal solutions of (H) at infinity in the sense of (1.3).

To our knowledge, the first use of Moore–Penrose pseudoinverses in the theory of linear Hamiltonian systems is documented in Reid's paper [30] mentioned above. After that, a long time elapsed until W. Kratz reintroduced them into this theory in his influential work [24]. In essence, it was that paper [24] by Kratz and the results in [39, 40] which motivated present appearance of the Moore–Penrose pseudoinverse of $\hat{X}(t)$ and $X(t)$ in (1.6) and (1.7), respectively. In [24, Theorem 3] it is proven that under (1.1) every conjoined basis $(X,U)$ of (H) has the kernel of $X$ piecewise constant on any compact subinterval of $[a,\infty)$. Furthermore, by [39, 40] the kernel of $X$ is eventually constant for any conjoined basis $(X,U)$ of (H) when the system is *nonoscillatory*. The nonoscillation or oscillation of system (H) or its conjoined bases is defined through the concept of *proper focal points* from [43, Definition 1.1]. These are the points where the (piecewise constant) kernel of $X$ increases. It is known that under (1.1) the proper focal points of any conjoined basis of (H) are isolated and that the Sturmian theory works as in the controllable case, see [39, 40]. Therefore, a conjoined basis $(X,U)$ of (H) is nonoscillatory, and then every conjoined basis of (H) is nonoscillatory as well, when the kernel of $X$ is eventually nonincreasing, hence eventually constant. This motivates the study of conjoined bases of (H) with constant kernel and leads to a proper concept of the (anti)principal solution at infinity for possibly abnormal linear Hamiltonian systems. This new concept of an (anti)principal solution can be then utilized in order to extend any result where the classical (anti)principal solution of (H) was used, see e.g. the applications of the principal solution, which we mention earlier in this section.

The results of this work reopen the very traditional theory of principal and antiprincipal solutions of (H) at infinity in [6, 16, 31] and they contribute in a significant way to the current research of possibly abnormal linear Hamiltonian systems [21–24, 26, 39, 40].

## 1.2 Review of matrices and matrix functions

In this section we summarize some notions from the matrix analysis, in particular about orthogonal projectors and Moore–Penrose pseudoinverses.

For any real matrix $M$ we denote by $\operatorname{Im} M$, $\operatorname{Ker} M$, $\operatorname{rank} M$, $M^T$, $M^{-1}$, and $M^{\dagger}$ the image of $M$, the kernel of $M$, the rank of $M$, the transpose of $M$, the classical inverse of $M$ when $M$ is square and invertible, and the Moore–Penrose pseudoinverse of $M$ (see Remark 1.2.3), respectively. Furthermore, for any square matrix $M \in \mathbb{R}^{n\times n}$ we denote by $(M)_k$ the $k$-th leading principal submatrix of $M$, i.e., $(M)_k$ is formed by the elements $m_{ij}$ of $M$ for $i, j = 1,\dots,k$. We write $M \geq 0$ and $M > 0$ when the symmetric matrix $M$ is nonnegative definite and positive definite, respectively. Moreover, $I_n$ and $0_n$ denote the identity and zero matrices of dimension $n$. When it is clear from the context, we just write $I$ and $0$ for the corresponding identity and zero matrices. Sometimes we use the notation $\operatorname{diag}\{M_1,\dots,M_k\}$ for the block-diagonal matrix with (block) entries $M_1,\dots,M_k$ on its diagonal.

For any linear subspace $\mathscr{V}$ of $\mathbb{R}^n$ we denote by $\dim \mathscr{V}$ and $\mathscr{V}^{\perp}$ the dimension of $\mathscr{V}$ and the orthogonal complement of $\mathscr{V}$ in $\mathbb{R}^n$ with respect to the canonical inner product. The following remark concerns the properties of orthogonal projectors (for further results we refer to Appendix A).

**Remark 1.2.1.** (i) According to [5, Section 0.2], if $\mathscr{V}$ is a linear subspace of $\mathbb{R}^n$, then a matrix $P_{\mathscr{V}} \in \mathbb{R}^{n\times n}$ is said to be an orthogonal projector onto $\mathscr{V}$ if $P_{\mathscr{V}}\, v = v$ for all $v \in \mathscr{V}$, and $P_{\mathscr{V}}\, v = 0$ for all $v \in \mathscr{V}^{\perp}$. The matrix $P_{\mathscr{V}}$ is uniquely determined by the subspace $\mathscr{V}$. The matrix $I - P_{\mathscr{V}}$ is then the orthogonal projector onto $\mathscr{V}^{\perp}$. Moreover, $P_{\mathscr{V}}$ is symmetric and

$$\operatorname{Im} P_{\mathscr{V}} = \operatorname{Ker}(I - P_{\mathscr{V}}) = \mathscr{V}, \quad \operatorname{Ker} P_{\mathscr{V}} = \operatorname{Im}(I - P_{\mathscr{V}}) = \mathscr{V}^{\perp}.$$

(ii) A matrix $P \in \mathbb{R}^{n\times n}$ is the orthogonal projector onto a subspace of $\mathbb{R}^n$ if and only if $P$ is symmetric and idempotent, i.e., $P^2 = P$. In this case $P$ is the orthogonal projector onto $\operatorname{Im} P$. Every orthogonal projector is diagonalizable matrix with the spectrum consisting of only two values, 0 and 1. More precisely, if $P \in \mathbb{R}^{n\times n}$ is an orthogonal projector and $p := \operatorname{rank} P$, then there exists an $n \times n$ orthogonal matrix $V$ such that

$$P = V \operatorname{diag}\{I_p, 0_{n-p}\}\, V^T. \tag{1.8}$$

Orthogonal projectors are frequently constructed by using Moore–Penrose pseudoinverses, as we comment in Remark 1.2.3 below. In this work we will use several important subsets of matrices associated with projectors. More precisely, let $P_{**}$, $P_*$ and $P$ be orthogonal projectors in $\mathbb{R}^n$ satisfying the inclusions $\operatorname{Im} P_{**} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P$. We define the following sets of matrices

$$\mathscr{M}(P_*) = \big\{E \in \mathbb{R}^{n\times n},\ E \text{ is invertible},\ EP_* = P_*\big\}, \tag{1.9}$$

$$\mathscr{A}(P_{**}, P_*) = \big\{F \in \mathbb{R}^{n\times n},\ F = F^T,\ P_{**}\, F\, P_* = 0\big\}, \tag{1.10}$$

$$\begin{aligned}\mathscr{B}(P_{**}, P_*, P) = \big\{&(G,H) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n},\ \operatorname{rank}(G^T, H^T, P_*) = n,\\ &P_{**}\, G = 0,\ PG = G,\ P_*\, G = G^T P_*,\ PH = 0\big\}.\end{aligned} \tag{1.11}$$

It follows that $\mathscr{M}(P_*)$ is a subgroup of the multiplicative matrix group $\mathrm{GL}(n,\mathbb{R})$ and $\mathscr{A}(P_{**}, P_*)$ is a subgroup of the additive matrix group $\mathrm{Mat}(n,\mathbb{R})$.

**Remark 1.2.2.** Note that the set $\mathscr{B}(P_{**},P_*,P)$ is always nonempty, because the pair $(G,H)$ with $G := P - P_*$ and $H := I - P$ belongs to $\mathscr{B}(P_{**},P_*,P)$. Moreover, it is easy to see that for $P = I$ and $(G,H) \in \mathscr{B}(P_{**},P_*,I)$ we have $H = 0$. This follows from the last equality in (1.11). Therefore, in order to shorten and simplify our notation, we identify the set $\mathscr{B}(P_{**},P_*,I)$ with the set of all matrices $\bar{G} \in \mathbb{R}^{n\times n}$ satisfying

$$P_{**}\bar{G} = 0, \quad P_*\bar{G} = \bar{G}^T P_*, \quad \operatorname{rank}(\bar{G}^T,\, P_*) = n. \tag{1.12}$$

The following remark collects important properties of the Moore–Penrose pseudoinverses. These results can be found in [5, Section 1.4] and [3, Chapter 6].

**Remark 1.2.3.** (i) For any matrix $M \in \mathbb{R}^{m\times n}$ there exists a unique matrix $M^\dagger \in \mathbb{R}^{n\times m}$, called the Moore–Penrose pseudoinverse of $M$, satisfying the equalities

$$MM^\dagger M = M, \quad M^\dagger MM^\dagger = M^\dagger, \quad MM^\dagger = (MM^\dagger)^T, \quad M^\dagger M = (M^\dagger M)^T. \tag{1.13}$$

(ii) The matrix $MM^\dagger$ is the orthogonal projector onto set $\operatorname{Im} M = \operatorname{Im} M^{\dagger T}$ and the matrix $M^\dagger M$ is the orthogonal projector onto set $\operatorname{Im} M^\dagger = \operatorname{Im} M^T$. Moreover, $\operatorname{rank} M = \operatorname{rank} MM^\dagger = \operatorname{rank} M^\dagger M$.

(iii) If $M \in \mathbb{R}^{m\times n}$ and if $V$ and $W$ are orthogonal matrices of suitable dimensions, then the formula $(VMW)^\dagger = W^T M^\dagger V^T$ holds.

(iv) If $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{n\times p}$, then $(MN)^\dagger = (P_{\operatorname{Im} M^T} N)^\dagger\, (MP_{\operatorname{Im} N})^\dagger$.

We denote vector functions by small letters, e.g. as $x,u : [a,\infty) \to \mathbb{R}^n$, and we use capital letters for matrix-valued functions, e.g. as $X,U : [a,\infty) \to \mathbb{R}^{n\times n}$. Limits, differentiation, and integration of matrix-valued functions are always understood elementwise. By $\mathrm{C_p}$ we denote the set of piecewise continuous (vector- or matrix-valued) functions on $[a,\infty)$, i.e., a function $f \in \mathrm{C_p}$ has finitely many discontinuities $t_1,\dots,t_m$ in every subinterval $[a,b] \subseteq [a,\infty)$ with finite one-sided limits at these points $t_1,\dots,t_m$. Moreover, by $\mathrm{C}^1_\mathrm{p}$ we denote the set of piecewise continuously differentiable functions on $[a,\infty)$, i.e., a function $f \in \mathrm{C}^1_\mathrm{p}$ is continuous on $[a,\infty)$ with $f' \in \mathrm{C_p}$. In particular, the one-sided derivatives $f'(t_0^+)$ and $f'(t_0^-)$ are finite at points $t_0$, where $f'(t)$ is not continuous. These values are then used by convention in all formulas involving $f'(t)$ without any further notice. We will need the following results on the differentiability of the Moore–Penrose pseudoinverse $M^\dagger(t)$ of a matrix-valued function $M(t)$. The corresponding results can be found in [5, Theorems 10.5.1 and 10.5.3].

**Remark 1.2.4.** (i) Let $\alpha \in [a,\infty)$. If $M(t)$ is a differentiable matrix-valued function defined on $[\alpha,\infty)$, then the following three conditions are equivalent: (a) $M^\dagger(t)$ is differentiable on $[\alpha,\infty)$, (b) $M^\dagger(t)$ is continuous on $[\alpha,\infty)$, (c) $\operatorname{rank} M(t)$ is constant on $[\alpha,\infty)$. In this case, the formula for the derivative of the pseudoinverse of $M(t)$ is (we omit the argument $t$)

$$(M^\dagger)' = -M^\dagger M' M^\dagger + (I - M^\dagger M)\,(M')^T M^{\dagger T} M^\dagger + M^\dagger M^{\dagger T} (M')^T (I - MM^\dagger). \tag{1.14}$$

Note that formula (1.14) holds when $\operatorname{Ker} M$ is constant on $[\alpha,\infty)$, which is a part of the statement in [24, Lemma 6] for $M$ piecewise continuously differentiable. In particular, when $\operatorname{Ker} M$ is constant on $[\alpha,\infty)$ we have $\operatorname{Ker} M \subseteq \operatorname{Ker} M'$ on $[\alpha,\infty)$ and the formula in (1.14) becomes

$$(M^\dagger)' = -M^\dagger M' M^\dagger + M^\dagger M^{\dagger T} (M')^T (I - MM^\dagger). \tag{1.15}$$

In addition, when $M$ is symmetric then we also have $\operatorname{Ker} M^T \subseteq \operatorname{Ker}(M')^T$ and equality (1.15) yields the standard formula $(M^\dagger)' = -M^\dagger M' M^\dagger$ on $[\alpha,\infty)$.

(ii) Let $M(t)$ be a matrix function such that $M(t) \to M$ for $t \to \infty$. Then by [5, Theorems 10.4.1 and 10.4.2] the function $M^\dagger(t)$ has a limit (say $N$) as $t \to \infty$ if and only if $\operatorname{rank} M(t) = \operatorname{rank} M$ for large $t$. In this case we have $N = M^\dagger$.

## 1.3 Linear Hamiltonian systems

In this section we recall some important properties of linear Hamiltonian systems and their solutions. By a vector or a matrix solution of (H) we mean a pair of functions $(x,u)$ such that $x,u : [a,\infty) \to \mathbb{R}^n$ or a pair of functions $(X,U)$ such that $X,U : [a,\infty) \to \mathbb{R}^{n\times n}$ with $x,u,X,U \in \mathrm{C}^1_{\mathrm{p}}$. In order to shorten the notation and the calculations, we suppress the argument $t$ in the solutions whenever it is possible. For any two matrix solutions $(X_1,U_1)$, $(X_2,U_2)$ of (H) their Wronskian $X_1^T U_2 - U_1^T X_2$ is a constant matrix on $[a,\infty)$, as can be verified by differentiation. A solution $(X,U)$ of (H) is called a *conjoined basis* if $\operatorname{rank}(X^T(t),\ U^T(t)) = n$ and $X^T(t)U(t)$ is symmetric at some and hence at any $t \in [a,\infty)$. Alternative terminology which is also used in the literature is an *isotropic basis* or a *prepared basis* of (H). It is well known that many results in the oscillation and spectral theory of linear Hamiltonian systems hold only for conjoined bases of (H), see e.g. [6, 24, 31, 39, 40].

For a fixed point $\alpha \in [a,\infty)$, the *principal solution at* $\alpha$ is the conjoined basis $(\hat{X}_\alpha, \hat{U}_\alpha)$ of (H) given by the initial conditions $\hat{X}_\alpha(\alpha) = 0$ and $\hat{U}_\alpha(\alpha) = I$. Since $\hat{X}_\alpha(\alpha) = 0$, the principal solution $(\hat{X}_\alpha, \hat{U}_\alpha)$ is the "smallest" solution at the point $\alpha$ among all conjoined bases $(X,U)$ of (H), measured by their first component $X$. This terminology is also used in the main results of this work, where we construct principal solutions of (H) at infinity.

Every conjoined basis $(X,U)$ of (H) forms one half of a fundamental system of (H). The conjoined basis $(X,U)$ can always be completed to a fundamental system of (H) by another conjoined basis $(\bar{X},\bar{U})$, as follows from [25, Corollary 3.3.9]. In addition, the conjoined basis $(\bar{X},\bar{U})$ can be chosen so that $(X,U)$ and $(\bar{X},\bar{U})$ are *normalized*, i.e., on $[a,\infty)$ we have

$$X^T\bar{U} - U^T\bar{X} = I. \tag{1.16}$$

Except of identity (1.16), the above normalized conjoined bases of (H) satisfy the equalities

$$X\bar{U}^T - \bar{X}U^T = I, \quad X\bar{X}^T = \bar{X}X^T, \quad U\bar{U}^T = \bar{U}U^T \tag{1.17}$$

on $[a,\infty)$. The formulas in (1.17) follow from the fact that the $2n\times 2n$ matrix-valued function

$$\Phi := \begin{pmatrix} X & \bar{X} \\ U & \bar{U} \end{pmatrix}$$

is a symplectic fundamental matrix of (H) with the inverse $\Phi^{-1}$ satisfying

$$\Phi^{-1} = \begin{pmatrix} \bar{U}^T & -\bar{X}^T \\ -U^T & X^T \end{pmatrix}. \tag{1.18}$$

Then any solution $(X_0, U_0)$ of (H) can be expressed as

$$\begin{pmatrix} X_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} X & \bar{X} \\ U & \bar{U} \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \quad \text{on } [a, \infty), \tag{1.19}$$

where the constant matrices $M, N \in \mathbb{R}^{n \times n}$ are uniquely determined by the solutions $(X_0, U_0)$, $(X, U)$ and $(\bar{X}, \bar{U})$. In particular, we have from (1.19) and (1.18) that

$$\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} \bar{U}^T & -\bar{X}^T \\ -U^T & X^T \end{pmatrix} \begin{pmatrix} X_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} \bar{U}^T X_0 - \bar{X}^T U_0 \\ X^T U_0 - U^T X_0 \end{pmatrix} \quad \text{on } [a, \infty). \tag{1.20}$$

Note that by (1.20) the matrix $-M$ is the Wronskian of $(\bar{X}, \bar{U})$ and $(X_0, U_0)$, while the matrix $N$ is the Wronskian of $(X, U)$ and $(X_0, U_0)$. In [25, Corollary 3.3.9] it is shown that, for the above given $(X, U)$ and $(\bar{X}, \bar{U})$, another conjoined basis $(\bar{X}_*, \bar{U}_*)$ of (H) satisfies identity (1.16) if and only if there exists a constant symmetric matrix $D$ with the property

$$\bar{X}_*(t) = \bar{X}(t) + X(t)D, \quad \bar{U}_*(t) = \bar{U}(t) + U(t)D, \qquad t \in [a, \infty). \tag{1.21}$$

Moreover, if the solution $(X_0, U_0)$ of (H) is expressed in terms of $(X, U)$ and $(\bar{X}_*, \bar{U}_*)$ via the matrices $M_*$ and $N_*$ in (1.19), then the formulas

$$M_* = M - DN \quad \text{and} \quad N_* = N, \tag{1.22}$$

hold, which one can easily verify by using (1.20) and (1.21).

## 1.4 Controllable systems

Traditionally, such as in [6, 16, 25, 31], system (H) is studied under the *complete controllability* (or *identical normality*) assumption. This means that if for a solution $(x, u)$ the function $x$ vanishes on a nondegenerate subinterval of $[a, \infty)$, then also $u$ vanishes on this subinterval, and hence $(x, u) \equiv (0, 0)$ by uniqueness of solutions. In this section we summarize some basic results about completely controllable linear Hamiltonian systems. The following key result is proven in [25, Theorem 4.1.3].

**Proposition 1.4.1.** *Assume that* (1.1) *holds. Then system* (H) *is completely controllable on* $[\alpha, \beta]$ *if and only if for every conjoined basis* $(X, U)$ *of* (H) *the matrix* $X(t)$ *is singular only at isolated points in* $[\alpha, \beta]$.

The result in Proposition 1.4.1 has motivated the definition of the *focal points* of a conjoined basis $(X, U)$ of (H) as the points $t_0 \in [a, \infty)$ for which

$$m(t_0) := \operatorname{def} X(t_0) \geq 1,$$

see [31, 33]. The number $m(t_0)$ is then called the multiplicity of the focal point $t_0$. Thus, under the Legendre condition (1.1) the focal points of a conjoined basis $(X, U)$ of a completely controllable system (H) are isolated and one can count them in any bounded subinterval of $[a, \infty)$. This then leads to the following classical result, which can be found e.g. in [31, Corollary 2 in Section VII.7] or [33, Corollary 1 in Section V.8].

**Proposition 1.4.2** (Sturmian separation theorem)**.** *Assume that* (1.1) *holds and that system* (H) *is completely controllable on* $[a,\infty)$*. Let* $[\alpha,\beta]$ *be a nondegenerate subinterval in* $[a,\infty)$*. Then the difference between the numbers of the focal points in* $(\alpha,\beta]$ *of any two conjoined bases of* (H) *is at most* $n$*.*

The result of Proposition 1.4.2 immediately implies that either every conjoined basis of (H) has finitely many focal points in $(a,\infty)$, or every conjoined basis of (H) has infinitely many focal points in $(a,\infty)$. In the former case the conjoined bases and hence the system (H) are said to be *nonoscillatory on* $[a,\infty)$, while in the latter case they are called *oscillatory on* $[a,\infty)$. An alternative (but equivalent) definition of the nonoscillation of (H) is that system (H) is nonoscillatory on $[a,\infty)$ if it is *disconjugate* for large $t$, i.e., if every vector solution $(x,u)$ of (H) satisfies for sufficiently large $t_1,t_2 \in [a,\infty)$ that

$$\text{if } x(t_1)=0=x(t_2) \text{ and } t_1<t_2, \text{ then } x(t)\equiv 0 \text{ on } [t_1,t_2]. \tag{1.23}$$

Note that condition (1.23) can be implemented even in the general abnormal case, see [30, Section III] and Section 1.5 below.

Let system (H) be completely controllable and nonoscillatory on $[a,\infty)$ and fix a conjoined basis $(X,U)$ of (H). Then there exists $\alpha\in[a,\infty)$ such that $X(t)$ is invertible on $[\alpha,\infty)$. In particular, we can associate with $(X,U)$ the matrix-valued function $S(t)$ defined by

$$S(t) := \int_\alpha^t X^{-1}(s)\,B(s)\,X^{T-1}(s)\,\mathrm{d}s, \quad t\in[\alpha,\infty). \tag{1.24}$$

In [6, Proposition 2 in Section 2.1] it is shown that under the Legendre condition (1.1) the matrix $S(t)$ in (1.24) is symmetric and strictly increasing on $[\alpha,\infty)$. And since $S(\alpha)=0$, we have that $S(t)$ is invertible for all $t\in(\alpha,\infty)$. The invertibility of $S(t)$ has been used in several results in the literature, such as in the proof of [6, Theorem 2, pg. 39]. At the same time, it is a central requirement for the traditional definition of the principal and antiprincipal solution at infinity, as we present in (1.2) and (1.4) for completely controllable systems (H).

**Definition 1.4.3** (Principal and antiprincipal solutions)**.** Let system (H) be completely controllable on $[a,\infty)$. A conjoined basis $(\hat{X},\hat{U})$ of (H) is said to be a *principal solution at infinity* if there exists $\alpha\in[a,\infty)$ such that the matrix $\hat{X}(t)$ is invertible on $[\alpha,\infty)$ and its corresponding matrix $\hat{S}(t)$ defined in (1.24) through $\hat{X}(t)$ satisfies

$$\lim_{t\to\infty}\hat{S}^{-1}(t)=0.$$

Further, a conjoined basis $(X,U)$ of (H) is called an *antiprincipal solution at infinity* if there exists $\alpha\in[a,\infty)$ such that the matrix $X(t)$ is invertible on $[\alpha,\infty)$ and the associated matrix $S(t)$ in (1.24) satisfies

$$\lim_{t\to\infty}S^{-1}(t)=T, \qquad \text{with } T \text{ invertible.}$$

The existence and the uniqueness of the principal solution is established in [6, Theorem 3, pg 43], [16, Theorem XI.10.5], [31], or Corollary 5.3.1.

**Proposition 1.4.4.** *Assume that (1.1) holds and that system (H) is completely controllable on $[a,\infty)$. Then (H) is nonoscillatory on $[a,\infty)$ if and only if there exists a principal solution $(\hat{X},\hat{U})$ of (H) at infinity. The principal solution $(\hat{X},\hat{U})$ is uniquely determined up to a right nonsingular multiple.*

Another classical result for completely controllable systems (H) is the limit characterization of the principal solution at infinity, which involves the concept of an antiprincipal solution at infinity in (1.4), see [6, Proposition 4, pg 43], [16, Theorem XI.10.5], [31, Theorem VII.3.2], or Corollary 6.4.6.

**Proposition 1.4.5.** *Assume that (1.1) holds and system (H) is nonoscillatory and completely controllable on $[a,\infty)$. Let $(\hat{X},\hat{U})$ and $(X,U)$ be two conjoined bases of (H) and let $\hat{N} := \hat{X}^T(t)\,U(t) - \hat{U}^T(t)\,X(t)$ be their (constant) Wronskian. Then $(\hat{X},\hat{U})$ is the principal solution of (H) at infinity and the matrix $\hat{N}$ is invertible if and only if*

$$\lim_{t\to\infty} X^{-1}(t)\,\hat{X}(t) = 0. \tag{1.25}$$

*In this case $(X,U)$ is an antiprincipal solution of (H) at infinity.*

## 1.5 Abnormal systems

A systematic study of general linear Hamiltonian systems without the complete controllability assumption was initiated by W. Kratz in [24], where the following fundamental result can be found.

**Proposition 1.5.1.** *Assume (1.1). Then for every conjoined basis $(X,U)$ of (H) the function $X(t)$ has the kernel piecewise constant on any compact subinterval of $[a,\infty)$.*

The result in Proposition 1.5.1 is a key tool for the definition of proper focal points of conjoined bases of (H). According to [43, Definition 1.1], a point $t_0 \in (a,\infty)$ is a *(left) proper focal point* of a conjoined basis $(X,U)$ if

$$\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0), \quad \text{i.e.,} \quad m(t_0) := \dim \operatorname{Ker} X(t_0) - \dim \operatorname{Ker} X(t_0^-) \geq 1,$$

and then $m(t_0)$ is its multiplicity. The notation $\operatorname{Ker} X(t_0^-)$ represents the left-hand limit of the constant kernel of $X(t)$ at the point $t_0$. From Proposition 1.5.1 it then follows that the proper focal points of any conjoined basis of (H) are isolated in $(a,\infty)$. In particular, every conjoined basis of (H) has finitely many proper focal points in each bounded subinterval of $[a,\infty)$, although in the whole interval $(a,\infty)$ it may have infinitely many proper focal points. The corresponding Sturmian separation theorem for abnormal linear Hamiltonian systems, being a generalization of Proposition 1.4.2, was proven in [39].

**Proposition 1.5.2** (Sturmian separation theorem)**.** *Assume (1.1) and let $[\alpha,\beta]$ be a nondegenerate subinterval in $[a,\infty)$. Suppose that the principal solution of (H) at the point $\alpha$ has $m$ proper focal points in $(\alpha,\beta]$. Then any conjoined basis of (H) has at least $m$ and at most $m+n$ proper focal points in $(\alpha,\beta]$.*

In view of Proposition 1.5.2 and similarly as in the controllable case, we then classify system (H) as *nonoscillatory* if every its conjoined basis $(X,U)$ is nonoscillatory, i.e., $(X,U)$ has finitely many proper focal points in $(a,\infty)$ or equivalently, the matrix $X(t)$ has eventually constant kernel on $[a,\infty)$. In the opposite case system (H) and its conjoined bases are called *oscillatory*. This classification of possibly abnormal linear Hamiltonian systems is justified by the following result, see [40, Theorem 2.2].

**Proposition 1.5.3.** *Assume* (1.1)*. Then the following statements are equivalent.*

(i) *There exists a conjoined basis of* (H) *which is (non)oscillatory.*

(ii) *Every conjoined basis of* (H) *is (non)oscillatory.*

As we mentioned in Section 1.3, the first concept of the principal solution at infinity for possibly abnormal linear Hamiltonian systems (H) was developed by W. T. Reid in his paper [30]. Under the Legendre condition (1.1) and the eventual disconjugacy of (H) defined in (1.23), he showed the existence of a conjoined basis $(X,U)$ of (H) with the invertible matrix $X(t)$ for large $t$. This result then allowed him to implement the definition of a principal solution at infinity in (1.5). The existence of such a principal solution was established in [30, Theorem 5.3].

**Proposition 1.5.4.** *Assume that* (1.1) *holds and that system* (H) *is eventually disconjugate. Then there exists a principal solution* $(\hat{X},\hat{U})$ *of* (H) *at infinity in* (1.5)*.*

In general, Reid's principal solution $(\hat{X},\hat{U})$ defined in (1.5) is not uniquely determined (see Remark 5.1.8 in Chapter 5) and to author's knowledge, there is no classification of all such principal solutions known in the literature. In our opinion, this is the primary reason why the above concept of Reid's principal solution at infinity for general linear Hamiltonian systems was not further developed in the previous years.

# Chapter 2

# Representability of conjoined bases

In the following chapter we develop elements of the theory of representation of solutions for possibly abnormal linear Hamiltonian systems. The new results generalize their analogies for controllable systems in [6, Proposition 1 in Section 2.1] and [31, Theorem 2.2 in Section VII.2]. The first section contains some basic properties of conjoined bases $(X,U)$ of (H) with constant kernel. In Section 2.2 we discuss normalized conjoined bases of (H). Finally, in the last section we introduce a concept of the representability of solutions of (H) as a key tool for a mutual comparison of two conjoined bases of (H) with constant kernel.

## 2.1 Conjoined bases with constant kernel

In this section we provide some basic properties of conjoined bases $(X,U)$ of (H), for which the kernel of the function $X$ is constant on a subinterval $[\alpha,\infty) \subseteq [a,\infty)$. In particular, we introduce the $S$-matrix for $(X,U)$, which generalizes its corresponding analogy for the controllable case in (1.24).

**Definition 2.1.1.** Let $(X,U)$ be a conjoined basis of (H). By the *kernel of* $(X,U)$ we mean the kernel of $X$. We say that $(X,U)$ has a constant kernel on an subinterval $\mathscr{I} \subseteq [a,\infty)$ if $\operatorname{Ker} X$ is constant on $\mathscr{I}$.

Following Remarks 1.2.1 and 1.2.3(i), for a conjoined basis $(X,U)$ of (H) we define the orthogonal projectors onto the subspaces $\operatorname{Im} X^T$ and $\operatorname{Im} X$ by

$$P(t) := P_{\operatorname{Im} X^T(t)} = X^\dagger(t)X(t), \quad R(t) := P_{\operatorname{Im} X(t)} = X(t)X^\dagger(t), \quad t \in [a,\infty). \tag{2.1}$$

Since $I-P(t)$ is the orthogonal projector onto the subspace $[\operatorname{Im} X^T(t)]^\perp = \operatorname{Ker} X(t)$, it follows from [24, Theorem 3] that under (1.1) the matrix function $P(t)$ is piecewise constant on any compact subinterval of $[a,\infty)$. And if $(X,U)$ has a constant kernel on $[\alpha,\infty) \subseteq [a,\infty)$, then $P(t)$ is a constant matrix on $[\alpha,\infty)$, i.e.,

$$P := P(t) \quad \text{is constant on } [\alpha,\infty). \tag{2.2}$$

In this case we have by Remark 1.2.3(ii) that

$$r := \operatorname{rank} X(t) = \operatorname{rank} P = \operatorname{rank} R(t) \quad \text{on } [\alpha,\infty), \tag{2.3}$$

and we say that $(X,U)$ has rank $r$ on $[\alpha,\infty)$. The following theorem states basic properties of conjoined bases with constant kernel on $[\alpha,\infty)$.

**Theorem 2.1.2.** *Let $(X,U)$ be a conjoined basis of* (H) *with constant kernel on $[\alpha,\infty)$ and let $P$ and $R(t)$ be the orthogonal projectors defined in* (2.2) *and* (2.1) *through the function $X(t)$. Then the following statements hold.*

(i) $\operatorname{Im}[U(t)(I-P)] = \operatorname{Ker} R(t) = \operatorname{Ker} X^T(t)$ *for each* $t \in [\alpha,\infty)$.

(ii) $B(t) = R(t)B(t) = B(t)R(t)$ *for each* $t \in [\alpha,\infty)$.

(iii) *The function $X^\dagger$ belongs to $\mathrm{C}^1_\mathrm{p}$ on $[\alpha,\infty)$ with*

$$(X^\dagger)' = X^\dagger A^T(I-R) - X^\dagger AR - X^\dagger BUX^\dagger \quad \text{on } [\alpha,\infty). \tag{2.4}$$

*Proof.* We suppress the argument $t$ whenever it is possible. For part (i) we note that the inclusion $\operatorname{Im}[U(I-P)] \subseteq \operatorname{Ker} R$ is equivalent with $\operatorname{Im} U^T R \subseteq \operatorname{Im} P = \operatorname{Im} X^T$, which obviously holds because of $U^T R = U^T X X^\dagger = X^T U X^\dagger$. On the other hand, we have $X(I-P) = 0$, and if $U(I-P)v = 0$ for some $v \in \mathbb{R}^n$, then $(I-P)v = 0$, because $(X,U)$ is a conjoined basis. This shows that $v \in \operatorname{Im} P$ and hence, $\operatorname{Ker}[U(I-P)] \subseteq \operatorname{Im} P$. The latter inclusion implies that $\operatorname{rank}[U(I-P)] = \operatorname{rank}[U(I-P)]^T \geq \dim\operatorname{Ker} P = \dim\operatorname{Ker} R$, which yields the assertion. Part (ii) is [24, Lemma 2 (iii)]. Finally, part (iii) follows from Remark 1.2.4(i), since in this case the kernel of $X(t)$ is constant on $[\alpha,\infty)$. Moreover, using formula (1.15) with $M := X$ together with the identity $B(I-R) = 0$ on $[\alpha,\infty)$ in part (ii) we obtain

$$\begin{aligned}(X^\dagger)' &= -X^\dagger(AX+BU)X^\dagger + X^\dagger X^{\dagger T}(X^T A^T + U^T B)(I-R) \\ &= X^\dagger A^T(I-R) - X^\dagger AR - X^\dagger BUX^\dagger\end{aligned}$$

on $[\alpha,\infty)$. Thus, the equality in (2.4) holds and the proof is complete. ∎

**Remark 2.1.3.** The result in Theorem 2.1.2(iii) implies that for a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ the *Riccati quotient*

$$Q := UX^\dagger + (UX^\dagger)^T(I-XX^\dagger) \tag{2.5}$$

is symmetric and piecewise continuously differentiable on $[\alpha,\infty)$, see also [34, pg. 23] or [41, formula (3.4)]. It then follows from the properties of the matrices $P$, $R(t)$ and $X^\dagger$ in (2.2), (2.1) and Remark 1.2.3(i) that on $[\alpha,\infty)$

$$QX = UP, \qquad RQ = X^{\dagger T}U^T. \tag{2.6}$$

By [41, Lemma 3.2], the above matrix $Q$ is a solution of a certain implicit Riccati equation. However, this observation is not needed in this work. Note that by the second equality in (2.6), the formula (2.4) can be effectively rewritten in the form

$$(X^\dagger)' = X^\dagger A^T(I-R) - X^\dagger(A+BQ)R \quad \text{on } [\alpha,\infty). \tag{2.7}$$

Given a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$, the result in Theorem 2.1.2(iii) allows to define for a fixed $\beta\in[\alpha,\infty)$ the matrix-valued function

$$S_\beta(t) := \int_\beta^t X^\dagger(s)\,B(s)\,X^{\dagger T}(s)\,\mathrm{d}s, \quad t\in[\alpha,\infty). \tag{2.8}$$

Indeed, the function $X^\dagger B X^{\dagger T}$ is piecewise continuous on $[\alpha,\infty)$, hence it is integrable on $[\beta,t]$ or $[t,\beta]$ for each $t\in[\alpha,\infty)$. The function $S_\beta(t)$ will be referred to as the *S-matrix* which corresponds to the conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$. From its definition it immediately follows that $S_\beta(t)$ is symmetric and $S_\beta\in\mathrm{C}^1_\mathrm{p}$ on $[\alpha,\infty)$. Moreover, under (1.1) the matrix $S_\beta(t)$ is nonnegative definite on $[\beta,\infty)$ and nonpositive definite on $[\alpha,\beta]$. In the next theorem we establish further basic properties of the $S$-matrices in (2.8) which correspond to a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$.

**Theorem 2.1.4.** *Assume* (1.1). *Let $(X,U)$ be a conjoined basis of* (H) *with constant kernel on $[\alpha,\infty)$ and let $P$ be its corresponding orthogonal projector in* (2.2). *For a given $\beta\in[\alpha,\infty)$ let $S_\beta(t)$ be defined in* (2.8). *Then the following statements hold.*

(i) *The matrix $S_\beta(t)$ is nondecreasing on $[\alpha,\infty)$.*

(ii) *The set $\operatorname{Im} S_\beta(t)$ is nonincreasing on $[\alpha,\beta]$, while it is nondecreasing on $[\beta,\infty)$. In particular, $\operatorname{Im} S_\beta(t)$ is eventually constant with*

$$\operatorname{Im} S_\beta(t)\subseteq \operatorname{Im} P \quad \textit{for each } t\in[\alpha,\infty). \tag{2.9}$$

(iii) *If $S_\beta(t)$ has constant kernel on some subinterval $\mathscr{I}\subseteq[\alpha,\infty)$, then $S^\dagger_\beta(t)$ is piecewise continuously differentiable and nonincreasing on $\mathscr{I}$. Consequently, the limit of $S^\dagger_\beta(t)$ as $t\to\infty$ exists.*

*Proof.* Let $(X,U)$ and $\beta$ be as in theorem. From the definition of $S_\beta(t)$ in (2.8) we have for any $t_1,t_2\in[\alpha,\infty)$

$$S_\beta(t_2)-S_\beta(t_1)=S_{t_1}(t_2). \tag{2.10}$$

The definiteness property of the matrix $S_{t_1}(t_2)$ and formula (2.10) then imply that $S_\beta(t_2)\geq S_\beta(t_1)$ if and only if $t_2\geq t_1$, which proves part (i). In particular, if $t_2\geq t_1\geq\beta$ then we have $S_\beta(t_2)\geq S_\beta(t_1)\geq S_\beta(\beta)=0$ and consequently, $\operatorname{Im} S_\beta(t_1)\subseteq\operatorname{Im} S_\beta(t_2)$. Therefore, the set $\operatorname{Im} S_\beta(t)$ is nondecreasing on $[\beta,\infty)$ and hence, eventually constant. For $\beta\geq t_2\geq t_1\geq\alpha$ we obtain by (2.8) that

$$S_{t_1}(\beta)=-S_\beta(t_1)\geq -S_\beta(t_2)=S_{t_2}(\beta)\geq S_{t_2}(t_2)=0. \tag{2.11}$$

The inequalities in (2.11) then yield $\operatorname{Im} S_\beta(t_2)=\operatorname{Im} S_{t_2}(\beta)\subseteq\operatorname{Im} S_{t_1}(\beta)=\operatorname{Im} S_\beta(t_1)$. Hence, the set $\operatorname{Im} S_\beta(t)$ is nonincreasing on $[\alpha,\beta]$. In addition, using the identity $PX^\dagger(t)=X^\dagger(t)$ on $[\alpha,\infty)$ the calculation

$$PS_\beta(t)=\int_\beta^t PX^\dagger(s)\,B(s)\,X^{\dagger T}(s)\,\mathrm{d}s=\int_\beta^t X^\dagger(s)\,B(s)\,X^{\dagger T}(s)\,\mathrm{d}s=S_\beta(t)$$

implies that $\operatorname{Im} S_\beta(t)\subseteq\operatorname{Im} P$ on $[\alpha,\infty)$ and the proof of part (ii) is complete.

(iii) By the definition of $S_\beta(t)$ in (2.8) we have $S'_\beta(t) = X^\dagger(t)B(t)X^{\dagger T}(t)$ on $[\alpha,\infty)$, which implies through the Legendre condition (1.1) that $S'_\beta(t) \geq 0$ on $[\alpha,\infty)$. Let $\mathscr{I} \subseteq [\alpha,\infty)$ be a subinterval such that $S_\beta(t)$ has constant kernel on $\mathscr{I}$. Then $S^\dagger_\beta \in \mathrm{C}^1_\mathrm{p}(\mathscr{I})$, $(S^\dagger_\beta)' = -S^\dagger_\beta S'_\beta S^\dagger_\beta$, and $S^\dagger_\beta(t)$ is nonincreasing on $\mathscr{I}$, all by Remark 1.2.4(i). In addition, from part (ii) we know that $\operatorname{Im} S_\beta(t)$ and hence $\operatorname{Ker} S_\beta(t)$ are constant for large $t$. Therefore, $S^\dagger_\beta(t)$ is nonincreasing for large $t$. Finally, since $S^\dagger_\beta(t)$ is nonnegative definite on $[\beta,\infty)$, it follows that the limit of $S^\dagger_\beta(t)$ exists as $t \to \infty$. ■

**Remark 2.1.5.** (i) For a given $\beta \in [\alpha,\infty)$ let $\mathscr{S}_\beta(t) := \operatorname{Im} S_\beta(t)$ and denote by $P_{\mathscr{S}_\beta}(t)$ the orthogonal projector onto the set $\mathscr{S}_\beta(t)$ for $t \in [\alpha,\infty)$. Then we have $P_{\mathscr{S}_\beta}(t) = S_\beta(t)S^\dagger_\beta(t) = S^\dagger_\beta(t)S_\beta(t)$, by the symmetry of $S_\beta(t)$. Moreover, the set $\operatorname{Im} P_{\mathscr{S}_\beta}(t)$ is nondecreasing on $[\beta,\infty)$ and hence eventually constant, by Theorem 2.1.4(ii). The constant orthogonal projector corresponding to this "maximal" set $\operatorname{Im} P_{\mathscr{S}_\beta}(t)$ will be denoted by

$$P_{\mathscr{S}_{\beta\infty}} := P_{\mathscr{S}_\beta}(t) \quad \text{for } t \to \infty. \tag{2.12}$$

From (2.9) we then have the inclusions

$$\operatorname{Im} S_\beta(t) = \operatorname{Im} P_{\mathscr{S}_\beta}(t) \subseteq \operatorname{Im} P_{\mathscr{S}_{\beta\infty}} \subseteq \operatorname{Im} P, \quad t \in [\beta,\infty). \tag{2.13}$$

(ii) The maximal set $\operatorname{Im} P_{S_{\beta\infty}}$ is nonincreasing in $\beta \in [\alpha,\infty)$, that is

$$\operatorname{Im} P_{\mathscr{S}_{\gamma\infty}} \subseteq \operatorname{Im} P_{\mathscr{S}_{\beta\infty}} \subseteq \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} \quad \text{when} \quad \alpha \leq \beta \leq \gamma. \tag{2.14}$$

Indeed, for any $\beta,\gamma \in [\alpha,\infty)$ the identity $S_\beta(t) - S_\gamma(t) = S_\beta(\gamma)$ on $[\alpha,\infty)$ implies that $S_\beta(t) \geq S_\gamma(t) \geq 0$ on $[\gamma,\infty)$ if and only if $\beta \leq \gamma$.

**Remark 2.1.6.** For a given $\beta \in [\alpha,\infty)$ denote by $T_\beta \in \mathbb{R}^{n\times n}$ the limit of $S^\dagger_\beta(t)$ for $t \to \infty$. The matrix $T_\beta$ will be referred to as the *$T$-matrix* corresponding to $S_\beta(t)$. Obviously, $T_\beta$ is symmetric, nonnegative definite, and $\operatorname{Im} T_\beta \subseteq \operatorname{Im} P_{\mathscr{S}_{\beta\infty}}$, by (2.12).

**Remark 2.1.7.** When $\beta = \alpha$ in (2.8), we will sometimes use for the matrices $S_\alpha(t)$, $P_{\mathscr{S}_\alpha}(t)$, $P_{\mathscr{S}_{\alpha\infty}}$, and $T_\alpha$ defined above the notations

$$S(t) := S_\alpha(t), \quad P_\mathscr{S}(t) := P_{\mathscr{S}_\alpha}(t), \quad P_{\mathscr{S}\infty} := P_{\mathscr{S}_{\alpha\infty}}, \quad T := T_\alpha. \tag{2.15}$$

These simplifications help us to avoid double indices in our notation.

## 2.2 Normalized conjoined bases

In this section we study in details a certain class of conjoined bases $(\bar{X}_\beta, \bar{U}_\beta)$ of (H), which are normalized with a given conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$. As the main result, we show that these conjoined bases are closely related to the function $S_\beta(t)$ defined in (2.8), see Theorem 2.2.5 and Remark 2.2.6. In the following lemma we derive some auxiliary results about conjoined bases $(X,U)$ of (H) with constant kernel.

**Lemma 2.2.1.** *Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $R(t)$ and $Q(t)$ be its corresponding matrices in (2.1) and (2.5). Then any solution $(X_0,U_0)$ of (H) satisfies on $[\alpha,\infty)$*

$$X_0' = (A+BQ)X_0 + BX^{\dagger T}W, \tag{2.16}$$
$$(X^\dagger X_0)' = X^\dagger BX^{\dagger T}W + X^\dagger(A^T+A+BQ)(I-R)X_0, \tag{2.17}$$

*where $W := X^TU_0 - U^TX_0$ is the (constant) Wronskian of $(X,U)$ and $(X_0,U_0)$.*

*Proof.* Let $(X,U)$ and $(X_0,U_0)$ be as in theorem. Using the identity $X^{\dagger T}X^T = R$ with the second formula in (2.6) we obtain

$$X^{\dagger T}(t)W = R(t)U_0(t) - R(t)Q(t)X_0(t) \tag{2.18}$$

on $[\alpha,\infty)$. With the aid of Theorem 2.1.2(ii) and (2.18) we now get

$$\begin{aligned} X_0' &= AX_0 + BU_0 = AX_0 + BRU_0 = AX_0 + B(RQX_0 + X^{\dagger T}W) \\ &= AX_0 + BQX_0 + BX^{\dagger T}W = (A+BQ)X_0 + BX^{\dagger T}W \end{aligned}$$

on $[\alpha,\infty)$. Thus, equality (2.16) holds. Moreover, by using formulas (2.7) and (2.16) we obtain the equality in (2.17), since on $[\alpha,\infty)$ we have

$$\begin{aligned} (X^\dagger X_0)' &= \Big[X^\dagger A^T(I-R) - X^\dagger(A+BQ)R\Big]X_0 + X^\dagger\Big[(A+BQ)X_0 + BX^{\dagger T}W\Big] \\ &= X^\dagger A^T(I-R)X_0 - X^\dagger(A+BQ)RX_0 + X^\dagger(A+BQ)X_0 + X^\dagger BX^{\dagger T}W \\ &= X^\dagger BX^{\dagger T}W + X^\dagger(A^T+A+BQ)(I-R)X_0. \end{aligned}$$

This completes the proof. ∎

**Remark 2.2.2.** The choice $(X_0,U_0) := (X,U)$ in Lemma 2.2.1 yields the formula

$$X' = (A+BQ)X \quad \text{on } [\alpha,\infty), \tag{2.19}$$

because in this case the Wronskian $W=0$. Let $\beta\in[\alpha,\infty)$ be fixed. By uniqueness of solutions of equation (2.19) we then obtain that

$$X(t) = \Phi(t,\beta)X(\beta) \quad \text{for all } t\in[\alpha,\infty), \tag{2.20}$$

where $\Phi(t,\beta)$ is the fundamental matrix of (2.19) satisfying $\Phi(\beta,\beta)=I$. Moreover, if $(X_0,U_0)$ is any solution of (H), then the variation constant formula applied to equation (2.16) yields that for $t\in[\alpha,\infty)$

$$X_0(t) = \Phi(t,\beta)X_0(\beta) + \Phi(t,\beta)\int_\beta^t \Phi^{-1}(s,\beta)B(s)X^{\dagger T}(s)\,\mathrm{d}s\,W. \tag{2.21}$$

With the aid of the equality in (2.20) we can write the function under the integral in (2.21) as

$$\begin{aligned} \Phi^{-1}(s,\beta)B(s)X^{\dagger T}(s) &= \Phi^{-1}(s,\beta)X(s)X^\dagger(s)B(s)X^{\dagger T}(s) \\ &= X(\beta)X^\dagger(s)B(s)X^{\dagger T}(s) = X(\beta)S_\beta'(s), \end{aligned}$$

where $S_\beta(t)$ is the $S$-matrix in (2.8) which corresponds to $(X,U)$. Inserting this into (2.21) and using $S_\beta(\beta)=0$ we obtain on $[\alpha,\infty)$ the formula

$$X_0(t) = \Phi(t,\beta)\,\big[X_0(\beta) + X(\beta)S_\beta(t)W\big]. \tag{2.22}$$

**Remark 2.2.3.** Formula (2.17) in Lemma 2.2.1 is a generalization of [25, Corollary 1.1.4] when the function $R(t) \not\equiv I$ on $[\alpha,\infty)$.

In the next result we show how to express the matrix $S_\beta(t)$ corresponding to the conjoined basis $(X,U)$ with constant kernel through the projector $P$ and a conjoined basis of (H), which is normalized with $(X,U)$.

**Theorem 2.2.4.** *Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $P$ and $S_\beta(t)$ be its corresponding matrices in (2.2) and (2.8) for a given $\beta \in [\alpha,\infty)$. Then for every conjoined basis $(\bar{X},\bar{U})$ satisfying (1.16) we have that*

$$S_\beta(t) = X^\dagger(t)\bar{X}(t)P - X^\dagger(\beta)\bar{X}(\beta)P \quad \textit{for all } t \in [\alpha,\infty). \tag{2.23}$$

*Proof.* Let $(\bar{X},\bar{U})$ be a conjoined basis of (H) satisfying (1.16), i.e. the equality $X^T\bar{U} - U^T\bar{X} = I$ holds on $[\alpha,\infty)$. In order to prove the equality (2.23), we need to calculate the derivative of $X^\dagger\bar{X}P$. Using formula (2.17) with $(X_0,U_0) := (\bar{X},\bar{U})$ and $W := I$ and the identity $X^{\dagger T}P = X^{\dagger T}$ we obtain

$$(X^\dagger\bar{X}P)' = X^\dagger BX^{\dagger T} + X^\dagger(A^T + A + BQ)(I-R)\bar{X}P \quad \text{on } [\alpha,\infty), \tag{2.24}$$

where the orthogonal projector $R$ is defined in (2.1). Observe that $\operatorname{Im}\bar{X}P \subseteq \operatorname{Im}X$ on $[\alpha,\infty)$, because by (1.17) we have $\bar{X}P = \bar{X}X^\dagger X = \bar{X}X^TX^{\dagger T} = X\bar{X}^TX^{\dagger T}$. Thus $(I-R)\bar{X}P = 0$ on $[\alpha,\infty)$ and formula (2.24) becomes $(X^\dagger\bar{X}P)' = X^\dagger BX^{\dagger T} = S'_\beta(t)$, by (2.8). Finally, by the integrating from $\beta$ to $t \in [\alpha,\infty)$ and using $S_\beta(\beta) = 0$ we get identity (2.23). The proof is complete. ■

In the next theorem we introduce a special class of conjoined bases of (H) satisfying (1.16) for a given conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$.

**Theorem 2.2.5.** *Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $\beta \geq \alpha$ be given. Then there exists a conjoined basis $(\bar{X}_\beta,\bar{U}_\beta)$ of (H) satisfying (1.16) and such that*

$$X^\dagger(\beta)\bar{X}_\beta(\beta) = 0. \tag{2.25}$$

*The matrix function $\bar{X}_\beta(t)$ is uniquely determined by $(X,U)$ on $[\alpha,\infty)$.*

*Proof.* Let $(\bar{X},\bar{U})$ be a conjoined basis of (H) satisfying (1.16), i.e., $(X,U)$ and $(\bar{X},\bar{U})$ are normalized. Consider the constant matrix

$$D_\beta := X^\dagger(\beta)\bar{X}(\beta)P - X^\dagger(\beta)\bar{X}(\beta) - \bar{X}^T(\beta)X^{\dagger T}(\beta), \tag{2.26}$$

where the matrix $P$ is defined in (2.2). The matrix $D_\beta$ is symmetric, because by (1.17) the matrix $X^\dagger\bar{X}P = X^\dagger\bar{X}X^\dagger X = X^\dagger\bar{X}X^TX^{\dagger T}$ is symmetric on $[\alpha,\infty)$. Furthermore, we have

$$PD_\beta = -X^\dagger(\beta)\bar{X}(\beta), \quad (I-P)D_\beta = -(I-P)\bar{X}^T(\beta)X^{\dagger T}(\beta). \tag{2.27}$$

According to (1.21), the solution $(\bar{X}_\beta,\bar{U}_\beta) := (\bar{X}+XD_\beta, \bar{U}+UD_\beta)$ is a conjoined basis of (H) satisfying (1.16) and also (2.25), since $X^\dagger(\beta)\bar{X}_\beta(\beta) = X^\dagger(\beta)\bar{X}(\beta) + PD_\beta = 0$, by the first equality in (2.27). The uniqueness of the function $\bar{X}_\beta(t)$ on $[\alpha,\infty)$ follows from (1.21), for if two conjoined bases $(\bar{X}_{\beta1},\bar{U}_{\beta1})$ and $(\bar{X}_{\beta2},\bar{U}_{\beta2})$ simultaneously satisfy (1.16) and (2.25), then there exists a constant symmetric matrix $D_{12}$ such that $\bar{X}_{\beta2} = \bar{X}_{\beta1} + X\bar{D}_{12}$ on $[a,\infty)$. The equality $X^\dagger(\beta)\bar{X}_{\beta1}(\beta) = X^\dagger(\beta)\bar{X}_{\beta2}(\beta)$ then implies $PD_{12} = 0$, which means that $\operatorname{Im}D_{12} \subseteq \operatorname{Ker}X$ on $[\alpha,\infty)$. Thus, $XD_{12} = 0$ and so $\bar{X}_{\beta2} = \bar{X}_{\beta1}$ on $[\alpha,\infty)$. ■

**Remark 2.2.6.** The class of conjoined bases $(\bar{X}_\beta, \bar{U}_\beta)$ of (H) introduced in Theorem 2.2.5 allows to express the matrix $S_\beta(t)$ in a particularly simple form. More precisely, when $(\bar{X}_\beta, \bar{U}_\beta)$ is any conjoined basis of (H) satisfying (1.16) and (2.25), then

$$S_\beta(t) = X^\dagger(t)\,\bar{X}_\beta(t)\,P \quad \text{for all } t \in [\alpha, \infty). \tag{2.28}$$

Formula (2.28) immediately follows from condition (2.25) and the identity in (2.23) with $(\bar{X}, \bar{U}) := (\bar{X}_\beta, \bar{U}_\beta)$.

The following theorem describes a construction of all the conjoined bases $(\bar{X}_\beta, \bar{U}_\beta)$ of (H) defined in Theorem 2.2.5.

**Theorem 2.2.7.** *Assume (1.1). Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $P$ and $S_\beta(t)$ be the matrices defined in (2.2) and (2.8) for a given $\beta \geq \alpha$. A solution $(\bar{X}_\beta, \bar{U}_\beta)$ of (H) satisfies (1.16) and (2.25) if and only if*

$$\bar{X}_\beta(t) = X(t)\,S_\beta(t) + \Big[\bar{X}(t) - X(t)\,X^\dagger(\beta)\,\bar{X}(\beta)\Big]\,(I-P), \tag{2.29}$$

$$\begin{aligned} \bar{U}_\beta(t) = {} & U(t)\,S_\beta(t) + X^{\dagger T}(t) + U(t)\,(I-P)\,\Big[X^\dagger(t)\,\bar{X}(t) - X^\dagger(\beta)\,\bar{X}(\beta)\Big]^T \\ & + \Big[\bar{U}(t) - U(t)\,X^\dagger(\beta)\,\bar{X}(\beta)\Big]\,(I-P), \quad t \in [\alpha,\infty), \end{aligned} \tag{2.30}$$

*for some conjoined basis $(\bar{X}, \bar{U})$ of (H) satisfying (1.16).*

*Proof.* Let $(X,U)$ and $\beta$ be as in theorem. We show that for any solution $(\bar{X}_\beta, \bar{U}_\beta)$ of (H) satisfying (1.16) and (2.25) the functions $\bar{X}_\beta(t)$ and $\bar{U}_\beta(t)$ have the forms displayed in (2.29) and (2.30) with $(\bar{X}, \bar{U}) := (\bar{X}_\beta, \bar{U}_\beta)$. By the condition $X^\dagger(\beta)\,\bar{X}_\beta(\beta) = 0$ the right side of (2.29) reads $X(t)\,S_\beta(t) + \bar{X}_\beta(t)\,(I-P)$. Consequently, using the identities $S_\beta(t) = X^\dagger(t)\,\bar{X}_\beta(t)\,P$ and $X(t)\,X^\dagger(t)\,\bar{X}_\beta(t)\,P = \bar{X}_\beta(t)\,P$ on $[\alpha,\infty)$ we get

$$\begin{aligned} X(t)\,S_\beta(t) + \bar{X}_\beta(t)\,(I-P) &= X(t)\,X^\dagger(t)\,\bar{X}_\beta(t)\,P + \bar{X}_\beta(t)\,(I-P) \\ &= \bar{X}_\beta(t)\,P + \bar{X}_\beta(t)\,(I-P) = \bar{X}_\beta(t) \end{aligned}$$

for all $t \in [\alpha,\infty)$. Thus, $\bar{X}_\beta(t)$ satisfies (2.29). Similarly, inserting $(\bar{X}, \bar{U}) = (\bar{X}_\beta, \bar{U}_\beta)$ into the right side of (2.30) we obtain for $t \in [\alpha,\infty)$ the expression

$$U(t)\,S_\beta(t) + X^{\dagger T}(t) + U(t)\,(I-P)\bar{X}_\beta^T(t)\,X^{\dagger T}(t) + \bar{U}_\beta(t)\,(I-P). \tag{2.31}$$

Since the matrix $S_\beta(t) = X^\dagger(t)\,\bar{X}_\beta(t)\,P$ is symmetric on $[\alpha,\infty)$, we obtain that $U(t)\,S_\beta(t) = U(t)\,P\bar{X}_\beta^T(t)\,X^{\dagger T}(t)$ on $[\alpha,\infty)$. Expression (2.31) then becomes

$$X^{\dagger T}(t) + U(t)\,\bar{X}_\beta^T(t)\,X^{\dagger T}(t) + \bar{U}_\beta(t)\,(I-P). \tag{2.32}$$

By applying the identities $U(t)\,\bar{X}_\beta^T(t) = \bar{U}_\beta(t)\,X^T(t) - I$ and $X^T(t)\,X^{\dagger T}(t) = P$ to (2.32) we get the expression

$$\bar{U}_\beta(t)\,X^T(t)\,X^{\dagger T}(t) + \bar{U}_\beta(t)\,(I-P) = \bar{U}_\beta(t)\,P + \bar{U}_\beta(t)\,(I-P) = \bar{U}_\beta(t).$$

Hence, the function $\bar{U}_\beta(t)$ satisfies (2.30). For the proof of the opposite implication consider a conjoined basis $(\bar{X},\bar{U})$ of (H) which satisfies condition (1.16). Then the solution $(\bar{X}_\beta,\bar{U}_\beta) := (\bar{X},\bar{U}) + (X,U)D_\beta$ with $D_\beta$ given by (2.26) is a conjoined basis of (H) satisfying (1.16) and (2.25), as we showed in the proof of Theorem 2.2.5. Moreover, using the symmetry of $X^\dagger(t)\bar{X}(t)P$ and the identity $X^T(t)X^{\dagger T}(t) = P$ on $[\alpha,\infty)$ together with formula (1.17) we have for $t \in [\alpha,\infty)$

$$\begin{aligned}
\bar{U}(t)P &= \bar{U}(t)X^T(t)X^{\dagger T}(t) \overset{(1.17)}{=} [U(t)\bar{X}^T(t)+I]X^{\dagger T}(t) \\
&= U(t)\bar{X}^T(t)X^{\dagger T}(t) + X^{\dagger T}(t) \\
&= U(t)P\bar{X}^T(t)X^{\dagger T}(t) + X^{\dagger T}(t) + U(t)(I-P)\bar{X}^T(t)X^{\dagger T}(t) \\
&= U(t)X^\dagger(t)\bar{X}(t)P + X^{\dagger T}(t) + U(t)(I-P)\bar{X}^T(t)X^{\dagger T}(t).
\end{aligned} \tag{2.33}$$

We show that the above functions $\bar{X}_\beta(t)$ and $\bar{U}_\beta(t)$ can be expressed as in (2.29) and (2.30). Fix $t \in [\alpha,\infty)$. By the equality $X(t) = X(t)P$ and (2.27) we have

$$\begin{aligned}
\bar{X}_\beta(t) &= \bar{X}(t) + X(t)D_\beta = \bar{X}(t) + X(t)PD_\beta \overset{(2.27)}{=} \bar{X}(t) - X(t)X^\dagger(\beta)\bar{X}(\beta) \\
&= \bar{X}(t)P + \bar{X}(t)(I-P) - X(t)X^\dagger(\beta)\bar{X}(\beta)P - X(t)X^\dagger(\beta)\bar{X}(\beta)(I-P) \\
&= \bar{X}(t)P - X(t)X^\dagger(\beta)\bar{X}(\beta)P + \left[\bar{X}(t) - X(t)X^\dagger(\beta)\bar{X}(\beta)\right](I-P).
\end{aligned} \tag{2.34}$$

Since $\bar{X}(t)P = X(t)X^\dagger(t)\bar{X}(t)P$, the first two terms in (2.34) together with (2.23) give $X(t)\,[X^\dagger(t)\bar{X}(t)P - X^\dagger(\beta)\bar{X}(\beta)P] = X(t)S_\beta(t)$. Thus, formula (2.34) yields equality (2.29). Similarly, the matrix $\bar{U}_\beta(t)$ satisfies

$$\begin{aligned}
\bar{U}_\beta(t) &= \bar{U}(t) + U(t)D_\beta = \bar{U}(t)P + \bar{U}(t)(I-P) + U(t)PD_\beta + U(t)(I-P)D_\beta \\
&\overset{(2.27)}{=} \bar{U}(t)P + \bar{U}(t)(I-P) - U(t)X^\dagger(\beta)\bar{X}(\beta) - U(t)(I-P)\bar{X}^T(\beta)X^{\dagger T}(\beta) \\
&= \bar{U}(t)P + \bar{U}(t)(I-P) - U(t)X^\dagger(\beta)\bar{X}(\beta)P - U(t)X^\dagger(\beta)\bar{X}(\beta)(I-P) \\
&\quad - U(t)(I-P)\bar{X}^T(\beta)X^{\dagger T}(\beta) \\
&= \bar{U}(t)P - U(t)X^\dagger(\beta)\bar{X}(\beta)P - U(t)(I-P)\bar{X}^T(\beta)X^{\dagger T}(\beta) \\
&\quad + \left[\bar{U}(t) - U(t)X^\dagger(\beta)\bar{X}(\beta)\right](I-P).
\end{aligned} \tag{2.35}$$

By using formulas (2.33) and (2.23) the first two terms in (2.35) reads

$$\begin{aligned}
\bar{U}(t)P - U(t)X^\dagger(\beta)\bar{X}(\beta)P = U(t)S_\beta(t) + X^{\dagger T}(t)& \\
+ U(t)(I-P)\bar{X}^T(t)X^{\dagger T}(t)&.
\end{aligned} \tag{2.36}$$

Finally, inserting equality (2.36) into (2.35) we get expression (2.30) for $\bar{U}_\beta(t)$. ■

**Remark 2.2.8.** (i) Let us analyze formulas (2.29) and (2.30). The terms $XS_\beta$ and $US_\beta + X^{\dagger T}$ are analogous to those ones, which occur in the corresponding formulas for completely controllable system (H), see e.g. [6, Proposition 1 in Section 2.1]. However, the terms

$$\left[\bar{X}(t) - X(t)X^\dagger(\beta)\bar{X}(\beta)\right](I-P),$$

$$U(t)(I-P)\left[X^\dagger(t)\bar{X}(t) - X^\dagger(\beta)\bar{X}(\beta)\right]^T + \left[\bar{U}(t) - U(t)X^\dagger(\beta)\bar{X}(\beta)\right](I-P)$$

have no analogy in the controllable theory. Their presence is a direct consequence of the abnormality of (H), since we may have the projector $P \neq I$ in the general abnormal case.

(ii) For $t \in [\alpha,\infty)$ the terms $\bar{X}(t) - X(t)X^{\dagger}(\beta)\bar{X}(\beta)$ and $X^{\dagger}(t)\bar{X}(t) - X^{\dagger}(\beta)\bar{X}(\beta)$ do not depend on a particular choice of the conjoined basis $(\bar{X},\bar{U})$. Indeed, if $(\bar{X}_*,\bar{U}_*)$ is another conjoined basis of (H) which satisfies condition (1.16), then $(\bar{X}_*,\bar{U}_*) = (\bar{X},\bar{U}) + (X,U)D$ on $[a,\infty)$ for some constant symmetric matrix $D$, by (1.21). Consequently, using the identity $X^{\dagger}(t)X(t) = P$ we have on $[\alpha,\infty)$

$$\begin{aligned}
\bar{X}_*(t) - X(t)X^{\dagger}(\beta)\bar{X}_*(\beta) &= \bar{X}(t) + X(t)D - X(t)X^{\dagger}(\beta)\,[\bar{X}(\beta) + X(\beta)D] \\
&= \bar{X}(t) + X(t)D - X(t)X^{\dagger}(\beta)\bar{X}(t) - X(t)D \\
&= \bar{X}(t) - X(t)X^{\dagger}(\beta)\bar{X}(t), \\
X^{\dagger}(t)\bar{X}_*(t) - X^{\dagger}(\beta)\bar{X}_*(\beta) &= X^{\dagger}(t)\,[\bar{X}(t) + X(t)D] - X^{\dagger}(\beta)\,[\bar{X}(\beta) + X(\beta)D] \\
&= X^{\dagger}(t)\bar{X}(t) - X^{\dagger}(\beta)\bar{X}(\beta).
\end{aligned}$$

Note that this observation is in a full agreement with the uniqueness of the function $\bar{X}_\beta(t)$ on $[\alpha,\infty)$ in Theorem 2.2.5. On the other hand, the same property does not need to be satisfied for the term $\bar{U}(t) - U(t)X^{\dagger}(\beta)\bar{X}(\beta)$, as we can see by the calculation

$$\bar{U}_*(t) - U(t)X^{\dagger}(\beta)\bar{X}_*(\beta) = \bar{U}(t) - U(t)X^{\dagger}(\beta)\bar{X}(\beta) + U(t)(I-P)D$$

on $[\alpha,\infty)$. Consequently, this causes a nonuniqueness of the function $\bar{U}_\beta(t)$ on $[\alpha,\infty)$. As we shall see, it is not $(\bar{X}_\beta,\bar{U}_\beta)$ itself but its constant multiple $(\bar{X}_\beta P, \bar{U}_\beta P)$, which plays an important role in this theory. The solution $(\bar{X}_\beta P, \bar{U}_\beta P)$ is then uniquely determined by $(X,U)$ in both components even on the whole interval $[a,\infty)$ and

$$\begin{gathered}
\bar{X}_\beta(t)P = X(t)S_\beta(t), \\
\bar{U}_\beta(t)P = U(t)S_\beta(t) + X^{\dagger T}(t) + U(t)(I-P)\left[X^{\dagger}(t)\bar{X}(t) - X^{\dagger}(\beta)\bar{X}(\beta)\right]^T
\end{gathered}$$

on $[\alpha,\infty)$ for any above $(\bar{X},\bar{U})$. Finally, by the choice $(\bar{X},\bar{U}) := (\bar{X}_\beta,\bar{U}_\beta)$ we obtain

$$\bar{X}_\beta P = XS_\beta, \qquad \bar{U}_\beta P = US_\beta + X^{\dagger T} + U(I-P)\bar{X}_\beta^T X^{\dagger T} \quad \text{on } [\alpha,\infty). \tag{2.37}$$

The next result provides a classification of all the conjoined bases $(\bar{X}_\beta,\bar{U}_\beta)$ of (H), which correspond to a given conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$ via Theorem 2.2.5.

**Theorem 2.2.9.** *Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $P$ be the matrix defined in (2.2). Moreover, let $(\bar{X}_\beta,\bar{U}_\beta)$ be a conjoined basis of (H) satisfying (1.16) and (2.25) for a given $\beta \geq \alpha$. Then a solution $(X_0,U_0)$ of (H) is a conjoined basis of the same type (i.e. as $(\bar{X}_\beta,\bar{U}_\beta)$) if and only if there exists unique $n \times n$ matrix $H$ such that*

$$X_0(t) = \bar{X}_\beta(t), \quad U_0(t) = \bar{U}_\beta(t) + U(t)H, \quad t \in [\alpha,\infty), \tag{2.38}$$

$$H \text{ is symmetric}, \quad \operatorname{Im} H \subseteq \operatorname{Im}(I-P). \tag{2.39}$$

*Proof.* Let $(X,U)$ and $(\bar{X}_\beta,\bar{U}_\beta)$ be as in theorem. From (1.21) we know that a solution $(X_0,U_0)$ of (H) is a conjoined basis satisfying condition (1.16) if and only if $(X_0,U_0) = (\bar{X}_\beta,\bar{U}_\beta)+(X,U)H$ on $[a,\infty)$ for some constant symmetric matrix $H$. At the same time, the solution $(X_0,U_0)$ satisfies (2.25) if and only if $X(t)H = 0$ on $[\alpha,\infty)$ or equivalently, $PH = 0$, see the proof of Theorem 2.2.5. ■

**Remark 2.2.10.** In the next chapter we will provide a proper interpretation of the formulas in (2.38) and (2.39). More precisely, as we will show in Remark 3.1.8, all conjoined bases $(\bar{X}_\beta,\bar{U}_\beta)$ of (H) satisfying (1.16) and (2.25) for a given $\beta \geq \alpha$ are mutually equivalent solutions of (H) on $[\alpha,\infty)$ in the terminology of Section 3.1.

In the last theorem of this section we display some important properties of the conjoined bases $(\bar{X}_\beta,\bar{U}_\beta)$ defined in Theorem 2.2.5. These results will be then effectively utilized in Chapter 6.

**Theorem 2.2.11.** *Assume* (1.1). *Let* $(X,U)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *and with corresponding matrices* $P$, $R(t)$, $S_\beta(t)$, *and* $P_{\mathscr{S}_\beta}(t)$ *in* (2.2), (2.1), (2.8), *and Remark* 2.1.5 *for a given* $\beta \geq \alpha$. *Let* $(\bar{X}_\beta,\bar{U}_\beta)$ *be a conjoined basis of* (H) *satisfying* (1.16) *and* (2.25). *Then the following statements hold.*

(i) $\operatorname{Ker}\bar{X}_\beta(t) = \operatorname{Im}[P-P_{\mathscr{S}_\beta}(t)]$ *for each* $t\in[\alpha,\infty)$.

(ii) $S_\beta^\dagger(t) = \bar{X}_\beta^\dagger(t)X(t)P_{\mathscr{S}_\beta}(t)$ *for each* $t\in[\alpha,\infty)$.

(iii) $\operatorname{Im}\bar{X}_\beta(\beta) = \operatorname{Im}[I-R(\beta)]$ *and* $\operatorname{Im}\bar{X}_\beta^T(\beta) = \operatorname{Im}(I-P)$.

(iv) *The matrix* $X(\beta)-\bar{X}_\beta(\beta)$ *is invertible with*

$$[X(\beta)-\bar{X}_\beta(\beta)]^{-1} = X^\dagger(\beta)-\bar{X}_\beta^\dagger(\beta). \tag{2.40}$$

(v) $\bar{X}_\beta^\dagger(\beta) = -(I-P)U^T(\beta)$.

*Proof.* (i) From the identity $X^T(t)\bar{U}_\beta(t)-U^T(t)\bar{X}_\beta(t) = I$ on $[\alpha,\infty)$ in (1.16) it follows that $\operatorname{Ker}\bar{X}_\beta(t)\subseteq\operatorname{Im}X^T(t)=\operatorname{Im}P$ on $[\alpha,\infty)$. Moreover, the equality in (2.28) then implies that $\operatorname{Ker}\bar{X}_\beta(t)\subseteq\operatorname{Im}P\cap\operatorname{Ker}S_\beta(t)=\operatorname{Im}P\cap\operatorname{Ker}P_{\mathscr{S}_\beta}(t)$ on $[\alpha,\infty)$, by the definition of $P_{\mathscr{S}_\beta}(t)$ in Remark 2.1.5. Since $PP_{\mathscr{S}_\beta}(t) = P_{\mathscr{S}_\beta}(t)$ for $t\in[\alpha,\infty)$ by (2.9), we have that $\operatorname{Ker}\bar{X}_\beta(t)\subseteq\operatorname{Im}[P-P_{\mathscr{S}_\beta}(t)]$ on $[\alpha,\infty)$. Conversely, fix $t\in[\alpha,\infty)$ and assume $v\in\operatorname{Im}[P-P_{\mathscr{S}_\beta}(t)]=\operatorname{Im}P\cap\operatorname{Ker}S_\beta(t)$. The first formula in (2.37) then implies that $\bar{X}_\beta(t)v=\bar{X}_\beta(t)Pv=X(t)S_\beta(t)v=0$, and hence $v\in\operatorname{Ker}\bar{X}_\beta(t)$. This shows the opposite inclusion $\operatorname{Im}[P-P_{\mathscr{S}_\beta}(t)]\subseteq\operatorname{Ker}\bar{X}_\beta(t)$. In addition, we note that the result of part (i) is equivalent with the fact that the matrix $I-P+P_{\mathscr{S}_\beta}(t)$ is the orthogonal projector onto $\operatorname{Im}\bar{X}_\beta^T(t)$ for each $t\in[\alpha,\infty)$, by Remark 1.2.1(i). Consequently, from Remark 1.2.3(ii) we then have the formula

$$\bar{X}_\beta^\dagger(t)\bar{X}_\beta(t) = I-P+P_{\mathscr{S}_\beta}(t) \quad \text{on } [\alpha,\infty). \tag{2.41}$$

(ii) By using the identities $P_{\mathscr{S}_\beta}(t) = S_\beta(t)\, S_\beta^\dagger(t)$, $X(t)\, S_\beta(t) = \bar{X}_\beta(t)\, P$ on $[\alpha,\infty)$ together with equality (2.41) we get for each $t \in [\alpha,\infty)$

$$\begin{aligned}\bar{X}_\beta^\dagger(t)\, X(t)\, P_{\mathscr{S}_\beta}(t) &= \bar{X}_\beta^\dagger(t)\, X(t)\, S_\beta(t)\, S_\beta^\dagger(t) = \bar{X}_\beta^\dagger(t)\, \bar{X}_\beta(t)\, P S_\beta^\dagger(t) \\ &\overset{(2.41)}{=} \left[I - P + P_{\mathscr{S}_\beta}(t)\right] P S_\beta^\dagger(t) = P_{\mathscr{S}_\beta}(t)\, P S_\beta^\dagger(t) = S_\beta^\dagger(t).\end{aligned}$$

(iii) The second identity immediately follows from part (i) for $t = \beta$, because in this case $S_\beta(\beta) = 0 = P_{\mathscr{S}_\beta}(\beta)$. Moreover, from condition (2.25) we have that $\operatorname{Im} \bar{X}_\beta(\beta) \subseteq \operatorname{Ker} X^\dagger(\beta) = \operatorname{Im}[I - R(\beta)]$. Since $\operatorname{rank} \bar{X}_\beta(\beta) = \operatorname{rank} \bar{X}_\beta^T(\beta) = \operatorname{rank}(I-P) = \operatorname{rank}[I - R(\beta)]$, by (2.3), we obtain the first equality $\operatorname{Im} \bar{X}_\beta(\beta) = \operatorname{Im}[I - R(\beta)]$.

(iv) From part (iii) and Remark 1.2.3(ii) it follows that

$$\bar{X}_\beta(\beta)\, \bar{X}_\beta^\dagger(\beta) = I - R(\beta), \quad \bar{X}_\beta^\dagger(\beta)\, \bar{X}_\beta(\beta) = I - P. \tag{2.42}$$

Furthermore, the identities $X(\beta)\, \bar{X}_\beta^\dagger(\beta) = X(\beta)\,(I-P)\, \bar{X}_\beta^\dagger(\beta) = 0$ and $\bar{X}_\beta(\beta)\, X^\dagger(\beta) = \bar{X}_\beta(\beta)\,[I-P]\, X^\dagger(\beta) = 0$ hold. These results then yield

$$[X(\beta) - \bar{X}_\beta(\beta)]\, [X^\dagger(\beta) - \bar{X}_\beta^\dagger(\beta)] = X(\beta)\, X^\dagger(\beta) + I - R(\beta) = I,$$

since $X(\beta)\, X^\dagger(\beta) = R(\beta)$, by (2.1). Thus, the matrix $X(\beta) - \bar{X}_\beta(\beta)$ is invertible and formula (2.40) holds.

(v) Set $M := \bar{X}_\beta(\beta)$ and $N := -(I-P)\, U^T(\beta)$. We show that the four equations in (1.13) are satisfied, i.e., $N = M^\dagger$. Equalities (2.42) and (1.17) together with the identity $R(\beta)\, X(\beta) = X(\beta)$ imply that

$$\begin{aligned}MN &= -\bar{X}_\beta(\beta)\,(I-P)\, U^T(\beta) = -\bar{X}_\beta(\beta)\, U^T(\beta) = -[I - R(\beta)]\, \bar{X}_\beta(\beta)\, U^T(\beta) \\ &\overset{(1.17)}{=} -[I - R(\beta)]\, [X(\beta)\, \bar{U}_\beta^T(\beta) - I] = I - R(\beta),\end{aligned} \tag{2.43}$$

while formula (1.16) and the identity $P X^T(\beta) = X^T(\beta)$ give

$$NM = -(I-P)\, U^T(\beta)\, \bar{X}_\beta(\beta) \overset{(1.16)}{=} -(I-P)\, [X^T(\beta)\, \bar{U}_\beta(\beta) - I] = I - P. \tag{2.44}$$

Thus, the matrices $MN$ and $NM$ are symmetric. Using (2.43) and (2.44) we obtain

$$\begin{aligned}MNM &= M\,(NM) = \bar{X}_\beta(\beta)\,(I-P) = \bar{X}_\beta(\beta) = M, \\ NMN &= (NM)\, N = -(I-P)^2\, U^T(\beta) = -(I-P)\, U^T(\beta) = N.\end{aligned}$$

It follows from Remark 1.2.3(i) that $M^\dagger = N$, which completes the proof. ■

## 2.3 Representation of conjoined bases

In this section we develop the tools for the representation of conjoined bases of (H) with constant kernel. More precisely, in Definitions 2.3.1 and 2.3.6 we introduce a concept of the representability of solutions of (H) and establish its basic properties (Theorems 2.3.3 and 2.3.8).

**Definition 2.3.1.** Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. We say that a solution $(X_0,U_0)$ of (H) is *representable by* $(X,U)$ *on* $[\alpha,\infty)$, or $(X,U)$ *represents* $(X_0,U_0)$ *on* $[\alpha,\infty)$, if for some $\beta\geq\alpha$ the matrices $M_\beta$ and $N_\beta$ defined by the relation

$$\begin{pmatrix} X_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} X & \bar{X}_\beta \\ U & \bar{U}_\beta \end{pmatrix} \begin{pmatrix} M_\beta \\ N_\beta \end{pmatrix} \quad \text{on } [a,\infty) \tag{2.45}$$

do not depend on the choice of the conjoined basis $(\bar{X}_\beta,\bar{U}_\beta)$ satisfying (1.16) and (2.25).

**Remark 2.3.2.** From (1.20) in Section 1.3 we know that the matrix $N_\beta$ in (2.45) is the Wronskian of $(X,U)$ and $(X_0,U_0)$. This means that $N_\beta$ does not depend on the particular choice of any conjoined basis $(\bar{X},\bar{U})$ of (H) satisfying (1.16), and hence on the particular choice of $\beta\geq\alpha$. Therefore, we will drop the index $\beta$ in the notation $N_\beta$ and use only $N$. On the other hand, the matrix $M_\beta$ in (2.45) depends on the choice of $\beta$ and in general, on the conjoined basis $(\bar{X}_\beta,\bar{U}_\beta)$ as well. In particular, by (1.20) we have

$$M_\beta = \bar{U}_\beta^T(\beta)\,X_0(\beta) - \bar{X}_\beta^T(\beta)\,U_0(\beta). \tag{2.46}$$

The first main result of this section provides a criterion for the representability of solutions of (H) by a given conjoined basis with constant kernel on $[\alpha,\infty)$.

**Theorem 2.3.3.** *Let $(X,U)$ be a conjoined basis of* (H) *with constant kernel on $[\alpha,\infty)$ and let $P$ be its corresponding matrix in* (2.2)*. Moreover, let $(X_0,U_0)$ be a solution of* (H)*. Then the following statements are equivalent.*

(i) *The solution $(X_0,U_0)$ is representable by $(X,U)$ on $[\alpha,\infty)$.*

(ii) *The (constant) Wronskian $N := X^T U_0 - U^T X_0$ of $(X,U)$ and $(X_0,U_0)$ satisfies*

$$\operatorname{Im} N \subseteq \operatorname{Im} P. \tag{2.47}$$

(iii) *The inclusion $\operatorname{Im} X_0(t) \subseteq \operatorname{Im} X(t)$ holds for each $t\in[\alpha,\infty)$.*

(iv) *The inclusion $\operatorname{Im} X_0(\beta) \subseteq \operatorname{Im} X(\beta)$ holds for some $\beta\in[\alpha,\infty)$.*

*Proof.* Let $(X,U)$ and $(X_0,U_0)$ be as in the theorem. Suppose that $(X_0,U_0)$ is representable by $(X,U)$ on $[\alpha,\infty)$. According to Definition 2.3.1 and Remark 2.3.2, this means that there exists $\beta\in[\alpha,\infty)$ so that the matrix $M_\beta$ in (2.46) does not depend on the choice of the conjoined basis $(\bar{X}_\beta,\bar{U}_\beta)$ introduced in Theorem 2.2.5. Moreover, by formulas (2.38) and (2.39) in Theorem 2.2.9 with $H := I-P$ we know that the conjoined basis $(\bar{X}_{\beta*},\bar{U}_{\beta*}) := (\bar{X}_\beta,\bar{U}_\beta) + (X,U)\,(I-P)$ satisfies conditions (1.16) and (2.25) for the point $\beta$. Denote by $M_{\beta*}$ and $N_*$ the matrices in (2.45) which correspond to $(\bar{X}_{\beta*},\bar{U}_{\beta*})$. The equalities in (1.22) with $D := H = I-P$ then imply that $M_{\beta*} = M_\beta - (I-P)\,N$ and $N_* = N$. But $M_{\beta*} = M_\beta$ in this case, and hence $(I-P)\,N = 0$ or $PN = N$. Since the matrix $N$ is the Wronskian of $(X,U)$ and $(X_0,U_0)$, the inclusion in (2.47) holds. Assume now (ii). Combining the identities $PN = N$, $N = X^T(t)\,U_0(t) - U^T(t)\,X_0(t)$, and $PX^T(t) = X^T(t)$ on $[\alpha,\infty)$ we obtain that $PU^T(t)\,X_0(t) = U^T(t)\,X_0(t)$ or equivalently $(I-P)\,U^T(t)\,X_0(t) = 0$ for all $t\in[\alpha,\infty)$. By using the result in Theorem 2.1.2(i) the last equality then yields $\operatorname{Im} X_0(t) \subseteq \operatorname{Ker}\,[U(t)\,(I-P)]^T = \operatorname{Im} X(t)$ on $[\alpha,\infty)$, showing (iii). Part (iii) implies (iv) trivially. Fix now $\beta \geq \alpha$ and suppose that $\operatorname{Im} X_0(\beta) \subseteq \operatorname{Im} X(\beta)$. In particular, this means that the orthogonal projector $R(t)$ in (2.1) satisfies $R(\beta)X_0(\beta) = X_0(\beta)$. Furthermore, define the matrix $L_\beta := X^\dagger(\beta)X_0(\beta)$. Then we have

$$X_0(\beta) = R(\beta)X_0(\beta) = X(\beta)X^\dagger(\beta)X_0(\beta) = X(\beta)L_\beta. \tag{2.48}$$

Let $(\bar{X}_\beta, \bar{U}_\beta)$ be the conjoined basis of (H) satisfying (1.16) and (2.25). We show that the matrix $M_\beta$ in (2.45) in Definition 2.3.1 does not depend on the choice of $(\bar{X}_\beta, \bar{U}_\beta)$. By inserting (2.48) into the formula in (2.46) and using (1.16) we obtain

$$\begin{aligned} M_\beta &= \bar{U}_\beta^T(\beta)X(\beta)L_\beta - \bar{X}_\beta^T(\beta)U_0(\beta) \overset{(1.16)}{=} \left[I + \bar{X}_\beta^T(\beta)U(\beta)\right]L_\beta - \bar{X}_\beta^T(\beta)U_0(\beta) \\ &= L_\beta + \bar{X}_\beta^T(\beta)\left[U(\beta)L_\beta - U_0(\beta)\right]. \end{aligned} \tag{2.49}$$

Hence, the matrix $M_\beta$ is independent on the choice of $(\bar{X}_\beta, \bar{U}_\beta)$, because only the solutions $(X,U)$, $(X_0,U_0)$ and the matrix $\bar{X}_\beta(\beta)$, which is unique by Theorem 2.2.5, are used in expression (2.49). Therefore, the solution $(X_0,U_0)$ is representable by $(X,U)$ on $[\alpha,\infty)$, by Definition 2.3.1 and Remark 2.3.2. The proof is complete. ∎

**Remark 2.3.4.** (i) From Theorem 2.3.3 and its proof it follows that the representability of the solution $(X_0,U_0)$ by $(X,U)$ on $[\alpha,\infty)$ does not depend on the particular choice of the point $\beta \geq \alpha$ in Definition 2.3.1. More precisely, $(X_0,U_0)$ is representable by $(X,U)$ on $[\alpha,\infty)$ if and only if the matrix $M_\beta$ in (2.46) does not depend on the choice of the conjoined basis $(\bar{X}_\beta, \bar{U}_\beta)$ for each $\beta \geq \alpha$.

(ii) Let $\beta \geq \alpha$ be fixed. If $(X_0,U_0)$ is representable by $(X,U)$ on $[\alpha,\infty)$, then expression (2.45) in Definition 2.3.1 together with (2.47) in Theorem 2.3.3(ii) yield

$$X_0 = XM_\beta + \bar{X}_\beta PN, \quad U_0 = UM_\beta + \bar{U}_\beta PN \quad \text{on } [\alpha,\infty).$$

Thus, representation (2.45) contains the uniquely determined solution $(\bar{X}_\beta P, \bar{U}_\beta P)$. Moreover, with the aid of formulas (2.37) we get on $[\alpha,\infty)$

$$X_0 = X(M_\beta + S_\beta N), \tag{2.50}$$

$$U_0 = U(M_\beta + S_\beta N) + X^{\dagger T}N + U(I-P)\bar{X}_\beta^T X^{\dagger T}N, \tag{2.51}$$

where $S_\beta$ is the $S$-matrix associated with $(X,U)$. These expressions generalize the corresponding formulas in [6, Proposition 1 in Chapter 2] and [31, Theorem VII.2.2] to abnormal systems (H). At the same time, they justify the notion "being representable" introduced in Definition 2.3.1, because the matrices $X_0$ and $U_0$ in (2.50) and (2.51) are expressed only in terms of conjoined basis $(X,U)$ on $[\alpha,\infty)$. Moreover, one can see that the last term in equality (2.51) arises from the abnormality of (H), which allows $P \neq I$, as we also comment in Remark 2.2.8(i). In addition, formulas (2.50) and (2.51) at $t = \beta$ with (2.25) and $S_\beta(\beta) = 0$ imply

$$X_0(\beta) = X(\beta)M_\beta, \quad U_0(\beta) = U(\beta)M_\beta + X^{\dagger T}(\beta)N. \tag{2.52}$$

The following corollary contains an important property of conjoined bases of (H) with eventually constant kernel.

**Corollary 2.3.5.** *Let $(X,U)$ be a conjoined basis of* (H) *with eventually constant kernel and $(X_0,U_0)$ be a solution of* (H). *Then the following statements are equivalent.*

(i) *The inclusion $\operatorname{Im} X_0(t) \subseteq \operatorname{Im} X(t)$ holds on some subinterval $[\alpha,\infty)$, where $(X,U)$ has constant kernel.*

(ii) *The inclusion $\operatorname{Im} X_0(t) \subseteq \operatorname{Im} X(t)$ holds on every subinterval $[\alpha,\infty)$, where $(X,U)$ has constant kernel.*

*Proof.* The assumptions of the theorem imply that we may choose $\alpha \geq a$ so that $(X,U)$ has constant kernel on $[\alpha,\infty)$. The equivalence of assertions (i) and (ii) then directly follows from Theorem 2.3.3. ■

In the following definition we extend the concept of representability of solutions of (H) in Definitions 2.3.1 to arbitrary interval $[\alpha,\infty)$, where the conjoined basis $(X,U)$ has constant kernel.

**Definition 2.3.6.** Let $(X,U)$ be a conjoined basis of (H). We say that a solution $(X_0,U_0)$ of (H) is *representable by* $(X,U)$, or $(X,U)$ *represents* $(X_0,U_0)$, if there exists $\alpha \in [a,\infty)$ such that $(X,U)$ has constant kernel on $[\alpha,\infty)$ and $(X_0,U_0)$ is representable by $(X,U)$ on $[\alpha,\infty)$ in the sense of Definition 2.3.1.

**Remark 2.3.7.** (i) Assume that the Legendre condition (1.1) holds and that system (H) is nonoscillatory. Then every conjoined basis of (H) has eventually constant kernel, see Section 1.5. From Theorem 2.3.3 and Corollary 2.3.5 it follows that in this case a solution $(X_0,U_0)$ of (H) is representable by a conjoined basis $(X,U)$ of (H) if and only if the inclusion $\operatorname{Im} X_0(t) \subseteq \operatorname{Im} X(t)$ is satisfied on some (and hence every) interval $[\alpha,\infty)$, where $(X,U)$ has constant kernel. Moreover, formulas (2.50) and (2.51) hold on every such interval $[\alpha,\infty)$.

(ii) In particular, when the function $X(t)$ is eventually invertible, the conjoined basis $(X,U)$ represents any solution $(X_0,U_0)$ of (H), because in this case we have $\operatorname{Im} X(t) = \mathbb{R}^n$ for large $t$.

In the second main result of this section we describe a mutual representability of conjoined bases $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$. In particular, this result is a generalization of [6, Propositions 1, 3 in Chapter 2] and [31, Theorem VII.2.2] to abnormal systems (H).

**Theorem 2.3.8.** *Let $(X_1,U_1)$ and $(X_2,U_2)$ be conjoined bases of* (H) *with constant kernels on $[\alpha,\infty)$ and let $P_1$ and $P_2$ be the projectors defined in (2.2) through the functions $X_1$ and $X_2$, respectively. For a given $\beta \geq \alpha$ let $(X_2,U_2)$ be expressed in terms of $(X_1,U_1)$ via matrices $M_{1\beta}$, $N_1$ and let $(X_1,U_1)$ be expressed in terms of $(X_2,U_2)$ via matrices $M_{2\beta}$, $N_2$, that is,*

$$\begin{pmatrix} X_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} X_1 & \bar{X}_{1\beta} \\ U_1 & \bar{U}_{1\beta} \end{pmatrix} \begin{pmatrix} M_{1\beta} \\ N_1 \end{pmatrix}, \quad \begin{pmatrix} X_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} X_2 & \bar{X}_{2\beta} \\ U_2 & \bar{U}_{2\beta} \end{pmatrix} \begin{pmatrix} M_{2\beta} \\ N_2 \end{pmatrix} \quad \text{on } [\alpha,\infty), \tag{2.53}$$

*where $(\bar{X}_{1\beta},\bar{U}_{1\beta})$ and $(\bar{X}_{2\beta},\bar{U}_{2\beta})$ are conjoined bases of* (H) *satisfying (1.16) and (2.25) with regard to conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ as in Theorem 2.2.5. Then $(X_1,U_1)$ and $(X_2,U_2)$ are mutually representable on $[\alpha,\infty)$ if and only if the equality $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ holds for some (and hence for every) $t \in [\alpha,\infty)$. Moreover, in this case we have that*

(i) $M_{1\beta}^T N_1$ *and* $M_{2\beta}^T N_2$ *are symmetric and* $N_1 + N_2^T = 0$,

(ii) $M_{1\beta}$ *and* $M_{2\beta}$ *are nonsingular and* $M_{1\beta} M_{2\beta} = M_{2\beta} M_{1\beta} = I$,

(iii) $\operatorname{Im} N_1 \subseteq \operatorname{Im} P_1$ *and* $\operatorname{Im} N_2 \subseteq \operatorname{Im} P_2$.

*Proof.* The first part of the theorem directly follows from Theorem 2.3.3 while part (i) is a consequence of the facts that $(X_1,U_1)$ and $(X_2,U_2)$ are conjoined bases and that the matrices $N_1$ and $N_2$ are the Wronskians of $(X_1,U_1)$, $(X_2,U_2)$ and $(X_2,U_2)$, $(X_1,U_1)$, respectively. Moreover, the equalities

$$X_2(\beta) = X_1(\beta) M_{1\beta}, \quad U_2(\beta) = U_1(\beta) M_{1\beta} + X_1^{\dagger T}(\beta) N_1, \tag{2.54}$$

$$X_1(\beta) = X_2(\beta) M_{2\beta}, \quad U_1(\beta) = U_2(\beta) M_{2\beta} + X_2^{\dagger T}(\beta) N_2, \tag{2.55}$$

hold, by (2.52). For part (ii) we calculate the product $M_{1\beta} M_{2\beta}$. From (2.46) we know that

$$M_{1\beta} = \bar{U}_{1\beta}^T(\beta) X_2(\beta) - \bar{X}_{1\beta}^T(\beta) U_2(\beta), \quad M_{2\beta} = \bar{U}_{2\beta}^T(\beta) X_1(\beta) - \bar{X}_{2\beta}^T(\beta) U_1(\beta). \tag{2.56}$$

By using the first equality in (2.56), condition (1.16), and formulas (2.55) we get

$$\begin{aligned} M_{1\beta} M_{2\beta} &= \bar{U}_{1\beta}^T(\beta) X_2(\beta) M_{2\beta} - \bar{X}_{1\beta}^T(\beta) U_2(\beta) M_{2\beta} \\ &= \bar{U}_{1\beta}^T(\beta) X_1(\beta) - \bar{X}_{1\beta}^T(\beta) \left[ U_1(\beta) - X_2^{\dagger T}(\beta) N_2 \right] \\ &= \left[ \bar{U}_{1\beta}^T(\beta) X_1(\beta) - \bar{X}_{1\beta}^T(\beta) U_1(\beta) \right] + \bar{X}_{1\beta}^T(\beta) X_2^{\dagger T}(\beta) N_2 \\ &\overset{(1.16)}{=} I + \bar{X}_{1\beta}^T(\beta) X_2^{\dagger T}(\beta) N_2. \end{aligned} \tag{2.57}$$

Since $\operatorname{Im} X_1(\beta) = \operatorname{Im} X_2(\beta)$, by the first part of the theorem, the orthogonal projectors $R_1(\beta)$ and $R_2(\beta)$ onto sets $\operatorname{Im} X_1(\beta)$ and $\operatorname{Im} X_2(\beta)$, which are are defined in (2.1), satisfy $R_1(\beta) = R_2(\beta)$. In patricular, we have that $X_2^\dagger(\beta) \bar{X}_{1\beta} = X_2^\dagger(\beta) R_2(\beta) \bar{X}_{1\beta} = X_2^\dagger(\beta) R_1(\beta) \bar{X}_{1\beta} = 0$, by Theorem 2.2.11(iii). Therefore, the equality in (2.57) then yields $M_{1\beta} M_{2\beta} = I$. Similarly, we obtain that $M_{2\beta} M_{1\beta} = I$, showing (ii). Finally, part (iii) follows from Theorem 2.3.3(ii). The proof is complete. ■

**Remark 2.3.9.** (i) If $S_{1\beta}(t)$ and $S_{2\beta}(t)$ are the $S$-matrices associated with the conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ in Theorem 2.3.8, then the formulas

$$X_2 = X_1(M_{1\beta} + S_{1\beta} N_1), \quad X_1 = X_2(M_{2\beta} + S_{2\beta} N_2) \tag{2.58}$$

hold on $[\alpha,\infty)$, by (2.50). Moreover, the matrices $M_{1\beta} + S_{1\beta} N_1$ and $M_{2\beta} + S_{2\beta} N_2$ are invertible on $[\alpha,\infty)$. Indeed, from conditions (i) and (iii) in Theorem 2.3.8 we obtain that $\operatorname{Im} N_1^T \subseteq \operatorname{Im} P_2$, so that $\operatorname{Ker} X_2 = \operatorname{Ker} P_2 \subseteq \operatorname{Ker} N_1$ on $[\alpha,\infty)$. If for some vector $v \in \mathbb{R}^n$ we have $(M_{1\beta} + S_{1\beta} N_1) v = 0$, then $v \in \operatorname{Ker} X_2$ by (2.58). In turn, $v \in \operatorname{Ker} N_1$, and so $v \in \operatorname{Ker} M_{1\beta}$. But since $M_{1\beta}$ is invertible by (ii) of Theorem 2.3.8, it follows that $v = 0$. Similarly, one has that $M_{2\beta} + S_{2\beta} N_2$ is invertible on $[\alpha,\infty)$.

(ii) Formulas (2.54) and (2.55) together with the identities $X_1^\dagger(\beta) X_1(\beta) = P_1$ and $X_2^\dagger(\beta) X_2(\beta) = P_2$ imply that $P_1 M_{1\beta} = X_1^\dagger(\beta) X_2(\beta)$ and $P_2 M_{2\beta} = X_2^\dagger(\beta) X_1(\beta)$. Consequently, the matrices $P_1 M_{1\beta}$ and $P_2 M_{2\beta}$ satisfy

$$\operatorname{Im}(P_1 M_{1\beta})^T = \operatorname{Im} P_2, \quad \operatorname{Im}(P_2 M_{2\beta})^T = \operatorname{Im} P_1, \quad P_2 M_{2\beta} = (P_1 M_{1\beta})^\dagger, \tag{2.59}$$

which one can verify by using the above expressions for $P_1M_{1\beta}$ and $P_2M_{2\beta}$ and the invertibility of the matrices $M_{1\beta}$ and $M_{2\beta}$. In addition we note that the matrix $N_1$ is the Wronskian of $(X_1,U_1)$ and $(X_2,U_2)$ while the matrix $N_2 = -N_1^T$ is the Wronskian of $(X_2,U_2)$ and $(X_1,U_1)$, as we comment in the proof of Theorem 2.3.8.

(iii) From the first part of Theorem 2.3.8 it follows that if the equality $\operatorname{Im} X_1(\beta) = \operatorname{Im} X_2(\beta)$ is satisfied for some $\beta \in [\alpha,\infty)$, then $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ holds for all $t \in [\alpha,\infty)$.

Combining the results in Theorem 2.3.8 and Remark 2.3.7(i) with Definition 2.3.6 we immediately obtain the following statement.

**Corollary 2.3.10.** *Assume that* (1.1) *holds and that system* (H) *is nonoscillatory. Let* $(X_1,U_1)$ *and* $(X_2,U_2)$ *be conjoined bases of* (H). *Then* $(X_1,U_1)$ *and* $(X_2,U_2)$ *are mutually representable if and only if the equality* $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ *holds on some (and hence on every) subinterval* $[\alpha,\infty)$, *where* $(X_1,U_1)$ *and* $(X_2,U_2)$ *have constant kernel.*

# Chapter 3

# Construction of conjoined bases with constant kernel

The following chapter is focused on a construction of conjoined bases of system (H). This topic is closely related with the abnormality of system (H), which discussed in Section 3.1. The main results of this chapter are formulated in terms of the relation "being contained" for conjoined bases of (H) established in Section 3.2. Finally, in Section 3.3 we introduce a special class of conjoined bases of (H) with constant kernel called minimal conjoined bases. We note that the construction of conjoined bases via the relation "being contained" together with the concept of minimal conjoined bases represent fundamental tools for the theory of principal and antiprincipal solutions of (H) at infinity.

## 3.1 Abnormality and equivalence of solutions

In this section we study in details the abnormality of system (H). Our results, in particular Theorem 3.1.2 and Remark 3.1.3, extend the theory established in [30, Section 3] and [31, Section 3 in Chapter VII].

We use a standard notation from [30, Section 3], that is, for a nondegenerate subinterval $\mathscr{I} \subseteq [a,\infty)$, the symbol $\Lambda(\mathscr{I})$ denotes the linear space of $n$-dimensional vector-valued functions $u \in \mathrm{C}^1_{\mathrm{p}}$ which satisfy the equations $u' = -A^T(t)\,u$ and $B(t)\,u = 0$ on $\mathscr{I}$. It is easy to see that $u \in \Lambda(\mathscr{I})$ if and only if the pair $(x \equiv 0, u)$ is a solution of (H) on $\mathscr{I}$. Obviously, $\Lambda(\mathscr{I})$ is finite-dimensional with $d(\mathscr{I}) := \dim \Lambda(\mathscr{I}) \leq n$. For completeness we put $\Lambda(\mathscr{I}) := \mathbb{R}^n$ and $d(\mathscr{I}) := n$, when the subinterval $\mathscr{I}$ is degenerate. The number $d(\mathscr{I})$ is called the *order of abnormality* of system (H) on $\mathscr{I}$. If $d(\mathscr{I}) = 0$, then (H) is said to be *normal* (or *controllable*) on $\mathscr{I}$. Moreover, the system (H) is said to be *identically normal* (or *completely controllable*) on $\mathscr{I}$, if $d(\mathscr{J}) = 0$ for every nondegenerate subinterval $\mathscr{J} \subseteq \mathscr{I}$. For brevity we write $\Lambda[\alpha,\beta]$ and $d[\alpha,\beta]$ instead of $\Lambda([\alpha,\beta])$ and $d([\alpha,\beta])$ when $\mathscr{I} = [\alpha,\beta]$. Similar notation is used for other types of bounded or unbounded intervals.

Let $\alpha \in [a,\infty)$. By $\Lambda_0[\alpha,\infty)$ we denote the subspace of $\mathbb{R}^n$ consisting of the initial values of functions $u \in \Lambda[\alpha,\infty)$, that is,

$$\Lambda_0[\alpha,\infty) := \{c \in \mathbb{R}^n, \quad u(\alpha) = c \quad \text{for some } u \in \Lambda[\alpha,\infty)\}. \tag{3.1}$$

Clearly, $\Lambda_0[\alpha,\infty)$ is finite-dimensional and $\dim\Lambda_0[\alpha,\infty)=d[\alpha,\infty)$. In [30, Section 3] it is shown that the integer-valued function $d[a,t]$ is nonincreasing and piecewise constant in $t$ on $[a,\infty)$ with at most $n$ points of discontinuity, at which $d[a,t]$ is left-continuous. On the other hand, the integer-valued function $d[t,\infty)$ is nondecreasing and piecewise constant in $t$ on $[a,\infty)$, and it has at most $n$ points of discontinuity, at which it is right-continuous. Therefore, there exists the maximal order of abnormality $d_\infty$ of (H) defined by

$$d_\infty := \lim_{t\to\infty} d[t,\infty) = \max_{t\in[a,\infty)} d[t,\infty). \tag{3.2}$$

Moreover, the above properties of the function $d[t,\infty)$ imply that $0\le d_\infty\le n$ and there exists a point $\alpha\in[a,\infty)$ such that

$$d[\alpha,\infty)=d_\infty. \tag{3.3}$$

The subintervals $[\alpha,\infty)$ with the property in (3.3) are extremely important for the study of the $S$-matrices and consequently, for the construction of (anti)principal solutions at infinity in Chapter 5.

The following theorem provides a basic connection between the subspaces $\Lambda_0[\alpha,t]$ for $t\ge\alpha$ and the principal solution $(\hat X_\alpha,\hat U_\alpha)$ of (H) at the point $\alpha$. We recall from Section 1.3 that $(\hat X_\alpha,\hat U_\alpha)$ is defined by the initial conditions $\hat X_\alpha(\alpha)=0$ and $\hat U_\alpha(\alpha)=I$.

**Theorem 3.1.1.** *Let $(\hat X_\alpha,\hat U_\alpha)$ be the principal solution of* (H) *at the point $\alpha$. Then*

$$\Lambda_0[\alpha,t]=\bigcap_{s\in[\alpha,t]}\operatorname{Ker}\hat X_\alpha(s) \quad \textit{for every } t\in[\alpha,\infty). \tag{3.4}$$

*Proof.* If $c\in\Lambda_0[\alpha,t]$, then $(x\equiv 0,u)$ is a solution of (H) for some $u\in\Lambda[\alpha,t]$ with $u(\alpha)=c$. By the uniqueness of solutions of system (H), it follows that $(x,u)=(\hat X_\alpha c,\hat U_\alpha c)$ on $[\alpha,t]$. Hence, $\hat X_\alpha(s)\,c=0$ for all $s\in[\alpha,t]$. The opposite direction is trivial. ∎

In the first main result of this section we establish an exact relation between the rank of the $S$-matrix corresponding to a conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$ and the order of abnormality of (H). We also derive a representation of the subspaces $\Lambda_0[\beta,t]$ for $t\ge\beta\ge\alpha$ in terms of some orthogonal subspaces associated with the initial values $X(\beta)$ and $U(\beta)$.

**Theorem 3.1.2.** *Assume* (1.1)*. Let $(X,U)$ be a conjoined basis of* (H) *with constant kernel on $[\alpha,\infty)$ and $S_\beta(t)$ be its corresponding $S$-matrix in* (2.8) *for a given $\beta\ge\alpha$. Furthermore, let $P$ and $P_{\mathscr{S}_\beta}(t)$ be the matrices defined in* (2.2) *and Remark 2.1.5(i). If $(\hat X_\beta,\hat U_\beta)$ is the principal solution at $\beta$, then the equality $\hat X_\beta(t)=X(t)\,S_\beta(t)\,X^T(\beta)$ holds on $[\alpha,\infty)$ and*

$$\operatorname{rank} S_\beta(t)=\operatorname{rank}\hat X_\beta(t)=\begin{cases} n-d[t,\beta], & t\in[\alpha,\beta] \\ n-d[\beta,t], & t\in[\beta,\infty). \end{cases} \tag{3.5}$$

*Moreover, for $t\in[\beta,\infty)$ we have*

$$\Lambda_0[\beta,t]=\operatorname{Im}[X^{\dagger T}(\beta)\,(I-P_{\mathscr{S}_\beta}(t))]\oplus\operatorname{Im}[U(\beta)\,(I-P)]. \tag{3.6}$$

*Proof.* Let $(X,U)$ be as in the theorem and for a fixed $\beta \geq \alpha$ consider the conjoined basis $(\bar{X}_\beta, \bar{U}_\beta)$ satisfying (1.16) and (2.25). Moreover, as in (2.45), assume that $(\hat{X}_\beta, \hat{U}_\beta)$ is represented in terms of $(X,U)$ and $(\bar{X}_\beta, \bar{U}_\beta)$ via the matrices $M_\beta$ and $N_\beta$, i.e.,

$$\begin{pmatrix} \hat{X}_\beta \\ \hat{U}_\beta \end{pmatrix} = \begin{pmatrix} X & \bar{X}_\beta \\ U & \bar{U}_\beta \end{pmatrix} \begin{pmatrix} M_\beta \\ N_\beta \end{pmatrix} \quad \text{on } [\alpha,\infty). \tag{3.7}$$

From (1.20) and (3.7) at $t = \beta$ we obtain

$$M_\beta = \bar{U}_\beta^T(\beta)\hat{X}_\beta(\beta) - \bar{X}_\beta^T(\beta)\hat{U}_\beta(\beta) = -\bar{X}_\beta^T(\beta), \tag{3.8}$$

$$N_\beta = X^T(\beta)\hat{U}_\beta(\beta) - U^T(\beta)\hat{X}_\beta(\beta) = X^T(\beta). \tag{3.9}$$

Since $\operatorname{Im} N_\beta = \operatorname{Im} X^T(\beta) = \operatorname{Im} P$, the principal solution $(\hat{X}_\beta, \hat{U}_\beta)$ is representable by $(X,U)$ on $[\alpha,\infty)$, by Theorem 2.3.3. Furthermore, inserting expressions (3.8) and (3.9) into the formula in (2.50) with $(X_0,U_0) := (\hat{X}_\beta, \hat{U}_\beta)$ and using the definition of $P$ and the first equality in (2.37) yield for all $t \in [\alpha,\infty)$

$$\begin{aligned} \hat{X}_\beta(t) &= X(t)\,[-\bar{X}_\beta^T(\beta) + S_\beta(t)X^T(\beta)] = -X(t)P\bar{X}_\beta^T(\beta) + X(t)S_\beta(t)X^T(\beta) \\ &\overset{(2.37)}{=} -X(t)S_\beta(\beta)X^T(\beta) + X(t)S_\beta(t)X^T(\beta) = X(t)S_\beta(t)X^T(\beta). \end{aligned} \tag{3.10}$$

On the other hand, since by (2.9) the equalities $PS_\beta(t) = S_\beta(t)P = S_\beta(t)$ hold, (3.10) gives

$$X^\dagger(t)\hat{X}_\beta(t)X^{\dagger T}(\beta) \overset{(3.10)}{=} X^\dagger(t)X(t)S_\beta(t)X^T(\beta)X^{\dagger T}(\beta) = PS_\beta(t)P = S_\beta(t) \tag{3.11}$$

for all $t \in [\alpha,\infty)$. The expressions in (3.10) and (3.11) then show that

$$\operatorname{rank} S_\beta(t) = \operatorname{rank} \hat{X}_\beta(t) \quad \text{on } [\alpha,\infty). \tag{3.12}$$

By Proposition 1.5.2, condition (1.1) and the constancy of the kernel of $(X,U)$ on $[\alpha,\infty)$ imply that the principal solution $(\hat{X}_\beta, \hat{U}_\beta)$ has no proper focal point in $(\beta,\infty)$. This means that the kernel of $\hat{X}_\beta(t)$ is nonincreasing on $[\beta,\infty)$, see also [24, Definition 1]. Using (3.4) we obtain

$$\Lambda_0[\beta,t] = \operatorname{Ker} \hat{X}_\beta(t) \quad \text{for all } t \in [\beta,\infty) \tag{3.13}$$

and hence, the formula $\operatorname{rank} S_\beta(t) = \operatorname{rank} \hat{X}_\beta(t) = n - d[\beta,t]$ holds on $[\beta,\infty)$, by (3.12). Now let $t \in [\alpha,\beta]$ be fixed and consider the principal solution $(\hat{X}_t, \hat{U}_t)$ of (H) at $t$. Since $(X,U)$ has constant kernel on the interval $[t,\infty)$, by mutual changing the arguments $\beta$ and $t$ in formulas (3.13) and (3.12) we get the equalities $\Lambda_0[t,\beta] = \operatorname{Ker} \hat{X}_t(\beta)$ and $\operatorname{rank} S_t(\beta) = \operatorname{rank} \hat{X}_t(\beta) = n - d[t,\beta]$. But $S_t(\beta) = -S_\beta(t)$ by (2.8) and hence, using (3.12) once more we obtain $\operatorname{rank} \hat{X}_\beta(t) = \operatorname{rank} S_\beta(t) = \operatorname{rank} S_t(\beta) = n - d[t,\beta]$, which completely establishes equality (3.5). In order to prove (3.6), fix $t \in [\beta,\infty)$ and let $v \in \Lambda_0[\beta,t]$. By (3.13) and (3.10) we then have $X(t)S_\beta(t)X^T(\beta)\,v = 0$ and using (2.9) we get

$$P_{\mathscr{S}_\beta}(t)X^T(\beta)\,v = S_\beta^\dagger(t)PS_\beta(t)X^T(\beta)\,v = S_\beta^\dagger(t)X^\dagger(t)X(t)S_\beta(t)X^T(\beta)\,v = 0.$$

Hence, $X^T(\beta)\,v = [I - P_{\mathscr{S}_\beta}(t)]\,v_*$ for some $v_* \in \mathbb{R}^n$. The vector $v$ can be uniquely decomposed as $v = v_1 + v_2$ with $v_1 \in \operatorname{Im} X(\beta) = \operatorname{Im} X^{\dagger T}(\beta)$ and $v_2 \in [\operatorname{Im} X(\beta)]^\perp =$

$\operatorname{Ker} X^T(\beta) = \operatorname{Ker} R(\beta) = \operatorname{Im}[U(\beta)(I-P)]$, by (2.1) and Theorem 2.1.2(i). Consequently, $X^T(\beta)\,v_1 = [I - P_{\mathscr{S}_\beta}(t)]\,v_*$ which implies that $v_1 \in \operatorname{Im}[X^{\dagger T}(\beta)\,(I - P_{\mathscr{S}_\beta}(t))]$. Thus, $v \in \operatorname{Im}[X^{\dagger T}(\beta)\,(I - P_{\mathscr{S}_\beta}(t))] \oplus \operatorname{Im}[U(\beta)\,(I-P)]$. Conversely, every $w \in \operatorname{Im}[X^{\dagger T}(\beta)\,(I - P_{\mathscr{S}_\beta}(t))] \oplus \operatorname{Im}[U(\beta)\,(I-P)]$ has the form

$$w = X^{\dagger T}(\beta)\,[I - P_{\mathscr{S}_\beta}(t)]\,w_1 + U(\beta)\,(I-P)\,w_2 \quad \text{for some } w_1, w_2 \in \mathbb{R}^n.$$

We show that $\hat{X}_\beta(t)\,w = 0$. This follows, by the aid of (3.10), from the equality

$$\begin{aligned}\hat{X}_\beta(t)\,w &= X(t)\,S_\beta(t)\,X^T(\beta)\,X^{\dagger T}(\beta)\,[I - P_{\mathscr{S}_\beta}(t)]\,w_1 \\ &\quad + X(t)\,S_\beta(t)\,X^T(\beta)\,U(\beta)\,(I-P)\,w_2.\end{aligned}$$

The first term in the last formula is zero, because $S_\beta(t)\,X^T(\beta)\,X^{\dagger T}(\beta)\,[I - P_{\mathscr{S}_\beta}(t)] = S_\beta(t)\,[I - P_{\mathscr{S}_\beta}(t)] = 0$. And the second term is zero too, because $X^T(\beta)\,U(\beta)\,(I-P) = U^T(\beta)\,X(\beta)\,(I-P) = 0$. Thus $w \in \operatorname{Ker}\hat{X}_\beta(t) = \Lambda_0[\beta,t]$ by (3.13) and the proof of (3.6) is complete. ■

**Remark 3.1.3.** Formula (3.5) shows that for a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ the rank of its corresponding matrix $S_\beta$ depends only on the rank of $\hat{X}_\beta$ and hence, on the abnormality of (H) on $[\alpha,\infty)$, but not on the choice of $(X,U)$ itself. This means that the changes in $\operatorname{Im}\hat{X}_\beta^T(t)$ and $\operatorname{Im} S_\beta(t)$ occur at the same points, i.e., according to Theorem 2.1.2(ii) there exists a finite partition $\alpha = \tau_{-k} < \tau_{-k+1} < \cdots < \beta = \tau_0 < \cdots < \tau_{l-1} < \tau_l < \infty$ of $[\alpha,\infty)$, which does not depend on $S_\beta$, such that $\operatorname{Im} S_\beta(t)$ is constant on each subinterval $(\tau_\nu, \tau_{\nu+1})$ and

$$\begin{aligned}\operatorname{Im} S_\beta(t) \equiv \operatorname{Im} S_\beta(\tau_{\nu+1}^-) \supsetneqq \operatorname{Im} S_\beta(\tau_{\nu+1}) \quad &\text{for all } t \in [\tau_\nu, \tau_{\nu+1}), \quad \nu \in \{-k,\dots,-1\},\\ \operatorname{Im} S_\beta(\tau_\nu) \subsetneqq \operatorname{Im} S_\beta(\tau_\nu^+) \equiv \operatorname{Im} S_\beta(t) \quad &\text{for all } t \in (\tau_\nu, \tau_{\nu+1}], \quad \nu \in \{0,\dots,l-1\},\\ \operatorname{Im} S_\beta(\tau_l) \subsetneqq \operatorname{Im} S_\beta(\tau_l^+) \equiv \operatorname{Im} S_\beta(t) \quad &\text{for all } t \in (\tau_l,\infty).\end{aligned}$$

On the last subinterval $(\tau_l,\infty)$ we then have $\operatorname{rank} S_\beta(t) = n - d[\beta,\infty)$. This means, by Remark 2.1.5(i), that the orthogonal projector $P_{\mathscr{S}_{\beta\infty}}$ defined in (2.12) satisfies

$$\operatorname{rank} P_{\mathscr{S}_{\beta\infty}} = n - d[\beta,\infty). \tag{3.14}$$

We stress that the partition $\{\tau_{-k}, \tau_{-k+1}, \dots, \tau_{l-1}, \tau_l\}$ of $[\alpha,\infty)$ defined above depends on the choice of the point $\beta \geq \alpha$. In addition, for $\beta = \alpha$ we have $k = 0$.

**Remark 3.1.4.** The results in Theorem 3.1.2 lead to a set of admissible values which may be attained by the rank of conjoined bases of (H) with constant kernel on $[\alpha,\infty)$. According to (3.10) for $\beta = \alpha$, any such a conjoined basis $(X,U)$ has the property $\operatorname{rank}\hat{X}_\alpha(t) \leq \operatorname{rank} X(t)$ for each $t \in [\alpha,\infty)$, where $(\hat{X}_\alpha, \hat{U}_\alpha)$ is the principal solution at $\alpha$. Since the rank of the matrix $X(t)$ is constant on $[\alpha,\infty)$ and since $n - d[\alpha,\infty)$ is the maximum of $\operatorname{rank}\hat{X}_\alpha(t)$ on $[\alpha,\infty)$ by (3.5), we get from the above the estimates

$$n - d[\alpha,\infty) \leq \operatorname{rank} X(t) \leq n \quad \text{on } [\alpha,\infty). \tag{3.15}$$

In addition, by using the maximal order of abnormality $d_\infty$ of (H) defined in (3.2) estimates (3.15) then yield

$$n - d_\infty \le \operatorname{rank} X(t) \le n \quad \text{on } [\alpha,\infty). \tag{3.16}$$

In the next section we will give a precise analysis of conjoined bases $(X,U)$ with constant kernel on $[\alpha,\infty)$ satisfying (3.15) and (3.16).

**Remark 3.1.5.** When the system (H) is completely controllable on $[\alpha,\infty)$, then the indices $k$ and $l$ from Remark 3.1.3 are necessarily zero, i.e., $\beta = \tau_{-k} = \tau_l$ for all $\beta \ge \alpha$. In this case $d(\mathscr{I}) = 0$ for every nondegenerate subinterval $\mathscr{I} \subseteq [\beta,\infty)$ and so the matrix $S_\beta(t)$ is invertible on $[\alpha,\beta)\cup(\beta,\infty)$. Moreover, from (3.15) it follows that the only conjoined bases $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ are those which have $X(t)$ invertible on $[\alpha,\infty)$.

In the following definition we introduce the concept of equivalent solutions of system (H), which can be viewed as an extension of the equality of two solutions with respect to the abnormality of (H), see Remark 3.1.9 below.

**Definition 3.1.6.** Let $(X_1,U_1)$ and $(X_2,U_2)$ be solutions of (H). We say that $(X_1,U_1)$ and $(X_2,U_2)$ are *equivalent* on $[\alpha,\infty)$ and write $(X_1,U_1) \sim (X_2,U_2)$ on $[\alpha,\infty)$ if $X_1(t) = X_2(t)$ for all $t \in [\alpha,\infty)$.

Obviously, the relation $\sim$ is an equivalence on the set of all solutions of (H). Moreover, it is straightforward to see from the definition of $\Lambda_0[\alpha,\infty)$ in (3.1) that two solutions $(X_1,U_1)$ and $(X_2,U_2)$ of (H) are equivalent on $[\alpha,\infty)$ if and only if

$$X_2(\alpha) = X_1(\alpha) \quad \text{and} \quad \operatorname{Im}[U_2(\alpha) - U_1(\alpha)] \subseteq \Lambda_0[\alpha,\infty). \tag{3.17}$$

In the next theorem we establish a criterion for the equivalence of two such solutions under an additional condition, which is satisfied for nonoscillatory (and possibly abnormal) systems (H).

**Theorem 3.1.7.** *Assume that the Legendre condition* (1.1) *holds and there exists a conjoined basis* $(X,U)$ *of* (H) *with constant kernel on* $[\alpha,\infty)$*. Let* $P$ *and* $P_{\mathscr{S}_{\alpha\infty}}$ *be the orthogonal projectors defined in* (2.2) *and* (2.12)*. Then two solutions* $(X_1,U_1)$ *and* $(X_2,U_2)$ *of* (H) *are equivalent on* $[\alpha,\infty)$ *if and only if there exist unique* $n\times n$ *matrices* $G$ *and* $H$ *such that*

$$X_2(\alpha) = X_1(\alpha) \quad \textit{and} \quad U_2(\alpha) - U_1(\alpha) = X^{\dagger T}(\alpha)\, G + U(\alpha)\, H, \tag{3.18}$$

$$\operatorname{Im} G \subseteq \operatorname{Im}(P - P_{S_{\alpha\infty}}) \quad \textit{and} \quad \operatorname{Im} H \subseteq \operatorname{Im}(I - P). \tag{3.19}$$

*Proof.* The existence of matrices $G$ and $H$ satisfying (3.18) follows from (3.17) and from the representation of $\Lambda_0[\alpha,\infty)$ in (3.6) in Theorem 3.1.2. The latter reference also gives the second property in (3.19) and the inclusion $\operatorname{Im} G \subseteq \operatorname{Im}(I - P_{S_{\alpha\infty}})$. However, the matrix $G$ can be chosen so that $\operatorname{Im} G \subseteq \operatorname{Im} P$, because in (3.18) we may take $X^{\dagger T} X^T X^{\dagger T} G = X^{\dagger T} P G$ instead of $X^{\dagger T} G$, by the properties of the Moore–Penrose pseudoinverse. Equation (3.19) then also implies the uniqueness of $G$ and $H$, because $(X,U)$ is a conjoined basis. Conversely, the conditions in (3.18) and (3.19) imply that (3.17) holds, and thus $(X_1,U_1)$ and $(X_2,U_2)$ are equivalent. ■

**Remark 3.1.8.** Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. From Theorem 2.2.5 it follows that conjoined bases of (H), which satisfy conditions (1.16) and (2.25) for a given $\beta \geq \alpha$, are mutually equivalent on $[\alpha,\infty)$ in the terminology of Definition 3.1.6, see also Remark 2.2.10. The corresponding classification of all such conjoined bases displayed in Theorem 2.2.9 then can be viewed as a special case of the result in Theorem 3.1.7, where the matrix $G=0$ and the matrix $H$ is symmetric.

**Remark 3.1.9.** Note that the system (H) is completely controllable on $[\alpha,\infty)$ if and only if the equivalence of solutions of (H) implies the equality of solutions of (H) on every subinterval $\mathscr{I} \subseteq [\alpha,\infty)$.

Based on the result in Theorem 3.1.7 we provides a classification of all mutually equivalent conjoined bases of (H) with constant kernel on $[\alpha,\infty)$.

**Corollary 3.1.10.** *Assume* (1.1). *Let $(X,U)$ be a conjoined basis of* (H) *with constant kernel on $[\alpha,\infty)$ and let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be its corresponding matrices defined in* (2.2) *and* (2.12). *Moreover, let $(\bar{X}_\alpha,\bar{U}_\alpha)$ be the conjoined basis of* (H) *satisfying* (1.16) *and* (2.25). *Then a solution $(X_0,U_0)$ of* (H) *is a conjoined basis which is equivalent with $(X,U)$ on $[\alpha,\infty)$ if and only if the matrices $M,N \in \mathbb{R}^{n\times n}$ defined by*

$$\begin{pmatrix} X_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} X & \bar{X}_\alpha \\ U & \bar{U}_\alpha \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \quad \text{on } [a,\infty), \tag{3.20}$$

*satisfy the properties*

$$M \text{ is nonsingular}, \quad PM=P, \quad M^TN=N^TM, \quad \operatorname{Im} N \subseteq \operatorname{Im}(P-P_{\mathscr{S}_{\alpha\infty}}). \tag{3.21}$$

*Proof.* If $(X_0,U_0)$ is a conjoined basis of (H) such that $(X,U) \sim (X_0,U_0)$, then the equality $X(t)=X_0(t)$ holds for all $t\in[\alpha,\infty)$, by Definition 3.1.6. In particular, this means that $(X_0,U_0)$ has constant kernel on $[\alpha,\infty)$ and that $(X,U)$ and $(X_0,U_0)$ are mutually representable on $[\alpha,\infty)$, by Theorem 2.3.8. From the same reference we also have that $M$ is nonsingular and $M^TN$ is symmetric. Moreover, according to Theorem 3.1.7 with $(X_1,U_1) := (X,U)$ and $(X_2,U_2) := (X_0,U_0)$ we have

$$X_0(\alpha)=X(\alpha), \quad U_0(\alpha)-U(\alpha)=X^{\dagger T}(\alpha)\,G+U(\alpha)\,H, \tag{3.22}$$

with $G,H \in \mathbb{R}^{n\times n}$ satisfying $\operatorname{Im} G \subseteq \operatorname{Im}(P-P_{\mathscr{S}_{\alpha\infty}})$ and $\operatorname{Im} H \subseteq \operatorname{Im}(I-P)$. We show that the identities $M=I+H$ and $N=G$ hold. Indeed, by using (1.20) and (1.16) at $t=\alpha$ together with equalities (3.22) and $X^{\dagger}(\alpha)\bar{X}_\alpha(\alpha)=0$ we have that

$$\begin{aligned} M &= \bar{U}_\alpha^T(\alpha)X_0(\alpha)-\bar{X}_\alpha^T(\alpha)U_0(\alpha) \\ &\overset{(3.22)}{=} \bar{U}_\alpha^T(\alpha)X(\alpha)-\bar{X}_\alpha^T(\alpha)\,[U(\alpha)+X^{\dagger T}(\alpha)\,G+U(\alpha)\,H] \\ &= \bar{U}_\alpha^T(\alpha)X(\alpha)-\bar{X}_\alpha^T(\alpha)U(\alpha)-\bar{X}_\alpha^T(\alpha)U(\alpha)H \overset{(1.16)}{=} I-\bar{X}_\alpha^T(\alpha)U(\alpha)H \\ &\overset{(1.16)}{=} I+[I-\bar{U}_\alpha^T(\alpha)X(\alpha)]\,H = I+H-\bar{U}_\alpha^T(\alpha)X(\alpha)H = I+H, \end{aligned} \tag{3.23}$$

since the identities $X(\alpha) = X(\alpha)P$ and $PH = 0$ hold. The similar arguments together with symmetry of $X^TU$ and the equalities $X^TX^{\dagger T} = P$ and $PG = G$ yield

$$\begin{aligned} N &= X^T(\alpha)U_0(\alpha) - U^T(\alpha)X_0(\alpha) \\ &\overset{(3.22)}{=} X^T(\alpha)[U(\alpha) + X^{\dagger T}(\alpha)G + U(\alpha)H] - U^T(\alpha)X(\alpha) \\ &= X^T(\alpha)X^{\dagger T}(\alpha)G + X^T(\alpha)U(\alpha)H = PG + U^T(\alpha)X(\alpha)H = PG = G. \end{aligned} \tag{3.24}$$

Formulas (3.23) and (3.24) then prove the second and fourth condition in (3.21), because $PM = P(I+H) = P + PH = P$ and $\operatorname{Im} N = \operatorname{Im} G \subseteq \operatorname{Im}(P - P_{\mathscr{S}_{\alpha\infty}})$. Conversely, suppose that the matrices $M$ and$N$ satisfy (3.21). The first and third condition in (3.21) then imply that the solution $(X_0, U_0)$ given by (3.20) is a conjoined basis of (H), while the last condition in (3.21) yields that $(X_0, U_0)$ is representable by $(X, U)$ on $[\alpha, \infty)$, by Theorem 2.3.3. Therefore, the equalities

$$X_0(\alpha) = X(\alpha)M, \quad U_0(\alpha) = U(\alpha)M + X^{\dagger T}(\alpha)N \tag{3.25}$$

hold, by (2.52). By using the second equality in (3.21) the first equation in (3.25) becomes $X_0(\alpha) = X(\alpha)PM = X(\alpha)$. Moreover, the second formula in (3.25) yields $U_0(\alpha) - U(\alpha) = X^{\dagger T}(\alpha)N + U(\alpha)(M - I)$. Set $G := N$ and $H := M - I$. Then $U_0(\alpha) - U(\alpha) = X^{\dagger T}(\alpha)G + U(\alpha)H$ and the properties of $M$ and $N$ in (3.21) imply that the matrices $G$ and $H$ satisfies inclusions (3.19), since $\operatorname{Im} G = \operatorname{Im} N \subseteq \operatorname{Im}(P - P_{\mathscr{S}_{\alpha\infty}})$ and $PH = PM - P = 0$. Thus, the conjoined basis $(X_0, U_0)$ is equivalent with $(X, U)$ on $[\alpha, \infty)$, by Theorem 3.1.7, and the proof is complete. ∎

## 3.2 Relation "being contained" for conjoined bases

In this section we develop tools for the construction of conjoined bases of (H) with constant kernel on a given interval $[\alpha, \infty)$. In particular, the main results of this section (Theorems 3.2.5, 3.2.7, 3.2.8, 3.2.10, and 3.2.11) are closely related with the relation "being contained" for conjoined bases of (H) with constant kernel on $[\alpha, \infty)$ presented in Definition 3.2.1. These results play a crucial role in the theory of principal and antiprincipal solutions of (H) at infinity discussed in Chapter 5. In the following definition we introduce one of the central notions of this chapter.

**Definition 3.2.1.** Let $(X, U)$ and $(X_*, U_*)$ be two conjoined bases of (H) such that $(X, U)$ has constant kernel on $[\alpha, \infty)$. Let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be the associated orthogonal projectors for $(X, U)$ defined in (2.2) and (2.12). We say that $(X_*, U_*)$ is *contained in* $(X, U)$ on $[\alpha, \infty)$ (or that $(X, U)$ *contains* the conjoined basis $(X_*, U_*)$ on $[\alpha, \infty)$) if there exists an orthogonal projector $P_*$ satisfying

$$\operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P \tag{3.26}$$

such that $(X_*, U_*) \sim (XP_*, UP_*)$ on $[\alpha, \infty)$. In this case we also say that the conjoined basis $(X_*, U_*)$ is contained in $(X, U)$ on $[\alpha, \infty)$ with respect to the projector $P_*$.

In the following theorem and remark we state a basic property of the relation "being contained" for conjoined bases of (H) with constant kernel from Definition 3.2.1.

**Theorem 3.2.2.** *Assume* (1.1)*. Let* $(X,U)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *and let* $P$ *and* $P_{\mathscr{S}_{\alpha\infty}}$ *be the matrices corresponding to* $(X,U)$ *in* (2.2) *and* (2.12)*. If* $(X_*,U_*)$ *is a conjoined basis of* (H) *which is contained in* $(X,U)$ *on* $[\alpha,\infty)$ *with respect to the orthogonal projector* $P_*$ *satisfying* (3.26)*, then*

$$X_*(t)=X(t)P_*,\quad \operatorname{Ker}X_*(t)=\operatorname{Ker}P_*,\quad X_*^{\dagger}(t)=X^{\dagger}(t)R_*(t)\quad \text{on } [\alpha,\infty), \tag{3.27}$$

*where* $R_*(t)$ *is the orthogonal projector associated with* $(X_*,U_*)$ *in* (2.1)*.*

*Proof.* Let $(X,U)$ and $(X_*,U_*)$ be as in the theorem. If $(X_*,U_*)$ is contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$, then we immediately have $X_*(t)=X(t)P_*$ for all $t\in[\alpha,\infty)$, by Definition 3.2.1. Moreover, by using the identities $X^{\dagger}(t)X(t)=P$ on $[\alpha,\infty)$ and $PP_*=P_*$ we obtain that

$$\operatorname{Ker}X_*(t)=\operatorname{Ker}X(t)P_*=\operatorname{Ker}X^{\dagger}(t)X(t)P_*=\operatorname{Ker}PP_*=\operatorname{Ker}P_*$$

for $t\in[\alpha,\infty)$. In particular, the last formula shows that $(X_*,U_*)$ has constant kernel on $[\alpha,\infty)$ with $P_*$ being its corresponding orthogonal projector in (2.2). Hence, the equality $X_*^{\dagger}(t)=P_*X_*^{\dagger}(t)$ holds on $[\alpha,\infty)$. Finally, the last formula in (3.27) is established by the calculation

$$X_*^{\dagger}=P_*X_*^{\dagger}=PP_*X_*^{\dagger}=X^{\dagger}XP_*X_*^{\dagger}=X^{\dagger}X_*X_*^{\dagger}=X^{\dagger}R_*\quad \text{on } [\alpha,\infty). \tag{3.28}$$

Note that in (3.28) we use only the fact that $(X,U)$ and $(X_*,U_*)$ have constant kernel on $[\alpha,\infty)$, the inclusion $\operatorname{Im}P_*\subseteq\operatorname{Im}P$, and the formula $XP_*=X_*$ on $[\alpha,\infty)$. ■

**Remark 3.2.3.** Formulas (3.27) in Theorem 3.2.2 imply that any conjoined basis $(X_*,U_*)$, which is contained in a given conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$, has constant kernel on $[\alpha,\infty)$ too. Moreover, we have the inclusions

$$\operatorname{Im}X_*(t)\subseteq\operatorname{Im}X(t),\quad \operatorname{Im}X_*^{T}(t)\subseteq\operatorname{Im}X^{T}(t),\quad t\in[\alpha,\infty).$$

These results reveal two significant interpretations of Definition 3.2.1. On the one hand, the relation "being contained" introduced therein induces an ordering on the set of all conjoined bases of (H) with constant kernel on $[\alpha,\infty)$. This topic will be discussed in more detailed and more general way in Chapter 6. On the other hand, Definition 3.2.1 provides a tool for a construction of new conjoined bases $(X_*,U_*)$ of (H) with constant kernel on $[\alpha,\infty)$ from a given conjoined basis $(X,U)$ of the same type with the aid of the orthogonal projectors $P_*$ satisfying (3.26). In addition, all such conjoined bases $(X_*,U_*)$ are representable by $(X,U)$ on $[\alpha,\infty)$, by Theorem 2.3.3.

In the next theorem we show that the relation "being contained" for conjoined bases of (H) preserves the corresponding $S$-matrices. This is one of the main properties needed for the construction of (anti)principal solutions of (H) at infinity in Chapter 5.

**Theorem 3.2.4.** *Assume* (1.1)*. Let* $(X,U)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *and let* $S_\beta(t)$ *be its corresponding matrix defined in* (2.8) *for a given* $\beta\geq\alpha$*. If* $(X_*,U_*)$ *is any conjoined basis of* (H) *which is contained in* $(X,U)$ *on* $[\alpha,\infty)$ *and if* $S_{*\beta}(t)$ *is its corresponding* $S$*-matrix, then* $S_{*\beta}(t)=S_\beta(t)$ *for all* $t\in[\alpha,\infty)$*.*

*Proof.* Suppose that $(X_*, U_*)$ is contained in $(X, U)$ on $[\alpha, \infty)$ and let $R_*(t)$ be the orthogonal projector onto $\operatorname{Im} X_*(t)$ in (2.1). Then $(X_*, U_*)$ has constant kernel on $[\alpha, \infty)$, by (3.27). Moreover, by using (2.8), the last formula in (3.27), and Theorem 2.1.2(ii), we get that for any given $\beta \geq \alpha$ the equality

$$S_{*\beta}(t) = \int_\beta^t X_*^\dagger(s)\, B(s)\, X_*^{\dagger T}(s)\, \mathrm{d}s \overset{(3.27)}{=} \int_\beta^t X^\dagger(s)\, R_*(s)\, B(s)\, R_*(s)\, X^{\dagger T}(s)\, \mathrm{d}s$$
$$= \int_\beta^t X^\dagger(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s = S_\beta(t)$$

holds on $[\alpha, \infty)$. The proof is complete. ■

The next result classifies all the conjoined bases of (H) which are contained in a given conjoined basis $(X, U)$ with constant kernel on $[\alpha, \infty)$ with respect to the fixed orthogonal projector $P_*$ in (3.26). Here we use the set $\mathscr{B}(P_{**}, P_*, P)$ from (1.11).

**Theorem 3.2.5.** *Assume (1.1). Let $(X, U)$ be a conjoined basis of (H) with constant kernel on $[\alpha, \infty)$ and let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be defined in (2.2) and (2.12). Consider an orthogonal projector $P_*$ satisfying (3.26). Then a solution $(X_*, U_*)$ of (H) is a conjoined basis which is contained in $(X, U)$ on $[\alpha, \infty)$ with respect to $P_*$ if and only if*

$$X_*(\alpha) = X(\alpha)\, P_* \quad \text{and} \quad U_*(\alpha) = U(\alpha)\, P_* + X^{\dagger T}(\alpha)\, G + U(\alpha)\, H \tag{3.29}$$

*for some $(G, H) \in \mathscr{B}(P_{\mathscr{S}_{\alpha\infty}}, P_*, P)$, where the set $\mathscr{B}(P_{\mathscr{S}_{\alpha\infty}}, P_*, P)$ is defined in (1.11).*

*Proof.* Let $(X_*, U_*)$ be a conjoined basis of (H) which is contained in $(X, U)$ on $[\alpha, \infty)$ with respect to $P_*$. Then $(X, U) \sim (XP_*, UP_*)$ on $[\alpha, \infty)$, by Definition 3.2.1. From Theorem 3.1.7 (with $(X_1, U_1) := (XP_*, UP_*)$ and $(X_2, U_2) := (X_*, U_*)$) it follows that $(X_*, U_*)$ satisfies the initial conditions in (3.29) with the matrices $G$ and $H$ such that $P_{\mathscr{S}_{\alpha\infty}} G = 0$, $PG = G$, and $PH = 0$. We will show that $(G, H) \in \mathscr{B}(P_{\mathscr{S}_{\alpha\infty}}, P_*, P)$. Multiplying (3.29) by $X_*^T(\alpha)$ we get

$$X_*^T(\alpha)\, U_*(\alpha) = P_* X^T(\alpha)\, U(\alpha)\, P_* + P_* X^T(\alpha)\, X^{\dagger T}(\alpha)\, G + P_* X^T(\alpha)\, U(\alpha)\, H. \tag{3.30}$$

The symmetry of $X^T(\alpha)\, U(\alpha)$ and the identities $X(\alpha)\, P = X(\alpha)$ and $PH = 0$ yield

$$P_* X^T(\alpha)\, U(\alpha)\, H = P_* U^T(\alpha)\, X(\alpha)\, H = P_* U^T(\alpha)\, X(\alpha)\, PH = 0. \tag{3.31}$$

Inserting (3.31) into (3.30) and using $X^T(\alpha)\, X^{\dagger T}(\alpha) = P$ and $PG = G$ we obtain

$$P_* G = P_* PG = P_* X^T(\alpha)\, X^{\dagger T}(\alpha)\, G = X_*^T(\alpha)\, U_*(\alpha) - P_* X^T(\alpha)\, U(\alpha)\, P_*.$$

This shows that the matrix $P_* G$ is symmetric, i.e., $P_* G = G^T P_*$. Furthermore, if $v \in \mathbb{R}^n$ such that $v \in \operatorname{Ker} G \cap \operatorname{Ker} H \cap \operatorname{Ker} P_*$, then (3.29) implies that $v \in \operatorname{Ker} X_*(\alpha) \cap \operatorname{Ker} U_*(\alpha) = \{0\}$, because $(X_*, U_*)$ is a conjoined basis. Therefore, $\operatorname{Ker} G \cap \operatorname{Ker} H \cap \operatorname{Ker} P_* = \{0\}$, which is equivalent with $\operatorname{rank}(G^T, H^T, P_*) = n$. The above properties of $G$ and $H$ imply that $(G, H) \in \mathscr{B}(P_{\mathscr{S}_{\alpha\infty}}, P_*, P)$. Conversely, it is easy to see that for any pair $(G, H) \in \mathscr{B}(P_{\mathscr{S}_{\alpha\infty}}, P_*, P)$ the solution $(X_*, U_*)$ of (H) satisfying the initial conditions in (3.29) is a conjoined basis, which is contained in $(X, U)$ on $[\alpha, \infty)$ with respect to $P_*$. ■

**Remark 3.2.6.** (i) It follows from Theorem 3.2.5 that there exists a one-to-one correspondence between the set of all conjoined bases of (H) which are contained in $(X,U)$ on $[\alpha,\infty)$ with respect to a fixed $P_*$ and the set $\mathscr{B}(P_{\mathscr{S}_{\alpha\infty}},P_*,P)$, because the matrices $G$ and $H$ in (3.29) are uniquely determined, by Theorem 3.1.7. We thus adopt the terminology that the conjoined basis $(X_*,U_*)$ is contained in $(X,U)$, or $(X,U)$ contains $(X_*,U_*)$, *through a pair* $(G,H)$ if the associated projector $P_*$ satisfies (3.26), the pair $(G,H)$ belongs to $\mathscr{B}(P_{\mathscr{S}_{\alpha\infty}},P_*,P)$, and (3.29) holds.

(ii) The matrix $G$ in (3.29) is in fact equal to the Wronskian of the conjoined bases $(X,U)$ and $(X_*,U_*)$, that is, $G = X^T(t)\,U_*(t) - U^T(t)\,X_*(t)$ on $[\alpha,\infty)$. Indeed, the Wronskian is constant on $[\alpha,\infty)$ and the latter equality is satisfied at $t=\alpha$, which follows from (3.29) and the properties of the matrices $G$ and $H$ in (1.11).

The following theorem guarantees the existence of a conjoined basis of (H), which is contained in a given conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$ with respect to any orthogonal projector $P_*$ satisfying (3.26).

**Theorem 3.2.7.** *Assume* (1.1) *and let* $(X,U)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$. *Furthermore, let* $P$ *and* $P_{\mathscr{S}_{\alpha\infty}}$ *be the orthogonal projectors which correspond to* $(X,U)$ *in* (2.2) *and* (2.12). *Then for any orthogonal projector* $P_*$ *satisfying* (3.26) *there exists a conjoined basis* $(X_*,U_*)$ *of* (H) *which is contained in* $(X,U)$ *on* $[\alpha,\infty)$ *with respect to* $P_*$.

*Proof.* Let $(X,U)$, $P$, $P_{\mathscr{S}_{\alpha\infty}}$, and $P_*$ be as in theorem. By Remark 1.2.2, the pair $(P-P_*,I-P)$ is an element of the set $\mathscr{B}(P_{\mathscr{S}_{\alpha\infty}},P_*,P)$. According to Remark 3.2.6(i), there then exists a conjoined basis of (H) which is contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$. Moreover, by Theorem 3.2.5 with $(G,H) := (P-P_*,I-P)$, the solution $(X_*,U_*)$ of (H) with the initial conditions

$$X_*(\alpha) = X(\alpha)\,P_*, \quad U_*(\alpha) = U(\alpha)\,P_* + X^{\dagger T}(\alpha)\,(P-P_*) + U(\alpha)\,(I-P)$$

is an example of such a conjoined basis of (H). ∎

In the next theorem we provide a construction of all conjoined bases of (H) with constant kernel on $[\alpha,\infty)$, which contain a given conjoined basis $(X_*,U_*)$ on $[\alpha,\infty)$. The corresponding result is formulated in terms of the solvability of the following system of five algebraic matrix equations

$$X_\alpha X_\alpha^\dagger = R_\alpha, \quad X_\alpha^\dagger X_\alpha = P_\alpha, \tag{3.32}$$

$$X_\alpha P_* = X_*(\alpha), \quad X_\alpha^T U_\alpha = U_\alpha^T X_\alpha, \quad X_\alpha^{\dagger T} G + U_\alpha\,(P_* + H) = U_*(\alpha) \tag{3.33}$$

for unknown matrices $X_\alpha, U_\alpha \in \mathbb{R}^{n\times n}$ (see also Remark 3.2.9 below). Here $P_\alpha$ and $R_\alpha$ are given orthogonal projectors with

$$\operatorname{Im} P_* \subseteq \operatorname{Im} P_\alpha, \quad \operatorname{Im} R_*(\alpha) \subseteq \operatorname{Im} R_\alpha, \quad \operatorname{rank} P_\alpha = \operatorname{rank} R_\alpha, \tag{3.34}$$

and the pair $(G,H) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ belongs to the set $\mathscr{B}(P_{\mathscr{S}_{*\infty}},P_*,P_\alpha)$.

**Theorem 3.2.8.** *Assume (1.1). Let $(X_*,U_*)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $P_*$, $R_*(t)$, and $P_{\mathscr{S}_{*\alpha\infty}}$ be its corresponding orthogonal projectors in (2.2), (2.1), and (2.12). Then a solution $(X,U)$ of (H) is a conjoined basis with constant kernel on $[\alpha,\infty)$ which contains $(X_*,U_*)$ on $[\alpha,\infty)$ if and only if the matrices $X(\alpha)$ and $U(\alpha)$ solve system (3.32)–(3.33) for some orthogonal projectors $P_\alpha$ and $R_\alpha$ satisfying (3.34) and a pair $(G,H)\in\mathscr{B}(P_{\mathscr{S}_{*\infty}},P_*,P_\alpha)$. In this case $(X,U)$ contains $(X_*,U_*)$ through the pair $(G,H)$ and the orthogonal projectors $P$ and $R(t)$ associated with $(X,U)$ in (2.2) and (2.1) satisfy $P=P_\alpha$ and $R(\alpha)=R_\alpha$.*

*Proof.* Let $(X_*,U_*)$ be as in the theorem. Assume that $(X,U)$ is a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ which contains $(X_*,U_*)$ on $[\alpha,\infty)$. Moreover, let $P$, $R(t)$, and $P_{\mathscr{S}_{\alpha\infty}}$ be the matrices in (2.2), (2.1), and (2.12) and set $P_\alpha:=P$ and $R_\alpha:=R(\alpha)$. The matrices $P_\alpha$ and $R_\alpha$ then satisfy conditions (3.34), by (2.3), (3.26) and Remark 3.2.3. Furthermore, from (3.29) in Theorem 3.2.5 and from the symmetry of $X^TU$ we obtain that the matrices $X(\alpha)$ and $U(\alpha)$ solve system (3.32)–(3.33) for some $(G,H)\in\mathscr{B}(P_{\mathscr{S}_{\alpha\infty}},P_*,P_\alpha)$. Consequently, by Theorem 3.2.4 the equality $P_{\mathscr{S}_{*\alpha\infty}}=P_{\mathscr{S}_{\alpha\infty}}$ holds, which gives that $(G,H)\in\mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}},P_*,P_\alpha)$. Conversely, let $P_\alpha$, $R_\alpha$, $G$, and $H$ be matrices satisfying (3.34) and $(G,H)\in\mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}},P_*,P_\alpha)$. Moreover, as in Theorem A.2.7 and (A.20), we set $\bar G:=G+H$ and $\bar G_\perp:=\bar G\,(I-P_*)$. Let $X_\alpha$ and $U_\alpha$ solve system (3.32)–(3.33) and consider the solution $(X,U)$ of (H) given by the initial conditions $X(\alpha)=X_\alpha$ and $U(\alpha)=U_\alpha$. In order to show that $(X,U)$ is a conjoined basis it suffices to check the two defining properties of a conjoined basis only at $t=\alpha$. As the symmetry of $X_\alpha^TU_\alpha$ is guaranteed by the second equation in (3.33), it remains to show that $\operatorname{rank}(X_\alpha^T,U_\alpha^T)=n$, or equivalently that $\operatorname{Ker}X_\alpha\cap\operatorname{Ker}U_\alpha=\{0\}$. If $v\in\mathbb{R}^n$ is such that $X_\alpha v=0$ and $U_\alpha v=0$, then $P_\alpha v=0$ and also $P_*v=0$, by the first inclusion in (3.34). Therefore, the vector $w:=\bar G_\perp^{\dagger}v$ satisfies $w\in\operatorname{Im}\bar G_\perp^{\dagger}=\operatorname{Im}(I-P_*)$, by (A.21), and $\bar G_\perp w=\bar G_\perp\bar G_\perp^{\dagger}v=(I-P_*)\,v=v$, by (A.22). Using the third equation in (3.33) and the identities $w=(I-P_*)\,w$ and $G=P_\alpha\bar G$ from (A.25) we get that

$$
\begin{aligned}
U_*(\alpha)\,w &= X_\alpha^{\dagger T}Gw+U_\alpha(P_*+H)\,w \overset{(A.25)}{=} X_\alpha^{\dagger T}P_\alpha\bar Gw+U_\alpha(P_*+H)\,w\\
&= X_\alpha^{\dagger T}P_\alpha\bar G\,(I-P_*)\,w+U_\alpha Hw = X_\alpha^{\dagger T}P_\alpha\bar G_\perp w+U_\alpha Hw\\
&= X_\alpha^{\dagger T}P_\alpha v+U_\alpha H\bar G_\perp^{\dagger}v = U_\alpha H\bar G_\perp^{\dagger}v=0,
\end{aligned}
\tag{3.35}
$$

because by (A.26) the last term becomes $U_\alpha H\bar G_\perp^{\dagger}v=U_\alpha(I-P_\alpha)\,v=U_\alpha v=0$. And since $X_*(\alpha)\,w=X_*(\alpha)\,P_*w=0$, the equality in (3.35) yields that $w\in\operatorname{Ker}X_*(\alpha)\cap\operatorname{Ker}U_*(\alpha)=\{0\}$. Thus, $v=\bar G_\perp w=0$, showing that $(X,U)$ is a conjoined basis. Further we note that the matrix $-G^T$ is equal to the Wronskian of $(X_*,U_*)$ and $(X,U)$, i.e., $-G^T=X_*^T(t)\,U(t)-U_*^T(t)\,X(t)$ on $[\alpha,\infty)$. Indeed, the Wronskian is constant on $[\alpha,\infty)$ and the latter equality is satisfied at $t=\alpha$, which follows from (3.32) and (3.33) and the properties of $G$ and $H$ in (1.11). Using this observation and formula (2.22) with $(X,U):=(X_*,U_*)$, $(X_0,U_0):=(X,U)$, and $W:=-G^T$ we obtain

$$
X(t)=\Phi_*(t,\alpha)\left[X_\alpha-X_*(\alpha)\,S_{*\alpha}(t)\,G^T\right]\quad\text{on }[\alpha,\infty),
\tag{3.36}
$$

where $S_{*\alpha}(t)$ is the $S$-matrix associated with $(X_*,U_*)$ and $\Phi_*(t,\alpha)$ is the fundamental matrix of the equation $Y'=(A+BQ_*)\,Y$ with $Q_*(t)$ defined in (2.5) by $(X_*,U_*)$ such that

$\Phi_*(\alpha,\alpha)=I$. Moreover, by using $X_*(\alpha)=X_\alpha P_*$ and $P_* S_{*\alpha}(t)=S_{*\alpha}(t)$ for $t\in[\alpha,\infty)$ the formula in (3.36) reads

$$X(t)=\Phi_*(t,\alpha)\left[X_\alpha - X(\alpha)P_* S_{*\alpha}(t)G^T\right]=\Phi_*(t,\alpha)X_\alpha\left[I-S_{*\alpha}(t)G^T\right] \tag{3.37}$$

on $[\alpha,\infty)$. By using (3.37), we now show that the matrix $X(t)$ has constant kernel on $[\alpha,\infty)$ and that $\operatorname{Ker}X(t)=\operatorname{Ker}P_\alpha$ on this interval. Fix $t\in[\alpha,\infty)$. The inclusion $\operatorname{Ker}P_\alpha\subseteq\operatorname{Ker}X(t)$ holds trivially, because $G^T P_\alpha=G^T$ by (1.11). Conversely, assume that $v\in\operatorname{Ker}X(t)$. Then equality (3.37) implies that $X_\alpha[P_\alpha - S_{*\alpha}(t)G^T]v=0$. Multiplying this equation by $X_\alpha^\dagger$ and using the inclusions $\operatorname{Im}S_{*\alpha}(t)\subseteq\operatorname{Im}P_*\subseteq P_\alpha$, we get $[P_\alpha - S_{*\alpha}(t)G^T]v=0$. This means that $P_\alpha v=S_{*\alpha}(t)G^T v=S_{*\alpha}(t)G^T P_\alpha v$. The vector $w:=P_\alpha v$ then satisfies $w=S_{*\alpha}(t)G^T w$, which shows that $w\in\operatorname{Im}S_{*\alpha}(t)\subseteq\operatorname{Im}P_{\mathscr{S}_{*\alpha\infty}}$, by (2.13). Thus, $P_{\mathscr{S}_{*\alpha\infty}}w=w$ and so

$$w=S_{*\alpha}(t)G^T P_{\mathscr{S}_{*\alpha\infty}}w=S_{*\alpha}(t)\,(P_{\mathscr{S}_{*\alpha\infty}}G)^T w=0,$$

since $P_{\mathscr{S}_{*\alpha\infty}}G=0$ by the second condition in (1.11). Therefore, we have $P_\alpha v=w=0$, which shows that $v\in\operatorname{Ker}P_\alpha$. In addition, the orthogonal projector $P$ in (2.2) in this case satisfies $\operatorname{Im}P=\operatorname{Im}X^T(t)=\operatorname{Im}P_\alpha$ on $[\alpha,\infty)$, which yields that $P=P_\alpha$. Similarly, for the orthogonal projector $R(t)$ in (2.1) we have $R(\alpha)=R_\alpha$, by the second equation in (3.32). In the remaining part of the proof we show that the conjoined basis $(X,U)$ contains $(X_*,U_*)$ through the pair $(G,H)$. Let $S_\alpha(t)$ be the $S$-matrix which corresponds to $(X,U)$. By Remark 3.2.6(i) and Theorem 3.2.5, it suffices to prove that the orthogonal projectors $P_{\mathscr{S}_{*\alpha\infty}}$ and $P_{\mathscr{S}_{\alpha\infty}}$ defined in (2.12) through the matrices $S_{*\alpha}(t)$ and $S_\alpha(t)$ satisfy $P_{\mathscr{S}_{*\alpha\infty}}=P_{\mathscr{S}_{\alpha\infty}}$. We will show a stronger statement that $S_{*\alpha}(t)=S_\alpha(t)$ on $[\alpha,\infty)$. First we establish the equalities

$$X(t)P_*=X_*(t),\qquad X^\dagger(t)R_*(t)=X_*^\dagger(t) \tag{3.38}$$

on $[\alpha,\infty)$. The first equality in (3.38) follows from formula (3.37) by using the properties of the matrix $G$ in (1.11). Namely, the equality $P_\alpha P_*=P_*$, the symmetry of $G^T P_*$, and the identity $S_{*\alpha}(t)P_*=S_{*\alpha}(t)$ on $[\alpha,\infty)$ imply that

$$X(t)P_*=\Phi_*(t,\alpha)X_\alpha[P_*-S_{*\alpha}(t)G^T P_*]=\Phi_*(t,\alpha)X_\alpha[P_*-S_{*\alpha}(t)G]. \tag{3.39}$$

Since $S_{*\alpha}(t)G=S_{*\alpha}(t)P_{\mathscr{S}_{*\alpha\infty}}G=0$, equation (3.39) and the first condition in (3.33) yield $X(t)P_*=\Phi_*(t,\alpha)X_\alpha P_*=\Phi_*(t,\alpha)X_*(\alpha)=X_*(t)$ on $[\alpha,\infty)$, by the first part of Remark 2.2.2 with $(X,U):=(X_*,U_*)$ and $\Phi(t,\alpha):=\Phi_*(t,\alpha)$. The second formula in (3.38) then follows from the first one and from $P=P_\alpha$, see formula (3.28) and the comment at the end of the proof of Theorem 3.2.2. Finally, by exactly the same calculation as in the proof of Theorem 3.2.4 we obtain that $S_{*\alpha}(t)=S_\alpha(t)$ for all $t\in[\alpha,\infty)$. Therefore, $P_{\mathscr{S}_{\alpha\infty}}=P_{\mathscr{S}_{*\alpha\infty}}$ and $\operatorname{Im}P_{\mathscr{S}_{\alpha\infty}}=\operatorname{Im}P_{\mathscr{S}_{*\alpha\infty}}\subseteq\operatorname{Im}P_*\subseteq\operatorname{Im}P_\alpha=\operatorname{Im}P$. This shows that $(G,H)\in\mathscr{B}(P_{\mathscr{S}_{\alpha\infty}},P_*,P)$. Moreover, the first and third equations in (3.33) mean that formulas (3.29) hold. By Theorem 3.2.5 and Remark 3.2.6(i) we may conclude that $(X_*,U_*)$ is contained in $(X,U)$ through the pair $(G,H)$. ∎

**Remark 3.2.9.** The result in Theorem 3.2.9 shows that the existence and the construction of all conjoined bases $(X,U)$ with constant kernel on $[\alpha,\infty)$, which contain a given conjoined basis $(X_*,U_*)$ of the same type on $[\alpha,\infty)$, is completely characterized by the set of all solutions of the algebraic system (3.32)–(3.33). In Chapter 6 we will provide a detailed analysis of the solvability of this system.

In the following theorem we establish the existence of a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$, which contains a given conjoined basis $(X_*,U_*)$ of the same type. Moreover, we show that $(X,U)$ can be chosen with an additional property on the sets $\operatorname{Im} X^T(\alpha)$ and $\operatorname{Im} X(\alpha)$. For the proof we refer to Section 6.2 in Chapter 6.

**Theorem 3.2.10.** *Assume* (1.1) *and let* $(X_*,U_*)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$*. Furthermore, let* $P_*$*,* $R_*(t)$*, and* $P_{\mathscr{S}_{*\alpha\infty}}$ *be the matrices which correspond to* $(X_*,U_*)$ *in* (2.2)*,* (2.1)*, and* (2.12)*. Then for any orthogonal projectors* $P_\alpha$ *and* $R_\alpha$ *satisfying* (3.34) *there exists a conjoined basis* $(X,U)$ *of* (H) *with constant kernel on* $[\alpha,\infty)$*, which contains* $(X_*,U_*)$ *on* $[\alpha,\infty)$ *such that the associated matrices* $P$ *and* $R(t)$ *defined in* (2.2) *and* (2.1) *satisfy* $P=P_\alpha$ *and* $R(\alpha)=R_\alpha$*.*

Combining the results in Theorems 3.2.7 and 3.2.10 with estimate (3.15) yields a construction of conjoined bases $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$, which have the rank of $X(t)$ equal to any integer value between $n-d[\alpha,\infty)$ and $n$. This construction is based on the suitable choice of the orthogonal projectors $P_*$ and $P_\alpha$ in (3.26) and (3.34).

**Theorem 3.2.11.** *Assume* (1.1) *and let* $(X_*,U_*)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$*. Then for any integer value* $r$ *between* $n-d[\alpha,\infty)$ *and* $n$ *there exists a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *and with rank of its first component equal to* $r$*. Moreover, such a conjoined basis can be chosen so that it is contained in* $(X_*,U_*)$ *on* $[\alpha,\infty)$ *when* $r\le r_*$ *or it contains* $(X_*,U_*)$ *on* $[\alpha,\infty)$ *when* $r\ge r_*$*, where* $r_*:=\operatorname{rank} X_*(t)$ *on* $[\alpha,\infty)$*.*

*Proof.* Let $P_*$, $R_*(t)$, and $P_{\mathscr{S}_{*\alpha\infty}}$ be the associated orthogonal projectors from (2.2), (2.1), and (2.12). If $r\le\operatorname{rank} P_*$, then we choose an orthogonal projector $P_{**}$ satisfying (3.26) (with $P:=P_*$ and $P_*:=P_{**}$) and $\operatorname{rank} P_{**}=r$. From Theorems 3.2.7 and 3.2.2 with $(X,U):=(X_*,U_*)$ it then follows that there exists a conjoined basis $(X_{**},U_{**})$ of (H) such that $(X_{**},U_{**})$ is contained in $(X_*,U_*)$ and $\operatorname{Ker} X_{**}(t)=\operatorname{Ker} P_{**}$ for $t\in[\alpha,\infty)$. Hence, $(X_{**},U_{**})$ has constant kernel on $[\alpha,\infty)$ and $\operatorname{rank} X_{**}(t)=\operatorname{rank} P_{**}=r$ on $[\alpha,\infty)$. Similarly, for $r\ge\operatorname{rank} P_*$ we choose orthogonal projectors $P_\alpha$ and $R_\alpha$ such that conditions (3.34) hold and $\operatorname{rank} P_\alpha=r$. According to Theorem 3.2.10 there exists a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$, which contains $(X_*,U_*)$ on $[\alpha,\infty)$ such that the equality $\operatorname{Ker} X(t)=\operatorname{Ker} P_\alpha$ holds for all $t\in[\alpha,\infty)$. Thus, $\operatorname{rank} X(t)=\operatorname{rank} P_\alpha=r$ on $[\alpha,\infty)$ and the proof is complete. ■

The following result shows that the relation "being contained" for conjoined bases of (H) with constant kernel on $[\alpha,\infty)$ is invariant under suitable change of the interval $[\alpha,\infty)$. Namely, the point $\alpha$ can always be moved forward, and under some additional conditions also backward. Here we use the maximal order of abnormality $d_\infty$ defined in (3.2). We recall from Section 3.1 that $d[t,\infty)$ is the order of abnormality of system (H) on the interval $[t,\infty)$ and that by [30, Section 3] the integer-valued function $d[t,\infty)$ is nondecreasing, piecewise constant, and right-continuous on $[a,\infty)$.

**Theorem 3.2.12.** *Assume* (1.1)*. Let* $(X,U)$ *and* $(X_*,U_*)$ *be two conjoined bases of* (H) *with constant kernel on* $[\alpha,\infty)$*. Then the following statements hold.*

(i) *If* $(X,U)$ *contains* $(X_*,U_*)$ *on* $[\alpha,\infty)$*, then* $(X,U)$ *contains* $(X_*,U_*)$ *also on* $[\beta,\infty)$ *for all* $\beta\ge\alpha$*.*

(ii) *Assume* (3.2)*, i.e., that* $d[\alpha,\infty) = d_\infty$*. If* $(X,U)$ *contains* $(X_*,U_*)$ *on* $[\beta,\infty)$ *for some* $\beta \ge \alpha$*, then* $(X,U)$ *contains* $(X_*,U_*)$ *also on* $[\alpha,\infty)$*, and hence on* $[\gamma,\infty)$ *for every* $\gamma \ge \alpha$*.*

*Proof.* Fix $\beta \in [\alpha,\infty)$. We denote by $S_\alpha(t)$, $S_\beta(t)$, resp. $S_{*\alpha}(t)$, $S_{*\beta}(t)$, the $S$-matrices corresponding to $(X,U)$, resp. $(X_*,U_*)$. Let $P$ and $P_*$ be the orthogonal projectors in (2.2) defined by the functions $X(t)$ and $X_*(t)$. Moreover, let $P_{\mathscr{S}_{\alpha}\infty}$, $P_{\mathscr{S}_{\beta}\infty}$, $P_{\mathscr{S}_{*\alpha}\infty}$, $P_{\mathscr{S}_{*\beta}\infty}$ be the orthogonal projectors associated with the matrices $S_\alpha(t)$, $S_\beta(t)$, $S_{*\alpha}(t)$, $S_{*\beta}(t)$ through (2.12). From (2.14) we then have

$$\operatorname{Im} P_{\mathscr{S}_{\beta}\infty} \subseteq \operatorname{Im} P_{\mathscr{S}_{\alpha}\infty}, \quad \operatorname{Im} P_{\mathscr{S}_{*\beta}\infty} \subseteq \operatorname{Im} P_{\mathscr{S}_{*\alpha}\infty}. \tag{3.40}$$

For part (i) we suppose that $(X,U)$ contains $(X_*,U_*)$ on $[\alpha,\infty)$, that is, (3.26) holds and $(X_*,U_*) \sim (XP_*,UP_*)$ on $[\alpha,\infty)$ by Definition 3.2.1. Then $\operatorname{Im} P_{\mathscr{S}_{\beta}\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P$ as well, by the first inclusion in (3.40), and $(X_*,U_*) \sim (XP_*,UP_*)$ on $[\beta,\infty)$, by the definition of $\sim$ in Definition 3.1.6. Therefore, $(X,U)$ contains $(X_*,U_*)$ also on $[\beta,\infty)$, by Definition 3.2.1. For the proof of part (ii) we assume that $d[\alpha,\infty) = d_\infty$. Then $d[\alpha,\infty) = d[\beta,\infty)$ and by (3.14) and (3.40),

$$P_{\mathscr{S}_{\beta}\infty} = P_{\mathscr{S}_{\alpha}\infty}, \quad P_{\mathscr{S}_{*\beta}\infty} = P_{\mathscr{S}_{*\alpha}\infty}. \tag{3.41}$$

Now suppose that $(X,U)$ contains $(X_*,U_*)$ on $[\beta,\infty)$. Moreover, let $(X_{**},U_{**})$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ such that $(X,U)$ contains $(X_{**},U_{**})$ on $[\alpha,\infty)$ with respect to the projector $P_*$. Such a conjoined basis always exists, by Theorem 3.2.7. According to part (i) of this theorem, $(X,U)$ contains $(X_{**},U_{**})$ also on $[\beta,\infty)$ with respect to $P_*$ and hence, $(X_*,U_*) \sim (X_{**},U_{**})$ on $[\beta,\infty)$, by the transitivity of $\sim$. This means that $X_*(t) = X_{**}(t)$ on $[\beta,\infty)$. We will show that the assumption $d[\alpha,\infty) = d_\infty$ allows to extend the latter equality to the whole interval $[\alpha,\infty)$. Let $N_*$ be the Wronskian of $(X_*,U_*)$ and $(X_{**},U_{**})$, that is, $N_* = X_*^T U_{**} - U_*^T X_{**}$. By Corollary 3.1.10 (with $\alpha := \beta$, $(X,U) := (X_*,U_*)$, $(X_0,U_0) := (X_{**},U_{**})$, $P := P_*$, $P_{\mathscr{S}_{\alpha}\infty} := P_{\mathscr{S}_{*\beta}\infty}$, and $N := N_*$), it follows that $\operatorname{Im} N_* \subseteq \operatorname{Im}(P_* - P_{\mathscr{S}_{*\beta}\infty})$. Consequently, $\operatorname{Im} N_* \subseteq \operatorname{Im}(P_* - P_{\mathscr{S}_{*\alpha}\infty})$, by the second equality in (3.41). On the other hand, the equality $X_*(t) = X_{**}(t)$ on $[\beta,\infty)$ implies that $\operatorname{Im} X_*(t) = \operatorname{Im} X_{**}(t)$ on $[\alpha,\infty)$, by Corollary 2.3.5. Therefore, from Theorem 2.3.8 it then follows that the conjoined bases $(X_*,U_*)$ and $(X_{**},U_{**})$ are mutually representable on $[\alpha,\infty)$. In particular, we have on $[\alpha,\infty)$

$$\begin{pmatrix} X_{**} \\ U_{**} \end{pmatrix} = \begin{pmatrix} X_* & \bar{X}_{*\alpha} \\ U_* & \bar{U}_{*\alpha} \end{pmatrix} \begin{pmatrix} M_{*\alpha} \\ N_* \end{pmatrix},$$

where $(\bar{X}_{*\alpha},\bar{U}_{*\alpha})$ is a conjoined basis of (H) satisfying (1.16) and (2.25) with respect to $(X_*,U_*)$ and where $M_{*\alpha} \in \mathbb{R}^{n\times n}$ is a constant invertible matrix. In addition, the matrix $M_{*\alpha}^T N_*$ is symmetric, because $(X_{**},U_{**})$ is a conjoined basis. By using (2.58) with $(X_1,U_1) := (X_*,U_*)$ and $(X_2,U_2) := (X_{**},U_{**})$ we obtain the formula

$$X_{**} = X_* \,(M_{*\alpha} + S_{*\alpha} N_*) \quad \text{on } [\alpha,\infty). \tag{3.42}$$

But $S_{*\alpha}(t) N_* = S_{*\alpha}(t) P_{\mathscr{S}_{*\alpha}\infty} N_* = 0$ on $[\alpha,\infty)$. Therefore, (3.42) becomes $X_{**}(t) = X_*(t) M_{*\alpha}$ on $[\alpha,\infty)$. At the same time we have $X_{**}(t) = X_*(t)$ on $[\beta,\infty)$, which gives the formula $X_*(t)M_{*\alpha} = X_*(t)$ on $[\beta,\infty)$. Multiplying the latter equation by $X_*^\dagger(t)$ from the left and using the identity $X_*^\dagger X_* = P_*$ on $[\beta,\infty)$, we get $P_* M_{*\alpha} = P_*$. Hence, $(X_*,U_*)$ and $(X_{**},U_{**})$ are equivalent on $[\alpha,\infty)$, by Corollary 3.1.10. Consequently, the relations $(X_*,U_*) \sim (X_{**},U_{**}) \sim (XP_*,UP_*)$ on $[\alpha,\infty)$ then yield that $(X,U)$ contains $(X_*,U_*)$ also on $[\alpha,\infty)$ with respect to $P_*$, by Definition 3.2.1. Finally, the fact that $(X,U)$ contains $(X_*,U_*)$ on $[\gamma,\infty)$ for every $\gamma \geq \alpha$ now follows from part (i) of this theorem. ■

The following result contains a similar statement as in Theorem 3.2.12 in a certain sense, namely it gives conditions which guarantee the invariance of the constant kernel of a conjoined basis of (H) with respect to the change of the interval $[\alpha,\infty)$. In the proof we utilize the properties of the set $\Lambda[\alpha,\infty)$ defined in Section 3.1. Namely, we recall that a vector function $u$ belongs to $\Lambda[\alpha,\infty)$ if and only if the pair $(x,u) \equiv (0,u)$ is a vector solution of (H) on $[\alpha,\infty)$. Moreover, $\dim \Lambda[\alpha,\infty) = d[\alpha,\infty)$ and for every $\beta \geq \alpha$ we have $\Lambda[\alpha,\infty) \subseteq \Lambda[\beta,\infty)$.

**Theorem 3.2.13.** *Assume* (1.1)*. Let* $(X_*,U_*)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *with* $d[\alpha,\infty) = d_\infty$*. Then the following statements hold for every* $\beta \in [\alpha,\infty)$*.*

(i) *If* $(X_{**},U_{**})$ *is a conjoined basis of* (H) *with constant kernel on* $[\beta,\infty)$ *and it is contained in* $(X_*,U_*)$ *on* $[\beta,\infty)$*, then* $(X_{**},U_{**})$ *has constant kernel also on* $[\alpha,\infty)$*.*

(ii) *If* $(X,U)$ *is a conjoined basis of* (H) *with constant kernel on* $[\beta,\infty)$ *and it contains* $(X_*,U_*)$ *on* $[\beta,\infty)$*, then* $(X,U)$ *has constant kernel also on* $[\alpha,\infty)$*.*

*Proof.* Fix $\beta \in [\alpha,\infty)$. The condition $d[\alpha,\infty) = d_\infty$ implies that $d[\alpha,\infty) = d[\beta,\infty)$ and consequently, the equality $\Lambda[\alpha,\infty) = \Lambda[\beta,\infty)$ holds by the properties of $\Lambda[t,\infty)$ discussed above. For part (i), let $(X_{**},U_{**})$ be a conjoined basis of (H) with constant kernel on $[\beta,\infty)$, which is contained in $(X_*,U_*)$ on $[\beta,\infty)$. If $P_{**}$ is the orthogonal projector in (2.2) defined by $X_{**}(t)$ on $[\beta,\infty)$, then $(X_{**},U_{**}) \sim (X_* P_{**},U_* P_{**})$ on $[\beta,\infty)$ by Definition 3.2.1 and Theorem 3.2.2 with $(X,U) := (X_*,U_*)$ and $(X_{**},U_{**}) := (X_*,U_*)$. This means that $X_{**}(t) = X_*(t)P_{**}$ on $[\beta,\infty)$. Now we put $(\tilde{X},\tilde{U}) := (X_{**} - X_* P_{**},\, U_{**} - U_* P_{**})$ and let $v \in \mathbb{R}^n$. Then $(\tilde{X}v,\tilde{U}v)$ is a vector solution of (H) satisfying $\tilde{X}(t)\,v \equiv 0$ on $[\beta,\infty)$. Therefore, $\tilde{U}v \in \Lambda[\beta,\infty) = \Lambda[\alpha,\infty)$. But this means that $\tilde{X}(t)\,v \equiv 0$ also on $[\alpha,\infty)$. Since the vector $v$ was chosen arbitrarily, the latter equality yields $\tilde{X}(t) \equiv 0$ on $[\alpha,\infty)$ and consequently, $X_{**}(t) = X_*(t)P_{**}$ on $[\alpha,\infty)$. Thus, according to Definition 3.2.1 the conjoined basis $(X_{**},U_{**})$ is contained in $(X_*,U_*)$ on $[\alpha,\infty)$ with respect to the orthogonal projector $P_{**}$. In turn, $(X_{**},U_{**})$ has constant kernel also on $[\alpha,\infty)$, by Theorem 3.2.2. For part (ii), let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\beta,\infty)$. We denote by $S_{*\alpha}(t)$ and $S_{*\beta}(t)$ the $S$-matrices corresponding to $(X_*,U_*)$, and by $S_\beta(t)$ the $S$-matrix corresponding to $(X,U)$ on $[\beta,\infty)$. Let $P_*$ and $P$ be the orthogonal projectors in (2.2) defined by $X_*(t)$ and $X(t)$. Moreover, let $P_{\mathscr{S}_{*\alpha}\infty}$, $P_{\mathscr{S}_{*\beta}\infty}$, and $P_{\mathscr{S}_{\beta}\infty}$ be the orthogonal projectors defined in (2.12) through $S_{*\alpha}(t)$, $S_{*\beta}(t)$, and $S_\beta(t)$. Then the second equality in (3.41) holds, that is $P_{\mathscr{S}_{*\beta}\infty} = P_{\mathscr{S}_{*\alpha}\infty}$. Now if $(X,U)$ contains $(X_*,U_*)$ on $[\beta,\infty)$, then $P_{\mathscr{S}_{\beta}\infty} = P_{\mathscr{S}_{*\beta}\infty}$ by Theorem 3.2.4. Furthermore, from Theorem 3.2.5 and Remark 3.2.6(ii) with $\alpha := \beta$ we know that the (constant) Wronskian $G := X^T(t)\,U_*(t) - U^T(t)\,X_*(t)$ of $(X,U)$ and $(X_*,U_*)$ satisfies $\operatorname{Im} G \subseteq \operatorname{Im}(P - P_{\mathscr{S}_{\beta}\infty})$ and hence, $\operatorname{Im} G \subseteq \operatorname{Im}(P - P_{\mathscr{S}_{*\beta}\infty}) = \operatorname{Im}(P - P_{\mathscr{S}_{*\alpha}\infty})$. Since

$(X_*,U_*)$ has constant kernel on $[\alpha,\infty)$ and since the matrix $-G^T$ is the Wronskian of $(X_*,U_*)$ and $(X,U)$, formula (2.22) and the equalities $X_*(\beta)=X(\beta)P_*$ and $P_*S_{*\beta}(t)=S_{*\beta}(t)$ yield

$$\begin{aligned} X(t) &\overset{(2.22)}{=} \Phi_*(t,\beta)\{X(\beta)-X_*(\beta)S_{*\beta}(t)G^T\} \\ &= \Phi_*(t,\beta)\{X(\beta)-X(\beta)P_*S_{*\beta}(t)G^T\} \\ &= \Phi_*(t,\beta)X(\beta)\{I-S_{*\beta}(t)G^T\} \\ &= \Phi_*(t,\beta)X(\beta)\{I-[S_{*\alpha}(t)-S_{*\alpha}(\beta)]G^T\} \quad \text{on } [\alpha,\infty), \end{aligned} \tag{3.43}$$

where $\Phi_*(t,\beta)$ is the fundamental matrix of the equation $Y'=(A+BQ_*)Y$ with $Q_*(t)$ defined in (2.5) by $(X_*,U_*)$ such that $\Phi_*(\beta,\beta)=I$. With the aid of (3.43) we show that $\operatorname{Ker}X(t)=\operatorname{Ker}P$ on whole interval $[\alpha,\infty)$. Fix $t\in[\alpha,\infty)$. Similarly, as in the proof of Theorem 3.2.8, we have that $\operatorname{Ker}P\subseteq\operatorname{Ker}X(t)$ and that $Pv=[S_{*\alpha}(t)-S_{*\alpha}(\beta)]G^TPv$ for every $v\in\operatorname{Ker}X(t)$. Now putting $w:=Pv$ and using the identity $P_{\mathscr{S}_{*\alpha\infty}}[S_{*\alpha}(t)-S_{*\alpha}(\beta)]=S_{*\alpha}(t)-S_{*\alpha}(\beta)$ yields that $w=[S_{*\alpha}(t)-S_{*\alpha}(\beta)]G^Tw$ and $w\in\operatorname{Im}P_{\mathscr{S}_{*\alpha\infty}}$. Consequently, $w=[S_{*\alpha}(t)-S_{*\alpha}(\beta)]G^TP_{\mathscr{S}_{*\alpha\infty}}w=0$, because $G^TP_{\mathscr{S}_{*\alpha\infty}}=(P_{\mathscr{S}_{*\alpha\infty}}G)^T=0$. Therefore, $Pv=0$ and hence, $\operatorname{Ker}X(t)\subseteq\operatorname{Ker}P$. Thus, $(X,U)$ has constant kernel on $[\alpha,\infty)$ and the proof is complete. ■

## 3.3 Minimal conjoined bases

The concept of minimal conjoined bases (Definition 3.3.1) is intimately connected to the abnormality of system (H) and it is essential for the subsequent development of the theory of (minimal) principal and antiprincipal solutions of (H) at infinity. One of the central results of this section is a generalization of the classical result in [6, Formula (12), pg. 41], which reads in the notation of Section 2.1 as

$$S_2^{-1}(t)=M_1^T\,[S_1^{-1}(t)M_1+N_1] \quad \text{for all } t\in[\alpha,\infty),$$

to the case of minimal conjoined bases of (H) and hence, to abnormal linear Hamiltonian systems, see formula (3.50) in Theorem 3.3.6. Motivated by Remark 3.1.4, we now define a minimal conjoined basis of (H) on $[\alpha,\infty)$. We shall see that any conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ determines some minimal conjoined basis and conversely, every minimal conjoined basis can be obtained from some conjoined basis of (H) with constant kernel on $[\alpha,\infty)$.

**Definition 3.3.1.** A conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ is called *minimal* on $[\alpha,\infty)$ if the equality $\operatorname{rank}X(t)=n-d[\alpha,\infty)$ holds on $[\alpha,\infty)$.

The notion "minimal" is natural, because in view of inequalities (3.15) the number $n-d[\alpha,\infty)$ is the minimal value which may be attained by the rank of any conjoined basis of (H) with constant kernel on $[\alpha,\infty)$.

**Remark 3.3.2.** Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be the orthogonal projectors defined in (2.2) and (2.12). It is straightforward to see that the conjoined basis $(X_*,U_*)$, which is contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_{\mathscr{S}_{\alpha\infty}}$, is a minimal conjoined basis of (H) on $[\alpha,\infty)$ according to Definition 3.3.1. This follows from the identity $\operatorname{Ker} X_*(t) = \operatorname{Ker} P_{\mathscr{S}_{\alpha\infty}}$ on $[\alpha,\infty)$ obtained from (3.27) in Theorem 3.2.2, and from formula (3.14) in Remark 3.1.3. On the other hand, every minimal conjoined basis $(X_*,U_*)$ is contained in itself, since by Definition 3.1.6 the solutions $(X_*,U_*)$ and $(X_*P_*,U_*P_*) = (X_*,U_*P_*)$ are equivalent on $[\alpha,\infty)$ where $P_*$ is the orthogonal projector defined in (2.2) through the function $X_*(t)$. Therefore, minimal conjoined bases of (H) can be effectively constructed via the relation of "being contained" in Definition 3.2.1 and Theorem 3.2.7.

**Remark 3.3.3.** Another equivalent definition of a minimal conjoined basis of (H) is the following. Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. Then $(X,U)$ is minimal if and only if $P_{\mathscr{S}_{\alpha\infty}} = P$. This result is a simple consequence of Remark 2.1.5.

The next result is essentially a consequence of Theorems 3.1.2 and 3.1.7 for the case of minimal conjoined bases of (H). It gives a characterization of the subspace $\Lambda_0[\alpha,\infty)$ and a criterion for the equivalence of two minimal conjoined bases of (H).

**Theorem 3.3.4.** *Assume* (1.1). *Let $(X,U)$ be a minimal conjoined basis of* (H) *on $[\alpha,\infty)$ and let $P$ be the orthogonal projector defined in* (2.2). *Then*

$$\Lambda_0[\alpha,\infty) = \operatorname{Im}\,[U(\alpha)\,(I-P)]. \tag{3.44}$$

*Furthermore, if $(X_1,U_1)$ and $(X_2,U_2)$ are two minimal conjoined bases of* (H) *on $[\alpha,\infty)$ with their corresponding orthogonal projectors $P_1$ and $P_2$ defined in* (2.2) *through the functions $X_1$ and $X_2$, respectively, then $(X_1,U_1)$ and $(X_2,U_2)$ are equivalent on $[\alpha,\infty)$ if and only if*

$$P_2 = P_1 \quad \text{and} \quad X_2(t) = X_1(t)\,M, \quad U_2(t) = U_1(t)\,M, \quad t \in [\alpha,\infty), \tag{3.45}$$

*where $M$ is a constant nonsingular matrix satisfying $P_1M = P_1$.*

*Proof.* Since $(X,U)$ is a minimal conjoined basis of (H) on $[\alpha,\infty)$, we have by Remark 3.3.3 that $P_{\mathscr{S}_{\alpha\infty}} = P$, where the matrix $P_{\mathscr{S}_{\alpha\infty}}$ is defined in (2.12). Therefore, $X^{\dagger T}(\alpha)\,(I - P_{\mathscr{S}_{\alpha\infty}}) = 0$ and using formula (3.6) from Theorem 3.1.2 we obtain equality (3.44). Using Theorem 3.1.7, the fact $(X_1,U_1) \sim (X_2,U_2)$ on $[\alpha,\infty)$ means that $X_2(\alpha) = X_1(\alpha)$ and $\operatorname{Im}\,[U_2(\alpha) - U_1(\alpha)] \subseteq \Lambda_0[\alpha,\infty)$, while from (3.44) we get $\Lambda_0[\alpha,\infty) = \operatorname{Im}\,[U_1(\alpha)\,(I-P_1)]$. Therefore, the projectors $P_1$ and $P_2$ satisfy $P_2 = P_1$ and $U_2(\alpha) - U_1(\alpha) = U_1(\alpha)\,H$ with $P_1H = 0$. Consequently, with $M := I + H$, we have $X_1(\alpha)\,M = X_2(\alpha)$ and $U_1(\alpha)\,M = U_2(\alpha)$. This completes the formulas in (3.45) by the uniqueness of solutions of (H). In addition, the constant matrix $M$ is nonsingular, because $\operatorname{rank}\,(X_2^T,\, U_2^T) = n$. Finally, $P_1M = P_1$ follows by the definition of $M$. Conversely, if the minimal conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ satisfy the equalities in (3.45) with a nonsingular matrix $M$ such that $P_1M = P_1$, then they are equivalent on $[\alpha,\infty)$. This follows from equalities $X_2(\alpha) = X_1(\alpha)\,P_1M = X_1(\alpha)$ and $U_2(\alpha) - U_1(\alpha) = U_1(\alpha)\,(M-I)$, where $\operatorname{Im}\,(M-I) \subseteq \operatorname{Im}\,(I-P_1)$. ■

**Remark 3.3.5.** According to Remark 3.3.2 and Theorem 3.2.4, every $S$-matrix corresponding to a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ arises from some minimal conjoined basis of (H) on $[\alpha,\infty)$. Furthermore, the formula in (3.44) implies that for any minimal conjoined basis $(X,U)$ the image of the matrix $X(\alpha)$ does not depend on the choice of $(X,U)$. More precisely, with the aid of Theorem 2.1.2(i), we obtain

$$\operatorname{Im} X(\alpha) = \operatorname{Im} R(\alpha) = \big(\operatorname{Im}[U(\alpha)(I-P)]\big)^{\perp} = \big(\Lambda_0[\alpha,\infty)\big)^{\perp}. \tag{3.46}$$

Now, Theorems 2.3.3 and 2.3.8 imply that every pair of minimal conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ of (H) on $[\alpha,\infty)$ is mutually representable on $[\alpha,\infty)$, i.e., $(X_1,U_1)$ and $(X_2,U_2)$ can be expressed in terms of one another, by Definition 2.3.1. As we will show in Theorem 3.3.6, the same can be done with the associated $S$-matrices $S_{1\beta}(t)$ and $S_{2\beta}(t)$, as well as with their Moore–Penrose pseudoinverses $S^{\dagger}_{1\beta}(t)$ and $S^{\dagger}_{2\beta}(t)$ for a given $\beta \geq \alpha$. Note that the condition $\operatorname{Im} X_1(\alpha) = \operatorname{Im} X_2(\alpha)$ required for Theorems 2.3.3 and 2.3.8 is satisfied as a consequence of (3.46). Hence, these results indicate that minimal conjoined bases of (H) represent one of the key tools for the comparison of the $S$-matrices.

The next theorem completes Theorem 2.3.8 in a sense that it provides a relationship between two $S$-matrices, which correspond to a pair of mutually representable conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ of (H) with constant kernel on $[\alpha,\infty)$. In addition, if $(X_1,U_1)$ and $(X_2,U_2)$ are minimal on $[\alpha,\infty)$, then one can also compare the Moore–Penrose pseudoinverses of its $S$-matrices on a certain subinterval of $[\alpha,\infty)$. The results generalize [6, Proposition 3 in Chapter 2] and [6, Corollary to Proposition 3, pg. 41] to abnormal linear Hamiltonian systems. They are also essential for the construction of the minimal principal solution of (H) at infinity in Chapter 5. Here we use the notation from Theorem 2.3.8 and its proof and from Remarks 2.3.9 and 3.1.3.

**Theorem 3.3.6.** *With the assumptions and notation of Theorem 2.3.8 and Remark 2.3.9, if the conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ are mutually representable on $[\alpha,\infty)$, then we have*

$$[P_1 M_{1\beta} + S_{1\beta}(t) N_1]^{\dagger} = P_2 M_{2\beta} + S_{2\beta}(t) N_2, \tag{3.47}$$

$$\operatorname{Im}[P_1 M_{1\beta} + S_{1\beta}(t) N_1] = \operatorname{Im} P_1, \quad \operatorname{Im}[P_2 M_{2\beta} + S_{2\beta}(t) N_2] = \operatorname{Im} P_2, \tag{3.48}$$

$$S_{2\beta}(t) = [P_1 M_{1\beta} + S_{1\beta}(t) N_1]^{\dagger} S_{1\beta}(t) M_{1\beta}^{T-1} P_2. \tag{3.49}$$

*for all $t \in [\alpha,\infty)$. Moreover, if $(X_1,U_1)$ and $(X_2,U_2)$ are minimal on $[\alpha,\infty)$ and $\alpha = \tau_{-k} < \tau_{-k+1} < \cdots < \beta = \tau_0 < \cdots < \tau_{l-1} < \tau_l < \infty$ is the partition of $[\alpha,\infty)$ from Remark 3.1.3, then*

$$S^{\dagger}_{2\beta}(t) = M^T_{1\beta} S^{\dagger}_{1\beta}(t) M_{1\beta} + M^T_{1\beta} N_1 \quad \textit{for all } t \in (\tau_l,\infty). \tag{3.50}$$

*Proof.* Fix $\beta, t \in [\alpha,\infty)$. From Theorem 2.3.8 we know that the equality $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ holds. This means that the orthogonal projectors $R_1(t)$ and $R_2(t)$ satisfy $R_1(t) = R_2(t)$. Using (2.9) in Theorem 2.1.4(ii) and (2.58) we have

$$\begin{aligned} P_1 M_{1\beta} + S_{1\beta}(t) N_1 &= P_1 [M_{1\beta} + S_{1\beta}(t) N_1] = X_1^{\dagger}(t) X_1(t) [M_{1\beta} + S_{1\beta}(t) N_1] \\ &= X_1^{\dagger}(t) X_2(t), \end{aligned} \tag{3.51}$$

$$\begin{aligned} P_2 M_{2\beta} + S_{2\beta}(t) N_2 &= P_2 [M_{2\beta} + S_{2\beta}(t) N_2] = X_2^{\dagger}(t) X_2(t) [M_{2\beta} + S_{2\beta}(t) N_2] \\ &= X_2^{\dagger}(t) X_1(t). \end{aligned} \tag{3.52}$$

Expressions (3.51) and (3.52) then imply formula (3.47) by the verification of the four equalities in (1.13). In particular, the third and fourth identity in (1.13) read as

$$[P_1 M_{1\beta} + S_{1\beta}(t) N_1] \, [P_2 M_{2\beta} + S_{2\beta}(t) N_2] = X_1^{\dagger}(t) R_2(t) X_1(t) = X_1^{\dagger}(t) R_1(t) X_1(t) = P_1, \tag{3.53}$$

$$[P_2 M_{2\beta} + S_{2\beta}(t) N_2] \, [P_1 M_{1\beta} + S_{1\beta}(t) N_1] = X_2^{\dagger}(t) R_1(t) X_2(t) = X_2^{\dagger}(t) R_2(t) X_2(t) = P_2, \tag{3.54}$$

which imply the relations in (3.48). The proof of formula (3.49) is slightly more complicated. It can be carried out by the same way as in [6, Proposition 3 in Chapter 2]. Since by (2.59) and Theorem 2.3.8(i),(iii) we have $P_1 M_{1\beta} P_2 M_{2\beta} = P_1$ and $N_1 P_2 = N_1$, from (3.53) we get $[P_1 M_{1\beta} + S_{1\beta}(t) N_1] \, S_{2\beta}(t) N_2 = -S_{1\beta}(t) N_1 M_{2\beta}$. Using (3.54), the fact $\operatorname{Im} S_{2\beta}(t) \subseteq \operatorname{Im} P_2$, and (3.47) we then obtain $S_{2\beta}(t) N_2 = -[P_1 M_{1\beta} + S_{1\beta}(t) N_1]^{\dagger} S_{1\beta}(t) N_1 M_{2\beta}$. From Theorem 2.3.8 we know that $N_2 = -N_1^T$, $M_{2\beta} = M_{1\beta}^{-1}$, and $N_2^T M_{2\beta}$ is symmetric. This implies $N_1 M_{2\beta} = -M_{1\beta}^{T-1} N_2$, so that

$$S_{2\beta}(t) N_1^T = [P_1 M_{1\beta} + S_{1\beta}(t) N_1]^{\dagger} S_{1\beta}(t) M_{1\beta}^{T-1} N_1^T. \tag{3.55}$$

We show that the matrix $N_1^T$ in (3.55) can be cancelled. Indeed, if $\operatorname{Im} N_1^T = \operatorname{Im} N_2 = \operatorname{Im} P_2$, then it suffices to multiply equality (3.55) from the right by the matrix $N_1^{\dagger T}$, because $N_1^T N_1^{\dagger T} = N_1^{\dagger} N_1 = P_2$ and $\operatorname{Im} S_{2\beta}(t) \subseteq \operatorname{Im} P_2$. But in general we only have $\operatorname{Im} N_1^T = \operatorname{Im} N_2 \subseteq \operatorname{Im} P_2$, which shows that more analysis is required in order to cancel $N_1^T$ in (3.55). Let us denote $G := M_{1\beta}^T N_1$. The matrix $G$ is symmetric, $\operatorname{Im} G \subseteq \operatorname{Im} P_2$, and $N_1 = M_{1\beta}^{T-1} G$. According to Lemma A.1.3 in Appendix A, there exists a sequence $\{G^{(\nu)}\}_{\nu=1}^{\infty}$ of symmetric matrices with $\operatorname{Im} G^{(\nu)} = \operatorname{Im} P_2$ for all $\nu \in \mathbb{N}$ such that $G^{(\nu)} \to G$ for $\nu \to \infty$. Furthermore, with $N^{(\nu)} := M_{1\beta}^{T-1} G^{(\nu)}$ we have $N^{(\nu)} \to M_{1\beta}^{T-1} G = N_1$ for $\nu \to \infty$ and in addition, $\operatorname{Im} N^{(\nu)} = \operatorname{Im} P_1$ and $\operatorname{Im} N^{(\nu)T} = \operatorname{Im} P_2$ for all $\nu \in \mathbb{N}$, because $P_1 M_{1\beta}^{T-1} P_2 = M_{1\beta}^{T-1} P_2$. By verifying the identities in (1.13) it follows that $N^{(\nu)\dagger} = G^{(\nu)\dagger} M_{1\beta}^T P_1$, and in particular $N^{(\nu)} N^{(\nu)\dagger} = P_1$ and $N^{(\nu)\dagger} N^{(\nu)} = P_2$ hold. Since $M_{1\beta}^T N^{(\nu)} = G^{(\nu)}$, the matrix $M_{1\beta}^T N^{(\nu)}$ is symmetric. For each $\nu \in \mathbb{N}$ we now define the solution $(X^{(\nu)}, U^{(\nu)})$ of system (H) by

$$\begin{pmatrix} X^{(\nu)} \\ U^{(\nu)} \end{pmatrix} := \begin{pmatrix} X_1 & \bar{X}_{1\beta} \\ U_1 & \bar{U}_{1\beta} \end{pmatrix} \begin{pmatrix} M_{1\beta} \\ N^{(\nu)} \end{pmatrix} \qquad \text{on } [\alpha, \infty). \tag{3.56}$$

Since $M_{1\beta}^T N^{(\nu)}$ is symmetric and $\operatorname{rank}(M_{1\beta}^T, \, (N^{(\nu)})^T) = n$ (as $M_{1\beta}$ is invertible by Theorem 2.3.8), it follows that $(X^{(\nu)}, U^{(\nu)})$ is a conjoined basis of (H). Moreover, the sequence $\{(X^{(\nu)}, U^{(\nu)})\}_{\nu=1}^{\infty}$ converges locally uniformly on $[\alpha, \infty)$ to the conjoined basis $(X_2, U_2)$, which follows from (2.53) and from the convergence of $\{N_1^{(\nu)}\}_{\nu=1}^{\infty}$ to $N_1$. Since $\operatorname{Im} N^{(\nu)} = \operatorname{Im} P_1$, the function $X^{(\nu)}$ in (3.56) will have the form as in (2.58). That is, we have $X^{(\nu)} = X_1 \, (M_{1\beta} + S_{1\beta} N^{(\nu)})$ on $[\alpha, \infty)$ for every $\nu \in \mathbb{N}$. Thus, for each $t \in [\alpha, \infty)$ we have $\operatorname{Im} X^{(\nu)}(t) \subseteq \operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ and $\operatorname{Im} X^{(\nu)T}(t) \subseteq \operatorname{Im} [M_{1\beta}^T P_1 + N^{(\nu)T} S_{1\beta}(t)] \subseteq$

$\operatorname{Im} P_2 = \operatorname{Im} X_2^T(t)$ for all $\nu \in \mathbb{N}$. Now fix $\gamma \geq \beta$. By Lemma A.1.5 and Theorem A.1.4 in Appendix A, there then exists $\nu_0 \in \mathbb{N}$ such that for all $\nu \geq \nu_0$

$$\operatorname{Im} X^{(\nu)}(t) = \operatorname{Im} X_2(t), \quad \operatorname{Im} X^{(\nu)T}(t) = \operatorname{Im} X_2^T(t) \quad \text{for all } t \in [\alpha, \gamma], \tag{3.57}$$

$$X^{(\nu)\dagger}(t) \to X_2^\dagger(t) \quad \text{for } \nu \to \infty \text{ uniformly on } [\alpha, \gamma]. \tag{3.58}$$

Fix now $\nu \geq \nu_0$. From the second equality in (3.57) it follows that $\operatorname{Ker} X^{(\nu)}(t) = \operatorname{Ker} X_2(t) = \operatorname{Ker} P_2$ on $[\alpha, \gamma]$, so that $(X^{(\nu)}, U^{(\nu)})$ is a conjoined basis of (H) with constant kernel on $[\alpha, \gamma]$ and $\operatorname{Im} X^{(\nu)}(\alpha) = \operatorname{Im} X_2(\alpha) = \operatorname{Im} X_1(\alpha)$. Hence, Theorem 2.3.8 can be applied and the first formula in (3.48) proven above holds for the pair $(X_1, U_1)$, $(X^{(\nu)}, U^{(\nu)})$, i.e.,

$$\operatorname{Im}\,[P_1 M_{1\beta} + S_{1\beta}(t) N^{(\nu)}] = \operatorname{Im} P_1 \quad \text{on } [\alpha, \gamma]. \tag{3.59}$$

Let $S_\beta^{(\nu)}(t)$ be the $S$-matrix corresponding to the conjoined basis $(X^{(\nu)}, U^{(\nu)})$. Then according to (3.55) we have $S_\beta^{(\nu)}(t) N^{(\nu)T} = [P_1 M_{1\beta} + S_{1\beta}(t) N^{(\nu)}]^\dagger S_{1\beta}(t) M_{1\beta}^{T-1} N^{(\nu)T}$ for all $t \in [\alpha, \gamma]$. Since $\operatorname{Im} N^{(\nu)T} = \operatorname{Im} P_2$, the matrix $N^{(\nu)T}$ can be cancelled as we showed above, and then we obtain

$$S_\beta^{(\nu)}(t) = [P_1 M_{1\beta} + S_{1\beta}(t) N^{(\nu)}]^\dagger S_{1\beta}(t) M_{1\beta}^{T-1} P_2 \qquad \text{for all } t \in [\alpha, \gamma]. \tag{3.60}$$

Assertion (3.58) yields that $S_\beta^{(\nu)}(t) \to S_{2\beta}(t)$ pointwise on $[\alpha, \gamma]$. Since we have that $P_1 M_{1\beta} + S_{1\beta}(t) N^{(\nu)} \to P_1 M_{1\beta} + S_{1\beta}(t) N_1$ for $\nu \to \infty$ uniformly on $[\alpha, \gamma]$ and (3.59) holds, it follows from Theorem A.1.4 in Appendix A that $[P_1 M_{1\beta} + S_{1\beta}(t) N^{(\nu)}]^\dagger \to [P_1 M_{1\beta} + S_{1\beta}(t) N_1]^\dagger$ for $\nu \to \infty$ uniformly on $[\alpha, \gamma]$. In turn, equation (3.60) implies that $S_\beta^{(\nu)}(t) \to S_{2\beta}(t)$ for $\nu \to \infty$ even uniformly on $[\alpha, \gamma]$. Upon taking $\nu \to \infty$ in (3.60) we obtain

$$S_{2\beta}(t) = [P_1 M_{1\beta} + S_{1\beta}(t) N_1]^\dagger S_{1\beta}(t) M_{1\beta}^{T-1} P_2 \quad \text{for all } t \in [\alpha, \gamma]. \tag{3.61}$$

Since $\gamma \geq \beta \geq \alpha$ was chosen arbitrarily, formula (3.61) immediately implies (3.49). For the proof of formula (3.50) suppose that $(X_1, U_1)$ and $(X_2, U_2)$ are minimal conjoined bases on $[\alpha, \infty)$. From Remarks 3.1.3 and 3.3.3 it then follows that

$$\operatorname{Im} S_{1\beta}(t) = \operatorname{Im} P_1 \quad \text{and} \quad \operatorname{Im} S_{2\beta}(t) = \operatorname{Im} P_2 \quad \text{for all } t \in (\tau_l, \infty). \tag{3.62}$$

Fix $t \in (\tau_l, \infty)$. Multiplying the equation in (3.49) by $M_{1\beta}^T P_1$ from the right and using the identities $M_{1\beta}^{T-1} P_2 M_{1\beta}^T P_1 = P_1$ and $S_{1\beta}(t) P_1 = S_{1\beta}(t)$ on $[\alpha, \infty)$ we get

$$S_{2\beta}(t) M_{1\beta}^T P_1 = [P_1 M_{1\beta} + S_{1\beta}(t) N_1]^\dagger S_{1\beta}(t). \tag{3.63}$$

Moreover, with the aid of Remark 1.2.3(iv), the equalities in (3.62) and $\operatorname{Im}\,[P_1 M_{1\beta} + S_{1\beta}(t) N_1]^{\dagger T} = \operatorname{Im} P_2$, and the properties of $P_1 M_{1\beta}$ and $P_2 M_{2\beta}$ in (2.59), the Moore–Penrose pseudoinverse of each side of (3.63) is equal to

$$\begin{aligned}
[S_{2\beta}(t)\, M_{1\beta}^T\, P_1]^\dagger &= (P_2\, M_{1\beta}^T\, P_1)^\dagger\, [S_{2\beta}(t)\, P_2]^\dagger = M_{2\beta}^T\, P_2 S_{2\beta}^\dagger(t) \\
&= M_{2\beta}^T\, S_{2\beta}^\dagger(t), \\
\left\{[P_1\, M_{1\beta} + S_{1\beta}(t)\, N_1]^\dagger\, S_{1\beta}(t)\right\}^\dagger &= [P_1 S_{1\beta}(t)]^\dagger \left\{[P_1\, M_{1\beta} + S_{1\beta}(t)\, N_1]^\dagger\, P_1\right\}^\dagger \\
&= S_{1\beta}^\dagger(t)\, M_{1\beta} + N_1.
\end{aligned}$$

Finally, combining the last two equalities with (3.63) yields the identity

$$M_{2\beta}^T\, S_{2\beta}^\dagger(t) = S_{1\beta}^\dagger(t)\, M_{1\beta} + N_1.$$

If we now multiply the latter equation by $M_{1\beta}^T$ from the left and use $M_{1\beta}^T M_{2\beta}^T = I$, by Theorem 2.3.8(ii), then formula (3.50) follows and the proof is complete. ∎

**Remark 3.3.7.** Let $T_{1\beta}$ and $T_{2\beta}$ be the $T$-matrices, which correspond to the matrices $S_{1\beta}(t)$ and $S_{2\beta}(t)$, by Remark 2.1.6. Formula (3.50) then allows to derive a mutual relationship between the matrices $T_{1\beta}$ and $T_{2\beta}$. More precisely, upon taking the limit as $t \to \infty$ in formula (3.50), we get

$$T_{2\beta} = M_{1\beta}^T\, T_{1\beta}\, M_{1\beta} + M_{1\beta}^T\, N_1. \tag{3.64}$$

Moreover, using identity (3.64) back in (3.50) yields that

$$S_{2\beta}^\dagger(t) - T_{2\beta} = M_{1\beta}^T\, [S_{1\beta}^\dagger(t) - T_{1\beta}]\, M_{1\beta} \quad \text{on } (\tau_l, \infty). \tag{3.65}$$

Since $M_{1\beta}$ is invertible, the equality in (3.65) then implies that $\operatorname{rank}[S_{1\beta}^\dagger(t) - T_{1\beta}] = \operatorname{rank}[S_{2\beta}^\dagger(t) - T_{2\beta}]$ on $(\tau_l, \infty)$. Thus, if $(X, U)$ is a minimal conjoined basis of (H) on $[\alpha, \infty)$ and $S_\beta(t)$ and $T_\beta$ are its corresponding matrices defined in (2.8) and Remark 2.1.6 for a given $\beta \geq \alpha$, then for $t \in (\tau_l, \infty)$ the quantity $\operatorname{rank}[S_\beta^\dagger(t) - T_\beta]$ does not depend for on the matrix $S_\beta(t)$ itself, and hence, it does not depend on the choice of $(X, U)$. As we shall see, this observation is crucial for a deeper analysis of the $S$-matrices in Chapter 4. We also note that by Theorem 3.2.4 and Remark 3.3.2 the statement holds even when the assumption that $(X, U)$ is minimal on $[\alpha, \infty)$ is dropped and we suppose only that $(X, U)$ has constant kernel on $[\alpha, \infty)$.

The last result of this section is based on the fact that by Remark 3.3.5 any two conjoined bases $(X_{*1}, U_{*1})$ and $(X_{*2}, U_{*2})$ of (H), which are minimal on $[\alpha, \infty)$, are mutually representable on $[\alpha, \infty)$. Here we use notation (2.15) introduced in Remark 2.1.7. Namely, if $P_{\mathscr{S}_{*1}\infty}$ and $P_{\mathscr{S}_{*2}\infty}$ are the orthogonal projectors in (2.12) defined by the functions $X_{*1}(t)$ and $X_{*2}(t)$, then there exist matrices $M_{*1}$, $N_{*1}$, $M_{*2}$, $N_{*2}$ such that the conclusions of Theorem 2.3.8 and Remark 2.3.9 hold with $\beta := \alpha$, $(X_i, U_i) := (X_{*i}, U_{*i})$, $P_i := P_{*i} = P_{\mathscr{S}_{*i}\infty}$, $M_{i\alpha} := M_{*i}$, and $N_i := N_{*i}$ for $i = 1, 2$, i.e.,

$$X_{*2}(\alpha) = X_{*1}(\alpha)\, M_{*1}, \quad U_{*2}(\alpha) = U_{*1}(\alpha)\, M_{*1} + X_{*1}^{\dagger T}(\alpha)\, N_{*1}, \tag{3.66}$$

$$X_{*1}(\alpha) = X_{*2}(\alpha)\, M_{*2}, \quad U_{*1}(\alpha) = U_{*2}(\alpha)\, M_{*2} + X_{*2}^{\dagger T}(\alpha)\, N_{*2}. \tag{3.67}$$

**Theorem 3.3.8.** *Assume (1.1). Let $(X_1,U_1)$ and $(X_2,U_2)$ be conjoined bases of (H) with constant kernel on $[\alpha,\infty)$ and let $P_1$, $P_2$ and $P_{\mathscr{S}_1\infty}$, $P_{\mathscr{S}_2\infty}$ be the corresponding orthogonal projectors defined in (2.2) and (2.12) through the functions $X_1(t)$ and $X_2(t)$. Moreover, let $(X_{*1},U_{*1})$ be a minimal conjoined basis of (H), which is contained in $(X_1,U_1)$ on $[\alpha,\infty)$ with respect to $P_{\mathscr{S}_1\infty}$ and similarly, let $(X_{*2},U_{*2})$ be a minimal conjoined basis of (H) which is contained in $(X_2,U_2)$ on $[\alpha,\infty)$ with respect to $P_{\mathscr{S}_2\infty}$. Suppose that $(X_1,U_1)$ and $(X_2,U_2)$ are mutually representable on $[\alpha,\infty)$ through the matrices $M_1$, $N_1$, $M_2$, $N_2$ as in Theorem 2.3.8 with $\beta:=\alpha$. If $M_{*1}$, $N_{*1}$ and $M_{*2}$, $N_{*2}$ are the matrices corresponding to $(X_{*1},U_{*1})$ and $(X_{*2},U_{*2})$ in (3.66)–(3.67), then*

(i) $P_1 M_1 P_{\mathscr{S}_2\infty} = P_{\mathscr{S}_1\infty} M_{*1}$ *and* $P_2 M_2 P_{\mathscr{S}_1\infty} = P_{\mathscr{S}_2\infty} M_{*2}$,

(ii) $N_{*1} M_{*1}^{-1} = P_{\mathscr{S}_1\infty} N_1 M_1^{-1} P_{\mathscr{S}_1\infty}$ *and* $N_{*2} M_{*2}^{-1} = P_{\mathscr{S}_2\infty} N_2 M_2^{-1} P_{\mathscr{S}_2\infty}$.

*Proof.* If $P_{\mathscr{S}_{*1}\infty}$ and $P_{\mathscr{S}_{*2}\infty}$ are the orthogonal projectors in (2.12) defined by $X_{*1}(t)$ and $X_{*2}(t)$, then $P_{\mathscr{S}_{*1}\infty} = P_{\mathscr{S}_1\infty}$ and $P_{\mathscr{S}_{*2}\infty} = P_{\mathscr{S}_2\infty}$, by Theorem 3.2.4. By using Theorem 3.2.5 we have

$$X_{*1}(\alpha) = X_1(\alpha) P_{\mathscr{S}_1\infty}, \quad U_{*1}(\alpha) = U_1(\alpha) P_{\mathscr{S}_1\infty} + X_1^{\dagger T}(\alpha) G_1 + U_1(\alpha) H_1, \tag{3.68}$$
$$X_{*2}(\alpha) = X_2(\alpha) P_{\mathscr{S}_2\infty}, \quad U_{*2}(\alpha) = U_2(\alpha) P_{\mathscr{S}_2\infty} + X_2^{\dagger T}(\alpha) G_2 + U_2(\alpha) H_2 \tag{3.69}$$

with $(G_i,H_i) \in \mathscr{B}(P_{\mathscr{S}_i\infty}, P_{\mathscr{S}_i\infty}, P_i)$ for $i\in\{1,2\}$. Inserting the first equality from (3.68) and from (3.69) into the first formula in (3.66) gives $X_2(\alpha) P_{\mathscr{S}_2\infty} = X_1(\alpha) P_{\mathscr{S}_1\infty} M_{*1}$, from which we obtain by (2.54) that $X_1(\alpha) M_1 P_{\mathscr{S}_2\infty} = X_1(\alpha) P_{\mathscr{S}_1\infty} M_{*1}$. Consequently, multiplying the latter equality by $X_1^{\dagger}(\alpha)$ from the left and using the identities $X_1^{\dagger}(\alpha) X_1(\alpha) = P_1$ and $P_1 P_{\mathscr{S}_1\infty} = P_{\mathscr{S}_1\infty}$ we get the first formula in part (i). The second one follows in a similar way upon considering $(X_2,U_2)$ instead of $(X_1,U_1)$. For the proof of part (ii) we recall from Remark 2.3.9(ii) that the matrices $N_1$ and $N_{*1}$ are the Wronskians of $(X_1,U_1)$, $(X_2,U_2)$ and of $(X_{*1},U_{*1})$, $(X_{*2},U_{*2})$, respectively. In particular, at the point $t=\alpha$ we have

$$N_1 = X_1^T(\alpha) U_2(\alpha) - U_1^T(\alpha) X_2(\alpha), \quad N_{*1} = X_{*1}^T(\alpha) U_{*2}(\alpha) - U_{*1}^T(\alpha) X_{*2}(\alpha). \tag{3.70}$$

Combining (3.70) with (3.68)–(3.69) leads to the expression

$$\begin{aligned} N_{*1} = P_{\mathscr{S}_1\infty} N_1 P_{\mathscr{S}_2\infty} + P_{\mathscr{S}_1\infty} X_1^T(\alpha) X_2^{\dagger T}(\alpha) G_2 + P_{\mathscr{S}_1\infty} X_1^T(\alpha) U_2(\alpha) H_2 \\ - G_1^T X_1^{\dagger}(\alpha) X_2(\alpha) P_{\mathscr{S}_2\infty} - H_1^T U_1^T(\alpha) X_2(\alpha) P_{\mathscr{S}_2\infty}. \end{aligned} \tag{3.71}$$

We now calculate the last four terms on the right hand side of (3.71) separately. By the first equality in (2.55) and part (i) and by the symmetry of $X_2^T U_2$ and the identities $X_2(\alpha) H_2 = 0$, $X_2^T(\alpha) X_2^{\dagger T}(\alpha) = P_2$, and $P_{\mathscr{S}_2\infty} G_2 = 0$, we have

$$\begin{aligned} P_{\mathscr{S}_1\infty} X_1^T(\alpha) X_2^{\dagger T}(\alpha) G_2 &= P_{\mathscr{S}_1\infty} M_2^T X_2^T(\alpha) X_2^{\dagger T}(\alpha) G_2 \\ &= P_{\mathscr{S}_1\infty} M_2^T P_2 G_2 = M_{*2}^T P_{\mathscr{S}_2\infty} G_2 = 0, \\ P_{\mathscr{S}_1\infty} X_1^T(\alpha) U_2(\alpha) H_2 &= P_{\mathscr{S}_1\infty} M_2^T X_2^T(\alpha) U_2(\alpha) H_2 \\ &= P_{\mathscr{S}_1\infty} M_2^T U_2^T(\alpha) X_2(\alpha) H_2 = 0. \end{aligned}$$

Similarly, it follows by using the first equality in (2.54) and part (i) and by the symmetry of $X_1^T U_1$ and the identities $X_1(\alpha) H_1 = 0$, $X_1^{\dagger}(\alpha) X_1(\alpha) = P_1$, and $P_{\mathscr{S}_1\infty} G_1 = 0$ that the last two terms in (3.71) are equal to zero. Therefore, we have $N_{*1} = P_{\mathscr{S}_1\infty} N_1 P_{\mathscr{S}_2\infty}$. Now we use the properties $M_{*2} = M_{*1}^{-1}$, $M_2 = M_1^{-1}$, $N_1 P_2 = N_1$ from Theorem 2.3.8 and part (i) to get

$$N_{*1} M_{*1}^{-1} = P_{\mathscr{S}_1\infty} N_1 P_{\mathscr{S}_2\infty} M_{*2} = P_{\mathscr{S}_1\infty} N_1 P_2 M_2 P_{\mathscr{S}_1\infty} = P_{\mathscr{S}_1\infty} N_1 M_1^{-1} P_{\mathscr{S}_1\infty}.$$

This shows the first equality in (ii). The second one can be proven analogously. ■

# Chapter 4

# Analysis of $S$-matrices

In this chapter we deal with the matrices $S_\alpha(t)$ in (2.8) associated with conjoined bases of (H) with constant kernel. In Section 4.1 we present some asymptotic properties of these matrices and consequently, in Section 4.2 we apply these results in order to obtain a classification of all conjoined bases of (H), which are minimal on given subinterval. In the last section we study some properties of corresponding $T$-matrices. All results obtained in this chapter will be effectively utilized in Chapter 5.

## 4.1 Asymptotic properties of $S$-matrices

In this section we discuss important properties of the $S$-matrix $S_\alpha(t)$ defined in (2.8) in a more detailed way. More precisely, we present several results concerning the asymptotic behavior of the matrix $S_\alpha^\dagger(t)$ and its relation with the abnormality of (H) on certain subintervals in $[\alpha,\infty)$. In addition, in Theorem 4.1.12 we derive a key result, which is utilized for the existence (and hence a correct definition) of the minimal principal solution at infinity in Chapter 5. Note that by replacing $\alpha$ by $\beta\in[\alpha,\infty)$, all statements remain true on the interval $[\beta,\infty)$. Moreover, we put

$$\begin{aligned} r(t) &:= \operatorname{rank} S_\alpha(t), \quad t\in[\alpha,\infty), \\ r_\infty &:= \operatorname{rank} P_{\mathscr{S}_{\alpha\infty}} = r(t), \quad t\in(\tau_l,\infty), \end{aligned}$$

where the matrix $P_{\mathscr{S}_{\alpha\infty}}$ is defined in (2.12) and where the point $\tau_l$ comes from the partition of the interval $[\alpha,\infty)$ for $\beta=\alpha$ in Remark 3.1.3. The following theorem completes the basic results about the $S$-matrices displayed in Sections 2.1 and 3.1.

**Theorem 4.1.1.** *Assume (1.1). Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $S_\alpha(t)$ be its corresponding S-matrix. Then there exists a constant orthogonal matrix $V\in\mathbb{R}^{n\times n}$ such that for all $t\in(\alpha,\infty)$*

$$S_\alpha(t)=V\begin{pmatrix}\Sigma_\alpha(t) & 0\\ 0 & 0_{n-r(t)}\end{pmatrix}V^T, \qquad S_\alpha^\dagger(t)=V\begin{pmatrix}\Sigma_\alpha^{-1}(t) & 0\\ 0 & 0_{n-r(t)}\end{pmatrix}V^T, \tag{4.1}$$

*where $\Sigma_\alpha(t)\in\mathbb{R}^{r(t)\times r(t)}$ is symmetric and positive definite.*

*Proof.* By Theorem 2.1.4(ii) and Remark 3.1.3, the set $\operatorname{Im} S_\alpha(t)$ is piecewise constant and nondecreasing on $[\alpha,\infty)$. Hence, there exists a constant orthogonal matrix $V\in\mathbb{R}^{n\times n}$ such that for each $t\in(\alpha,\infty)$ its first $r(t)$ columns form an orthonormal basis of $\operatorname{Im} S_\alpha(t)$. This means that for each $t\in(\alpha,\infty)$ the matrix $V$ has the block form $\begin{pmatrix} V_{r(t)} & V_{n-r(t)}\end{pmatrix}$, where $V_{r(t)}\in\mathbb{R}^{n\times r(t)}$ and $V_{n-r(t)}\in\mathbb{R}^{n\times(n-r(t))}$ are such that $\operatorname{Im} S_\alpha(t)=\operatorname{Im} V_{r(t)}$ and $\operatorname{Ker} S_\alpha(t)=\operatorname{Im} V_{n-r(t)}$. Then

$$V^T S_\alpha(t)\,V=\begin{pmatrix} V^T_{r(t)}S_\alpha(t)\,V_{r(t)} & V^T_{r(t)}S_\alpha(t)\,V_{n-r(t)}\\ V^T_{n-r(t)}S_\alpha(t)\,V_{r(t)} & V^T_{n-r(t)}S_\alpha(t)\,V_{n-r(t)}\end{pmatrix}=\begin{pmatrix}\Sigma_\alpha(t) & 0\\ 0 & 0_{n-r(t)}\end{pmatrix}, \tag{4.2}$$

where $\Sigma_\alpha(t):=V^T_{r(t)}S_\alpha(t)\,V_{r(t)}\in\mathbb{R}^{r(t)\times r(t)}$ is symmetric and positive definite with nondecreasing rank (i.e., dimension). The formulas in (4.1) now follow from (4.2) and Remark 1.2.3(iii). ∎

**Remark 4.1.2.** (i) Using the identities in (4.1), the orthogonal projectors $P_{\mathscr{S}_\alpha}(t)$ and $P_{\mathscr{S}_{\alpha\infty}}$ defined in Remark 2.1.5 and (2.12) can be expressed as

$$P_{\mathscr{S}_\alpha}(t)=V\begin{pmatrix} I_{r(t)} & 0\\ 0 & 0_{n-r(t)}\end{pmatrix}V^T,\quad t\in(\alpha,\infty),\quad P_{\mathscr{S}_{\alpha\infty}}=V\begin{pmatrix} I_{r_\infty} & 0\\ 0 & 0_{n-r_\infty}\end{pmatrix}V^T. \tag{4.3}$$

Similarly, the second equality in (4.1) implies that the matrix $T_\alpha$ introduced in Remark 2.1.6 as the limit of $S^\dagger_\alpha(t)$ for $t\to\infty$, has the form

$$T_\alpha=V\begin{pmatrix} T^\star_\alpha & 0\\ 0 & 0\end{pmatrix}V^T,\quad \text{where}\quad T^\star_\alpha:=\lim_{t\to\infty}\Sigma^{-1}_\alpha(t), \tag{4.4}$$

with $T^\star_\alpha\in\mathbb{R}^{r_\infty\times r_\infty}$ being symmetric and nonnegative definite.

(ii) From Remark 3.1.3 with $\beta=\alpha$, we know that $\operatorname{Im} S_\alpha(t)$ and hence, $\operatorname{Ker} S_\alpha(t)$ is constant on $(\tau_l,\infty)$ and $\operatorname{rank} S_\alpha(t)=r_\infty$ for $t\in(\tau_l,\infty)$, see the notation at the beginning of this section. Thus, the matrix $S^\dagger_\alpha(t)$ is nonincreasing on $(\tau_l,\infty)$, by Theorem 2.1.4(iii), and the definition of $T_\alpha$ in Remark 2.1.6 then yields

$$S^\dagger_\alpha(t)\geq T_\alpha\quad \text{for all}\quad t\in(\tau_l,\infty). \tag{4.5}$$

This result can be also expressed in terms of the $r_\infty\times r_\infty$ matrices $\Sigma^{-1}_\alpha(t)$ and $T^\star_\alpha$ defined in (4.1) and (4.4). More precisely, inequality (4.5) is equivalent with

$$\Sigma^{-1}_\alpha(t)\geq T^\star_\alpha\quad \text{for all}\quad t\in(\tau_l,\infty). \tag{4.6}$$

Next we discuss the monotonicity properties of the matrix functions $\Sigma_\alpha(t)$ and $\Sigma^{-1}_\alpha(t)$ from Theorem 4.1.1. We recall the notation $(M)_k$ for the $k$-th leading principal submatrix of the matrix $M$, which we introduced at the beginning of Section 1.2.

**Theorem 4.1.3.** *Assume (1.1). Let $S_\alpha(t)$ be the S-matrix corresponding to a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $\Sigma_\alpha(t)$ be the function defined in Theorem 4.1.1. Then for all $t_1,t_2\in(\alpha,\infty)$ such that $t_1\leq t_2$ we have*

$$\big(\Sigma_\alpha(t_2)\big)_{r(t_1)}\geq\Sigma_\alpha(t_1)>0, \tag{4.7}$$

$$\Sigma^{-1}_\alpha(t_1)\geq\big(\Sigma^{-1}_\alpha(t_2)\big)_{r(t_1)}\geq\big(\Sigma_\alpha(t_2)\big)^{-1}_{r(t_1)}>0. \tag{4.8}$$

*Proof.* Fix $t_1, t_2 \in (\alpha, \infty)$ with $t_1 \le t_2$. The functions $S_\alpha(t)$ and $V^T S_\alpha(t) V$ with $V$ defined in Theorem 4.1.1 are nondecreasing on $(\alpha, \infty)$. Using the first equality in (4.1) we then get

$$\begin{pmatrix} \Sigma_\alpha(t_2) & 0 \\ 0 & 0_{n-r(t_2)} \end{pmatrix} \ge \begin{pmatrix} \Sigma_\alpha(t_1) & 0 \\ 0 & 0_{n-r(t_1)} \end{pmatrix}. \tag{4.9}$$

In particular, all corresponding leading principal submatrices in (4.9) satisfy the same inequality. Since the function $r(t)$ is nondecreasing, by Theorem 2.1.4(ii), we have $1 \le r(t_1) \le r(t_2) \le n$. Consequently, inequality (4.7) follows directly from (4.9) by considering the leading principal submatrices of the size $r(t_1)$. Similarly, considering the leading principal submatrices of the size $r(t_2)$ we obtain from (4.9) that

$$\Sigma_\alpha(t_2) \ge \begin{pmatrix} \Sigma_\alpha(t_1) & 0 \\ 0 & 0_{n-r(t_1)} \end{pmatrix}_{r(t_2)} = \begin{pmatrix} \Sigma_\alpha(t_1) & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \ge 0. \tag{4.10}$$

From Theorem 4.1.1 we know that the matrix $\Sigma_\alpha(t_2)$ is invertible and the matrix

$$O := \begin{pmatrix} \Sigma_\alpha(t_1) & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \begin{pmatrix} \Sigma_\alpha^{-1}(t_1) & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} = \begin{pmatrix} I_{r(t_1)} & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix}$$

is the orthogonal projector onto $\operatorname{diag}\{\Sigma_\alpha(t_1), 0_{r(t_2)-r(t_1)}\}$, by Remark 1.2.3(ii). According to Lemma A.1.2 in Appendix A with $G := \operatorname{diag}\{\Sigma_\alpha(t_1), 0_{r(t_2)-r(t_1)}\}$, $H := \Sigma_\alpha(t_2)$, and $R := O$, the inequalities in (4.10) then imply that

$$\begin{pmatrix} \Sigma_\alpha^{-1}(t_1) & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \ge \begin{pmatrix} I_{r(t_1)} & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \Sigma_\alpha^{-1}(t_2) \begin{pmatrix} I_{r(t_1)} & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \ge 0.$$

Performing the matrix multiplication in the last inequalities we get

$$\begin{pmatrix} \Sigma_\alpha^{-1}(t_1) & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \ge \begin{pmatrix} \left(\Sigma_\alpha^{-1}(t_2)\right)_{r(t_1)} & 0 \\ 0 & 0_{r(t_2)-r(t_1)} \end{pmatrix} \ge 0. \tag{4.11}$$

Consequently, by taking the leading principal submatrices of size $r(t_1)$ in equality (4.11) we obtain $\Sigma_\alpha^{-1}(t_1) \ge \left(\Sigma_\alpha^{-1}(t_2)\right)_{r(t_1)}$. This proves the first inequality in (4.8). The second inequality in (4.8) follows from Lemma A.1.1 in Appendix A. ■

Motivated by the results in Theorem 4.1.3, we now define for each fixed $t \in (\alpha, \infty)$ the matrix functions $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\text{inv}}(t,s)$ by

$$\Sigma_\alpha(t,s) := \left(\Sigma_\alpha(s)\right)_{r(t)} \quad \text{and} \quad \Sigma_\alpha^{\text{inv}}(t,s) := \left(\Sigma_\alpha^{-1}(s)\right)_{r(t)}, \quad s \in [t, \infty). \tag{4.12}$$

As we shall see, these functions can be effectively utilized for a deeper study of the $S$-matrices. The subscript "inv" in the notation $\Sigma_\alpha^{\text{inv}}(t,s)$ refers to a certain connection with the inverse of $\Sigma_\alpha(t,s)$. More precisely, according to Lemma A.1.1 the functions $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\text{inv}}(t,s)$ satisfy the relation $\Sigma_\alpha^{\text{inv}}(t,s) \ge \Sigma_\alpha^{-1}(t,s)$ on $[t, \infty)$. The following theorem provides basic properties of the functions $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\text{inv}}(t,s)$.

**Theorem 4.1.4.** *The functions $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\rm inv}(t,s)$ defined in (4.12) are symmetric, invertible, monotone, and continuous in s on $[t,\infty)$. In particular, $\Sigma_\alpha(t,s)$ is nondecreasing and $\Sigma_\alpha^{\rm inv}(t,s)$ is nonincreasing in s on $[t,\infty)$ and the inequalities*

$$\Sigma_\alpha^{\rm inv}(t,s) \geq \big(T_\alpha^\star\big)_{r(t)} \geq 0 \tag{4.13}$$

*hold in s on $[t,\infty)$ for each $t \in (\alpha,\infty)$, where the matrix $T_\alpha^\star$ is defined in (4.4).*

*Proof.* The symmetry and the invertibility of the matrices $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\rm inv}(t,s)$ follows from Theorems 4.1.1 and 4.1.3 and from the definitions in (4.12). For the proof of monotonicity of $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\rm inv}(t,s)$ fix $t \in (\alpha,\infty)$ and let $s_1,s_2 \in [t,\infty)$ be such that $s_1 \leq s_2$. According to the first inequality in (4.7) we then have that

$$\big(\Sigma_\alpha(s_2)\big)_{r(s_1)} \geq \Sigma_\alpha(s_1). \tag{4.14}$$

Since $1 \leq r(t) \leq r(s_1)$, by taking the leading principal submatrices od size $r(t)$ in inequality (4.14) and using (4.12) we get

$$\Sigma_\alpha(t,s_2) = \big(\Sigma_\alpha(s_2)\big)_{r(t)} = \Big(\big(\Sigma_\alpha(s_2)\big)_{r(s_1)}\Big)_{r(t)} \overset{(4.14)}{\geq} \big(\Sigma_\alpha(s_1)\big)_{r(t)} = \Sigma_\alpha(t,s_1). \tag{4.15}$$

Similarly, the first inequality in (4.8) with $t_1 := s_1$ and $t_2 := s_2$ implies the inequality

$$\Sigma_\alpha^{-1}(s_1) \geq \big(\Sigma_\alpha^{-1}(s_2)\big)_{r(s_1)}, \tag{4.16}$$

which by taking the leading principal submatrices od size $r(t)$ with (4.12) yields

$$\Sigma_\alpha^{\rm inv}(t,s_1) = \big(\Sigma_\alpha^{-1}(s_1)\big)_{r(t)} \overset{(4.16)}{\geq} \Big(\big(\Sigma_\alpha^{-1}(s_2)\big)_{r(s_1)}\Big)_{r(t)} = \big(\Sigma_\alpha^{-1}(s_2)\big)_{r(t)} = \Sigma_\alpha^{\rm inv}(t,s_2). \tag{4.17}$$

Inequality (4.15) then shows that the function $\Sigma_\alpha(t,s)$ is nondecreasing in $s$ on $[t,\infty)$, while inequality (4.17) proves that the function $\Sigma_\alpha^{\rm inv}(t,s)$ is nonincreasing in $s$ on $[t,\infty)$. For the proof of continuity of the functions $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\rm inv}(t,s)$ we note that the matrix $S_\alpha(t)$ is continuous on $[\alpha,\infty)$ and the matrix $S_\alpha^\dagger(t)$ is left-continuous on $(\alpha,\infty)$, by Remarks 3.1.3 and 1.2.4(i). From Theorem 4.1.1 and from the left-continuity of the function $r(t)$ on $(\alpha,\infty)$ it then follows that the matrices $\Sigma_\alpha(t)$ and $\Sigma_\alpha^{-1}(t)$ are left-continuous on $(\alpha,\infty)$. Moreover, the right-continuity of $S_\alpha(t)$ and the fact that $r(t)$ is nondecreasing yield for all $t \in (\alpha,\infty)$ the equality

$$\lim_{\tau\to t^+} \Sigma_\alpha(\tau) = \begin{cases} \Sigma_\alpha(t), & \text{if} \quad r(t) = r(t^+), \\[2ex] \begin{pmatrix} \Sigma_\alpha(t) & 0 \\ 0 & 0_{r(t^+)-r(t)} \end{pmatrix}, & \text{if} \quad r(t) < r(t^+), \end{cases} \tag{4.18}$$

where $r(t^+)$ represents the right-hand limit of the function $r(\tau)$ at the point $t$. Therefore, considering the leading principal submatrices od size $r(t)$ in (4.18) we obtain

$$\lim_{\tau\to t^+} \big(\Sigma_\alpha(\tau)\big)_{r(t)} = \Sigma_\alpha(t) \quad \text{and} \quad \lim_{\tau\to t^+} \big(\Sigma_\alpha(\tau)\big)_{r(t)}^{-1} = \Sigma_\alpha^{-1}(t), \tag{4.19}$$

because the matrix $\Sigma_\alpha(t)$ is invertible on $(\alpha,\infty)$. Now let $t\in(\alpha,\infty)$ and $s\geq t$ be fixed. By using the first formula in (4.19) with $t:=s$ together with (4.12) we get

$$\begin{aligned}\lim_{\tau\to s^+}\Sigma_\alpha(t,\tau)&\overset{(4.12)}{=}\lim_{\tau\to s^+}\big(\Sigma_\alpha(\tau)\big)_{r(t)}=\lim_{\tau\to s^+}\Big(\big(\Sigma_\alpha(\tau)\big)_{r(s)}\Big)_{r(t)}\\&=\Big(\lim_{\tau\to s^+}\big(\Sigma_\alpha(\tau)\big)_{r(s)}\Big)_{r(t)}\overset{(4.19)}{=}\big(\Sigma_\alpha(s)\big)_{r(t)}=\Sigma_\alpha(t,s).\end{aligned}\tag{4.20}$$

On the other hand, from (4.8) we have that the inequalities

$$\Sigma_\alpha^{-1}(s)\geq\big(\Sigma_\alpha^{-1}(\tau)\big)_{r(s)}\geq\big(\Sigma_\alpha(\tau)\big)_{r(s)}^{-1}\tag{4.21}$$

hold for every $\tau>s$. Upon taking $\tau\to s^+$ the second formula in (4.19) with $t:=s$ implies $\big(\Sigma_\alpha(\tau)\big)_{r(s)}^{-1}\to\Sigma_\alpha^{-1}(s)$ and by the aid of (4.21) we may conclude that

$$\lim_{\tau\to s^+}\big(\Sigma_\alpha^{-1}(\tau)\big)_{r(s)}=\Sigma_\alpha^{-1}(s).\tag{4.22}$$

Combining (4.12) and the equality in (4.22) yields that

$$\begin{aligned}\lim_{\tau\to s^+}\Sigma_\alpha^{\text{inv}}(t,\tau)&\overset{(4.12)}{=}\lim_{\tau\to s^+}\big(\Sigma_\alpha^{-1}(\tau)\big)_{r(t)}=\lim_{\tau\to s^+}\Big(\big(\Sigma_\alpha^{-1}(\tau)\big)_{r(s)}\Big)_{r(t)}\\&=\Big(\lim_{\tau\to s^+}\big(\Sigma_\alpha^{-1}(\tau)\big)_{r(s)}\Big)_{r(t)}\overset{(4.22)}{=}\big(\Sigma_\alpha^{-1}(s)\big)_{r(t)}=\Sigma_\alpha^{\text{inv}}(t,s).\end{aligned}\tag{4.23}$$

Thus, formulas (4.20) and (4.23) show the continuity of the functions $\Sigma_\alpha(t,s)$ and $\Sigma_\alpha^{\text{inv}}(t,s)$ in $s$. Finally, for any $\tau\in[s,\infty)\cap(\tau_l,\infty)$ the monotonicity of $\Sigma_\alpha^{\text{inv}}(t,s)$, inequality (4.6), and the nonnegative definiteness of $T_\alpha^\star$, by Remark 4.1.2(i), imply

$$\Sigma_\alpha^{\text{inv}}(t,s)\geq\Sigma_\alpha^{\text{inv}}(t,\tau)=\big(\Sigma_\alpha^{-1}(\tau)\big)_{r(t)}\overset{(4.6)}{\geq}\big(T_\alpha^\star\big)_{r(t)}\geq 0,$$

which establishes (4.13). The proof is complete. ■

**Remark 4.1.5.** The results in Theorem 4.1.4 allow to extend inequality (4.6) to the whole interval $(\alpha,\infty)$. Namely, by the aid of (4.12), the choice $s=t$ in (4.13) immediately implies the inequality

$$\Sigma_\alpha^{-1}(t)\geq\big(T_\alpha^\star\big)_{r(t)}\quad\text{for all}\quad t\in(\alpha,\infty).\tag{4.24}$$

In Section 3.1 we introduced the maximal order of abnormality $d_\infty$ by (3.2) and showed that there exists a point $\alpha\geq a$, for which the order of abnormality of (H) on $[\alpha,\infty)$ is $d_\infty$, i.e., condition (3.3) holds. Thus, in the remaining part of this section we consider the interval $[\alpha,\infty)$ with such a property. In particular, one can easily see that condition (3.3), that is, $d[\alpha,\infty)=d_\infty$, is equivalent with

$$d[\alpha,\infty)=d[t,\infty)\quad\text{for all } t\in[\alpha,\infty).\tag{4.25}$$

This is in a similar spirit as in Reid's construction of the principal solution in [30, pg. 402], in which the common quantity in (4.25) is also denoted by $d_\infty$. The next theorem, which is the first main result of this section, shows how condition (4.25) affects the asymptotic properties of the $S$-matrix corresponding to a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$.

**Theorem 4.1.6.** *Assume (1.1) and (4.25). Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and let $S_\alpha(t)$, $P_{\mathscr{S}_{\alpha\infty}}$, and $T_\alpha$ be its corresponding matrices in (2.8), (2.12), and (2.1.6). Then there exists $\beta \geq \alpha$ such that*

$$\operatorname{Im}\left[S_\alpha^\dagger(t) - T_\alpha\right] = \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} \quad \text{for all } t \in [\beta,\infty). \tag{4.26}$$

*Proof.* The results in Theorem 3.2.4 and Remark 3.3.2 imply that without loss of generality we may assume that $(X,U)$ is a minimal conjoined basis of (H) on $[\alpha,\infty)$. Let $P$ be the orthogonal projectors defined in (2.2). Then $\operatorname{rank} P = n - d[\alpha,\infty)$ and $P_{\mathscr{S}_{\alpha\infty}} = P$, by Definition 3.3.1 and Remark 3.3.3. Furthermore, consider a conjoined basis $(\hat{X},\hat{U})$ defined by

$$\begin{pmatrix} \hat{X} \\ \hat{U} \end{pmatrix} := \begin{pmatrix} X & \bar{X}_\alpha \\ U & \bar{U}_\alpha \end{pmatrix} \begin{pmatrix} I \\ -T_\alpha \end{pmatrix} = \begin{pmatrix} X - \bar{X}_\alpha T_\alpha \\ U - \bar{U}_\alpha T_\alpha \end{pmatrix} \qquad \text{on } [\alpha,\infty), \tag{4.27}$$

where $(\bar{X}_\alpha,\bar{U}_\alpha)$ is a conjoined basis associated with $(X,U)$ through Theorem 2.2.5. By Remark 2.1.6, the inclusion $\operatorname{Im} T_\alpha \subseteq \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} = \operatorname{Im} P$ holds. Hence, from Theorem 2.3.3 we conclude that $(\hat{X},\hat{U})$ is representable by $(X,U)$ on $[\alpha,\infty)$ and its representation in (4.27) does not depend on the choice of $(\bar{X}_\alpha,\bar{U}_\alpha)$, by Definition 2.3.1. Consequently, from (4.27), (2.37), and the definition of $P$ in (2.1) we obtain the equality $\hat{X} = X(P - S_\alpha T_\alpha)$ on $[\alpha,\infty)$. This implies that $\operatorname{Ker}(P - S_\alpha T_\alpha) \subseteq \operatorname{Ker} \hat{X}$ on $[\alpha,\infty)$. The opposite inclusion follows from the identities $X^\dagger X = P$ and $PS_\alpha = S_\alpha$, so that $\operatorname{Ker} \hat{X} = \operatorname{Ker}(P - S_\alpha T_\alpha)$ on $[\alpha,\infty)$. Moreover, recalling Remarks 3.1.3 and 4.1.2, we have $\operatorname{Im} S_\alpha(t) = \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} = \operatorname{Im} P$ for all $t > \tau_l$. This yields that $P = S_\alpha S_\alpha^\dagger = S_\alpha^\dagger S_\alpha$ on $(\tau_l,\infty)$ and consequently,

$$\operatorname{Ker}[P - S_\alpha(t)\, T_\alpha] = \operatorname{Ker}[S_\alpha^\dagger(t) - T_\alpha] \quad \text{for all } t \in (\tau_l,\infty).$$

Since $(X,U)$ has constant kernel on $[\alpha,\infty)$, it has no proper focal point in $(\alpha,\infty)$. By Proposition 1.5.2, the conjoined basis $(\hat{X},\hat{U})$ has only finitely many proper focal points in $(\alpha,\infty)$ and hence, the kernel of $\hat{X}(t)$ is constant on $[\beta,\infty)$ for some $\beta > \tau_l$. The above analysis then implies that

$$\operatorname{Ker} \hat{X}(t) = \operatorname{Ker}[S_\alpha^\dagger(t) - T_\alpha] \quad \text{for all } t \in [\beta,\infty). \tag{4.28}$$

Let $\hat{P}$ be the orthogonal projector defined in (2.2) through the conjoined basis $(\hat{X},\hat{U})$ on the interval $[\beta,\infty)$. We show that $\operatorname{Im} \hat{P} = \operatorname{Im} P$, i.e., $\hat{P} = P$, by the uniqueness of orthogonal projectors. One inclusion follows from the fact that for $t \in [\beta,\infty)$ we have $\operatorname{Im} \hat{P} = \operatorname{Im} \hat{X}^T(t) = \operatorname{Im}[S_\alpha^\dagger(t) - T_\alpha] \subseteq \operatorname{Im} P$, because $\operatorname{Im} S_\alpha^\dagger(t) = \operatorname{Im} P$ and $\operatorname{Im} T_\alpha \subseteq \operatorname{Im} P$. On the other hand, by assumption (4.25) and the first inequality in (3.15) with $X := \hat{X}$ on $[\beta,\infty)$, we obtain that

$$\operatorname{rank} P = n - d[\alpha,\infty) \overset{(4.25)}{=} n - d[\beta,\infty) \overset{(3.15)}{\leq} \operatorname{rank} \hat{P}.$$

Thus, $\operatorname{Im} P = \operatorname{Im} \hat{P}$ follows. Since $\hat{P}$ is the orthogonal projector onto $\operatorname{Im} \hat{X}^T(t)$ on $[\beta,\infty)$ and since $\hat{P} = P$ as we just proved, the equality in (4.28) reads as $\operatorname{Ker}[S_\alpha^\dagger(t) - T_\alpha] = \operatorname{Ker} P$ for all $t \in [\beta,\infty)$ or equivalently, $\operatorname{Im}[S_\alpha^\dagger(t) - T_\alpha] = \operatorname{Im} P$. Finally, substituting $P = P_{\mathscr{S}_{\alpha\infty}}$ into the last equality we obtain formula (4.26) and the proof is complete. ∎

**Remark 4.1.7.** Since $\operatorname{rank} P_{\mathscr{S}_{\alpha\infty}} = n - d[\alpha,\infty)$, by (3.14), the result in Theorem 4.1.6, i.e., equality (4.26), can be equivalently formulated as

$$\operatorname{rank}\left[S_\alpha^\dagger(t) - T_\alpha\right] = n - d[\alpha,\infty) \quad \text{for all } t \in [\beta,\infty). \tag{4.29}$$

Moreover, from Remark 3.3.7 we know that for $t \in (\tau_l,\infty)$ the rank of the matrix $S_\alpha^\dagger(t) - T_\alpha$ does not depend on the choice of the conjoined basis $(X,U)$ with constant kernel on $[\alpha,\infty)$. Therefore, we may conclude that under condition (4.25) formula (4.29) holds for every $S$-matrix $S_\alpha(t)$.

In the following theorem we interpret the result in Theorem 4.1.6 in terms of the matrices $\Sigma_\alpha^{\mathrm{inv}}(t,s)$ and $T_\alpha^\star$ from (4.12) and (4.4).

**Theorem 4.1.8.** *With the assumption and notation of Theorems 4.1.5 and 4.1.6, if condition (4.25) is satisfied, then for $t \in (\alpha,\infty)$ we have*

$$\Sigma_\alpha^{\mathrm{inv}}(t,s) > \left(T_\alpha^\star\right)_{r(t)} \quad \textit{for all } s \in [t,\infty). \tag{4.30}$$

*Proof.* According to (4.4) and the second formula in (4.1) on $[\beta,\infty)$, we have

$$S_\alpha^\dagger(t) - T_\alpha = V \begin{pmatrix} \Sigma_\alpha^{-1}(t) - T_\alpha^\star & 0 \\ 0 & 0_{n-r_\infty} \end{pmatrix} V^T \quad \text{for all } t \in [\beta,\infty).$$

Since $\beta > \tau_l$, formula (4.29) and inequality (4.6) then imply that the $r_\infty \times r_\infty$ matrix $\Sigma_\alpha^{-1}(t) - T_\alpha^\star$ is positive definite on $[\beta,\infty)$, i.e., $\Sigma_\alpha^{-1}(t) > T_\alpha^\star$ for all $t \in [\beta,\infty)$. This property means, in terms of the function $\Sigma_\alpha^{\mathrm{inv}}(t,s)$ introduced in (4.12), that for every $t \in (\alpha,\infty)$ we have

$$\Sigma_\alpha^{\mathrm{inv}}(t,s) > \left(T_\alpha^\star\right)_{r(t)} \quad \text{for all } s \in [t,\infty) \cap [\beta,\infty). \tag{4.31}$$

Moreover, from Theorem 4.1.4 we know that the function $\Sigma_\alpha^{\mathrm{inv}}(t,s)$ is nonincreasing in $s$ on $[t,\infty)$, which allows to extend (4.31) to the whole interval $[t,\infty)$. ∎

**Remark 4.1.9.** (i) From Theorem 4.1.8 it follows that condition (4.25) strengthens inequality (4.24) in Remark 4.1.5. That is, if (4.25) holds, then

$$\Sigma_\alpha^{-1}(t) > \left(T_\alpha^\star\right)_{r(t)} \quad \text{for all} \quad t \in (\alpha,\infty). \tag{4.32}$$

This is a direct consequence of inequality (4.30) in Theorem 4.1.8 for $s = t$.

(ii) When the system (H) is completely controllable on $[\alpha,\infty)$, then the matrix $S_\alpha(t)$ is invertible on $(\alpha,\infty)$, by [6, Proposition 2, pg. 38]. Therefore, in this case $r(t) \equiv n$ and we may take $V = I$. Consequently, $\Sigma_\alpha(t,s) = S_\alpha(s)$ and $\Sigma_\alpha^{\mathrm{inv}}(t,s) = S_\alpha^{-1}(s)$ on $[t,\infty)$, and $T_\alpha^\star = T_\alpha$. In addition, the inequality $S_\alpha^{-1}(t) > T_\alpha$ holds for all $t \in (\alpha,\infty)$ in this case.

The next results extend inequality (4.5) and Theorem 4.1.6 in the sense that the maximal orthogonal projector $P_{\mathscr{S}_{\alpha\infty}}$ is replaced by the orthogonal projector $P_{\mathscr{S}_\alpha}(t)$ and the statements are considered for $t$ in the whole interval $[\alpha,\infty)$ instead only for large $t$, see also Remark 4.1.11 below.

**Theorem 4.1.10.** *Assume (1.1). Let $S_\alpha(t)$, $P_{\mathscr{S}_\alpha}(t)$, $P_{\mathscr{S}_{\alpha\infty}}$, and $T_\alpha$ be the matrices which correspond to a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ in (2.8), Remark 2.1.5(i), (2.12), and Remark 2.1.6. Then*

$$S_\alpha^\dagger(t) - P_{\mathscr{S}_\alpha}(t)\, T_\alpha\, P_{\mathscr{S}_\alpha}(t) \geq 0, \quad \text{on } [\alpha,\infty). \tag{4.33}$$

*Moreover, if the condition in (4.25) holds, then for all $t \in [\alpha,\infty)$ we have*

$$\operatorname{Im}\left[S_\alpha^\dagger(t) - P_{\mathscr{S}_\alpha}(t)\, T_\alpha\, P_{\mathscr{S}_\alpha}(t)\right] = \operatorname{Im} P_{\mathscr{S}_\alpha}(t), \tag{4.34}$$

$$\operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(t)\, T_\alpha\right] = \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} = \operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(t)\, T_\alpha\right]^T. \tag{4.35}$$

*Proof.* When $t = \alpha$, all the formulas in (4.33), (4.34), and (4.35) hold trivially, because $S_\alpha(\alpha) = 0 = P_{\mathscr{S}_\alpha}(\alpha)$ and the matrix $P_{\mathscr{S}_{\alpha\infty}}$ is symmetric. Fix now $t \in (\alpha,\infty)$. With the aid of the expressions in (4.1), (4.3), and (4.4) we have

$$S_\alpha^\dagger(t) - P_{\mathscr{S}_\alpha}(t)\, T_\alpha\, P_{\mathscr{S}_\alpha}(t) = V \begin{pmatrix} \Sigma_\alpha^{-1}(t) - \left(T_\alpha^\star\right)_{r(t)} & 0 \\ 0 & 0_{n-r(t)} \end{pmatrix} V^T. \tag{4.36}$$

By (4.36) and (4.24) we get inequality (4.33). Moreover, if condition (4.25) is satisfied, then the equality in (4.36) and (4.32) imply that

$$\operatorname{rank}\left[S_\alpha^\dagger(t) - P_{\mathscr{S}_\alpha}(t)\, T_\alpha\, P_{\mathscr{S}_\alpha}(t)\right] = \operatorname{rank}\left[\Sigma_\alpha^{-1}(t) - \left(T_\alpha^\star\right)_{r(t)}\right] = r(t).$$

This equality together with the identities $S_\alpha^\dagger(t) = S_\alpha^\dagger(t)\, P_{\mathscr{S}_\alpha}(t)$ and $\operatorname{rank} P_{\mathscr{S}_\alpha}(t) = r(t)$ then yields the formula in (4.34). In order to prove (4.35), we partition the $r_\infty \times r_\infty$ matrix $T_\alpha^\star$ into the refined block structure in the dimension $r(t) + q(t)$, where $q(t) := r_\infty - r(t) \geq 0$. From (4.1), (4.3), and (4.4) we then obtain

$$\begin{aligned} P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(t)\, T_\alpha = {} & V \begin{pmatrix} I_{r(t)} & 0 & 0 \\ 0 & I_{q(t)} & 0 \\ 0 & 0 & 0_{n-r_\infty} \end{pmatrix} V^T \\ & - V \begin{pmatrix} \Sigma_\alpha(t) & 0 & 0 \\ 0 & 0_{q(t)} & 0 \\ 0 & 0 & 0_{n-r_\infty} \end{pmatrix} \begin{pmatrix} \left(T_\alpha^\star\right)_{r(t)} & E & 0 \\ E^T & F & 0 \\ 0 & 0 & 0_{n-r_\infty} \end{pmatrix} V^T, \end{aligned}$$

where $E \in \mathbb{R}^{r(t)\times q(t)}$ and $F \in \mathbb{R}^{q(t)\times q(t)}$. Performing the matrix multiplication, we get

$$P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(t)\, T_\alpha = V \begin{pmatrix} I_{r(t)} - \Sigma_\alpha(t) \left(T_\alpha^\star\right)_{r(t)} & -\Sigma_\alpha(t)\, E & 0 \\ 0 & I_{q(t)} & 0 \\ 0 & 0 & 0_{n-r_\infty} \end{pmatrix} V^T. \tag{4.37}$$

From (4.32) we know that $\Sigma_\alpha^{-1}(t) - \left(T_\alpha^\star\right)_{r(t)}$ is positive definite and hence, the matrix $I_{r(t)} - \Sigma_\alpha(t) \left(T_\alpha^\star\right)_{r(t)} = \Sigma_\alpha(t) \left(\Sigma_\alpha^{-1}(t) - \left(T_\alpha^\star\right)_{r(t)}\right)$ in (4.37) is invertible. Equality (4.37) then implies that $\operatorname{rank}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(t)\, T_\alpha\right] = n - d[\alpha,\infty)$, which together with the identities $S_\alpha(t) = P_{\mathscr{S}_{\alpha\infty}} S_\alpha(t)$, $T_\alpha = P_{\mathscr{S}_{\alpha\infty}} T_\alpha$, and $\operatorname{rank} P_{\mathscr{S}_{\alpha\infty}} = n - d[\alpha,\infty)$ yields formulas (4.35). The proof is complete. ∎

**Remark 4.1.11.** We note that inequality (4.33) in Theorem 4.1.10 is an extension of (4.5) in Remark 4.1.2 to the whole interval $[\alpha,\infty)$, because $P_{\mathscr{S}_\alpha}(t)=P_{\mathscr{S}_{\alpha\infty}}$ and $P_{\mathscr{S}_\alpha}(t)\,T_\alpha\,P_{\mathscr{S}_\alpha}(t)=P_{\mathscr{S}_{\alpha\infty}}T_\alpha P_{\mathscr{S}_{\alpha\infty}}=T_\alpha$ on $(\tau_l,\infty)$, by Remarks 3.1.3 and 2.1.6. In a similar way, formula (4.34) extends equality (4.26) to the interval $[\alpha,\infty)$.

In the last result of this section we prove the equivalence of the condition on the maximal order of abnormality in (4.25) and conditions (4.34) and (4.35).

**Theorem 4.1.12.** *Assume* (1.1) *and let there exists a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$. *Then each of the equalities* (4.34) *and* (4.35) *holds for some (and hence for any) S-matrix* $S_\alpha(t)$ *associated with a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *if and only if condition* (4.25) *is satisfied.*

*Proof.* We have already proven in Theorem 4.1.10 that (4.25) implies (4.34) for any $S$-matrix $S_\alpha(t)$ which corresponds to a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. We now assume that (4.34) holds for some such a matrix $S_\alpha(t)$ associated with a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. We will prove that (4.35) is satisfied, by showing the equality

$$\operatorname{Ker}[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]=\operatorname{Ker}P_{\mathscr{S}_{\alpha\infty}}\quad\text{on } [\alpha,\infty). \tag{4.38}$$

First we observe that $\operatorname{Ker}P_{\mathscr{S}_{\alpha\infty}}\subseteq\operatorname{Ker}[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]$ on $[\alpha,\infty)$, because $T_\alpha=T_\alpha P_{\mathscr{S}_{\alpha\infty}}$, by Remark 2.1.6 and the symmetry of $T_\alpha$ and $P_{\mathscr{S}_{\alpha\infty}}$. In order to show the opposite inclusion, fix $t\in[\alpha,\infty)$ and let $v\in\operatorname{Ker}[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]$. Then the vector $w:=P_{\mathscr{S}_{\alpha\infty}}v$ satisfies $w=S_\alpha(t)\,T_\alpha\,v=S_\alpha(t)\,T_\alpha\,P_{\mathscr{S}_{\alpha\infty}}v=S_\alpha(t)\,T_\alpha\,w$, which yields that $w\in\operatorname{Im}S_\alpha(t)=\operatorname{Im}P_{\mathscr{S}_\alpha}(t)$, by Remark 2.1.5. Moreover, multiplying the last equality by $S_\alpha^\dagger(t)$ from the left and using the identities $S_\alpha^\dagger(t)\,S_\alpha(t)=P_{\mathscr{S}_\alpha}(t)$ and $w=P_{\mathscr{S}_\alpha}(t)\,w$ we get $S_\alpha^\dagger(t)\,w=P_{\mathscr{S}_\alpha}(t)\,T_\alpha\,P_{\mathscr{S}_\alpha}(t)\,w$. Therefore, the vector $w$ satisfies $w\in\operatorname{Ker}\left[S_\alpha^\dagger(t)-P_{\mathscr{S}_\alpha}(t)\,T_\alpha\,P_{\mathscr{S}_\alpha}(t)\right]=\operatorname{Ker}P_{\mathscr{S}_\alpha}(t)$, by taking the orthogonal complements on both sides of equality (4.34). Thus, $w\in\operatorname{Im}P_{\mathscr{S}_\alpha}(t)\cap\operatorname{Ker}P_{\mathscr{S}_\alpha}(t)=\{0\}$. This implies that $v\in\operatorname{Ker}P_{\mathscr{S}_{\alpha\infty}}$. Consequently, the inclusion $\operatorname{Ker}[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]\subseteq\operatorname{Ker}P_{\mathscr{S}_{\alpha\infty}}$ holds and the equality in (4.38) is established. One can easily check that formula (4.38) is equivalent with the equality $\operatorname{Im}[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]^T=\operatorname{Im}P_{\mathscr{S}_{\alpha\infty}}$ on $[\alpha,\infty)$, as well as with $\operatorname{Im}[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]=\operatorname{Im}P_{\mathscr{S}_{\alpha\infty}}$ on $[\alpha,\infty)$. The second equivalence follows from the identity $P_{\mathscr{S}_{\alpha\infty}}S_\alpha(t)=S_\alpha(t)$ on $[\alpha,\infty)$ and from the fact that the matrices $P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha$ and $[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(t)\,T_\alpha]^T$ have the same ranks. Therefore, the formulas in (4.35) holds.

We now suppose that (4.35) holds for some matrix $S_\alpha(t)$, which corresponds to a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$. Fix any $\gamma\in[\alpha,\infty)$. We will prove that $d[\gamma,\infty)=d[\alpha,\infty)$, by showing that the $S$-matrix $S_\gamma(t)$ satisfies $\operatorname{Im}S_\gamma(t)=\operatorname{Im}S_\alpha(t)$ for large $t$. Clearly, the formula $S_\gamma(t)=S_\alpha(t)-S_\alpha(\gamma)$ holds on $[\alpha,\infty)$, by (2.8). From inclusions (2.13) in Remark 2.1.5(i) we get $\operatorname{Im}S_\alpha(\gamma)\subseteq\operatorname{Im}P_{\mathscr{S}_{\alpha\infty}}$ and $\operatorname{Im}S_\alpha(t)=\operatorname{Im}P_{\mathscr{S}_{\alpha\infty}}$ for large $t$. This implies that $S_\alpha(\gamma)=P_{\mathscr{S}_{\alpha\infty}}S_\alpha(\gamma)=S_\alpha(\gamma)\,P_{\mathscr{S}_{\alpha\infty}}$, by the symmetry of $S_\alpha(\gamma)$ and $P_{\mathscr{S}_{\alpha\infty}}$, and $P_{\mathscr{S}_{\alpha\infty}}S_\alpha(t)=S_\alpha(t)$ with $P_{\mathscr{S}_{\alpha\infty}}=S_\alpha^\dagger(t)\,S_\alpha(t)=S_\alpha(t)\,S_\alpha^\dagger(t)$ for large $t$. Consequently, for large $t$ we have

$$S_\gamma(t)=P_{\mathscr{S}_{\alpha\infty}}S_\alpha(t)-S_\alpha(\gamma)\,S_\alpha^\dagger(t)\,S_\alpha(t)=\left[P_{\mathscr{S}_{\alpha\infty}}-S_\alpha(\gamma)\,S_\alpha^\dagger(t)\right]S_\alpha(t). \tag{4.39}$$

If we now let $t \to \infty$, then $P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t) \to P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, T_\alpha$. By using assumption (4.35), this limiting matrix then satisfies

$$\operatorname{Im}\,[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, T_\alpha] = \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} \quad \text{and} \quad \operatorname{Im}\,[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, T_\alpha]^T = \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}}.$$

Moreover, we have that $\operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right] \subseteq \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} = \operatorname{Im}\,[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, T_\alpha]$ and $\operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]^T \subseteq \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} = \operatorname{Im}\,[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, T_\alpha]^T$ for all $t \in [\alpha,\infty)$. Therefore, by Lemma A.1.5 and Corollary A.1.7 in Appendix A, we obtain that

$$\operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right] = \operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]^T = \operatorname{Im} P_{\mathscr{S}_{\alpha\infty}} \quad \text{for large } t, \tag{4.40}$$

as well as we get the convergence

$$\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]^\dagger \to [P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, T_\alpha]^\dagger \quad \text{for } t \to \infty.$$

With the aid of Remark 1.2.3(iv) and equalities (4.39) and (4.40) we now calculate

$$\begin{aligned} S_\gamma^\dagger(t) &= [P_{\mathscr{S}_{\alpha\infty}} S_\alpha(t)]^\dagger \left\{ [P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)]\, P_{\mathscr{S}_{\alpha\infty}} \right\}^\dagger \\ &= S_\alpha^\dagger(t) \left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]^\dagger \end{aligned} \tag{4.41}$$

for large $t$. By using Remark 1.2.3(ii), the matrix $S_\gamma(t)\, S_\gamma^\dagger(t)$ is the orthogonal projector onto $\operatorname{Im} S_\gamma(t)$. Thus, by (4.39) and (4.41) we have for large $t$ that

$$\begin{aligned} S_\gamma(t)\, S_\gamma^\dagger(t) &= \left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right] S_\alpha(t)\, S_\alpha^\dagger(t) \left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]^\dagger \\ &= \left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right] \left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]^\dagger, \end{aligned} \tag{4.42}$$

where we used the identities $S_\alpha(t) S_\alpha^\dagger(t) = P_{\mathscr{S}_{\alpha\infty}}$ and $S_\alpha^\dagger(t)\, P_{\mathscr{S}_{\alpha\infty}} = S_\alpha^\dagger(t)$ for large $t$. But since by Remark 1.2.3(ii) the matrix in (4.42) is the orthogonal projector onto $\operatorname{Im}\left[P_{\mathscr{S}_{\alpha\infty}} - S_\alpha(\gamma)\, S_\alpha^\dagger(t)\right]$, we conclude from (4.40) and (4.42) that $S_\gamma(t)\, S_\gamma^\dagger(t) = P_{\mathscr{S}_{\alpha\infty}}$ for large $t$. This means that the two projectors onto $\operatorname{Im} S_\gamma(t)$ and $\operatorname{Im} S_\alpha(t)$ are the same for large $t$ (they are equal to $P_{\mathscr{S}_{\alpha\infty}}$). Therefore, $\operatorname{Im} S_\gamma(t) = \operatorname{Im} S_\alpha(t)$ for large $t$, which implies through Remark 3.1.3 that $n - d[\gamma,\infty) = \operatorname{rank} S_\gamma(t) = \operatorname{rank} S_\alpha(t) = n - d[\alpha,\infty)$ for large $t$. This shows that $d[\alpha,\infty) = d[\gamma,\infty)$. Since the point $\gamma \in [\alpha,\infty)$ was chosen arbitrarily, condition (4.25) holds and the proof is complete. ∎

## 4.2 Classification of minimal conjoined bases

The content of this section is a complete classification of all conjoined bases of (H) which are minimal on given subinterval $[\alpha,\infty)$ with maximal order of abnormality. This turns out to be one of the crucial results of this chapter. It will be utilized in the characterization of the matrices $T_\alpha$ in the next section, as well as in the construction of minimal antiprincipal solutions of (H) in Chapter 5. We recall that by Definition 3.3.1 a conjoined basis $(X,U)$ of (H) is minimal on $[\alpha,\infty)$ if the matrix $X(t)$ has constant kernel on $[\alpha,\infty)$ and its rank is equal to $n - d[\alpha,\infty)$, see estimate (3.15).

**Theorem 4.2.1.** *Assume (1.1). Let $(X,U)$ be a minimal conjoined basis of (H) on $[\alpha,\infty)$ with its corresponding matrices $P_{\mathscr{S}_{\alpha\infty}}$ and $T_\alpha$ in (2.12) and Remark 2.1.6. Moreover, assume that $d[\alpha,\infty)=d_\infty$. Then a solution $(\tilde{X},\tilde{U})$ of (H) is a minimal conjoined basis on $[\alpha,\infty)$ if and only if there exist matrices $M,N\in\mathbb{R}^{n\times n}$ such that*

$$\tilde{X}(\alpha)=X(\alpha)M, \quad \tilde{U}(\alpha)=U(\alpha)M+X^{\dagger T}(\alpha)N, \tag{4.43}$$

$$M \text{ is nonsingular}, \quad M^TN=N^TM, \quad \operatorname{Im}N\subseteq\operatorname{Im}P_{\mathscr{S}_{\alpha\infty}}, \quad NM^{-1}+T_\alpha\geq 0. \tag{4.44}$$

*Proof.* Let $(X,U)$ and $\alpha$ be as in the theorem. Since $(X,U)$ is a minimal conjoined basis on $[\alpha,\infty)$, from Remark 3.3.3 it follows that the orthogonal projector $P$ defined in (2.2) satisfies $P=P_{\mathscr{S}_{\alpha\infty}}$. Let $(\tilde{X},\tilde{U})$ be another minimal conjoined basis on $[\alpha,\infty)$. Then, according to Remark 3.3.5, $(X,U)$ and $(\tilde{X},\tilde{U})$ are mutually representable on $[\alpha,\infty)$. Therefore, by Theorem 2.3.8 and its proof with $\beta:=\alpha$, $(X_1,U_1):=(X,U)$, and $(X_2,U_2):=(\tilde{X},\tilde{U})$, there exist matrices $M,N\in\mathbb{R}^{n\times n}$ such that (4.43) and the first three conditions in (4.44) hold. Moreover, let $\tilde{T}_\alpha$ be the $T$-matrix defined in Remark 2.1.6 through the function $\tilde{S}_\alpha(t)$ in (2.8), which is associated with $(\tilde{X},\tilde{U})$. By using formula (3.64) with $T_{1\beta}:=T_\alpha$, $T_{2\beta}:=\tilde{T}_\alpha$, $M_{1\beta}:=M$, and $N_1:=N$, we have

$$\tilde{T}_\alpha=M^TT_\alpha M+M^TN, \quad \text{i.e.,} \quad NM^{-1}+T_\alpha=M^{T-1}\tilde{T}_\alpha M^{-1}\geq 0, \tag{4.45}$$

since $\tilde{T}_\alpha\geq 0$. This shows the fourth condition in (4.44). Conversely, let $(\tilde{X},\tilde{U})$ be a solution of (H) satisfying (4.43) and (4.44). The first three conditions in (4.44) together with the identity $X^T(\alpha)X^{\dagger T}(\alpha)=P=P_{\mathscr{S}_{\alpha\infty}}$ and the fact that $(X,U)$ is a conjoined basis imply that $(\tilde{X},\tilde{U})$ is also a conjoined basis of (H) and that the matrix $N$ is (constant) Wronskian of $(X,U)$ and $(\tilde{X},\tilde{U})$. In addition, $(\tilde{X},\tilde{U})$ is representable by $(X,U)$ on $[\alpha,\infty)$, by Theorem 2.3.3. Let $S_\alpha(t)$ be the $S$-matrix in (2.8) corresponding to $(X,U)$. By using formula (2.50) in Remark 2.3.4(ii) with $\beta:=\alpha$, $X_0:=\tilde{X}$, $M_\beta:=M$ and the identities $X(t)=X(t)P=X(t)P_{\mathscr{S}_{\alpha\infty}}$ and $P_{\mathscr{S}_{\alpha\infty}}S_\alpha(t)=S_\alpha(t)$ for $t\in[\alpha,\infty)$, we obtain that

$$\tilde{X}(t)=X(t)\,[P_{\mathscr{S}_{\alpha\infty}}M+S_\alpha(t)N] \quad \text{on } [\alpha,\infty). \tag{4.46}$$

We will show that $(\tilde{X},\tilde{U})$ has constant kernel on $[\alpha,\infty)$ with $\operatorname{Ker}\tilde{X}(t)=\operatorname{Ker}P_{\mathscr{S}_{\alpha\infty}}M$ on $[\alpha,\infty)$. First we note that the symmetry of $M^TN$ and the identity $P_{\mathscr{S}_{\alpha\infty}}N=N$ give $NM^{-1}P_{\mathscr{S}_{\alpha\infty}}=M^{T-1}N^TP_{\mathscr{S}_{\alpha\infty}}=M^{T-1}N^T=NM^{-1}$. Hence, by (4.46),

$$\tilde{X}(t)=X(t)\left[P_{\mathscr{S}_{\alpha\infty}}M+S_\alpha(t)NM^{-1}M\right]=X(t)\left[I+S_\alpha(t)NM^{-1}\right]P_{\mathscr{S}_{\alpha\infty}}M \tag{4.47}$$

on $[\alpha,\infty)$. Therefore, $\operatorname{Ker}P_{\mathscr{S}_{\alpha\infty}}M\subseteq\operatorname{Ker}\tilde{X}(t)$ on $[\alpha,\infty)$. Fix now $t\in[\alpha,\infty)$, $v\in\operatorname{Ker}\tilde{X}(t)$, and set $w:=P_{\mathscr{S}_{\alpha\infty}}Mv$. Then $X(t)\,[w+S_\alpha(t)NM^{-1}w]=0$ by (4.47). Multiplying the latter equality by $X^\dagger(t)$ from the left and using the identities $X^\dagger(t)X(t)=P_{\mathscr{S}_{\alpha\infty}}$, $P_{\mathscr{S}_{\alpha\infty}}S_\alpha(t)=S_\alpha(t)$ and $w=P_{\mathscr{S}_{\alpha\infty}}w$, we get $w=-S_\alpha(t)NM^{-1}w$. This implies that $w\in\operatorname{Im}S_\alpha(t)=\operatorname{Im}P_{\mathscr{S}_\alpha}(t)$, by (2.13) in Remark 2.1.5(i), and consequently,

$$w^TS_\alpha^\dagger(t)\,w=-w^TS_\alpha^\dagger(t)\,S_\alpha(t)\,NM^{-1}w=-w^TP_{\mathscr{S}_\alpha}(t)\,NM^{-1}P_{\mathscr{S}_\alpha}(t)\,w. \tag{4.48}$$

Combining the equality in (4.48) and the last condition in (4.44) then yields the inequality $w^TS_\alpha^\dagger(t)\,w\leq w^TP_{\mathscr{S}_\alpha}(t)\,T_\alpha P_{\mathscr{S}_\alpha}(t)\,w$, or equivalently

$$w^T\left[S_\alpha^\dagger(t)-P_{\mathscr{S}_\alpha}(t)\,T_\alpha P_{\mathscr{S}_\alpha}(t)\right]w\leq 0.$$

But $S^{\dagger}_{\alpha}(t) - P_{\mathscr{S}_\alpha}(t)\, T_\alpha\, P_{\mathscr{S}_\alpha}(t) \geq 0$ according to (4.33) in Theorem 4.1.10 and thus, $w \in \operatorname{Ker}\left[S^{\dagger}_{\alpha}(t) - P_{\mathscr{S}_\alpha}(t)\, T_\alpha\, P_{\mathscr{S}_\alpha}(t)\right] = \operatorname{Ker} P_{\mathscr{S}_\alpha}(t)$, by the formula in (4.34). Hence, we obtain that $w \in \operatorname{Ker} P_{\mathscr{S}_\alpha}(t) \cap \operatorname{Im} P_{\mathscr{S}_\alpha}(t) = \{0\}$. This shows that $v \in \operatorname{Ker} P_{\mathscr{S}_{\alpha\infty}} M$, i.e., $\operatorname{Ker} \tilde{X}(t) \subseteq \operatorname{Ker} P_{\mathscr{S}_{\alpha\infty}} M$. Finally, the first formula in (4.43) and the invertibility of $M$ imply that $\operatorname{rank} \tilde{X}(t) = \operatorname{rank} \tilde{X}(\alpha) = \operatorname{rank} X(\alpha) = n - d[\alpha,\infty)$ on $[\alpha,\infty)$. Consequently, $(\tilde{X},\tilde{U})$ is a minimal conjoined basis of (H) on $[\alpha,\infty)$ and the proof is complete. ■

**Remark 4.2.2.** From the proof of Theorem 4.2.1 it follows that the matrices $\tilde{T}_\alpha$ and $T_\alpha$, which correspond to the conjoined bases $(\tilde{X},\tilde{U})$ and $(X,U)$ in Remark 2.1.6, satisfy (4.45). In particular, this implies that

$$\operatorname{rank} \tilde{T}_\alpha = \operatorname{rank}(NM^{-1} + T_\alpha). \tag{4.49}$$

Formula (4.49) will be important for the construction of principal and antiprincipal solutions of (H) at infinity in Chapter 5.

**Remark 4.2.3.** Let $(X,U)$ be a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. From Remark 2.1.5(ii) we know that the set $\operatorname{Im} P_{\mathscr{S}_{\beta\infty}}$, where $P_{\mathscr{S}_{\beta\infty}}$ is the associated orthogonal projector defined in (2.12), is nonincreasing in $\beta \in [\alpha,\infty)$. Moreover, by the aid of equality (3.14), we obtain that the set $\operatorname{Im} P_{\mathscr{S}_{\beta\infty}}$ is constant in $\beta \in [\alpha,\infty)$ if and only if condition (4.25) holds. In this case, the matrix $P_{\mathscr{S}_{\beta\infty}}$ is the same for all points $\beta \in [\alpha,\infty)$. Therefore, under the condition in (4.25) we will sometimes drop the index $\beta$ in the notation $P_{\mathscr{S}_{\beta\infty}}$ and use only $P_{\mathscr{S}\infty}$, highlighting the uniqueness of the orthogonal projector $P_{\mathscr{S}\infty}$ on $[\alpha,\infty)$.

## 4.3 Properties of $T$-matrices

In this section we complete the information about the matrices $T_\beta$ in Remark 2.1.6, which correspond to conjoined bases of (H) with constant kernel on $[\alpha,\infty)$. In particular, in the following theorem we derive monotonicity properties of the set $\operatorname{Im} T_\beta$ for $\beta \in [\alpha,\infty)$.

**Theorem 4.3.1.** *Assume* (1.1). *Let* $(X,U)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$ *and let* $T_\beta$ *be its corresponding matrix defined in Remark* 2.1.6 *for* $\beta \in [\alpha,\infty)$. *Then the following statements hold.*

(i) *The set* $\operatorname{Im} T_\beta$ *is nondecreasing in* $\beta$ *on* $[\alpha,\infty)$.

(ii) *The set* $\operatorname{Im} T_\beta$ *is constant in* $\beta$ *on* $[\alpha,\infty)$ *if and only if condition* (4.25) *holds.*

*Proof.* Let $(X,U)$ be as in the theorem. Fix $\beta,\gamma \in [\alpha,\infty)$ so that $\beta \leq \gamma$. Let $S_\beta(t)$, $S_\gamma(t)$, $P_{\mathscr{S}_{\beta\infty}}$, $P_{\mathscr{S}_{\gamma\infty}}$, and $T_\beta$, $T_\gamma$ be the matrices in (2.8), (2.12), and Remark 2.1.6, respectively, which correspond to $(X,U)$. Then $S_\gamma(t) = S_\beta(t) - S_\beta(\gamma)$ on $[\alpha,\infty)$ and the inclusion $\operatorname{Im} P_{\mathscr{S}_{\gamma\infty}} \subseteq \operatorname{Im} P_{\mathscr{S}_{\beta\infty}}$ holds, by Remark 2.1.5(ii). Moreover, choose $\delta \geq \alpha$ so that the equalities $\operatorname{Im} S_\beta(t) = \operatorname{Im} P_{\mathscr{S}_{\beta\infty}}$ and $\operatorname{Im} S_\gamma(t) = \operatorname{Im} P_{\mathscr{S}_{\gamma\infty}}$ are satisfied for all $t \in [\delta,\infty)$. This yields the identities $S_\gamma(t) = S_\gamma(t)\, P_{\mathscr{S}_{\gamma\infty}} = P_{\mathscr{S}_{\gamma\infty}} S_\gamma(t)$ and $P_{\mathscr{S}_{\gamma\infty}} = S^{\dagger}_{\gamma}(t)\, S_\gamma(t)$ on $[\delta,\infty)$. Consequently, we have

$$S_\beta(t)\, P_{\mathscr{S}_{\gamma\infty}} = S_\gamma(t)\, P_{\mathscr{S}_{\gamma\infty}} + S_\beta(\gamma)\, P_{\mathscr{S}_{\gamma\infty}} = \left[P_{\mathscr{S}_{\gamma\infty}} + S_\beta(\gamma)\, S^{\dagger}_{\gamma}(t)\right] S_\gamma(t) \tag{4.50}$$

for all $t \in [\delta,\infty)$. Furthermore, by using (4.50) together with the equalities $S_\gamma^\dagger(t) = P_{\mathscr{S}_\gamma\infty} S_\gamma^\dagger(t) = S_\gamma^\dagger(t) P_{\mathscr{S}_\gamma\infty}$, $P_{\mathscr{S}_\gamma\infty} = P_{\mathscr{S}_\beta\infty} P_{\mathscr{S}_\gamma\infty}$, $P_{\mathscr{S}_\beta\infty} = S_\beta^\dagger(t) S_\beta(t)$, and $S_\gamma(t) S_\gamma^\dagger(t) = P_{\mathscr{S}_\gamma\infty}$ on $[\delta,\infty)$, we get that

$$\begin{aligned} S_\gamma^\dagger(t) &= P_{\mathscr{S}_\gamma\infty} S_\gamma^\dagger(t) = P_{\mathscr{S}_\beta\infty} P_{\mathscr{S}_\gamma\infty} S_\gamma^\dagger(t) = S_\beta^\dagger(t) S_\beta(t) P_{\mathscr{S}_\gamma\infty} S_\gamma^\dagger(t) \\ &\overset{(4.50)}{=} S_\beta^\dagger(t) \left[P_{\mathscr{S}_\gamma\infty} + S_\beta(\gamma) S_\gamma^\dagger(t)\right] S_\gamma(t) S_\gamma^\dagger(t) \\ &= S_\beta^\dagger(t) \left[P_{\mathscr{S}_\gamma\infty} + S_\beta(\gamma) S_\gamma^\dagger(t)\right] \end{aligned} \tag{4.51}$$

for all $t \in [\delta,\infty)$. Finally, upon taking $t \to \infty$ in (4.51) we obtain

$$T_\gamma = T_\beta \left[P_{\mathscr{S}_\gamma\infty} + S_\beta(\gamma) T_\gamma\right], \tag{4.52}$$

which yields $\operatorname{Im} T_\gamma \subseteq \operatorname{Im} T_\beta$, showing (i). For the proof of part (ii) we note that equality (4.52) is equivalent with the formula

$$T_\gamma \left[P_{\mathscr{S}_\gamma\infty} - S_\beta(\gamma) T_\beta\right] = P_{\mathscr{S}_\gamma\infty} T_\beta, \tag{4.53}$$

as one can easily verify by using the symmetry of $P_{\mathscr{S}_\gamma\infty}$, $T_\beta$, and $T_\gamma$ together with the identity $T_\gamma = T_\gamma P_{\mathscr{S}_\gamma\infty}$. Now, if the set $\operatorname{Im} T_s$ is constant in $s$ on $[\alpha,\infty)$, then $\operatorname{Im} T_\beta = \operatorname{Im} T_\gamma$. In particular, we have the inclusions $\operatorname{Im} T_\beta = \operatorname{Im} T_\gamma \subseteq \operatorname{Im} P_{\mathscr{S}_\gamma\infty}$ and $\operatorname{Im} T_\gamma = \operatorname{Im} T_\beta \subseteq \operatorname{Im} P_{\mathscr{S}_\beta\infty}$ and consequently, the identities $P_{\mathscr{S}_\gamma\infty} T_\beta = T_\beta P_{\mathscr{S}_\gamma\infty} = T_\beta$ and $P_{\mathscr{S}_\beta\infty} T_\gamma = T_\gamma P_{\mathscr{S}_\beta\infty} = T_\gamma$. Combining these observations with (4.53) we get that

$$\begin{aligned} T_\gamma \left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right] &= T_\gamma P_{\mathscr{S}_\beta\infty} - T_\gamma S_\beta(\gamma) T_\beta = T_\gamma - T_\gamma S_\beta(\gamma) T_\beta \\ &= T_\gamma \left[P_{\mathscr{S}_\gamma\infty} - S_\beta(\gamma) T_\beta\right] \overset{(4.53)}{=} P_{\mathscr{S}_\gamma\infty} T_\beta = T_\beta. \end{aligned} \tag{4.54}$$

We will show that $\operatorname{Ker}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right] = \operatorname{Ker} P_{\mathscr{S}_\beta\infty}$. Indeed, the identity $T_\beta = T_\beta P_{\mathscr{S}_\beta\infty}$ yields the inclusion $\operatorname{Ker} P_{\mathscr{S}_\beta\infty} \subseteq \operatorname{Ker}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right]$. On the other hand, every vector $v \in \operatorname{Ker}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right]$ satisfies $T_\beta v = 0$, by (4.54) and consequently, $v \in \operatorname{Ker} P_{\mathscr{S}_\beta\infty}$. And since $S_\beta(\gamma) = S_\beta(\gamma) P_{\mathscr{S}_\beta\infty}$ and $\operatorname{def}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right]^T = \operatorname{def}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right]$, we may conclude that

$$\operatorname{Ker}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right] = \operatorname{Ker} P_{\mathscr{S}_\beta\infty} = \operatorname{Ker}\left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right]^T. \tag{4.55}$$

With the choice $\beta := \alpha$ and arbitrary $\gamma \geq \alpha$, the condition in (4.55) is then equivalent with (4.35) in Theorem 4.1.10. Thus, condition (4.25) holds, by Theorem 4.1.12. Conversely, suppose that (4.25) is satisfied. Then the equality $P_{\mathscr{S}_\gamma\infty} = P_{\mathscr{S}_\beta\infty}$ holds, by Remark 4.2.2. Moreover, the formula in (4.53) then reads as $T_\gamma \left[P_{\mathscr{S}_\beta\infty} - S_\beta(\gamma) T_\beta\right] = P_{\mathscr{S}_\beta\infty} T_\beta = T_\beta$, because $P_{\mathscr{S}_\beta\infty} T_\beta = T_\beta$. Therefore, the inclusion $\operatorname{Im} T_\beta \subseteq \operatorname{Im} T_\gamma$ holds, which together with the result of part (i) gives the equality $\operatorname{Im} T_\gamma = \operatorname{Im} T_\beta$. Since $\beta$ and $\gamma$ were chosen arbitrarily so that $\beta \geq \gamma \geq \alpha$, the last equality implies that the set $\operatorname{Im} T_s$ is constant in $s$ on $[\alpha,\infty)$. ■

In the main result of this section we establish a criterion for the classification of all $T$-matrices, which correspond to conjoined bases of (H) with constant kernel on $[\alpha,\infty)$ satisfying condition (4.25).

**Theorem 4.3.2.** *Assume that (1.1) holds and that system (H) is nonoscillatory. Then a matrix $D \in \mathbb{R}^{n\times n}$ is a $T$-matrix of a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$ if and only if*

$$D \text{ is symmetric}, \quad D \geq 0, \quad \operatorname{rank} D \leq n - d_\infty. \tag{4.56}$$

*Proof.* Let $(X,U)$ be a conjoined basis of (H) with constant kernel on an interval $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$. Fix $\beta \in [\alpha,\infty)$ and let $S_\beta(t)$, $P_{\mathscr{S}_\beta\infty}$, and $T_\beta$ be the matrices in (2.8), (2.12), and Remark 2.1.6, which correspond to $(X,U)$. We will show that the matrix $D := T_\beta$ satisfies conditions (4.56). From Remark 2.1.6 we obtain that $D$ is symmetric, nonnegative definite, and $\operatorname{Im} D \subseteq \operatorname{Im} P_{\mathscr{S}_\beta\infty}$. But since $\operatorname{rank} P_{\mathscr{S}_\beta\infty} = n - d[\beta,\infty) = n - d[\alpha,\infty) = n - d_\infty$, by (3.14) and (4.25), the condition $\operatorname{rank} D \leq n - d_\infty$ follows. Conversely, assume that $D \in \mathbb{R}^{n\times n}$ satisfies (4.56). From the third condition in (4.56) we have that there exists an orthogonal projector $O$ such that $\operatorname{Im} D \subseteq \operatorname{Im} O$ and $\operatorname{rank} O = n - d_\infty$. Furthermore, let $(X,U)$ be a conjoined basis of (H). The nonoscillation of (H) implies that $(X,U)$ has eventually constant kernel, i.e., there exists an interval $[\alpha,\infty)$ such that $(X,U)$ has constant kernel on $[\alpha,\infty)$. Without loss of generality, we may assume that $d[\alpha,\infty) = d_\infty$ and that $(X,U)$ is a minimal conjoined basis on $[\alpha,\infty)$. Let $S_\alpha(t)$, $P_{\mathscr{S}_\alpha\infty}$, and $T_\alpha$ be the matrices associated with $(X,U)$ in (2.8), (2.12), and Remark 2.1.6. Since $d[\alpha,\infty) = d_\infty$, we have $\operatorname{rank} P_{\mathscr{S}_\alpha\infty} = n - d_\infty = \operatorname{rank} O$ and hence, there exists an invertible matrix $E$ satisfying $\operatorname{Im} E P_{\mathscr{S}_\alpha\infty} = \operatorname{Im} O$. The matrix $E$ can be obtained e.g. from the diagonalization of $P_{\mathscr{S}_\alpha\infty}$ and $O$ or from Theorem A.2.2 in Appendix A with $P_* := 0$. In particular, we then have $\operatorname{Im} E^{-1} O = \operatorname{Im} P_{\mathscr{S}_\alpha\infty}$, i.e., $P_{\mathscr{S}_\alpha\infty} E^{-1} O = E^{-1} O$. Define now the matrices $M, N \in \mathbb{R}^{n\times n}$ by

$$M := E^T, \quad N := E^{-1} D - T_\alpha E^T. \tag{4.57}$$

We show that these matrices satisfy conditions (4.44) in Theorem 4.2.1. The matrix $M$ is invertible by its definition. The symmetry of $D$ and $T_\alpha$ implies that $M^T N = D - E\, T_\alpha E^T$ is also symmetric. Moreover, the equalities $OD = D$, $P_{\mathscr{S}_\alpha\infty} E^{-1} O = E^{-1} O$, and $P_{\mathscr{S}_\alpha\infty} T_\alpha = T_\alpha$ yield

$$P_{\mathscr{S}_\alpha\infty} N = P_{\mathscr{S}_\alpha\infty} E^{-1} OD - T_\alpha E^T = E^{-1} OD - T_\alpha E^T = E^{-1} D - T_\alpha E^T = N.$$

This means that $\operatorname{Im} N \subseteq \operatorname{Im} P_{\mathscr{S}_\alpha\infty}$. Furthermore, the inequality $D \geq 0$ implies the fourth condition in (4.44), since $NM^{-1} + T_\alpha = (E^{-1} D - T_\alpha E^T) E^{T-1} + T_\alpha = E^{-1} D E^{T-1} \geq 0$. Therefore, we proved that for a given $D$ satisfying (4.56) and for any minimal conjoined basis $(X,U)$ on $[\alpha,\infty)$ the matrices $M$ and $N$ in (4.57) satisfy the conditions in (4.44). Consider now the solution $(\tilde{X},\tilde{U})$ of (H) given by the initial conditions (4.43). From Theorem 4.2.1 it follows that $(\tilde{X},\tilde{U})$ is a minimal conjoined basis on $[\alpha,\infty)$. Moreover, the matrix $\tilde{T}_\alpha$ associated with $(\tilde{X},\tilde{U})$ in Remark 2.1.6 then satisfies $\tilde{T}_\alpha = M^T T_\alpha M + M^T N$, by (4.45). Finally, by using (4.57) we then obtain that $\tilde{T}_\alpha = D$. Therefore, the matrix $D$ is a $T$-matrix associated with the minimal conjoined basis $(\tilde{X},\tilde{U})$ on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$. ■

**Remark 4.3.3.** The result in Theorem 4.3.2 implies that the property of $D$ being a $T$-matrix for a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty)=d_\infty$ does not depend on the particular choice of such a point $\alpha\in[a,\infty)$. This follows from the fact that the conditions in (4.56) do not depend on $\alpha$. At the same time, the proof of Theorem 4.3.2 shows that the existence of a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty)=d_\infty$ is equivalent with the existence of a conjoined basis of (H) of the same type, which is associated with any $T$-matrix satisfying conditions (4.56).

# Chapter 5

# Principal and antiprincipal solutions at infinity

In this chapter we introduce the concepts of principal and antiprincipal solutions at infinity for possibly abnormal linear Hamiltonian systems and establish their existence, construction, and basic properties. In addition, in Section 5.3 we provide some applications of these new results.

## 5.1 Principal solutions at infinity

In this section we define the principal solution at infinity for a nonoscillatory system (H). We prove two main results about the principal solutions of (H) at infinity: (i) the existence of principal solutions and their classification depending on their rank (Theorem 5.1.5), and (ii) the construction of principal solutions (Theorem 5.1.10). The minimal possible rank in the first item above then corresponds to minimal principal solutions of (H) at infinity, which are discussed in Theorems 5.1.6 and 5.1.7. In contrast with the commonly accepted fact, the principal solution is now not unique (up to a right nonsingular multiple), when its rank is strictly greater than the rank of the minimal principal solution. By (3.16), the latter quantity is equal to $n-d_\infty$, where $d_\infty$ is the maximal order of abnormality of (H) on $[a,\infty)$ defined in (3.2).

**Definition 5.1.1** (Principal solution at infinity)**.** A conjoined basis $(\hat{X},\hat{U})$ of (H) is said to be a *principal solution at infinity* if there exists $\alpha\in[a,\infty)$ such that $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$ and its corresponding $S$-matrix $\hat{S}_\alpha(t)$ defined in (2.8) through the function $\hat{X}(t)$ satisfies $\hat{S}_\alpha^\dagger(t)\to 0$ as $t\to\infty$, that is,

$$\lim_{t\to\infty}\hat{S}_\alpha^\dagger(t)=0, \quad \text{where } \hat{S}_\alpha(t):=\int_\alpha^t \hat{X}^\dagger(s)\,B(s)\,\hat{X}^{\dagger T}(s)\,\mathrm{d}s. \tag{5.1}$$

When it is clear from the context, we will drop the term “at infinity”. The two properties of $(\hat{X},\hat{U})$ in Definition 5.1.1, namely that $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$ and that $\hat{S}_\alpha^\dagger(t)\to 0$ as $t\to\infty$ with $\hat{S}_\alpha(t)$ in (5.1), are required to hold simultaneously.

**Remark 5.1.2.** Let $(\hat{X},\hat{U})$ be a principal solution of (H) at infinity and let $r$ be its rank from (2.3). If $r=n-d_\infty$, then the principal solution $(\hat{X},\hat{U})$ is called *minimal*, while if $r=n$, then

$(\hat{X},\hat{U})$ is called *maximal*. This terminology corresponds to the two extreme cases in formula (3.16), which holds for the rank of any conjoined basis of (H) with constant kernel. We will use a special notation $(\hat{X}_{\min},\hat{U}_{\min})$ and $(\hat{X}_{\max},\hat{U}_{\max})$ for the principal solutions of (H) which are according to the above definition minimal and maximal, respectively. Moreover, if $n-d_\infty < r < n$, then the principal solution $(\hat{X},\hat{U})$ is called *intermediate* (of the rank $r$). We note that the principal solution $(\hat{X}_{\max},\hat{U}_{\max})$ corresponds to the principal solution developed by Reid in [30]. For this reason, the maximal principal solution $(\hat{X}_{\max},\hat{U}_{\max})$ is sometimes also denoted as $(\hat{X}_R,\hat{U}_R)$.

The following theorem shows that the property of being a principal solution of (H) is inherited (in both directions) by the relation "being contained" introduced in Definition 3.2.1. This fact will later be utilized for the construction of principal solutions of (H) at infinity.

**Theorem 5.1.3.** *Assume* (1.1). *Let* $(\hat{X},\hat{U})$ *be a principal solution of* (H) *at infinity with respect to the interval* $[\alpha,\infty)$. *Then every conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$, *which is either contained in* $(\hat{X},\hat{U})$ *on* $[\alpha,\infty)$ *or which contains* $(\hat{X},\hat{U})$ *on* $[\alpha,\infty)$, *is also a principal solution of* (H) *at infinity with respect to the interval* $[\alpha,\infty)$.

*Proof.* The statement follows directly from Theorem 3.2.4 and Definition 5.1.1, since the relation "being contained" for conjoined bases of (H) with constant kernel on $[\alpha,\infty)$ preserves the corresponding $S$-matrices. ∎

In the next result we show that a principal solution of (H) is necessarily associated with an interval $[\alpha,\infty)$, on which the order of abnormality of (H) is maximal and that the initial point $\alpha$ can be moved to the right side.

**Theorem 5.1.4.** *Assume* (1.1). *Let* $(\hat{X},\hat{U})$ *be a principal solution of* (H) *at infinity with respect to the interval* $[\alpha,\infty)$. *Then* $d[\alpha,\infty)=d_\infty$ *and* $(\hat{X},\hat{U})$ *is a principal solution also with respect to the interval* $[\beta,\infty)$ *for every* $\beta>\alpha$.

*Proof.* Let $(\hat{X},\hat{U})$ be as in the theorem and let $\hat{S}_\alpha(t)$, $\hat{T}_\alpha$ and $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ be the matrices in (2.8), Remark 2.1.6, and (2.12) associated with $(\hat{X},\hat{U})$ on $[\alpha,\infty)$. Then $\hat{T}_\alpha=0$ and hence, $\operatorname{Im}[P_{\hat{\mathscr{S}}_{\alpha\infty}}-\hat{S}_\alpha(t)\,\hat{T}_\alpha]=\operatorname{Im}[P_{\hat{\mathscr{S}}_{\alpha\infty}}-\hat{S}_\alpha(t)\,\hat{T}_\alpha]^T=\operatorname{Im}P_{\hat{\mathscr{S}}_{\alpha\infty}}$ for all $t\in[\alpha,\infty)$. Therefore, the condition in (4.35) is satisfied for conjoined basis $(\hat{X},\hat{U})$ on $[\alpha,\infty)$. Consequently, we get $d[\alpha,\infty)=d[t,\infty)$ for all $t\in[\alpha,\infty)$, or equivalently, $d[\alpha,\infty)=d_\infty$, by Theorem 4.1.12. Finally, from Theorem 4.3.1(ii) we then know that the set $\operatorname{Im}\hat{T}_\beta$ is constant in $\beta$ on $[\alpha,\infty)$. Since $\hat{T}_\alpha=0$, we have $\hat{T}_\beta=0$ for every $\beta>\alpha$ as well. Thus, $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity with respect to $[\beta,\infty)$ for every $\beta>\alpha$, by Definition 5.1.1. ∎

In one of the main results of this work we establish the existence of principal solutions of a nonoscillatory system (H). In fact, the nonoscillation of (H) is equivalent with the existence of a principal solution with some (and hence with any) rank $r$ between $n-d_\infty$ and $n$.

**Theorem 5.1.5.** *Assume* (1.1). *Then the following statements are equivalent.*

(i) *System* (H) *is nonoscillatory.*

(ii) *There exists a principal solution of* (H) *at infinity.*

(iii) *For any integer $r$ satisfying $n-d_\infty \le r \le n$ there exists a principal solution of* (H) *at infinity with rank equal to $r$.*

*Proof.* If (H) is nonoscillatory and Legendre condition (1.1) holds, then every conjoined basis of (H) has eventually constant kernel. Without loss of generality, we assume that $\alpha \in [a,\infty)$ is such that condition (3.3) holds and there exists a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. The results in Theorem 4.3.2 and Remark 4.3.3 then guarantee the existence of a conjoined basis $(\hat{X},\hat{U})$ of (H) with constant kernel on $[\alpha,\infty)$ such that its corresponding matrix $\hat{T}_\alpha$ in Remark 2.1.6 satisfies $\hat{T}_\alpha = 0$. Thus, $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity, by Definition 5.1.1. Assume (ii) and let $(\hat{X},\hat{U})$ be a principal solution of (H) at infinity. According to Definition 5.1.1, there exists $\alpha \in [a,\infty)$ such that $(\hat{X},\hat{U})$ is a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$ and its associated $S$-matrix $\hat{S}_\alpha(t)$ satisfies (5.1). From Theorem 5.1.4 we get $d[\alpha,\infty) = d_\infty$. Thus, by Theorem 3.2.11 for any integer $r$ between $n-d_\infty$ and $n$ there exists a conjoined basis $(X,U)$ of (H) with constant kernel and $\operatorname{rank} X(t) = r$ on $[\alpha,\infty)$, such that $(X,U)$ is either contained or contains the principal solution $(\hat{X},\hat{U})$ on $[\alpha,\infty)$. In turn, Theorem 5.1.3 implies that $(X,U)$ is also a principal solution of (H), i.e., condition (iii) holds. Finally, condition (iii) yields the existence of a conjoined basis of (H) with eventually constant kernel, i.e., nonoscillatory conjoined basis. This implies that system (H) is nonoscillatory as well. ■

The following two theorems contain basic results about the minimal principal solutions of (H) at infinity, which is a proper generalization of Reid's (or Hartman's, or Coppel's) principal solution to possibly abnormal linear Hamiltonian systems. In particular, in Theorem 5.1.6 we show the uniqueness of the minimal principal solution. We remark that this property is in a full agreement with the previous result in Theorem 3.3.4, namely with (3.45), where we considered minimal conjoined bases of (H). Furthermore, in Theorem 5.1.7 we provide a construction of the minimal principal solution at infinity from a minimal conjoined basis of (H) on interval $[\alpha,\infty)$ satisfying condition (3.3).

**Theorem 5.1.6.** *The minimal principal solution of* (H) *at infinity is unique up to a right nonsingular constant multiple. More precisely, if $(\hat{X}_{\min},\hat{U}_{\min})$ is a minimal principal solution of* (H) *at infinity, then a solution $(X,U)$ of* (H) *is also a minimal principal solution of* (H) *at infinity if and only if there exists a constant nonsingular matrix $\hat{M} \in \mathbb{R}^{n\times n}$ such that $X(t) = \hat{X}(t)\hat{M}$ and $U(t) = \hat{U}(t)\hat{M}$ on $[a,\infty)$.*

*Proof.* Let $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X,U)$ be minimal principal solutions of (H) at infinity. Let $\alpha,\alpha_0 \in [a,\infty)$ be such that $(\hat{X}_{\min},\hat{U}_{\min})$ is a minimal principal solution with respect to $[\alpha,\infty)$ and $(X,U)$ is a minimal principal solution with respect to $[\alpha_0,\infty)$. Without loss of generality we may assume that $\alpha_0 = \alpha$, because shifting the initial point to the right preserves the property of being a minimal principal solution at infinity, by Theorem 5.1.4. In particular, $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X,U)$ are minimal conjoined bases on $[\alpha,\infty)$. Let $\hat{P}$ and $P$ be the corresponding orthogonal projectors defined in (2.2) through $\hat{X}_{\min}$ and $X$, respectively. By (3.46), we have $\operatorname{Im}\hat{X}_{\min}(\alpha) = \operatorname{Im} X(\alpha)$, which implies through Theorem 2.3.8 (with $(X_1,U_1) = (\hat{X}_{\min},\hat{U}_{\min})$ and $(X_2,U_2) = (X,U)$) that $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X,U)$ are mutually representable on $[\alpha,\infty)$ and

$$\begin{pmatrix} X \\ U \end{pmatrix} = \begin{pmatrix} \hat{X}_{\min} & \bar{X}_{1\alpha} \\ \hat{U}_{\min} & \bar{U}_{1\alpha} \end{pmatrix} \begin{pmatrix} \hat{M} \\ \hat{N} \end{pmatrix} \quad \text{on } [\alpha,\infty),$$

where the matrix $\hat{M}$ is constant and nonsingular, and $\operatorname{Im}\hat{N} \subseteq \operatorname{Im}\hat{P}$. Now, if $\hat{S}_{\min}(t)$ and $S(t)$ are the $S$-matrices corresponding to $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X,U)$ on $[\alpha,\infty)$, then (3.50) becomes

$$S^{\dagger}(t) = \hat{M}^T \hat{S}^{\dagger}_{\min}(t)\,\hat{M} + \hat{M}^T\hat{N} \quad \text{for all } t \in (\tau_l,\infty). \tag{5.2}$$

Upon taking the limit as $t \to \infty$ in (5.2) and using the fact that $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X,U)$ are minimal principal solutions at infinity, we get $\hat{M}^T\hat{N} = 0$ and consequently, $\hat{N} = 0$. This means that $X(t) = \hat{X}_{\min}(t)\,\hat{M}$ and $U(t) = \hat{U}_{\min}(t)\,\hat{M}$ on $[\alpha,\infty)$ and hence, on $[a,\infty)$ by uniqueness of solutions. Conversely, the solution $X(t) := \hat{X}_{\min}(t)\,\hat{M}$ and $U(t) := \hat{U}_{\min}(t)\,\hat{M}$ with $\hat{M}$ constant and nonsingular is obviously a minimal conjoined basis on $[\alpha,\infty)$ with $\operatorname{Im}X(\alpha) = \operatorname{Im}\hat{X}_{\min}(\alpha)$. In particular, the (constant) Wronskian $\hat{N}$ of $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X,U)$ satisfies $\hat{N} = 0$. In turn, by (3.50) in Theorem 3.3.6 with $\beta := \alpha$, $(X_1,U_1) = (\hat{X}_{\min},\hat{U}_{\min})$, $(X_2,U_2) = (X,U)$, $M_{1\beta} := \hat{M}$, and $N_1 := \hat{N}$, we get that $S^{\dagger}(t) = \hat{M}^T\hat{S}^{\dagger}_{\min}(t)\,\hat{M}$ for $t \in (\tau_l,\infty)$. Here $S(t)$ is the matrix in (2.8) corresponding to $(X,U)$ on $[\alpha,\infty)$. Since $\hat{S}^{\dagger}_{\min}(t) \to 0$ as $t \to \infty$, we get that $S^{\dagger}(t) \to 0$ as $t \to \infty$ as well. Therefore, $(X,U)$ is a minimal principal solution of (H) at infinity. ■

**Theorem 5.1.7.** *Assume that condition (1.1) holds and system (H) is nonoscillatory. Let $\alpha \in [a,\infty)$ be such that $d[\alpha,\infty) = d_\infty$ and there exists a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$. Then a solution $(\hat{X},\hat{U})$ of (H) is a minimal principal solution at infinity with respect to the interval $[\alpha,\infty)$ if and only if*

$$\begin{pmatrix} \hat{X} \\ \hat{U} \end{pmatrix} = \begin{pmatrix} X & \bar{X}_\alpha \\ U & \bar{U}_\alpha \end{pmatrix} \begin{pmatrix} I \\ -T_\alpha \end{pmatrix} \quad \text{on } [\alpha,\infty), \tag{5.3}$$

*for some minimal conjoined basis $(X,U)$ of (H) on $[\alpha,\infty)$. Here $(\bar{X}_\alpha,\bar{U}_\alpha)$ is a conjoined basis of (H) satisfying (1.16) and (2.25) with $\beta := \alpha$ and the matrix $T_\alpha$ is defined in Remark 2.1.6.*

*Proof.* Let $\alpha$ be as in the theorem. If $(\hat{X},\hat{U})$ is a minimal principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$, then it is a minimal conjoined basis on $[\alpha,\infty)$, by Definitions 5.1.1 and 3.3.1 and Remark 5.1.2. Moreover, the associated matrix $\hat{T}_\alpha$ in Remark 2.1.6 satisfies $\hat{T}_\alpha = 0$. Formula (5.3) then holds trivially with $(X,U) := (\hat{X},\hat{U})$. Conversely, let $(X,U)$ be a minimal conjoined basis of (H) on $[\alpha,\infty)$ and let $S_\alpha(t)$ and $T_\alpha$ be its corresponding matrices in (2.8) and Remark 2.1.6. Furthermore, let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be the associated orthogonal projectors defined in (2.2) and (2.12). In particular, $P_{\mathscr{S}_{\alpha\infty}} = P$, by Remark 3.3.3. Consider the solution $(\hat{X},\hat{U})$ of (H) in (5.3). As we showed in the proof of Theorem 4.1.6, $(\hat{X},\hat{U})$ is a conjoined basis of (H) and $\operatorname{Ker}\hat{X}(t) = \operatorname{Ker}[P - S_\alpha(t)\,T_\alpha]$ on $[\alpha,\infty)$. Moreover, from Theorem 4.1.10 we then know that $\operatorname{Ker}[P - S_\alpha(t)\,T_\alpha] = \operatorname{Ker}P$ on $[\alpha,\infty)$, because $d[\alpha,\infty) = d_\infty$. Therefore, $\operatorname{Ker}\hat{X} = \operatorname{Ker}P$ on $[\alpha,\infty)$ and hence, the conjoined basis $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$ and the corresponding orthogonal projector $\hat{P}$ onto $\operatorname{Im}\hat{X}^T$ on $[\alpha,\infty)$ satisfies $\hat{P} = P$. Consequently, $(\hat{X},\hat{U})$ is a minimal conjoined basis of (H) on $[\alpha,\infty)$. Let $\hat{S}_\alpha(t)$ be its corresponding $S$-matrix. From equation (3.50) in Theorem 3.3.6 (with $\beta := \alpha$, $(X_1,U_1) := (X,U)$, $(X_2,U_2) := (\hat{X},\hat{U})$, $M_{1\beta} = I$, and $N_1 = -T_\alpha$) we obtain the equality

$$\hat{S}^\dagger_\alpha(t) = S^\dagger_\alpha(t) - T_\alpha \quad \text{for large } t. \tag{5.4}$$

Finally, formula (5.4) implies that $\hat{S}^\dagger_\alpha(t) \to 0$ for $t \to \infty$. This shows that $(\hat{X},\hat{U})$ is a minimal principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$. ■

In Theorem 5.1.6 we guarantee the uniqueness of the minimal principal solution of (H) at infinity. In the following remark we show that the minimal principal solution is the only one for which this property is satisfied. This result also indicates that nonunique principal solutions of (H) will always exist as long as $d_\infty \geq 1$.

**Remark 5.1.8.** Let $(\hat{X},\hat{U})$ be a principal solution of (H) with rank $r$ satisfying $n - d_\infty \leq r \leq n$. Then $(\hat{X},\hat{U})$ is unique up to a right nonsingular multiple if and only if $r = n - d_\infty$, that is, $(\hat{X},\hat{U})$ is a minimal principal solution of (H). We shall prove by construction the implication "⇒", as the opposite direction "⇐" is contained in Theorem 5.1.6. Let $(\hat{X},\hat{U})$ be a principal solution of (H) at infinity with respect to the interval $[\alpha,\infty)$ with the projectors $\hat{P}$ and $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ in (2.2) and (2.12). Then $d[\alpha,\infty) = d_\infty$, by Theorem 5.1.4. Set $\hat{M} := 2I - \hat{P}$ and $\hat{N} := \hat{P} - P_{\hat{\mathscr{S}}_{\alpha\infty}}$ and define the solution $(X,U) := (\hat{X},\hat{U})\hat{M} + (\bar{X}_\alpha,\bar{U}_\alpha)\hat{N}$ on $[\alpha,\infty)$, where $(\bar{X}_\alpha,\bar{U}_\alpha)$ is the conjoined basis of (H) associated with $(\hat{X},\hat{U})$ in Theorem 2.2.5. Obviously, the matrix $\hat{M}$ is nonsingular and $\hat{P}\hat{M} = \hat{P}$. Since $\hat{P}P_{\hat{\mathscr{S}}_{\alpha\infty}} = P_{\hat{\mathscr{S}}_{\alpha\infty}}$ by (2.13), it follows that $\hat{M}^T\hat{N} = \hat{P} - P_{\hat{\mathscr{S}}_{\alpha\infty}}$ is symmetric. Thus, the matrices $\hat{M}$ and $\hat{N}$ satisfy conditions (3.21). This shows that the solutions $(\hat{X},\hat{U})$ and $(X,U)$ are equivalent on $[\alpha,\infty)$, by Corollary 3.1.10 with $(X,U) := (\hat{X},\hat{U})$ and $(X_0,U_0) := (X,U)$. Therefore, $(X,U)$ is also a principal solution of (H) with respect to $[\alpha,\infty)$ with the same rank $r$. Now if $(\hat{X},\hat{U})$ is unique up to a right nonsingular multiple, then necessarily $\hat{N} = 0$. This means that $\hat{P} = P_{\hat{\mathscr{S}}_{\alpha\infty}}$ and hence, $(\hat{X},\hat{U})$ is a minimal conjoined basis on $[\alpha,\infty)$, by Remark 3.3.3. But then $d[\alpha,\infty) = d_\infty$, which yields that $(\hat{X},\hat{U})$ is a minimal principal solution of (H) at infinity.

In the remaining part of this section we will present a construction of principal solutions of (H) with all admissible ranks from the minimal principal solutions. This method utilizes the properties of the relation "being contained" derived in Sections 3.2. Thus, we now assume that system (H) is nonoscillatory and that $(\hat{X}_{\min},\hat{U}_{\min})$ is a minimal principal solution of (H) at infinity. Define the point $\hat{\alpha}_{\min} \in [a,\infty)$ associated with $(\hat{X}_{\min},\hat{U}_{\min})$ by

$$\hat{\alpha}_{\min} := \inf\big\{\alpha \in [a,\infty),\ (\hat{X}_{\min},\hat{U}_{\min}) \text{ has constant kernel on } [\alpha,\infty)\big\}. \tag{5.5}$$

It is obvious that $(\hat{X}_{\min},\hat{U}_{\min})$ has constant kernel on the open interval $(\hat{\alpha}_{\min},\infty)$ and since by Theorem 5.1.6 the minimal principal solution $(\hat{X}_{\min},\hat{U}_{\min})$ is unique up to a right nonsingular multiple, the point $\hat{\alpha}_{\min}$ does not depend on the particular choice of $(\hat{X}_{\min},\hat{U}_{\min})$.

The construction of principal solutions $(\hat{X},\hat{U})$ of (H) is based on the choice of a point $\alpha \in (\hat{\alpha}_{\min},\infty)$, for which $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$, as it is required in Section 3.2. However, we will show that the outcome of this construction, i.e., the principal solution $(\hat{X},\hat{U})$, is independent of $\alpha$.

First we prove that the property of being a minimal principal solution with respect to the interval $[\alpha,\infty)$ (see Definition 5.1.1) is preserved within the interval $(\hat{\alpha}_{\min},\infty)$.

**Theorem 5.1.9.** *Assume* (1.1). *Let* $(\hat{X}_{\min}, \hat{U}_{\min})$ *be a minimal principal solution of* (H) *at infinity with* $\hat{\alpha}_{\min}$ *defined in* (5.5). *Then* $d[\alpha,\infty) = d_\infty$ *and* $(\hat{X}_{\min}, \hat{U}_{\min})$ *is a minimal principal solution with respect to the interval* $[\alpha,\infty)$ *for every* $\alpha \in (\hat{\alpha}_{\min},\infty)$.

*Proof.* In order to simplify the notation and avoid double indices, we put $(X,U) := (\hat{X}_{\min}, \hat{U}_{\min})$. Fix $\alpha \in (\hat{\alpha}_{\min},\infty)$. According to (5.5), the minimal principal solution $(X,U)$ has constant kernel on $[\alpha,\infty)$. Then $n - d[\alpha,\infty) \leq \operatorname{rank} X(t)$ holds on $[\alpha,\infty)$, by (3.15). On the other hand, since $(X,U)$ is a minimal principal solution, we have by (3.16) that $\operatorname{rank} X(t) = n - d_\infty$ on $[\alpha,\infty)$. Thus, $d_\infty \leq d[\alpha,\infty)$, which implies that $d_\infty = d[\alpha,\infty)$ by the definition of $d_\infty$ in (3.2). Furthermore, let $S_\alpha(t)$ be the $S$-matrix in (2.8) corresponding to $(X,U)$ on $[\alpha,\infty)$ and let $T_\alpha \in \mathbb{R}^{n\times n}$ be such that $S_\alpha^\dagger(t) \to T_\alpha$ for $t \to \infty$. The equality $d[\alpha,\infty) = d_\infty$ implies that $(X,U)$ is a minimal conjoined basis on $[\alpha,\infty)$. This means by Remark 3.3.3 that the orthogonal projectors and $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ in (2.2) and (2.12) satisfy $P_{\mathscr{S}_{\alpha\infty}} = P$. Consider the solution $(\hat{X}, \hat{U})$ defined in (5.3). From Theorem 5.1.7 and its proof it then follows that $(\hat{X}, \hat{U})$ is a minimal principal solution of (H) with respect to $[\alpha,\infty)$ and that $\hat{X}(t) = X(t)\,[P - S_\alpha(t)\,T_\alpha]$ on $[\alpha,\infty)$. On the other hand, from Theorem 5.1.6 it follows that $\hat{X}(t) = X(t)\,M$ and $\hat{U}(t) = U(t)\,M$ on $[\alpha,\infty)$ for some constant and invertible matrix $M$. Thus, we obtain $X(t)\,[P - S(t)\,T] = X(t)\,M$ on $[\alpha,\infty)$. Multiplying the latter equality by $X^\dagger(t)$ from the left and using the identities $X^\dagger(t)\,X(t) = P$ and $PS_\alpha(t) = S_\alpha(t)$ we get the equality $P - S_\alpha(t)\,T_\alpha = PM$ on $[\alpha,\infty)$. Since $S_\alpha(\alpha) = 0$, we have $P = PM$. But then $\hat{X}(t) = X(t)\,M = X(t)\,PM = X(t)$ on $[\alpha,\infty)$. Therefore, the $S$-matrices for $(\hat{X}, \hat{U})$ and $(X,U)$ coincide on $[\alpha,\infty)$, which yields that $T_\alpha = 0$. Hence, $(X,U)$ is a minimal principal solution of (H) with respect to $[\alpha,\infty)$. ■

The following theorem describes the construction of principal solutions of (H) from the minimal principal solutions.

**Theorem 5.1.10.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (5.5). *A solution* $(X,U)$ *of* (H) *is a principal solution at infinity if and only if* $(X,U)$ *is a conjoined basis of* (H) *with constant kernel on* $(\hat{\alpha}_{\min},\infty)$*, which contains some minimal principal solution of* (H) *on* $[\alpha,\infty)$ *for some (and hence every)* $\alpha \in (\hat{\alpha}_{\min},\infty)$.

*Proof.* Let $(X,U)$ be a principal solution of (H) at infinity. According to Definition 5.1.1, there exists $\alpha \in [a,\infty)$ such that $(X,U)$ has constant kernel on $[\alpha,\infty)$ and the corresponding matrix $S_\alpha(t)$ defined in (2.8) satisfies $S_\alpha^\dagger(t) \to 0$ for $t \to \infty$. From Theorem 5.1.4 we know that this property of $S_\alpha(t)$ is preserved under shifting the point $\alpha$ to the right. Therefore, we may assume that $\alpha > \hat{\alpha}_{\min}$. By using Theorem 5.1.4 again we have $d[\alpha,\infty) = d_\infty$. Consequently, by Theorem 3.2.11 there exists a conjoined basis $(X_*, U_*)$ of (H) with constant kernel on $[\alpha,\infty)$ and with $\operatorname{rank} X_*(t) = n - d_\infty$ on $[\alpha,\infty)$ such that $(X,U)$ contains $(X_*, U_*)$ on $[\alpha,\infty)$. In turn, Theorem 5.1.3 and Remark 5.1.2 imply that $(X_*, U_*)$ is a minimal principal solution of (H) with respect to the interval $[\alpha,\infty)$. From Theorem 3.2.12(i) we then obtain that $(X,U)$ contains $(X_*, U_*)$ also on $[\beta,\infty)$ for all $\beta \geq \alpha$. It remains to show that $(X,U)$ contains $(X_*, U_*)$ on $[\beta,\infty)$ for all $\beta \in (\hat{\alpha}_{\min},\alpha)$. Let us fix such a point $\beta$. By Theorem 5.1.9, we know that $(X_*, U_*)$ is a minimal principal solution with respect to the interval $[\beta,\infty)$. On the one hand, this means that $d[\beta,\infty) = d_\infty$ (by Theorem 5.1.4) and that $(X_*, U_*)$ has constant kernel on $[\beta,\infty)$ (by Definition 5.1.1). Consequently, $(X,U)$ has constant kernel on $[\beta,\infty)$ according to Theorem 3.2.13(ii). On the other hand, Theorem 3.2.12(ii) implies that $(X,U)$ contains $(X_*,U_*)$ also on $[\beta,\infty)$. This completes the proof of the first implication. Conversely, suppose that $(X,U)$ is a conjoined basis of (H) with constant kernel on $(\hat{\alpha}_{\min},\infty)$. Let $(\hat{X}_{\min},\hat{U}_{\min})$ be a minimal principal solution of (H) at infinity, which is contained in $(X,U)$ on $[\alpha_0,\infty)$ for some $\alpha_0 > \hat{\alpha}_{\min}$. Since $(\hat{X}_{\min},\hat{U}_{\min})$ is a minimal principal solution with respect to $[\alpha,\infty)$ for every $\alpha \in (\hat{\alpha}_{\min},\infty)$, it follows that $(\hat{X}_{\min},\hat{U}_{\min})$ has constant kernel on $(\hat{\alpha}_{\min},\infty)$ and $d[\alpha,\infty) = d_\infty$, by Theorem 5.1.4. Consequently, Theorem 3.2.12 implies that $(\hat{X}_{\min},\hat{U}_{\min})$ is contained in $(X,U)$ on $[\alpha,\infty)$ for every $\alpha \in (\hat{\alpha}_{\min},\infty)$. The fact that $(X,U)$ is a principal solution of (H) at infinity now follows from Theorem 5.1.3. ∎

**Remark 5.1.11.** From Theorem 5.1.10 it follows that every principal solution $(\hat{X},\hat{U})$ of (H) at infinity is a principal solution with respect to $[\alpha,\infty)$ for every $\alpha \in (\hat{\alpha}_{\min},\infty)$. In particular, this means that $(\hat{X},\hat{U})$ has constant kernel on $(\hat{\alpha}_{\min},\infty)$, by Definition 5.1.1. In addition, the orthogonal projector $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ in (2.12) associated with $(\hat{X},\hat{U})$ through the function $\hat{S}(t)$ in (5.1) is the same for all initial points $\alpha \in (\hat{\alpha}_{\min},\infty)$, compare with the first equality in (3.41).

## 5.2 Antiprincipal solutions at infinity

In this section we introduce the antiprincipal solutions of (H) at infinity and study their properties. In particular, similarly as for principal solutions, we prove the existence of all antiprincipal solutions at infinity with their rank between $n-d_\infty$ and $n$ for a nonoscillatory system (H), and provide a construction of all antiprincipal solutions from the minimal antiprincipal solutions (see Theorems 5.2.7 and 5.2.8).

**Definition 5.2.1** (Antiprincipal solution at infinity)**.** A conjoined basis $(X,U)$ of (H) is said to be an *antiprincipal solution at infinity* if there exists $\alpha \in [a,\infty)$ with $d[\alpha,\infty) = d_\infty$ such that $(X,U)$ has constant kernel on $[\alpha,\infty)$ and its corresponding matrix $T_\alpha$ defined in Remark 2.1.6 satisfies $\operatorname{rank} T_\alpha = n-d_\infty$.

Similarly, as for principal solutions of (H), we will often drop the term "at infinity" in the terminology of antiprincipal solutions. The properties of $(X,U)$ in Definition 5.2.1, namely that $(X,U)$ has constant kernel on $[\alpha,\infty)$ with $d[\alpha,\infty) = d_\infty$ and that $\operatorname{rank} T_\alpha = n-d_\infty$ with $T_\alpha$ in Remark 2.1.6, are required to hold simultaneously. We can see from Theorem 4.3.2 that the antiprincipal solutions of (H) are defined by the maximal possible rank of the associated matrix $T_\alpha$, while the principal solutions of (H) in Definition 5.1.1 were defined by the minimal possible rank of $T_\alpha$ (hence $T_\alpha = 0$).

In the following remark we introduce an analogous terminology and notation as in Section 5.1 for principal solutions at infinity.

**Remark 5.2.2.** Let $(X,U)$ be an antiprincipal solution of (H) at infinity and let $r$ be its rank from (2.3). If $r = n-d_\infty$, then $(X,U)$ is called a *minimal antiprincipal solution* at infinity, while if $r = n$, then $(X,U)$ is called a *maximal antiprincipal solution* at infinity. This terminology corresponds to the two extreme cases in formula (3.16). As before, we will use the notation $(X_{\min},U_{\min})$ and $(X_{\max},U_{\max})$ for the minimal and maximal antiprincipal solutions of (H). Moreover, if $n-d_\infty < r < n$, then the antiprincipal solution $(X,U)$ is called *intermediate* (of the rank $r$).

The first result of this section contains a characterization of the antiprincipal solutions of (H) in terms of the limit of $S_\alpha(t)$ as $t \to \infty$.

**Theorem 5.2.3.** *Assume* (1.1). *Let* $(X,U)$ *be a conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty) \subseteq (\hat{\alpha}_{\min},\infty)$ *with* $\hat{\alpha}_{\min}$ *defined in* (5.5). *Let* $S_\alpha(t)$ *and* $T_\alpha$ *be the matrices defined in* (2.8) *and Remark* 2.1.6. *Then* $(X,U)$ *is an antiprincipal solution of* (H) *at infinity if and only if*

$$\lim_{t\to\infty} S_\alpha(t) = T_\alpha^\dagger. \tag{5.6}$$

*Proof.* Let $(X,U)$ and $\alpha$ be as in the theorem. From the definition of $\hat{\alpha}_{\min}$ in (5.5) it follows that condition (3.3) holds, that is $d[\alpha,\infty) = d_\infty$. Since $S_\alpha^\dagger(t) \to T_\alpha$ as $t \to \infty$ and $\operatorname{rank} S_\alpha^\dagger(t) = \operatorname{rank} S_\alpha(t) = n - d_\infty$ for large $t$ by (2.12) and (3.14), it follows from Remark 1.2.4(ii), in which we take $M(t) := S_\alpha^\dagger(t)$ and $M := T_\alpha$, that $(X,U)$ is an antiprincipal solution of (H) at infinity if and only if (5.6) holds. ∎

**Remark 5.2.4.** Condition (5.6) in Theorem 5.2.3 can be replaced by the weaker (but equivalent) condition, which is only the existence of the limit of $S_\alpha(t)$ for $t \to \infty$. This can be also seen from Remark 1.2.4(ii).

In the next statement we show that the initial point $\alpha$ in Definition 5.2.1 can be arbitrarily moved to the right side. This corresponds to the situation with the principal solutions of (H) at infinity in Theorem 5.1.4.

**Theorem 5.2.5.** *Assume* (1.1) *and let* $(X,U)$ *be an antiprincipal solution of* (H) *at infinity with respect to the interval* $[\alpha,\infty)$. *Then* $(X,U)$ *is an antiprincipal solution also with respect to the interval* $[\beta,\infty)$ *for every* $\beta > \alpha$.

*Proof.* Let $(X,U)$ be as in the theorem and let $T_\alpha$ be the matrix in Remark 2.1.6 associated with $(X,U)$ on $[\alpha,\infty)$. By Definition 5.2.1, we have that condition (3.3) holds and $\operatorname{rank} T_\alpha = n - d_\infty$. Using similar arguments as in the second part of the proof of Theorem 5.1.4 we obtain that $\operatorname{rank} T_\beta = \operatorname{rank} T_\alpha = n - d_\infty$ for all $\beta \in [\alpha,\infty)$. This then shows that $(X,U)$ is an antiprincipal solution of (H) at infinity also with respect to $[\beta,\infty)$ for every $\beta > \alpha$, by Definition 5.2.1. ∎

Next we present an analogue of Theorem 5.1.3 for the antiprincipal solutions of (H) at infinity. We omit the proof because it is based on exactly the same arguments as the proof of Theorem 5.1.3.

**Theorem 5.2.6.** *Assume* (1.1). *Let* $(X,U)$ *be an antiprincipal solution of* (H) *at infinity with respect to the interval* $[\alpha,\infty)$. *Then every conjoined basis of* (H) *with constant kernel on* $[\alpha,\infty)$, *which is either contained in* $(X,U)$ *on* $[\alpha,\infty)$ *or which contains* $(X,U)$ *on* $[\alpha,\infty)$, *is also an antiprincipal solution of* (H) *at infinity with respect to the interval* $[\alpha,\infty)$.

In the following result we characterize the nonoscillation of system (H) in terms of the existence of antiprincipal solutions of (H) at infinity with any rank between $n - d_\infty$ and $n$ in the same spirit as in Theorem 5.1.5 for the principal solutions. We note that in contrast with the minimal principal solutions of (H) in Theorem 5.1.6, the minimal antiprincipal solutions of (H) are not in general unique (up to a right nonsingular multiple), see Remark 5.2.11 below.

**Theorem 5.2.7.** *Assume* (1.1). *Then the following statements are equivalent.*

(i) *System* (H) *is nonoscillatory.*

(ii) *There exists an antiprincipal solution of* (H) *at infinity.*

(iii) *For any integer $r$ satisfying $n-d_\infty \leq r \leq n$ there exists an antiprincipal solution of* (H) *at infinity with rank equal to $r$.*

*Proof.* If (H) is nonoscillatory, then by Theorem 4.3.2 for any symmetric and nonnegative definite matrix $D$ with $\operatorname{rank} D = n - d_\infty$ there exists a conjoined basis $(X,U)$ of (H) with constant kernel on $[\alpha,\infty)$ for some $\alpha \in [a,\infty)$ with $d[\alpha,\infty) = d_\infty$ so that its corresponding matrix $T_\alpha$ in Remark 2.1.6 satisfies $T_\alpha = D$, i.e., $\operatorname{rank} T_\alpha = n - d_\infty$. By Definition 5.2.1, we then have that $(X,U)$ is an antiprincipal solution of (H) at infinity. Suppose now that (ii) holds and let $(X,U)$ be an antiprincipal solution of (H) at infinity, i.e., there exists $\alpha \in [a,\infty)$ such that (3.3) holds, $(X,U)$ is a conjoined basis of (H) with constant kernel on $[\alpha,\infty)$, and its associated matrix $T_\alpha$ satisfies $\operatorname{rank} T_\alpha = n - d_\infty$. By Theorem 3.2.11, for any integer $r$ between $n-d_\infty$ and $n$ there exists a conjoined basis $(\tilde{X},\tilde{U})$ of (H) with constant kernel and $\operatorname{rank} \tilde{X}(t) = r$ on $[\alpha,\infty)$ and such that $(\tilde{X},\tilde{U})$ is either contained or contains $(X,U)$ on $[\alpha,\infty)$. Therefore, $(\tilde{X},\tilde{U})$ is also an antiprincipal solution of (H), by Theorem 5.2.6, showing part (iii). Finally, if (iii) is satisfied, then system (H) is nonoscillatory, by Proposition 1.5.3. ∎

In the next result we provide a construction of all antiprincipal solutions of (H) at infinity from minimal antiprincipal solutions. This corresponds to Theorem 5.1.10, where the principal solutions of (H) were considered.

**Theorem 5.2.8.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with $\hat{\alpha}_{\min}$ defined in* (5.5). *A solution $(X,U)$ of* (H) *is an antiprincipal solution at infinity if and only if $(X,U)$ is a conjoined basis of* (H)*, which contains some minimal antiprincipal solution of* (H) *at infinity on $[\alpha,\infty)$ for some $\alpha \in (\hat{\alpha}_{\min},\infty)$.*

*Proof.* Let $(X,U)$ be an antiprincipal solution of (H) at infinity. This means by Definition 5.2.1 that $(X,U)$ is a conjoined basis with constant kernel on $[\alpha,\infty)$ for some $\alpha \in [a,\infty)$ satisfying (3.3) and the corresponding matrix $T_\alpha$ in Remark 2.1.6 satisfies $\operatorname{rank} T_\alpha = n - d_\infty$. By Theorem 5.2.5, we may assume that $\alpha > \hat{\alpha}_{\min}$. From Theorem 3.2.11 we know that there exists a conjoined basis $(X_*,U_*)$ of (H) with constant kernel on $[\alpha,\infty)$ and with $\operatorname{rank} X_*(t) = n - d_\infty$ on $[\alpha,\infty)$ such that $(X,U)$ contains $(X_*,U_*)$ on $[\alpha,\infty)$. In turn, by Theorem 5.2.6 and Remark 5.2.2, we have that $(X_*,U_*)$ is a minimal antiprincipal solution of (H) with respect to the interval $[\alpha,\infty)$. Conversely, if $(X,U)$ is a conjoined basis of (H) with constant kernel on $[\alpha,\infty) \subseteq (\hat{\alpha}_{\min},\infty)$ and such that $(X,U)$ contains some minimal antiprincipal solution of (H) on $[\alpha,\infty)$, then $(X,U)$ is also an antiprincipal solution of (H), by Theorem 5.2.6. ∎

In the following result we present an interesting class of antiprincipal solutions at infinity. In particular, the principal solutions at the points $\alpha > \hat{\alpha}_{\min}$ are examples of minimal antiprincipal solutions at infinity (see Examples 7.1.1–7.1.3). This observation also reveals the complicated structure of the set of all antiprincipal solutions at infinity, see Remark 5.2.11 below.

**Proposition 5.2.9.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (5.5)*. Then for every* $\alpha > \hat{\alpha}_{\min}$ *the principal solution* $(\hat{X}_\alpha, \hat{U}_\alpha)$ *at the point* $\alpha$ *is a minimal antiprincipal solution of* (H) *at infinity.*

*Proof.* Let $\alpha > \hat{\alpha}_{\min}$ be fixed. From Theorem 5.1.5 we know that there exists the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (H) at infinity with constant kernel on the interval $(\hat{\alpha}_{\min}, \infty)$. In order to simplify the notation, we put $(X,U) := (\hat{X}_\alpha, \hat{U}_\alpha)$ and $(\hat{X}, \hat{U}) := (\hat{X}_{\min}, \hat{U}_{\min})$. Let $\hat{P}$, $\hat{R}(t)$, $\hat{S}_\alpha(t)$, and $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ be the matrices in (2.2), (2.1), (2.8), (2.12), and Remark 5.1.11 defined through the function $\hat{X}(t)$ on $[\alpha, \infty)$. In particular, $\hat{P} = P_{\hat{\mathscr{S}}_{\alpha\infty}}$. By Theorem 3.1.2 with $\beta := \alpha$, we have

$$X(t) = \hat{X}(t)\, \hat{S}_\alpha(t)\, \hat{X}^T(\alpha), \quad \operatorname{rank} \hat{S}_\alpha(t) = \operatorname{rank} X(t) = n - d[\alpha, t], \quad t \in [\alpha, \infty). \tag{5.7}$$

Let $\beta \geq \alpha$ be such that $(X,U)$ has constant kernel on $[\beta, \infty)$ and $d[\alpha,t] = d[\alpha, \infty)$ for all $t \geq \beta$. Then $\operatorname{rank} X(t) = n - d[\alpha, \infty] = n - d_\infty$ on $[\beta, \infty)$, by Theorem 5.1.9. Therefore, from the second formula in (5.7) we obtain that $\operatorname{rank} \hat{S}_\alpha(t) = n - d_\infty$ on $[\beta, \infty)$. Consequently, $\operatorname{Im} \hat{S}_\alpha(t) = \operatorname{Im} P_{\hat{\mathscr{S}}_{\alpha\infty}}$ on $[\beta, \infty)$, by the definition of $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ in (2.12). We will show that

$$X^\dagger(t) = \hat{X}^{\dagger T}(\alpha)\, \hat{S}_\alpha^\dagger(t)\, \hat{X}^\dagger(t), \quad t \in [\beta, \infty). \tag{5.8}$$

Setting $M := \hat{X}(t)\, \hat{S}_\alpha(t)\, \hat{X}^T(\alpha)$ and $N := \hat{X}^{\dagger T}(\alpha)\, \hat{S}_\alpha^\dagger(t)\, \hat{X}^\dagger(t)$ for fixed $t \in [\beta, \infty)$, we verify that the four equations in (1.13) are satisfied. The identities $\hat{X}^\dagger(t)\hat{X}(t) = \hat{P}$, $\hat{X}(t)\hat{X}^\dagger(t) = \hat{R}(t)$, and $\hat{S}_\alpha^\dagger(t)\,\hat{S}_\alpha(t) = \hat{S}_\alpha(t)\,\hat{S}_\alpha^\dagger(t) = P_{\hat{\mathscr{S}}_{\alpha\infty}}$ imply that $NM = \hat{R}(\alpha)$ and $MN = \hat{R}(t)$ are symmetric. Moreover,

$$\begin{aligned} NMN &= (NM)N = \hat{R}(\alpha)\hat{X}^{\dagger T}(\alpha)\,\hat{S}_\alpha^\dagger(t)\,\hat{X}^\dagger(t) = \hat{X}^{\dagger T}(\alpha)\,\hat{S}_\alpha^\dagger(t)\,\hat{X}^\dagger(t) = N, \\ MNM &= (MN)M = \hat{R}(t)\,\hat{X}(t)\,\hat{S}_\alpha(t)\,\hat{X}^T(\alpha) = \hat{X}(t)\,\hat{S}_\alpha(t)\,\hat{X}^T(\alpha) = M. \end{aligned}$$

It follows from Remark 1.2.3(i) that $M^\dagger = N$ and hence, formula (5.8) holds. Now we construct the matrix $S_\beta(t)$ in (2.8) through the function $X(t)$, that is,

$$S_\beta(t) = \int_\beta^t X^\dagger(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s, \quad t \in [\beta, \infty). \tag{5.9}$$

Inserting (5.8) into (5.9) and using the equality $\hat{S}_\alpha'(t) = \hat{X}^\dagger(t)\, B(t)\, \hat{X}^{\dagger T}(t)$ on $[\beta, \infty)$ and Remark 1.2.4(i) with $M(t) := \hat{S}_\alpha(t)$, we obtain

$$\begin{aligned} S_\beta(t) &= \hat{X}^{\dagger T}(\alpha) \int_\beta^t \hat{S}_\alpha^\dagger(s)\, \hat{S}_\alpha'(s)\, \hat{S}_\alpha^\dagger(s)\, \mathrm{d}s\, \hat{X}^\dagger(\alpha) \\ &= -\hat{X}^{\dagger T}(\alpha) \int_\beta^t [\hat{S}_\alpha^\dagger(s)]'\, \mathrm{d}s\, \hat{X}^\dagger(\alpha) \end{aligned} \tag{5.10}$$

on $[\beta, \infty)$. Performing the integration in (5.10) yields the formula

$$S_\beta(t) = \hat{X}^{\dagger T}(\alpha)\, [\hat{S}_\alpha^\dagger(\beta) - \hat{S}_\alpha^\dagger(t)]\, \hat{X}^\dagger(\alpha), \quad t \in [\beta, \infty). \tag{5.11}$$

Finally, since $\hat{S}_\alpha^\dagger(t) \to 0$ for $t \to \infty$, equality (5.11) implies that the function $S_\beta(t)$ has the limit $\hat{X}^{\dagger T}(\alpha)\, \hat{S}_\alpha^\dagger(\beta)\, \hat{X}^\dagger(\alpha)$ as $t \to \infty$. Thus, according to Remark 5.2.4 and Theorem 5.2.3 the conjoined basis $(X,U)$ is an antiprincipal solution of (H) at infinity. And since the matrix $X(t)$ has the minimal rank $n - d_\infty$ on $[\beta, \infty)$, $(X,U)$ is a minimal antiprincipal solution, by Remark 5.2.2. ∎

**Remark 5.2.10.** Note that by (5.6) the matrix $T_\beta$ in Remark 2.1.6 associated with $(X,U)$ satisfies $T_\beta^\dagger = \hat{X}^{\dagger T}(\alpha)\,\hat{S}_\alpha^\dagger(\beta)\,\hat{X}^\dagger(\alpha)$ and hence, $T_\beta = \hat{X}(\alpha)\,\hat{S}_\alpha(\beta)\,\hat{X}^T(\alpha)$ by Remark 1.2.3(i). This additional information is however not needed in the proof of Proposition 5.2.9.

**Remark 5.2.11.** The result of Proposition 5.2.9 shows that in contrast to the minimal principal solution of (H) at infinity (Theorem 5.1.6), a minimal antiprincipal solution of (H) at infinity is not uniquely determined. Thus, one cannot expect to have a unifying classification of all minimal antiprincipal solutions at infinity in the spirit of Theorem 5.1.6. Moreover, the nonuniqueness of minimal antiprincipal solutions implies the same property for all antiprincipal solutions with higher ranks.

## 5.3 Applications

In the last section of this chapter we present several applications of the results about (anti)principal solutions of (H) at infinity. First, we discuss a completely controllable and nonoscillatory system (H). In this case $d_\infty = 0$ in (3.2) and from (3.16) it follows that the rank of conjoined bases of (H) with constant kernel can be equal to $r = n$ only, compare also with Remark 3.1.5. Thus, the notions of the minimal and maximal (anti)principal solutions of (H) at infinity coincide. This means that there is only one type of (anti)principal solutions of (H), i.e., the (anti)principal solutions $(X,U)$ with $X(t)$ invertible for large $t$, see also Example 7.1.1. In this case we obtain the traditional concept of the principal and antiprincipal solutions of a nonoscillatory system (H) displayed in Definition 1.4.3 in Section 1.4, see Reid's or Hartman's or Coppel's principal solution at infinity in [29], [31, Section VII.3], [16, Section XI.10], or [6, Section 2.2] and Ahlbrandt's antiprincipal solution at infinity in [1], as well as the result in Proposition 1.4.4.

**Corollary 5.3.1.** *Assume that* (1.1) *holds and* (H) *is completely controllable. Then the following statements are equivalent.*

(i) *System* (H) *is nonoscillatory.*

(ii) *There exists a principal solution* $(\hat{X},\hat{U})$ *of* (H) *at infinity with rank equal to n, i.e., with* $\hat{X}(t)$ *eventually invertible.*

(iii) *There exists an antiprincipal solution* $(X,U)$ *of* (H) *at infinity with rank equal to n, i.e., with* $X(t)$ *eventually invertible.*

*In this case, all principal solutions, as well as all antiprincipal solutions of* (H) *have rank equal to n. The principal solution* $(\hat{X},\hat{U})$ *is unique up to a right nonsingular multiple.*

*Proof.* The results follow directly from Theorems 5.1.5, 5.2.7, and 5.1.6. ∎

The second application of Theorems 5.1.5 and 5.2.7 concerns the existence of principal and antiprincipal solutions for two linear Hamiltonian systems. With system (H) we consider another system

$$x' = \tilde{A}(t)\,x + \tilde{B}(t)\,u, \quad u' = \tilde{C}(t)\,x - \tilde{A}^T(t)\,u, \qquad t \in [a,\infty), \tag{$\tilde{\mathrm{H}}$}$$

where $\tilde{A},\tilde{B},\tilde{C}:[a,\infty)\to\mathbb{R}^{n\times n}$ are piecewise continuous functions such that $\tilde{B}(t)$ and $\tilde{C}(t)$ are symmetric on $[a,\infty)$. We define on $[a,\infty)$ the symmetric $2n\times 2n$ matrices

$$\mathscr{H}(t):=\begin{pmatrix}-C(t) & A^T(t)\\ A(t) & B(t)\end{pmatrix},\quad \tilde{\mathscr{H}}(t):=\begin{pmatrix}-\tilde{C}(t) & \tilde{A}^T(t)\\ \tilde{A}(t) & \tilde{B}(t)\end{pmatrix}.$$

**Corollary 5.3.2.** *Assume* (1.1) *and* $\mathscr{H}(t)\le\tilde{\mathscr{H}}$ *for all* $t\in[a,\infty)$. *If system* ($\tilde{\text{H}}$) *has an (anti)principal solution at infinity, then for every integer* $r$ *between* $n-d_\infty$ *and* $n$ *also system* (H) *has an (anti)principal solution at infinity with rank* $r$.

*Proof.* From (1.1) and the inequality $\mathscr{H}(t)\le\tilde{\mathscr{H}}(t)$ on $[a,\infty)$ it follows that $\tilde{B}(t)\ge 0$ on $[a,\infty)$. By Theorems 5.1.5 and 5.2.7 applied to ($\tilde{\text{H}}$), we know that system ($\tilde{\text{H}}$) is nonoscillatory. In turn, by [40, Theorem 2.6], we get that system (H) is nonoscillatory as well. The conclusion now follows from Theorems 5.1.5 and 5.2.7 applied to (H). ∎

The last result of this section contains a construction of (anti)principal solutions of (H) with given rank from the (anti)principal solutions of systems with lower dimensions (see Example 7.1.5). Therefore, with system (H) we consider another linear Hamiltonian system

$$x'=\underline{A}(t)\,x+\underline{B}(t)\,u,\quad u'=\underline{C}(t)\,x-\underline{A}^T(t)\,u,\qquad t\in[a,\infty),\tag{$\underline{\text{H}}$}$$

where $\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$ are given $\underline{n}\times\underline{n}$ piecewise continuous matrices on $[a,\infty)$ such that $\underline{B}(t)$ and $\underline{C}(t)$ are symmetric and

$$\underline{B}(t)\ge 0\quad\text{on }[a,\infty).\tag{5.12}$$

From systems (H) and ($\underline{\text{H}}$) we construct the "augmented" linear Hamiltonian system

$$x_*'=A_*(t)\,x_*+B_*(t)\,u_*,\quad u_*'=C_*(t)\,x_*-A_*^T(t)\,u_*,\qquad t\in[a,\infty),\tag{$\text{H}_*$}$$

where $A_*,B_*,C_*\in\mathrm{C_p}$ are $(n+\underline{n})\times(n+\underline{n})$ matrices defined on $[a,\infty)$ by

$$A_*(t):=\begin{pmatrix}A(t) & 0\\ 0 & \underline{A}(t)\end{pmatrix},\quad B_*(t):=\begin{pmatrix}B(t) & 0\\ 0 & \underline{B}(t)\end{pmatrix},\quad C_*(t):=\begin{pmatrix}C(t) & 0\\ 0 & \underline{C}(t)\end{pmatrix}.\tag{5.13}$$

**Theorem 5.3.3.** *Assume that the Legendre conditions* (1.1) *and* (5.12) *hold and that the systems* (H) *and* ($\underline{\text{H}}$) *are nonoscillatory. If* $(\hat{X},\hat{U})$ *and* $(\underline{\hat{X}},\underline{\hat{U}})$ *are principal solutions of* (H) *of* ($\underline{\text{H}}$) *at infinity with rank equal to* $r$ *and* $\underline{r}$, *respectively, then the pair* $(\hat{X}_*,\hat{U}_*)$ *defined by*

$$\hat{X}_*(t):=\begin{pmatrix}\hat{X}(t) & 0\\ 0 & \underline{\hat{X}}(t)\end{pmatrix},\quad \hat{U}_*(t):=\begin{pmatrix}\hat{U}(t) & 0\\ 0 & \underline{\hat{U}}(t)\end{pmatrix},\quad t\in[a,\infty),\tag{5.14}$$

*is a principal solution of system* ($\text{H}_*$) *at infinity with rank equal to* $r+\underline{r}$. *Similarly, if* $(X,U)$ *and* $(\underline{X},\underline{U})$ *are antiprincipal solutions of* (H) *of* ($\underline{\text{H}}$) *at infinity with rank equal to* $r$ *and* $\underline{r}$, *respectively, then the pair* $(X_*,U_*)$ *defined by*

$$X_*(t):=\begin{pmatrix}X(t) & 0\\ 0 & \underline{X}(t)\end{pmatrix},\quad U_*(t):=\begin{pmatrix}U(t) & 0\\ 0 & \underline{U}(t)\end{pmatrix},\quad t\in[a,\infty),\tag{5.15}$$

*is an antiprincipal solution of system* ($\text{H}_*$) *at infinity with rank equal to* $r+\underline{r}$. *Moreover, the principal and antiprincipal solutions* $(\hat{X}_*,\hat{U}_*)$ *and* $(X_*,U_*)$ *constructed in* (5.14) *and* (5.15) *are minimal (maximal) if and only if the principal and antiprincipal solutions* $(\hat{X},\hat{U})$, $(\underline{\hat{X}},\underline{\hat{U}})$ *and* $(X,U)$, $(\underline{X},\underline{U})$ *are minimal (maximal).*

*Proof.* We will focus only on the proof of the case of principal solutions, since for the case of antiprincipal solutions one can use exactly the same arguments. Let $(\hat{X},\hat{U})$ and $(\underline{\hat{X}},\underline{\hat{U}})$ be as in the theorem. By Definition 5.1.1 and Theorem 5.1.4, there exists $\alpha \geq a$ such that $(\hat{X},\hat{U})$ and $(\underline{\hat{X}},\underline{\hat{U}})$ are principal solutions of (H) and ($\underline{\text{H}}$) with respect to the interval $[\alpha,\infty)$ and that $\operatorname{rank}\hat{X}(t) = r$ and $\operatorname{rank}\underline{\hat{X}}(t) = \underline{r}$ on $[\alpha,\infty)$. It is easy to see that the pair $(\hat{X}_*,\hat{U}_*)$ defined in (5.14) is a conjoined basis of ($\text{H}_*$) with constant kernel on $[\alpha,\infty)$ and with $\operatorname{rank}\hat{X}_*(t) = r+\underline{r}$ on $[\alpha,\infty)$. Now, if $\hat{S}(t)$, $\underline{\hat{S}}(t)$, $\hat{S}_*(t)$ are the matrices in (2.8) with $\beta := \alpha$ corresponding to $(\hat{X},\hat{U})$, $(\underline{\hat{X}},\underline{\hat{U}})$, $(\hat{X}_*,\hat{U}_*)$, respectively, then

$$\hat{S}_*(t) = \begin{pmatrix} \hat{S}(t) & 0 \\ 0 & \underline{\hat{S}}(t) \end{pmatrix}, \quad \hat{S}_*^{\dagger}(t) = \begin{pmatrix} \hat{S}^{\dagger}(t) & 0 \\ 0 & \underline{\hat{S}}^{\dagger}(t) \end{pmatrix}, \quad t \in [\alpha,\infty). \tag{5.16}$$

Since $\hat{S}^{\dagger}(t) \to 0$ and $\underline{\hat{S}}^{\dagger}(t) \to 0$ for $t \to \infty$, by Definition 5.1.1, it follows from (5.16) that $\hat{S}_*^{\dagger}(t) \to 0$ for $t \to \infty$ as well. Therefore, $(\hat{X}_*,\hat{U}_*)$ is a principal solution of system ($\text{H}_*$) at infinity. Moreover, if $d[\alpha,\infty)$ and $\underline{d}[\alpha,\infty)$ are respectively the orders of abnormality of systems (H) and ($\underline{\text{H}}$) on $[\alpha,\infty)$, then the order of abnormality $d_*[\alpha,\infty)$ of system ($\text{H}_*$) on $[\alpha,\infty)$ is equal to $d[\alpha,\infty)+\underline{d}[\alpha,\infty)$. This follows from the block structure of the coefficients $A_*(t)$ and $B_*(t)$ in (5.13), which implies that the space $\Lambda_*[\alpha,\infty)$, defined in Section 3.1 for system ($\text{H}_*$), consists of the function $u_* = (u^T,\underline{u}^T)^T$ with $u \in \Lambda[\alpha,\infty)$ and $\underline{u} \in \underline{\Lambda}[\alpha,\infty)$. In particular, the maximal order of abnormality of ($\text{H}_*$) is equal to the sum of the maximal orders of abnormality of systems (H) and ($\underline{\text{H}}$), i.e., $d_{*\infty} = d_\infty + \underline{d}_\infty$. Therefore, the conjoined basis $(\hat{X}_*,\hat{U}_*)$ of ($\text{H}_*$) defined in (5.14) is a minimal (maximal) principal solution of ($\text{H}_*$) if and only if $(\hat{X},\hat{U})$ and $(\underline{\hat{X}},\underline{\hat{U}})$ are minimal (maximal) principal solutions of (H) and ($\underline{\text{H}}$). ■

**Remark 5.3.4.** Note that the converse in Theorem 5.3.3 does not hold in general, i.e., principal and antiprincipal solutions of ($\text{H}_*$) at infinity do not need to have the block diagonal form displayed in (5.14) and (5.15), see Example 7.1.4.

# Chapter 6

# Genera of conjoined bases and limit properties of principal solutions

In this chapter we provide a completion of the theory of (anti)principal solutions of (H) at infinity. In particular, a deeper study of the relation "being contained" in Section 6.1 reveals an ordering on the set of all conjoined bases of (H) with constant kernel. Furthermore, in Section 6.2 we discuss the solvability of a certain system of algebraic matrix equations, which can be viewed as a continuation of the study in Section 3.2. Finally, in Section 6.3 we introduce a new concept of a genus of conjoined bases of (H), as a fundamental tool for the classification of conjoined bases, as well as (anti)principal solutions of (H) according to the image of their first component. These results are then utilized in a limit characterization of principal solutions of (H) at infinity in Section 6.4.

## 6.1 Ordering between equivalence classes

In this section we study the relation "being contained" for conjoined bases of (H) in Definition 3.2.1 in a more detailed way. In particular, we develop an extension of this relation to certain equivalence classes of conjoined bases of (H) and provide its basic properties. We show that the extended relation is an ordering (Theorem 6.1.9) and that the minimal elements of this ordering correspond to equivalence classes of minimal conjoined bases, while the maximal elements of this ordering are determined by the conjoined bases $(X,U)$ of (H) with invertible $X(t)$ (Theorem 6.1.14).

**Definition 6.1.1.** Let $(X,U)$ be a solution of (H). By the *equivalence class* corresponding to $(X,U)$ on $[\alpha,\infty)$ we mean the set denoted by $[(X,U)]$ of all conjoined bases of (H) which are equivalent with $(X,U)$ on $[\alpha,\infty)$. By the *kernel* of the nonempty equivalence class $[(X,U)]$ we mean the kernel of the solution $(X,U)$ on $[\alpha,\infty)$. The set of all nonempty equivalence classes of (H) with constant kernel on $[\alpha,\infty)$ will be denoted by $\mathscr{E}[\alpha,\infty)$.

**Remark 6.1.2.** (i) It is easy to see that a solution $(X,U)$ of (H) belongs to the equivalence class $[(X,U)]$ if and only if $(X,U)$ is a conjoined basis.

(ii) If $[(X,U)]$ is a nonempty equivalence class corresponding to a solution $(X,U)$ of (H) on $[\alpha,\infty)$, then $[(X,U)]$ corresponds also to each conjoined basis $(X_0,U_0)\in[(X,U)]$, because in this case $X_0(t)\equiv X(t)$ on $[\alpha,\infty)$. Therefore, we may assume without loss of generality that nonempty equivalence classes always correspond to conjoined bases of (H). The kernel of the equivalence class $[(X,U)]$ then obviously does not depend on the choice of $(X_0,U_0)$ in $[(X,U)]$.

(iii) If $[(X,U)] \in \mathscr{E}[\alpha,\infty)$, then we associate with $[(X,U)]$ the unique orthogonal projectors $P$ and $R(t)$ in (2.2) and (2.1), the function $S_\alpha(t)$ in (2.8), and the orthogonal projector $P_{\mathscr{S}_{\alpha\infty}}$ in (2.12), which are defined through any conjoined basis $(X_0,U_0)$ in $[(X,U)]$.

The following lemma contains some auxiliary results about the equivalence of two solutions of (H). In particular, we show the invariance of this relation with respect to the multiplication from the right by a constant square matrix.

**Lemma 6.1.3.** *Let $(X_1,U_1)$ and $(X_2,U_2)$ be two solutions of (H) with the corresponding equivalence classes $[(X_1,U_1)]$ and $[(X_2,U_2)]$ on $[\alpha,\infty)$. Then for any constant matrix $M \in \mathbb{R}^{n\times n}$ the following statements hold.*

(i) *If $(X_1,U_1) \sim (X_2,U_2)$ on $[\alpha,\infty)$, then also $(X_1M,U_1M) \sim (X_2M,U_2M)$ on $[\alpha,\infty)$.*

(ii) *If $[(X_1,U_1)] = [(X_2,U_2)] \neq \emptyset$, then also $[(X_1M,U_1M)] = [(X_2M,U_2M)]$.*

*Proof.* Part (i) follows trivially from the definition of the relation $\sim$ (see Definition 3.1.6). In part (ii) there is a conjoined basis $(X,U)$ such that $(X,U) \sim (X_1,U_1)$ and $(X,U) \sim (X_2,U_2)$ on $[\alpha,\infty)$. But since $\sim$ is an equivalence, we get $(X_1,U_1) \sim (X_2,U_2)$ on $[\alpha,\infty)$. Hence, $(X_1M,U_1M) \sim (X_2M,U_2M)$ on $[\alpha,\infty)$, by part (i), and the equality $[(X_1M,U_1M)] = [(X_2M,U_2M)]$ follows from Definition 6.1.1. ■

Note that in part (ii) of Lemma 6.1.3 it is sufficient to assume that $[(X_1,U_1)] \cap [(X_2,U_2)] \neq \emptyset$ for the same conclusion. In the next theorem we prove that the relation "being contained" for conjoined bases of (H) is preserved for the equivalence classes of (H) with constant kernel on $[\alpha,\infty)$, i.e., for the set $\mathscr{E}[\alpha,\infty)$.

**Theorem 6.1.4.** *Assume (1.1). Let $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ and let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be the orthogonal projectors defined in (2.2) and (2.12). Consider an orthogonal projector $P_*$ satisfying (3.26). Then a conjoined basis $(X_*,U_*)$ of (H) is contained in the conjoined basis $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$ if and only if $(X_*,U_*)$ is contained in each element of $[(X,U)]$ with respect to $P_*$. In addition, all conjoined bases of (H) which are contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$ form a unique equivalence class in $\mathscr{E}[\alpha,\infty)$.*

*Proof.* Let $(X_*,U_*)$ be a conjoined basis of (H) which is contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$. This means, by Definition 3.2.1, that $(X_*,U_*) \sim (XP_*,UP_*)$ on $[\alpha,\infty)$. If $(X_0,U_0) \in [(X,U)]$ is any conjoined basis, then $(X_0,U_0) \sim (X,U)$ on $[\alpha,\infty)$ and consequently, $(X_0P_*,U_0P_*) \sim (XP_*,UP_*)$ on $[\alpha,\infty)$ by using Lemma 6.1.3(i). Therefore, $(X_*,U_*)$ is equivalent with the solution $(X_0P_*,U_0P_*)$ on $[\alpha,\infty)$, which implies that $(X_*,U_*)$ is contained in $(X_0,U_0)$ on $[\alpha,\infty)$ with respect to $P_*$. The opposite direction follows trivially from Remark 6.1.2(i). Furthermore, the transitivity of the relation $\sim$ implies that if $(X,U)$ contains a conjoined basis $(X_*,U_*)$ on $[\alpha,\infty)$ with respect to $P_*$, then $(X,U)$ contains with respect to $P_*$ every conjoined basis of (H) which is equivalent with $(X_*,U_*)$ on $[\alpha,\infty)$. This fact also implies that every two conjoined bases of (H) which are contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$ are equivalent on $[\alpha,\infty)$. Therefore, all conjoined bases of (H) contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$ form a unique equivalence class of (H) on $[\alpha,\infty)$. According to Theorem 3.2.7, this class is nonempty and has a constant kernel on $[\alpha,\infty)$ and hence, it belongs to $\mathscr{E}[\alpha,\infty)$, by Definition 6.1.1. ■

**Remark 6.1.5.** The proof of Theorem 6.1.4 shows that the equivalence class of all conjoined bases of (H) contained in $(X,U)$ on $[\alpha,\infty)$ with respect to $P_*$ is equal to $[(XP_*,UP_*)] \in \mathscr{E}[\alpha,\infty)$. Note that in the sense of Remark 6.1.2(iii) the orthogonal projector $P_*$ is associated with the class $[(XP_*,UP_*)]$ by (3.27) and (2.2), as well as the orthogonal projector $P_{\mathscr{S}_{\alpha\infty}}$ is associated with $[(XP_*,UP_*)]$ by Theorem 3.2.4.

The results in Theorem 6.1.4 allow to extend the relation "being contained" to the set $\mathscr{E}[\alpha,\infty)$ in Definition 6.1.1.

**Definition 6.1.6.** Let $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ and let $P$ and $P_{\mathscr{S}_{\alpha\infty}}$ be the orthogonal projectors in (2.2) and (2.12), which are associated with $[(X,U)]$. Consider an orthogonal projector $P_*$ satisfying (3.26). We say that the equivalence class $[(X_*,U_*)] \in \mathscr{E}[\alpha,\infty)$ is *contained* in the class $[(X,U)]$ with respect to $P_*$, if $[(X_*,U_*)] = [(XP_*,UP_*)]$. In this case we write $[(X_*,U_*)] \preceq [(X,U)]$.

Alternatively, we say that the class $[(X,U)]$ *contains* the class $[(X_*,U_*)]$ with respect to $P_*$. We will also drop the orthogonal projector $P_*$ when it is clear from the context which projector $P_*$ is considered, as we comment in Remark 6.1.7(ii) below.

**Remark 6.1.7.** (i) Let $(X,U)$, $P$, $P_{\mathscr{S}_{\alpha\infty}}$ be as in Definition 6.1.6. Then for every orthogonal projector $P_*$ satisfying (3.26) there exists a unique equivalence class $[(X_*,U_*)] \in \mathscr{E}[\alpha,\infty)$ such that $[(X_*,U_*)] \preceq [(X,U)]$ with respect to $P_*$. This is a direct consequence of Definition 6.1.6 together with Remark 6.1.5 and Theorem 6.1.4. These results also imply that with respect to Remark 6.1.2(iii) the orthogonal projectors $P_*$ and $P_{\mathscr{S}_{*\alpha\infty}} = P_{\mathscr{S}_{\alpha\infty}}$ are associated with the equivalence class $[(X_*,U_*)]$. In particular, for $P_* = P$ we have $[(X_*,U_*)] = [(X,U)]$, since in this case the solutions $(XP_*,UP_*) = (XP,UP)$ and $(X,U)$ are equivalent on $[\alpha,\infty)$, i.e., they determine the same equivalence class in $\mathscr{E}[\alpha,\infty)$.

(ii) On the other hand, if $[(X_*,U_*)]$ is any equivalence class from $\mathscr{E}[\alpha,\infty)$ with the corresponding orthogonal projector $P_*$ defined in (2.2) through the function $X_*(t)$, then the relation $[(X_*,U_*)] \preceq [(X,U)]$ means that $P_*$ satisfies (3.26) and $[(X_*,U_*)] = [(XP_*,UP_*)]$, i.e. the equivalence class $[(X_*,U_*)]$ is contained in $[(X,U)]$ with respect to the projector $P_*$.

**Remark 6.1.8.** The result in Theorem 3.2.10 in Section 3.2 implies that every equivalence class from $\mathscr{E}[\alpha,\infty)$ is contained in some element of $\mathscr{E}[\alpha,\infty)$. More precisely, if $[(X_*,U_*)]$ is a nonempty equivalence class with constant kernel on $[\alpha,\infty)$ and $P_*$ and $R_*(t)$ are the associated matrices in (2.2) and (2.1), then for any orthogonal projectors $P_\alpha$ and $R_\alpha$ satisfying (3.34) there exists an equivalence class $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ such that $[(X_*,U_*)] \preceq [(X,U)]$ and its corresponding matrices $P$ and $R(t)$ defined in (2.2) and (2.1) satisfy $P = P_\alpha$ and $R(\alpha) = R_\alpha$.

The next theorem shows that the relation "being contained" for equivalence classes of conjoined bases of (H) introduced in Definition 6.1.6 is an ordering on $\mathscr{E}[\alpha,\infty)$.

**Theorem 6.1.9.** *Assume* (1.1). *The relation* $\preceq$ *from Definition 6.1.6 is an ordering on the set* $\mathscr{E}[\alpha,\infty)$*, i.e., it is reflexive, antisymmetric, and transitive on* $\mathscr{E}[\alpha,\infty)$.

*Proof. Reflexivity:* Let $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ and let $P$ be the orthogonal projector defined in (2.2). Then $XP = X$, so that $(X,U) \sim (X,UP) = (XP,UP)$ and hence $[(X,U)] = [(XP,UP)]$. According to Definition 6.1.6 we then have $[(X,U)] \preceq [(X,U)]$ with respect to projector $P$.

*Antisymmetry:* Let $[(X_1,U_1)]$, $[(X_2,U_2)] \in \mathscr{E}[\alpha,\infty)$ and let $P_1$ and $P_2$ be the orthogonal projectors defined in (2.2) through the functions $X_1(t)$ and $X_2(t)$, respectively. Suppose that $[(X_1,U_1)] \preceq [(X_2,U_2)]$ and $[(X_2,U_2)] \preceq [(X_1,U_1)]$. From Definition 6.1.6 and Remark 6.1.7(ii) we get that $P_1 = P_2$ and consequently $[(X_1,U_1)] = [(X_2,U_2)]$, by Remark 6.1.7(i).

*Transitivity:* Let $[(X_1,U_1)]$, $[(X_2,U_2)]$, $[(X_3,U_3)] \in \mathscr{E}[\alpha,\infty)$ and let $P_1$, $P_2$, $P_3$ and $P_{\mathscr{S}_{1\alpha\infty}}$, $P_{\mathscr{S}_{2\alpha\infty}}$, $P_{\mathscr{S}_{3\alpha\infty}}$ be their associated orthogonal projectors as in (2.2) and (2.12) with the functions $X_1(t)$, $X_2(t)$, $X_3(t)$, respectively. Suppose that $[(X_1,U_1)] \preceq [(X_2,U_2)]$ and $[(X_2,U_2)] \preceq [(X_3,U_3)]$. Then by Definition 6.1.6 and Remark 6.1.7 we have

$$\operatorname{Im} P_{\mathscr{S}_{1\alpha\infty}} = \operatorname{Im} P_{\mathscr{S}_{2\alpha\infty}} = \operatorname{Im} P_{\mathscr{S}_{3\alpha\infty}} \subseteq \operatorname{Im} P_1 \subseteq \operatorname{Im} P_2 \subseteq \operatorname{Im} P_3. \tag{6.1}$$

This implies that $P_1 = P_2 P_1$. In addition, from Remark 6.1.5 we obtain the equalities $[(X_1,U_1)] = [(X_2P_1,U_2P_1)]$ and $[(X_2,U_2)] = [(X_3P_2,U_3P_2)]$, which, by Lemma 6.1.3(ii) with $M = P_1$, yield that $[(X_2P_1,U_2P_1)] = [(X_3P_2P_1,U_3P_2P_1)]$. Therefore,

$$[(X_1,U_1)] = [(X_2P_1,U_2P_1)] = [(X_3P_2P_1,U_3P_2P_1)] = [(X_3P_1,U_3P_1)].$$

And since $\operatorname{Im} P_{\mathscr{S}_{3\alpha\infty}} \subseteq \operatorname{Im} P_1 \subseteq \operatorname{Im} P_3$ by (6.1), we have that $[(X_1,U_1)] \preceq [(X_3,U_3)]$ with respect to projector $P_1$, compare with Remark 6.1.7(ii). ■

**Remark 6.1.10.** (i) Every principal solution $(\hat{X},\hat{U})$ of (H) with respect to the interval $[\alpha,\infty)$ defines a nonempty equivalence class $[(\hat{X},\hat{U})] \in \mathscr{E}[\alpha,\infty)$. Such equivalence classes in $\mathscr{E}[\alpha,\infty)$ will be called *principal*. The result in Theorem 5.1.3 then implies that the equivalence classes from $\mathscr{E}[\alpha,\infty)$, which are either contained in $[(\hat{X},\hat{U})]$ or which contain $[(\hat{X},\hat{U})]$, are also principal equivalence classes. Therefore, if we denote by $\mathscr{E}_P[\alpha,\infty)$ the set of all principal equivalence classes from $\mathscr{E}[\alpha,\infty)$, then $\mathscr{E}_P[\alpha,\infty)$ is an isolated ordered component of $\mathscr{E}[\alpha,\infty)$.

(ii) Similarly, all antiprincipal solutions $(X,U)$ of (H) at infinity with respect to $[\alpha,\infty)$ generate *antiprincipal* equivalence classes in $\mathscr{E}[\alpha,\infty)$ which, by Theorem 5.2.6, form an isolated ordered component $\mathscr{E}_A[\alpha,\infty)$.

The following theorem shows that the ordering of two classes of the set $\mathscr{E}[\alpha,\infty)$ with common upper bound is determined only by the ordering of the images of their corresponding orthogonal projectors defined in (2.2) with respect to the inclusion.

**Theorem 6.1.11.** *Let $[(X,U)]$, $[(X_*,U_*)]$, $[(X_{**},U_{**})] \in \mathscr{E}[\alpha,\infty)$ and let $P$, $P_*$ and $P_{**}$ be the orthogonal projectors defined in (2.2) through the functions $X(t)$, $X_*(t)$ and $X_{**}(t)$, respectively. Furthermore, suppose that*

$$[(X_*,U_*)] \preceq [(X,U)] \quad \text{and} \quad [(X_{**},U_{**})] \preceq [(X,U)]. \tag{6.2}$$

*Then $[(X_{**},U_{**})] \preceq [(X_*,U_*)]$ if and only if $\operatorname{Im} P_{**} \subseteq \operatorname{Im} P_*$.*

*Proof.* According to Definition 6.1.6 and Remark 6.1.5 the inequalities in (6.2) yield

$$[(X_*,U_*)] = [(XP_*,UP_*)] \quad \text{and} \quad [(X_{**},U_{**})] = [(XP_{**},UP_{**})]. \tag{6.3}$$

Now if $[(X_{**},U_{**})] \preceq [(X_*,U_*)]$ then the inclusion $\operatorname{Im} P_{**} \subseteq \operatorname{Im} P_*$ holds, by Definition 6.1.6. Conversely, the inclusion $\operatorname{Im} P_{**} \subseteq \operatorname{Im} P_*$ means that $P_{**} = P_* P_{**}$. By using Lemma 6.1.3(ii), the equalities in (6.3) then imply that

$$[(X_{**},U_{**})] = [(XP_{**},UP_{**})] = [(XP_*P_{**},UP_*P_{**})] = [(X_*P_{**},U_*P_{**})]. \tag{6.4}$$

Finally, formula (6.4) means that $[(X_{**},U_{**})] \preceq [(X_*,U_*)]$, by Definition 6.1.6. ■

It is convenient to introduce the following notion.

**Definition 6.1.12.** An equivalence class $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ is said to be *minimal* (or it is a *minimal equivalence class*) if $(X,U)$ is a minimal conjoined basis of (H) on $[\alpha,\infty)$.

**Remark 6.1.13.** Alternatively, an equivalence class $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ is minimal if and only if the projectors $P$ and $P_{\mathscr{S}_\alpha\infty}$ associated with $(X,U)$ through (2.2) and (2.12) satisfy $P_{\mathscr{S}_\alpha\infty} = P$. This observation follows from Remark 3.3.3.

We now recall a standard terminology for ordered sets $(\mathscr{M},\preceq)$, see e.g. [27, Section XIV.1]. Namely, an element $x \in \mathscr{M}$ is minimal with respect to the ordering $\preceq$ if $y \in \mathscr{M}$ and $y \preceq x$ imply $y = x$. Similarly, an element $x \in \mathscr{M}$ is maximal with respect to the ordering $\preceq$ if $y \in \mathscr{M}$ and $x \preceq y$ imply $x = y$. In this context, the following theorem shows that minimal equivalence classes correspond to the minimal elements of $\mathscr{E}[\alpha,\infty)$, while the equivalence classes $[(X,U)]$ with $X(t)$ invertible on $[\alpha,\infty)$ correspond to the maximal elements of $\mathscr{E}[\alpha,\infty)$ with respect to the ordering $\preceq$.

**Theorem 6.1.14.** *Assume* (1.1). *Then the following statements hold.*

(i) *An equivalence class $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ is minimal if and only if it is a minimal element of the set $\mathscr{E}[\alpha,\infty)$ with respect to the ordering $\preceq$. Moreover, for every equivalence class $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ there exists a unique minimal equivalence class $[(X_*,U_*)] \in \mathscr{E}[\alpha,\infty)$ such that $[(X_*,U_*)] \preceq [(X,U)]$.*

(ii) *An equivalence class $[(\tilde{X},\tilde{U})] \in \mathscr{E}[\alpha,\infty)$ is a maximal element of $\mathscr{E}[\alpha,\infty)$ with respect to the ordering $\preceq$ if and only if the function $\tilde{X}(t)$ is invertible on $[\alpha,\infty)$. Moreover, for every equivalence class $[(X_*,U_*)] \in \mathscr{E}[\alpha,\infty)$ there exists a maximal element $[(\tilde{X},\tilde{U})] \in \mathscr{E}[\alpha,\infty)$ such that $[(X_*,U_*)] \preceq [(\tilde{X},\tilde{U})]$.*

*Proof.* (i) Let $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ be a minimal equivalence class with the projectors $P$ and $P_{\mathscr{S}\infty}$ defined in (2.2) and (2.12). Then $P_{\mathscr{S}_\alpha\infty} = P$, by Remark 6.1.13. Furthermore, let $[(X_*,U_*)] \in \mathscr{E}[\alpha,\infty)$ be such that $[(X_*,U_*)] \preceq [(X,U)]$ and let $P_*$ and $P_{\mathscr{S}_{*\alpha}\infty}$ be its associated orthogonal projectors defined in (2.2) and (2.12) through the function $X_*(t)$. Then $\operatorname{Im} P_{\mathscr{S}_{*\alpha}\infty} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P = \operatorname{Im} P_{\mathscr{S}_\alpha\infty}$, by Remark 6.1.7(ii). Since $\operatorname{rank} P_{\mathscr{S}_{*\alpha}\infty} = \operatorname{rank} P_{\mathscr{S}_\alpha\infty}$, we have $\operatorname{Im} P_* = \operatorname{Im} P$ and consequently $P_* = P$. According to Remark 6.1.7(i), the last equality implies that $[(X_*,U_*)] = [(X,U)]$. Therefore, $[(X,U)]$ is a minimal element of the ordered set $\mathscr{E}[\alpha,\infty)$. Moreover, from Definition 6.1.6 and Remarks 6.1.5 and 6.1.13

it follows that for any $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ the equivalence class $[(X_*,U_*)] \in \mathscr{E}[\alpha,\infty)$ which is contained in $[(X,U)]$ with respect to the projector $P_{\mathscr{S}_{\alpha\infty}}$ is a minimal equivalence class. The uniqueness of $[(X_*,U_*)]$ is then guaranteed by Remark 6.1.7(i). In particular, if $[(X,U)]$ is a minimal element of the ordered set $\mathscr{E}[\alpha,\infty)$, then necessarily $[(X_*,U_*)] = [(X,U)]$ and hence, $[(X,U)]$ is a minimal equivalence class according to Definition 6.1.12.

(ii) Let $[(\tilde{X},\tilde{U})] \in \mathscr{E}[\alpha,\infty)$ be a maximal element of $\mathscr{E}[\alpha,\infty)$. Set $P_\alpha := I$ and $R_\alpha := I$. By Remark 6.1.8 with $(X_*,U_*) := (\tilde{X},\tilde{U})$, there exists an equivalence class $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ such that the associated orthogonal projectors $P$ and $R(t)$ in (2.2) and (2.1) satisfy $P = P_\alpha = I$, $R(\alpha) = R_\alpha = I$, and $[(\tilde{X},\tilde{U})] \preceq [(X,U)]$. From (2.3) we then obtain $\operatorname{rank} X(t) = \operatorname{rank} P_\alpha = n$, i.e., the function $X(t)$ is invertible on $[\alpha,\infty)$. Since $[(\tilde{X},\tilde{U})]$ is a maximal element of $\mathscr{E}[\alpha,\infty)$, it follows that $[(\tilde{X},\tilde{U})] = [(X,U)]$ and hence, the function $\tilde{X}(t) = X(t)$ is also invertible on $[\alpha,\infty)$. Conversely, suppose that the equivalence class $[(\tilde{X},\tilde{U})] \in \mathscr{E}[\alpha,\infty)$ has $\tilde{X}(t)$ invertible on $[\alpha,\infty)$ and let $[(X,U)] \in \mathscr{E}[\alpha,\infty)$ be such that $[(\tilde{X},\tilde{U})] \preceq [(X,U)]$. Denote by $\tilde{P}$ and $P$ the corresponding orthogonal projectors in (2.2), which are defined through $\tilde{X}(t)$ and $X(t)$. Then $\tilde{P} = I$ and $\mathbb{R}^n = \operatorname{Im}\tilde{P} \subseteq \operatorname{Im} P$, by Remark 6.1.7(ii) with $(X_*,U_*) := (\tilde{X},\tilde{U})$. Consequently, $\operatorname{Im} P = \operatorname{Im}\tilde{P}$ and thus $P = \tilde{P}$. This implies by Remark 6.1.7(i) that $[(\tilde{X},\tilde{U})] = [(X,U)]$ and hence, the class $[(\tilde{X},\tilde{U})]$ is a maximal element of $\mathscr{E}[\alpha,\infty)$ with respect to $\preceq$. Therefore, the maximal elements $(\tilde{X},\tilde{U})$ of $\mathscr{E}[\alpha,\infty)$ are exactly the classes with $\tilde{X}(t)$ invertible on $[\alpha,\infty)$. The remaining part of the proof follows from Remark 6.1.8 with $P_\alpha := I$ and $R_\alpha := I$ and from the just established property of the maximal elements of $\mathscr{E}[\alpha,\infty)$. ■

**Remark 6.1.15.** The proof of Theorem 6.1.14(i) provides a construction of minimal equivalence classes in $\mathscr{E}[\alpha,\infty)$ via the relation "being contained" in Definition 6.1.6. This situation is analogous to the corresponding one in Remark 3.3.2 for minimal conjoined bases of (H).

## 6.2 Equivalence classes with given rank

In this section we discuss the solvability of system (3.32)–(3.33) of algebraic matrix equations. As we showed in Section 3.2 (Theorem 3.2.8), this system is closely related with the construction of all conjoined bases of (H) with constant kernel on $[\alpha,\infty)$, which contain a given conjoined basis of the same type, see also Remark 3.2.9. In particular, we prove that system (3.32)–(3.33) is always solvable for a suitable choice of its coefficients (Theorem 6.2.5). This result is then utilized to prove Theorem 3.2.10. Thus, throughout this section we fix a conjoined basis $(X_*,U_*)$ of (H) with constant kernel on $[\alpha,\infty)$ and denote by $P_*$, $R_*(t)$, and $P_{\mathscr{S}_{*\alpha\infty}}$ the associated orthogonal projectors defined in (2.2), (2.1), and (2.12) through the function $X_*(t)$. Moreover, let $Q_*(t)$ be the Riccati quotient in (2.5) corresponding to $(X_*,U_*)$. We also fix a conjoined basis $(\bar{X}_{*\alpha},\bar{U}_{*\alpha})$ of (H), for which (1.16), (1.17), and (2.25) with $\beta := \alpha$ hold. In order to simplify the notation we will drop the index $\alpha$ and use only $(\bar{X}_*,\bar{U}_*)$. Thus, we have

$$X_*^T\bar{U}_* - U_*^T\bar{X}_* = I = X_*\bar{U}_*^T - \bar{X}_*U_*^T \quad \text{on } [\alpha,\infty), \quad X_*^\dagger(\alpha)\bar{X}_*(\alpha) = 0.$$

We note that by Theorem 2.2.5 such a conjoined basis $(\bar{X}_*, \bar{U}_*)$ always exists. Moreover, the function $\bar{X}_*(t)$ is unique on $[\alpha, \infty)$ and the formulas

$$\bar{X}_*^T(\alpha)\,\bar{X}_*^{\dagger T}(\alpha) = I - P_*, \quad \bar{X}_*(\alpha)\,\bar{X}_*^{\dagger}(\alpha) = I - R_*(\alpha), \tag{6.5}$$

$$[X_*(\alpha) - \bar{X}_*(\alpha)]^{-1} = X_*^{\dagger}(\alpha) - \bar{X}_*^{\dagger}(\alpha), \quad \bar{X}_*^{\dagger}(\alpha) = -(I - P_*)\,U_*^T(\alpha) \tag{6.6}$$

hold, by Theorem 2.2.11(iii)–(v). In addition, for a given orthogonal projector $R_\alpha$ satisfying (3.34) we let $\tilde{P}_\alpha$ to be the orthogonal projector defined by

$$\operatorname{Im} \tilde{P}_\alpha := \operatorname{Im}\,[X_*^{\dagger}(\alpha) - \bar{X}_*^{\dagger}(\alpha)]\,R_\alpha. \tag{6.7}$$

In the next two lemmas we derive some additional properties of the coefficients of system (3.32)–(3.33).

**Lemma 6.2.1.** *Let $P_\alpha$, $R_\alpha$, and $\tilde{P}_\alpha$ be orthogonal projectors satisfying* (3.34) *and* (6.7). *For $\bar{G} \in \mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, I)$ define the matrix $\bar{G}_\perp := \bar{G}\,(I - P_*)$, see* (A.20) *in Appendix* A. *Then*

(i) $\operatorname{Im} P_* \subseteq \operatorname{Im} \tilde{P}_\alpha$ *and* $\operatorname{rank} \tilde{P}_\alpha = \operatorname{rank} P_\alpha$,

(ii) $\tilde{P}_\alpha\,\bar{X}_*^{\dagger}(\alpha)\,R_\alpha = \bar{X}_*^{\dagger}(\alpha)\,R_\alpha$ *and* $R_\alpha\,\bar{X}_*(\alpha)\,\tilde{P}_\alpha = \bar{X}_*(\alpha)\,\tilde{P}_\alpha$,

(iii) $[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^{\dagger} = \bar{G}_\perp^{\,\dagger T}\,\bar{X}_*^{\dagger}(\alpha)$,

(iv) $\operatorname{Im} \bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} = \operatorname{Im}\,[I - R_*(\alpha)]$ *and* $\operatorname{Im}\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^T = \operatorname{Im}\,(I - P_*)$.

*Proof.* (i) Using the properties $\operatorname{Im} \bar{X}_*^{\dagger T}(\alpha) = \operatorname{Im}\,[I - R_*(\alpha)]$ and $R_*(\alpha) = R_\alpha\,R_*(\alpha)$ from Theorem 2.2.11(iii) and (3.34), we get $X_*^{\dagger}(\alpha) = [X_*^{\dagger}(\alpha) - \bar{X}_*^{\dagger}(\alpha)]\,R_*(\alpha) = [X_*^{\dagger}(\alpha) - \bar{X}_*^{\dagger}(\alpha)]\,R_\alpha\,R_*(\alpha)$. Thus, $\operatorname{Im} P_* = \operatorname{Im} X_*^{\dagger}(\alpha) \subseteq \operatorname{Im} \tilde{P}_\alpha$, by the definition of $\tilde{P}_\alpha$ in (6.7). Since the matrix $X_*^{\dagger}(\alpha) - \bar{X}_*^{\dagger}(\alpha)$ is invertible by Theorem 2.2.11(iii), we have $\operatorname{rank} \tilde{P}_\alpha = \operatorname{rank} R_\alpha = \operatorname{rank} P_\alpha$, by the last condition in (3.34).

(ii) The definition of $\tilde{P}_\alpha$ in (6.7) implies that $\tilde{P}_\alpha\,[X_*^{\dagger}(\alpha) - \bar{X}_*^{\dagger}(\alpha)]\,R_\alpha = X_*^{\dagger}(\alpha)\,R_\alpha - \bar{X}_*^{\dagger}(\alpha)\,R_\alpha$. Since $\tilde{P}_\alpha\,X_*^{\dagger}(\alpha) = X_*^{\dagger}(\alpha)$ by the proof of part (i), we have $\tilde{P}_\alpha\,\bar{X}_*^{\dagger}(\alpha)\,R_\alpha = \bar{X}_*^{\dagger}(\alpha)\,R_\alpha$. On the other hand, (2.40) implies that $\operatorname{Im} R_\alpha = \operatorname{Im}\,[X_*(\alpha) - \bar{X}_*(\alpha)]\,\tilde{P}_\alpha$. Then $R_\alpha\,[X_*(\alpha) - \bar{X}_*(\alpha)]\,\tilde{P}_\alpha = X_*(\alpha)\,\tilde{P}_\alpha - \bar{X}_*(\alpha)\,\tilde{P}_\alpha$ and using the identity $R_\alpha\,X_*(\alpha) = X_*(\alpha)$ we obtain $R_\alpha\,\bar{X}_*(\alpha)\,\tilde{P}_\alpha = \bar{X}_*(\alpha)\,\tilde{P}_\alpha$.

(iii) The statement follows from Remark 1.2.3(iii), in which $M := \bar{X}_*(\alpha)$ and $N := \bar{G}_\perp^{\,T}$. Indeed, by the properties of $\bar{X}_*(\alpha)$ and $\bar{G}_\perp$ in Theorem 2.2.11(iii) and Remark A.2.4 in Appendix A, we have $P_{\operatorname{Im} M^T} = I - P_* = P_{\operatorname{Im} N}$, so that in this case $(MN)^{\dagger} = N^{\dagger} M^{\dagger}$.

(iv) From Remark 1.2.3(i) with $M := \bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}$ and part (iii) we know that the matrices

$$MM^{\dagger} = \bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}\,\bar{G}_\perp^{\,\dagger T}\,\bar{X}_*^{\dagger}(\alpha) = \bar{X}_*(\alpha)\,(I - P_*)\,\bar{X}_*^{\dagger}(\alpha) = \bar{X}_*(\alpha)\,\bar{X}_*^{\dagger}(\alpha) = I - R_*(\alpha),$$
$$M^{\dagger}M = \bar{G}_\perp^{\,\dagger T}\,\bar{X}_*^{\dagger}(\alpha)\,\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} = \bar{G}_\perp^{\,\dagger T}\,(I - P_*)\,\bar{G}_\perp^{\,T} = \bar{G}_\perp^{\,\dagger T}\,\bar{G}_\perp^{\,T} = I - P_*$$

are the orthogonal projectors onto $\operatorname{Im} M$ and $\operatorname{Im} M^T$, which completes the proof. ■

In the following lemma we utilize the result from Theorem A.2.7 in Appendix A.

**Lemma 6.2.2.** *Let $P_\alpha$ and $R_\alpha$ be the orthogonal projectors satisfying (3.34) and let $(G,H) \in \mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, P_\alpha)$. Moreover, let $\bar{G} \in \mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, I)$ be the matrix associated with $(G,H)$ through Theorem A.2.7 in Appendix A and set $\bar{G}_\perp := \bar{G}\,(I-P_*)$. If the matrices $X_\alpha$, $U_\alpha$ solve system (3.32)–(3.33), then*

$$[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger = -(P_\alpha - P_*)\,X_\alpha^\dagger - (I-P_\alpha)\,U_\alpha^T. \tag{6.8}$$

*Proof.* Let $X_\alpha$ and $U_\alpha$ solve system (3.32)–(3.33). By inserting the formulas $G = P_\alpha\,\bar{G}$ and $H = (I-P_\alpha)\,\bar{G}$ from (A.25) in Appendix A into the third equation in (3.33) and using the identity $X_\alpha^{\dagger T} P_\alpha = X_\alpha^{\dagger T}$, we get $X_\alpha^{\dagger T}\bar{G} + U_\alpha\,[P_* + (I-P_\alpha)\,\bar{G}] = U_*(\alpha)$. Multiplying the latter equation by $I-P_*$ from the right gives $X_\alpha^{\dagger T}\bar{G}\,(I-P_*) + U_\alpha\,(I-P_\alpha)\,\bar{G}\,(I-P_*) = U_*(\alpha)\,(I-P_*)$. From the second formula in (6.6) we then get

$$[X_\alpha^{\dagger T} + U_\alpha\,(I-P_\alpha)]\,\bar{G}_\perp = -\bar{X}_*^{\dagger T}(\alpha). \tag{6.9}$$

Now the result of Lemma 6.2.1(iii), equality (6.9), and the properties of the matrix $\bar{G}_\perp$ in (A.22) in Appendix A together with the identities $(I-P_*)\,(I-P_\alpha) = I-P_\alpha$ and $(I-P_*)\,P_\alpha = P_\alpha - P_*$ imply

$$\begin{aligned}[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger &= \bar{G}_\perp^{\dagger T}\bar{X}_*^\dagger(\alpha) \overset{(6.9)}{=} -\bar{G}_\perp^{\dagger T}\bar{G}_\perp^{\,T}\,[X_\alpha^\dagger + (I-P_\alpha)\,U_\alpha^T] \\ &\overset{(A.22)}{=} -(I-P_*)\,[P_\alpha X_\alpha^\dagger + (I-P_\alpha)\,U_\alpha^T] = -(P_\alpha - P_*)\,X_\alpha^\dagger - (I-P_\alpha)\,U_\alpha^T.\end{aligned}$$

Therefore, formula (6.8) holds. ∎

In the next theorem we present an equivalent condition to the solvability of system (3.32)–(3.33). This result then allows to find the solutions of this system.

**Theorem 6.2.3.** *Let $P_\alpha$ and $R_\alpha$ be the orthogonal projectors satisfying (3.34). For a pair $(G,H) \in \mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, P_\alpha)$ denote by $\bar{G} \in \mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, I)$ the matrix, which is associated with $(G,H)$ through Theorem A.2.7 in Appendix A, and set $\bar{G}_\perp := \bar{G}\,(I-P_*)$. Then system (3.32)–(3.33) is solvable if and only if*

$$\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} P_\alpha = R_\alpha\,\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}. \tag{6.10}$$

*Proof.* Suppose that system (3.32)–(3.33) is solvable and let $X_\alpha$, $U_\alpha$ be its solution. It follows by (6.8) that $P_\alpha\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger = -(P_\alpha - P_*)\,X_\alpha^\dagger$ and $[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger R_\alpha = -(P_\alpha - P_*)\,X_\alpha^\dagger$, since $X_\alpha^\dagger R_\alpha = X_\alpha^\dagger$ and $(I-P_\alpha)\,U_\alpha^T R_\alpha = (I-P_\alpha)\,U_\alpha^T X_\alpha X_\alpha^\dagger = 0$, by (3.32) and the symmetry of $U_\alpha^T X_\alpha$. Hence, $P_\alpha\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger = [\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger R_\alpha$. We will show by Lemma 6.2.1(iv) and the identities $(I-P_*)\,P_\alpha = P_\alpha\,(I-P_*)$ and $[I-R_*(\alpha)]\,R_\alpha = R_\alpha\,[I-R_*(\alpha)]$ that (6.10) holds. We have

$$\begin{aligned}\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} P_\alpha &= [\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}(I-P_*)]\,P_\alpha = \bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} P_\alpha\,(I-P_*) \\ &= \bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} P_\alpha\,\big([\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger \bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}\big) \\ &= \big(\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}[\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}]^\dagger\big)\,R_\alpha\,\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} = [I-R_*(\alpha)]\,R_\alpha\,\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} \\ &= R_\alpha\,[I-R_*(\alpha)]\,\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T} = R_\alpha\,\bar{X}_*(\alpha)\,\bar{G}_\perp^{\,T}.\end{aligned}$$

Conversely, assume that condition (6.10) is satisfied and we put $\bar{G}_{\shortparallel} := \bar{G}P_*$ as in (A.20). We will show that

$$\left.\begin{aligned} X_\alpha &:= X_*(\alpha) - \bar{X}_*(\alpha)\,\bar{G}_\perp^T P_\alpha, \\ U_\alpha &:= Q_*(\alpha)\,X_\alpha - X_*^{\dagger T}(\alpha)\,\bar{G}_{\shortparallel}^T P_\alpha - [\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T}(I - P_\alpha - \bar{G}_{\shortparallel}) \end{aligned}\right\} \tag{6.11}$$

is a solution of system (3.32)–(3.33). By using (3.34), (6.10), and the properties of $\bar{X}_*(\alpha)$ and $\bar{G}_\perp$ in Theorem 2.2.11(iii) and Remark A.2.4 in Appendix A, it can be verified by direct calculations that

$$X_\alpha^\dagger = X_*^\dagger(\alpha) - P_\alpha\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^\dagger, \quad X_\alpha X_\alpha^\dagger = R_\alpha, \quad X_\alpha^\dagger X_\alpha = P_\alpha, \tag{6.12}$$

i.e., (3.32) holds. Moreover, by the same arguments as above, we get

$$X_\alpha P_* = [X_*(\alpha) - \bar{X}_*(\alpha)\,\bar{G}_\perp^T P_\alpha]\,P_* = X_*(\alpha)\,P_* - \bar{X}_*(\alpha)\,\bar{G}_\perp^T P_* = X_*(\alpha), \tag{6.13}$$

showing the first equation in (3.33). By using (6.11) the matrix $X_\alpha^T U_\alpha$ becomes

$$X_\alpha^T U_\alpha = X_\alpha^T Q_*(\alpha)\,X_\alpha - P_*\,\bar{G}_{\shortparallel}^T P_\alpha + P_\alpha\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^T\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T}(I - P_\alpha - \bar{G}_{\shortparallel}). \tag{6.14}$$

By (A.20), we have $P_*\,\bar{G}_{\shortparallel}^T = \bar{G}_{\shortparallel}^T$, while from Lemma 6.2.1(iv) and $P_\alpha P_* = P_*$ we obtain that $P_\alpha\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^T\,[\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T} = P_\alpha\,(I - P_*) = P_\alpha - P_*$. Hence, equation (6.14) becomes $X_\alpha^T U_\alpha = X_\alpha^T\,Q_*(\alpha)\,X_\alpha - \bar{G}_{\shortparallel}^T P_\alpha + (P_\alpha - P_*)\,(I - P_\alpha - \bar{G}_{\shortparallel})$ and consequently,

$$X_\alpha^T U_\alpha = X_\alpha^T\,Q_*(\alpha)\,X_\alpha - \bar{G}_{\shortparallel}^T P_\alpha - P_\alpha\,\bar{G}_{\shortparallel} + P_*\,\bar{G}_{\shortparallel}. \tag{6.15}$$

Since the matrix $Q_*(\alpha)$ is symmetric by (2.5) and the matrix $P_*\,\bar{G}_{\shortparallel}$ is symmetric by Theorem A.2.3 in Appendix A, it follows from (6.15) that $X_\alpha^T U_\alpha$ is also symmetric, i.e., the second equation in (3.33) holds. Finally, we note that by Theorem A.2.7 in Appendix A we have the representation $G = P_\alpha\,\bar{G} = P_\alpha\,(\bar{G}_\perp + \bar{G}_{\shortparallel})$ and $H = (I - P_\alpha)\,\bar{G} = (I - P_\alpha)\,(\bar{G}_\perp + \bar{G}_{\shortparallel})$. Combining this with formulas (6.11), (6.12), (6.13) and with the equalities $P_* P_\alpha = P_*$, $P_*\,\bar{G}_\perp = 0$ and $\bar{G}_{\shortparallel}^T P_* = P_*\,\bar{G}_{\shortparallel}$ (from Theorem A.2.3 in Appendix A), $X_*^{\dagger T}(\alpha)\,P_* = X_*^{\dagger T}(\alpha)$, and $\bar{G}_{\shortparallel} P_* = \bar{G}_{\shortparallel}$ (from (A.20) in Appendix A) we obtain

$$\begin{aligned} X_\alpha^{\dagger T} G &= X_*^{\dagger T}(\alpha)\,\bar{G}_{\shortparallel} - [\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T} P_\alpha\,(\bar{G}_\perp + \bar{G}_{\shortparallel}), \\ U_\alpha\,(P_* + H) &= Q_*(\alpha)\,X_*(\alpha) - X_*^{\dagger T}(\alpha)\,\bar{G}_{\shortparallel} - [\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T}\,\bar{G}_\perp \\ &\quad + [\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T} P_\alpha\,(\bar{G}_\perp + \bar{G}_{\shortparallel}). \end{aligned}$$

Moreover, from Lemma 6.2.1(iii) and identities (A.22) in Appendix A we know that $[\bar{X}_*(\alpha)\,\bar{G}_\perp^T]^{\dagger T}\bar{G}_\perp = \bar{X}_*^{\dagger T}(\alpha)$. Consequently,

$$X_\alpha^{\dagger T} G + U_\alpha\,(P_* + H) = Q_*(\alpha)\,X_*(\alpha) - \bar{X}_*^{\dagger T}(\alpha) = U_*(\alpha)\,P_* + [(I - P_*)\,U_*^T(\alpha)]^T = U_*(\alpha),$$

where in the middle step we used the first equality in (2.6) and the second identity in (6.6). Therefore, the third equation in (3.33) is also satisfied. ∎

**Remark 6.2.4.** The representation of the matrix $\bar{G}_\perp \in \mathscr{B}(P_{\mathscr{S}_{*\alpha}\infty}, P_*, I)$ in the first equation in (A.24) in Appendix A provides an equivalent expression of condition (6.10) in Theorem 6.2.3. More precisely, if $E \in \mathscr{M}(P_*)$ is a matrix such that $\bar{G}_\perp = E^T(I - P_*)$, then condition (6.10) is satisfied if and only if the formula

$$\bar{X}_*(\alpha)EP_\alpha = R_\alpha \bar{X}_*(\alpha)E, \tag{6.16}$$

holds. This immediately follows from the calculation $\bar{X}_*(\alpha)\bar{G}_\perp{}^T = \bar{X}_*(\alpha)(I - P_*)E = \bar{X}_*(\alpha)E$, where we use the identity $\bar{X}_*(\alpha)(I - P_*) = \bar{X}_*(\alpha)$, by Theorem 2.2.11(iii). Moreover, by using the equality $\bar{X}_*^\dagger(\alpha)\bar{X}_*(\alpha) = I - P_*$ and formula (6.16) we get

$$\begin{aligned} EP_\alpha &= (I - P_*)EP_\alpha + P_*EP_\alpha = \bar{X}_*^\dagger(\alpha)\bar{X}_*(\alpha)EP_\alpha + P_*EP_\alpha \\ &\overset{(6.16)}{=} \bar{X}_*^\dagger(\alpha)R_\alpha\bar{X}_*(\alpha)E + P_*EP_\alpha. \end{aligned}$$

Combining last equality with Lemma 6.2.1(i)–(ii) we obtain that formula (6.16) implies the inclusion $\operatorname{Im} EP_\alpha \subseteq \operatorname{Im} \tilde{P}_\alpha$. And since the matrix $E$ is invertible and $\operatorname{rank} \tilde{P}_\alpha = \operatorname{rank} P_\alpha$, we have the equality

$$\operatorname{Im} EP_\alpha = \operatorname{Im} \tilde{P}_\alpha. \tag{6.17}$$

We stress that the above results, that is, formulas (6.16) and (6.17), hold for any matrix $E \in \mathscr{M}(P_*)$, which represents the matrix $\bar{G}_\perp$ in (A.24) whenever condition (6.10) is satisfied.

In the following main result of this section we establish the solvability of system (3.32)–(3.33).

**Theorem 6.2.5.** *Let $P_\alpha$ and $R_\alpha$ be as in Theorem 6.2.3. Then there exists a pair $(G,H) \in \mathscr{B}(P_{\mathscr{S}_{*\alpha}\infty}, P_*, P_\alpha)$ so that system (3.32)–(3.33) is solvable.*

*Proof.* The proof is based on the result in Theorem 6.2.3 and representation (A.27) of the set $\mathscr{B}(P_{\mathscr{S}_{*\alpha}\infty}, P_*, P_\alpha)$ in Remark A.2.8(ii) in Appendix A. Let $\tilde{P}_\alpha$ be the orthogonal projector defined in (6.7). Then the first condition in (3.34), Lemma 6.2.1(i) (which yields (A.16) in Appendix A), and Theorem A.2.2 in Appendix A imply that there exists a matrix $E_0 \in \mathscr{M}(P_*)$ satisfying $\operatorname{Im} E_0P_\alpha = \operatorname{Im} \tilde{P}_\alpha$, i.e., $\tilde{P}_\alpha E_0 P_\alpha = EP_\alpha$. In addition, by Lemma 6.2.1(ii), we have the identity $R_\alpha \bar{X}_*(\alpha)\tilde{P}_\alpha = \bar{X}_*(\alpha)\tilde{P}_\alpha$, so that

$$R_\alpha \bar{X}_*(\alpha)E_0P_\alpha = R_\alpha \bar{X}_*(\alpha)\tilde{P}_\alpha E_0 P_\alpha = \bar{X}_*(\alpha)\tilde{P}_\alpha E_0 P_\alpha = \bar{X}_*(\alpha)E_0P_\alpha. \tag{6.18}$$

We will show that the matrix $E \in \mathbb{R}^{n\times n}$ defined by

$$E := E_0P_\alpha + \bar{X}_*^\dagger(\alpha)(I - R_\alpha)\bar{X}_*(\alpha)E_0 \tag{6.19}$$

belongs to the set $\mathscr{M}(P_*)$ introduced in (1.9). Since $P_\alpha P_* = P_*$, $E_0P_* = P_*$, and $\bar{X}_*(\alpha)P_* = 0$, it follows from (6.19) that

$$EP_* = E_0P_* + \bar{X}_*^\dagger(\alpha)(I - R_\alpha)\bar{X}_*(\alpha)E_0P_* = P_*.$$

Moreover, if $v \in \operatorname{Ker} E^T$, then by (6.19) we have

$$P_\alpha E_0^T v + E_0^T \bar{X}_*^T(\alpha)(I - R_\alpha)\bar{X}_*^{\dagger T}(\alpha)v = 0. \tag{6.20}$$

Multiplying the equality in (6.20) by $P_\alpha$ from the left and using (6.18) we get

$$\begin{aligned} 0 &= P_\alpha E_0^T v + P_\alpha E_0^T \bar{X}_*^T(\alpha)(I - R_\alpha)\bar{X}_*^{\dagger T}(\alpha) v \\ &= P_\alpha E_0^T v + P_\alpha E_0^T \bar{X}_*^T(\alpha) R_\alpha (I - R_\alpha)\bar{X}_*^{\dagger T}(\alpha) v = P_\alpha E_0^T v. \end{aligned}$$

Thus, $v \in \operatorname{Ker} P_\alpha E_0^T = \operatorname{Ker} \tilde{P}_\alpha \subseteq \operatorname{Ker} P_*$, by Lemma 6.2.1(i), and in turn, identity (6.20) becomes

$$0 = E_0^T \bar{X}_*^T(\alpha)(I - R_\alpha)\bar{X}_*^{\dagger T}(\alpha) v = E_0^T (I - P_*) v - E_0^T \bar{X}_*^T(\alpha) R_\alpha \bar{X}_*^{\dagger T}(\alpha) v, \qquad (6.21)$$

where we used the first equality $\bar{X}_*^T(\alpha)\bar{X}_*^{\dagger T}(\alpha) = I - P_*$ from (6.5). From Lemma 6.2.1(ii) we know that $R_\alpha \bar{X}_*^{\dagger T}(\alpha) v = R_\alpha \bar{X}_*^{\dagger T}(\alpha)\tilde{P}_\alpha v = 0$, so that $E_0^T (I - P_*) v = 0$, by (6.21). But we also have $P_* v = 0$ and hence, $E_0^T v = 0$. The invertibility of $E_0$ then yields $v = 0$. Therefore, the matrix $E^T$ and consequently, also the matrix $E$ is nonsingular, which completes the proof of $E \in \mathscr{M}(P_*)$. In addition, using the identities $\bar{X}_*(\alpha)\bar{X}_*^\dagger(\alpha) = I - R_*(\alpha)$ and $\bar{X}_*(\alpha)(I - P_*) = \bar{X}_*(\alpha)$ from (6.5) together with equality $[I - R_*(\alpha)](I - R_\alpha) = I - R_\alpha$ and formulas (6.18) and (6.19) yields

$$\begin{aligned} \bar{X}_*(\alpha) E P_\alpha &\overset{(6.19)}{=} \bar{X}_*(\alpha) E_0 P_\alpha + [I - R_*(\alpha)](I - R_\alpha)\bar{X}_*(\alpha) E_0 P_\alpha \\ &= \bar{X}_*(\alpha) E_0 P_\alpha + (I - R_\alpha)\bar{X}_*(\alpha) E_0 P_\alpha \overset{(6.18)}{=} \bar{X}_*(\alpha) E_0 P_\alpha, \\ R_\alpha \bar{X}_*(\alpha) E &\overset{(6.19)}{=} R_\alpha \bar{X}_*(\alpha) E_0 P_\alpha + R_\alpha [I - R_*(\alpha)](I - R_\alpha)\bar{X}_*(\alpha) E_0 \\ &= R_\alpha \bar{X}_*(\alpha) E_0 P_\alpha \overset{(6.18)}{=} \bar{X}_*(\alpha) E_0 P_\alpha. \end{aligned}$$

Therefore, condition (6.16) holds. Define now the matrices $G, H \in \mathbb{R}^{n\times n}$ by

$$G := P_\alpha E^T (I - P_*), \quad H := (I - P_\alpha) E^T (I - P_*). \qquad (6.22)$$

From Remark A.2.8(ii) with $F := 0 \in \mathscr{A}(P_{\mathscr{S}_{*\alpha\infty}}, P_*)$ it then follows that the pair $(G, H)$ in (6.22) belongs to the set $\mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, P_\alpha)$. Moreover, by setting $\bar{G} := G + H$ and $\bar{G}_\perp := \bar{G}(I - P_*)$, as in Theorem A.2.7 and formula (A.20) in Appendix A, the condition in (6.10) is satisfied, by Remark 6.2.4. According to Theorem 6.2.3 this means that system (3.32)–(3.33) is solvable. ■

Based on the result in Theorem 6.2.5 we are now ready to prove Theorem 3.2.10 in Chapter 3.

*Proof of Theorem 3.2.10.* According to Theorem 6.2.5, for any orthogonal projectors $P_\alpha$ and $R_\alpha$ satisfying (3.34) there exists a pair $(G, H)$ belonging to $\mathscr{B}(P_{\mathscr{S}_{*\alpha\infty}}, P_*, P_\alpha)$ such that system (3.32)–(3.33) has a solution $X_\alpha$, $U_\alpha$. From Theorem 3.2.8 we then know that the solution $(X, U)$ of (H) given by the initial conditions $X(\alpha) = X_\alpha$ and $U(\alpha) = U_\alpha$ is a conjoined basis of (H) with constant kernel on $[\alpha, \infty)$ and the projectors $P$ and $R(t)$ in (2.2) and (2.1) satisfy $P = P_\alpha$ and $R(\alpha) = R_\alpha$. Moreover, the conjoined basis $(X_*, U_*)$ is contained in $(X, U)$ on $[\alpha, \infty)$ with respect to $P_*$. ■

## 6.3 Genus of conjoined bases

In this section we develop the tools for a classification of conjoined bases of (H) with constant kernel for large $t$. In particular, we introduce a concept of a genus of conjoined bases of (H), which allows to classify the conjoined bases $(X,U)$ of (H) according to the image of $X(t)$ for large $t$. The main results of this section concern a particular application of this new concept to principal and antiprincipal solutions of (H) at infinity.

**Definition 6.3.1** (Genus of conjoined bases)**.** Let $(X_1,U_1)$ and $(X_2,U_2)$ be two conjoined bases of (H). We say that $(X_1,U_1)$ and $(X_2,U_2)$ *have the same genus* (or they *belong to the same genus*) if there exists $\alpha \in [a,\infty)$ such that $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ on $[\alpha,\infty)$.

We note that the terminology in Definition 6.3.1 is motivated by the theory of binary quadratic forms with integer-valued coefficients, where a similar concept is used for the classification of these quadratic forms. More details in this direction can be found e.g. in [7].

**Remark 6.3.2.** (i) From Definition 6.3.1 it follows that the relation "having (or belonging to) the same genus" is an equivalence on the set of all conjoined bases of (H). Therefore, there exists a partition of this set into disjoint classes of conjoined bases of (H) with the same genus. This allows to interpret each such an equivalence class $\mathcal{G}$ as a genus itself. It is obvious that if a conjoined basis $(X,U)$ of (H) belongs to some genus $\mathcal{G}$, then every conjoined basis $(X_0,U_0)$ of (H), such that $(X_0,U_0) \sim (X,U)$ on some subinterval $[\beta,\infty)$ according to Section 3.1, also belongs to the same genus $\mathcal{G}$.

(ii) When the Legendre condition (1.1) holds and system (H) is nonoscillatory, then the relation "having (or belonging to) the same genus" coincides with the relation of mutual representability of conjoined bases of (H) in Definition 2.3.6. More precisely, according to Corollary 2.3.10 and Definition 6.3.1, two conjoined bases of (H) are mutually representable if and only if they belong to the same genus. Therefore, in this case each genus of conjoined bases of (H) contains all mutual representable conjoined bases of (H).

**Remark 6.3.3.** We note that there is only one genus of conjoined bases (denoted by $\mathcal{G}_{\min}$) containing all conjoined bases of (H) with the minimal rank in (3.16), i.e., with rank equal to $n-d_\infty$. The reason is that any two conjoined bases $(X_1,U_1)$ and $(X_2,U_2)$ of (H) with rank equal to $n-d_\infty$ satisfy $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ for large $t$. The proof of the latter equality is based on the following arguments. If $(X_1,U_1)$ and $(X_2,U_2)$ have constant kernel on $[\alpha,\infty)$, then $d[\alpha,\infty) = d_\infty$, by (3.15) and (3.2). Therefore, $(X_1,U_1)$ and $(X_2,U_2)$ are minimal conjoined bases on $[\alpha,\infty)$ and, by Remark 3.3.5, they are mutually representable on $[\alpha,\infty)$. In turn, Definition 2.3.6 and Corollary 2.3.10 imply that $\operatorname{Im} X_1(t) = \operatorname{Im} X_2(t)$ on $[\alpha,\infty)$. In particular, all minimal (anti)principal solutions of (H) at infinity belong to the minimal genus $\mathcal{G}_{\min}$.

**Remark 6.3.4.** Similarly to Remark 6.3.3, there is only one genus of conjoined bases (denoted by $\mathcal{G}_{\max}$) containing all conjoined bases $(X,U)$ of (H) with the function $X(t)$ eventually invertible, since in this case $\operatorname{Im} X(t) = \mathbb{R}^n$ for large $t$. The maximal genus $\mathcal{G}_{\max}$ then contains all maximal (anti)principal solutions of (H) at infinity.

We recall the definition of the point $\hat{\alpha}_{\min}$ in (5.5), which determines the maximal interval $(\hat{\alpha}_{\min},\infty)$ on which the minimal principal solutions of (H) have constant kernel. In our first result we prove that a conjoined basis from the minimal genus $\mathscr{G}_{\min}$ can have constant kernel only on a subinterval of $(\hat{\alpha}_{\min},\infty)$.

**Theorem 6.3.5.** *Assume* (1.1) *and let* $(X,U)$ *be a conjoined basis of* (H) *belonging to the minimal genus* $\mathscr{G}_{\min}$*. If* $(X,U)$ *has constant kernel on* $[\alpha,\infty)$*, then* $\alpha \geq \hat{\alpha}_{\min}$ *and* (3.3) *holds.*

*Proof.* Let $(X,U) \in \mathscr{G}_{\min}$ be as in the theorem. Since the rank of $(X,U)$ is $n-d_\infty$, we have $n-d[\alpha,\infty) \leq n-d_\infty$, by (3.15). This implies that $d[\alpha,\infty) \geq d_\infty$. The definition of $d_\infty$ in (3.2) then implies that $d[\alpha,\infty) = d_\infty$, i.e., condition (3.3) holds. In particular, $(X,U)$ is a minimal conjoined basis on the interval $[\alpha,\infty)$. Let $S_\alpha(t)$ be the $S$-matrix corresponding to $(X,U)$ in (2.8) and let $T_\alpha$ be given in Remark 2.1.6. Consider a solution $(\hat{X},\hat{U})$ of (H) defined by $(\hat{X},\hat{U}) := (X,U) - (\bar{X}_\alpha,\bar{U}_\alpha)\,T_\alpha$ on $[\alpha,\infty)$, where $(\bar{X}_\alpha,\bar{U}_\alpha)$ is the conjoined basis associated with $(X,U)$ through Theorem 2.2.5. From Theorem 5.1.7 it follows that $(\hat{X},\hat{U})$ is the minimal principal solution of (H) with respect to the interval $[\alpha,\infty)$. In particular, $(\hat{X},\hat{U})$ has constant kernel on $[\alpha,\infty)$. Thus, the inequality $\alpha \geq \hat{\alpha}_{\min}$ holds, by (5.5). ■

**Remark 6.3.6.** It follows from Theorem 6.3.5 that $(\hat{\alpha}_{\min},\infty)$ is the maximal open interval for which there exists a conjoined basis $(X,U) \in \mathscr{G}_{\min}$ with constant kernel on this interval. More precisely, the point $\hat{\alpha}_{\min}$ in (5.5) has the equivalent expression

$$\hat{\alpha}_{\min} := \inf\big\{\alpha \in [a,\infty),\ (X,U) \in \mathscr{G}_{\min} \text{ has constant kernel on } [\alpha,\infty)\big\}. \tag{6.23}$$

In Remark 5.1.11 we showed that the orthogonal projector $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ in (2.12), which is associated with a principal solution $(\hat{X},\hat{U})$ of (H) at infinity, is the same for all initial points $\alpha \in (\hat{\alpha}_{\min},\infty)$. Formula (6.23) now yields that the same property holds for any conjoined basis $(X,U)$ and an interval, where $(X,U)$ has constant kernel. More precisely, if $(X,U)$ is a conjoined basis of (H) with constant kernel on $[\alpha,\infty) \subseteq (\hat{\alpha}_{\min},\infty)$, then the associated orthogonal projector $P_{\mathscr{S}_{\beta\infty}}$ defined in (2.12) is the same for all initial points $\beta \in [\alpha,\infty)$.

The following theorem can be viewed as a corollary to Theorem 4.2.1, in which the minimal principal solution $(\hat{X}_{\min},\hat{U}_{\min})$ of (H) at infinity is considered. We also utilize the observation that the orthogonal projector $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ in (2.12) associated with $(\hat{X}_{\min},\hat{U}_{\min})$ is the same for all initial points $\alpha \in (\hat{\alpha}_{\min},\infty)$ with $\hat{\alpha}_{\min}$ defined in (5.5).

**Theorem 6.3.7.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory. Let* $(\hat{X}_{\min},\hat{U}_{\min})$ *be a minimal principal solution of* (H) *at infinity and let* $P_{\hat{\mathscr{S}}_{\alpha\infty}}$ *be defined in* (2.12) *through the function* $\hat{X}_{\min}(t)$ *on* $(\hat{\alpha}_{\min},\infty)$*. Then a solution* $(X,U)$ *of* (H) *belongs to the minimal genus* $\mathscr{G}_{\min}$ *if and only if for some* $\alpha \in (\hat{\alpha}_{\min},\infty)$ *there exist matrices* $\hat{M},\hat{N} \in \mathbb{R}^{n\times n}$ *such that*

$$X(\alpha) = \hat{X}_{\min}(\alpha)\,\hat{M}, \quad U(\alpha) = \hat{U}_{\min}(\alpha)\,\hat{M} + \hat{X}^{\dagger T}_{\min}(\alpha)\,\hat{N}, \tag{6.24}$$

$$\hat{M} \text{ is nonsingular}, \quad \hat{M}^T\hat{N} = \hat{N}^T\hat{M}, \quad \operatorname{Im}\hat{N} \subseteq \operatorname{Im}P_{\hat{\mathscr{S}}_{\alpha\infty}}, \quad \hat{N}\hat{M}^{-1} \geq 0. \tag{6.25}$$

*Proof.* First we note that according to Theorem 5.1.9, $(\hat{X}_{\min},\hat{U}_{\min})$ is a minimal principal solution with respect to the interval $[\alpha,\infty)$ for every $\alpha\in(\hat{\alpha}_{\min},\infty)$. In particular, this means that $(\hat{X}_{\min},\hat{U}_{\min})$ has constant kernel on $(\hat{\alpha}_{\min},\infty)$ and the function $\hat{S}_\alpha(t)$ defined in (2.8) through $\hat{X}_{\min}(t)$ satisfies $\hat{S}^{\dagger}_\alpha(t)\to\hat{T}_\alpha=0$ as $t\to\infty$ for every initial point $\alpha\in(\hat{\alpha}_{\min},\infty)$. Assume that $(X,U)$ belongs to $\mathcal{G}_{\min}$ and has constant kernel on a given interval $[\alpha,\infty)$. From Theorem 6.3.5 we know that $\alpha\geq\hat{\alpha}_{\min}$. Without loss of generality we assume $\alpha>\hat{\alpha}_{\min}$. By Theorem 4.2.1, with $(\tilde{X},\tilde{U}):=(X,U)$, $(X,U):=(\hat{X}_{\min},\hat{U}_{\min})$, $P_{\mathcal{S}_{\alpha\infty}}:=P_{\hat{\mathcal{S}}_{\alpha\infty}}$, and $T_\alpha:=\hat{T}_\alpha=0$, there exist matrices $\hat{M},\hat{N}\in\mathbb{R}^{n\times n}$ such that (6.24) and (6.25) hold. The opposite implication is a consequence of Theorem 4.2.1 with $(X,U):=(\hat{X}_{\min},\hat{U}_{\min})$, since for every $\alpha\in(\hat{\alpha}_{\min},\infty)$ a solution $(X,U)$ of (H) satisfying (6.24) and (6.25) is a conjoined basis, which belongs to the minimal genus $\mathcal{G}_{\min}$ and which has constant kernel on $[\alpha,\infty)$. ∎

The next result provides an additional information about the structure of the minimal genus $\mathcal{G}_{\min}$, as we comment in Remark 6.3.9 below. We recall that for a fixed $\alpha\in[a,\infty)$ the principal solution $(\hat{X}_\alpha,\hat{U}_\alpha)$ at the point $\alpha$ is defined as the conjoined basis of (H) satisfying the initial conditions $\hat{X}_\alpha(\alpha)=0$ and $\hat{U}_\alpha(\alpha)=I$.

**Proposition 6.3.8.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (5.5). *The for every* $\alpha>\hat{\alpha}_{\min}$ *the principal solution* $(\hat{X}_\alpha,\hat{U}_\alpha)$ *at* $\alpha$ *belongs to* $\mathcal{G}_{\min}$.

*Proof.* The result is a direct consequence of Proposition 5.2.9 and Remark 6.3.3. ∎

**Remark 6.3.9.** Proposition 6.3.8 shows that there is no universal interval $[\alpha,\infty)$ in $(\hat{\alpha}_{\min},\infty)$ such that all the conjoined bases $(X,U)$ in the minimal genus $\mathcal{G}_{\min}$ would have its kernel constant on this universal interval.

The next theorem describes the distribution of principal and antiprincipal solutions of (H) at infinity into the genera of conjoined bases (see Definition 6.3.1). In particular, we prove that every genus does contain some principal solution, as well as some antiprincipal solution of (H), see Examples 7.1.4 and 7.1.5.

**Theorem 6.3.10.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory. Let* $\mathcal{G}$ *be a genus of conjoined bases of* (H). *Then there exists a principal solution and an antiprincipal solution of* (H), *which belong to* $\mathcal{G}$.

*Proof.* First we focus on the case of principal solutions. From Theorem 5.1.5 we know that there exists a minimal principal solution $(\hat{X}_{\min},\hat{U}_{\min})$ of (H). Let $\hat{\alpha}_{\min}$ be defined in (5.5). Suppose that $(X,U)$ is a conjoined basis of (H), which belongs to the genus $\mathcal{G}$. Then there exists $\alpha>\hat{\alpha}_{\min}$ such that $(X,U)$ has constant kernel on $[\alpha,\infty)$. By Theorem 5.1.4 we may assume that the point $\alpha$ is such that $(\hat{X}_{\min},\hat{U}_{\min})$ is a minimal principal solution with respect to $[\alpha,\infty)$. In particular, $(\hat{X}_{\min},\hat{U}_{\min})$ has constant kernel on $[\alpha,\infty)$ and $d[\alpha,\infty)=d_\infty$. Furthermore, by Theorem 3.2.11 (with $(X_*,U_*):=(X,U)$) there exists a conjoined basis $(X_*,U_*)$ of (H) with constant kernel on $[\alpha,\infty)$ and with rank $X_*(t)=n-d_\infty$ on $[\alpha,\infty)$ such that $(X,U)$ contains $(X_*,U_*)$ on $[\alpha,\infty)$. Therefore, $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X_*,U_*)$ are minimal conjoined bases of (H) on $[\alpha,\infty)$ and thus, $\operatorname{Im}\hat{X}_{\min}(\alpha)=\operatorname{Im}X_*(\alpha)$, by Remark 3.3.5. Denote by $\hat{R}_{\min}(t)$, $R_*(t)$, and $R(t)$ the orthogonal projectors in (2.1) defined by $\hat{X}_{\min}(t)$, $X_*(t)$, and $X(t)$, respectively. Then $\hat{R}_{\min}(\alpha) = R_*(\alpha)$. In turn, Theorem 3.2.8 and the second condition in (3.34) (with $R_\alpha := R(\alpha)$) imply that $\operatorname{Im}\hat{R}_{\min}(\alpha) = \operatorname{Im}R_*(\alpha) \subseteq \operatorname{Im}R(\alpha)$. From Theorem 3.2.10 (with $R_\alpha := R(\alpha)$ and $P_\alpha$ arbitrary satisfying (3.34)) we know that there exists a conjoined basis $(\hat{X},\hat{U})$ of (H) with constant kernel on $[\alpha,\infty)$, which contains $(\hat{X}_{\min},\hat{U}_{\min})$ on $[\alpha,\infty)$ and $\operatorname{Im}\hat{X}(\alpha) = \operatorname{Im}R(\alpha)$. Consequently, according to Theorem 5.1.3 the conjoined basis $(\hat{X},\hat{U})$ is a principal solution of (H) with respect to $[\alpha,\infty)$ satisfying $\operatorname{Im}\hat{X}(\alpha) = \operatorname{Im}R(\alpha) = \operatorname{Im}X(\alpha)$. Therefore, the conjoined bases $(\hat{X},\hat{U})$ and $(X,U)$ are mutually representable on $[\alpha,\infty)$, by Theorem 2.3.3. Moreover, Theorem 2.3.8 implies that $\operatorname{Im}\hat{X}(t) = \operatorname{Im}X(t)$ on $[\alpha,\infty)$. It now follows from Definition 6.3.1 that the principal solution $(\hat{X},\hat{U})$ belongs to the genus $\mathscr{G}$. The case of antiprincipal solutions can be carried out by exactly the same arguments, considering a minimal antiprincipal solution $(X_{\min},U_{\min})$ instead of $(\hat{X}_{\min},\hat{U}_{\min})$ in the above proof. ■

In the remaining part of this section we will use notation (2.15) introduced in Remark 2.1.7. The following theorem provides a classification of all antiprincipal solutions in terms of their Wronskian with a given principal solution within one genus $\mathscr{G}$.

**Theorem 6.3.11.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory. Let $\mathscr{G}$ be a genus of conjoined bases of* (H)*. Let $(\hat{X},\hat{U})$ be a principal solution of* (H) *at infinity belonging to $\mathscr{G}$ and let $(X,U)$ be a conjoined basis from $\mathscr{G}$. Denote by $P_{\hat{\mathscr{S}}\infty}$ and $P_{\mathscr{S}\infty}$ their associated projectors in* (2.12) *and Remark 6.3.6. Then $(X,U)$ is an antiprincipal solution of* (H) *at infinity if and only if the (constant) Wronskian $\hat{N} := \hat{X}^T(t)U(t) - \hat{U}^T(t)X(t)$ of $(\hat{X},\hat{U})$ and $(X,U)$ satisfies*

$$\operatorname{rank} P_{\hat{\mathscr{S}}\infty}\hat{N}P_{\mathscr{S}\infty} = n - d_\infty. \tag{6.26}$$

*Proof.* Let $(\hat{X},\hat{U})$ and $(X,U)$ be as in the theorem. Then there exists $\alpha > \hat{\alpha}_{\min}$ such that $(\hat{X},\hat{U})$ and $(X,U)$ have constant kernel on $[\alpha,\infty)$. Let $\hat{P}$ and $P$ be the orthogonal projectors in (2.2) associated with $(\hat{X},\hat{U})$ and $(X,U)$. By (5.5) and Theorem 5.1.9 we have $d[\alpha,\infty) = d_\infty$. Since $(\hat{X},\hat{U})$ and $(X,U)$ belong to the same genus $\mathscr{G}$, we may assume without loss of generality that $\operatorname{Im}\hat{X}(t) = \operatorname{Im}X(t)$ on $[\alpha,\infty)$. Thus by Theorem 2.3.8, the conjoined bases $(\hat{X},\hat{U})$ and $(X,U)$ are mutually representable on $[\alpha,\infty)$. Furthermore, denote by $(\hat{X}_*,\hat{U}_*)$ and $(X_*,U_*)$ the minimal conjoined bases of (H) on $[\alpha,\infty)$, which are contained in $(\hat{X},\hat{U})$ and $(X,U)$ on $[\alpha,\infty)$, respectively. In particular, $(\hat{X}_*,\hat{U}_*)$ is a minimal principal solution of (H) and hence, the matrix $\hat{T}_* = 0$ in Remark 2.1.6. We apply the representations of $(\hat{X},\hat{U})$, $(X,U)$ and of $(\hat{X}_*,\hat{U}_*)$, $(X_*,U_*)$ in Theorems 2.3.8 and 3.3.8. More precisely, with $(X_1,U_1) := (\hat{X},\hat{U})$, $(X_2,U_2) := (X,U)$, $(X_{*1},U_{*1}) := (\hat{X}_*,\hat{U}_*)$, $(X_{*2},U_{*2}) := (X_*,U_*)$, $P_1 := \hat{P}$, $P_2 := P$, $P_{\mathscr{S}_1\infty} := P_{\hat{\mathscr{S}}\infty}$, $P_{\mathscr{S}_2\infty} := P_{\mathscr{S}\infty}$, and $N_1 := \hat{N}$, the Wronskian $\hat{N}$ satisfies $\hat{P}\hat{N} = \hat{N}$ and $P\hat{N}^T = \hat{N}^T$, and there exist matrices $\hat{M}$, $\hat{M}_*$, $\hat{N}_*$ such that $\hat{M}$ and $\hat{M}_*$ are invertible, $\hat{M}^T\hat{N}$ and $\hat{M}_*^T\hat{N}_*$ are symmetric, and

$$X_*(\alpha) = \hat{X}_*(\alpha)\hat{M}_*, \quad \hat{U}(\alpha) = \hat{U}_*(\alpha)\hat{M}_* + \hat{X}_*^{\dagger T}(\alpha)\hat{N}_*, \tag{6.27}$$

$$\hat{P}\hat{M}P_{\mathscr{S}\infty} = P_{\hat{\mathscr{S}}\infty}\hat{M}_*, \quad P\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty} = P_{\mathscr{S}\infty}\hat{M}_*^{-1}, \quad \hat{N}_*\hat{M}_*^{-1} = P_{\hat{\mathscr{S}}\infty}\hat{N}\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty}. \tag{6.28}$$

By using (6.28) and the equality $\hat{N}P = \hat{N}$ we then obtain

$$\hat{N}_*\hat{M}_*^{-1} = P_{\hat{\mathscr{S}}\infty}\hat{N}P\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty} = P_{\hat{\mathscr{S}}\infty}\hat{N}P_{\mathscr{S}\infty}\hat{M}_*^{-1}. \tag{6.29}$$

Now let $(X,U)$ be an antiprincipal solution of (H) at infinity. Then also $(X_*,U_*)$ is an antiprincipal solution, by Theorem 5.2.6. From (4.49) we have that the matrix $T_*$ in Remark 2.1.6 defined through $(X_*,U_*)$ satisfies $\operatorname{rank} T_* = \operatorname{rank}(\hat{N}_*\hat{M}_*^{-1}+\hat{T}_*) = \operatorname{rank}\hat{N}_*\hat{M}_*^{-1}$. Since $\operatorname{rank} T_* = n-d_\infty$ by Definition 5.2.1, we get from (6.29) that $\operatorname{rank} P_{\hat{\mathscr{S}}\infty}\hat{N}P_{\mathscr{S}\infty}\hat{M}_*^{-1} = \operatorname{rank}\hat{N}_*\hat{M}_*^{-1} = n-d_\infty$, i.e., formula (6.26) holds. Conversely, if (6.26) is satisfied, then from (6.29) we have $\operatorname{rank}\hat{N}_*\hat{M}_*^{-1} = n-d_\infty$. Therefore, $\operatorname{rank} T_* = n-d_\infty$, by (4.49), and so $(X_*,U_*)$ is an antiprincipal solution of (H). Finally, Theorem 5.2.6 implies that $(X,U)$ is an antiprincipal solution as well. ■

When system (H) is completely controllable, we obtain from Theorem 6.3.11 and Proposition 5.2.9 an interesting characterization of its antiprincipal solutions at infinity. This result is new even in this controllable case.

**Corollary 6.3.12.** *Assume that (1.1) holds and system (H) is nonoscillatory and completely controllable. Let $(\hat{X},\hat{U})$ be the principal solution of (H) at infinity. Then a conjoined basis $(X,U)$ is an antiprincipal solution of (H) at infinity if and only if the (constant) Wronskian $\hat{N} := \hat{X}^T(t)U(t)-\hat{U}^T(t)X(t)$ of $(\hat{X},\hat{U})$ and $(X,U)$ is invertible. In particular, for every $\alpha > \hat{\alpha}_{\min}$ the principal solution $(\hat{X}_\alpha,\hat{U}_\alpha)$ at the point $\alpha$ is antiprincipal at infinity. Or more generally, for $\alpha\in[a,\infty)$ the principal solution $(\hat{X}_\alpha,\hat{U}_\alpha)$ at the point $\alpha$ is antiprincipal at infinity if and only if $\hat{X}(\alpha)$ is invertible.*

*Proof.* If (H) is completely controllable, then $d_\infty = 0$ and for every conjoined basis $(X,U)$ of (H) the function $X(t)$ is eventually invertible. This means that there is only one (minimal/maximal) genus of conjoined bases, see also Remark 6.3.4. The orthogonal projectors $P_{\mathscr{S}\infty}$ and $P_{\hat{\mathscr{S}}\infty}$ in (2.8) and Remark 6.3.6 associated with $(X,U)$ and $(\hat{X},\hat{U})$ in this case satisfy $P_{\mathscr{S}\infty} = I = P_{\hat{\mathscr{S}}\infty}$. The first part of the statement now follows directly from Theorem 6.3.11, while the second part follows from Proposition 5.2.9. Finally, if $(\hat{X}_\alpha,\hat{U}_\alpha)$ is the principal solution of (H) at some point $\alpha\in[a,\infty)$, then $\hat{X}_\alpha(\alpha)=0$ and $\hat{U}_\alpha(\alpha)=I$ and hence, $\hat{N}=\hat{X}^T(\alpha)$. This means, by the first part, that $(\hat{X}_\alpha,\hat{U}_\alpha)$ is an antiprincipal solution at infinity if and only if $\hat{X}(\alpha)$ is invertible. ■

From Definition 6.3.1 it follows that all conjoined bases $(X,U)$ of (H), which belong to a given genus $\mathscr{G}$, have the same rank (of $X(t)$ for large $t$). In particular, all principal solutions in $\mathscr{G}$ have the same rank. The following theorem then describes a complete classification of all principal solutions within the genus $\mathscr{G}$.

**Theorem 6.3.13.** *Assume that (1.1) holds and system (H) is nonoscillatory with $\hat{\alpha}_{\min}$ defined in (5.5). Let $(\hat{X},\hat{U})$ be a principal solution of (H) at infinity, which belongs to a genus $\mathscr{G}$. Moreover, let $\hat{P}$ and $P_{\hat{\mathscr{S}}\infty}$ be the orthogonal projectors defined through the function $\hat{X}(t)$ on $(\hat{\alpha}_{\min},\infty)$ in (2.2), (2.12), and Remark 5.1.11. Then a solution $(X,U)$ of (H) is a principal solution belonging to the genus $\mathscr{G}$ if and only if for some (and hence for every) $\alpha\in(\hat{\alpha}_{\min},\infty)$ there exist matrices $\hat{M},\hat{N}\in\mathbb{R}^{n\times n}$ such that*

$$X(\alpha)=\hat{X}(\alpha)\hat{M}, \quad U(\alpha)=\hat{U}(\alpha)\hat{M}+\hat{X}^{\dagger T}(\alpha)\hat{N}, \tag{6.30}$$

$$\hat{M} \text{ is invertible}, \quad \hat{M}^T\hat{N}=\hat{N}^T\hat{M}, \quad \operatorname{Im}\hat{N}\subseteq\operatorname{Im}\hat{P}, \quad P_{\hat{\mathscr{S}}\infty}\hat{N}\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty}=0. \tag{6.31}$$

*Proof.* Let $(X,U)$ be a principal solution of (H) belonging to the genus $\mathscr{G}$. From Theorem 5.1.10 we know that $(\hat{X},\hat{U})$ and $(X,U)$ have constant kernel on $(\hat{\alpha}_{\min},\infty)$ and consequently, according to Remark 6.3.2(ii) and Corollary 2.3.10 they satisfy $\operatorname{Im}\hat{X}(t) = \operatorname{Im}X(t)$ on $(\hat{\alpha}_{\min},\infty)$. Therefore, by Theorem 2.3.8 and its proof with $(X_1,U_1) := (\hat{X},\hat{U})$ and $(X_2,U_2) := (X,U)$, for every $\alpha \in (\hat{\alpha}_{\min},\infty)$ there exist $\hat{M},\hat{N} \in \mathbb{R}^{n\times n}$ such that (6.30) and the first three conditions in (6.31) hold. We prove the last condition in (6.31). Fix $\alpha \in (\hat{\alpha}_{\min},\infty)$. Denote by $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X_*,U_*)$ respectively the minimal principal solutions of (H) from Theorem 5.1.10, which are contained in $(\hat{X},\hat{U})$ and $(X,U)$ on $[\alpha,\infty)$. By Theorem 5.1.6 we know that $X_* = \hat{X}_{\min}\hat{M}_{\min}$ and $U_* = \hat{U}_{\min}\hat{M}_{\min}$ on $[a,\infty)$ for some constant and nonsingular matrix $\hat{M}_{\min}$. This means that the Wronskian $\hat{N}_{\min}$ of $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X_*,U_*)$ is $\hat{N}_{\min} = 0$. On the other hand, the conjoined bases $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X_*,U_*)$ are minimal on $[\alpha,\infty)$ and hence, (3.66) holds with $(X_{*1},U_{*1}) := (\hat{X}_{\min},\hat{U}_{\min})$, $(X_{*2},U_{*2}) := (X_*,U_*)$, $M_{*1} := \hat{M}_{\min}$, and $N_{*1} := \hat{N}_{\min} = 0$. Consequently, by Theorem 3.3.8(ii) with $P_{\mathscr{S}_1\infty} := P_{\hat{\mathscr{S}}\infty}$, we obtain that $P_{\hat{\mathscr{S}}\infty}\hat{N}\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty} = 0$ holds. Conversely, fix $\alpha \in (\hat{\alpha}_{\min},\infty)$ and suppose that a solution $(X,U)$ of (H) satisfies (6.30) and (6.31). From Theorem 5.1.9 we have $d[\alpha,\infty) = d_\infty$. The first three conditions in (6.31), the fact that $(\hat{X},\hat{U})$ is a conjoined basis of (H), and the identity $\hat{X}^T(\alpha)\hat{X}^{\dagger T}(\alpha) = \hat{P}$ imply that $(X,U)$ is a conjoined basis of (H). Let $\hat{S}(t)$ be the $S$-matrix in (2.8) corresponding to $(\hat{X},\hat{U})$ on $[\alpha,\infty)$ and let $(\bar{X}_\alpha,\bar{U}_\alpha)$ be a conjoined basis of (H) satisfying (1.16) and (2.25) with $(X,U) := (\hat{X},\hat{U})$ and $\beta := \alpha$. Then (2.37) holds, that is,

$$\bar{X}_\alpha\hat{P} = \hat{X}\hat{S}, \quad \bar{U}_\alpha\hat{P} = \hat{U}\hat{S} + \hat{X}^{\dagger T} + \hat{U}(I-\hat{P})\bar{X}_\alpha^T\hat{X}^{\dagger T} \quad \text{on } [\alpha,\infty). \tag{6.32}$$

Consequently, by using the identities $\hat{P}\hat{N} = \hat{N}$, $\hat{X}^\dagger(\alpha)\bar{X}_\alpha(\alpha) = 0$, and (6.32) at $t = \alpha$, we can rewrite (6.30) as $X(\alpha) = \hat{X}(\alpha)\hat{M} + \bar{X}_\alpha(\alpha)\hat{N}$ and $U(\alpha) = \hat{U}(\alpha)\hat{M} + \bar{U}_\alpha(\alpha)\hat{N}$. Hence, $X(t) = \hat{X}(t)\hat{M} + \bar{X}_\alpha(t)\hat{N}$ and $U(t) = \hat{U}(t)\hat{M} + \bar{U}_\alpha(t)\hat{N}$ on $[\alpha,\infty)$, by the uniqueness of solutions of (H). In particular, by (6.32) and $\hat{X}(t)\hat{P} = \hat{X}(t)$ on $[\alpha,\infty)$ we get

$$X(t) = \hat{X}(t)\hat{M} + \hat{X}(t)\hat{S}(t)\hat{N} = \hat{X}(t)[\hat{P}\hat{M} + \hat{S}(t)\hat{N}] \quad \text{on } [\alpha,\infty). \tag{6.33}$$

We show that $(X,U)$ has constant kernel on $[\alpha,\infty)$, namely that $\operatorname{Ker}X(t) = \operatorname{Ker}\hat{P}\hat{M}$ on $[\alpha,\infty)$. First we note that the symmetry of $\hat{M}^T\hat{N}$ implies the symmetry of $\hat{N}\hat{M}^{-1}$ and that $\hat{N}\hat{M}^{-1}\hat{P} = \hat{M}^{T-1}\hat{N}^T\hat{P} = \hat{M}^{T-1}\hat{N}^T = \hat{N}\hat{M}^{-1}$ holds. Hence, by (6.33),

$$X(t) = \hat{X}(t)[\hat{P}\hat{M} + \hat{S}(t)\hat{N}\hat{M}^{-1}\hat{M}] = \hat{X}(t)[\hat{P}\hat{M} + \hat{S}(t)\hat{N}\hat{M}^{-1}\hat{P}\hat{M}] \quad \text{on } [\alpha,\infty).$$

Then $\operatorname{Ker}\hat{P}\hat{M} \subseteq \operatorname{Ker}X(t)$ for all $t \in [\alpha,\infty)$. Fix now $t \in [\alpha,\infty)$, $v \in \operatorname{Ker}X(t)$, and put $w := \hat{P}\hat{M}v$. Then $\hat{X}(t)[w + \hat{S}(t)\hat{N}\hat{M}^{-1}w] = 0$. Multiplying the latter equality by $\hat{X}^\dagger(t)$ from the left and using the identities $\hat{P}\hat{S}(t) = \hat{S}(t)$ and $w = \hat{P}w$ we get $w = -\hat{S}(t)\hat{N}\hat{M}^{-1}w$. Therefore, $w \in \operatorname{Im}\hat{S}(t) \subseteq \operatorname{Im}P_{\hat{\mathscr{S}}\infty}$, by (2.13), and thus, $w = -\hat{S}(t)P_{\hat{\mathscr{S}}\infty}\hat{N}\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty}w = 0$, by the last condition in (6.31). This shows that $v \in \operatorname{Ker}\hat{P}\hat{M}$, i.e., $\operatorname{Ker}X(t) \subseteq \operatorname{Ker}\hat{P}\hat{M}$. The fact that $(X,U)$ belongs to the genus $\mathscr{G}$ follows from Definition 6.3.1 and Remark 2.3.9(iii), since by (6.30) and the invertibility of $\hat{M}$, we have $\operatorname{Im}X(\alpha) = \operatorname{Im}\hat{X}(\alpha)$. In addition, $(X,U)$ and $(\hat{X},\hat{U})$ are mutually representable on $[\alpha,\infty)$ in the sense of Theorem 2.3.8. By Theorem 3.2.11, there exists a minimal conjoined basis $(X_*,U_*)$ of (H), which is contained in $(X,U)$ on $[\alpha,\infty)$, i.e., $(X_*,U_*)$ has constant kernel on $[\alpha,\infty)$ and $\operatorname{rank}X_*(t) = n - d_\infty$ on $[\alpha,\infty)$. Furthermore, if we denote by $(\hat{X}_{\min},\hat{U}_{\min})$ the minimal principal solution of (H) from Theorem 5.1.10, which is contained in $(\hat{X},\hat{U})$ on $[\alpha,\infty)$, then $(\hat{X}_{\min},\hat{U}_{\min})$ is also a minimal conjoined basis of (H) on $[\alpha,\infty)$. According to Remark 3.3.5, $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X_*,U_*)$ are mutually representable on $[\alpha,\infty)$. This means that there exist $\hat{M}_{\min},\hat{N}_{\min}\in\mathbb{R}^{n\times n}$ such that the equalities in (3.66) are satisfied with $(X_{*1},U_{*1}):=(\hat{X}_{\min},\hat{U}_{\min})$, $(X_{*2},U_{*2}):=(X_*,U_*)$, $M_{*1}:=\hat{M}_{\min}$, and $N_{*1}:=\hat{N}_{\min}$. In particular, $\hat{M}_{\min}$ is invertible and $\hat{N}_{\min}$ is the Wronskian of $(\hat{X}_{\min},\hat{U}_{\min})$ and $(X_*,U_*)$, by Remark 2.3.9. Consequently, by the last condition in (6.31) and Theorem 3.3.8(ii) with $(X_1,U_1):=(\hat{X},\hat{U})$, $(X_2,U_2):=(X,U)$, $M_1:=\hat{M}$, $N_1:=\hat{N}$, and $P_{\mathscr{S}_1\infty}:=P_{\hat{\mathscr{S}}\infty}$, we get $\hat{N}_{\min}\hat{M}_{\min}^{-1}=0$. Hence, $\hat{N}_{\min}=0$, which implies by (3.66) that $X_*(\alpha)=\hat{X}_{\min}(\alpha)\hat{M}_{\min}$ and $U_*(\alpha)=\hat{U}_{\min}(\alpha)\hat{M}_{\min}$. This gives that $X_*=\hat{X}_{\min}\hat{M}_{\min}$ and $U_*=\hat{U}_{\min}\hat{M}_{\min}$ on $[a,\infty)$, by the uniqueness of solutions of (H). According to Theorem 5.1.6, $(X_*,U_*)$ is a minimal principal solution of (H) with respect to the interval $[\alpha,\infty)$. Finally, Theorem 5.1.3 then yields that $(X,U)$ is a principal solution of (H) at infinity. ∎

**Remark 6.3.14.** (i) The result in Theorem 6.3.13 applied to the genus $\mathscr{G}_{\min}$ yields the classification of the minimal principal solutions of (H) in Theorem 5.1.6. Indeed, in this case we have $P_{\hat{\mathscr{S}}\infty}=\hat{P}$ in Theorem 6.3.13, which implies through the third and fourth condition in (6.31) that $\hat{N}\hat{M}^{-1}=0$. Therefore, $\hat{N}=0$ and (6.30) yields that $X(\alpha)=\hat{X}(\alpha)\hat{M}$ and $U(\alpha)=\hat{U}(\alpha)\hat{M}$ with invertible $\hat{M}$. By the uniqueness of solutions of (H) we then obtain $(X,U)=(\hat{X}\hat{M},\hat{U}\hat{M})$ on $[a,\infty)$, as we claim in Theorem 5.1.6. On the other hand, we note that the uniqueness of the minimal principal solution in Theorem 5.1.6 together with Theorem 6.3.10 also imply that all minimal principal solutions of (H) belong to the same genus $\mathscr{G}_{\min}$.

(ii) Similarly, all maximal principal solutions of (H) are simultaneously classified through Theorem 6.3.13 with $\hat{P}=I$, when it is applied to the genus $\mathscr{G}_{\max}$, see Corollary 6.3.15 below. This result then completes Reid's concept of principal solutions at infinity in [30].

**Corollary 6.3.15.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (5.5). *Let* $(\hat{X},\hat{U})$ *be a maximal principal solution of* (H) *at infinity. Moreover, let* $P_{\hat{\mathscr{S}}\infty}$ *be the orthogonal projector defined through the invertible function* $\hat{X}(t)$ *on* $(\hat{\alpha}_{\min},\infty)$ *in* (2.12) *and Remark* 5.1.11. *Then a solution* $(X,U)$ *of* (H) *is a maximal principal solution if and only if for some (and hence for every)* $\alpha\in(\hat{\alpha}_{\min},\infty)$ *there exist matrices* $\hat{M},\hat{N}\in\mathbb{R}^{n\times n}$ *such that*

$$X(\alpha)=\hat{X}(\alpha)\hat{M},\quad U(\alpha)=\hat{U}(\alpha)\hat{M}+\hat{X}^{T^{-1}}(\alpha)\hat{N},$$
$$\hat{M} \text{ is nonsingular},\quad \hat{M}^T\hat{N}=\hat{N}^T\hat{M},\quad P_{\hat{\mathscr{S}}\infty}\hat{N}\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty}=0.$$

## 6.4 Limit characterization of principal solutions

In this section we derive a limit characterization of principal solutions of (H) at infinity in the sense of (1.3), see Theorems 6.4.1 and 6.4.5. This can be regarded as a generalization of the classical limit result for principal solutions at infinity of controllable linear Hamiltonian systems in Proposition 1.4.5. Below we use the same notation as in Theorem 6.3.11 and its proof.

**Theorem 6.4.1.** *Assume that (1.1) holds and system (H) is nonoscillatory with $\hat{\alpha}_{\min}$ defined in (5.5). Let $(\hat{X},\hat{U})$ and $(X,U)$ be two conjoined bases of (H) from a given genus $\mathscr{G}$ and let $\alpha > \hat{\alpha}_{\min}$ be such that $(\hat{X},\hat{U})$ and $(X,U)$ have constant kernel on $[\alpha,\infty)$. Denote by $\hat{P}$, $P_{\hat{\mathscr{S}}\infty}$, and $P_{\mathscr{S}\infty}$ their associated orthogonal projectors in (2.2), (2.12), and Remark 6.3.6. Moreover, let $\hat{N} := \hat{X}^T(t)U(t) - \hat{U}^T(t)X(t)$ be the (constant) Wronskian of $(\hat{X},\hat{U})$ and $(X,U)$. Then $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity and $\operatorname{rank} P_{\hat{\mathscr{S}}\infty}\hat{N}P_{\mathscr{S}\infty} = n - d_\infty$ if and only if*

$$\lim_{t\to\infty} X^\dagger(t)\hat{X}(t) = L \quad \text{with} \quad \operatorname{Im} L^T = \operatorname{Im}(\hat{P} - P_{\hat{\mathscr{S}}\infty}). \tag{6.34}$$

*In this case $(X,U)$ is an antiprincipal solution of (H) at infinity.*

*Proof.* With $\alpha$ as in the theorem, let $\hat{S}(t)$ and $S(t)$ be the $S$-matrices in (2.8) which are associated with $(\hat{X},\hat{U})$ and $(X,U)$ on $[\alpha,\infty)$. By Remark 2.3.9(ii), we have on $[\alpha,\infty)$ the representation

$$X(t) = \hat{X}(t)\,[\hat{M} + \hat{S}(t)\hat{N}], \quad \hat{X}(t) = X(t)\,[\hat{M}^{-1} - S(t)\hat{N}^T], \tag{6.35}$$

where $\hat{M}$ is invertible, see the proof of Theorem 6.3.11. By using (6.35) and the identities $X^\dagger(t)X(t) = P$ and $PS(t) = S(t)$ on $[\alpha,\infty)$, we obtain

$$X^\dagger(t)\hat{X}(t) = P\hat{M}^{-1} - S(t)\hat{N}^T \quad \text{on } [\alpha,\infty). \tag{6.36}$$

Let $\hat{T}_*$ and $T_*$ be the matrices in Remark 2.1.6 defined through the minimal conjoined bases $(\hat{X}_*,\hat{U}_*)$ and $(X_*,U_*)$ from the proof of Theorem 6.3.11. It follows by (3.64) that

$$T_* = \hat{M}_*^T\hat{T}_*\hat{M}_* + \hat{M}_*^T\hat{N}_*. \tag{6.37}$$

Suppose now that $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity and (6.26) holds. Then $\hat{T}_* = 0$ and $(X,U)$ is an antiprincipal solution of (H) at infinity, by Theorem 6.3.11. This means that $\operatorname{Im} T_* = \operatorname{Im} P_{\mathscr{S}\infty}$, because $\operatorname{Im} T_* \subseteq \operatorname{Im} P_{\mathscr{S}\infty}$ and $\operatorname{rank} T_* = n - d_\infty = \operatorname{rank} P_{\mathscr{S}\infty}$, by Remark 2.1.6, Definition 5.2.1, and (3.14). Formula (6.37) then becomes $T_* = \hat{M}_*^T\hat{N}_* = \hat{N}_*^T\hat{M}_*$. Multiplying this equality by $T_*^\dagger$ from the left and by $\hat{M}_*^{-1}$ from the right and using $T_*^\dagger T_* = P_{\mathscr{S}\infty}$ yields

$$P_{\mathscr{S}\infty}\hat{M}_*^{-1} = T_*^\dagger\hat{N}_*^T. \tag{6.38}$$

Furthermore, by Theorem 5.2.3 and (6.36) we know that

$$\lim_{t\to\infty} X^\dagger(t)\hat{X}(t) = \lim_{t\to\infty}[P\hat{M}^{-1} - S(t)\hat{N}^T] = L := P\hat{M}^{-1} - T_*^\dagger\hat{N}^T. \tag{6.39}$$

We show that $\operatorname{Im} L^T = \operatorname{Im}(\hat{P} - P_{\hat{\mathscr{S}}\infty}) = \operatorname{Im}\hat{P} \cap \operatorname{Ker} P_{\hat{\mathscr{S}}\infty}$. By using (6.39) and the identities $P\hat{M}^{-1}\hat{P} = P\hat{M}^{-1}$ and $\hat{N}^T\hat{P} = \hat{N}^T$, we get $L\hat{P} = L$, i.e., $\operatorname{Im} L^T \subseteq \operatorname{Im}\hat{P}$. Moreover, the equality $T_*^\dagger P_{\mathscr{S}\infty} = T_*^\dagger$ and the formulas (6.28), (6.29), and (6.38) imply that

$$LP_{\hat{\mathscr{S}}\infty} \overset{(6.39)}{=} P\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty} - T_*^\dagger P_{\mathscr{S}\infty}\hat{N}^T P_{\hat{\mathscr{S}}\infty} = P_{\mathscr{S}\infty}\hat{M}_*^{-1} - T_*^\dagger\hat{N}_*^T \overset{(6.38)}{=} 0. \tag{6.40}$$

Thus, $\operatorname{Im} L^T \subseteq \operatorname{Ker} P_{\hat{\mathscr{S}}\infty}$. Hence, we proved that $\operatorname{Im} L^T \subseteq \operatorname{Im}\hat{P} \cap \operatorname{Ker} P_{\hat{\mathscr{S}}\infty}$. Now we show the opposite inclusion $\operatorname{Im}\hat{P} \cap \operatorname{Ker} P_{\hat{\mathscr{S}}\infty} \subseteq \operatorname{Im} L^T$, which is equivalent with $\operatorname{Ker} L \subseteq \operatorname{Im} P_{\hat{\mathscr{S}}\infty} \oplus \operatorname{Ker}\hat{P}$. Let $v \in \operatorname{Ker} L$. Then $v$ can be uniquely decomposed as $v = v_1 + v_2$ with $v_1 \in \operatorname{Im}\hat{P}$ and

$v_2 \in \operatorname{Ker}\hat{P}$. The identity $L\hat{P} = L$ then implies that $(P\hat{M}^{-1} - T_*^{\dagger}\hat{N}^T)\, v_1 = Lv_1 = 0$ and hence, $P\hat{M}^{-1}v_1 = T_*^{\dagger}\hat{N}^T v_1$. The vector $w := P\hat{M}^{-1}v_1$ therefore satisfies $w \in \operatorname{Im} T_*^{\dagger} = \operatorname{Im} P_{\mathscr{S}\infty}$. By using the equalities $\hat{P}\hat{M}P\hat{M}^{-1} = \hat{P}$, $\hat{P}v_1 = v_1$, $P_{\mathscr{S}\infty}w = w$, and the first formula in (6.28), we get $v_1 = \hat{P}\hat{M}P\hat{M}^{-1}v_1 = \hat{P}\hat{M}w = \hat{P}\hat{M}P_{\mathscr{S}\infty}w = P_{\hat{\mathscr{S}}\infty}\hat{M}_*w$ and hence, $v_1 \in \operatorname{Im} P_{\hat{\mathscr{S}}\infty}$. This shows that $v = v_1 + v_2 \in \operatorname{Im} P_{\hat{\mathscr{S}}\infty} \oplus \operatorname{Ker}\hat{P}$, which completes the proof in this direction.

Conversely, assume that (6.34) is satisfied. Denote by $L_0 := P\hat{M}^{-1} - L$, where $L$ is given in (6.34). Then by (6.36) we get $S(t)\hat{N}^T \to L_0$ as $t \to \infty$. The equality $S(t) = S(t)P_{\mathscr{S}\infty}$ now implies that $\operatorname{Ker} P_{\mathscr{S}\infty}\hat{N}^T \subseteq \operatorname{Ker} L_0$ and similarly, the equality $S(t) = P_{\mathscr{S}\infty}S(t)$ implies that $\operatorname{Im} L_0 \subseteq \operatorname{Im} P_{\mathscr{S}\infty}$. In particular, we have $\operatorname{rank} L_0 \le \operatorname{rank} P_{\mathscr{S}\infty}\hat{N}^T$. Moreover, by using the identities $P\hat{M}^{-1}P_{\hat{\mathscr{S}}\infty} = P_{\mathscr{S}\infty}\hat{M}_*^{-1}$ and $LP_{\hat{\mathscr{S}}\infty} = 0$, we get $L_0P_{\hat{\mathscr{S}}\infty} = P_{\mathscr{S}\infty}\hat{M}_*^{-1}$, which implies that $\operatorname{Im} P_{\mathscr{S}\infty} \subseteq \operatorname{Im} L_0$. Hence, we have $\operatorname{Im} L_0 = \operatorname{Im} P_{\mathscr{S}\infty}$ and $\operatorname{rank} L_0 = \operatorname{rank} P_{\mathscr{S}\infty} = n - d_\infty$. In turn, the inequality $n - d_\infty = \operatorname{rank} L_0 \le \operatorname{rank} P_{\mathscr{S}\infty}\hat{N}^T$ holds. On the other hand, we have $\operatorname{rank} P_{\mathscr{S}\infty}\hat{N}^T \le \operatorname{rank} P_{\mathscr{S}\infty} = n - d_\infty$. Thus, we conclude that $\operatorname{rank} P_{\mathscr{S}\infty}\hat{N}^T = n - d_\infty$. The definition of $T_*$ in Remark 2.1.6 now yields

$$P_{\mathscr{S}\infty}\hat{N}^T = \lim_{t\to\infty} S^{\dagger}(t)\,S(t)\,\hat{N}^T = \lim_{t\to\infty} S^{\dagger}(t) \times \lim_{t\to\infty} S(t)\,\hat{N}^T = T_*L_0. \tag{6.41}$$

We thus obtain from (6.41) the inequality $n - d_\infty = \operatorname{rank} P_{\mathscr{S}\infty}\hat{N}^T = \operatorname{rank} T_*L_0 \le \operatorname{rank} T_*$. Using the third condition in (4.56) we obtain that $\operatorname{rank} T_* = n - d_\infty$. This shows that $(X,U)$ is an antiprincipal solution of (H) at infinity. Moreover, by using (6.29), (6.41), the symmetry of $\hat{N}_*\hat{M}_*^{-1}$, and the equalities $L_0P_{\hat{\mathscr{S}}\infty} = P_{\mathscr{S}\infty}\hat{M}_*^{-1}$ and $T_*P_{\mathscr{S}\infty} = T_*$, we get

$$\hat{N}_*\hat{M}_*^{-1} = \hat{M}_*^{T-1}P_{\mathscr{S}\infty}\hat{N}^TP_{\hat{\mathscr{S}}\infty} = \hat{M}_*^{T-1}T_*L_0P_{\hat{\mathscr{S}}\infty} = \hat{M}_*^{T-1}T_*P_{\mathscr{S}\infty}\hat{M}_*^{-1} = \hat{M}_*^{T-1}T_*\hat{M}_*^{-1}.$$

This implies that $T_* = \hat{M}_*^T\hat{N}_*$. From (6.37) we now obtain that $\hat{M}_*^T\hat{T}_*\hat{M}_* = 0$, i.e., $\hat{T}_* = 0$ as the matrix $\hat{M}_*$ is invertible. Therefore, $(\hat{X},\hat{U})$ is a principal solution of (H) at infinity. Finally, Theorem 6.3.11 yields the equality $\operatorname{rank} P_{\hat{\mathscr{S}}\infty}\hat{N}P_{\mathscr{S}\infty} = n - d_\infty$. ■

**Remark 6.4.2.** From the proof of Theorem 6.4.1 it follows that the equality in the second condition in (6.34) can be equivalently replaced by the inclusion $\operatorname{Im} L^T \subseteq \operatorname{Im}(\hat{P} - P_{\hat{\mathscr{S}}\infty})$.

The following result shows that the limit in (6.34) always exists for any conjoined basis $(\tilde{X},\tilde{U})$ from $\mathscr{G}$ instead of the principal solution $(\hat{X},\hat{U})$ at infinity, when $(X,U)$ happens to be an antiprincipal solution at infinity from $\mathscr{G}$. In this case we have an additional information about the structure of the space $\operatorname{Im} L^T$ in (6.34).

**Theorem 6.4.3.** *Assume that* (1.1) *holds and system* (H) *is nonoscillatory with* $\hat{\alpha}_{\min}$ *defined in* (5.5). *Let* $(\tilde{X},\tilde{U})$ *and* $(X,U)$ *be two conjoined bases from a given genus* $\mathscr{G}$, *such that* $(X,U)$ *is an antiprincipal solution of* (H) *at infinity and such that* $(\tilde{X},\tilde{U})$ *and* $(X,U)$ *have constant kernel on* $[\alpha,\infty)$ *with some* $\alpha > \hat{\alpha}_{\min}$. *Let* $\tilde{P}$, $P_{\tilde{\mathscr{S}}\infty}$, *and* $\tilde{T}$ *be the matrices in* (2.2), (2.12), *and Remark* 2.1.6 *defined through the function* $\tilde{X}(t)$. *Then the limit of* $X^{\dagger}(t)\tilde{X}(t)$ *as* $t \to \infty$ *exists and satisfies*

$$\lim_{t\to\infty} X^{\dagger}(t)\tilde{X}(t) = L \quad \textit{with} \quad \operatorname{Im} L^T = \operatorname{Im}\tilde{T} \oplus \operatorname{Im}(\tilde{P} - P_{\tilde{\mathscr{S}}\infty}). \tag{6.42}$$

*Proof.* We proceed similarly as in the proof of Theorem 6.4.1 with $(\tilde{X},\tilde{U})$ instead of $(\hat{X},\hat{U})$, since some of those arguments were independent of the fact that $(\hat{X},\hat{U})$ was the principal solution at infinity. Let $\tilde{N} := \tilde{X}^T(t)\,U(t) - \tilde{U}^T(t)\,X(t)$ be the (constant) Wronskian of $(\tilde{X},\tilde{U})$ and $(X,U)$. Then as in (6.35) and (6.36) we have on $[\alpha,\infty)$

$$X(t) = \tilde{X}(t)\,[\tilde{P}\tilde{M} + \tilde{S}(t)\,\tilde{N}], \quad \tilde{X}(t) = X(t)\,[P\tilde{M}^{-1} - S(t)\,\tilde{N}^T],$$
$$X^\dagger(t)\,\tilde{X}(t) = P\tilde{M}^{-1} - S(t)\,\tilde{N}^T,$$

where $\tilde{M}$ is invertible. Let $\tilde{T}_*$ and $T_*$ be the matrices in Remark 2.1.6 defined through the minimal conjoined bases $(\tilde{X}_*,\tilde{U}_*)$ and $(X_*,U_*)$, which are contained in $(\tilde{X},\tilde{U})$ and $(X,U)$ on $[\alpha,\infty)$, respectively, see the proof of Theorem 6.3.11. Then by (3.64) we have

$$T_* = \tilde{M}_*^T\,\tilde{T}_*\,\tilde{M}_* + \tilde{M}_*^T\,\tilde{N}_*. \tag{6.43}$$

Since $(X,U)$ is an antiprincipal solution at infinity, $\operatorname{Im} T_* = \operatorname{Im} P_{\mathscr{S}\infty}$ and as in (6.39) we get

$$\lim_{t\to\infty} X^\dagger(t)\,\tilde{X}(t) = \lim_{t\to\infty} [P\tilde{M}^{-1} - S(t)\,\tilde{N}^T] = L := P\tilde{M}^{-1} - T_*^\dagger\,\tilde{N}^T \tag{6.44}$$

and $L\tilde{P} = L$. Moreover, $\operatorname{Ker} L \subseteq \operatorname{Im} P_{\tilde{\mathscr{S}}\infty} \oplus \operatorname{Ker} \tilde{P}$, which shows that every vector $v \in \operatorname{Ker} L$ can be uniquely decomposed as $v = v_1 + v_2$, where $v_1 \in \operatorname{Im} P_{\tilde{\mathscr{S}}\infty}$ and $v_2 \in \operatorname{Ker} \tilde{P}$. The vector $w := P\tilde{M}^{-1} v_1$ then satisfies $w \in \operatorname{Im} P_{\mathscr{S}\infty}$, $v_1 = P_{\tilde{\mathscr{S}}\infty}\,\tilde{M}_*\,w$, and $w = T_*^\dagger\,\tilde{N}^T v_1$, see the paragraph following formula (6.40). In particular, by combining the last two equalities and by using the identities $T_*^\dagger P_{\mathscr{S}\infty} = T_*^\dagger$ and $P_{\mathscr{S}\infty}\,\tilde{N}^T P_{\tilde{\mathscr{S}}\infty} = \tilde{N}_*^T$ we obtain

$$w = T_*^\dagger\,\tilde{N}^T P_{\tilde{\mathscr{S}}\infty}\,\tilde{M}_*\,w = T_*^\dagger P_{\mathscr{S}\infty}\,\tilde{N}^T P_{\tilde{\mathscr{S}}\infty}\,\tilde{M}_*\,w = T_*^\dagger\,\tilde{N}_*^T\,\tilde{M}_*\,w. \tag{6.45}$$

We shall derive some additional properties of the matrix $L$, which are needed for the statement of this theorem. In particular, we prove the formula

$$\operatorname{Ker} L = \big(\operatorname{Ker} \tilde{T}_* \cap \operatorname{Im} P_{\tilde{\mathscr{S}}\infty}\big) \oplus \operatorname{Ker} \tilde{P}. \tag{6.46}$$

Let $v \in \operatorname{Ker} L$ and let $v_1$, $v_2$, and $w$ be its associated vectors defined above. Multiplying formula (6.45) by $T_*$ from the left together with the identities $T_*\,T_*^\dagger = P_{\mathscr{S}\infty}$ and $P_{\mathscr{S}\infty}\,\tilde{N}_*^T = \tilde{N}_*^T$ yields $T_*\,w = \tilde{N}_*^T\,\tilde{M}_*\,w = \tilde{M}_*^T\,\tilde{N}_*\,w$. By using (6.43) in the last equality, we get $\tilde{M}_*^T\,\tilde{T}_*\,\tilde{M}_*\,w = 0$. The invertibility of $\tilde{M}_*$ and the equality $\tilde{T}_* = \tilde{T}_*\,P_{\tilde{\mathscr{S}}\infty}$ then imply that $\tilde{T}_*\,P_{\tilde{\mathscr{S}}\infty}\,\tilde{M}_*\,w = 0$. Therefore, the vector $v_1 = P_{\tilde{\mathscr{S}}\infty}\,\tilde{M}_*\,w$ satisfies $v_1 \in \operatorname{Ker} \tilde{T}_* \cap \operatorname{Im} P_{\tilde{\mathscr{S}}\infty}$. Hence, the inclusion $\subseteq$ in (6.46) holds. Conversely, assume that $v \in \big(\operatorname{Ker} \tilde{T}_* \cap \operatorname{Im} P_{\tilde{\mathscr{S}}\infty}\big) \oplus \operatorname{Ker} \tilde{P}$. Then we can write $v = v_1 + v_2$ with $v_1 \in \operatorname{Ker} \tilde{T}_* \cap \operatorname{Im} P_{\tilde{\mathscr{S}}\infty}$ and $v_2 \in \operatorname{Ker} \tilde{P}$. Since $L\tilde{P} = L$, it follows from (6.44) that $Lv = Lv_1 = (P\tilde{M}^{-1} - T_*^\dagger\tilde{N}^T)\,v_1$. By using the identities $v_1 = P_{\tilde{\mathscr{S}}\infty} v_1$, $P\tilde{M}^{-1} P_{\tilde{\mathscr{S}}\infty} = P_{\mathscr{S}\infty}\,\tilde{M}_*^{-1}$, $T_*^\dagger P_{\mathscr{S}\infty} = T_*^\dagger$, $P_{\mathscr{S}\infty}\,\tilde{N}^T P_{\tilde{\mathscr{S}}\infty} = \tilde{N}_*^T$, and $T_*^\dagger\,T_* = P_{\mathscr{S}\infty}$, we then get

$$Lv = (P\tilde{M}^{-1} - T_*^\dagger\tilde{N}^T)\,P_{\tilde{\mathscr{S}}\infty} v_1 = (P_{\mathscr{S}\infty}\tilde{M}_*^{-1} - T_*^\dagger\tilde{N}_*^T)\,v_1 = T_*^\dagger\,(T_*\tilde{M}_*^{-1} - \tilde{N}_*^T)\,v_1. \tag{6.47}$$

Moreover, equality (6.43), the invertibility of $\tilde{M}_*$, and the symmetry of $\tilde{M}_*^T\tilde{N}_*$ imply that $T_*\,\tilde{M}_*^{-1} - \tilde{N}_*^T = \tilde{M}_*^T\,\tilde{T}_*$. Therefore, formula (6.47) yields that $Lv = T_*^\dagger\,\tilde{M}_*^T\,\tilde{T}_*\,v_1 = 0$, because $v_1 \in \operatorname{Ker} \tilde{T}_*$. This shows that $v \in \operatorname{Ker} L$, i.e., the inclusion $\supseteq$ in (6.46) is satisfied as well. Therefore, (6.46) is proven. According to Theorem 3.2.4, we have $\tilde{S}(t) = \tilde{S}_*(t)$ on $[\alpha,\infty)$ and hence, $\tilde{T} = \tilde{T}_*$. This means that the matrix $\tilde{T}_*$ in (6.46) can be replaced by $\tilde{T}$. Finally, by using $\operatorname{Im} \tilde{T} \cap \operatorname{Ker} P_{\tilde{\mathscr{S}}\infty} \subseteq \operatorname{Im} P_{\tilde{\mathscr{S}}\infty} \cap \operatorname{Ker} P_{\tilde{\mathscr{S}}\infty} = \{0\}$ and $\operatorname{Im} \tilde{T} \subseteq \operatorname{Im} \tilde{P}$, we get

$$\operatorname{Im} L^T = (\operatorname{Ker} L)^\perp = \big(\operatorname{Im} \tilde{T} \oplus \operatorname{Ker} P_{\tilde{\mathscr{S}}\infty}\big) \cap \operatorname{Im} \tilde{P} = \operatorname{Im} \tilde{T} \oplus \big(\operatorname{Ker} P_{\tilde{\mathscr{S}}\infty} \cap \operatorname{Im} \tilde{P}\big),$$

which is the second condition in (6.42). The proof is complete. ∎

**Remark 6.4.4.** If (H) is nonoscillatory, we introduce for every genus $\mathscr{G}$ its rank and defect as follows. The number $\operatorname{rank}\mathscr{G}$ is defined as the rank of $(\tilde{X},\tilde{U})$, where $(\tilde{X},\tilde{U})$ is any conjoined basis from $\mathscr{G}$. This quantity is well defined, since any two conjoined bases from $\mathscr{G}$ have eventually the same image of their first components. Then $n-d_\infty \leq \operatorname{rank}\mathscr{G} \leq n$. Also, we define $\operatorname{def}\mathscr{G} := n - \operatorname{rank}\mathscr{G}$, for which $0 \leq \operatorname{def}\mathscr{G} \leq d_\infty$. From Theorem 6.4.3 it follows that

$$\operatorname{rank} L = \operatorname{rank}\tilde{T} + d_\infty - \operatorname{def}\mathscr{G},$$

since by (6.42) we have $\operatorname{rank} L = \operatorname{rank}\tilde{T} + \operatorname{rank}\tilde{P} - \operatorname{rank} P_{\tilde{\mathscr{S}}\infty}$, while $\operatorname{rank}\tilde{P} = \operatorname{rank}\mathscr{G}$ and $\operatorname{rank} P_{\tilde{\mathscr{S}}\infty} = n - d_\infty$. Therefore, the actual value of the rank of $L$ depends primarily on the rank of $\tilde{T}$. In particular, the rank of $L$ is minimal if and only if the conjoined basis $(\tilde{X},\tilde{U})$ is a principal solution of (H) at infinity. This property is well known in the controllable case, for which $d_\infty = 0 = \operatorname{def}\mathscr{G}$ and hence, $\operatorname{rank} L = \operatorname{rank}\tilde{T}$, compare also with Proposition 1.4.5 in Section 1.4.

The statement of Theorem 6.4.1 is particularly simple for the minimal genus $\mathscr{G}_{\min}$.

**Theorem 6.4.5.** *Assume that (1.1) holds and system (H) is nonoscillatory. Let $(\hat{X},\hat{U})$ and $(X,U)$ be two conjoined bases of (H) from the minimal genus $\mathscr{G}_{\min}$. Moreover, denote by $\hat{N} := \hat{X}^T(t)\,U(t) - \hat{U}^T(t)\,X(t)$ the (constant) Wronskian of $(\hat{X},\hat{U})$ and $(X,U)$. Then $(\hat{X},\hat{U})$ is a minimal principal solution of (H) at infinity and* $\operatorname{rank}\hat{N} = n - d_\infty$ *if and only if*

$$\lim_{t\to\infty} X^\dagger(t)\,\hat{X}(t) = 0. \tag{6.48}$$

*In this case $(X,U)$ is a minimal antiprincipal solution of (H) at infinity.*

*Proof.* Let $(\hat{X},\hat{U})$ and $(X,U)$ be as in the corollary and let $\alpha > \hat{\alpha}_{\min}$ be such that $(\hat{X},\hat{U})$ and $(X,U)$ have constant kernel on $[\alpha,\infty)$. Then $(\hat{X},\hat{U})$ and $(X,U)$ are minimal conjoined bases on $[\alpha,\infty)$. Moreover, let $\hat{P}$, $P$, $P_{\hat{\mathscr{S}}\infty}$, and $P_{\mathscr{S}\infty}$ be the corresponding matrices in (2.2), (2.12), and Remark 6.3.6. Then $\hat{P} = P_{\hat{\mathscr{S}}\infty}$, $P = P_{\mathscr{S}\infty}$, and $P_{\hat{\mathscr{S}}\infty}\hat{N}P_{\mathscr{S}\infty} = \hat{N}$. The statement now follows from Theorem 6.4.1. ∎

The result in Theorem 6.4.5 gives the classical limit characterization of the principal solutions at infinity of a completely controllable system (H), see [6, Proposition 4, pg 43], [31, Theorem VII.3.2], [16, Theorem XI.10.5], and Proposition (1.4.5) in Section 1.4. In this case $d_\infty = 0$ and there is only one (that is, minimal) genus of conjoined bases of (H). We recall from Theorem 5.1.6 that the (minimal) principal solution at infinity is in this case unique up to a right nonsingular multiple.

**Corollary 6.4.6.** *Assume that (1.1) holds and system (H) is nonoscillatory and completely controllable. Let $(\hat{X},\hat{U})$ and $(X,U)$ be two conjoined bases of (H) with $\hat{N}$ being their Wronskian. Then $(\hat{X},\hat{U})$ is the principal solution of (H) at infinity and $\hat{N}$ is invertible if and only if*

$$\lim_{t\to\infty} X^{-1}(t)\,\hat{X}(t) = 0.$$

*In this case $(X,U)$ is an antiprincipal solution of (H) at infinity.*

## 6.5 Principal and antiprincipal solutions at minus infinity

In this section we complete the theory of principal and antiprincipal solutions by providing the corresponding notions at minus infinity. Throughout this section we assume that system (H) is defined on the interval $(-\infty,c]$ for some $c\in\mathbb{R}$ and that

$$B(t)\geq 0, \quad \text{on } (-\infty,c]. \tag{6.49}$$

First, we remark that under condition (6.49) the system (H) is defined to be nonoscillatory (at minus infinity) if every its conjoined basis $(X,U)$ has constant kernel near $-\infty$. Moreover, the order of abnormality $d(-\infty,t]$ of (H) is nonincreasing in $t$ on $(-\infty,c]$ and hence, there exists the maximal order of abnormality $d_{-\infty}$ (at minus infinity) defined by

$$d_{-\infty} := \lim_{t\to-\infty} d(-\infty,t] = \max_{t\in(-\infty,c]} d(-\infty,t]. \tag{6.50}$$

These observations then allow to define a principal and antiprincipal solution of (H) at minus infinity in a similar way as in Definitions 5.1.1 and 5.2.1.

**Definition 6.5.1** (Principal and antiprincipal solution at minus infinity)**.** A conjoined basis $(\tilde{X},\tilde{U})$ of (H) is said to be a *principal solution at minus infinity* if there exists $\gamma\in(-\infty,c]$ such that $(\tilde{X},\tilde{U})$ has constant kernel on $(-\infty,\gamma]$ and

$$\lim_{t\to-\infty} \tilde{S}_{\gamma}^{\dagger}(t)=0, \quad \text{where } \tilde{S}_{\gamma}(t):=\int_{\gamma}^{t}\tilde{X}^{\dagger}(s)\,B(s)\,\tilde{X}^{\dagger T}(s)\,\mathrm{d}s.$$

Further, a conjoined basis $(X,U)$ of (H) is called an *antiprincipal solution at minus infinity* if there exists $\gamma\in(-\infty,c]$ with $d(-\infty,\gamma]=d_{-\infty}$ such that $(X,U)$ has constant kernel on $(-\infty,\gamma]$ and

$$\lim_{t\to-\infty}\left(\int_{\gamma}^{t}X^{\dagger}(s)\,B(s)\,X^{\dagger T}(s)\,\mathrm{d}s\right)^{\dagger}=T_{\gamma}, \qquad \text{with } \operatorname{rank} T_{\gamma}=n-d_{-\infty}.$$

**Remark 6.5.2.** We note that all the results about principal and antiprincipal solutions at infinity, which were derived in the previous sections, remain (in appropriate interpretation) true also for principal and antiprincipal solutions at minus infinity introduced in Definition 6.5.1. More precisely, similarly as in Theorems 5.1.5 and 5.2.7, the nonoscillation of system (H) is equivalent with the existence of (anti)principal solutions of (H) at minus infinity with any rank of their first component between $n-d_{-\infty}$ and $n$. Here the number $d_{-\infty}$ is defined in (6.50). In particular, the minimal principal solution at minus infinity (with the rank $n-d_{-\infty}$) is uniquely determined up to a right nonsingular multiple, compare with the result in Theorem 5.1.6. Furthermore, principal and antiprincipal solutions at minus infinity are distributed into genera of conjoined bases, that is, they can be classified according to the image of their first component near $-\infty$. The corresponding results are analogous to those ones, which are presented in Section 6.3 for (anti)principal solutions at infinity. Finally, a limit characterization of the principal solutions of (H) at minus infinity can be derived in a sense of the criteria in Theorems 6.4.1 and 6.4.5.

# Chapter 7

# Examples

In this chapter we present several examples, which illustrate the theory of principal and antiprincipal solutions at infinity.

## 7.1 Principal and antiprincipal solutions

In agreement with the notation in Remarks 5.1.2 and 5.2.2, the principal solutions at infinity will be denoted by $(\hat{X},\hat{U})$, in the special case of minimal and maximal principal solutions at infinity by $(\hat{X}_{\min},\hat{U}_{\min})$ and $(\hat{X}_{\max},\hat{U}_{\max})$. Similarly, the antiprincipal solutions at infinity will be denoted by $(X,U)$, in the special case of minimal and maximal antiprincipal solutions at infinity by $(X_{\min},U_{\min})$ and $(X_{\max},U_{\max})$.

**Example 7.1.1.** In the first example we discuss a controllable linear Hamiltonian system. Let $n=1$, $a=0$, $A(t)=0$, $B(t)=1+t^2$, and $C(t)=-2/(1+t^2)^2$, which implies that this system corresponds to the second order Sturm–Liouville equation $[y'/(1+t^2)]'+2y/(1+t^2)^2=0$. Since $B(t)>0$ on $[0,\infty)$, system (H) is completely controllable on $[0,\infty)$ and $d[0,\infty)=d_\infty=0$. Therefore, there is only one (minimal/maximal) genus $\mathcal{G}$ of conjoined bases with the rank $r=n=1$, i.e., the minimal and maximal (anti)principal solutions at infinity coincide. The principal solutions at infinity are nonzero multiples of

$$\big(\hat{X}(t),\hat{U}(t)\big)=\big(t,\,1/(1+t^2)\big), \tag{7.1}$$

with $\hat{\alpha}_{\min}=0$, by (5.5). On the other hand, by Corollary 6.3.12, nonzero multiples of the principal solutions at the points $\alpha>0$ are antiprincipal solutions at infinity. For example,

$$\big(X(t),U(t)\big)=\big(t^2-1,\,2t/(1+t^2)\big), \tag{7.2}$$

is an antiprincipal solution at infinity, being at the same time the principal solution at the point $\alpha=1$ (see Corollary 6.3.12). Moreover, the solutions in (7.1) and (7.2) satisfy $X^{-1}(t)\hat{X}(t)=t/(t^2-1)\to 0$ as $t\to\infty$, as we claim in formula (1.25) of Proposition 1.4.5.

**Example 7.1.2.** We consider the so-called zero system (H) with $n\times n$ coefficient matrices $A(t)=B(t)=C(t)\equiv 0$ on $[a,\infty)$. This system is extremely abnormal, because $d[a,\infty)=d_\infty=n$. In this case every conjoined basis of (H) is constant on $[a,\infty)$ and all conjoined bases are simultaneously both principal and antiprincipal solutions at infinity with respect to the interval $[a,\infty)$, so that $\hat{\alpha}_{\min} = a$. Moreover, for any genus $\mathscr{G}$ there exists a unique orthogonal projector $P \in \mathbb{R}^{n\times n}$ such that the conjoined basis $(X,U) = (\hat{X},\hat{U}) = (P, I-P)$ is a principal and antiprincipal solution at infinity belonging to $\mathscr{G}$. In particular, if $P = 0$, then $(X_{\min},U_{\min}) = (\hat{X}_{\min},\hat{U}_{\min}) = (0,I)$, while if $P = I$, then $(X_{\max},U_{\max}) = (X_{\max},U_{\max}) = (I,0)$. In addition, $X^{\dagger}(t)\hat{X}(t) = P^{\dagger}P \to L := P$ as $t \to \infty$. This is in full agreement with Theorem 6.4.1, because in this case $\hat{P} = P$ and $P_{\hat{\mathscr{S}}_\infty} = 0$. Note that for any $\alpha \geq a$ the principal solution at the point $\alpha$ is equal to $(0,I)$, which is at the same time a minimal antiprincipal solution at infinity, as we claim in Proposition 5.2.9.

In the following we extend Example 7.1.2 to a system with variable $A(t)$.

**Example 7.1.3.** Let $B(t) = C(t) \equiv 0 \in \mathbb{R}^{n\times n}$ on $[a,\infty)$. Then (H) has the form $X' = A(t)X$ and $U' = -A^T(t)U$ and, as in Example 7.1.2, we have $d[a,\infty) = d_\infty = n$. Therefore, the principal and antiprincipal solutions at infinity coincide and they can be constructed from the fundamental matrix $\Psi(t,a)$ of system $U' = -A^T(t)U$ on $[a,\infty)$ satisfying $\Psi(a,a) = I$. More precisely, if $P \in \mathbb{R}^{n\times n}$ is an orthogonal projector, then

$$\big(\hat{X}(t),\hat{U}(t)\big) = \big(X(t),U(t)\big) = \big(\Psi^{T-1}(t,a)\,P,\ \Psi(t,a)\,(I-P)\big) \tag{7.3}$$

is an (anti)principal solution at infinity with the rank equal to $\operatorname{rank} P$. In particular, we have

$$\begin{aligned} \big(\hat{X}_{\min}(t),\hat{U}_{\min}(t)\big) &= \big(X_{\min}(t),U_{\min}(t)\big) = \big(0,\ \Psi(t,a)\big), \\ \big(\hat{X}_{\max}(t),\hat{U}_{\max}(t)\big) &= \big(X_{\max}(t),U_{\max}(t)\big) = \big(\Psi^{T-1}(t,a),\, 0\big). \end{aligned} \tag{7.4}$$

Note that in this case the solutions in (7.3) satisfy $X^{\dagger}(t)\hat{X}(t) \to L := P$ as $t \to \infty$, which we also guarantee in Theorem 6.4.1 with $\hat{P} = P$ and $P_{\hat{\mathscr{S}}_\infty} = 0$. The equalities in (7.4) then illustrate the statement of Proposition 5.2.9.

In the last two examples we utilize the construction of principal and antiprincipal solutions at infinity through Theorem 5.3.3. We also demonstrate the variability of the genera of conjoined bases with different ranks.

**Example 7.1.4.** Let $n = 2$ and $A(t) = \operatorname{diag}\{0,1\}$, $B(t) = \operatorname{diag}\{1+t^2,0\}$, and $C(t) = \operatorname{diag}\{-2/(1+t^2)^2,0\}$ on $[0,\infty)$. One can see that system (H) has the form of system ($\text{H}_*$) in Section 5.3 with the block structure coming from the scalar systems in Example 7.1.1 and in Example 7.1.3 with $\Psi(t,0) = \mathrm{e}^{-t}$. In this case we have $d[0,\infty) = d_\infty = 1$ and $\hat{\alpha}_{\min} = 0$. Therefore, there exist only two genera of conjoined bases, i.e., the minimal genus $\mathscr{G}_{\min}$ with the rank $r = n - d_\infty = 1$ and the maximal genus $\mathscr{G}_{\max}$ with the rank $r = n = 2$. By Theorem 5.3.3, the solutions

$$\big(\hat{X}_{\min}(t),\hat{U}_{\min}(t)\big) = \left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/(1+t^2) & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\right),$$

$$\big(X_{\min}(t),U_{\min}(t)\big) = \left(\begin{pmatrix} t^2-1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2t/(1+t^2) & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\right)$$

are minimal (anti)principal solutions at infinity while

$$\big(\hat{X}_{\max}(t),\hat{U}_{\max}(t)\big)=\left(\begin{pmatrix} t & 0\\ 0 & \mathrm{e}^{t}\end{pmatrix},\begin{pmatrix} 1/(1+t^2) & 0\\ 0 & 0\end{pmatrix}\right),$$

$$\big(X_{\max}(t),U_{\max}(t)\big)=\left(\begin{pmatrix} t^2-1 & 0\\ 0 & \mathrm{e}^{t}\end{pmatrix},\begin{pmatrix} 2t/(1+t^2) & 0\\ 0 & 0\end{pmatrix}\right).$$

are maximal (anti)principal solutions. Note that the functions in the left upper corner and the functions in the right lower corner of the above four solutions constitute (anti)principal solutions of the systems from Examples 7.1.1 and 7.1.3, respectively, as Theorem 5.3.3 guarantees. When we multiply $(\hat{X}_{\min},\hat{U}_{\min})$ by the invertible matrix $M$ given below, then the resulting solution $(\hat{X}_0,\hat{U}_0):=(\hat{X}_{\min}M,\hat{U}_{\min}M)$ is also a minimal principal solution of (H) at infinity by Theorem 5.1.6, where

$$M=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\qquad \big(\hat{X}_0(t),\hat{U}_0(t)\big)=\left(\begin{pmatrix} 0 & t\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 1/(1+t^2)\\ \mathrm{e}^{-t} & 0\end{pmatrix}\right).$$

However, as we comment in Remark 5.3.4, the above minimal principal solution $(\hat{X}_0,\hat{U}_0)$ does not have the form displayed in equation (5.14).

**Example 7.1.5.** Let $n=3$ and $a=0$. We consider system (H) with $A(t)=\operatorname{diag}\{0,0,1\}$, $B(t)=\operatorname{diag}\{1+t^2,0,0\}$, and $C(t)=\operatorname{diag}\{-2/(1+t^2)^2,0,0\}$ on $[0,\infty)$. It is easy to see that system (H) comes from the scalar systems in Examples 7.1.1, 7.1.2, and 7.1.3 with $\Psi(t,0)=\mathrm{e}^{-t}$. In this case we have $d_\infty=2$ and $\hat{\alpha}_{\min}=0$. First we examine the minimal genus $\mathcal{G}_{\min}$, whose rank is $r=n-d_\infty=1$. By Theorem 5.3.3, we have

$$\big(\hat{X}_{\min}(t),\hat{U}_{\min}(t)\big)=\left(\begin{pmatrix} t & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix} 1/(1+t^2) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{-t}\end{pmatrix}\right),$$

$$\big(X_{\min}(t),U_{\min}(t)\big)=\left(\begin{pmatrix} t^2-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix} 2t/(1+t^2) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{-t}\end{pmatrix}\right).$$

Moreover, $X_{\min}^{\dagger}(t)\hat{X}_{\min}(t)=\operatorname{diag}\{t/(t^2-1),0,0\}\to 0$ as $t\to\infty$, as we claim in formula (6.48) of Theorem 6.4.5.

Now we discuss the maximal genus $\mathcal{G}_{\max}$, whose rank is $r=n=3$. From Theorem 5.3.3 we obtain

$$\big(\hat{X}_{\max}(t),\hat{U}_{\max}(t)\big)=\left(\begin{pmatrix} t & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{t}\end{pmatrix},\begin{pmatrix} 1/(1+t^2) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right),$$

$$\big(X_{\max}(t),U_{\max}(t)\big)=\left(\begin{pmatrix} t^2-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{t}\end{pmatrix},\begin{pmatrix} 2t/(1+t^2) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right).$$

In this case $X_{\max}^{\dagger}(t)\hat{X}_{\max}(t) = \operatorname{diag}\{t/(t^2-1),1,1\} \to L = \operatorname{diag}\{0,1,1\}$ as $t \to \infty$, and $\hat{P} = I$ and $P_{\hat{\mathscr{S}}\infty} = \operatorname{diag}\{1,0,0\}$ in formula (6.34) of Theorem 6.4.1.

In the remaining part of this example we analyze three different genera with rank equal to $r = 2$. Observe that only two of them arise from the diagonal construction in Theorem 5.3.3. Consider the genus $\mathscr{G}_1$ with rank $r = 2$, which is given by the principal and antiprincipal solutions at infinity

$$(\hat{X}_1(t),\hat{U}_1(t)) = \left( \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^t \end{pmatrix}, \begin{pmatrix} 1/(1+t^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

$$(X_1(t),U_1(t)) = \left( \begin{pmatrix} t^2-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^t \end{pmatrix}, \begin{pmatrix} 2t/(1+t^2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Therefore, in Theorem 6.4.1 we now have $X_1^{\dagger}(t)\hat{X}_1(t) = \operatorname{diag}\{t/(t^2-1),0,1\} \to L = \operatorname{diag}\{0,0,1\}$, and $\hat{P} = \operatorname{diag}\{1,0,1\}$ and $P_{\hat{\mathscr{S}}\infty} = \operatorname{diag}\{1,0,0\}$. In the genus $\mathscr{G}_2$ with rank $r = 2$ defined by the principal and antiprincipal solutions at infinity

$$(\hat{X}_2(t),\hat{U}_2(t)) = \left( \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/(1+t^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-t} \end{pmatrix} \right),$$

$$(X_2(t),U_2(t)) = \left( \begin{pmatrix} t^2-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2t/(1+t^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-t} \end{pmatrix} \right)$$

we have that $X_2^{\dagger}(t)\hat{X}_2(t) = \operatorname{diag}\{t/(t^2-1),1,0\} \to L = \operatorname{diag}\{0,1,0\}$, and $\hat{P} = \operatorname{diag}\{1,1,0\}$ and $P_{\hat{\mathscr{S}}\infty} = \operatorname{diag}\{1,0,0\}$. Note that in all the above genera $\mathscr{G}_{\min}$, $\mathscr{G}_{\max}$, $\mathscr{G}_1$, $\mathscr{G}_2$ the matrix $L$ satisfies $L^T = \hat{P} - P_{\hat{\mathscr{S}}\infty}$, which will not be the case of the following nondiagonal genus. Let $\mathscr{G}_3$ be the genus with rank $r = 2$ defined by the principal and antiprincipal solutions at infinity

$$(\hat{X}_3(t),\hat{U}_3(t)) = \left( \begin{pmatrix} t & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & \mathrm{e}^t/2 & \mathrm{e}^t/2 \end{pmatrix}, \begin{pmatrix} 1/(1+t^2) & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -\mathrm{e}^{-t}/2 & \mathrm{e}^{-t}/2 \end{pmatrix} \right),$$

$$(X_3(t),U_3(t)) = \left( \begin{pmatrix} t^2-1 & 0 & 0 \\ 0 & 1/4 & -1/4 \\ 0 & \mathrm{e}^t/4 & -\mathrm{e}^t/4 \end{pmatrix}, \begin{pmatrix} 2t/(1+t^2) & 0 & 0 \\ 0 & 1/4 & 1/4 \\ 0 & -\mathrm{e}^{-t}/4 & -\mathrm{e}^{-t}/4 \end{pmatrix} \right).$$

In this case we have in Theorem 6.4.1

$$X_3^{\dagger}(t)\hat{X}_3(t) = \begin{pmatrix} t/(t^2-1) & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \to L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \quad \text{as } t \to \infty.$$

The matrices $\hat{P}$ and $P_{\hat{\mathscr{S}}_\infty}$ in (6.34) now have the form

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \quad P_{\hat{\mathscr{S}}_\infty} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and we can see that $\operatorname{Im} L^T = \operatorname{Im}(\hat{P} - P_{\hat{\mathscr{S}}_\infty})$, although $L^T \neq \hat{P} - P_{\hat{\mathscr{S}}_\infty}$.

# Chapter 8

# Conclusions and open problems

In this work we defined concepts of principal and antiprincipal solutions at infinity for nonoscillatory and possibly abnormal linear Hamiltonian systems (H). This new notions generalize the traditional principal (or recessive) and antiprincipal (or dominant) solutions, which were originally defined in the works by Reid, Hartman, Coppel, and Ahlbrandt for completely controllable linear Hamiltonian systems, see [1, 6, 16, 29, 31]. We proved the equivalence of the nonoscillation of system (H) with the existence of (anti)principal solutions with any rank of their first component between $n-d_\infty$ and $n$. Here $d_\infty$ is the maximal order of abnormality of (H). The minimal rank $n-d_\infty$ corresponds to the minimal principal solution, which is uniquely determined (up to a right nonsingular multiple). For the maximal rank $n$ we have the maximal principal solution, which corresponds to the earlier Reid's principal solution for abnormal systems (H) in [30]. Moreover, we also classified the principal and antiprincipal solutions at infinity, as well as all conjoined bases of a nonoscillatory system (H), according to their image. This gave rise to a new concept, called a genus of conjoined bases, which has no analogy in the theory of controllable systems (H). The classification of all (anti)principal solutions within one genus allowed us to derive the proper limit characterization of principal solutions of (H) at infinity. At the same time, we completed the work by Reid on the invertible principal solutions.

All the results mentioned above are based on a detailed analysis of conjoined bases of (H) with constant kernel, the abnormality of (H), and asymptotic properties of the $S$-matrices. Altogether, this work opens a completely new direction in the study of linear Hamiltonian systems and their solutions.

Some of our results are new even in the controllable case. For example, in Corollary 6.3.12 we describe a rich class of antiprincipal solutions at infinity of a controllable system (H). Moreover, in Theorem 4.3.2 we show that the conjoined bases $(X,U)$ of (H) lead to matrices $T$ in (1.4) with $\operatorname{rank} T$ between 0 and $n$. The antiprincipal solutions of (H) at infinity then correspond to the maximal value of $\operatorname{rank} T$ (i.e., $\operatorname{rank} T = n$), while the principal solutions of (H) at infinity correspond to the minimal value of $\operatorname{rank} T$ (i.e., $\operatorname{rank} T = 0$). The values of the rank of $T$ strictly between 0 and $n$ then lead to a class of "nonstandard" solutions of (H), i.e., to solutions which are neither principal nor antiprincipal at infinity, see also item (ii) below.

There are several topics and open problems, which are closely related to the results of this work or to the subsequent research in this area.

(i) In Theorem 6.3.7 we studied the structure of the minimal genus $\mathcal{G}_{\min}$. We believe that similar properties can be derived for an arbitrary genus $\mathcal{G}$.

(ii) By Theorems 5.1.5 and 5.2.7, the nonoscillation of system (H) is equivalent with the existence of principal solutions at infinity (corresponding to the minimal rank of $T$) or antiprincipal solutions at infinity (corresponding to the maximal rank of $T$), and these solutions have any rank between $n-d_\infty$ and $n$. We believe that this property holds also for the conjoined bases of (H), whose matrix $T$ satisfies $0 < \operatorname{rank} T < n-d_\infty$. Such intermediate or "nonstandard" solutions are present even in the controllable case ($d_\infty = 0$) when $n \geq 2$, but were never considered in the literature.

(iii) The limit characterization of principal solutions of (H) at infinity in Theorems 6.4.1 and 6.4.3 uses the conjoined bases from the same genus $\mathcal{G}$. We expect that it is possible to derive a limit property in the spirit of (6.34) or (6.42) for conjoined bases belonging to two different genera $\mathcal{G}_1$ and $\mathcal{G}_2$.

(iv) We are also working on the corresponding theory of minimal (distinguished) solutions of the associated Riccati matrix differential equation for possibly abnormal linear Hamiltonian systems.

(iv) We expect that this theory can be implemented also to discrete symplectic systems. In this direction we refer to our initial paper [38].

(v) The unified theory of continuous and discrete linear Hamiltonian and symplectic systems on time scales can be used as a platform for extending the concepts of principal and antiprincipal solutions infinity to more general systems, see [18, 19].

We find these research directions to be very interesting for the theory of differential equations, because they bring not only new results, but also reveal new methods for their analysis. We believe that the present work is successful in both of these two issues.

# Appendix A

# Auxiliary results from matrix analysis

The following appendix contains several results from matrix analysis needed in this work. In particular, we present results about limit behavior of sequences of Moore–Penrose pseudoinverses of matrices and results about orthogonal projectors.

## A.1 Results about matrix functions

In this section we present some auxiliary results mainly about sequences of matrices or matrix-valued functions. We recall the notation $(G)_k$ for the $k$-th leading principal submatrix of the matrix $G$, see Section 1.2. The following lemma relates the inverse of $(G)_k$ with the corresponding leading principal submatrix of the inverse of $G$. Its proof follows from [3, Propositions 8.2.3(v) and 8.2.4].

**Lemma A.1.1.** *Let $G \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix and $1 \leq k \leq n$. Then the inequalities $(G^{-1})_k \geq (G)_k^{-1} > 0$ hold.*

The next lemma is equivalent with [25, Lemma 3.1.9(ii)].

**Lemma A.1.2.** *Let $G, H \in \mathbb{R}^{n\times n}$ be symmetric matrices such that $H$ is invertible and the relations $H \geq G \geq 0$ are satisfied. Denote by $R$ the orthogonal projector onto $\operatorname{Im} G$. Then the inequalities $G^\dagger \geq RH^{-1}R \geq 0$ hold.*

*Proof.* Let $G$ and $H$ be as in the lemma. From [25, Lemma 3.1.9(ii)] it follows that $G \geq GH^{-1}G \geq 0$. The matrix $R$ is symmetric and the equalities $GG^\dagger = R = G^\dagger G$ and $G^\dagger = G^\dagger GG^\dagger$ hold, by Remarks 1.2.1(ii) and 1.2.3(i)–(ii). Therefore, we have that $G^\dagger = G^\dagger GG^\dagger \geq G^\dagger GH^{-1}GG^\dagger = RH^{-1}R \geq 0$ and the proof is complete. ■

In the next result we generalize the statement that every symmetric matrix $G$ is a limit of a sequence of invertible symmetric matrices $G_\nu$, compare with [6, pg. 40]. In the present context the matrices $G_\nu$ are no longer invertible, but their image is equal to the image of some fixed orthogonal projector.

**Lemma A.1.3.** *Let $G \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $R$ be an orthogonal projector with $\operatorname{Im} G \subseteq \operatorname{Im} R$. Then there exists a sequence $\{G_\nu\}_{\nu=1}^{\infty}$ of symmetric matrices such that $\operatorname{Im} G_\nu = \operatorname{Im} R$ for all $\nu \in \mathbb{N}$, and $G_\nu \to G$ as $\nu \to \infty$.*

*Proof.* Let $g := \operatorname{rank} G$ and $r := \operatorname{rank} R$, so that $g \le r$. If $g = r$, then $\operatorname{Im} G = \operatorname{Im} R$ and we may take the constant sequence $G_\nu := G$ for all $\nu \in \mathbb{N}$. Suppose now that $g < r$. Then there exists an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ such that its first $g$ columns form an orthonormal basis of $\operatorname{Im} G$ and at the same time, its first $r$ columns form an orthonormal basis of $\operatorname{Im} R$. This means that $V^T GV = \operatorname{diag}\{\Gamma_g, 0_{n-g}\}$ and $V^T RV = \operatorname{diag}\{\Theta_r, 0_{n-r}\}$, where $\Gamma_g \in \mathbb{R}^{g\times g}$ and $\Theta_r \in \mathbb{R}^{r\times r}$ are symmetric and nonsingular. Since $R$ is an orthogonal projector, $R^2 = R$ according to Remark 1.2.1(ii). It follows that the matrix $\Theta_r$ satisfies $\Theta_r^2 = \Theta_r$, which gives $\Theta_r = I_r$. Therefore, we have $G = V \operatorname{diag}\{\Gamma_g, 0_{n-g}\}V^T$ and $R = V \operatorname{diag}\{I_r, 0_{n-r}\}V^T$. Consider the sequence $\{G_\nu\}_{\nu=1}^{\infty}$ of matrices, where

$$G_\nu := V \begin{pmatrix} \Gamma_g & 0 & 0 \\ 0 & \frac{1}{\nu} I_{r-g} & 0 \\ 0 & 0 & 0_{n-r} \end{pmatrix} V^T \quad \text{for all } \nu \in \mathbb{N}. \tag{A.1}$$

It is obvious that for each $\nu \in \mathbb{N}$ the matrix $G_\nu$ is symmetric and $RG_\nu = G_\nu$. Since $\operatorname{rank} G_\nu = r$, we have $\operatorname{Im} G_\nu = \operatorname{Im} R$. Moreover, from (A.1) it follows that $\lim_{\nu\to\infty} G_\nu = G$. ∎

As a main result of this section we present a criterion for uniform convergence of a sequence of pseudoinverses of matrix-valued functions. This generalizes the corresponding statement for invertible matrices used in [6, pp. 40–41]. The proof of Theorem A.1.4 below is presented at the end of this section.

**Theorem A.1.4.** *Let $\mathscr{I}$ be a compact real interval and $\{G_\nu(t)\}_{\nu=1}^{\infty}$ be a sequence of $m \times n$ matrix-valued functions defined on $\mathscr{I}$. Assume that*

*(i) $G_\nu(t) \to G(t)$ for $\nu \to \infty$ uniformly on $\mathscr{I}$,*

*(ii) $G^\dagger(t)$ is continuous on $\mathscr{I}$,*

*(iii) there exists $\nu_0 \in \mathbb{N}$ such that for all $\nu \ge \nu_0$ and for all $t \in \mathscr{I}$*

$$\operatorname{Im} G_\nu(t) \subseteq \operatorname{Im} G(t), \quad \operatorname{Im} G_\nu^T(t) \subseteq \operatorname{Im} G^T(t). \tag{A.2}$$

*Then $G_\nu^\dagger(t) \to G^\dagger(t)$ for $\nu \to \infty$ uniformly on $\mathscr{I}$.*

We note that when $G(t)$ is continuous on $\mathscr{I}$, then assumption (ii) in Theorem A.1.4 is equivalent with $\operatorname{rank} G(t)$ constant on $\mathscr{I}$, by [5, Theorem 10.5.1]. The following two results are key tools for the proof of Theorem A.1.4.

**Lemma A.1.5.** *Let $\mathscr{I}$ be a compact real interval and $\{G_\nu(t)\}_{\nu=1}^{\infty}$ be a sequence of $m \times n$ matrix-valued functions defined on $\mathscr{I}$ satisfying conditions (i)–(ii) in Theorem A.1.4. Then there exists $\mu_1 \in \mathbb{N}$ such that*

$$\operatorname{Im} G_\nu(t) \subseteq \operatorname{Im} G(t) \quad \text{for all } \nu \ge \mu_1 \text{ and } t \in \mathscr{I} \tag{A.3}$$

*if and only if there exists $\mu_2 \in \mathbb{N}$ such that*

$$\operatorname{Im} G_\nu(t) = \operatorname{Im} G(t) \quad \text{for all } \nu \ge \mu_2 \text{ and } t \in \mathscr{I}. \tag{A.4}$$

*Proof.* First observe that (A.4) implies (A.3) trivially by taking $\mu_1 := \mu_2$. Conversely, assume that (A.3) holds for some $\mu_1 \in \mathbb{N}$, but (A.4) does not hold, i.e., there exist sequences $\{\nu_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ and $\{t_i\}_{i=1}^{\infty} \subseteq \mathscr{I}$ such that, by assumption (A.3),

$$\operatorname{Im} G_{\nu_i}(t_i) \subsetneqq \operatorname{Im} G(t_i) \quad \text{for all } i \in \mathbb{N}.$$

This means that $\operatorname{Ker} G^T(t_i) \subsetneqq \operatorname{Ker} G^T_{\nu_i}(t_i)$ for all $i \in \mathbb{N}$. Fix $i \in \mathbb{N}$. Then there exists a nonzero vector $v_{\nu_i} \in \mathbb{R}^m$ such that $G^T_{\nu_i}(t_i)\, v_{\nu_i} = 0$ and $G^T(t_i)\, v_{\nu_i} \neq 0$. Without loss of generality we may assume that $v_{\nu_i} \in \operatorname{Im} G(t_i)$ and $\|v_{\nu_i}\| = 1$. Here $\|\cdot\|$ is the Euclidean norm for vectors and at the same time, it denotes the corresponding induced norm for matrices. We show that

$$\|G^{\dagger T}(t_i)\|^{-1} \leq \|G^T(t_i)\, v_{\nu_i}\|. \tag{A.5}$$

Estimate (A.5) follows from the fact that the matrix $G(t_i)\, G^{\dagger}(t_i)$ is the orthogonal projector onto $\operatorname{Im} G(t_i)$ and from the inequality

$$1 = \|v_{\nu_i}\| = \|G(t_i)\, G^{\dagger}(t_i)\, v_{\nu_i}\| = \|G^{\dagger T}(t_i)\, G^T(t_i)\, v_{\nu_i}\| \leq \|G^{\dagger T}(t_i)\| \times \|G^T(t_i)\, v_{\nu_i}\|.$$

On the other hand, since $v_{\nu_i} \in \operatorname{Ker} G^T_{\nu_i}(t_i)$, we have

$$\begin{aligned}\|G^T(t_i)\, v_{\nu_i}\| &= \|G^T_{\nu_i}(t_i)\, v_{\nu_i} - G^T(t_i)\, v_{\nu_i}\| \leq \|G^T_{\nu_i}(t_i) - G^T(t_i)\| \times \|v_{\nu_i}\| \\ &= \|G^T_{\nu_i}(t_i) - G^T(t_i)\|.\end{aligned}$$

From (A.5) we then get

$$\|G^{\dagger T}(t_i)\|^{-1} \leq \|G^T_{\nu_i}(t_i) - G^T(t_i)\|. \tag{A.6}$$

Since the function $G^{\dagger T}(t)$ is continuous on the compact interval $\mathscr{I}$, its norm $\|G^{\dagger T}(t)\|$ is bounded on $\mathscr{I}$. Therefore, there exists $h > 0$ such that $\|G^{\dagger T}(t)\| \leq (1/h)$ for all $t \in \mathscr{I}$. Consequently,

$$h \leq \|G^{\dagger T}(t_i)\|^{-1} \overset{\text{(A.6)}}{\leq} \|G^T_{\nu_i}(t_i) - G^T(t_i)\|. \tag{A.7}$$

Thus, we proved that inequality (A.7) holds for every $i \in \mathbb{N}$. But this contradicts assumption (i), which implies that $G^T_{\nu}(t) \to G^T(t)$ uniformly on $\mathscr{I}$. ■

The result of the following lemma can be obtained from [5, Theorem 10.4.4].

**Lemma A.1.6.** *Let $E, G \in \mathbb{R}^{m \times n}$ with $R$ and $P$ being the orthogonal projector onto $\operatorname{Im} G$ and $\operatorname{Im} G^T$, respectively. Define the matrices $E_{ij}$ for $i, j \in \{1,2\}$ by*

$$E_{11} := REP, \quad E_{12} := RE(I-P), \quad E_{21} := (I-R)EP, \quad E_{12} := (I-R)E(I-P).$$

*Moreover, suppose that $\|G^{\dagger}\| \times \|E\| < 1$ and that $\operatorname{rank}(G+E) = \operatorname{rank} G$. Then*

$$\|(G+E)^{\dagger} - G^{\dagger}\| \leq \|G^{\dagger}\| \left( \beta_{11} + \gamma \sum_{(i,j) \neq (1,1)} \frac{\beta_{ij}^2}{1 + \beta_{ij}^2} \right)^{1/2}, \tag{A.8}$$

*where $\gamma := \left(1 - \|G^{\dagger}\| \times \|E_{11}\|\right)^{-1}$ and $\beta_{ij} := \gamma \|G^{\dagger}\| \times \|E_{ij}\|$.*

Based on the above two lemmas, we are now ready to establish Theorem A.1.4.

*Proof of Theorem A.1.4.* Assumptions (i)–(ii) in the theorem and the first condition in (A.2) imply, by Lemma A.1.5, that the equality in (A.4) holds for some $\mu_2 \in \mathbb{N}$. The same conclusion is valid also for the transposed matrices $G_\nu^T(t)$ and $G^T(t)$ with the aid of the second part of (A.2), i.e., there exists $\mu_3 \in \mathbb{N}$ such that

$$\operatorname{Im} G_\nu^T(t) = \operatorname{Im} G^T(t) \quad \text{for all } \nu \geq \mu_3 \text{ and } t \in \mathscr{I}. \tag{A.9}$$

Let $\mu_4 := \max\{\mu_2, \mu_3\}$. Then for all $\nu \geq \mu_4$ the equalities in (A.4) and (A.9) are satisfied. Denote by $E_\nu(t) := G_\nu(t) - G(t)$ on $\mathscr{I}$. According to Lemma A.1.6 in which $G := G(t)$ and $E := E_\nu$, we have $E_{11} = E_\nu(t)$ and $E_{12} = E_{21} = E_{22} = 0_{m\times n}$. Moreover, from (A.8) we then obtain the estimate

$$\|G_\nu^\dagger(t) - G^\dagger(t)\| \leq \|G^\dagger(t)\| \sqrt{\frac{\|E_\nu(t)\| \times \|G^\dagger(t)\|}{1 - \|G^\dagger(t)\| \times \|E_\nu(t)\|}} \tag{A.10}$$

for every $t \in \mathscr{I}$ satisfying $\|G^\dagger(t)\| \times \|E_\nu(t)\| < 1$. Since the function $G^\dagger(t)$ is continuous on the compact interval $\mathscr{I}$, its norm $\|G^\dagger(t)\|$ is bounded on $\mathscr{I}$. Hence, there exists $h_* > 0$ such that $\|G^\dagger(t)\| \leq h_*$ for all $t \in \mathscr{I}$. Furthermore, for sufficiently large $\nu$ we have $\|E_\nu(t)\| < 3/(4h_*)$ for all $t \in \mathscr{I}$. Therefore, $\|G^\dagger(t)\| \times \|E_\nu(t)\| < 3/4 < 1$ for all $t \in \mathscr{I}$. Using this in formula (A.10) we get for large $\nu$ the estimate

$$\|G_\nu^\dagger(t) - G^\dagger(t)\| \leq 2h_*^{3/2} \sqrt{\|E_\nu(t)\|} \quad \text{for all } t \in \mathscr{I}. \tag{A.11}$$

But since $\|E_\nu(t)\| \to 0$ for $\nu \to \infty$ uniformly on $\mathscr{I}$, we get from (A.11) that $\|G_\nu^\dagger(t) - G^\dagger(t)\| \to 0$ for $\nu \to \infty$ uniformly on $\mathscr{I}$. The proof of Theorem A.1.4 is complete. ∎

When each of the functions $G_\nu(t) \equiv G_\nu$ is constant on the interval $\mathscr{I}$, we obtain from Theorem A.1.4 the following statement. Note that the same conclusion regarding the convergence of $G_\nu^\dagger$ to $G^\dagger$ can be obtained also from [5, Theorem 10.4.1] via the result in Lemma A.1.5, which yields that $\operatorname{rank} G_\nu = \operatorname{rank} G$ for large $\nu$.

**Corollary A.1.7.** *Let $\{G_\nu\}_{\nu=1}^\infty$ be a sequence of matrices. Assume that*

*(i) $G_\nu \to G$ for $\nu \to \infty$,*

*(ii) there exists an index $\nu_0 \in \mathbb{N}$ such that $\operatorname{Im} G_\nu \subseteq \operatorname{Im} G$ and $\operatorname{Im} G_\nu^T \subseteq \operatorname{Im} G^T$ for all $\nu \geq \nu_0$.*

*Then $G_\nu^\dagger \to G^\dagger$ for $\nu \to \infty$.*

## A.2 Results about orthogonal projectors

In this section we state some needed results about orthogonal projectors. We also derive a representation of the set $\mathscr{B}(P_{**}, P_*, P)$ introduced in (1.11).

**Lemma A.2.1.** *Let $P_* \in \mathbb{R}^{n\times n}$ be an orthogonal projector and $p_* := \operatorname{rank} P_*$. Furthermore, let $V_* \in \mathbb{R}^{n\times n}$ be the corresponding orthogonal matrix from (1.8), i.e.*

$$P_* = V_* \operatorname{diag}\{I_{p_*}, 0_{n-p_*}\} V_*{}^T. \tag{A.12}$$

*Let $p \in \mathbb{N}$ satisfy $p_* \le p \le n$. Then a matrix $P \in \mathbb{R}^{n\times n}$ is an orthogonal projector with*

$$\operatorname{Im} P_* \subseteq \operatorname{Im} P \quad \text{and} \quad \operatorname{rank} P = p \tag{A.13}$$

*if and only if $P$ has the form*

$$P = V_* \operatorname{diag}\{I_{p_*}, R_*\} V_*{}^T, \tag{A.14}$$

*where $R_* \in \mathbb{R}^{(n-p_*)\times(n-p_*)}$ is an orthogonal projector with rank equal to $p - p_*$.*

*Proof.* It is easy to see that every matrix $P$ of the form (A.14) is symmetric and idempotent (i.e., it is an orthogonal projector) and (A.13) holds. Conversely, suppose that $P \in \mathbb{R}^{n\times n}$ is an orthogonal projector satisfying (A.13). Then we may write

$$P = V_* \begin{pmatrix} K_* & L_* \\ L_*^T & R_* \end{pmatrix} V_*{}^T, \tag{A.15}$$

where $K_* \in \mathbb{R}^{p_*\times p_*}$ is symmetric, $L_* \in \mathbb{R}^{p_*\times(n-p_*)}$, and $R_* \in \mathbb{R}^{(n-p_*)\times(n-p_*)}$ is symmetric. The first condition in (A.13) is equivalent with $PP_* = P_*$, from which we get by using the representations in (A.12) and (A.15) that $K_* = I_{p_*}$ and $L_* = 0_{p_*\times(n-p_*)}$. Thus, $P = V_* \operatorname{diag}\{I_{p_*}, R_*\} V_*{}^T$, where $\operatorname{rank} R_* = \operatorname{rank} P - p_* = p - p_*$ according to (A.13). Finally, the idempotence of $P$ now implies the idempotence of $R_*$, showing that $R_*$ is an orthogonal projector. ∎

In the following theorem we use the set $\mathscr{M}(P_*)$ defined in (1.9).

**Theorem A.2.2.** *Let $P_*, P, \tilde{P} \in \mathbb{R}^{n\times n}$ be orthogonal projectors satisfying*

$$\operatorname{Im} P_* \subseteq \operatorname{Im} P, \quad \operatorname{Im} P_* \subseteq \operatorname{Im} \tilde{P}, \quad \operatorname{rank} P = \operatorname{rank} \tilde{P}. \tag{A.16}$$

*Then there exists a matrix $E \in \mathscr{M}(P_*)$ such that* $\operatorname{Im} EP = \operatorname{Im} \tilde{P}$.

*Proof.* Let $p_* := \operatorname{rank} P_*$ and $p := \operatorname{rank} P = \operatorname{rank} \tilde{P}$. Then obviously $p \ge p_*$. Let $V_* \in \mathbb{R}^{n\times n}$ be the orthogonal matrix in (1.8) associated with projector $P_*$, that is, (A.12) holds. According to Lemma A.2.1 there exist orthogonal projectors $R_*, \tilde{R}_* \in \mathbb{R}^{(n-p_*)\times(n-p_*)}$ such that

$$P = V_* \operatorname{diag}\{I_{p_*}, R_*\} V_*{}^T, \quad \tilde{P} = V_* \operatorname{diag}\{I_{p_*}, \tilde{R}_*\} V_*{}^T, \tag{A.17}$$

and $\operatorname{rank} R_* = \operatorname{rank} \tilde{R}_* = p - p_*$. Let $Z_*, \tilde{Z}_* \in \mathbb{R}^{(n-p_*)\times(n-p_*)}$ be orthogonal matrices in (1.8) associated with the projectors $R_*$ and $\tilde{R}_*$, that is, $R_* = Z_* \operatorname{diag}\{I_{p-p_*}, 0_{n-p}\} Z_*^T$ and $\tilde{R}_* = \tilde{Z}_* \operatorname{diag}\{I_{p-p_*}, 0_{n-p}\} \tilde{Z}_*^T$. It follows that

$$\tilde{Z}_* Z_*^T R_* = \tilde{Z}_* \operatorname{diag}\{I_{p-p_*}, 0_{n-p}\} Z_*^T = \tilde{R}_* \tilde{Z}_* Z_*^T. \tag{A.18}$$

We set $E := V_* \operatorname{diag}\{I_{p_*}, \tilde{Z}_* Z_*^T\} V_*{}^T \in \mathbb{R}^{n\times n}$. Then $E$ is nonsingular and from (A.12) it follows that $EP_* = P_*$. Thus, $E \in \mathscr{M}(P_*)$. Finally, by (A.17) and (A.18) we obtain

$$EP = V_* \operatorname{diag}\{I_{p_*}, \tilde{Z}_* Z_*^T R_*\} V_*{}^T = V_* \operatorname{diag}\{I_{p_*}, \tilde{R}_* \tilde{Z}_* Z_*^T\} V_*{}^T = \tilde{P} E,$$

which shows that $\operatorname{Im} EP = \operatorname{Im} \tilde{P}E = \operatorname{Im} \tilde{P}$. The proof is complete. ∎

In the following (see Theorems A.2.5 and A.2.7) we provide a certain representation of the set $\mathscr{B}(P_{**},P_*,P)$ defined in (1.11), where $P_{**}$, $P_*$, and $P$ are $n\times n$ orthogonal projectors such that

$$\operatorname{Im} P_{**} \subseteq \operatorname{Im} P_* \subseteq \operatorname{Im} P. \tag{A.19}$$

First we consider the special case with $P=I$. In this case the elements of $\mathscr{B}(P_{**},P_*,I)$ are characterized by (1.12). For $\bar{G}\in\mathbb{R}^{n\times n}$ we consider the matrices $\bar{G}_\perp$ and $\bar{G}_{\shortparallel}$ defined by

$$\bar{G}_\perp := \bar{G}(I-P_*) \quad \text{and} \quad \bar{G}_{\shortparallel} := \bar{G}P_*. \tag{A.20}$$

Then $\bar{G}=\bar{G}_\perp+\bar{G}_{\shortparallel}$ holds. The next theorem provides a characterization of the elements $\bar{G}\in\mathscr{B}(P_{**},P_*,I)$ via the matrices $\bar{G}_\perp$ and $\bar{G}_{\shortparallel}$ associated with $\bar{G}$ through (A.20). For convenience we will use the notation

$$\bar{G}_\perp{}^T := (\bar{G}_\perp)^T, \quad \bar{G}_{\shortparallel}{}^T := (\bar{G}_{\shortparallel})^T, \quad \bar{G}_\perp{}^\dagger := (\bar{G}_\perp)^\dagger, \quad \bar{G}_\perp{}^{\dagger T} := (\bar{G}_\perp)^{\dagger T}.$$

**Theorem A.2.3.** *Let $P_{**}$ and $P_*$ be orthogonal projectors with $\operatorname{Im} P_{**}\subseteq \operatorname{Im} P_*$. Then $\bar{G}\in\mathscr{B}(P_{**},P_*,I)$ if and only if its corresponding matrices $\bar{G}_\perp$ and $\bar{G}_{\shortparallel}$ in (A.20) satisfy*

$$\operatorname{Im}\bar{G}_\perp = \operatorname{Im}(I-P_*), \quad P_{**}\bar{G}_{\shortparallel}=0, \quad P_*\bar{G}_{\shortparallel}=\bar{G}_{\shortparallel}{}^T P_*. \tag{A.21}$$

*Proof.* Let $\bar{G}\in\mathscr{B}(P_{**},P_*,I)$. First we show that $\operatorname{Im}\bar{G}_\perp=\operatorname{Im}(I-P_*)=\operatorname{Ker}P_*$. Since by (1.12) the matrix $P_*\bar{G}$ is symmetric, we have $P_*\bar{G}_\perp = P_*\bar{G}(I-P_*)=\bar{G}^T P_*(I-P_*)=0$. This shows that $\operatorname{Im}\bar{G}_\perp\subseteq\operatorname{Ker}P_*$. Moreover, $\operatorname{rank}(\bar{G}_\perp{}^T, P_*)=\operatorname{rank}((I-P_*)\bar{G}^T, P_*)=\operatorname{rank}(\bar{G}^T, P_*)=n$, by the last condition in (1.12). But the definition of $\bar{G}_\perp$ in (A.20) yields that $P_*\bar{G}_\perp{}^T=0$, so that $\operatorname{Im}\bar{G}_\perp{}^T\subseteq\operatorname{Ker}P_*$. It follows that $\operatorname{rank}\bar{G}_\perp=\operatorname{rank}\bar{G}_\perp{}^T=\operatorname{def}P_*$ and thus, $\operatorname{Im}\bar{G}_\perp=\operatorname{Im}(I-P_*)$. Furthermore, from (1.12) we get $P_{**}\bar{G}_{\shortparallel}=P_{**}\bar{G}P_*=0$. The last condition in (A.21) follows again from (1.12) and from the idempotence of $P_*$, because $P_*\bar{G}_{\shortparallel}=P_*\bar{G}P_*=P_*P_*\bar{G}P_*=P_*\bar{G}^TP_*P_*=\bar{G}_{\shortparallel}{}^T P_*$. Conversely, let $\bar{G}\in\mathbb{R}^{n\times n}$ be such that the corresponding matrices $\bar{G}_\perp$ and $\bar{G}_{\shortparallel}$ defined in (A.20) satisfy (A.21). We shall prove that (1.12) holds. The first equality in (1.12) follows from (A.21) and from the identity $P_{**}=P_{**}P_*$, since $P_{**}\bar{G}=P_{**}(\bar{G}_\perp+\bar{G}_{\shortparallel})=P_{**}P_*\bar{G}_\perp=0$. The symmetry of $P_*\bar{G}$ follows from $P_*\bar{G}=P_*(\bar{G}_\perp+\bar{G}_{\shortparallel})=P_*\bar{G}_{\shortparallel}$ and from the symmetry of $P_*\bar{G}_{\shortparallel}$. Finally, $P_*\bar{G}_\perp{}^T=0$ by (A.20) and $\operatorname{rank}\bar{G}_\perp{}^T=\operatorname{rank}\bar{G}_\perp=\operatorname{def}P_*$ by the first condition in (A.21). Hence, $\operatorname{rank}(\bar{G}^T, P_*)=\operatorname{rank}((I-P_*)\bar{G}^T, P_*)=\operatorname{rank}(\bar{G}_\perp{}^T, P_*)=n$, which completes the proof. ∎

**Remark A.2.4.** If $\bar{G}\in\mathscr{B}(P_{**},P_*,I)$, then the first conditions in (A.20) and (A.21) imply that

$$\bar{G}_\perp\bar{G}_\perp{}^\dagger = I-P_* = \bar{G}_\perp{}^\dagger\bar{G}_\perp. \tag{A.22}$$

In the next theorem we derive the form of the matrices $\bar{G}\in\mathscr{B}(P_{**},P_*,I)$. Here we use the notation $\mathscr{M}(P_*)$ and $\mathscr{A}(P_{**},P_*)$ from (1.9) and (1.10).

**Theorem A.2.5.** *Let $P_{**}$ and $P_*$ be orthogonal projectors satisfying $\operatorname{Im} P_{**}\subseteq\operatorname{Im} P_*$. Then $\bar{G}\in\mathscr{B}(P_{**},P_*,I)$ if and only if for some matrices $E\in\mathscr{M}(P_*)$ and $F\in\mathscr{A}(P_{**},P_*)$ we have*

$$\bar{G}=E^T(I-P_*)-FP_*. \tag{A.23}$$

*Proof.* Let $\bar{G} \in \mathscr{B}(P_{**}, P_*, I)$ and define the matrices $E, F \in \mathbb{R}^{n\times n}$ by $E := \bar{G}_\perp{}^T + P_*$ and $F := \bar{G}_\parallel{}^T P_* - \bar{G}_\parallel - \bar{G}_\parallel{}^T$, where $\bar{G}_\perp$ and $\bar{G}_\parallel$ are defined in (A.20). We will show that $E \in \mathscr{M}(P_*), F \in \mathscr{A}(P_{**}, P_*)$, and equality (A.23) holds. The first condition in (A.21) implies that $\bar{G}_\perp{}^T P_* = 0 = P_* \bar{G}_\perp{}^{\dagger T}$. By (A.22) we then obtain $E(\bar{G}_\perp{}^{\dagger T} + P_*) = \bar{G}_\perp{}^T \bar{G}_\perp{}^{\dagger T} + P_* = I$, which shows that the matrix $E$ is nonsingular with $E^{-1} = \bar{G}_\perp{}^{\dagger T} + P_*$. In addition, $EP_* = (\bar{G}_\perp{}^T + P_*)P_* = P_*$, so that $E \in \mathscr{M}(P_*)$ according to (1.9). The last equality in (A.21) implies that $F$ is symmetric, while from the second equality in (A.21) we get $P_{**}FP_* = P_{**}\bar{G}_\parallel{}^T P_* - P_{**}\bar{G}_\parallel P_* - P_{**}\bar{G}_\parallel{}^T P_* = 0$. Hence, $F \in \mathscr{A}(P_{**}, P_*)$ according to (1.10). Moreover, from (A.20) we obtain

$$E^T(I-P_*) - FP_* = (\bar{G}_\perp + P_*)(I-P_*) - (\bar{G}_\parallel{}^T P_* - \bar{G}_\parallel - \bar{G}_\parallel{}^T)P_* = \bar{G}(I-P_*) + \bar{G}P_* = \bar{G},$$

as we claim in (A.23). Conversely, let $E \in \mathscr{M}(P_*)$ and $F \in \mathscr{A}(P_{**}, P_*)$ be given and consider the matrices $\bar{G} := E^T(I-P_*) - FP_*$ and $\bar{G}_\perp, \bar{G}_\parallel$ from (A.20). It follows that $\bar{G}_\perp = E^T(I-P_*)$ and $\bar{G}_\parallel = -FP_*$. The nonsingularity of $E$ implies that $\operatorname{rank} \bar{G}_\perp = n - \operatorname{rank} P_* = \operatorname{def} P_*$, while $P_* \bar{G}_\perp = P_* E^T(I-P_*) = 0$ because $P_* E^T = (EP_*)^T = P_*$. Therefore, the first equality in (A.21) holds. The symmetry of $F$ implies the symmetry of $P_* \bar{G}_\parallel = -P_* FP_*$ and finally, $P_{**}\bar{G}_\parallel = -P_{**}FP_* = 0$. Therefore, the second and third conditions in (A.21) are satisfied as well. We conclude from Theorem A.2.3 that $\bar{G} \in \mathscr{B}(P_{**}, P_*, I)$. ■

**Remark A.2.6.** The representation of $\bar{G} \in \mathscr{B}(P_{**}, P_*, I)$ in (A.23) and the proof of Theorem A.2.5 allow to express the matrices $\bar{G}_\perp$, $\bar{G}_\perp{}^\dagger$, $\bar{G}_\parallel$ in terms of $E \in \mathscr{M}(P_*)$ and $F \in \mathscr{A}(P_{**}, P_*)$ as

$$\bar{G}_\perp = E^T(I-P_*), \quad \bar{G}_\perp{}^\dagger = E^{T-1}(I-P_*), \quad \bar{G}_\parallel = -FP_*. \tag{A.24}$$

In the last result of this section we provide a representation of the set $\mathscr{B}(P_{**}, P_*, P)$ in (1.11) in terms of the elements $\bar{G} \in \mathscr{B}(P_{**}, P_*, I)$.

**Theorem A.2.7.** *Let $P_{**}$, $P_*$, and $P$ be orthogonal projectors satisfying (A.19). Then $(G,H) \in \mathscr{B}(P_{**}, P_*, P)$ if and only if for some $\bar{G} \in \mathscr{B}(P_{**}, P_*, I)$ we have*

$$G = P\bar{G} \quad \text{and} \quad H = (I-P)\bar{G}. \tag{A.25}$$

*In this case, the matrix $\bar{G}$ is uniquely determined by $(G,H)$ with $\bar{G} = G + H$.*

*Proof.* Let $(G,H) \in \mathscr{B}(P_{**}, P_*, P)$ and set $\bar{G} := G + H$. Then clearly $G = P\bar{G}$ and $H = (I-P)\bar{G}$, by the third and fifth properties in (1.11). We show that the matrix $\bar{G}$ satisfies (1.12). Using the second and fifth equalities in (1.11) together with the identity $P_{**} = P_{**}P$ we get $P_{**}\bar{G} = P_{**}G + P_{**}H = P_{**}PH = 0$. Moreover, the identities $P_* = P_*P$ and $PH = 0$ and the symmetry of $P_* G$, i.e. the fourth equality in (1.11), imply the symmetry of $P_* \bar{G}$, because $P_* \bar{G} = P_* G + P_* H = P_* G + P_* PH = P_* G$. For the last equality in (1.12) it suffices to prove that $\operatorname{Ker} \bar{G} \cap \operatorname{Ker} P_* = \{0\}$. Let $v \in \mathbb{R}^n$ be such that $\bar{G}v = 0$ and $P_* v = 0$. Then also $Gv = P\bar{G}v = 0$ and $Hv = (I-P)\bar{G}v = 0$. Thus, $v \in \operatorname{Ker} G \cap \operatorname{Ker} H \cap \operatorname{Ker} P_*$ and consequently, $v = 0$ by the first equality in (1.11). Therefore, the matrix $\bar{G}$ belongs to the set $\mathscr{B}(P_{**}, P_*, I)$. Conversely, for any $\bar{G} \in \mathscr{B}(P_{**}, P_*, I)$ the pair $(G,H) := (P\bar{G}, (I-P)\bar{G})$ satisfies conditions (1.11), as can be directly verified by (1.12). Hence, $(G,H) \in \mathscr{B}(P_{**}, P_*, P)$. Finally, we have $G + H = \bar{G}$ and so the matrix $\bar{G}$ is unique. The proof is complete. ■

**Remark A.2.8.** (i) The characterization of the set $\mathscr{B}(P_{**},P_*,P)$ displayed in Theorem A.2.7 reveals some additional properties of its elements. Namely, for every $(G,H) \in \mathscr{B}(P_{**},P_*,P)$ the formulas

$$H\bar{G}_{\perp}{}^{\dagger} = I - P, \quad \text{and} \quad \operatorname{Im} H = \operatorname{Im}(I-P) \tag{A.26}$$

hold, where the matrix $\bar{G}_{\perp}{}^{\dagger} = (I-P_*)\,\bar{G}$ with $\bar{G} = G + H$ from Theorem A.2.7. Indeed, the representation of the matrix $H$ in (A.25) and the first equality in (A.21) imply that $H\bar{G}_{\perp}{}^{\dagger} = (I-P)\,\bar{G}\bar{G}_{\perp}{}^{\dagger} = (I-P)\,\bar{G}(I-P_*)\,\bar{G}_{\perp}{}^{\dagger} = (I-P)\,\bar{G}_{\perp}\,\bar{G}_{\perp}{}^{\dagger}$. Consequently, by using (A.22) and the identity $PP_* = P_*$ we obtain $H\bar{G}_{\perp}{}^{\dagger} = (I-P)\,(I-P_*) = I-P$, showing the first equality in (A.26). In turn, we have the inclusion $\operatorname{Im}(I-P) \subseteq \operatorname{Im} H$, which immediately yields the second equality in (A.26), since the opposite inclusion $\operatorname{Im} H \subseteq \operatorname{Im}(I-P)$ follows from (A.25).

(ii) Combining the results of Theorems A.2.5 and A.2.7 we may conclude that $(G,H) \in \mathscr{B}(P_{**},P_*,P)$ if and only if for some $E \in \mathscr{M}(P_*)$ and $F \in \mathscr{A}(P_{**},P_*)$

$$G = PE^T(I-P_*) - PFP_*, \quad H = (I-P)\,E^T(I-P_*) - (I-P)\,FP_*. \tag{A.27}$$

We note that for a given pair $(G,H) \in \mathscr{B}(P_{**},P_*,P)$ the matrices $E$ and $F$ in representation (A.27) are not uniquely determined.

# Appendix B

# Author's publications and curriculum vitae

## B.1 Current list of author's publications

The publications are completed with the corresponding impact factor (IF).

**Published**

- P. Šepitka, R. Šimon Hilscher, Minimal principal solution at infinity for nonoscillatory linear Hamiltonian systems, *J. Dynam. Differential Equations* **26** (2014), no. 1, 57–91. IF 2013: 1.0.

**Accepted**

- P. Šepitka, R. Šimon Hilscher, Principal solution at infinity of given ranks for nonoscillatory linear Hamiltonian systems, *J. Dynam. Differential Equations*, to appear (2014). DOI: 10.1007/s10884-014-9389-7. IF 2013: 1.0.

**Submitted**

- P. Šepitka, R. Šimon Hilscher, Principal and antiprincipal solutions at infinity of linear Hamiltonian systems, submitted (August 2014).

- P. Šepitka, R. Šimon Hilscher, Recessive solutions for nonoscillatory discrete symplectic systems, submitted (September 2014).

## B.2 Author's curriculum vitae

**Personal data:** Peter Šepitka, born on April 3, 1986, in Martin (Slovakia).

**University education:**

*Since 2010* – Ph.D. student of the degree program "Mathematics" with the field of the study "Mathematical analysis" (Faculty of Science, Masaryk University),

*June 2, 2010* – MSc. in the study program "Applied Mathematics" with the field of the study "Applied Mathematics" (Faculty of Science, University of Žilina) and with the master thesis "Dynamic Equations on Time Scales" (in Slovak, supervisor prof. RNDr. Josef Diblík, DrSc.),

*June 10, 2008* – BSc. in the study program "Applied Mathematics" with the field of the study "Applied Mathematics" (Faculty of Science, University of Žilina) and with the bachelor thesis "The Quadratic Reciprocity Law" (in Slovak, supervisor doc. RNDr. Ľudovít Tománek, CSc.).

**Pedagogical activities:** Seminars at Faculty of Science and Faculty of Informatics, Masaryk University, with

- mathematical analysis in $\mathbb{R}$,
- mathematical analysis in $\mathbb{C}$,
- differential equations.

**Academical stays:** March–May 2013, University Ulm (Ulm, Germany).

**Seminar talks:** The following 60-minute seminar talks were given.

- Seminar on differential equations, Masaryk University (Brno, Czech Republic, October 13, 2014), the title of the talk *Principal and Antiprincipal Solutions of Linear Hamiltonian Systems*,
- "Forschungsseminar", University Ulm (Ulm, Germany, May 8, 2013), the title of the talk *Principal Solutions of Nonoscillatory Self-adjoint Linear Differential Systems*.

**International conferences:** The following 20-minute conference talks were given.

- Symposium on Differential and Difference Equations 2014 (Homburg/Saar, Germany, September 5–8, 2014), the title of the talk *Principal and Antiprincipal Solutions at Infinity of Linear Hamiltonian Systems*,
- Conference on Differential and Difference Equations and Applications 2014 (Jasná, Slovakia, June 23–27, 2014), the title of the talk *Principal Solutions at Infinity of Linear Hamiltonian Systems*,
- Conference on Differential and Difference Equations and Applications 2012 (Terchová, Slovakia, June 25–29, 2012), the title of the talk *Principal Solutions of Nonoscillatory Self-adjoint Linear Differential Systems*.

**Appreciation of Science Community:** The Prize of the Dean of the Faculty of Science (Masaryk University) for excellent research results (in 2014),

The Prize of the Rector of University of Žilina for excellent study results (in 2010),

The Prize of the Dean of the Faculty of Science (University of Žilina) for excellent study results (in 2008).

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