MASARYK UNIVERSITY
FACULTY OF SCIENCE
Department of mathematics and statistics

New Results in Theory of Symplectic Systems on Time Scales

Doctoral Dissertation

Petr Zemánek

$$\begin{aligned} x^{\Delta} &= \mathcal{A}(t)\,x + \mathcal{B}(t)\,u \\ u^{\Delta} &= \mathcal{C}(t)\,x + \mathcal{D}(t)\,u \end{aligned}$$

$$\begin{aligned} x' &= A(t)\,x + B(t)\,u \\ u' &= C(t)\,x - A^{*}(t)\,u \end{aligned} \qquad\qquad \begin{aligned} x_{k+1} &= \mathcal{A}_k x_k + \mathcal{B}_k u_k \\ u_{k+1} &= \mathcal{C}_k x_k + \mathcal{D}_k u_k \end{aligned}$$

$$\sum_{j=0}^{n} \left(-\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{j} \left\{ r_j(t)\, x^{(j)}(t) \right\} = 0 \qquad\qquad \sum_{j=0}^{n} (-\Delta)^{j} \left\{ r_k^{[j]}\, \Delta^{j} x_{k+1-j} \right\} = 0$$

Brno 2011

New Results in Theory of Symplectic Systems on Time Scales

Doctoral Dissertation

Petr Zemánek

Masaryk University
Brno 2011

The research was supported by Masaryk University under grant MUNI/A/0964/2009 and by the Czech Science Foundation under grants P201/10/1032, 201/09/J009.

The dissertation is available at the Office for Research, Development, International Relations, and Doctoral Study, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic, or see http://is.muni.cz/th/78442/prif_d/.

ISBN 978-80-210-5515-5

Bibliographic entry

Author: MSc. Petr Zemánek
Department of Mathematics and Statistics
Faculty of Science, Masaryk University

Title of dissertation: New Results in Theory of Symplectic Systems on Time Scales

Degree program: Mathematics

Field of study: Mathematical Analysis

Advisor: Assoc. Prof. RNDr. Roman Šimon Hilscher, DSc.
Department of Mathematics and Statistics
Faculty of Science, Masaryk University

Opponent #1: Assoc. Prof. MSc. Pavel Řehák, Ph.D.
Institute of Mathematics
Academy of Sciences of the Czech Republic

Opponent #2: ao. Univ.-Prof. Dipl.-Ing. Dr. Gerald Teschl
Faculty of Mathematics
The University of Vienna

Year of defense: 2011

Keywords: Time scale; Symplectic system; Sturm–Liouville equation; Trigonometric system; Sine function; Cosine function; Tangent function; Cotangent function; Hyperbolic system; Hyperbolic sine function; Hyperbolic cosine function; Hyperbolic tangent function; Hyperbolic cotangent function; Weyl–Titchmarsh theory; $M(\lambda)$-function; Weyl disk; Weyl circle; Limit circle case; Limit point case; Green function; Krein–von Neumann extension; Friedrichs extension; Critical, subcritical, and supercritical operators

Acknowledgement

I would like to express gratitude and appreciation to my advisor, Assoc. Prof. Roman Šimon Hilscher, for his guidance, patience, and encouragement. He learned me a lot of my current mathematical knowledge over the course of very fruitful discussions and enabled me to meet many mathematicians from the whole world. I am grateful to Prof. Ondřej Došlý for his support through my doctoral study starting from his report on my diploma thesis, which motivated my first paper. My special thanks belong to Prof. Werner Kratz for very interesting discussions and for the opportunity to speak about my research on their seminar and to the Department of Applied Analysis at the University of Ulm for a very pleasant environment during my stay (my first stay abroad ever). I am also indebted to Prof. Stephen Clark for helpful discussions and guidance provided while visiting the Missouri University of Science and Technology, to the Department of Mathematics and Statistics for hosting my visit, and to Prof. Martin Bohner for his questions inspiring the second half of Section 3.4.

Finally, I dedicate this work (with many words of thanks) to my family for everything, especially for the constant support during the years of my study, and to Péťa with

$$r(\theta) = \frac{\pi}{2} - 2\sin\theta + \frac{\sin\theta\sqrt{|\cos\theta|}}{\sin\theta + 1.55}, \quad \theta \in [0, 2\pi]\,.$$

Brno, June 2011 Petr Zemánek

Abstract

In this dissertation we present new results in the theory of symplectic systems on time scales (also symplectic dynamic systems) obtained and published by the author (jointly with collaborators) during his doctoral study between the years 2007 and 2011.

The dissertation is organized into five chapters. The study of symplectic systems is motivated in the introductory chapter, where an overview of the new results contained in the text is also given. In the second chapter, the reader will find fundamental parts of the time scale calculus indispensable for the understanding of the subsequent chapters.

The main body of the text is represented by the following chapters. In Chapter 3, we define trigonometric and hyperbolic systems on time scales and study their properties. Solutions of these systems generalize the well known trigonometric functions sine, cosine, tangent, cotangent, and their hyperbolic analogies. They also satisfy formulas generalizing some of the known trigonometric and hyperbolic identities from the scalar continuous case (e.g., Pythagorean trigonometric identity, double angle, product-to-sum, and sum-to-product formulas). In the following Chapter 4, the Weyl–Titchmarsh theory for symplectic dynamic systems is established. We generalize results for linear Hamiltonian differential systems obtained particularly during the second half of the 20th century. The theory given in both of these chapters is new even for symplectic difference systems, which are a special case of the symplectic systems on time scales. In the final chapter, we pay our attention to the most special case of the symplectic systems on time scales, namely to the Sturm–Liouville dynamic equations of the second order. For operators associated with these equations we characterize the domains of their Krein–von Neumann and Friedrichs extensions and also introduce the concept of the critical, subcritical, and supercritical operators. Some results obtained in Chapter 4 are also new in this special case, therefore the most important results of the Weyl–Titchmarsh theory for the second order Sturm–Liouville dynamic equations are given in the last part of this chapter.

For completeness, this dissertation is finished with a sketch of a further research in the presented theory, author's current list of publications, and his curriculum vitae.

2010 Mathematics Subject Classification:	*Primary 34N05; 34B20; 34B24.* *Secondary 26E70; 39A12; 34C99; 34B27; 47B25.*

Table of Contents

We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations.

Richard P. Feynman, see [70, Chapter 17]

List of Notation

For reader's convenience, in the following table we present a list of symbols (followed by an explanation of their meaning) appearing in this dissertation.

$\mathbb{C}$	the set of all complex numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{Z}$	the set of all integers
$\mathbb{N}$	the set of all natural numbers including 0
$\mathbb{T}$	a time scale
$[a,b]$	an interval of real numbers
$[a,b]_{\mathbb{Z}}$	a discrete interval
$[a,b]_{\mathbb{T}}$	a bounded time scale interval
$[a,\infty)_{\mathbb{T}}$	a time scale interval, which is unbounded above
$(-\infty,\infty)_{\mathbb{T}}$	an unbounded time scale
$\mathbb{R}^{n\times n}$	the set of all real $n \times n$ matrices
$\mathbb{C}^{n\times n}$	the set of all complex $n \times n$ matrices
$\mathcal{I}$	the identity matrix or operator of an appropriate dimension
$\mathcal{J}$	the matrix $\begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix}$
0	the zero matrix of an appropriate dimension
M	an $n \times n$ matrix M
M^T	the transpose of the matrix M
M^*	the conjugate transpose of the matrix M
M^{-1}	the inverse matrix of the square matrix M
$M^{*-1} = M^{-1*}$	the matrix $[M^*]^{-1} = \big[M^{-1}\big]^*$
$M^*(\cdot)$	the value $[M(\cdot)]^*$
$M^{-1}(\cdot)$	the value $[M(\cdot)]^{-1}$
$M > 0$	positive definiteness of the matrix M

$M \geq 0$	positive semidefiniteness of the matrix M
$M < 0$	negative definiteness of the matrix M
$M \leq 0$	negative semidefiniteness of the matrix M
$\operatorname{rank} M$	the rank of the matrix M
$\operatorname{Ker} M$	the kernel of the matrix M
$\operatorname{Im} M$	the image of the matrix M
$\operatorname{def} M$	the defect (i.e., the dimension of the kernel) of the matrix M
$\operatorname{Re}(M)$	the Hermitian component of the matrix M, i.e., $(M + M^*)/2$
$\operatorname{Im}(M)$	the Hermitian component of the matrix M, i.e., $(M - M^*)/(2i)$
$\overline{\lambda}$	the complex conjugate of the number λ
$\operatorname{Re}(\lambda)$	the real part of the number λ
$\operatorname{Im}(\lambda)$	the imaginary part of the number λ
$\delta(\lambda)$	the value $\operatorname{sgn}(\operatorname{Im} \lambda)$
Δy_k	the forward difference operator, i.e., the value $y_{k+1} - y_k$
$\sigma(\cdot)$	the forward jump operator on $\mathbb{T}$
$\rho(\cdot)$	the backward jump operator on $\mathbb{T}$
$\mu(\cdot)$ and $\nu(\cdot)$	the graininess functions on $\mathbb{T}$
$f^{\sigma}(t)$	the value $f(\sigma(t))$
$f^{\rho}(t)$	the value $f(\rho(t))$
$f^{\Delta}(t)$	the Δ-derivative of the function f at the point t
$f^{\nabla}(t)$	the ∇-derivative of the function f at the point t
$f^*(\cdot)$	the conjugate transpose of the function $f(\cdot)$
$f^{*\sigma}(t) = f^{\sigma *}(t)$	the value $[f^*(t)]^{\sigma} = [f^{\sigma}(t)]^*$
$f^{*\Delta}(t) = f^{\Delta *}(t)$	the value $[f^*(t)]^{\Delta} = \left[f^{\Delta}(t)\right]^*$
$f(t^{\pm})$	the right/left-hand limit of the function f at the point t
$[f(t)]_a^b$	the value $f(b) - f(a)$
$\mathrm{C_{rd}}$	the set of all rd-continuous functions
$\mathrm{C_{prd}}$	the set of all piecewise rd-continuous functions
$\mathrm{C_{rd}^1}$	the set of all rd-continuously Δ-differentiable functions
$\mathrm{C_{prd}^1}$	the set of all piecewise rd-continuously Δ-differentiable functions
$\mathrm{C_{ld}}$	the set of all ld-continuous functions
$\mathrm{C_{pld}}$	the set of all piecewise ld-continuous functions
$\mathrm{C_{ld}^1}$	the set of all ld-continuously ∇-differentiable functions
$\mathrm{C_{pld}^1}$	the set of all piecewise ld-continuously ∇-differentiable functions
(X.Y)(ii)	we refer to the second identity in (X.Y)

The approach turns out to be fruitful and successful, and leads to the effective construction as well as the theoretical understanding of an abundance of what we call symplectic difference scheme, or symplectic algorithms, or simply Hamiltonian algorithms, since they present the proper way, i.e., the Hamiltonian way for computing Hamiltonian dynamics.

KANG FENG, SEE [69]

Chapter 1

INTRODUCTION

Discrete symplectic systems

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in I \subseteq \mathbb{N}, \tag{1.1}$$

where the coefficient matrix

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix} \text{ is } \textit{symplectic}\text{, i.e.,} \quad \mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{for all } k \in I \text{ and } \quad \mathcal{J} := \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix},$$

were initiated as the proper discrete analogy (because systems (1.1) and (1.2) below have symplectic transition matrices) of *linear Hamiltonian differential systems*

$$x'(t) = A(t)\, x(t) + B(t)\, u(t), \quad u'(t) = C(t)\, x(t) - A^*(t)\, u(t), \quad t \in I \subseteq \mathbb{R}, \tag{1.2}$$

where $B(t)$ and $C(t)$ are Hermitian matrices for all $t \in I$.

Unfortunately, the terminology "symplectic" and "Hamiltonian" can be for the reader confusing because there were also introduced *discrete linear Hamiltonian systems* as

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^* u_k, \quad k \in I \subseteq \mathbb{N} \tag{1.3}$$

with Hermitian matrices B_k and C_k and the invertible matrix $\mathcal{I} - A_k$ for all $k \in I$ in [63,64]. Nevertheless, if we rewrite system (1.3) into the form

$$\begin{aligned} x_{k+1} &= (I - A_k)^{-1} x_k + (I - A_k)^{-1} B_k u_k, \\ u_{k+1} &= C_k (I - A_k)^{-1} x_k + \left[I - A_k^T + C_k (I - A_k)^{-1} B_k \right] u_k, \end{aligned}$$

we obtain a symplectic system, see [3, Theorem 3].

In the unifying theory for differential and difference equations – the theory of time scales – the theory of *symplectic systems on time scales*, i.e.,

$$x^\Delta(t) = \mathcal{A}(t)\, x(t) + \mathcal{B}(t)\, u(t), \quad u^\Delta(t) = \mathcal{C}(t)\, x(t) + \mathcal{D}(t)\, u(t), \quad t \in \mathbb{T}, \tag{1.4}$$

originated in [58]. These systems generalize and unify a large spectrum of differential and difference equations and systems, in particular any even order Sturm–Liouville differential and difference equations, systems (1.2), (1.1), and consequently (1.3). Let us note that, in analogy with the discrete case, dynamic systems in the form

$$x^{\Delta}(t) = A(t)\,x^{\sigma}(t) + B(t)\,u(t), \quad u^{\Delta}(t) = C(t)\,x^{\sigma}(t) - A^{*}(t)\,u(t), \quad t \in \mathbb{T}, \tag{1.5}$$

where the matrices $B(t)$ and $C(t)$ are Hermitian and $\mathcal{I}-\mu(t)\,A(t)$ is invertible on $\mathbb{T}$, are also studied in the literature starting in [25,88–90]. Such systems are called *linear Hamiltonian dynamic systems* and were developed as the dynamic analogy of (1.3). Similarly to the discrete case, it can be shown that (1.5) is a special case of symplectic system (1.4).

In recent years, an increasing attention has been paid for the development of the theory for symplectic systems on time scales. In this dissertation we present new contributions to this theory. The text consists of five chapters (including this chapter) which are organized as follows. In the next chapter we recall fundamental notions and necessary parts from the time scale theory. In Chapter 3 we introduce and study the trigonometric and hyperbolic systems on time scales and in Chapter 4 we establish the Weyl–Titchmarsh theory for symplectic systems on time scales. Moreover, the results presented in both of these chapters are not only a unification of the discrete and continuous theory, but they are new even in the discrete case. Finally, new results for the Sturm–Liouville dynamic equations of the second order are given in Chapter 5. We characterize the domains of the Krein–von Neumann and Friedrichs extensions and introduce the concept of critical operators on time scales. We also show the main parts of the Weyl–Titchmarsh theory for these equations.

The motivation for the study of the topics presented in this dissertation and their connection with the current literature are given in the introductory part of each of the chapters.

1.1 Overview of author's new results

This dissertation comprises of results which the author achieved as the PhD student (jointly with his collaborators) in the years 2007–2011. More specifically, his new results are the following:

- the qualitative theory of discrete trigonometric and hyperbolic systems, see [163], and of trigonometric and hyperbolic systems on time scales (jointly with R. Šimon Hilscher), see [100] and Chapter 3,
- the Weyl–Titchmarsh theory for discrete symplectic systems with a spectral parameter appearing in the second equation (jointly with S. L. Clark), see [45], and for symplectic systems on time scales (jointly with R. Šimon Hilscher), see [145] and Chapter 4,
- the characterization of the domains of the Krein–von Neumann and Friedrichs extensions for second order Sturm–Liouville dynamic equations, see [164] and Section 5.1,
- the critical, subcritical, and supercritical operators of the second order Sturm–Liouville equations on time scales (jointly with P. Hasil), see [83] and Section 5.2.

Barring the results mentioned above, the author published (jointly with R. Šimon Hilscher) also a survey paper concerning the definiteness of the quadratic functionals associated with symplectic systems, and a paper with a characterization of the Friedrichs extension for the operators associated with the linear Hamiltonian differential systems, see [A3, A4] on page 96.

A major task of mathematics to-day is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

Elaine T. Bell, see [19, p. 13]

Chapter 2

Time scale theory

The time scale calculus was established in Hilger's doctoral dissertation [85] and published (first time in English) in his paper [86]. His work dealt with the so-called measure chains, which are ordered topological objects equipped with a measure. However, with respect to [86, Theorem 2.1] any measure chain is isomorphic to some nonempty closed subset of $\mathbb{R}$, i.e., to a *time scale*, which is therefore the most illustrative and most appropriate form of measure chains, see also [17, p. 241]. Fundamental results of the time scale theory are presented in the following sections.

This theory unifies particularly the continuous and discrete calculi but also the quantum calculus (q-calculus), the calculus on the Cantor set, and (generally) a calculus on a set represented by a union of disjoint closed intervals. Consequently, it provides suitable tools for a study of differential, difference, and (generally) dynamic equations and their systems under the unified framework. Exempli gratia, the coexistence of a union of closed continuous intervals appears in hybrid dynamic systems (with applications in engineering, see [78] and the references therein) or in impulsive differential equations (developed in modeling impulsive problems, e.g., in physics, population dynamics, biotechnology, pharmacokinetics, and industrial robotics, see [21,118]). Some applications of the time scale calculus can also be found in economics, see, e.g., [13,15,29,152]. Moreover, the study of the time scale theory can motivate (and really motivates) results being new even in special cases of time scales (in particular in the continuous and discrete cases), see, e.g., [91,95,96].

2.1 Basic notation

By definition, a *time scale* $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. A bounded time scale $\mathbb{T}$ can be identified with the time scale interval $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$, where $a := \min \mathbb{T}$, $b := \max \mathbb{T}$, and $[a,b]$ is the usual interval of real numbers. A time scale unbounded above and below can be written as $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$ and $(-\infty,b]_{\mathbb{T}} := (-\infty,b] \cap \mathbb{T}$, respectively, and an unbounded time scale is denoted by $(-\infty,\infty)_{\mathbb{T}} := \mathbb{R} \cap \mathbb{T}$. Similarly, we use the notation $[a,b]_{\mathbb{Z}}$ for a discrete interval, where $a,b \in \mathbb{Z}$, i.e., $[a,b]_{\mathbb{Z}} :=$

$[a, b] \cap \mathbb{Z}$. Open and half-open time scale intervals are defined accordingly.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$$

(and simultaneously we put $\inf \emptyset := \sup \mathbb{T}$). The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}$$

(simultaneously we put $\sup \emptyset := \inf \mathbb{T}$).

Let $t \in \mathbb{T}$. A point $t > \inf \mathbb{T}$ is said to be *left-dense* and *left-scattered* if $\rho(t) = t$ and $\rho(t) < t$, respectively, while a point $t < \sup \mathbb{T}$ is said to be *right-dense* and *right-scattered* if $\sigma(t) = t$ and $\sigma(t) > t$, respectively, see also Figure 2.1. In addition, if a is a minimum of $\mathbb{T}$, then $\rho(a) = a$, and if b is a maximum of $\mathbb{T}$, then $\sigma(b) = b$. The point t is called *isolated* if it is right-scattered and left-scattered at the same time, and it is called simply *dense* if it is either right-dense or left-dense (compare to [32, p. 2] and [33, p. 2]). The *forward graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$ and the *backward graininess* function $\nu : \mathbb{T} \to [0, \infty)$ by $\nu(t) := t - \rho(t)$.

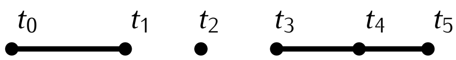

Figure 2.1: Illustration of time scale points.

2.2 Time scale derivative

For a better arrangement, we introduce for any time scale $\mathbb{T}$ the following notation

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus \{b\}, & \text{if the point } b \text{ is a left-scattered maximum of } \mathbb{T}, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

For a function $f : \mathbb{T} \to \mathbb{C}$ it is possible to define the Δ-*derivative* of f at $t \in \mathbb{T}^\kappa$ (denoted by $f^\Delta(t)$) in the following way

$$f^\Delta(t) := \begin{cases} \lim_{s \to t} \frac{f(s) - f(t)}{s - t}, & \text{if } \mu(t) = 0, \\ \frac{f^\sigma(t) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0. \end{cases} \tag{2.1}$$

Let us note that the value $f^\Delta(b)$ is not well defined if $b = \max \mathbb{T}$ exists and is left-scattered. The usual differential rules take the form

$$(f \pm g)^\Delta(t) = f^\Delta(t) \pm g^\Delta(t), \tag{2.2}$$

$$(fg)^\Delta(t) = f^\Delta(t)\, g(t) + f^\sigma(t)\, g^\Delta(t) = f^\Delta(t)\, g^\sigma(t) + f(t)\, g^\Delta(t). \tag{2.3}$$

We say that a function $f(t)$ is Δ-differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The special cases of the Δ-derivative for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are presented in Remark 2.2 below.

A function $f(t)$ is said to be *regressive* on an interval $I \subseteq \mathbb{T}^\kappa$ if

$$1 + \mu(t)\, f(t) \neq 0 \quad \text{for all } t \in I,$$

and an $n \times n$ matrix-valued function $A : \mathbb{T} \to \mathbb{C}^{n\times n}$ is called *regressive* on $I \subseteq \mathbb{T}^\kappa$ if

$$\mathcal{I} + \mu(t)\, A(t) \text{ is invertible for all } t \in I,$$

where $\mathcal{I}$ denotes an appropriate identity matrix. Analogously, we can also define ν-regressive scalar and matrix-valued functions. If an $n \times n$ matrix-valued function A is Δ-differentiable and such that AA^σ is invertible, then the differentiation of the identity $AA^{-1} = \mathcal{I}$ yields

$$(A^{-1})^\Delta = -(A^\sigma)^{-1} A^\Delta\, A^{-1} = -A^{-1} A^\Delta (A^\sigma)^{-1}. \tag{2.4}$$

A function $f : [a,b]_\mathbb{T} \to \mathbb{C}^{n\times n}$ is called *regulated* provided its right-hand limit $f(t^+)$ exists (finite) at all right-dense points $t \in [a,b]_\mathbb{T}$ and the left-hand limit $f(t^-)$ exists (finite) at all left-dense points $t \in [a,b]_\mathbb{T}$. A function f is called *rd-continuous* (we write $f \in \mathrm{C}_{\mathrm{rd}}$) on $[a,b]_\mathbb{T}$ if it is regulated and if it is continuous at each right-dense point $t \in [a,b)_\mathbb{T}$. A function f is said to be *piecewise rd-continuous* ($f \in \mathrm{C}_{\mathrm{prd}}$) on $[a,b]_\mathbb{T}$ if it is regulated and if f is rd-continuous at all but possibly finitely many right-dense points $t \in [a,b)_\mathbb{T}$. A function f is said to be *rd-continuously Δ-differentiable* ($f \in \mathrm{C}^1_{\mathrm{rd}}$) on $[a,b]_\mathbb{T}$ if f^Δ exists for all $t \in [a,\rho(b)]_\mathbb{T}$ and $f^\Delta \in \mathrm{C}_{\mathrm{rd}}$ on $[a,\rho(b)]_\mathbb{T}$. A function f is said to be *piecewise rd-continuously Δ-differentiable* ($f \in \mathrm{C}^1_{\mathrm{prd}}$) on $[a,b]_\mathbb{T}$ if f is continuous on $[a,b]_\mathbb{T}$ and $f^\Delta(t)$ exists at all except of possibly finitely many points $t \in [a,\rho(b)]_\mathbb{T}$, and $f^\Delta \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_\mathbb{T}$. As a consequence we have that the finitely many points t_i, at which $f^\Delta(t_i)$ does not exist, belong to $(a,b)_\mathbb{T}$ and these points t_i are necessarily right-dense and left-dense at the same time. Also, since we know that $f^\Delta(t_i^+)$ and $f^\Delta(t_i^-)$ exist finite at those points, we replace the quantity $f^\Delta(t_i)$ by $f^\Delta(t_i^\pm)$ in any formula involving $f^\Delta(t)$ for all $t \in [a,\rho(b)]_\mathbb{T}$.

The introduced notation is possible to extend for an unbounded time scale $[a,\infty)_\mathbb{T}$, if the conditions are satisfied on $[a,b]_\mathbb{T}$ for every $b \in (a,\infty)_\mathbb{T}$. It is known that a composition of a continuous function f with an rd-continuous (or piecewise rd-continuous) function, is an rd-continuous (or piecewise rd-continuous) function. We note that if $f^\Delta(t)$ exists, then

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t). \tag{2.5}$$

Remark 2.1. For a fixed $t_0 \in [a,b]_\mathbb{T}$ and an $n \times n$ matrix-valued function $A \in \mathrm{C}_{\mathrm{prd}}$ on $[a,b]_\mathbb{T}$ which is regressive on $[a,t_0)_\mathbb{T}$, the initial value problem

$$y^\Delta = A(t)\, y \quad \text{and} \quad y(t_0) = y_0 \text{ for } t \in \mathbb{T}^\kappa$$

has a unique solution $y \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_\mathbb{T}$ for any $y_0 \in \mathbb{C}^n$. Similarly, this result holds on $[a,\infty)_\mathbb{T}$.

If not specified otherwise, we use a common agreement that vector-valued solutions of a system of dynamic equations and matrix-valued solutions of a system of dynamic equations are denoted by small letters and capital letters, respectively, typically by $z(\cdot)$ or $\tilde{z}(\cdot)$ and $Z(\cdot)$ or $\widetilde{Z}(\cdot)$, respectively.

2.3 Nabla calculus on time scales

It was shown in [39] that statements known in delta calculus can be equivalently formulated for nabla calculus on time scales and vice versa via the so-called *duality principle*. Hence, in this section we present fundamental parts of the nabla calculus in analogy of the corresponding results presented in the previous sections for the delta calculus.

For brevity, we define for any time scale $\mathbb{T}$ the set

$$\mathbb{T}_\kappa := \begin{cases} \mathbb{T} \setminus \{a\}, & \text{if the point } a \text{ is a right-scattered minimum of } \mathbb{T}, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

For a function $f : \mathbb{T} \to \mathbb{C}$ we introduce the ∇-*derivative* of f at $t \in \mathbb{T}_\kappa$ (denoted by $f^\nabla(t)$) as

$$f^\nabla(t) := \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t}, & \text{if } \nu(t) = 0, \\ \frac{f(t)-f^\rho(t)}{\nu(t)}, & \text{if } \nu(t) > 0. \end{cases} \tag{2.6}$$

Analogously, we note that the value $f^\nabla(a)$ is not well defined if $a = \min \mathbb{T}$ exists and is right-scattered. The fundamental differential rules for nabla calculus take the form

$$(f \pm g)^\nabla(t) = f^\nabla(t) \pm g^\nabla(t), \tag{2.7}$$

$$(fg)^\nabla(t) = f^\nabla(t)\, g(t) + f^\rho(t)\, g^\nabla(t) = f^\nabla(t)\, g^\rho(t) + f(t)\, g^\nabla(t). \tag{2.8}$$

We say that a function f is ∇-differentiable on $\mathbb{T}_\kappa$, if $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Remark 2.2. One can easily see that for $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = t = \rho(t),\ \mu(t) = \nu(t) \equiv 0,\ \text{and } f^\Delta(t) = f^\nabla(t) = f'(t).$$

On the other hand, for $\mathbb{T} = \mathbb{Z}$ the relations

$$\sigma(t) = t + 1,\ \rho(t) = t - 1,\ \mu(t) = \nu(t) \equiv 1,\ f^\Delta = f(t+1) - f(t),\ \text{and } f^\nabla = f(t) - f(t-1)$$

hold true.

With respect to the definitions in the delta calculus, we can introduce the sets of *ld-continuous*, *piecewise ld-continuous*, *ld-continuously ∇-differentiable*, and *piecewise ld-continuously ∇-differentiable* functions on $[a, b]_\mathbb{T}$ and write $f \in \mathrm{C}_{\mathrm{ld}}$, $f \in \mathrm{C}_{\mathrm{pld}}$, $f \in \mathrm{C}^1_{\mathrm{ld}}$, and $f \in \mathrm{C}^1_{\mathrm{pld}}$, respectively, on bounded or unbounded time scales. We note that if $f^\nabla(t)$ exists, then

$$f^\rho(t) = f(t) - \nu(t) f^\nabla(t). \tag{2.9}$$

The following identities show the possibility how to interchange the ∇- and Δ-derivatives. If $f \in \mathrm{C}^1_{\mathrm{prd}}$ on $\mathbb{T}^\kappa$, then the function f is also ∇-differentiable on $\mathbb{T}_\kappa$ and it holds

$$f^\nabla(t) = \begin{cases} \lim_{s\to t^-} f^\Delta(s), & \text{if } t \text{ is left-dense and right-scattered point,} \\ f^\Delta(\rho(t)), & \text{otherwise.} \end{cases} \tag{2.10}$$

Similarly, if $f \in \mathrm{C}^1_{\mathrm{pld}}$ on $\mathbb{T}_\kappa$, then the function f is as well as Δ-differentiable on $\mathbb{T}^\kappa$ and we have

$$f^\Delta(t) = \begin{cases} \lim_{s\to t^+} f^\nabla(s), & \text{if } t \text{ is right-dense and left-scattered point,} \\ f^\nabla(\sigma(t)), & \text{otherwise.} \end{cases} \tag{2.11}$$

Especially, if f^Δ and f^∇ are continuous, we obtain $f^\Delta(t) = f^\nabla(\sigma(t))$ and $f^\nabla(t) = f^\Delta(\rho(t))$.

2.4 Integration on time scales

Now, let $c, d \in \mathbb{T}$ and $c < d$. The Δ-integral and ∇-integral are defined in such a way that they reduce to the usual Riemann integral in the continuous time case and to the Riemann sum in the discrete time case, i.e.,

$$\int_c^d f(t)\,\Delta t = \int_c^d f(t)\,\nabla t = \int_c^d f(t)\,\mathrm{d}t \quad \text{if} \quad \mathbb{T} = \mathbb{R},$$
$$\int_c^d f(t)\,\Delta t = \sum_{t=c}^{d-1} f(t) \quad \text{and} \quad \int_c^d f(t)\,\nabla t = \sum_{t=c+1}^{d} f(t) \quad \text{if} \quad \mathbb{T} = \mathbb{Z}.$$

The basic rules for the time scale Δ-integral have the standard form

$$\int_c^d f(s)\,\Delta s = \int_c^e f(s)\,\Delta s + \int_e^d f(s)\,\Delta s, \qquad \int_c^d f(s)\,\Delta s = -\int_d^c f(s)\,\Delta s, \tag{2.12}$$

where $c \leq e \leq d$. Analogous properties hold true for the time scale ∇-integral. The fundamental result from the theory of time scale integrals says that for every piecewise rd-continuous (or ld-continuous) function there exists a Δ-antiderivative (or a ∇-antiderivative). The rule for the integration by parts takes the following form. If $f, g \in \mathrm{C}^1_{\mathrm{prd}}$, then we have

$$\int_c^d f(t) g^{\Delta}(t)\,\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t) g^{\sigma}(t)\,\Delta t \tag{2.13}$$

and, if $f, g \in \mathrm{C}^1_{\mathrm{pld}}$, then

$$\int_c^d f(t) g^{\nabla}(t)\,\nabla t = [f(t)g(t)]_c^d - \int_c^d f^{\nabla}(t) g^{\rho}(t)\,\nabla t. \tag{2.14}$$

Moreover, if f and g are Δ- and ∇-differentiable functions, respectively, with continuous derivatives, the formulas $\int_c^d h^{\rho}(t)\,\nabla t = \int_c^d h(t)\,\Delta t$, $\int_c^d h^{\sigma}(t)\,\Delta t = \int_c^d h(t)\,\nabla t$ and identities (2.14), (2.13) yield

$$\int_c^d f(t) g^{\Delta}(t)\,\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\nabla}(t) g(t)\,\nabla t, \tag{2.15}$$
$$\int_c^d f(t) g^{\nabla}(t)\,\nabla t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t) g(t)\,\Delta t. \tag{2.16}$$

The *Cauchy–Schwarz inequality* is an important tool in the proofs of some statements in the Weyl–Titchmarsh theory for symplectic systems presented in Chapter 4. For $f, g \in \mathrm{C}_{\mathrm{prd}}$ we have

$$\int_c^d |f(t)\,g(t)|\,\Delta t \leq \left\{\int_c^d |f(t)|^2\,\Delta t\right\}^{1/2} \left\{\int_c^d |g(t)|^2\,\Delta t\right\}^{1/2}. \tag{2.17}$$

Finally, it is a known fact that for any function f and $s \in \mathbb{T}_{\kappa}$ the following identity

$$\int_{\rho(s)}^s f(t)\nabla t = \nu(s) f(s) \tag{2.18}$$

holds true.

2.5 Bibliographical notes

Excluding Hilger's doctoral dissertation and his first paper, the books [32, 33] are the fundamental references for theory of time scales. In addition, the concept of piecewise rd-continuous functions and rd-continuously Δ-differentiable functions on time scales was initiated in [92]. Special cases of (2.10) and (2.11) were proven in [14, Theorem 2.5 and Theorem 2.6], see also [32, Theorem 8.49] and [121, Theorem 4.8]. The statement of Remark 2.1 is known from [86, Theorem 5.7] or [32, Theorem 5.8] and was also discussed in [96, Remark 3.8]. The existence of an antiderivative is known from [32, Theorem 1.74 and Theorem 8.45]. Identity (2.14) was proven in [32, Theorem 8.47(vi)] and identity (2.13) in [32, Theorem 1.77(vi)]. For more details about the time scale integrals see, e.g., [17, 30, 79]. The proofs of identities (2.15) and (2.16) follow from [33, Corollaries 4.10 and 4.11]. Many classical inequalities (Hölder, Cauchy–Schwarz, Minkowski, Jensen etc.) were generalized on time scales in [1]. The proof of identity (2.18) can be found in [33, Lemma 4.13]. Moreover, similar identities also hold for the Δ-integral, and for the Δ- and ∇-integrals over $[s, \sigma(s)]$, see [33, Lemma 4.13].

One should always generalize.

Carl G. J. Jacobi, see [48]

Chapter 3

Trigonometric and hyperbolic systems on time scales

In this chapter we study trigonometric and hyperbolic systems on time scales and properties of their solutions, the time scale matrix trigonometric functions Sin, Cos, Tan, Cotan, and time scale matrix hyperbolic functions Sinh, Cosh, Tanh, Cotanh, which are all properly defined in this chapter. These trigonometric and hyperbolic systems generalize and unify their corresponding continuous time and discrete time analogies, namely the systems known in the literature as trigonometric and hyperbolic linear Hamiltonian systems and discrete symplectic systems. More precisely, the system of the form

$$X' = Q(t)\,U, \quad U' = -Q(t)\,X, \tag{3.1}$$

where $t \in [a,b]$, $X(t)$, $U(t)$, and $Q(t)$ are $n \times n$ complex-valued matrices and additionally the matrix $Q(t)$ is Hermitian for all $t \in [a,b]$, is called a *continuous trigonometric system*. Basic properties of this system can be found in [18, 65, 134].

The discrete counterpart of (3.1) has the form

$$X_{k+1} = P_k X_k + Q_k U_k, \quad U_{k+1} = -Q_k X_k + P_k U_k, \tag{3.2}$$

where $k \in [a,b]_{\mathbb{Z}}$, X_k, U_k, P_k, Q_k are $n \times n$ complex matrices and, additionally, for all $k \in [a,b]_{\mathbb{Z}}$ the following holds

$$P_k^* P_k + Q_k^* Q_k = \mathcal{I} = P_k P_k^* + Q_k Q_k^*, \tag{3.3}$$

$$P_k^* Q_k \quad \text{and} \quad P_k Q_k^* \quad \text{are Hermitian.} \tag{3.4}$$

System (3.2) is called a *discrete trigonometric system* and its basic properties can be found in [5, 26, 157, 162].

In a similar way we can define a *continuous hyperbolic system* as

$$X' = Q(t)\,U, \quad U' = Q(t)\,X, \tag{3.5}$$

where $t \in [a,b]$, $X(t)$, $U(t)$ and $Q(t)$ are $n \times n$ complex-valued matrices and, additionally, the matrix $Q(t)$ is Hermitian for all $t \in [a,b]$. A system of this form was first studied in [71].

A *discrete hyperbolic system* is defined as

$$X_{k+1} = P_k X_k + Q_k U_k, \quad U_{k+1} = Q_k X_k + P_k U_k, \tag{3.6}$$

where $k \in [a,b]_{\mathbb{Z}}$, X_k, U_k, P_k, Q_k are $n \times n$ complex matrices and, in addition to (3.4),

$$P_k^* P_k - Q_k^* Q_k = \mathcal{I} = P_k P_k^* - Q_k Q_k^*$$

holds for $k \in [a,b]_{\mathbb{Z}}$. The reader can get acquainted with these systems in [61, 162].

The conditions for the coefficient matrices in (3.1), (3.5) or (3.2), (3.6) are set in such a way so that the considered system is Hamiltonian or symplectic, respectively. That is, for the relevant matrices

$$\mathcal{S}(t) = \begin{pmatrix} 0 & Q(t) \\ -Q(t) & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{S}(t) = \begin{pmatrix} 0 & Q(t) \\ Q(t) & 0 \end{pmatrix} \quad \text{and}$$

$$\mathcal{S}_k = \begin{pmatrix} P_k & Q_k \\ -Q_k & P_k \end{pmatrix} \quad \text{or} \quad \mathcal{S}_k = \begin{pmatrix} P_k & Q_k \\ Q_k & P_k \end{pmatrix}$$

we have the identities

$$\mathcal{S}^*(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) = 0 \quad \text{and} \quad \mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J},$$

respectively, i.e., the matrix $\mathcal{S}(t)$ is *Hamiltonian* and $\mathcal{S}_k$ is symplectic.

The aim of this chapter is to unify and generalize the theories of continuous and discrete trigonometric systems, as well as the theories of continuous and discrete hyperbolic systems. This will be done within the theory of symplectic dynamic systems defined in the next section. We derive for general time scales $\mathbb{T}$ the same identities which are known for the special cases of the continuous time $\mathbb{T} = \mathbb{R}$ or the discrete time $\mathbb{T} = \mathbb{Z}$.

In the continuous time case the study of elementary properties of scalar and matrix trigonometric functions goes back to the paper [24] of Bohl and to the works of Barrett, Etgen, Došlý, and Reid, see [18, 50–53, 65, 66, 134]. Discrete time scalar and matrix trigonometric functions were studied by Anderson, Bohner, and Došlý in [5, 26–28], and more recently by Došlá, Došlý, Pechancová, and Škrabáková in [49, 60]. Parallel considerations but for the hyperbolic systems, both continuous and discrete, can be found in the works [61, 71, 162] by Došlý, Filakovský, Pospíšil, and the author. As for the general time scale setting, scalar trigonometric and hyperbolic functions were defined in [32, Chapter 3] by Bohner and Peterson and in [130] by Pospíšil. Some properties of the matrix analogs of the time scale trigonometric and hyperbolic functions were established in the papers [54, 131, 132] by Došlý and Pospíšil.

By the same technique as in [52], namely considering two different systems with the same initial conditions, we establish additive and difference formulas for trigonometric and hyperbolic systems on time scales. In particular, utilizing these identities in the continuous time we derive n-dimensional analogies of many classical formulas which are known for trigonometric and hyperbolic systems in the scalar case. The second purpose of this chapter is to provide a concise but complete treatment of properties of time scale matrix trigonometric and hyperbolic functions, as well as to point out to the analogies between them.

3.1 Symplectic dynamic systems on time scales

A *symplectic dynamic system* on a time scale $\mathbb{T}$ is the first order linear system

$$X^{\Delta} = \mathcal{A}(t)\,X + \mathcal{B}(t)\,U, \quad U^{\Delta} = \mathcal{C}(t)\,X + \mathcal{D}(t)\,U, \tag{$\mathcal{S}$}$$

where $X, U : \mathbb{T} \to \mathbb{C}^{n\times n}$, the coefficients are $n \times n$ complex-valued matrices such that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathrm{C_{prd}}$ on $\mathbb{T}^{\kappa}$, and the matrix

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix} \tag{3.7}$$

satisfies

$$\mathcal{S}^{*}(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^{*}(t)\,\mathcal{J}\mathcal{S}(t) = 0 \tag{3.8}$$

for all $t \in \mathbb{T}^{\kappa}$. This identity implies that the matrix $\mathcal{I} + \mu(t)\,\mathcal{S}(t)$ is symplectic. Since every symplectic matrix is invertible, it follows that the matrix function $\mathcal{S}(\cdot)$ is regressive on $\mathbb{T}^{\kappa}$. Consequently, the existence of a unique solution for any (vector or matrix) initial value problem follows by Remark 2.1.

Analogously, we can define *nabla time scale symplectic systems*. Such systems were studied in [97] with a surprising outcome that some results known for system ($\mathcal{S}$) do not coincide with parallel results obtained for nabla time scale symplectic systems even in the special cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

If $\mathbb{T} = \mathbb{R}$, then with $A(t) := \mathcal{A}(t)$, $B(t) := \mathcal{B}(t)$, and $C(t) := \mathcal{C}(t)$ system ($\mathcal{S}$) corresponds to linear Hamiltonian system (1.2) and the coefficient matrix

$$S(t) := \begin{pmatrix} A(t) & B(t) \\ C(t) & -A^{*}(t) \end{pmatrix} \text{ satisfies now } \mathcal{J}S(t) + S^{*}(t)\,\mathcal{J} = 0 \text{ for all } t \in [a,b],$$

i.e., the matrix $S(\cdot)$ is Hamiltonian. If $\mathbb{T} = \mathbb{Z}$, then system ($\mathcal{S}$) with

$$\mathcal{A}_k := \mathcal{I} + \mathcal{A}(k), \quad \mathcal{B}_k := \mathcal{B}(k), \quad \mathcal{C}_k := \mathcal{C}(k), \quad \text{and} \quad \mathcal{D}_k := \mathcal{I} + \mathcal{D}(k)$$

is discrete symplectic system (1.1) and the matrix $\mathcal{S}_k := \left(\begin{smallmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{smallmatrix}\right)$ is symplectic.

Identity (3.8) is in the block notation equivalent to (we omit the argument $t \in \mathbb{T}$)

$$\begin{aligned} \mathcal{B}^{*} - \mathcal{B} + \mu\,(\mathcal{B}^{*}\mathcal{D} - \mathcal{D}^{*}\mathcal{B}) &= 0, \\ \mathcal{C}^{*} - \mathcal{C} + \mu\,(\mathcal{C}^{*}\mathcal{A} - \mathcal{A}^{*}\mathcal{C}) &= 0, \\ \mathcal{A}^{*} + \mathcal{D} + \mu\,(\mathcal{A}^{*}\mathcal{D} - \mathcal{C}^{*}\mathcal{B}) &= 0. \end{aligned}$$

This implies that the matrices $\mathcal{B}^{*}(\mathcal{I} + \mu\mathcal{D})$ and $\mathcal{C}^{*}(\mathcal{I} + \mu\mathcal{A})$ are Hermitian. By using the fact that $\mathcal{I} + \mu(t)\,\mathcal{S}^{*}(t)$ is symplectic as well, we can derive other equivalent identities

$$\begin{aligned} \mathcal{C} - \mathcal{C}^{*} + \mu\,(\mathcal{C}\mathcal{D}^{*} - \mathcal{D}\mathcal{C}^{*}) &= 0, \\ \mathcal{B} - \mathcal{B}^{*} + \mu\,(\mathcal{B}\mathcal{A}^{*} - \mathcal{A}\mathcal{B}^{*}) &= 0, \\ \mathcal{D} + \mathcal{A}^{*} + \mu\,(\mathcal{D}\mathcal{A}^{*} - \mathcal{C}\mathcal{B}^{*}) &= 0. \end{aligned}$$

If $Z = \left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right)$ and $\widetilde{Z} = \left(\begin{smallmatrix} \widetilde{X} \\ \widetilde{U} \end{smallmatrix}\right)$ are any solutions of system ($\mathcal{S}$), then their *Wronskian matrix* is defined on $\mathbb{T}$ as

$$W[Z,\widetilde{Z}](t) := X^{*}(t)\,\widetilde{U}(t) - U^{*}(t)\,\widetilde{X}(t)$$

and the following is a simple consequence of the fact $W^{\Delta}[Z,\widetilde{Z}](t) \equiv 0$.

Proposition 3.1. *Let $Z = \binom{X}{U}$ and $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ be any solutions of ($\mathcal{S}$). Then the Wronskian $W[Z, \tilde{Z}](t) \equiv W$ is constant on $\mathbb{T}$.*

A solution $Z = \binom{X}{U}$ of ($\mathcal{S}$) is said to be a *conjoined solution* if $W[Z, Z](t) \equiv 0$, i.e., $X^*(t)U(t)$ is Hermitian at one and hence at any $t \in \mathbb{T}$. Two solutions Z and $\tilde{Z}$ are *normalized* if $W[Z, \tilde{Z}](t) \equiv \mathcal{I}$. A solution Z is said to be a *basis* if $\operatorname{rank} Z(t) \equiv n$ on $\mathbb{T}$. It is well known fact that for any conjoined basis Z there always exists another conjoined basis $\tilde{Z}$ such that Z and $\tilde{Z}$ are normalized.

Proposition 3.2. *Let Z be any solution of ($\mathcal{S}$). Then $\operatorname{rank} Z(t) \equiv r$ is constant on $\mathbb{T}$.*

Proof. Let $\Phi(t)$ be a fundamental matrix of system ($\mathcal{S}$), i.e., $\Phi = \begin{pmatrix} Z & \tilde{Z} \end{pmatrix}$, where Z and $\tilde{Z}$ are normalized solutions. Then every solution of ($\mathcal{S}$) is a constant multiple of $\Phi(t)$, that is, $Z(t) = \Phi(t)\, M$ on $\mathbb{T}$ for some $M \in \mathbb{C}^{2n \times n}$. If $\operatorname{rank} Z(t_0) = r$ at some $t_0 \in \mathbb{T}$, then $\operatorname{rank} M = r$. Consequently, $\operatorname{rank} Z(t) = r$ for all $t \in \mathbb{T}$. ∎

From Propositions 3.1 and 3.2 we can see that the defining properties of conjoined bases of ($\mathcal{S}$) can be prescribed just at one point $t_0 \in \mathbb{T}$, for example by the initial condition $Z(t_0) = Z_0$ with $Z_0^* \mathcal{J} Z_0 = 0$ and $\operatorname{rank} Z_0 = n$.

Proposition 3.3. *Two solutions Z and $\tilde{Z}$ of system ($\mathcal{S}$) are normalized conjoined bases if and only if the $2n \times 2n$ matrix $\Phi(t) := \begin{pmatrix} Z(t) & \tilde{Z}(t) \end{pmatrix}$ is symplectic for all $t \in \mathbb{T}$.*

It follows that $Z = \binom{X}{U}$ and $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ are normalized conjoined bases if and only if (suppressing the argument $t \in \mathbb{T}$)

$$\left.\begin{array}{c} X^*\tilde{U} - U^*\tilde{X} = \mathcal{I} = X\tilde{U}^* - U\tilde{X}^*, \\ X^*U = U^*X, \quad \tilde{X}^*\tilde{U} = \tilde{U}^*\tilde{X}, \quad X\tilde{X}^* = \tilde{X}X^*, \quad U\tilde{U}^* = \tilde{U}U^*. \end{array}\right\} \tag{3.9}$$

The fact that the matrix Φ is symplectic for all $t \in \mathbb{T}$ implies that $\Phi^{-1} = \mathcal{J}^*\Phi^*\mathcal{J}$, and thus from $\Phi^\sigma = (\mathcal{I} + \mu\mathcal{S})\,\Phi$ we get $\Phi^\sigma\mathcal{J}^*\Phi^*\mathcal{J} = \mathcal{I} + \mu\mathcal{S}$ for $t \in \mathbb{T}^\kappa$. That is (suppressing the argument $t \in \mathbb{T}^\kappa$),

$$\left.\begin{array}{ll} X^\sigma\tilde{U}^* - \tilde{X}^\sigma U^* = \mathcal{I} + \mu\mathcal{A}, & \tilde{X}^\sigma X^* - X^\sigma\tilde{X}^* = \mu\mathcal{B}, \\ \tilde{U}^\sigma X^* - U^\sigma\tilde{X}^* = \mathcal{I} + \mu\mathcal{D}, & U^\sigma\tilde{U}^* - \tilde{U}^\sigma U^* = \mu\mathcal{C}. \end{array}\right\} \tag{3.10}$$

For a given point $t_0 \in \mathbb{T}$, the conjoined basis $\binom{\hat{X}}{\hat{U}}$ of ($\mathcal{S}$) determined by the initial conditions $\hat{X}(t_0) = 0$ and $\hat{U}(t_0) = \mathcal{I}$ is called the *principal solution at t_0*.

3.2 Time scale trigonometric systems

In this section we consider the system ($\mathcal{S}$) on $[a, b]_\mathbb{T}$, where the coefficient matrix takes the form

$$\mathcal{S}(t) = \begin{pmatrix} \mathcal{P}(t) & \mathcal{Q}(t) \\ -\mathcal{Q}(t) & \mathcal{P}(t) \end{pmatrix}$$

with $n \times n$ complex-valued matrices $\mathcal{P}, \mathcal{Q} \in \mathrm{C_{prd}}$ on $[a, \rho(b)]_\mathbb{T}$. Therefore, from (3.8) we get that the matrices $\mathcal{P}$ and $\mathcal{Q}$ satisfy the identities (we omit the argument t)

$$\mathcal{Q}^* - \mathcal{Q} + \mu\,(\mathcal{Q}^*\mathcal{P} - \mathcal{P}^*\mathcal{Q}) = 0, \tag{3.11}$$

$$\mathcal{P}^* + \mathcal{P} + \mu\,(\mathcal{Q}^*\mathcal{Q} + \mathcal{P}^*\mathcal{P}) = 0 \tag{3.12}$$

for all $t \in [a, \rho(b)]_\mathbb{T}$, see also [32, p. 312] and [87, Theorem 7].

Definition 3.4 (Time scale trigonometric system)**.** The system

$$X^{\Delta} = \mathcal{P}(t)\,X + \mathcal{Q}(t)\,U, \quad U^{\Delta} = -\mathcal{Q}(t)\,X + \mathcal{P}(t)\,U, \tag{3.13}$$

where the coefficient matrices satisfy identities (3.11) and (3.12) for all $t \in [a, \rho(b)]_{\mathbb{T}}$, is called a *time scale trigonometric system.*

Remark 3.5. System ($\mathcal{S}$) is trigonometric if its coefficients satisfy, in addition to (3.8) the identity $\mathcal{J}^*\mathcal{S}(t)\,\mathcal{J} = \mathcal{S}(t)$ for all $t \in [a, \rho(b)]_{\mathbb{T}}$. Therefore, trigonometric systems are also called *self-reciprocal*. Moreover, any symplectic system ($\mathcal{S}$) can be transformed into a trigonometric system.

Remark 3.6. Now, we compare the continuous time trigonometric system arising from Definition 3.4, with the system (3.1) introduced at the beginning of this chapter. For $[a, b]_{\mathbb{T}} = [a, b]$, the time scale trigonometric system takes the form

$$X' = \mathcal{P}(t)\,X + \mathcal{Q}(t)U, \quad U' = -\mathcal{Q}(t)\,X + \mathcal{P}(t)\,U, \tag{3.14}$$

where $\mathcal{Q}(t)$ is Hermitian and $\mathcal{P}(t)$ is skew-Hermitian, see (3.11) and (3.12) with $\mu = 0$. Now we use the special transformation to reduce the system (3.14) into (3.1), see [28,134].

More precisely, let $H(t)$ be a solution of the system $H' = \mathcal{P}(t)\,H$ with the initial condition $H^*(a)\,H(a) = \mathcal{I}$, i.e., the matrix $H(a)$ is unitary. Now, we consider the transformation $\underline{X} := H^{-1}(t)\,X$ and $\underline{U} := H^*(t)\,U$, which yields

$$\underline{X}' = H^{-1}(t)\,\mathcal{Q}(t)\,H^{*-1}(t)\,\underline{U}, \quad \underline{U}' = -H^*(t)\,\mathcal{Q}(t)\,H(t)\,\underline{X}.$$

Hence, this resulting system will be of the form (3.1) once we show that $H^*(t) = H^{-1}(t)$ for all $t \in [a, b]$. But this follows from the calculation $(H^*H)' = 0$ and from the initial condition on $H(a)$. Now, we put $\widetilde{Q}(t) := H^*(t)\,\mathcal{Q}(t)\,H(t)$ which is Hermitian, so that

$$\underline{X}' = \widetilde{Q}(t)\,\underline{U}, \quad \underline{U}' = -\widetilde{Q}(t)\,\underline{X}.$$

Remark 3.7. Analogously, we consider the discrete case and show that the time scale trigonometric system reduces for $[a, b]_{\mathbb{T}} = [a, b]_{\mathbb{Z}}$ to system (3.2) introduced at the beginning of this chapter. Upon setting $P_k := \mathcal{I} + \mathcal{P}(k)$ and $Q_k := \mathcal{Q}(k)$ one can easily see that identities (3.11) and (3.12) are in this case equivalent to the properties of P_k and Q_k in (3.3)–(3.4).

Now, we turn our attention to solutions of the general time scale trigonometric system.

Lemma 3.8. *The pair* $\binom{X}{U}$ *solves the time scale trigonometric system in* (3.13) *if and only if the pair* $\binom{U}{-X}$ *solves the same system. Equivalently* $\binom{U}{X}$ *solves* (3.13) *if and only if* $\binom{-X}{U}$ *does so.*

The following definition extends to time scales the matrix sine and cosine functions known in the continuous time from [18, p. 511] and in the discrete case from [5, p. 39].

Definition 3.9. Let $s \in [a, b]_{\mathbb{T}}$ be fixed. We define the $n \times n$ matrix-valued functions *sine* (denoted by Sin_s) and *cosine* (denoted by Cos_s) as

$$\mathrm{Sin}_s(t) := X(t) \quad \text{and} \quad \mathrm{Cos}_s(t) := U(t),$$

respectively, where the pair $\binom{X}{U}$ is the principal solution of system (3.13) at s, i.e., it is given by the initial conditions $X(s) = 0$ and $U(s) = \mathcal{I}$. We suppress the index s when $s = a$, i.e., we denote $\mathrm{Sin} := \mathrm{Sin}_a$ and $\mathrm{Cos} := \mathrm{Cos}_a$.

Remark 3.10. (i) The matrix functions Sin_s and Cos_s are n-dimensional analogs of the scalar trigonometric functions $\sin(t-s)$ and $\cos(t-s)$.

(ii) When $n = 1$ and $\mathcal{P} = 0$ and $\mathcal{Q} = p$ with $p \in \mathrm{C_{rd}}$, the matrix functions $\mathrm{Sin}_s(t)$ and $\mathrm{Cos}_s(t)$ reduce exactly to the scalar time scale trigonometric functions $\sin_p(t,s)$ and $\cos_p(t,s)$ from [32, Definition 3.25].

(iii) In the continuous time scalar case and when $\mathcal{P} = 0$, i.e., system (3.13) is the same as (3.1), the solutions $\mathrm{Sin}(t) = \sin \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$ and $\mathrm{Cos}(t) = \cos \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$. Similar formulas hold for the discrete scalar case, see [5, p. 40].

Remark 3.11. By using Lemma 3.8, the above matrix sine and cosine functions can be alternatively defined as $\mathrm{Cos}_s(t) := \widetilde{X}(t)$ and $\mathrm{Sin}_s(t) := -\widetilde{U}(t)$, where $\binom{\widetilde{X}}{\widetilde{U}}$ is the solution of system (3.13) with the initial conditions $\widetilde{X}(s) = \mathcal{I}$ and $\widetilde{U}(s) = 0$.

By definition, the Wronskian of the two solutions $\binom{\mathrm{Cos}}{\mathrm{Sin}}$ and $\binom{-\mathrm{Sin}}{\mathrm{Cos}}$ is $W(t) \equiv W(a) = \mathcal{I}$. Hence, $\binom{\mathrm{Cos}}{\mathrm{Sin}}$ and $\binom{-\mathrm{Sin}}{\mathrm{Cos}}$ form normalized conjoined bases of system (3.13) and

$$\hat{\Phi}(t) := \begin{pmatrix} \mathrm{Cos}(t) & -\mathrm{Sin}(t) \\ \mathrm{Sin}(t) & \mathrm{Cos}(t) \end{pmatrix} \tag{3.15}$$

is a fundamental matrix of (3.13). Therefore, every solution $\binom{X}{U}$ of (3.13) has the form

$$X(t) = \mathrm{Cos}(t)\,X(a) - \mathrm{Sin}(t)\,U(a) \quad \text{and} \quad U(t) = \mathrm{Sin}(t)\,X(a) + \mathrm{Cos}(t)\,U(a)$$

for all $t \in [a,b]_{\mathbb{T}}$. As a consequence of formulas (3.9) and (3.10) we get the following.

Corollary 3.12. *For all $t \in [a,b]_{\mathbb{T}}$ the identities*

$$\mathrm{Cos}^*\,\mathrm{Cos} + \mathrm{Sin}^*\,\mathrm{Sin} = \mathcal{I} = \mathrm{Cos}\,\mathrm{Cos}^* + \mathrm{Sin}\,\mathrm{Sin}^*, \tag{3.16}$$
$$\mathrm{Cos}^*\,\mathrm{Sin} = \mathrm{Sin}^*\,\mathrm{Cos}, \quad \mathrm{Cos}\,\mathrm{Sin}^* = \mathrm{Sin}\,\mathrm{Cos}^* \tag{3.17}$$

hold, while for all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have the identities

$$\mathrm{Cos}^\sigma\,\mathrm{Cos}^* + \mathrm{Sin}^\sigma\,\mathrm{Sin}^* = \mathcal{I} + \mu\mathcal{P}, \quad \mathrm{Cos}^\sigma\,\mathrm{Sin}^* - \mathrm{Sin}^\sigma\,\mathrm{Cos}^* = \mu\mathcal{Q}.$$

The following result is a matrix analog of the fundamental formula $\cos^2(t)+\sin^2(t) = 1$ for scalar continuous time trigonometric functions, see also [32, Exercise 3.30]. Here $\|\cdot\|_F$ is the usual Frobenius norm, i.e., $\|V\|_F = \big(\sum_{i,j=1}^n v_{ij}^2\big)^{\frac{1}{2}}$, see [23, p. 346].

Corollary 3.13. *For all $t \in [a,b]_{\mathbb{T}}$ we have the identity*

$$\|\mathrm{Cos}\|_F^2 + \|\mathrm{Sin}\|_F^2 = n. \tag{3.18}$$

Proof. Since for arbitrary matrix $V \in \mathbb{C}^{n\times n}$ the identity $\mathrm{tr}\,(V^*V) = \|V\|_F^2$ holds, equation (3.18) follows directly from (3.16). ∎

Corollary 3.14. *For all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have*

$$\mathrm{Cos}^\Delta\,\mathrm{Cos}^* + \mathrm{Sin}^\Delta\,\mathrm{Sin}^* = \mathcal{P}, \tag{3.19}$$
$$\mathrm{Sin}^\Delta\,\mathrm{Cos}^* - \mathrm{Cos}^\Delta\,\mathrm{Sin}^* = \mathcal{Q}. \tag{3.20}$$

Proof. Since $\binom{\mathrm{Sin}}{\mathrm{Cos}}$ is the solution of system (3.13), we have

$$\mathrm{Sin}^{\Delta} = \mathcal{P}\,\mathrm{Sin} + \mathcal{Q}\,\mathrm{Cos} \quad \text{and} \quad \mathrm{Cos}^{\Delta} = -\mathcal{Q}\,\mathrm{Sin} + \mathcal{P}\,\mathrm{Cos}\,.$$

If we now multiply the first of these two identities by the matrix Sin^* from the right and the second one by Cos^* from the right, and if we add the two obtained equations, then formula (3.19) follows. In these computations we also used (3.16)(ii) and (3.17)(ii). Similar calculations lead to formula (3.20). ■

Remark 3.15. If the matrix Cos and/or Sin is invertible at some point $t \in [a,b]_{\mathbb{T}}$, then, by (3.16) and (3.17), we can write

$$\mathrm{Cos}^{-1} = \mathrm{Cos}^* + \mathrm{Sin}^*\,\mathrm{Cos}^{*-1}\,\mathrm{Sin}^*, \quad \mathrm{Sin}^{-1} = \mathrm{Sin}^* + \mathrm{Cos}^*\,\mathrm{Sin}^{*-1}\,\mathrm{Cos}^*\,. \tag{3.21}$$

Next we present additive formulas for matrix trigonometric functions on time scales. This result generalizes its continuous time counterpart in [66, Theorem 1.1] to time scales.

Theorem 3.16. *For $t, s \in [a,b]_{\mathbb{T}}$ we have*

$$\mathrm{Sin}_s(t) = \mathrm{Sin}(t)\,\mathrm{Cos}^*(s) - \mathrm{Cos}(t)\,\mathrm{Sin}^*(s), \tag{3.22}$$
$$\mathrm{Cos}_s(t) = \mathrm{Cos}(t)\,\mathrm{Cos}^*(s) + \mathrm{Sin}(t)\,\mathrm{Sin}^*(s), \tag{3.23}$$
$$\mathrm{Sin}(t) = \mathrm{Sin}_s(t)\,\mathrm{Cos}(s) + \mathrm{Cos}_s(t)\,\mathrm{Sin}(s), \tag{3.24}$$
$$\mathrm{Cos}(t) = \mathrm{Cos}_s(t)\,\mathrm{Cos}(s) - \mathrm{Sin}_s(t)\,\mathrm{Sin}(s). \tag{3.25}$$

Proof. We set

$$V(t) := \mathrm{Sin}(t)\,\mathrm{Cos}^*(s) - \mathrm{Cos}(t)\,\mathrm{Sin}^*(s), \quad Y(t) := \mathrm{Cos}(t)\,\mathrm{Cos}^*(s) + \mathrm{Sin}(t)\,\mathrm{Sin}^*(s).$$

Then we calculate

$$V^{\Delta}(t) = \mathrm{Sin}^{\Delta}(t)\,\mathrm{Cos}^*(s) - \mathrm{Cos}^{\Delta}(t)\,\mathrm{Sin}^*(s) = \mathcal{P}(t)V(t) + \mathcal{Q}(t)Y(t),$$
$$Y^{\Delta}(t) = \mathrm{Cos}^{\Delta}(t)\,\mathrm{Cos}^*(s) + \mathrm{Sin}^{\Delta}(t)\,\mathrm{Sin}^*(s) = -\mathcal{Q}(t)V(t) + \mathcal{P}(t)Y(t),$$

where we used (3.16)(ii) and (3.17)(ii) at t. The initial values are $V(s) = 0$ and $Y(s) = \mathcal{I}$, where we used (3.16)(ii) and (3.17)(ii) at s. Hence, equations (3.22) and (3.23) follow from the uniqueness of solutions of time scale symplectic systems. That is $V(t) = \mathrm{Sin}_s(t)$ and $Y(t) = \mathrm{Cos}_s(t)$. Note that equations (3.22) and (3.23) can be written as

$$\begin{pmatrix}\mathrm{Sin}_s(t) & \mathrm{Cos}_s(t)\end{pmatrix} = \begin{pmatrix}\mathrm{Sin}(t) & \mathrm{Cos}(t)\end{pmatrix}\begin{pmatrix}\mathrm{Cos}^*(s) & \mathrm{Sin}^*(s)\\ -\mathrm{Sin}^*(s) & \mathrm{Cos}^*(s)\end{pmatrix}, \tag{3.26}$$

where the $2n \times 2n$ matrix on the right-hand side equals to $\hat{\Phi}^{-1}(s)$ and the matrix $\hat{\Phi}(s)$ is defined in (3.15). Multiplying equality (3.26) by $\hat{\Phi}(s)$ from the right, identities (3.24) and (3.25) follow. ■

Remark 3.17. With respect to Remark 3.10 for the scalar continuous time case, identities (3.22)–(3.23) are matrix analogues of

$$\sin(t-s) = \sin(t)\cos(s) - \cos(t)\sin(s), \quad \cos(t-s) = \cos(t)\cos(s) + \sin(t)\sin(s),$$

while identities (3.24)–(3.25) are matrix analogues of

$$\sin(t) = \sin(t-s)\cos(s) + \cos(t-s)\sin(s), \quad \cos(t) = \cos(t-s)\cos(s) - \sin(t-s)\sin(s).$$

Interchanging the parameters t and s in (3.22) and (3.23) yields expected properties of the matrix trigonometric functions.

Corollary 3.18. *Let $t, s \in [a, b]_{\mathbb{T}}$. Then*

$$\mathrm{Sin}_s(t) = -\,\mathrm{Sin}_t^*(s) \quad \textit{and} \quad \mathrm{Cos}_s(t) = \mathrm{Cos}_t^*(s). \tag{3.27}$$

Remark 3.19. In the scalar continuous time case with $\mathcal{Q}(t) \equiv 1$, the formulas in (3.27) have the form

$$\sin(t - s) = -\sin(s - t) \quad \text{and} \quad \cos(t - s) = \cos(s - t).$$

Consequently, if we let $s = 0$, we obtain

$$\sin(t) = -\sin(-t) \quad \text{and} \quad \cos(t) = \cos(-t),$$

so that Corollary 3.18 is the matrix analogue of the statement about the parity for the scalar functions sine and cosine.

Next we wish to generalize the sum and difference formulas for solutions of two time scale symplectic systems. This can be done via the approach from [52]. This leads to a generalization of several formulas known in the scalar continuous case. Observe that, comparing to Theorem 3.16 in which we consider *one system* and solutions with *different initial conditions*, we shall now deal with *two systems* and solutions with the *same initial conditions*. Consider the following two time scale trigonometric systems

$$X^\Delta = \mathcal{P}_{(i)}(t)\, X + \mathcal{Q}_{(i)}(t)\, U, \quad U^\Delta = -\mathcal{Q}_{(i)}(t)\, X + \mathcal{P}_{(i)}(t)\, U \tag{3.28}$$

with initial conditions $X_{(i)}(a) = 0$ and $U_{(i)}(a) = \mathcal{I}$, where $i = 1, 2$. Denote by $\mathrm{Sin}_{(i)}(t)$ and $\mathrm{Cos}_{(i)}(t)$ the corresponding matrix sine and cosine functions from Definition 3.9. Put

$$\mathrm{Sin}^\pm(t) := \mathrm{Sin}_{(1)}(t)\, \mathrm{Cos}_{(2)}^*(t) \pm \mathrm{Cos}_{(1)}(t)\, \mathrm{Sin}_{(2)}^*(t), \tag{3.29}$$

$$\mathrm{Cos}^\pm(t) := \mathrm{Cos}_{(1)}(t)\, \mathrm{Cos}_{(2)}^*(t) \mp \mathrm{Sin}_{(1)}(t)\, \mathrm{Sin}_{(2)}^*(t). \tag{3.30}$$

Theorem 3.20. *Assume that $\mathcal{P}_{(i)}$ and $\mathcal{Q}_{(i)}$ satisfy (3.11) and (3.12). The pair $\mathrm{Sin}^\pm$ and $\mathrm{Cos}^\pm$ solves the system*

$$\left.\begin{aligned} X^\Delta &= \mathcal{P}_{(1)}(t)\, X + \mathcal{Q}_{(1)}(t)\, U + X\mathcal{P}_{(2)}^*(t) \pm U\mathcal{Q}_{(2)}^*(t) \\ &\quad + \mu(t) \left[\mathcal{P}_{(1)}(t) \left(X\mathcal{P}_{(2)}^*(t) \pm U\mathcal{Q}_{(2)}^*(t)\right) + \mathcal{Q}_{(1)}(t) \left(\mp X\mathcal{Q}_{(2)}^*(t) + U\mathcal{P}_{(2)}^*(t)\right)\right], \\ U^\Delta &= -\mathcal{Q}_{(1)}(t)\, X + \mathcal{P}_{(1)}(t)\, U \mp X\mathcal{Q}_{(2)}^*(t) + U\mathcal{P}_{(2)}^*(t) \\ &\quad + \mu(t) \left[-\mathcal{Q}_{(1)}(t) \left(X\mathcal{P}_{(2)}^*(t) \pm U\mathcal{Q}_{(2)}^*(t)\right) + \mathcal{P}_{(1)}(t) \left(\mp X\mathcal{Q}_{(2)}^*(t) + U\mathcal{P}_{(2)}^*(t)\right)\right] \end{aligned}\right\} \tag{3.31}$$

with the initial conditions $X(a) = 0$ and $U(a) = \mathcal{I}$. Moreover, for all $t \in [a, b]_{\mathbb{T}}$ we have

$$\mathrm{Sin}^\pm \left(\mathrm{Sin}^\pm\right)^* + \mathrm{Cos}^\pm \left(\mathrm{Cos}^\pm\right)^* = \mathcal{I} = \left(\mathrm{Sin}^\pm\right)^* \mathrm{Sin}^\pm + \left(\mathrm{Cos}^\pm\right)^* \mathrm{Cos}^\pm, \tag{3.32}$$

$$\mathrm{Sin}^\pm \left(\mathrm{Cos}^\pm\right)^* = \mathrm{Cos}^\pm \left(\mathrm{Sin}^\pm\right)^*, \quad \left(\mathrm{Sin}^\pm\right)^* \mathrm{Cos}^\pm = \left(\mathrm{Cos}^\pm\right)^* \mathrm{Sin}^\pm. \tag{3.33}$$

Proof. All the statements in this theorem are proven by straightforward calculations. In these we use the identities, see (2.5),

$$\mathrm{Sin}_{(1)}^\sigma = \mathrm{Sin}_{(1)} + \mu\, \mathrm{Sin}_{(1)}^\Delta = \mathrm{Sin}_{(1)} + \mu \left(\mathcal{P}_{(1)}\, \mathrm{Sin}_{(1)} + \mathcal{Q}_{(1)}\, \mathrm{Cos}_{(1)}\right),$$

$$\mathrm{Cos}_{(1)}^\sigma = \mathrm{Cos}_{(1)} + \mu\, \mathrm{Cos}_{(1)}^\Delta = \mathrm{Cos}_{(1)} + \mu \left(-\mathcal{Q}_{(1)}\, \mathrm{Sin}_{(1)} + \mathcal{P}_{(1)}\, \mathrm{Cos}_{(1)}\right),$$

time scale product rule (2.3), and system (3.28) for $i = 1, 2$. Then it follows that the pair $\mathrm{Sin}^{\pm}$ and $\mathrm{Cos}^{\pm}$ solves the system (3.31) and $\mathrm{Sin}^{\pm}(a) = 0$, $\mathrm{Cos}^{\pm}(a) = \mathcal{I}$.

Next we show identity (3.32). From the definitions of Sin^{+} and Cos^{+}, from the first identity in (3.16) for $i = 1$, and from the second identity in (3.17) for $i = 2$ we get

$$\begin{aligned} \mathrm{Sin}^{+}\left(\mathrm{Sin}^{+}\right)^{*} + \mathrm{Cos}^{+}\left(\mathrm{Cos}^{+}\right)^{*} &= \mathrm{Sin}_{(1)}\left(\mathrm{Cos}^{*}_{(2)}\,\mathrm{Cos}_{(2)} + \mathrm{Sin}^{*}_{(2)}\,\mathrm{Sin}_{(2)}\right)\mathrm{Sin}^{*}_{(1)} \\ &\quad + \mathrm{Cos}_{(1)}\left(\mathrm{Cos}^{*}_{(2)}\,\mathrm{Cos}_{(2)} + \mathrm{Sin}^{*}_{(2)}\,\mathrm{Sin}_{(2)}\right)\mathrm{Cos}^{*}_{(1)} \\ &= \mathrm{Cos}_{(1)}\,\mathrm{Cos}^{*}_{(1)} + \mathrm{Sin}_{(1)}\,\mathrm{Sin}^{*}_{(1)} = \mathcal{I}. \end{aligned}$$

The other identities in (3.32) are shown in analogous way. Similarly, by using (3.16) and (3.17) for $i = 1, 2$ one can show that all the identities in (3.33) hold true. ■

Remark 3.21. The properties in (3.32) and (3.33) of solutions $\mathrm{Sin}^{\pm}$ and $\mathrm{Cos}^{\pm}$ of system (3.31) mirror the properties in (3.16) and (3.17) of normalized conjoined bases of ($\mathcal{S}$). However, the two pairs $\begin{pmatrix} \mathrm{Sin}^{+} \\ \mathrm{Cos}^{+} \end{pmatrix}$ and $\begin{pmatrix} \mathrm{Sin}^{-} \\ \mathrm{Cos}^{-} \end{pmatrix}$ are not conjoined bases of their corresponding systems, because these systems are not symplectic.

Remark 3.22. In the continuous time case the assertion of Theorem 3.20 was proven in [52, Theorem 1]. On the other hand, the discrete form is new. The details can be found in [163, Theorem 3.14].

When the two systems in (3.28) are the same, Theorem 3.20 yields the following.

Corollary 3.23. *Assume that $\mathcal{P}$ and $\mathcal{Q}$ satisfy (3.11) and (3.12). Then the system*

$$\begin{aligned} X^{\Delta} &= \mathcal{P}(t)\,X + \mathcal{Q}(t)\,U + X\mathcal{P}^{*}(t) + U\mathcal{Q}^{*}(t) \\ &\qquad + \mu(t)\Big[\mathcal{P}(t)\big(X\mathcal{P}^{*}(t) + U\mathcal{Q}^{*}(t)\big) + \mathcal{Q}(t)\big(-X\mathcal{Q}^{*}(t) + U\mathcal{P}^{*}(t)\big)\Big], \\ U^{\Delta} &= -\mathcal{Q}(t)\,X + \mathcal{P}(t)\,U - X\mathcal{Q}^{*}(t) + U\mathcal{P}^{*}(t) \\ &\qquad + \mu(t)\Big[-\mathcal{Q}(t)\big(X\mathcal{P}^{*}(t) + U\mathcal{Q}^{*}(t)\big) + \mathcal{P}(t)\big(-X\mathcal{Q}^{*}(t) + U\mathcal{P}^{*}(t)\big)\Big] \end{aligned}$$

with the initial conditions $X(a) = 0$ and $U(a) = \mathcal{I}$ possesses the solution

$$X = 2\,\mathrm{Sin}\,\mathrm{Cos}^{*} \quad \text{and} \quad U = \mathrm{Cos}\,\mathrm{Cos}^{*} - \mathrm{Sin}\,\mathrm{Sin}^{*},$$

where Sin *and* Cos *are the matrix functions in Definition 3.9. Moreover, the above matrices X and U commute, i.e., $XU = UX$.*

Proof. The statement follows from Theorem 3.20 in which we take $\mathcal{P}_{(1)} = \mathcal{P}_{(2)} = \mathcal{P}$, $\mathcal{Q}_{(1)} = \mathcal{Q}_{(2)} = \mathcal{Q}$, and $\mathrm{Sin}_{(1)} = \mathrm{Sin}_{(2)} = \mathrm{Sin}$, $\mathrm{Cos}_{(1)} = \mathrm{Cos}_{(2)} = \mathrm{Cos}$. Finally, from (3.16) and (3.17) we get that $XU - UX = 0$. ■

Remark 3.24. The previous corollary can be viewed as the $n-$dimensional analogy of the double angle formulas for scalar continuous time goniometric functions

$$\sin(2t) = 2\sin(t)\cos(t) \quad \text{and} \quad \cos(2t) = \cos^{2}(t) - \sin^{2}(t).$$

In the continuous time case, the content of Corollary 3.23 coincides with [65, Theorem 1.1]. On the other hand, this result is new in the discrete case, see [163, Corollary 3.16].

Corollary 3.25. *For all $t \in [a, b]_{\mathbb{T}}$ we have the identities*

$$\mathrm{Sin}_{(1)}\,\mathrm{Sin}^*_{(2)} = \frac{1}{2}\left(\mathrm{Cos}^- - \mathrm{Cos}^+\right), \tag{3.34}$$

$$\mathrm{Cos}_{(1)}\,\mathrm{Cos}^*_{(2)} = \frac{1}{2}\left(\mathrm{Cos}^- + \mathrm{Cos}^+\right), \tag{3.35}$$

$$\mathrm{Sin}_{(1)}\,\mathrm{Cos}^*_{(2)} = \frac{1}{2}\left(\mathrm{Sin}^- + \mathrm{Sin}^+\right). \tag{3.36}$$

Proof. Subtracting the two equations in (3.30) we obtain $\mathrm{Cos}^- - \mathrm{Cos}^+ = 2\,\mathrm{Sin}_{(1)}\,\mathrm{Sin}^*_{(2)}$ from which formula (3.34) follows. Similarly, from identities

$$\mathrm{Cos}^- + \mathrm{Cos}^+ = 2\,\mathrm{Cos}_{(1)}\,\mathrm{Cos}^*_{(2)} \quad \text{and} \quad \mathrm{Sin}^- + \mathrm{Sin}^+ = 2\,\mathrm{Sin}_{(1)}\,\mathrm{Cos}^*_{(2)}$$

we obtain (3.35) and (3.36). ■

Remark 3.26. In the scalar continuous time case identities (3.34)–(3.36) correspond to

$$\sin(t)\sin(s) = \frac{1}{2}\left[\cos(t-s) - \cos(t+s)\right],$$
$$\cos(t)\cos(s) = \frac{1}{2}\left[\cos(t-s) + \cos(t+s)\right],$$
$$\sin(t)\cos(s) = \frac{1}{2}\left[\sin(t-s) + \sin(t+s)\right].$$

The next definition is a natural time scale matrix extension of the scalar trigonometric tangent and cotangent functions. It extends the discrete matrix tangent and cotangent functions known from [5, p. 42] to time scales.

Definition 3.27. Whenever $\mathrm{Cos}(t)$ and $\mathrm{Sin}(t)$ is invertible, we define the matrix-valued functions *tangent* (we write Tan) and *cotangent* (we write Cotan), by

$$\mathrm{Tan}(t) := \mathrm{Cos}^{-1}(t)\ \mathrm{Sin}(t) \quad \text{and} \quad \mathrm{Cotan}(t) := \mathrm{Sin}^{-1}(t)\ \mathrm{Cos}(t), \ \text{respectively.}$$

Remark 3.28. Analogous results concerning $\mathrm{Tan}(t)$ and $\mathrm{Cotan}(t)$ which are presented below, we can get by using the definitions

$$\widetilde{\mathrm{Tan}}(t) := \mathrm{Sin}(t)\ \mathrm{Cos}^{-1}(t) \quad \text{and} \quad \widetilde{\mathrm{Cotan}}(t) := \mathrm{Cos}(t)\ \mathrm{Sin}^{-1}(t).$$

Theorem 3.29. *Whenever* $\mathrm{Tan}(t)$ *is defined we get*

$$\mathrm{Tan}^*(t) = \mathrm{Tan}(t), \tag{3.37}$$

$$\mathrm{Cos}^{-1}(t)\,\mathrm{Cos}^{*-1}(t) - \mathrm{Tan}^2(t) = \mathcal{I}. \tag{3.38}$$

Moreover, if $\mathrm{Cos}(t)$ *and* $\mathrm{Cos}^\sigma(t)$ *are invertible, then*

$$\mathrm{Tan}^\Delta(t) = [\mathrm{Cos}^\sigma(t)]^{-1}\ \mathcal{Q}(t)\ \mathrm{Cos}^{*-1}(t). \tag{3.39}$$

Proof. From (3.17) it follows that

$$\mathrm{Tan}^* - \mathrm{Tan} = \mathrm{Cos}^{-1}\left(\mathrm{Cos}\,\mathrm{Sin}^* - \mathrm{Sin}\,\mathrm{Cos}^*\right)\mathrm{Cos}^{*-1} = 0,$$

while from (3.16) and (3.37) we get

$$\mathcal{I} = \mathrm{Cos}\left(\mathrm{Cos}^{-1}\,\mathrm{Sin}\,\mathrm{Sin}^*\,\mathrm{Cos}^{*-1} + \mathcal{I}\right)\mathrm{Cos}^* = \mathrm{Cos}\left(\mathrm{Tan}^2 + \mathcal{I}\right)\mathrm{Cos}^*,$$

which can be written as equation (3.38). In order to show (3.39) we note that if $\mathrm{Cos}(t_0)$ and $\mathrm{Cos}^\sigma(t_0)$ are invertible, then $\mathrm{Tan}^\Delta(t_0)$ exists and, by (3.16), (2.4), (3.21), and (3.37), we obtain

$$\begin{aligned}\mathrm{Tan}^\Delta &= \left(\mathrm{Cos}^{-1}\mathrm{Sin}\right)^\Delta = -(\mathrm{Cos}^\sigma)^{-1}\mathrm{Cos}^\Delta\,\mathrm{Cos}^{-1}\mathrm{Sin} + (\mathrm{Cos}^\sigma)^{-1}\mathrm{Sin}^\Delta \\ &= (\mathrm{Cos}^\sigma)^{-1}\mathcal{Q}\left(-\mathrm{Sin}\,\mathrm{Cos}^{-1}\mathrm{Sin} + \mathrm{Cos}\right) = (\mathrm{Cos}^\sigma)^{-1}\mathcal{Q}\,\mathrm{Cos}^{*-1}.\end{aligned}$$

Therefore (3.39) is established. ■

Similar results as in Theorem 3.29 can be shown for the matrix function cotangent.

Theorem 3.30. *Whenever* $\mathrm{Cotan}(t)$ *is defined we get*

$$\mathrm{Cotan}^*(t) = \mathrm{Cotan}(t), \tag{3.40}$$

$$\mathrm{Sin}^{-1}(t)\,\mathrm{Sin}^{*-1}(t) - \mathrm{Cotan}^2(t) = \mathcal{I}. \tag{3.41}$$

Moreover, if $\mathrm{Sin}(t)$ *and* $\mathrm{Sin}^\sigma(t)$ *are invertible, then*

$$\mathrm{Cotan}^\Delta(t) = -[\mathrm{Sin}^\sigma(t)]^{-1}\,\mathcal{Q}(t)\,\mathrm{Sin}^{*-1}(t). \tag{3.42}$$

Proof. It is analogous to the proof of Theorem 3.29. ■

Remark 3.31. In the scalar case $n = 1$ identities (3.37) and (3.40) are trivial. In the scalar continuous time case, identities (3.38), (3.41), (3.39), and (3.42) take the form

$$\frac{1}{\cos^2(s)} - \tan^2(s) = 1, \quad \frac{1}{\sin^2(s)} - \cot\mathrm{an}^2(s) = 1, \quad \text{with} \quad s = \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau,$$

$$\left(\tan\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau\right)' = \frac{\mathcal{Q}(t)}{\cos^2 s}, \quad \left(\mathrm{cotan}\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau\right)' = \frac{-\mathcal{Q}(t)}{\sin^2 s},$$

compare with Remark 3.10 (iii). The discrete versions of these identities can be found in [5, Corollary 6 and Lemma 12].

Remark 3.32. In the continuous time case with $\mathcal{Q}(t) \equiv \mathcal{I}$, i.e., when system (3.1) is $X' = U$, $U' = -X$ and hence it represents the second order matrix equation $X'' + X = 0$, the matrix functions Sin, Cos, Tan, and Cotan satisfy

$$\mathrm{Sin}' = \mathrm{Cos}, \quad \mathrm{Cos}' = -\mathrm{Sin}, \quad \mathrm{Tan}' = \mathrm{Cos}^{-1}\mathrm{Cos}^{*-1}, \quad \mathrm{Cotan}' = -\mathrm{Sin}^{-1}\mathrm{Sin}^{*-1}.$$

The first two equalities follow from the definition of Sin and Cos, while the last two equalities are simple consequences of (3.39) and (3.42).

Next, similarly to the definitions of the time scale matrix functions $\mathrm{Sin}_{(i)}$, $\mathrm{Cos}_{(i)}$, for $i = 1, 2$, $\mathrm{Sin}^\pm$, and $\mathrm{Cos}^\pm$ from (3.28)–(3.30) we define the following functions

$$\mathrm{Tan}_{(i)}(t) := \mathrm{Cos}_{(i)}^{-1}(t)\,\mathrm{Sin}_{(i)}(t), \qquad \mathrm{Cotan}_{(i)}(t) := \mathrm{Sin}_{(i)}^{-1}(t)\,\mathrm{Cos}_{(i)}(t),$$

$$\mathrm{Tan}^\pm(t) := \left[\mathrm{Cos}^\pm(t)\right]^{-1}\mathrm{Sin}^\pm(t), \qquad \mathrm{Cotan}^\pm(t) := \left[\mathrm{Sin}^\pm(t)\right]^{-1}\mathrm{Cos}^\pm(t).$$

Remark 3.33. It is natural that the matrix-valued functions $\mathrm{Tan}^\pm$ have similar properties as the function Tan. In particular, the first identity in (3.33) implies that $\mathrm{Tan}^\pm$ are Hermitian. Similarly, the functions $\mathrm{Cotan}^\pm$ are also Hermitian.

The results of the following theorem are new even in the special case of continuous and discrete time, see [163, Theorem 3.26].

Theorem 3.34. *For all $t \in [a,b]_{\mathbb{T}}$ such that all involved functions are defined we have (suppressing the argument t)*

$$\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)} = \mathrm{Tan}_{(1)} \left(\mathrm{Cotan}_{(2)} \pm \mathrm{Cotan}_{(1)}\right) \mathrm{Tan}_{(2)}, \tag{3.43}$$

$$\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)} = \mathrm{Cos}_{(1)}^{-1}\ \mathrm{Sin}^{\pm}\ \mathrm{Cos}_{(2)}^{*-1}, \tag{3.44}$$

$$\mathrm{Cotan}_{(1)} \pm \mathrm{Cotan}_{(2)} = \mathrm{Cotan}_{(1)} \left(\mathrm{Tan}_{(2)} \pm \mathrm{Tan}_{(1)}\right) \mathrm{Cotan}_{(2)}, \tag{3.45}$$

$$\mathrm{Cotan}_{(1)} \pm \mathrm{Cotan}_{(2)} = \pm \mathrm{Sin}_{(1)}^{-1}\ \mathrm{Sin}^{\pm}\ \mathrm{Sin}_{(2)}^{*-1}, \tag{3.46}$$

$$\mathrm{Tan}^{\pm} = \mathrm{Cos}_{(2)}^{*-1} \left(\mathcal{I} \mp \mathrm{Tan}_{(1)}\, \mathrm{Tan}_{(2)}\right)^{-1} \left(\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)}\right) \mathrm{Cos}_{(2)}^{*}, \tag{3.47}$$

$$\begin{aligned} \mathrm{Cotan}^{\pm} &= \mathrm{Sin}_{(2)}^{*-1} \left(\mathrm{Cotan}_{(2)} \pm \mathrm{Cotan}_{(1)}\right)^{-1} \\ &\qquad \times \left(\mathrm{Cotan}_{(1)}\ \mathrm{Cotan}_{(2)} \mp \mathcal{I}\right) \mathrm{Sin}_{(2)}^{*}. \end{aligned} \tag{3.48}$$

Proof. For identity (3.43) we have

$$\begin{aligned} \mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)} &= \mathrm{Cos}_{(1)}^{-1}\, \mathrm{Sin}_{(1)} \left(\mathrm{Sin}_{(2)}^{-1}\, \mathrm{Cos}_{(2)} \pm \mathrm{Sin}_{(1)}^{-1}\, \mathrm{Cos}_{(1)}\right) \mathrm{Cos}_{(2)}^{-1}\, \mathrm{Sin}_{(2)} \\ &= \mathrm{Tan}_{(1)} \left(\mathrm{Cotan}_{(2)} \pm \mathrm{Cotan}_{(1)}\right) \mathrm{Tan}_{(2)}\,. \end{aligned}$$

The equations in (3.44) follow from the fact that $\mathrm{Tan}_{(2)}$ is Hermitian, i.e.,

$$\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)} = \mathrm{Cos}_{(1)}^{-1} \left(\mathrm{Sin}_{(1)}\, \mathrm{Cos}_{(2)}^{*} \pm \mathrm{Cos}_{(1)}\, \mathrm{Sin}_{(2)}^{*}\right) \mathrm{Cos}_{(2)}^{*-1} = \mathrm{Cos}_{(1)}^{-1}\ \mathrm{Sin}^{\pm}\ \mathrm{Cos}_{(2)}^{*-1}\,.$$

The proofs of identities (3.45) and (3.46) are similar to the proofs of (3.43) and (3.44). Next, by using the fact that $\mathrm{Tan}_{(i)}$ are Hermitian, we obtain from (3.44) the identity

$$\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)} = \mathrm{Cos}_{(2)}^{-1} \left(\mathrm{Tan}^{\pm}\right)^{*} \left(\mathrm{Cos}^{\pm}\right)^{*} \mathrm{Cos}_{(1)}^{*-1},$$

from which we eliminate $\mathrm{Tan}^{\pm}$. That is, with $\left(\mathrm{Tan}^{\pm}\right)^{*} = \mathrm{Tan}^{\pm}$ and $\mathrm{Tan}_{(i)}^{*} = \mathrm{Tan}_{(i)}$ we have

$$\begin{aligned} \mathrm{Tan}^{\pm} &= \left(\mathrm{Cos}^{\pm}\right)^{-1} \mathrm{Cos}_{(1)} \left(\mathrm{Tan}_{(1)}^{*} \pm \mathrm{Tan}_{(2)}^{*}\right) \mathrm{Cos}_{(2)}^{*} \\ &= \left[\mathrm{Cos}_{(1)} \left(\mathcal{I} \mp \mathrm{Cos}_{(1)}^{-1}\, \mathrm{Sin}_{(1)}\, \mathrm{Sin}_{(2)}^{*}\, \mathrm{Cos}_{(2)}^{*-1}\right) \mathrm{Cos}_{(2)}^{*}\right]^{-1} \mathrm{Cos}_{(1)} \left(\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)}\right) \mathrm{Cos}_{(2)}^{*} \\ &= \mathrm{Cos}_{(2)}^{*-1} \left(\mathcal{I} \mp \mathrm{Tan}_{(1)}\, \mathrm{Tan}_{(2)}\right)^{-1} \left(\mathrm{Tan}_{(1)} \pm \mathrm{Tan}_{(2)}\right) \mathrm{Cos}_{(2)}^{*}\,. \end{aligned}$$

Therefore, the formulas in (3.47) are established. The identities in (3.48) follow from (3.47) by noticing that $\mathrm{Tan}^{\pm}\ \mathrm{Cotan}^{\pm} = \mathcal{I}$ and by using $\mathrm{Cotan}_{(i)}^{*} = \mathrm{Cotan}_{(i)}$. ∎

Remark 3.35. Consider the system (3.13) in the scalar continuous time case with $\mathcal{P}(t) \equiv 0$ and $\mathcal{Q}(t) \equiv 1$, or equivalently system (3.1) with $Q(t) \equiv 1$. Then the identities in (3.43) and (3.44) have the form

$$\tan(t) \pm \tan(s) = \frac{\mathrm{cotan}(s) \pm \mathrm{cotan}(t)}{\mathrm{cotan}(t)\ \mathrm{cotan}(s)} = \frac{\sin(t \pm s)}{\cos(t)\ \cos(s)},$$

identities (3.45) and (3.46) reduce to

$$\mathrm{cotan}(t) \pm \mathrm{cotan}(s) = \frac{\tan(s) \pm \tan(t)}{\tan(t)\ \tan(s)} = \frac{\sin(s \pm t)}{\sin(t)\ \sin(s)}.$$

In addition, it is common to write (3.43) and (3.45) as

$$\tan(t)\tan(s) = \pm\frac{\tan(t) \pm \tan(s)}{\cot(t) \pm \cot(s)}.$$

Finally, the identities in (3.47) and (3.48) correspond in this case to

$$\tan(t \pm s) = \frac{\tan(t) \pm \tan(s)}{1 \mp \tan(t)\,\tan(s)} \quad \text{and} \quad \cot(t \pm s) = \frac{\cot(t)\,\cot(s) \mp 1}{\cot(s) \pm \cot(t)}.$$

3.3 Time scale hyperbolic systems

In this section we define time scale matrix hyperbolic functions and prove analogous results as for the trigonometric functions in the previous section. In particular, we derive time scale matrix extensions of several identities which are known for the continuous time scalar hyperbolic functions. The proofs are similar to the corresponding proofs for the trigonometric case and therefore they will be omitted. We wish to remark that some results from this section have previously been derived in the unpublished paper [131] by Z. Pospíšil. We now present these results for completeness and clear comparison with the corresponding trigonometric results established in Section 3.2, as well as we derive several new formulas for time scale matrix hyperbolic functions.

Consider system ($\mathcal{S}$) on $[a,b]_{\mathbb{T}}$ with the matrix

$$\mathcal{S}(t) = \begin{pmatrix} \mathcal{P}(t) & \mathcal{Q}(t) \\ \mathcal{Q}(t) & \mathcal{P}(t) \end{pmatrix},$$

where $\mathcal{P}, \mathcal{Q} \in \mathrm{C_{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$ are $n \times n$ complex-valued matrices satisfying for all $t \in [a,\rho(b)]_{\mathbb{T}}$ the following identities

$$\mathcal{Q}^* - \mathcal{Q} + \mu\,(\mathcal{Q}^*\mathcal{P} - \mathcal{P}^*\mathcal{Q}) = 0, \tag{3.49}$$

$$\mathcal{P} + \mathcal{P}^* + \mu\,(\mathcal{P}^*\mathcal{P} - \mathcal{Q}^*\mathcal{Q}) = 0, \tag{3.50}$$

see also [131, p. 9] and [87, Theorem 8].

Definition 3.36 (Time scale hyperbolic system)**.** The system

$$X^{\Delta} = \mathcal{P}(t)\,X + \mathcal{Q}(t)\,U, \quad U^{\Delta} = \mathcal{Q}(t)\,X + \mathcal{P}(t)\,U, \tag{3.51}$$

where the matrices $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ satisfy identities (3.49) and (3.50) for all $t \in [a,\rho(b)]_{\mathbb{T}}$, is called a *time scale hyperbolic system*.

Remark 3.37. The above time scale hyperbolic system is in general defined through two coefficient matrices $\mathcal{P}$ and $\mathcal{Q}$. However, in the continuous time case we can use the same transformation as in Remark 3.6 and write the hyperbolic system from (3.51) in the form of (3.5). Similarly, by using the same arguments as in Remark 3.7, in the discrete case we can write the above hyperbolic system in the form (3.6).

Remark 3.38. In the discrete case it is known that the matrix $\mathcal{P}_k$ is necessarily invertible for all $k \in [a,b]_{\mathbb{Z}}$, see [61, identity (12)] or [162, Remark 67]. Similarly, in the general time scale setting we have that identity (3.50) implies $(\mathcal{I} + \mu\mathcal{P}^*)\,(\mathcal{I} + \mu\mathcal{P}) = \mathcal{I} + \mu^2\,\mathcal{Q}^*\mathcal{Q} > 0$, that is, the matrix $\mathcal{I} + \mu\mathcal{P}$ is invertible. And then (3.49) yields that $\mathcal{Q}\,(\mathcal{I} + \mu\mathcal{P})^{-1}$ is Hermitian.

Remark 3.39. Similarly to Remark 3.5, symplectic system ($\mathcal{S}$) can be written as a time scale hyperbolic system if there exist normalized conjoined bases $Z = \binom{X}{U}$ and $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ of system ($\mathcal{S}$) such that $X\tilde{X}^*$ is positive definite.

Lemma 3.40. *The pair $\binom{X}{U}$ solves system (3.51) if and only if the pair $\binom{U}{X}$ solves the same hyperbolic system.*

Following [131, Definition 2.1], we define the time scale matrix hyperbolic functions. See also the discrete version in [61, Definition 3.1] or [162, Definition 32].

Definition 3.41. Let $s \in [a,b]_{\mathbb{T}}$ be fixed. We define the $n \times n$ matrix valued functions *hyperbolic sine* (denoted by Sinh_s) and *hyperbolic cosine* (denoted by Cosh_s) as

$$\mathrm{Sinh}_s(t) := X(t) \quad \text{and} \quad \mathrm{Cosh}_s(t) := U(t),$$

respectively, where the pair $\binom{X}{U}$ is the principal solution of system (3.51) at s, i.e., it is given by the initial conditions $X(s) = 0$ and $U(s) = \mathcal{I}$. We suppress the index s when $s = a$, i.e., we denote $\mathrm{Sinh} := \mathrm{Sinh}_a$ and $\mathrm{Cosh} := \mathrm{Cosh}_s$.

Remark 3.42. (i) The matrix functions Sinh_s and Cosh_s are n-dimensional analogs of the scalar hyperbolic functions $\sinh(t-s)$ and $\cosh(t-s)$.

(ii) When $n = 1$ and $\mathcal{P} = 0$ and $\mathcal{Q} = p$ with $p \in \mathrm{C_{rd}}$, the matrix functions $\mathrm{Sinh}_s(t)$ and $\mathrm{Cosh}_s(t)$ reduce exactly to the scalar time scale hyperbolic functions $\sinh_p(t,s)$ and $\cosh_p(t,s)$ from [32, Definition 3.17].

(iii) In the continuous time scalar case and when $\mathcal{P} = 0$, i.e., system (3.51) is the same as (3.5), we have $\mathrm{Sinh}(t) = \sinh \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$ and $\mathrm{Cosh}(t) = \cosh \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau$, see [71, p. 12]. Similar formulas hold for the discrete scalar case, see [61, equations (27)–(28)].

Since the solutions $\binom{\mathrm{Cosh}}{\mathrm{Sinh}}$ and $\binom{\mathrm{Sinh}}{\mathrm{Cosh}}$ form normalized conjoined bases of (3.51),

$$\hat{\Psi}(t) := \begin{pmatrix} \mathrm{Cosh}(t) & \mathrm{Sinh}(t) \\ \mathrm{Sinh}(t) & \mathrm{Cosh}(t) \end{pmatrix}$$

is a fundamental matrix of (3.51). Therefore, every solution $\binom{X}{U}$ of (3.51) has the form

$$X(t) = \mathrm{Cosh}(t)\,X(a) + \mathrm{Sinh}(t)\,U(a) \quad \text{and} \quad U(t) = \mathrm{Sinh}(t)\,X(a) + \mathrm{Cosh}(t)\,U(a)$$

for all $t \in [a,b]_{\mathbb{T}}$. As a consequence of formulas (3.9) and (3.10) we get for solutions of time scale hyperbolic systems the following, see also [131, Theorem 2.1].

Corollary 3.43. *For all $t \in [a,b]_{\mathbb{T}}$ the identities*

$$\mathrm{Cosh}^*\mathrm{Cosh} - \mathrm{Sinh}^*\mathrm{Sinh} = \mathcal{I} = \mathrm{Cosh}\,\mathrm{Cosh}^* - \mathrm{Sinh}\,\mathrm{Sinh}^*, \tag{3.52}$$

$$\mathrm{Cosh}^*\mathrm{Sinh} = \mathrm{Sinh}^*\mathrm{Cosh}, \quad \mathrm{Cosh}\,\mathrm{Sinh}^* = \mathrm{Sinh}\,\mathrm{Cosh}^* \tag{3.53}$$

hold, while for all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have the identities

$$\mathrm{Cosh}^\sigma\,\mathrm{Cosh}^* - \mathrm{Sinh}^\sigma\,\mathrm{Sinh}^* = \mathcal{I} + \mu\mathcal{P}, \quad \mathrm{Sinh}^\sigma\,\mathrm{Cosh}^* - \mathrm{Cosh}^\sigma\,\mathrm{Sinh}^* = \mu\mathcal{Q}.$$

Now we establish a matrix analog of the formula $\cosh^2(t) - \sinh^2(t) = 1$, see also [131, Theorem 2.1], as well as the formulas from [131, Theorem 2.5].

Corollary 3.44. *For all $t \in [a,b]_{\mathbb{T}}$ the identity*

$$\|\mathrm{Cosh}\|_F^2 - \|\mathrm{Sinh}\|_F^2 = n$$

holds, while for all $t \in [a,\rho(b)]_{\mathbb{T}}$ we have

$$\mathrm{Cosh}^\Delta \mathrm{Cosh}^* - \mathrm{Sinh}^\Delta \mathrm{Sinh}^* = \mathcal{P}, \quad \mathrm{Sinh}^\Delta \mathrm{Cosh}^* - \mathrm{Cosh}^\Delta \mathrm{Sinh}^* = \mathcal{Q}.$$

Remark 3.45. It follows from identity (3.52) that the matrix $\mathrm{Cosh}(t)$ is invertible for all $t \in [a,b]_{\mathbb{T}}$. Moreover, if $\mathrm{Sinh}(t)$ is invertible at some t, then from (3.52) and (3.53) we obtain

$$\mathrm{Cosh}^{-1} = \mathrm{Cosh}^* - \mathrm{Sinh}^* \mathrm{Cosh}^{*-1} \mathrm{Sinh}^*, \quad \mathrm{Sinh}^{-1} = \mathrm{Cosh}^* \mathrm{Sinh}^{*-1} \mathrm{Cosh}^* - \mathrm{Sinh}^*.$$

The following additive formulas are established in [131, Theorem 2.2]. They are proven in a similar way as in Theorem 3.16.

Theorem 3.46. *For $t,s \in [a,b]_{\mathbb{T}}$ we have*

$$\mathrm{Sinh}_s(t) = \mathrm{Sinh}(t)\,\mathrm{Cosh}^*(s) - \mathrm{Cosh}(t)\,\mathrm{Sinh}^*(s), \tag{3.54}$$
$$\mathrm{Cosh}_s(t) = \mathrm{Cosh}(t)\,\mathrm{Cosh}^*(s) - \mathrm{Sinh}(t)\,\mathrm{Sinh}^*(s), \tag{3.55}$$
$$\mathrm{Sinh}(t) = \mathrm{Sinh}_s(t)\,\mathrm{Cosh}(s) + \mathrm{Cosh}_s(t)\,\mathrm{Sinh}(s), \tag{3.56}$$
$$\mathrm{Cosh}(t) = \mathrm{Cosh}_s(t)\,\mathrm{Cosh}(s) + \mathrm{Sinh}_s(t)\,\mathrm{Sinh}(s). \tag{3.57}$$

Remark 3.47. With respect to Remark 3.42 for the scalar continuous time case, identities (3.54)–(3.55) are matrix analogues of

$$\sinh(t-s) = \sinh(t)\cosh(s) - \cosh(t)\sinh(s),$$
$$\cosh(t-s) = \cosh(t)\cosh(s) - \sinh(t)\sinh(s),$$

while identities (3.56)–(3.57) are matrix analogues of

$$\sinh(t) = \sinh(t-s)\cosh(s) + \cosh(t-s)\sinh(s),$$
$$\cosh(t) = \cosh(t-s)\cosh(s) + \sinh(t-s)\sinh(s).$$

Interchanging the parameters t and s in (3.54) and (3.55) yields expected properties of the time scale matrix hyperbolic functions, see also [131, formula (34)].

Corollary 3.48. *Let $t,s \in [a,b]_{\mathbb{T}}$. Then*

$$\mathrm{Sinh}_s(t) = -\mathrm{Sinh}_t^*(s) \quad \text{and} \quad \mathrm{Cosh}_s(t) = \mathrm{Cosh}_t^*(s). \tag{3.58}$$

Remark 3.49. In the scalar continuous time case and when $\mathcal{Q}(t) \equiv 1$ and $s = 0$, the formulas in (3.58) show that $\sinh(t) = -\sinh(-t)$ and $\cosh(t) = \cosh(-t)$. So we can see that Corollary 3.48 gives the matrix analogies of the statement about the parity for the scalar functions hyperbolic sine and hyperbolic cosine.

Now we use the same approach as for the time scale trigonometric functions to obtain generalized sum and difference formulas for solutions of two time scale hyperbolic systems. Hence, we consider the following two time scale hyperbolic systems

$$X^\Delta = \mathcal{P}_{(i)}(t)\,X + \mathcal{Q}_{(i)}(t)\,U, \quad U^\Delta = \mathcal{Q}_{(i)}(t)\,X + \mathcal{P}_{(i)}(t)\,U \tag{3.59}$$

with initial conditions $X_{(i)}(a) = 0$ and $U_{(i)}(a) = \mathcal{I}$, where $i = 1, 2$. Denote by $\mathrm{Sinh}_{(i)}(t)$ and $\mathrm{Cosh}_{(i)}(t)$ the corresponding matrix-valued hyperbolic sine and hyperbolic cosine functions from Definition 3.41. If we set

$$\mathrm{Sinh}^{\pm}(t) := \mathrm{Sinh}_{(1)}(t)\,\mathrm{Cosh}^{*}_{(2)}(t) \pm \mathrm{Cosh}_{(1)}(t)\,\mathrm{Sinh}^{*}_{(2)}(t), \tag{3.60}$$

$$\mathrm{Cosh}^{\pm}(t) := \mathrm{Cosh}_{(1)}(t)\,\mathrm{Cosh}^{*}_{(2)}(t) \pm \mathrm{Sinh}_{(1)}(t)\,\mathrm{Sinh}^{*}_{(2)}(t), \tag{3.61}$$

then similarly to Theorem 3.20 we have the following.

Theorem 3.50. *Assume that $\mathcal{P}_{(i)}$ and $\mathcal{Q}_{(i)}$ satisfy (3.49) and (3.50). The pair $\mathrm{Sinh}^{\pm}$ and $\mathrm{Cosh}^{\pm}$ solves the system*

$$\begin{aligned} X^{\Delta} &= \mathcal{P}_{(1)}(t)\,X + \mathcal{Q}_{(1)}(t)\,U + X\mathcal{P}^{*}_{(2)}(t) \pm U\mathcal{Q}^{*}_{(2)}(t) \\ &\qquad + \mu(t)\Big[\mathcal{P}_{(1)}(t)\Big(X\mathcal{P}^{*}_{(2)}(t) \pm U\mathcal{Q}^{*}_{(2)}(t)\Big) + \mathcal{Q}_{(1)}(t)\Big(\pm X\mathcal{Q}^{*}_{(2)}(t) + U\mathcal{P}^{*}_{(2)}(t)\Big)\Big], \\ U^{\Delta} &= \mathcal{Q}_{(1)}(t)\,X + \mathcal{P}_{(1)}(t)\,U \pm X\mathcal{Q}^{*}_{(2)}(t) + U\mathcal{P}^{*}_{(2)}(t) \\ &\qquad + \mu(t)\Big[\mathcal{Q}_{(1)}(t)\Big(X\mathcal{P}^{*}_{(2)}(t) \pm U\mathcal{Q}^{*}_{(2)}(t)\Big) + \mathcal{P}_{(1)}(t)\Big(\pm X\mathcal{Q}^{*}_{(2)}(t) + U\mathcal{P}^{*}_{(2)}(t)\Big)\Big] \end{aligned}$$

with the initial conditions $X(a) = 0$ and $U(a) = \mathcal{I}$. Moreover, for all $t \in [a, b]_{\mathbb{T}}$ we have

$$\mathrm{Cosh}^{\pm}(\mathrm{Cosh}^{\pm})^{*} - \mathrm{Sinh}^{\pm}\left(\mathrm{Sinh}^{\pm}\right)^{*} = \mathcal{I} = \left(\mathrm{Cosh}^{\pm}\right)^{*}\mathrm{Cosh}^{\pm} - \left(\mathrm{Sinh}^{\pm}\right)^{*}\mathrm{Sinh}^{\pm}, \tag{3.62}$$

$$\mathrm{Sinh}^{\pm}(\mathrm{Cosh}^{\pm})^{*} = \mathrm{Cosh}^{\pm}\left(\mathrm{Sinh}^{\pm}\right)^{*}, \quad \left(\mathrm{Sinh}^{\pm}\right)^{*}\mathrm{Cosh}^{\pm} = \left(\mathrm{Cosh}^{\pm}\right)^{*}\mathrm{Sinh}^{\pm}. \tag{3.63}$$

Remark 3.51. An analogous statement as in Remark 3.21 now applies to the solutions $\begin{pmatrix} \mathrm{Sinh}^{+} \\ \mathrm{Cosh}^{+} \end{pmatrix}$ and $\begin{pmatrix} \mathrm{Sinh}^{-} \\ \mathrm{Cosh}^{-} \end{pmatrix}$. Namely, these two pairs are not conjoined bases of their corresponding systems, because these systems are not symplectic.

Remark 3.52. In the continuous time case, the assertion of Theorem 3.50 can be found in [71, Theorem 4.2]. For the discrete time hyperbolic systems this result is new, see the details in [163, Theorem 4.13].

When the two systems in (3.59) are the same, Theorem 3.50 yields the following.

Corollary 3.53. *Assume that $\mathcal{P}$ and $\mathcal{Q}$ satisfy (3.49) and (3.50). Then the system*

$$\begin{aligned} X^{\Delta} &= \mathcal{P}(t)\,X + \mathcal{Q}(t)\,U + X\mathcal{P}^{*}(t) + U\mathcal{Q}^{*}(t) \\ &\qquad + \mu(t)\Big[\mathcal{P}(t)\big(X\mathcal{P}^{*}(t) + U\mathcal{Q}^{*}(t)\big) + \mathcal{Q}(t)\big(X\mathcal{Q}^{*}(t) + U\mathcal{P}^{*}(t)\big)\Big], \\ U^{\Delta} &= \mathcal{Q}(t)\,X + \mathcal{P}(t)\,U + X\mathcal{Q}^{*}(t) + U\mathcal{P}^{*}(t) \\ &\qquad + \mu(t)\Big[\mathcal{Q}(t)\big(X\mathcal{P}^{*}(t) + U\mathcal{Q}^{*}(t)\big) + \mathcal{P}(t)\big(X\mathcal{Q}^{*}(t) + U\mathcal{P}^{*}(t)\big)\Big] \end{aligned}$$

with the initial conditions $X(a) = 0$ and $U(a) = \mathcal{I}$ possesses the solution

$$X = 2\,\mathrm{Sinh}\,\mathrm{Cosh}^{*} \quad \text{and} \quad U = \mathrm{Cosh}\,\mathrm{Cosh}^{*} + \mathrm{Sinh}\,\mathrm{Sinh}^{*},$$

where Sinh *and* Cosh *are the matrix functions in Definition 3.41. Moreover, the above matrices X and U commute, i.e., $XU = UX$.*

Remark 3.54. The previous corollary can be viewed as the $n-$dimensional analogy of the double angle formulas for scalar continuous time hyperbolic functions, i.e.,

$$\sinh(2t) = 2\sinh(t)\cosh(t) \quad \text{and} \quad \cosh(2t) = \cosh^2(t) + \sinh^2(t).$$

In the continuous time case the content of Corollary 3.53 can be found in [71, Corollary 1]. In the discrete case we get a new result, namely [163, Corollary 4.15].

Now we can prove as in Corollary 3.25 the following identities.

Corollary 3.55. *For all $t \in [a, b]_{\mathbb{T}}$ we have the identities*

$$\mathrm{Sinh}_{(1)}\ \mathrm{Sinh}^*_{(2)} = \frac{1}{2}\left(\mathrm{Cosh}^+ - \mathrm{Cosh}^-\right), \tag{3.64}$$

$$\mathrm{Cosh}_{(1)}\ \mathrm{Cosh}^*_{(2)} = \frac{1}{2}\left(\mathrm{Cosh}^+ + \mathrm{Cosh}^-\right), \tag{3.65}$$

$$\mathrm{Sinh}_{(1)}\ \mathrm{Cosh}^*_{(2)} = \frac{1}{2}\left(\mathrm{Sinh}^+ + \mathrm{Sinh}^-\right). \tag{3.66}$$

Remark 3.56. In the scalar continuous time case identities (3.64)–(3.66) have the form

$$\begin{aligned} \sinh(t)\sinh(s) &= \frac{1}{2}[\cosh(t+s) - \cosh(t-s)], \\ \cosh(t)\cosh(s) &= \frac{1}{2}[\cosh(t+s) + \cosh(t-s)], \\ \sinh(t)\cosh(s) &= \frac{1}{2}[\sinh(t+s) + \sinh(t-s)]. \end{aligned}$$

The next definition of time scale matrix hyperbolic tangent and cotangent functions is from [131, Definition 2.2]. It extends the discrete matrix hyperbolic tangent and hyperbolic cotangent functions known in [61, Definition 3.2] to time scales. Recall that the matrix function Cosh is invertible for all $t \in [a, b]_{\mathbb{T}}$, see Remark 3.45.

Definition 3.57. We define the matrix-valued function *hyperbolic tangent* (we write Tanh) and, whenever $\mathrm{Sinh}(t)$ is invertible, the matrix-valued function *hyperbolic cotangent* (we write Cotanh) in the form

$$\mathrm{Tanh}(t) := \mathrm{Cosh}^{-1}(t)\ \mathrm{Sinh}(t) \quad \text{and} \quad \mathrm{Cotanh}(t) := \mathrm{Sinh}^{-1}(t)\ \mathrm{Cosh}(t), \ \text{respectively.}$$

Remark 3.58. Analogous results concerning $\mathrm{Tanh}(t)$ and $\mathrm{Cotanh}(t)$ which are presented below, can be obtained by using the definitions

$$\widetilde{\mathrm{Tanh}}(t) := \mathrm{Sinh}(t)\ \mathrm{Cosh}^{-1}(t) \quad \text{and} \quad \widetilde{\mathrm{Cotanh}}(t) := \mathrm{Cosh}(t)\ \mathrm{Sinh}^{-1}(t).$$

Similarly to Theorems 3.29 and 3.30 we can establish that the functions Tanh and Cotanh are Hermitian. The following two results can be found in [131, Theorems 2.4, 2.5].

Theorem 3.59. *The following identities hold true*

$$\mathrm{Tanh}^*(t) = \mathrm{Tanh}(t), \tag{3.67}$$

$$\mathrm{Cosh}^{-1}(t)\,\mathrm{Cosh}^{*-1}(t) + \mathrm{Tanh}^2(t) = \mathcal{I}, \tag{3.68}$$

$$\mathrm{Tanh}^{\Delta}(t) = [\mathrm{Cosh}^{\sigma}(t)]^{-1}\ \mathcal{Q}(t)\ \mathrm{Cosh}^{*-1}(t). \tag{3.69}$$

Theorem 3.60. *Whenever* $\mathrm{Cotanh}(t)$ *is defined we obtain*

$$\mathrm{Cotanh}^*(t) = \mathrm{Cotanh}(t), \tag{3.70}$$
$$\mathrm{Cotanh}^2(t) - \mathrm{Sinh}^{-1}(t)\,\mathrm{Sinh}^{*-1}(t) = \mathcal{I}. \tag{3.71}$$

Moreover, if $\mathrm{Sinh}(t)$ *and* $\mathrm{Sinh}^\sigma(t)$ *are invertible, then*

$$\mathrm{Cotanh}^\Delta(t) = -[\mathrm{Sinh}^\sigma(t)]^{-1}\,\mathcal{Q}(t)\,\mathrm{Sinh}^{*-1}(t). \tag{3.72}$$

Remark 3.61. In the scalar case $n = 1$ identities (3.67) and (3.70) are trivial. In the scalar continuous time case identities (3.68), (3.71), (3.69), and (3.72) correspond to

$$\frac{1}{\cosh^2(s)} + \tanh^2(s) = 1, \quad \mathrm{cotanh}^2(s) - \frac{1}{\sinh^2(s)} = 1, \quad \text{with } s = \int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau,$$
$$\left(\tanh\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau\right)' = \frac{\mathcal{Q}(t)}{\cosh^2 s}, \quad \left(\mathrm{cotanh}\int_a^t \mathcal{Q}(\tau)\,\mathrm{d}\tau\right)' = \frac{-\mathcal{Q}(t)}{\sinh^2 s},$$

compare with Remark 3.42 (iii). The discrete versions of these identities can be found in [61, Theorem 3.4] or [162, Theorem 89].

Next, similarly to the definitions of the time scale matrix functions $\mathrm{Sinh}_{(i)}$, $\mathrm{Cosh}_{(i)}$, for $i = 1, 2$, $\mathrm{Sinh}^\pm$, and $\mathrm{Cosh}^\pm$ from (3.59)–(3.61) we define

$$\mathrm{Tanh}_{(i)}(t) := \mathrm{Cosh}_{(i)}^{-1}(t)\,\mathrm{Sinh}_{(i)}(t), \qquad \mathrm{Cotanh}_{(i)}(t) := \mathrm{Sinh}_{(i)}^{-1}(t)\,\mathrm{Cosh}_{(i)}(t),$$
$$\mathrm{Tanh}^\pm(t) := \left[\mathrm{Cosh}^\pm(t)\right]^{-1}\mathrm{Sinh}^\pm(t), \qquad \mathrm{Cotanh}^\pm(t) := \left[\mathrm{Sinh}^\pm(t)\right]^{-1}\mathrm{Cosh}^\pm(t).$$

Remark 3.62. As in Remark 3.33 we conclude that the first identities from (3.63) imply $\left(\mathrm{Tanh}^\pm\right)^* = \mathrm{Tanh}^\pm$. Similarly, the functions $\mathrm{Cotanh}^\pm$ are also Hermitian.

As it was the case for the trigonometric functions in Theorem 3.34, the results of the following theorem are new even in the special case of continuous and discrete time, see also [163, Theorem 4.25].

Theorem 3.63. *For all* $t \in [a, b]_{\mathbb{T}}$ *such that all involved functions are defined we have (suppressing the argument* t*)*

$$\mathrm{Tanh}_{(1)} \pm \mathrm{Tanh}_{(2)} = \mathrm{Tanh}_{(1)}\left(\mathrm{Cotanh}_{(2)} \pm \mathrm{Cotanh}_{(1)}\right)\mathrm{Tanh}_{(2)}, \tag{3.73}$$
$$\mathrm{Tanh}_{(1)} \pm \mathrm{Tanh}_{(2)} = \mathrm{Cosh}_{(1)}^{-1}\,\mathrm{Sinh}^\pm\,\mathrm{Cosh}_{(2)}^{*-1}, \tag{3.74}$$
$$\mathrm{Cotanh}_{(1)} \pm \mathrm{Cotanh}_{(2)} = \mathrm{Cotanh}_{(1)}\left(\mathrm{Tanh}_{(2)} \pm \mathrm{Tanh}_{(1)}\right)\mathrm{Cotanh}_{(2)}, \tag{3.75}$$
$$\mathrm{Cotanh}_{(1)} \pm \mathrm{Cotanh}_{(2)} = \pm\,\mathrm{Sinh}_{(1)}^{-1}\,\mathrm{Sinh}^\pm\,\mathrm{Sinh}_{(2)}^{*-1}, \tag{3.76}$$
$$\mathrm{Tanh}^\pm = \mathrm{Cosh}_{(2)}^{*-1}\left(\mathcal{I} \pm \mathrm{Tanh}_{(1)}\,\mathrm{Tanh}_{(2)}\right)^{-1} \times \left(\mathrm{Tanh}_{(1)} \pm \mathrm{Tanh}_{(2)}\right)\mathrm{Cosh}_{(2)}^*, \tag{3.77}$$
$$\mathrm{Cotanh}^\pm = \mathrm{Sinh}_{(2)}^{*-1}\left(\mathrm{Cotanh}_{(2)} \pm \mathrm{Cotanh}_{(1)}\right)^{-1} \times \left(\mathrm{Cotanh}_{(1)}\,\mathrm{Cotanh}_{(2)} \pm \mathcal{I}\right)\mathrm{Sinh}_{(2)}^*. \tag{3.78}$$

Remark 3.64. Consider now the system (3.51) in the scalar continuous time case with $\mathcal{P}(t) \equiv 0$ and $\mathcal{Q}(t) \equiv 1$, or equivalently system (3.5) with $Q(t) \equiv 1$. Then the identities in (3.73) and (3.74) have the form

$$\tanh(t) \pm \tanh(s) = \frac{\mathrm{cotanh}(s) \pm \mathrm{cotanh}(t)}{\mathrm{cotanh}(t)\,\mathrm{cotanh}(s)} = \frac{\sinh(t \pm s)}{\cosh(t)\,\cosh(s)},$$

identities (3.75) and (3.76) reduce to

$$\operatorname{cotanh}(t) \pm \operatorname{cotanh}(s) = \frac{\tanh(s) \pm \tanh(t)}{\tanh(t)\,\tanh(s)} = \frac{\sinh(s \pm t)}{\sinh(t)\,\sinh(s)}.$$

In addition, it is common to write (3.73) and (3.75) as

$$\tanh(t)\,\tanh(s) = \frac{\tanh(t) \pm \tanh(s)}{\operatorname{cotanh}(s) \pm \operatorname{cotan}(t)}.$$

Finally, the identities in (3.77) and (3.78) correspond in this case to

$$\tanh(t \pm s) = \frac{\tanh(t) \pm \tanh(s)}{1 \pm \tanh(t)\,\tanh(s)} \quad \text{and} \quad \operatorname{cotanh}(t \pm s) = \frac{1 \pm \operatorname{cotanh}(t)\,\operatorname{cotanh}(s)}{\operatorname{cotanh}(t) \pm \operatorname{cotanh}(s)}.$$

3.4 Concluding remarks

In this chapter we extended to the time scale matrix case several identities known in particular for the scalar continuous time trigonometric and hyperbolic functions. Namely, for trigonometric functions these are the identity $\cos^2(t) + \sin^2(t) = 1$ in Corollary 3.13, and the identities displayed in Theorems 3.16, 3.29, and 3.34, Remarks 3.19 and 3.31, and Corollaries 3.23 and 3.25. For hyperbolic functions these are the identity $\cosh^2(t) - \sinh^2(t) = 1$ in Corollary 3.44, and the identities displayed in Theorems 3.46 and 3.63, Remarks 3.49 and 3.61, and Corollary 3.55.

On the other hand, there are still several trigonometric and hyperbolic identities which we could not extend to the general time scale matrix case. For example, these are the identities

$$\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2},$$

as well as other corresponding identities for the sum or difference of scalar trigonometric and hyperbolic functions. When $y = 0$ in the above identity, we get

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}.$$

The right-hand side is similar to the solution $X(t)$ in Corollary 3.23, but the left-hand side is not the matrix function Sin, because the corresponding system is not a trigonometric system from (3.13), see Remark 3.21.

Furthermore, as for the time scale versions of the identities

$$\left.\begin{aligned} \sin(x+y)\sin(x-y) &= \sin^2 x - \sin^2 y, \\ \cos(x+y)\cos(x-y) &= \cos^2 x - \sin^2 y, \\ \sinh(x+y)\sinh(x-y) &= \sinh^2 x - \sinh^2 y, \\ \cosh(x+y)\cosh(x-y) &= \sinh^2 x + \cosh^2 y, \end{aligned}\right\} \tag{3.79}$$

in the *scalar case* on an arbitrary time scale we can calculate the products

$$\left.\begin{aligned} \mathrm{Sin}^+\ \mathrm{Sin}^- &= \mathrm{Sin}^2_{(1)} - \mathrm{Sin}^2_{(2)}, & \mathrm{Sinh}^+\ \mathrm{Sinh}^- &= \mathrm{Sinh}^2_{(1)} - \mathrm{Sinh}^2_{(2)}, \\ \mathrm{Cos}^+\ \mathrm{Cos}^- &= \mathrm{Cos}^2_{(1)} - \mathrm{Sin}^2_{(2)}, & \mathrm{Cosh}^+\ \mathrm{Cosh}^- &= \mathrm{Sinh}^2_{(1)} + \mathrm{Cosh}^2_{(2)}, \end{aligned}\right\} \tag{3.80}$$

because the cross terms cancel due to the commutativity. However, in the *general case* the matrix products $\mathrm{Sin}^+\,\mathrm{Sin}^-$, $\mathrm{Cos}^+\,\mathrm{Cos}^-$, $\mathrm{Sinh}^+\,\mathrm{Sinh}^-$, and $\mathrm{Cosh}^+\,\mathrm{Cosh}^-$ corresponding to the left-hand side of (3.79) do not simplify as in (3.80), since the matrix multiplication is not commutative.

By straightforward calculations we can verify that

$$(\mathrm{Cos}_s \pm i\,\mathrm{Sin}_s)^\Delta = (\mathcal{P} \pm i\mathcal{Q})\,(\mathrm{Cos}_s \pm i\,\mathrm{Sin}_s)\,,$$
$$(\mathrm{Sinh}_s \pm \mathrm{Cosh}_s)^\Delta = (\mathcal{P} \pm \mathcal{Q})\,(\mathrm{Sinh}_s \pm \mathrm{Cosh}_s)$$

hold. Hence, with using the exponential function introduced in [32, Section 5.1], see also [32, Section 2.2], we can define the trigonometric and hyperbolic functions alternatively in the form

$$\mathrm{Sin}_s(t) = \frac{e_{\mathcal{P}+i\mathcal{Q}}(t,s) - e_{\mathcal{P}-i\mathcal{Q}}(t,s)}{2i}, \quad \mathrm{Cos}_s(t) = \frac{e_{\mathcal{P}+i\mathcal{Q}}(t,s) + e_{\mathcal{P}-i\mathcal{Q}}(t,s)}{2},$$
$$\mathrm{Sinh}_s(t) = \frac{e_{\mathcal{P}+\mathcal{Q}}(t,s) - e_{\mathcal{P}-\mathcal{Q}}(t,s)}{2}, \quad \mathrm{Cosh}_s(t) = \frac{e_{\mathcal{P}+\mathcal{Q}}(t,s) + e_{\mathcal{P}-\mathcal{Q}}(t,s)}{2},$$

where the matrices $\mathcal{P} \pm i\mathcal{Q}$ or equivalently $2\mathcal{P} + \mu\left(\mathcal{P}^2 + \mathcal{Q}^2 + i\,(\mathcal{Q}\mathcal{P} - \mathcal{P}\mathcal{Q})\right)$ and matrices $\mathcal{P} \pm \mathcal{Q}$ or equivalently $2\mathcal{P} + \mu\left(\mathcal{P}^2 + \mathcal{Q}^2 + \mathcal{Q}\mathcal{P} - \mathcal{P}\mathcal{Q}\right)$ are regressive, compare also to Remarks 3.10(ii) and 3.42(ii).

Finally, for any time scale the following pairs of $2n \times 2n$ and $(2n+1)\times(2n+1)$ matrices

$$\mathcal{P} = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 \\ -\mathcal{I} & 0 & 0 \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} 0 & 0 & \mathcal{I} \\ 0 & 0 & 0 \\ \mathcal{I} & 0 & 0 \end{pmatrix}$$

determine $4n \times 4n$ and $(4n+2)\times(4n+2)$ hyperbolic systems, respectively. On the other hand, the problem of finding similar coefficients for trigonometric systems remains unsolved (but we conjecture that they do not exist in such simple form).

3.5 Bibliographical notes

The basic references for symplectic systems on time scales are [57, 58] and [32, Chapter 7]. The existence of a solution of the initial value problems connected with symplectic system ($\mathcal{S}$) was proven in [57, Corollary 7.12]. The proof of the existence of a conjoined basis completing a given conjoined basis to a normalized pair can the reader find in [32, Lemma 7.29]. The statement of Proposition 3.3 comes from [32, Lemma 7.27]. For the definition of the self-reciprocal systems and transformation mentioned in Remark 3.5 see [58, Definition 4] and [58, Theorem 2], respectively. The statement of Remark 3.39 corresponds to [57, Theorem 10.56].

The results presented in this chapter were published by R. Šimon Hilscher and the author in [100], and their special case (for $\mathbb{T} = \mathbb{Z}$) by the author in [163].

A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.

H. WEYL

Chapter 4

WEYL–TITCHMARSH THEORY FOR SYMPLECTIC DYNAMIC SYSTEMS

In this chapter we develop systematically the Weyl–Titchmarsh theory for time scale symplectic systems. As the research in the Weyl–Titchmarsh theory has been very active in the last years, we contribute to this development by presenting a theory which directly generalizes and unifies the results in several recent papers, such as [34, 45, 150, 166], and partly in [6, 38, 40, 43, 44, 106, 112, 122, 138, 142].

It is well known that the second order Sturm–Liouville differential equations

$$-\big(p(t)\,x'\big)' + q(t)\,x = \lambda\,w(t)\,x, \quad t \in [a, \infty), \tag{4.1}$$

can be divided into two cases depending on the count of its square-integrable solutions. Namely, in the *limit point case* there is exactly one (up to a multiplicative constant) square-integrable solution, and in the *limit circle case* there are two linearly independent square-integrable solutions. This dichotomy was initially investigated (by using a geometrical approach) by Weyl in his paper [160] from 1910. One of the most important contributions in extending this theory was made by Titchmarsh in the series of papers from 1939–1945 (especially in papers [153–155] from 1941), which were summarized in his book [156]. He re-proved Weyl's results by using an alternative method and established many properties of the fundamental function appearing in this theory, the so-called $m(\lambda)$-function. Hence in honor of the pioneers of this theory, it is called the *Weyl–Titchmarsh theory*. We refer to [22, 67, 158] for an overview of the original contributions to the Weyl–Titchmarsh theory for equation (4.1)

Their results were extended in many ways. First of all, there were weakened conditions put on the coefficients of the differential equation. For an overview of the progress in this way we refer to [67, Section 1]. In addition, Sims discussed in [147] equation (4.1) with $p(\cdot) \equiv 1$, $w(\cdot) \equiv 1$, and he allowed $q(\cdot)$ to be a complex function. This change gives (surprisingly) a new limit point behavior which does not occur when $q(\cdot)$ is real-valued, namely, there are two linearly independent square-integrable solutions while the equation is in the limit point case. Therefore, paper [147] started the development of the so-called *Titchmarsh–Sims–Weyl theory*.

According to [67], the investigation of the Weyl–Titchmarsh theory for the second order difference equations

$$b_{k-1}y_{k-1} + a_k y_k + b_k y_{k+1} = \lambda y_k, \quad k \in \mathbb{N} \setminus \{0\},$$

where $a_k \in \mathbb{R}$ and $b_k > 0$ for all $k \in \mathbb{N}$, was initiated by Hellinger and Nevanlinna in their independent papers from 1922, see [84,124]. Since then a long time elapsed until the theory of difference equations attracted more attention. Some results concerning the Weyl–Titchmarsh theory for the second order difference equations can be found, e.g., in [12,36,37,41,148]. In particular, in [36,37] the second order difference equation

$$-\Delta\,(p_k\,\Delta x_k) + q_k x_{k+1} = \lambda\, w_k x_{k+1}, \quad k \in \mathbb{N}$$

was investigated, where p_k, q_k, w_k are real and satisfy $p_k \neq 0$ and $w_k > 0$. A comprehensive summary of the history of the Weyl–Titchmarsh theory for the second order differential and difference equations can be found in the expository paper [67] by Everitt.

Extensions of the Weyl–Titchmarsh theory to more general equations, namely to the linear Hamiltonian differential systems

$$z'(t) = [\mathcal{H}(t) + \lambda\, \mathcal{H}_0(t)]\; z(t), \quad t \in [0, \infty),$$

where $\mathcal{H}(\cdot)$ and $\mathcal{H}_0(\cdot)$ are $2n \times 2n$ complex-valued Hermitian matrices, was initiated in [16] and developed further in [38,42,43,68,101–108,111–114,119,133,137,141].

For higher order Sturm–Liouville difference equations and linear Hamiltonian difference systems, such as

$$\Delta x_k = A_k x_{k+1} + (B_k + \lambda W_k^{[2]})\, u_k, \quad \Delta u_k = (C_k - \lambda W_k^{[1]})\, x_{k+1} - A_k^* u_k, \quad k \in [0, \infty)_{\mathbb{Z}},$$

where A_k, B_k, C_k, $W_k^{[1]}$, $W_k^{[2]}$ are complex $n \times n$ matrices such that B_k and C_k are Hermitian and $W_k^{[1]}$ and $W_k^{[2]}$ are Hermitian and nonnegative definite, the Weyl–Titchmarsh theory was studied in [44,142,149]. Recently, the results for linear Hamiltonian difference systems were generalized in [34,45] to discrete symplectic systems

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k + \lambda\, \mathcal{W}_k x_{k+1}, \quad k \in [0, \infty)_{\mathbb{Z}}, \tag{4.2}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, $\mathcal{W}_k$ are complex $n \times n$ matrices such that $\mathcal{W}_k$ is Hermitian and nonnegative definite and the $2n \times 2n$ transition matrix in (4.2) is symplectic.

The classification of second order Sturm–Liouville dynamic equations to be of the limit point or limit circle type is given in [109,159,166] and we refer to Section 5.3 for more details about this special case.

Another way of generalizing the Weyl–Titchmarsh theory for continuous and discrete Hamiltonian systems was presented in [6,150]. In these references the authors consider the linear Hamiltonian system

$$\left.\begin{aligned} x^{\Delta}(t) &= A(t)\, x^{\sigma}(t) + [B(t) + \lambda W_2(t)]\, u(t),\\ u^{\Delta}(t) &= [C(t) - \lambda W_1(t)]\, x^{\sigma}(t) - A^*(t)\, u(t), \end{aligned}\right\} \quad t \in [a, \infty)_{\mathbb{T}} \tag{4.3}$$

on the so-called Sturmian or general time scales, respectively.

In the present chapter we develop the Weyl–Titchmarsh theory for more general linear dynamic systems, namely the time scale symplectic systems

$$\left.\begin{aligned} x^{\Delta}(t) &= \mathcal{A}(t)\,x(t) + \mathcal{B}(t)\,u(t), \\ u^{\Delta}(t) &= \mathcal{C}(t)\,x(t) + \mathcal{D}(t)\,u(t) - \lambda\,\mathcal{W}(t)\,x^{\sigma}(t), \end{aligned}\right\} \quad t \in [a,\infty)_{\mathbb{T}}, \tag{$\mathcal{S}_\lambda$}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{W}$ are complex $n \times n$ matrix functions on $[a,\infty)_{\mathbb{T}}$, $\mathcal{W}(t)$ is Hermitian and nonnegative definite, $\lambda \in \mathbb{C}$, and the $2n \times 2n$ coefficient matrix $\mathcal{S}(t) = \left(\begin{smallmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{smallmatrix}\right)$ in system ($\mathcal{S}_\lambda$) satisfies (3.8) for all $t \in [a,\infty)_{\mathbb{T}}$. The spectral parameter λ appears only in the second equation of system ($\mathcal{S}_\lambda$). This system was introduced in [116] and it naturally unifies the previously mentioned continuous, discrete, and time scale linear Hamiltonian systems (having the spectral parameter in the second equation only) and discrete symplectic systems into one framework. Our main results are the properties of the $M(\lambda)$ function, the geometric description of the Weyl disks, and characterizations of the limit point and limit circle cases for the time scale symplectic system ($\mathcal{S}_\lambda$). In addition, we give a formula for the $L^2_{\mathcal{W}}$ solutions of a nonhomogeneous time scale symplectic system in terms of its Green function. These results generalize and unify in particular all the results in [34, 45, 150, 166] and some results from [6, 38, 40, 43, 44, 106, 112, 122, 138, 142]. The theory of time scale symplectic systems or Hamiltonian systems is a topic with active research in recent years, see, e.g., [4, 57, 93–95, 98, 100, 116]. This chapter can be regarded not only as a completion of these papers by establishing the Weyl–Titchmarsh theory for time scale symplectic systems, but also as a comparison of the corresponding continuous and discrete time results. The references to particular statements in the literature are displayed throughout the text.

In the next section we present fundamental properties of perturbed time scale symplectic systems with complex coefficients, including the important Lagrange identity (Theorem 4.5) and other formulas involving their solutions. In Section 4.2 we define the time scale $M(\lambda)$-function for system ($\mathcal{S}_\lambda$) and establish its basic properties in the case of the regular spectral problem. In Section 4.3 we introduce the Weyl disks and circles for system ($\mathcal{S}_\lambda$) and describe their geometric structure in terms of contractive matrices in $\mathbb{C}^{n\times n}$. The properties of the limiting Weyl disk and Weyl circle are then studied in Section 4.4, where we also prove that system ($\mathcal{S}_\lambda$) has at least n linearly independent solutions in the space $L^2_{\mathcal{W}}$ (see Theorem 4.41). In Section 4.5 we define the system ($\mathcal{S}_\lambda$) to be in the limit point and limit circle case and prove several characterizations of these properties. In the final section we consider the system ($\mathcal{S}_\lambda$) with a nonhomogeneous term. We construct its Green function, discuss its properties, and characterize the $L^2_{\mathcal{W}}$ solutions of this nonhomogeneous system in terms of the Green function (Theorem 4.55). A certain uniqueness result is also proven for the limit point case.

4.1 Perturbed symplectic systems on time scales

Let $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, $\mathcal{C}(\cdot)$, $\mathcal{D}(\cdot)$, $\mathcal{W}(\cdot)$ be $n \times n$ piecewise rd-continuous functions on $[a,\infty)_{\mathbb{T}}$ such that $\mathcal{W}(t) \geq 0$ for all $t \in [a,\infty)_{\mathbb{T}}$, i.e., $\mathcal{W}(t)$ is Hermitian and nonnegative definite, and the coefficient matrix $\mathcal{S}(t)$ satisfy identity (3.8). In this chapter we consider system ($\mathcal{S}_\lambda$) introduced in the introduction of this chapter. This system can be written as

$$z^{\Delta}(t,\lambda) = \mathcal{S}(t)\,z(t,\lambda) + \lambda\,\mathcal{J}\,\widetilde{\mathcal{W}}(t)\,z^{\sigma}(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{$\mathcal{S}_\lambda$}$$

where the $2n \times 2n$ matrix $\widetilde{\mathcal{W}}(t)$ is defined and has the property

$$\widetilde{\mathcal{W}}(t) := \begin{pmatrix} \mathcal{W}(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{J}\,\widetilde{\mathcal{W}}(t) = \begin{pmatrix} 0 & 0 \\ -\mathcal{W}(t) & 0 \end{pmatrix}. \tag{4.4}$$

System ($\mathcal{S}_\lambda$) can be also written in the equivalent form

$$z^{\Delta}(t,\lambda) = \mathbb{S}(t,\lambda)\, z(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.5}$$

where the matrix $\mathbb{S}(t,\lambda)$ is defined through the matrices $\mathcal{S}(t)$ and $\widetilde{\mathcal{W}}(t)$ from (3.7) and (4.4) by

$$\begin{aligned} \mathbb{S}(t,\lambda) &:= \mathcal{S}(t) + \lambda\,\mathcal{J}\,\widetilde{\mathcal{W}}(t)\,[\mathcal{I} + \mu(t)\,\mathcal{S}(t)] \qquad (4.6) \\ &= \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) - \lambda\,\mathcal{W}(t)\,[\mathcal{I} + \mu(t)\,\mathcal{A}(t)] & \mathcal{D}(t) - \lambda\,\mu(t)\,\mathcal{W}(t)\,\mathcal{B}(t) \end{pmatrix}. \end{aligned}$$

By using the identity in (3.8), a direct calculation shows that the matrix function $\mathbb{S}(\cdot,\cdot)$ satisfies

$$\mathbb{S}^*(t,\lambda)\,\mathcal{J} + \mathcal{J}\mathbb{S}(t,\bar\lambda) + \mu(t)\,\mathbb{S}^*(t,\lambda)\,\mathcal{J}\mathbb{S}(t,\bar\lambda) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \ \lambda \in \mathbb{C}. \tag{4.7}$$

Remark 4.1. The name time scale *symplectic system* and *Hamiltonian system* have been reserved in the literature for the systems of the form ($\mathcal{S}$) and (1.5), respectively. Since for a fixed $\lambda, \nu \in \mathbb{C}$ the matrix $\mathbb{S}(t,\lambda)$ from (4.6) satisfies

$$\begin{aligned} &\mathbb{S}^*(t,\lambda)\,\mathcal{J} + \mathcal{J}\mathbb{S}(t,\nu) + \mu(t)\,\mathbb{S}^*(t,\lambda)\,\mathcal{J}\mathbb{S}(t,\nu) \\ &\qquad = \left(\bar\lambda - \nu\right)\,[\mathcal{I} + \mu(t)\,\mathcal{S}^*(t)]\,\widetilde{\mathcal{W}}(t)\,[\mathcal{I} + \mu(t)\,\mathcal{S}(t)], \end{aligned} \tag{4.8}$$

it follows that the system ($\mathcal{S}_\lambda$) is a true time scale symplectic system according to the above terminology only for $\lambda \in \mathbb{R}$, while strictly speaking ($\mathcal{S}_\lambda$) *is not* a time scale symplectic system for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. However, since ($\mathcal{S}_\lambda$) is a perturbation of system ($\mathcal{S}$) and since the important properties of time scale symplectic systems needed in the presented Weyl–Titchmarsh theory, such as (4.7) or (4.10), are satisfied in an appropriate modification, we accept with the above understanding the same terminology for the system ($\mathcal{S}_\lambda$) for any $\lambda \in \mathbb{C}$.

Equation (4.7) represents a fundamental identity for the theory of time scale symplectic systems ($\mathcal{S}_\lambda$). Some important properties of the matrix $\mathbb{S}(t,\lambda)$ are displayed below. Note that formula (4.9) is a generalization of [57, Equation (10.4)] to complex values of λ.

Lemma 4.2. *Identity* (4.7) *is equivalent to the identity*

$$\mathbb{S}(t,\bar\lambda)\,\mathcal{J} + \mathcal{J}\mathbb{S}^*(t,\lambda) + \mu(t)\,\mathbb{S}(t,\bar\lambda)\,\mathcal{J}\mathbb{S}^*(t,\lambda) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \ \lambda \in \mathbb{C}. \tag{4.9}$$

In this case for any $\lambda \in \mathbb{C}$ we have

$$[\mathcal{I} + \mu(t)\,\mathbb{S}^*(t,\lambda)]\,\mathcal{J}\left[\mathcal{I} + \mu(t)\,\mathbb{S}(t,\bar\lambda)\right] = \mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.10}$$

$$\left[\mathcal{I} + \mu(t)\,\mathbb{S}(t,\bar\lambda)\right]\mathcal{J}\,[\mathcal{I} + \mu(t)\,\mathbb{S}^*(t,\lambda)] = \mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.11}$$

and the matrices $\mathcal{I} + \mu(t)\,\mathbb{S}(t,\lambda)$ and $\mathcal{I} + \mu(t)\,\mathbb{S}(t,\bar\lambda)$ are invertible with

$$[\mathcal{I} + \mu(t)\,\mathbb{S}(t,\lambda)]^{-1} = -\mathcal{J}\left[\mathcal{I} + \mu(t)\,\mathbb{S}^*(t,\bar\lambda)\right]\mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.12}$$

Proof. Let $t \in [a,\infty)_{\mathbb{T}}$ and $\lambda \in \mathbb{C}$ be fixed. If t is right-dense, i.e., $\mu(t) = 0$, then identity (4.7) reduces to $\mathcal{S}^*(t,\lambda)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t,\bar{\lambda}) = 0$. Upon multiplying this equation by $\mathcal{J}$ from the left and right side we get identity (4.9) with $\mu(t) = 0$. If t is right-scattered, i.e., $\mu(t) > 0$, then (4.7) is equivalent to (4.10). It follows that the determinants of $\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)$ and $\mathcal{I}+\mu(t)\,\mathcal{S}(t,\bar{\lambda})$ are nonzero proving that these matrices are invertible with the inverse given by (4.12). Upon multiplying equation (4.10) by the invertible matrices $[\mathcal{I} + \mu(t)\,\mathcal{S}(t,\bar{\lambda})]\,\mathcal{J}$ from the left and $-[\mathcal{I} + \mu(t)\,\mathcal{S}(t,\bar{\lambda})]^{-1}\mathcal{J}$ from the right and by using $\mathcal{J}^2 = -\mathcal{I}$, we get formula (4.11), which is equivalent to (4.9) due to $\mu(t) > 0$. ∎

Remark 4.3. Equation (4.12) allows to write system $(\mathcal{S}_\lambda)$ in the equivalent adjoint form

$$z^{\Delta}(t,\lambda) = \mathcal{J}\mathcal{S}^*(t,\bar{\lambda})\,\mathcal{J}z^{\sigma}(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.13}$$

System (4.13) can be found, e.g., in [93, Remark 3.1(iii)] or [98, Equation (7)] in the connection with optimality conditions for variational problems over time scales.

In the following result we show that equation (4.7) guarantees, among other properties, the existence and uniqueness of solutions of the initial value problems associated with $(\mathcal{S}_\lambda)$.

Theorem 4.4 (Existence and uniqueness theorem)**.** *Let $\lambda \in \mathbb{C}$, $t_0 \in [a,\infty)_{\mathbb{T}}$, and $z_0 \in \mathbb{C}^{2n}$ be given. Then the initial value problem $(\mathcal{S}_\lambda)$ with $z(t_0) = z_0$ has a unique solution $z(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}}$ on the interval $[a,\infty)_{\mathbb{T}}$.*

Proof. The coefficient matrix of system $(\mathcal{S}_\lambda)$, or equivalently of system (4.5), is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$. By Lemma 4.2, the matrix $\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)$ is invertible for all $t \in [a,\infty)_{\mathbb{T}}$, which proves that the function $\mathcal{S}(\cdot,\lambda)$ is regressive on $[a,\infty)_{\mathbb{T}}$. Hence, the result follows from Remark 2.1. ∎

Next we establish several identities involving solutions of system $(\mathcal{S}_\lambda)$ or solutions of two such systems with different spectral parameters. The first result is the Lagrange identity known in the special cases of continuous time linear Hamiltonian systems in [112, Theorem 4.1] or [43, Equation (2.23)], discrete linear Hamiltonian systems in [44, Equation (2.55)] or [142, Lemma 2.2], discrete symplectic systems in [34, Lemma 2.6] or [45, Lemma 2.3], and time scale linear Hamiltonian systems in [150, Lemma 3.5] and [6, Theorem 2.2].

Theorem 4.5 (Lagrange identity)**.** *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $z(\cdot,\nu)$ are $2n \times m$ solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, then*

$$[z^*(t,\lambda)\,\mathcal{J}z(t,\nu)]^{\Delta} = (\bar{\lambda} - \nu)\,z^{\sigma *}(t,\lambda)\,\widetilde{\mathcal{W}}(t)\,z^{\sigma}(t,\nu), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.14}$$

Proof. Formula (4.14) follows from the time scales product rule (2.3) by using the relation

$$z^{\sigma}(t,\lambda) = [\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)]\,z(t,\lambda)$$

and identity (4.8). ∎

As direct consequences of Theorem 4.5 we obtain the following.

Corollary 4.6. *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $z(\cdot,\nu)$ are $2n \times m$ solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, then for all $t \in [a,\infty)_{\mathbb{T}}$ we have*

$$z^*(t,\lambda)\,\mathcal{J}\,z(t,\nu) = z^*(a,\lambda)\,\mathcal{J}z(a,\nu) + (\bar{\lambda} - \nu)\int_a^t z^{\sigma *}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,z^{\sigma}(s,\nu)\,\Delta s.$$

One can easily see that if $z(\cdot,\lambda)$ is a solution of system ($\mathcal{S}_\lambda$), then $z(\cdot,\bar\lambda)$ is a solution of system ($\mathcal{S}_{\bar\lambda}$). Therefore, Theorem 4.5 with $\nu=\bar\lambda$ yields a Wronskian-type property of solutions of ($\mathcal{S}_\lambda$).

Corollary 4.7. *Let $\lambda\in\mathbb{C}$ and $m\in\mathbb{N}$ be given. For any $2n\times m$ solution $z(\cdot,\lambda)$ of system ($\mathcal{S}_\lambda$)*

$$z^*(t,\lambda)\,\mathcal{J}z(t,\bar\lambda)\equiv z^*(a,\lambda)\,\mathcal{J}z(a,\bar\lambda)\quad \textit{is constant on } [a,\infty)_{\mathbb{T}}.$$

The following result gives another interesting property of solutions of system ($\mathcal{S}_\lambda$) and ($\mathcal{S}_{\bar\lambda}$).

Lemma 4.8. *Let $\lambda\in\mathbb{C}$ and $m\in\mathbb{N}$ be given. For any $2n\times m$ solutions $z(\cdot,\lambda)$ and $\tilde z(\cdot,\lambda)$ of system ($\mathcal{S}_\lambda$) the $2n\times 2n$ matrix function $K(\cdot,\lambda)$ defined by*

$$K(t,\lambda):=z(t,\lambda)\,\tilde z^*(t,\bar\lambda)-\tilde z(t,\lambda)\,z^*(t,\bar\lambda),\quad t\in[a,\infty)_{\mathbb{T}},$$

satisfies the dynamic equation

$$K^{\Delta}(t,\lambda)=\mathcal{S}(t,\lambda)\,K(t,\lambda)+[\mathcal{I}+\mu(t)\,\mathcal{S}(t,\lambda)]\,K(t,\lambda)\,\mathcal{S}^*(t,\bar\lambda),\quad t\in[a,\infty)_{\mathbb{T}},$$

and the identities $K^*(t,\lambda)=-K(t,\bar\lambda)$ and

$$K^{\sigma}(t,\lambda)=[\mathcal{I}+\mu(t)\,\mathcal{S}(t,\lambda)]\,K(t,\lambda)\,[\mathcal{I}+\mu(t)\,\mathcal{S}^*(t,\bar\lambda)],\quad t\in[a,\infty)_{\mathbb{T}}. \tag{4.15}$$

Proof. Having that $z(\cdot,\lambda)$ and $\tilde z(\cdot,\lambda)$ are solutions of system ($\mathcal{S}_\lambda$), it follows that $z(\cdot,\bar\lambda)$ and $\tilde z(\cdot,\bar\lambda)$ are solutions of system ($\mathcal{S}_{\bar\lambda}$). The results then follow by direct calculations. ∎

Remark 4.9. The content of Lemma 4.8 appears to be new both in the continuous and discrete time cases. Moreover, when the matrix function $K(\cdot,\lambda)\equiv K(\lambda)$ is constant, identity (4.15) yields for any right-scattered $t\in[a,\infty)_{\mathbb{T}}$ that

$$\mathcal{S}(t,\lambda)\,K(\lambda)+K(\lambda)\,\mathcal{S}^*(t,\bar\lambda)+\mu(t)\,\mathcal{S}(t,\lambda)\,K(\lambda)\,\mathcal{S}^*(t,\bar\lambda)=0. \tag{4.16}$$

It is interesting to note that this formula is very much like equation (4.9). More precisely, identity (4.9) is a consequence of equation (4.16) for the case of $K(\lambda)\equiv\mathcal{J}$.

Next we present properties of certain fundamental matrices $\Psi(\cdot,\lambda)$ of system ($\mathcal{S}_\lambda$), which are generalizations of the corresponding results in [57, Section 10.2] and displayed in (3.9) and (3.10) to complex λ. Some of these results can be proven under the weaker condition that the initial value of $\Psi(a,\lambda)$ does depend on λ and satisfies $\Psi^*(a,\lambda)\,\mathcal{J}\,\Psi(a,\bar\lambda)=\mathcal{J}$. However, these more general results will not be needed in this chapter.

Lemma 4.10. *Let $\lambda\in\mathbb{C}$ be fixed. If $\Psi(\cdot,\lambda)$ is a fundamental matrix of system ($\mathcal{S}_\lambda$) such that $\Psi(a,\lambda)$ is symplectic and independent of λ, then for any $t\in[a,\infty)_{\mathbb{T}}$ we have*

$$\Psi^*(t,\lambda)\,\mathcal{J}\,\Psi(t,\bar\lambda)=\mathcal{J},\quad \Psi^{-1}(t,\lambda)=-\mathcal{J}\,\Psi^*(t,\bar\lambda)\,\mathcal{J},\quad \Psi(t,\lambda)\,\mathcal{J}\,\Psi^*(t,\bar\lambda)=\mathcal{J}. \tag{4.17}$$

Proof. Identity (4.17)(i) is a consequence of Corollary 4.7, in which we use the fact that $\Psi(a,\lambda)$ is symplectic and independent of λ. The second identity in (4.17) follows from the first one, while the third identity is obtained from the equation $\Psi(t,\lambda)\,\Psi^{-1}(t,\lambda)=\mathcal{I}$. ∎

Remark 4.11. If the fundamental matrix $\Psi(\cdot,\lambda) = \begin{pmatrix} Z(\cdot,\lambda) & \widetilde{Z}(\cdot,\lambda)\end{pmatrix}$ in Lemma 4.10 is partitioned into two $2n \times n$ blocks, then (4.17)(i) and (4.17)(iii) have respectively the form

$$Z^*(t,\lambda)\,\mathcal{J}Z(t,\bar\lambda) = 0, \quad Z^*(t,\lambda)\,\mathcal{J}\widetilde{Z}(t,\bar\lambda) = \mathcal{I}, \quad \widetilde{Z}^*(t,\lambda)\,\mathcal{J}\widetilde{Z}(t,\bar\lambda) = 0, \tag{4.18}$$

$$Z(t,\lambda)\,\widetilde{Z}^*(t,\bar\lambda) - \widetilde{Z}(t,\lambda)\,Z^*(t,\bar\lambda) = \mathcal{J}. \tag{4.19}$$

Observe that the matrix on the left-hand side of equation (4.19) represents a constant matrix $K(t,\lambda)$ from Lemma 4.8 and Remark 4.9.

Corollary 4.12. *Under the conditions of Lemma 4.10, for any $t \in [a,\infty)_{\mathbb{T}}$ we have*

$$\Psi^\sigma(t,\lambda)\,\mathcal{J}\,\Psi^*(t,\bar\lambda) = [\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)]\,\mathcal{J}, \tag{4.20}$$

which in the notation of Remark 4.11 has the form

$$Z^\sigma(t,\lambda)\,\widetilde{Z}^*(t,\bar\lambda) - \widetilde{Z}^\sigma(t,\lambda)\,Z^*(t,\bar\lambda) = [\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)]\,\mathcal{J}.$$

Proof. Identity (4.20) follows from the equation $\Psi^\sigma(t,\lambda) = [\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)]\,\Psi(t,\lambda)$ by applying formula (4.17)(ii). ∎

4.2 $M(\lambda)$-function for regular spectral problem

In this section we consider the regular spectral problem on the time scale interval $[a,b]_{\mathbb{T}}$ with some fixed $b \in (a,\infty)_{\mathbb{T}}$. We shall specify the corresponding boundary conditions in terms of complex $n \times 2n$ matrices from the set

$$\Gamma := \{\, \alpha \in \mathbb{C}^{n\times 2n},\ \alpha\alpha^* = \mathcal{I},\ \alpha\,\mathcal{J}\alpha^* = 0 \,\}. \tag{4.21}$$

The two defining conditions for $\alpha \in \mathbb{C}^{n\times 2n}$ in (4.21) imply that the $2n \times 2n$ matrix $\begin{pmatrix}\alpha^* & -\mathcal{J}\alpha^*\end{pmatrix}$ is unitary and symplectic. This yields the identity

$$\begin{pmatrix}\alpha^* & -\mathcal{J}\alpha^*\end{pmatrix}\begin{pmatrix}\alpha \\ \alpha\mathcal{J}\end{pmatrix} = \mathcal{I} \in \mathbb{C}^{2n\times 2n}, \qquad \text{i.e.,} \qquad \alpha^*\alpha - \mathcal{J}\alpha^*\alpha\,\mathcal{J} = \mathcal{I}. \tag{4.22}$$

The last equation also implies, compare with [115, Remark 2.1.2], that

$$\operatorname{Ker}\alpha = \operatorname{Im}\mathcal{J}\alpha^*. \tag{4.23}$$

Let $\alpha,\beta \in \Gamma$ be fixed and consider the boundary value problem

$$(\mathcal{S}_\lambda), \qquad \alpha\, z(a,\lambda) = 0, \qquad \beta\, z(b,\lambda) = 0. \tag{4.24}$$

Our first result shows that the boundary conditions in (4.24) are equivalent with the boundary conditions phrased in terms of the images of the $2n \times 2n$ matrices

$$R_a := \begin{pmatrix}\mathcal{J}\alpha^* & 0\end{pmatrix}, \qquad R_b := \begin{pmatrix}0 & -\mathcal{J}\beta^*\end{pmatrix},$$

which satisfy $R_a^*\mathcal{J}R_a = 0$, $R_b^*\mathcal{J}R_b = 0$, and $\operatorname{rank}\begin{pmatrix}R_a^* & R_b^*\end{pmatrix} = 2n$.

Lemma 4.13. *Let $\alpha,\beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be fixed. A solution $z(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$ satisfies the boundary conditions in (4.24) if and only if there exists a unique vector $\xi \in \mathbb{C}^{2n}$ such that*

$$z(a,\lambda) = R_a\,\xi, \qquad z(b,\lambda) = R_b\,\xi. \tag{4.25}$$

Proof. Assume that (4.24) holds. Identity (4.23) implies the existence of vectors $\xi_a, \xi_b \in \mathbb{C}^n$ such that $z(a,\lambda) = -\mathcal{J}\alpha^*\,\xi_a$ and $z(b,\lambda) = -\mathcal{J}\beta^*\,\xi_b$. It follows that $z(\cdot,\lambda)$ satisfies (4.25) with $\xi := \begin{pmatrix} -\xi_a^* & \xi_b^* \end{pmatrix}^*$. It remains to prove that ξ is unique such a vector. If $z(\cdot,\lambda)$ satisfies (4.25) and also $z(a,\lambda) = R_a\,\zeta$ and $z(b,\lambda) = R_b\,\zeta$ for some $\xi,\zeta \in \mathbb{C}^{2n}$, then $R_a\,(\xi-\zeta) = 0$ and $R_b\,(\xi-\zeta) = 0$. Hence, $\mathcal{J}\alpha^*\begin{pmatrix}\mathcal{I} & 0\end{pmatrix}(\xi-\zeta) = 0$ and $-\mathcal{J}\beta^*\begin{pmatrix}0 & \mathcal{I}\end{pmatrix}(\xi-\zeta) = 0$. If we multiply the latter two equalities by $\alpha\mathcal{J}$ and $\beta\mathcal{J}$ and use $\alpha\alpha^* = \mathcal{I} = \beta\beta^*$, we obtain $\begin{pmatrix}\mathcal{I} & 0\end{pmatrix}(\xi-\zeta) = 0$ and $\begin{pmatrix}0 & \mathcal{I}\end{pmatrix}(\xi-\zeta) = 0$, respectively. This yields $\xi-\zeta = 0$, which shows that the vector ξ in (4.25) is unique. The opposite direction, i.e., that (4.25) implies (4.24), is trivial. ∎

Following the standard terminology, see, e.g., [2,31], a number $\lambda \in \mathbb{C}$ is an *eigenvalue* of (4.24) if this boundary value problem has a solution $z(\cdot,\lambda) \not\equiv 0$. In this case the function $z(\cdot,\lambda)$ is called the *eigenfunction* corresponding to the eigenvalue λ and the dimension of the space of all eigenfunctions corresponding to λ (together with the zero function) is called the *geometric multiplicity* of λ.

Given $\alpha \in \Gamma$, we will utilize from now on the fundamental matrix $\Psi(\cdot,\lambda,\alpha)$ of system $(\mathcal{S}_\lambda)$ satisfying the initial condition from (4.24), that is,

$$\Psi^\Delta(t,\lambda,\alpha) = \mathcal{S}(t,\lambda)\,\Psi(t,\lambda,\alpha), \quad t \in [a,\rho(b)]_{\mathbb{T}}, \quad \Psi(a,\lambda,\alpha) = \begin{pmatrix}\alpha^* & -\mathcal{J}\alpha^*\end{pmatrix}. \tag{4.26}$$

Then $\Psi(a,\lambda,\alpha)$ does not depend on λ and it is symplectic and unitary with the inverse given by $\Psi^{-1}(a,\lambda,\alpha) = \Psi^*(a,\lambda,\alpha)$. Hence, the properties of fundamental matrices derived earlier in Lemma 4.10, Remark 4.11, and Corollary 4.12 apply for the matrix function $\Psi(\cdot,\lambda,\alpha)$.

The following assumption will be imposed in this section when studying the regular spectral problem.

Hypothesis 4.14. *For every $\lambda \in \mathbb{C}$ we have*

$$\int_a^b \Psi^{\sigma*}(t,\lambda,\alpha)\,\widetilde{\mathcal{W}}(t)\,\Psi^\sigma(t,\lambda,\alpha)\,\Delta t > 0. \tag{4.27}$$

Condition (4.27) can be written in the equivalent form as

$$\int_a^b z^{\sigma*}(t,\lambda)\,\widetilde{\mathcal{W}}(t)\,z^\sigma(t,\lambda)\,\Delta t > 0 \tag{4.28}$$

for every nontrivial solution $z(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$. Assumptions (4.27) and (4.28) are equivalent by a simple argument using the uniqueness of solutions of system $(\mathcal{S}_\lambda)$. The latter form (4.28) has been widely used in the literature, such as in the continuous time case in [43, Hypothesis 2.2], [107, Equation (1.3)], [102, Equation (2.3)], in the discrete time case in [44, Condition (2.16)], [142, Equation (1.7)], [34, Assumption 2.2], [45, Hypothesis 2.4], and in the time scale Hamiltonian case in [150, Assumption 3] and [6, Condition (3.9)].

Following Remark 4.11, we partition the fundamental matrix $\Psi(\cdot,\lambda,\alpha)$ as

$$\Psi(\cdot,\lambda,\alpha) = \begin{pmatrix} Z(\cdot,\lambda,\alpha) & \widetilde{Z}(\cdot,\lambda,\alpha)\end{pmatrix}, \tag{4.29}$$

where $Z(\cdot,\lambda,\alpha)$ and $\widetilde{Z}(\cdot,\lambda,\alpha)$ are the $2n\times n$ solutions of $(\mathcal{S}_\lambda)$ satisfying $Z(a,\lambda,\alpha) = \alpha^*$ and $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^*$. With the notation

$$\Lambda(\lambda,\alpha,\beta) := \Psi(b,\lambda,\alpha)\,\Psi^*(a,\lambda,\alpha)\,R_a - R_b = \begin{pmatrix} -\widetilde{Z}(b,\lambda,\alpha) & \mathcal{J}\beta^*\end{pmatrix} \tag{4.30}$$

we have the classical characterization of the eigenvalues of (4.24), see, e.g., the continuous time in [135, Chapter 4], the discrete time in [142, Theorem 2.3, Lemma 2.4], [45, Lemma 2.9, Theorem 2.11], and in the time scale case in [2, Lemma 3], [31, Corollary 1].

Proposition 4.15. *For $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$ we have with notation (4.30) the following.*

(i) *The number λ is an eigenvalue of (4.24) if and only if* $\det \Lambda(\lambda, \alpha, \beta) = 0$.

(ii) *The algebraic multiplicity of the eigenvalue λ, i.e., the number* $\operatorname{def} \Lambda(\lambda, \alpha, \beta)$ *is equal to the geometric multiplicity of λ.*

(iii) *Under Hypothesis 4.14, the eigenvalues of (4.24) are real and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product*

$$\langle z(\cdot, \lambda), z(\cdot, \nu) \rangle_{\mathcal{W}, b} := \int_a^b z^{\sigma *}(t, \lambda)\, \widetilde{\mathcal{W}}(t)\, z^{\sigma}(t, \nu)\, \Delta t.$$

The next algebraic characterization of the eigenvalues of (4.24) is more appropriate for the development of the Weyl–Titchmarsh theory for (4.24), since it uses the matrix $\beta \widetilde{Z}(b, \lambda, \alpha)$ which has dimension n instead of using the matrix $\Lambda(\lambda, \alpha, \beta)$ which has dimension $2n$. Results of this type can be found in special cases of system ($\mathcal{S}_\lambda$) in [43, Lemma 2.5], [112, Theorem 4.1], [44, Lemma 2.8], [142, Lemma 3.1], [34, Lemma 2.5], [150, Theorem 3.4], and [45, Lemma 3.1].

Lemma 4.16. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be fixed. Then λ is an eigenvalue of (4.24) if and only if* $\det \beta \widetilde{Z}(b, \lambda, \alpha) = 0$. *In this case the algebraic and geometric multiplicities of λ are equal to* $\operatorname{def} \beta \widetilde{Z}(b, \lambda, \alpha)$.

Proof. One can follow the same arguments as in the proof of the corresponding discrete symplectic case in [45, Lemma 3.1]. However, having the result of Proposition 4.15, we can proceed directly by the methods of linear algebra. In this proof we abbreviate $\Lambda := \Lambda(\lambda, \alpha, \beta)$ and $\widetilde{Z} := \widetilde{Z}(b, \lambda, \alpha)$. Assume that Λ is singular, i.e., $-\widetilde{Z}c + \mathcal{J}\beta^* d = 0$ for some vectors $c, d \in \mathbb{C}^n$, not both zero. Then $\widetilde{Z}c = \mathcal{J}\beta^* d$, which yields that $\beta\widetilde{Z}c = 0$. If $c = 0$, then $\mathcal{J}\beta^* d = 0$, which implies upon the multiplication by $\beta\mathcal{J}$ from the left that $d = 0$. Since not both c and d can be zero, it follows that $c \neq 0$ and the matrix $\beta\widetilde{Z}$ is singular.

Conversely, if $\beta\widetilde{Z}c = 0$ for some nonzero vector $c \in \mathbb{C}^n$, then $-\widetilde{Z}c + \mathcal{J}\beta^* d = 0$, i.e., Λ is singular, with the vector $d := -\beta\mathcal{J}\widetilde{Z}c$. Indeed, by using identity (4.22) we have $\mathcal{J}\beta^* d = -\mathcal{J}\beta^*\beta\mathcal{J}\widetilde{Z}c = (\mathcal{I} - \beta^*\beta)\widetilde{Z}c = \widetilde{Z}c$. From the above we can also see that the number of linearly independent vectors in $\operatorname{Ker} \beta\widetilde{Z}$ is the same as the number of linearly independent vectors in $\operatorname{Ker} \Lambda$. Therefore, by Proposition 4.15(ii), the algebraic and geometric multiplicities of λ as an eigenvalue of (4.24) is equal to $\operatorname{def} \beta\widetilde{Z}$. ∎

Since the eigenvalues of (4.24) are real, it follows that the matrix $\beta\widetilde{Z}(b, \lambda, \alpha)$ is invertible for every $\lambda \in \mathbb{C}$ except of at most n real numbers. This motivates the definition of the $M(\lambda)$-function for the regular spectral problem.

Definition 4.17 ($M(\lambda)$-function)**.** Let $\alpha, \beta \in \Gamma$. Whenever the matrix $\beta\widetilde{Z}(b, \lambda, \alpha)$ is invertible for some value $\lambda \in \mathbb{C}$, we define the *Weyl–Titchmarsh $M(\lambda)$-function* as the $n \times n$ matrix

$$M(\lambda) = M(\lambda, b) = M(\lambda, b, \alpha, \beta) := -\left[\beta\widetilde{Z}(b, \lambda, \alpha)\right]^{-1} \beta Z(b, \lambda, \alpha). \tag{4.31}$$

The above definition of the $M(\lambda)$-function is a generalization of the corresponding definitions for the continuous and discrete linear Hamiltonian and symplectic systems in [43, Definition 2.6], [44, Definition 2.9], [142, Equation (3.10)], [34, p. 2859], [45, Definition 3.2] and time scale linear Hamiltonian systems in [150, Equation (4.1)]. The dependence of the $M(\lambda)$-function on b, α, and β will be suppressed in the notation and $M(\lambda, b)$ or $M(\lambda, b, \alpha, \beta)$ will be used only in few situations when we emphasize the dependence on b (such as at the end of Section 4.3) or on α and β (as in Lemma 4.26). By [99, Corollary 4.5], see also [116, Remark 2.2], the $M(\cdot)$-function is an entire function in λ. Another important property of the $M(\lambda)$-function is established below.

Lemma 4.18. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$M^*(\lambda) = M(\bar{\lambda}). \tag{4.32}$$

Proof. We abbreviate $Z(\lambda) := Z(b, \lambda, \alpha)$ and $\widetilde{Z}(\lambda) := \widetilde{Z}(b, \lambda, \alpha)$. By using the definition of $M(\lambda)$ in (4.31) and identity (4.19) we have

$$\begin{aligned} M^*(\lambda) - M(\bar{\lambda}) &= [\beta \widetilde{Z}(\bar{\lambda})]^{-1} \beta \, [Z(\bar{\lambda})\, \widetilde{Z}^*(\lambda) - \widetilde{Z}(\bar{\lambda})\, Z^*(\lambda)]\, \beta^* \, [\beta \widetilde{Z}(\lambda)]^{*-1} \\ &\overset{(4.19)}{=} [\beta \widetilde{Z}(\bar{\lambda})]^{-1} \beta \, \mathcal{J} \beta^* \, [\beta \widetilde{Z}(\lambda)]^{*-1} = 0, \end{aligned}$$

because $\beta \in \Gamma$. Hence, equality (4.32) holds true. ∎

The following solution plays an important role in particular in the results concerning the square-integrable solutions of system ($\mathcal{S}_\lambda$).

Definition 4.19 (Weyl solution)**.** For any matrix $M \in \mathbb{C}^{n \times n}$ we define the so-called *Weyl solution* of system ($\mathcal{S}_\lambda$) by

$$\mathcal{X}(\cdot, \lambda, \alpha, M) := \Psi(\cdot, \lambda, \alpha) \begin{pmatrix} \mathcal{I} & M^* \end{pmatrix}^* = Z(\cdot, \lambda, \alpha) + \widetilde{Z}(\cdot, \lambda, \alpha)\, M, \tag{4.33}$$

where $Z(\cdot, \lambda, \alpha)$ and $\widetilde{Z}(\cdot, \lambda, \alpha)$ are defined in (4.29).

The function $\mathcal{X}(\cdot, \lambda, \alpha, M)$, being a linear combination of two solutions of system ($\mathcal{S}_\lambda$), is also a solution of this system. Moreover, $\alpha\, \mathcal{X}(a, \lambda, \alpha, M) = \mathcal{I}$ and, if $\beta \widetilde{Z}(b, \lambda, \alpha)$ is invertible, then $\beta\, \tilde{\mathcal{X}}(b, \lambda, \alpha, M) = \beta \widetilde{Z}(b, \lambda, \alpha)\, [M - M(\lambda)]$. Consequently, if we take $M := M(\lambda)$ in Definition 4.19, then $\beta\, \mathcal{X}(b, \lambda, \alpha, M(\lambda)) = 0$, i.e., the Weyl solution $\mathcal{X}(\cdot, \lambda, \alpha, M(\lambda))$ satisfies the right endpoint boundary condition in (4.24).

Following the corresponding notions in [43, Equation (2.18)], [44, Equation (2.51)], [142, p. 471], [34, p. 2859], [45, Equation (3.13)], and [150, Equation (4.2)], we define the Hermitian $n \times n$ matrix function $\mathcal{E}(M)$ for system ($\mathcal{S}_\lambda$).

Definition 4.20. For a fixed $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we define the matrix function

$$\mathcal{E} : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}, \quad \mathcal{E}(M) = \mathcal{E}(M, b) := i\, \delta(\lambda)\, \mathcal{X}^*(b, \lambda, \alpha, M)\, \mathcal{J} \mathcal{X}(b, \lambda, \alpha, M),$$

where $\delta(\lambda) := \operatorname{sgn}(\operatorname{Im}(\lambda))$.

For brevity we suppress the dependence of the function $\mathcal{E}(\cdot)$ on b and λ. In few cases we will need $\mathcal{E}(M)$ depending on b (as in Theorem 4.28 and Definition 4.36) and in such situations we will use the notation $\mathcal{E}(M, b)$. Since $(i\mathcal{J})^* = i\mathcal{J}$, it follows that $\mathcal{E}(M)$ is a Hermitian matrix for any $M \in \mathbb{C}^{n \times n}$. Moreover, from Corollary 4.6 we obtain

$$\mathcal{E}(M) = -2\, \delta(\lambda) \operatorname{Im}(M) + 2\, |\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma *}(t, \lambda, \alpha, M)\, \widetilde{\mathcal{W}}(t)\, \mathcal{X}^{\sigma}(t, \lambda, \alpha, M)\, \Delta t, \tag{4.34}$$

where we used the fact

$$\mathcal{X}^*(a,\lambda,\alpha,M)\,\mathcal{J}\mathcal{X}(a,\lambda,\alpha,M) \overset{(4.26)}{=} M - M^* = 2i\,\mathrm{Im}(M). \tag{4.35}$$

Next we define the Weyl disk and Weyl circle for the regular spectral problem. The geometric characterizations of the Weyl disk and Weyl circle in terms of the contractive or unitary matrices which justify the terminology “disk” or “circle” will be presented in Section 4.3.

Definition 4.21 (Weyl disk and Weyl circle)**.** For a fixed $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the set

$$D(\lambda) = D(\lambda, b) := \{\, M \in \mathbb{C}^{n\times n},\ \mathcal{E}(M) \le 0 \,\}$$

is called the *Weyl disk* and the set

$$C(\lambda) = C(\lambda, b) := \partial D(\lambda) = \{\, M \in \mathbb{C}^{n\times n},\ \mathcal{E}(M) = 0 \,\}$$

is called the *Weyl circle*.

The dependence of the Weyl disk and Weyl circle on b will be again suppressed. In the following result we show that the Weyl circle consists of precisely those matrices $M(\lambda)$ with $\beta \in \Gamma$. This result generalizes the corresponding statements in [43, Lemma 2.8], [44, Lemma 2.13], [142, Lemma 3.3], [34, Theorem 3.1], [45, Theorem 3.6], and [150, Theorem 4.2].

Theorem 4.22. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M belongs to the Weyl circle $C(\lambda)$ if and only if there exists $\beta \in \Gamma$ such that $\beta\,\mathcal{X}(b,\lambda,\alpha,M) = 0$. In this case and under Hypothesis 4.14, we have with such a matrix β that $M = M(\lambda)$ as defined in* (4.31).

Proof. Assume that $M \in C(\lambda)$, i.e., $\mathcal{E}(M) = 0$. Then with the vector

$$\beta := \mathcal{X}^*(b)\,\mathcal{J} = \begin{pmatrix} \mathcal{I} & M^* \end{pmatrix} \Psi^*(b,\lambda,\alpha)\,\mathcal{J} \in \mathbb{C}^{n\times 2n},$$

where $\mathcal{X}(b)$ denotes $\mathcal{X}(b,\lambda,\alpha,M)$, we have

$$\beta\,\mathcal{X}(b) = \mathcal{X}^*(b)\,\mathcal{J}\mathcal{X}(b) = [1/(i\,\delta(\lambda))]\mathcal{E}(M) = 0. \tag{4.36}$$

Moreover, $\operatorname{rank}\beta = n$, because the matrices $\Psi(b,\lambda,\alpha)$ and $\mathcal{J}$ are invertible and it holds $\operatorname{rank}\begin{pmatrix} \mathcal{I} & M^* \end{pmatrix} = n$. In addition, the identity $\mathcal{J}^* = \mathcal{J}^{-1}$ yields

$$\beta\,\mathcal{J}\beta^* = \mathcal{X}^*(b)\,\mathcal{J}\mathcal{X}(b) \overset{(4.36)}{=} 0. \tag{4.37}$$

Now, if the condition $\beta\beta^* = \mathcal{I}$ is not satisfied, then we replace β by $\tilde\beta := (\beta\beta^*)^{-1/2}\beta$ (note that $\beta\beta^* > 0$, so that $(\beta\beta^*)^{-1/2}$ is well defined) and in this case

$$\begin{aligned} \tilde\beta\,\mathcal{X}(b) &= (\beta\beta^*)^{-1/2}\beta\,\mathcal{X}(b) \overset{(4.36)}{=} 0, \\ \tilde\beta\,\mathcal{J}\tilde\beta^* &= (\beta\beta^*)^{-1/2}\beta\,\mathcal{J}\beta^*(\beta\beta^*)^{-1/2} \overset{(4.37)}{=} 0, \\ \tilde\beta\tilde\beta^* &= (\beta\beta^*)^{-1/2}\beta\beta^*(\beta\beta^*)^{-1/2} = \mathcal{I}. \end{aligned}$$

Conversely, suppose that for a given $M \in \mathbb{C}^{n\times n}$ there exists $\beta \in \Gamma$ such that $\beta\,\mathcal{X}(b) = 0$. Then from (4.23) it follows that $\mathcal{X}(b) = \mathcal{J}\beta^* P$ for the matrix $P := -\beta\,\mathcal{J}\mathcal{X}(b) \in \mathbb{C}^{n\times n}$. Hence,

$$\mathcal{E}(M) = i\,\delta(\lambda)\,P^*\beta\,\mathcal{J}^*\mathcal{J}\mathcal{J}\beta^*P = i\,\delta(\lambda)\,P^*\beta\,\mathcal{J}\beta^*P = 0,$$

i.e., $M \in C(\lambda)$. Finally, since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then by Proposition 4.15(iii) the number λ is not an eigenvalue of (4.24), which by Lemma 4.16 shows that the matrix $\beta\tilde{Z}(b,\lambda,\alpha)$ is invertible. The definition of the Weyl solution in (4.33) then yields

$$\beta Z(b,\lambda,\alpha) + \beta\tilde{Z}(b,\lambda,\alpha)\,M = \beta\,\mathcal{X}(b,\lambda,\alpha,M) = 0, \tag{4.38}$$

which implies that $M = -[\beta\tilde{Z}(b,\lambda,\alpha)]^{-1}\beta Z(b,\lambda,\alpha) = M(\lambda)$. ■

Remark 4.23. (i) The matrix $P := -\beta\,\mathcal{J}\mathcal{X}(b,\lambda,\alpha,M) \in \mathbb{C}^{n\times n}$ from the proof of Theorem 4.22 is invertible. This fact was not needed in that proof. However, we show that P is invertible because this argument will be used in the proof of Lemma 4.26 below. First we prove that $\operatorname{Ker} P = \operatorname{Ker} \mathcal{X}(b,\lambda,\alpha,M)$. For if $Pd = 0$ for some $d \in \mathbb{C}^n$, then from identity (4.22) we get $\mathcal{X}(b,\lambda,\alpha,M)\,d = (\mathcal{I} - \beta^*\beta)\,\mathcal{X}(b,\lambda,\alpha,M)\,d = \mathcal{J}\beta^* P d = 0$. Therefore, $\operatorname{Ker} P \subseteq \operatorname{Ker} \mathcal{X}(b,\lambda,\alpha,M)$. The opposite inclusion follows by the definition of P. And since, by (4.33), $\operatorname{rank} \mathcal{X}(b,\lambda,\alpha,M) = \operatorname{rank} \begin{pmatrix} \mathcal{I} & M^* \end{pmatrix}^* = n$, it follows that $\operatorname{Ker} \mathcal{X}(b,\lambda,\alpha,M) = \{0\}$. Hence, $\operatorname{Ker} P = \{0\}$ as well, i.e., the matrix P is invertible.

(ii) For the proof of Theorem 4.24 below we need to recall the fact that if the matrix $\operatorname{Im}(M)$ is positive or negative definite, then the matrix M is necessarily invertible. The proof of this fact can be found, e.g., in [45, Remark 2.6].

The next result contains a characterization of the matrices $M \in \mathbb{C}^{n\times n}$ which lie "inside" the Weyl disk $D(\lambda)$. In the previous result (Theorem 4.22) we have characterized the elements of the boundary of the Weyl disk $D(\lambda)$, i.e., the elements of the Weyl circle $C(\lambda)$, in terms of the matrices $\beta \in \Gamma$. For such β we have $\beta\mathcal{J}\beta^* = 0$, which yields $i\,\delta(\lambda)\,\beta\mathcal{J}\beta^* = 0$. Comparing with that statement we now utilize the matrices $\beta \in \mathbb{C}^{n\times 2n}$ which satisfy $i\,\delta(\lambda)\,\beta\mathcal{J}\beta^* > 0$. In the special cases of the discrete and continuous time this result can be found in [44, Lemma 2.18], [45, Theorem 3.13] and [43, Lemma 2.13].

Theorem 4.24. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M satisfies $\mathcal{E}(M) < 0$ if and only if there exists $\beta \in \mathbb{C}^{n\times 2n}$ such that $i\,\delta(\lambda)\,\beta\mathcal{J}\beta^* > 0$ and $\beta\,\mathcal{X}(b,\lambda,\alpha,M) = 0$. In this case and under Hypothesis 4.14, we have with such a matrix β that $M = M(\lambda)$ as defined in (4.31) and β may be chosen so that $\beta\beta^* = \mathcal{I}$.*

Proof. For $M \in \mathbb{C}^{n\times n}$ consider on $[a,b]_{\mathbb{T}}$ the Weyl solution

$$\mathcal{X}(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M) = \begin{pmatrix} \mathcal{X}_1(\cdot) \\ \mathcal{X}_2(\cdot) \end{pmatrix} \quad \text{with } n\times n \text{ blocks } \mathcal{X}_1(\cdot) \text{ and } \mathcal{X}_2(\cdot). \tag{4.39}$$

Suppose first that $\mathcal{E}(M) < 0$. Then the matrices $\mathcal{X}_j(b)$, $j \in \{1,2\}$, are invertible. Indeed, if one of them is singular, then there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\mathcal{X}_1(b)\,v = 0$ or $\mathcal{X}_2(b)\,v = 0$. Then we get

$$v^*\,\mathcal{E}(M)\,v = i\,\delta(\lambda)\,v^*\,\mathcal{X}^*(b)\,\mathcal{J}\mathcal{X}(b)\,v = i\,\delta(\lambda)\,v^*\,[\,\mathcal{X}_1^*(b)\,\mathcal{X}_2(b) - \mathcal{X}_2^*(b)\,\mathcal{X}_1(b)\,]\,v = 0, \tag{4.40}$$

which contradicts $\mathcal{E}(M) < 0$. Now we set $\beta_1 := \mathcal{I}$, $\beta_2 := -\mathcal{X}_1(b)\,\mathcal{X}_2^{-1}(b)$, and $\beta := \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$. Then for this $2n \times n$ matrix β we have $\beta\,\mathcal{X}(b) = 0$ and, by a similar calculation as in (4.40),

$$\begin{aligned} \mathcal{E}(M) &= i\,\delta(\lambda)\,\mathcal{X}^*(b)\,\mathcal{J}\mathcal{X}(b) = i\,\delta(\lambda)\,\mathcal{X}_2^*(b)\,(\beta_2\beta_1^* - \beta_1\beta_2^*)\,\mathcal{X}_2(b) \\ &= 2\,\delta(\lambda)\,\mathcal{X}_2^*(b)\,\operatorname{Im}(\beta_1\beta_2^*)\,\mathcal{X}_2(b) = -i\,\delta(\lambda)\,\mathcal{X}_2^*(b)\,\beta\mathcal{J}\beta^*\,\mathcal{X}_2(b), \end{aligned}$$

where we used the equality $\beta\,\mathcal{J}\beta^* = 2i\,\operatorname{Im}(\beta_1\beta_2^*)$. Since $\mathcal{E}(M) < 0$ and $\mathcal{X}_2(b)$ is invertible, it follows that $i\,\delta(\lambda)\,\beta\,\mathcal{J}\beta^* > 0$. Conversely, assume that for a given matrix $M \in \mathbb{C}^{n\times n}$ there is $\beta = \begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} \in \mathbb{C}^{n\times 2n}$ satisfying $i\,\delta(\lambda)\,\beta\,\mathcal{J}\beta^* > 0$ and $\beta\,\mathcal{X}(b) = 0$. Condition $i\,\delta(\lambda)\,\beta\,\mathcal{J}\beta^* > 0$ is equivalent to $\operatorname{Im}(\beta_1\beta_2^*) < 0$ when $\operatorname{Im}(\lambda) > 0$ and to $\operatorname{Im}(\beta_1\beta_2^*) > 0$ when $\operatorname{Im}(\lambda) < 0$. The positive or negative definiteness of $\operatorname{Im}(\beta_1\beta_2^*)$ implies the invertibility of β_1 and β_2, see Remark 4.23(ii). Therefore, from the equality $\beta_1\,\mathcal{X}_1(b) + \beta_2\,\mathcal{X}_2(b) = \beta\,\mathcal{X}(b) = 0$ we obtain $\mathcal{X}_1(b) = -\beta_1^{-1}\beta_2\,\mathcal{X}_2(b)$, and so it holds

$$\begin{aligned}\mathcal{E}(M) &= i\,\delta(\lambda)\,[\mathcal{X}_1^*(b)\,\mathcal{X}_2(b) - \mathcal{X}_2^*(b)\,\mathcal{X}_1(b)] = i\,\delta(\lambda)\,\mathcal{X}_2^*(b)\,\beta_1^{-1}(\beta_2\beta_1^* - \beta_1\beta_2^*)\,\beta_1^{*-1}\mathcal{X}_2(b)\\ &= -i\,\delta(\lambda)\,\mathcal{X}_2^*(b)\,\beta_1^{-1}\,\beta\,\mathcal{J}\beta^*\,\beta_1^{*-1}\mathcal{X}_2(b).\end{aligned} \tag{4.41}$$

The matrix $\mathcal{X}_2(b)$ is invertible, because if $\mathcal{X}_2(b)\,d = 0$ for some nonzero vector $d \in \mathbb{C}^n$, then $\mathcal{X}_1(b)\,d = -\beta_1^{-1}\beta_2\,\mathcal{X}_2(b)\,d = 0$, showing that $\operatorname{rank}\mathcal{X}(b) < n$. This however contradicts $\operatorname{rank}\mathcal{X}(b) = n$ which we have from the definition of the Weyl solution $\mathcal{X}(\cdot)$ in (4.33). Consequently, equation (4.41) yields through $i\,\delta(\lambda)\,\beta\,\mathcal{J}\beta^* > 0$ that $\mathcal{E}(M) < 0$.

If the matrix β does not satisfy $\beta\beta^* = \mathcal{I}$, then we modify it according to the procedure described in the proof of Theorem 4.22. Finally, since $\lambda \in \mathbb{C}\setminus\mathbb{R}$, we get from Proposition 4.15(iii) and Lemma 4.16 that the matrix $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible which in turn implies through the calculation in (4.38) that $M = -[\beta\widetilde{Z}(b,\lambda,\alpha)]^{-1}\beta Z(b,\lambda,\alpha) = M(\lambda)$. ■

In the following lemma we derive some additional properties of the Weyl disk and the $M(\lambda)$-function. Special cases of this statement can be found in [43, Lemma 2.9], [113, Theorem 3.1], [44, Lemma 2.14], [142, Lemma 3.2(ii)], [34, Theorem 3.7], [45, Lemma 3.7], and [150, Theorem 4.13].

Theorem 4.25. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. For any matrix $M \in D(\lambda)$ we have*

$$\delta(\lambda)\,\operatorname{Im}(M) \geq |\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\,\widetilde{\mathcal{W}}(t)\,\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\,\Delta t \geq 0. \tag{4.42}$$

In addition, under Hypothesis 4.14 we have $\delta(\lambda)\,\operatorname{Im}(M) > 0$.

Proof. By identity (4.34), for any matrix $M \in D(\lambda)$ we have

$$\begin{aligned}2\,\delta(\lambda)\,\operatorname{Im}(M) &= -\mathcal{E}(M) + 2\,|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\,\widetilde{\mathcal{W}}(t)\,\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\,\Delta t\\ &\geq 2\,|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\,\widetilde{\mathcal{W}}(t)\,\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\,\Delta t,\end{aligned}$$

which yields together with $\widetilde{\mathcal{W}}(t) \geq 0$ on $[a,\rho(b)]_{\mathbb{T}}$ the inequalities in (4.42). The last assertion in Theorem 4.25 is a simple consequence of Hypothesis 4.14. ■

In the last part of this section we wish to study the effect of changing α, which is one of the parameters of the $M(\lambda)$-function and the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$, when α varies within the set Γ. For this purpose we shall use the $M(\lambda)$-function with all its arguments in the two statements below.

Lemma 4.26. *Let $\alpha,\beta,\gamma \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$M(\lambda,b,\alpha,\beta) = [\,\alpha\mathcal{J}\gamma^* + \alpha\gamma^*\,M(\lambda,b,\gamma,\beta)\,][\,\alpha\gamma^* - \alpha\mathcal{J}\gamma^*\,M(\lambda,b,\gamma,\beta)\,]^{-1}. \tag{4.43}$$

Proof. Let $M(b,\lambda,\alpha,\beta)$ and $M(b,\lambda,\gamma,\beta)$ be given via (4.31) and consider the Weyl solutions

$$\mathcal{X}_\alpha(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M(b,\lambda,\alpha,\beta)) \quad \text{and} \quad \mathcal{X}_\gamma(\cdot) := \mathcal{X}(\cdot,\lambda,\gamma,M(b,\lambda,\gamma,\beta))$$

defined by (4.33) with $M = M(b,\lambda,\alpha,\beta)$ and $M = M(b,\lambda,\gamma,\beta)$, respectively. First we prove that the two Weyl solutions $\mathcal{X}_\alpha(\cdot)$ and $\mathcal{X}_\gamma(\cdot)$ differ by a constant nonsingular multiple. By definition, $\beta\,\mathcal{X}_\alpha(b) = 0$ and $\beta\,\mathcal{X}_\gamma(b) = 0$, which implies through (4.23) that $\mathcal{X}_\alpha(b) = \mathcal{J}\beta^* P_\alpha$ and $\mathcal{X}_\gamma(b) = \mathcal{J}\beta^* P_\gamma$ for some matrices $P_\alpha, P_\gamma \in \mathbb{C}^{n\times n}$, which are invertible by Remark 4.23(i), i.e., it holds $\mathcal{X}_\alpha(b)\,P_\alpha^{-1} = \mathcal{J}\beta^* = \mathcal{X}_\gamma(b)\,P_\gamma^{-1}$. Consequently, $\mathcal{X}_\alpha(b) = \mathcal{X}_\gamma(b)\,P$, where $P := P_\gamma^{-1}P_\alpha$. By the uniqueness of solutions of system ($\mathcal{S}_\lambda$), see Theorem 4.4, we obtain that $\mathcal{X}_\alpha(\cdot) = \mathcal{X}_\gamma(\cdot)\,P$ on $[a,b]_{\mathbb{T}}$. Upon the evaluation at $t=a$ we get

$$\Psi(a,\lambda,\alpha)\begin{pmatrix} \mathcal{I} \\ M(\lambda,b,\alpha,\beta)\end{pmatrix} = \Psi(a,\lambda,\gamma)\begin{pmatrix} \mathcal{I} \\ M(\lambda,b,\gamma,\beta)\end{pmatrix} P. \tag{4.44}$$

Since the matrices $\Psi(a,\lambda,\alpha) = \begin{pmatrix}\alpha^* & -\mathcal{J}\alpha^*\end{pmatrix}$ and $\Psi(a,\lambda,\gamma) = \begin{pmatrix}\gamma^* & -\mathcal{J}\gamma^*\end{pmatrix}$ are unitary, it follows from (4.44) that

$$\begin{aligned}\begin{pmatrix} \mathcal{I} \\ M(\lambda,b,\alpha,\beta)\end{pmatrix} &= \begin{pmatrix}\alpha \\ \alpha\mathcal{J}\end{pmatrix}\begin{pmatrix}\gamma^* & -\mathcal{J}\gamma^*\end{pmatrix}\begin{pmatrix} \mathcal{I} \\ M(\lambda,b,\gamma,\beta)\end{pmatrix} P \\ &= \begin{pmatrix}\alpha\gamma^* - \alpha\,\mathcal{J}\gamma^*\,M(\lambda,b,\gamma,\beta) \\ \alpha\,\mathcal{J}\gamma^* + \alpha\gamma^*\,M(\lambda,b,\gamma,\beta)\end{pmatrix} P.\end{aligned}$$

The first row above yields that $P = [\,\alpha\gamma^* - \alpha\,\mathcal{J}\gamma^*\,M(\lambda,b,\gamma,\beta)\,]^{-1}$, while the second row is then written as identity (4.43). ∎

Corollary 4.27. *Let $\alpha,\beta,\gamma \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. With notation (4.33) and (4.31) we have*

$$\mathcal{X}(\cdot,\lambda,\alpha,M(\lambda,b,\alpha,\beta)) = \mathcal{X}(\cdot,\lambda,\gamma,M(\lambda,b,\gamma,\beta))\,[\,\alpha\gamma^* - \alpha\,\mathcal{J}\gamma^*\,M(\lambda,b,\gamma,\beta)\,]^{-1}.$$

Proof. The above identity follows from (4.44) and the formula for the matrix P from the end of the proof of Lemma 4.26. ∎

4.3 Geometric properties of Weyl disks

In this section we study the geometric properties of the Weyl disks as the point b moves through the interval $[a,\infty)_{\mathbb{T}}$. Our first result shows that the Weyl disks $D(\lambda,b)$ are nested. This statement generalizes the results in [112, Theorem 4.5], [127, Section 3.2.1], [44, Equation (2.70)], [142, Theorem 3.1], [150, Theorem 4.4] and [6, Theorem 3.3(i)].

Theorem 4.28 (Nesting property of Weyl disks). *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$D(\lambda,b_2) \subseteq D(\lambda,b_1) \quad \textit{for every } b_1,b_2 \in [a,\infty)_{\mathbb{T}},\ b_1 < b_2.$$

Proof. Let $b_1,b_2 \in [a,\infty)_{\mathbb{T}}$ with $b_1 < b_2$ and take $M \in D(\lambda,b_2)$, i.e., $\mathcal{E}(M,b_2) \le 0$. From identity (4.34) with $b=b_1$ and later with $b=b_2$ and by using $\widetilde{\mathcal{W}}(\cdot) \ge 0$ we have

$$\begin{aligned}\mathcal{E}(M,b_1) &\overset{(4.34)}{=} -2\,\delta(\lambda)\,\operatorname{Im}(M) + 2\,|\operatorname{Im}(\lambda)| \int_a^{b_1} \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\,\widetilde{\mathcal{W}}(t)\,\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\,\Delta t \\ &\le -2\,\delta(\lambda)\,\operatorname{Im}(M) + 2\,|\operatorname{Im}(\lambda)| \int_a^{b_2} \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\,\widetilde{\mathcal{W}}(t)\,\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\,\Delta t \\ &\overset{(4.34)}{=} \mathcal{E}(M,b_2) \le 0.\end{aligned}$$

Therefore, by Definition 4.21, the matrix M belongs to $D(\lambda,b_1)$, which shows the result. ∎

Similarly for the regular case (Hypothesis 4.14) we now introduce the following assumption.

Hypothesis 4.29. *There exists $b_0 \in (a, \infty)_{\mathbb{T}}$ such that Hypothesis 4.14 is satisfied with $b = b_0$. That is, inequality (4.27) holds with $b = b_0$ for every $\lambda \in \mathbb{C}$.*

From Hypothesis 4.29 it follows by $\widetilde{\mathcal{W}}(\cdot) \geq 0$ that inequality (4.27) holds for every $b \in [b_0, \infty)_{\mathbb{T}}$. For the study of the geometric properties of Weyl disks we shall use the following representation

$$\mathcal{E}(M, b) = i\,\delta(\lambda)\,\mathcal{X}^*(b, \lambda, \alpha, M)\,\mathcal{J}\mathcal{X}(b, \lambda, \alpha, M) = \begin{pmatrix} \mathcal{I} & M^* \end{pmatrix} \begin{pmatrix} \mathcal{F}(b, \lambda, \alpha) & \mathcal{G}^*(b, \lambda, \alpha) \\ \mathcal{G}(b, \lambda, \alpha) & \mathcal{H}(b, \lambda, \alpha) \end{pmatrix} \begin{pmatrix} \mathcal{I} \\ M \end{pmatrix}$$

of the matrix $\mathcal{E}(M, b)$, where we define on $[a, \infty)_{\mathbb{T}}$ the $n \times n$ matrices

$$\left.\begin{aligned} \mathcal{F}(\cdot, \lambda, \alpha) &:= i\,\delta(\lambda)\,Z^*(\cdot, \lambda, \alpha)\,\mathcal{J}Z(\cdot, \lambda, \alpha), \\ \mathcal{G}(\cdot, \lambda, \alpha) &:= i\,\delta(\lambda)\,\widetilde{Z}^*(\cdot, \lambda, \alpha)\,\mathcal{J}Z(\cdot, \lambda, \alpha), \\ \mathcal{H}(\cdot, \lambda, \alpha) &:= i\,\delta(\lambda)\,\widetilde{Z}^*(\cdot, \lambda, \alpha)\,\mathcal{J}\widetilde{Z}(\cdot, \lambda, \alpha). \end{aligned}\right\} \tag{4.45}$$

Since $\mathcal{E}(M, b)$ is Hermitian, it follows that $\mathcal{F}(\cdot, \lambda, \alpha)$ and $\mathcal{H}(\cdot, \lambda, \alpha)$ are also Hermitian. Moreover, by (4.26) we have $\mathcal{H}(a, \lambda, \alpha) = 0$. In addition, if $b \in [b_0, \infty)_{\mathbb{T}}$, then Corollary 4.7 and Hypothesis 4.29 yield for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\mathcal{H}(b, \lambda, \alpha) = 2\,|\mathrm{Im}(\lambda)| \int_a^b \widetilde{Z}^{\sigma*}(t, \lambda, \alpha)\,\widetilde{\mathcal{W}}(t)\,\widetilde{Z}^{\sigma}(t, \lambda, \alpha)\,\Delta t > 0. \tag{4.46}$$

Therefore, $\mathcal{H}(b, \lambda, \alpha)$ is invertible (positive definite) for all $b \in [b_0, \infty)_{\mathbb{T}}$ and monotone nondecreasing as $b \to \infty$, with a consequence that $\mathcal{H}^{-1}(b, \lambda, \alpha)$ is monotone nonincreasing as $b \to \infty$. The following factorization of $\mathcal{E}(M, b)$ holds true, see also [45, Equation (4.11)].

Lemma 4.30. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By using the notation in (4.45), for any $M \in \mathbb{C}^{n \times n}$ and $b \in [a, \infty)_{\mathbb{T}}$ we have*

$$\begin{aligned} \mathcal{E}(M, b) = {} & \mathcal{F}(b, \lambda, \alpha) - \mathcal{G}^*(b, \lambda, \alpha)\,\mathcal{H}^{-1}(b, \lambda, \alpha)\,\mathcal{G}(b, \lambda, \alpha) \\ & + [\mathcal{G}^*(b, \lambda, \alpha)\,\mathcal{H}^{-1}(b, \lambda, \alpha) + M^*]\,\mathcal{H}(b, \lambda, \alpha)\,[\mathcal{H}^{-1}(b, \lambda, \alpha)\,\mathcal{G}(b, \lambda, \alpha) + M], \end{aligned}$$

whenever the matrix $\mathcal{H}(b, \lambda, \alpha)$ is invertible.

Proof. The result is shown by a direct calculation. ■

The following identity is a generalization of its corresponding versions published in [112, Lemma 4.3], [34, Lemma 3.3], [142, Proposition 3.2], [45, Lemma 4.2], [150, Lemma 4.6], and [6, Theorem 5.6].

Lemma 4.31. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By using the notation in (4.45), for any $b \in [a, \infty)_{\mathbb{T}}$ we have*

$$\mathcal{G}^*(b, \lambda, \alpha)\,\mathcal{H}^{-1}(b, \lambda, \alpha)\,\mathcal{G}(b, \lambda, \alpha) - \mathcal{F}(b, \lambda, \alpha) = \mathcal{H}^{-1}(b, \bar{\lambda}, \alpha), \tag{4.47}$$

whenever the matrices $\mathcal{H}(b, \lambda, \alpha)$ and $\mathcal{H}(b, \bar{\lambda}, \alpha)$ are invertible.

Proof. In order to simplify and abbreviate the notation we introduce the matrices

$$\left.\begin{array}{lll} \mathcal{F} := \mathcal{F}(b,\lambda,\alpha), & \mathcal{G} := \mathcal{G}(b,\lambda,\alpha), & \mathcal{H} := \mathcal{H}(b,\lambda,\alpha), \\ \widetilde{\mathcal{F}} := \mathcal{F}(b,\bar{\lambda},\alpha), & \widetilde{\mathcal{G}} := \mathcal{G}(b,\bar{\lambda},\alpha), & \widetilde{\mathcal{H}} := \mathcal{H}(b,\bar{\lambda},\alpha), \end{array}\right\} \tag{4.48}$$

and use the notation $Z(\lambda)$ and $\widetilde{Z}(\lambda)$ for $Z(b,\lambda,\alpha)$ and $\widetilde{Z}(b,\lambda,\alpha)$, respectively. Then, since $\mathcal{F}^* = \mathcal{F}$ and $\delta(\lambda)\,\delta(\bar{\lambda}) = -1$, we get the identities

$$\mathcal{G}^*\widetilde{\mathcal{F}} - \mathcal{F}^*\widetilde{\mathcal{G}} = Z^*(\lambda)\,\mathcal{J}\,[\widetilde{Z}(\lambda)\,Z^*(\bar{\lambda}) - Z(\lambda)\,\widetilde{Z}^*(\bar{\lambda})]\,\mathcal{J}\,Z(\bar{\lambda}) \overset{(4.19)}{=} Z^*(\lambda)\,\mathcal{J}\,Z(\bar{\lambda}) \overset{(4.18)}{=} 0, \tag{4.49}$$

$$\mathcal{H}\,\widetilde{\mathcal{G}}^* - \mathcal{G}\,\mathcal{H}^* = \widetilde{Z}^*(\lambda)\,\mathcal{J}\,[\widetilde{Z}(\lambda)\,Z^*(\bar{\lambda}) - Z(\lambda)\,\widetilde{Z}^*(\bar{\lambda})]\,\mathcal{J}\,\widetilde{Z}(\bar{\lambda}) \overset{(4.19)}{=} \widetilde{Z}^*(\lambda)\,\mathcal{J}\,\widetilde{Z}(\bar{\lambda}) \overset{(4.18)}{=} 0, \tag{4.50}$$

$$\mathcal{G}\,\widetilde{\mathcal{G}} - \mathcal{H}\,\widetilde{\mathcal{F}} = \widetilde{Z}^*(\lambda)\,\mathcal{J}\,[Z(\lambda)\,\widetilde{Z}^*(\bar{\lambda}) - \widetilde{Z}(\lambda)\,Z^*(\bar{\lambda})]\,\mathcal{J}\,Z(\bar{\lambda}) \overset{(4.19)}{=} -\widetilde{Z}^*(\lambda)\,\mathcal{J}\,Z(\bar{\lambda}) \overset{(4.18)}{=} \mathcal{I}, \tag{4.51}$$

$$\mathcal{G}^*\widetilde{\mathcal{G}}^* - \mathcal{F}\,\widetilde{\mathcal{H}} = Z^*(\lambda)\,\mathcal{J}\,[\widetilde{Z}(\lambda)\,Z^*(\bar{\lambda}) - Z(\lambda)\,\widetilde{Z}^*(\bar{\lambda})]\,\mathcal{J}\,\widetilde{Z}(\bar{\lambda}) \overset{(4.19)}{=} Z^*(\lambda)\,\mathcal{J}\,\widetilde{Z}(\bar{\lambda}) \overset{(4.18)}{=} \mathcal{I}. \tag{4.52}$$

Hence, by using that $\widetilde{\mathcal{H}}$ is Hermitian we see that

$$\widetilde{\mathcal{H}}^{-1} \overset{(4.52)}{=} \mathcal{G}^*\widetilde{\mathcal{G}}^*\,\widetilde{\mathcal{H}}^{-1} - \mathcal{F} = \mathcal{G}^*\widetilde{\mathcal{G}}^*\,\widetilde{\mathcal{H}}^{*-1} - \mathcal{F} \overset{(4.50)}{=} \mathcal{G}^*\,\mathcal{H}^{-1}\mathcal{G} - \mathcal{F}.$$

which proves identity (4.47). ■

Corollary 4.32. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Under Hypothesis 4.29, the matrix $\mathcal{H}(b,\lambda,\alpha)$ is invertible for every $b \in [b_0,\infty)_{\mathbb{T}}$ and for these values of b we have*

$$\mathcal{G}^*(b,\lambda,\alpha)\,\mathcal{H}^{-1}(b,\lambda,\alpha)\,\mathcal{G}(b,\lambda,\alpha) - \mathcal{F}(b,\lambda,\alpha) > 0. \tag{4.53}$$

Proof. Since $b \in [b_0,\infty)_{\mathbb{T}}$, then identity (4.46) yields that $\mathcal{H}(b,\lambda,\alpha) > 0$ and $\mathcal{H}(b,\bar{\lambda},\alpha) > 0$. Consequently, inequality (4.53) follows from equation (4.47) of Lemma 4.31. ■

In the next result we justify the terminology for the sets $D(\lambda,b)$ and $C(\lambda,b)$ in Definition 4.21 to be called a "disk" and a "circle". It is a generalization of [142, Theorem 3.1], [45, Theorem 5.4], [6, Theorem 3.3(iii)], see also [127, Theorem 3.5], [102, pp. 70–71], [43, p. 3485], [142, Proposition 3.3], [34, Theorem 3.3], [150, Theorem 4.8]. Consider the sets $\mathcal{V}$ and $\mathcal{U}$ of contractive and unitary matrices in $\mathbb{C}^{n\times n}$, respectively, i.e.,

$$\mathcal{V} := \{\, V \in \mathbb{C}^{n\times n},\ V^*V \le \mathcal{I}\,\}, \quad \mathcal{U} := \partial\mathcal{V} = \{\, U \in \mathbb{C}^{n\times n},\ U^*U = \mathcal{I}\,\}. \tag{4.54}$$

The set $\mathcal{V}$ is known to be closed (in fact compact, since $\mathcal{V}$ is bounded) and convex.

Theorem 4.33. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Under Hypothesis 4.29, for every $b \in [b_0,\infty)_{\mathbb{T}}$ the Weyl disk and Weyl circle have the representations*

$$D(\lambda,b) = \{\, P(\lambda,b) + R(\lambda,b)\ V\,R(\bar{\lambda},b),\ V \in \mathcal{V}\,\}, \tag{4.55}$$

$$C(\lambda,b) = \{\, P(\lambda,b) + R(\lambda,b)\ U\,R(\bar{\lambda},b),\ U \in \mathcal{U}\,\}, \tag{4.56}$$

where, with the notation (4.45),

$$P(\lambda,b) := -\mathcal{H}^{-1}(\lambda,b,\alpha)\,\mathcal{G}(\lambda,b,\alpha), \quad R(\lambda,b) := \mathcal{H}^{-1/2}(\lambda,b,\alpha). \tag{4.57}$$

Consequently, for every $b \in [b_0,\infty)_{\mathbb{T}}$ the sets $D(\lambda,b)$ are closed and convex.

The representations of $D(\lambda, b)$ and $C(\lambda, b)$ in (4.55) and (4.56) can be written as $D(\lambda, b) = P(\lambda, b) + R(\lambda, b)\mathcal{V}\, R(\bar\lambda, b)$ and $C(\lambda, b) = P(\lambda, b) + R(\lambda, b)\mathcal{U}\, R(\bar\lambda, b)$. The importance of the matrices $P(\lambda, b)$ and $R(\lambda, b)$ is justified in the following.

Definition 4.34. For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $b \in [a, \infty)_{\mathbb{T}}$ such that $\mathcal{H}(\lambda, b, \alpha)$ and $\mathcal{H}(\bar\lambda, b, \alpha)$ are positive definite, the matrix $P(\lambda, b)$ is called the *center* of the Weyl disk or the Weyl circle. The matrices $R(\lambda, b)$ and $R(\bar\lambda, b)$ are called the *matrix radii* of the Weyl disk or the Weyl circle.

Proof of Theorem 4.33. By (4.46) and for any $b \in [b_0, \infty)_{\mathbb{T}}$, the matrices $\mathcal{H} := \mathcal{H}(\lambda, b, \alpha)$ and $\widetilde{\mathcal{H}} := \mathcal{H}(\bar\lambda, b, \alpha)$ are positive definite, so that the matrices $P := P(\lambda, b)$, $R(\lambda) := R(\lambda, b)$, and $R(\bar\lambda) := R(\bar\lambda, b)$ are well defined. By Definition 4.21, for $M \in D(\lambda, b)$ we have $\mathcal{E}(M, b) \le 0$, which in turn with notation (4.48) implies by Lemmas 4.30 and 4.31 that

$$\begin{aligned} &-R^2(\bar\lambda) + (M^* - P^*)\, R^{-2}(\lambda)\, (M - P) \\ &\quad \overset{(4.47)}{=} \mathcal{F} - \mathcal{G}^* \mathcal{H}^{-1} \mathcal{G} + (\mathcal{H}^{-1}\mathcal{G} + M)^* \mathcal{H}\, (\mathcal{H}^{-1}\mathcal{G} + M) = \mathcal{E}(M, b) \le 0. \end{aligned} \tag{4.58}$$

Therefore, the matrix

$$V := R^{-1}(\lambda)\, (M - P)\, R^{-1}(\bar\lambda) \tag{4.59}$$

satisfies $V^* V \le \mathcal{I}$. This relation between the matrices $M \in D(\lambda, b)$ and $V \in \mathcal{V}$ is bijective (more precisely, it is a homeomorphism) and the inverse to (4.59) is given by $M = P + R(\lambda)\, V R(\bar\lambda)$. The latter formula proves that the Weyl disk $D(\lambda, b)$ has the representation in (4.55). Moreover, since by the definition $M \in C(\lambda, b)$ means that $\mathcal{E}(M, b) = 0$, it follows that the elements of the Weyl circle $C(\lambda, b)$ are in one-to-one correspondence with the matrices V defined in (4.59) which, similarly as in (4.58), now satisfy $V^* V = \mathcal{I}$. Hence, the representation of $C(\lambda, b)$ in (4.56) follows. The fact that for $b \in [b_0, \infty)_{\mathbb{T}}$ the sets $D(\lambda, b)$ are closed and convex follows from the same properties of the set $\mathcal{V}$, being homeomorphic to $D(\lambda, b)$. ■

4.4 Limiting Weyl disk and Weyl circle

In this section we study the limiting properties of the Weyl disk, Weyl circle, and their center and matrix radii. Since, under Hypothesis 4.29 the matrix function $\mathcal{H}(\cdot, \lambda, \alpha)$ is monotone nondecreasing as $b \to \infty$, it follows from the definition of $R(\lambda, b)$ and $R(\bar\lambda, b)$ in (4.57) that the two matrix functions $R(\lambda, \cdot)$ and $R(\bar\lambda, \cdot)$ are monotone nonincreasing for $b \to \infty$. Furthermore, since $R(\lambda, b)$ and $R(\bar\lambda, b)$ are Hermitian and positive definite for $b \in [b_0, \infty)_{\mathbb{T}}$, the limits

$$R_+(\lambda) := \lim_{b \to \infty} R(\lambda, b), \quad R_+(\bar\lambda) := \lim_{b \to \infty} R(\bar\lambda, b) \tag{4.60}$$

exist and satisfy $R_+(\lambda) \ge 0$ and $R_+(\bar\lambda) \ge 0$. The index "+" in the above notation as well as in Definition 4.36 refers to the limiting disk at $+\infty$. In the following result we shall see that the center $P(\lambda, b)$ also converges to a limiting matrix when $b \to \infty$. This is a generalization of [112, Theorem 4.7], [34, Theorem 3.5], [142, Proposition 3.5], [45, Theorem 4.5], [150, Theorem 4.10].

Theorem 4.35. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 4.29, the center $P(\lambda, b)$ converges as $b \to \infty$ to a limiting matrix $P_+(\lambda) \in \mathbb{C}^{n \times n}$, that is,*

$$P_+(\lambda) := \lim_{b \to \infty} P(\lambda, b). \tag{4.61}$$

Proof. We prove that the matrix function $P(\lambda,\cdot)$ satisfies the Cauchy convergence criterion. Let $b_1, b_2 \in [b_0,\infty)_{\mathbb{T}}$ be given with $b_1 < b_2$. By Theorem 4.28, we have that $D(\lambda, b_2) \subseteq D(\lambda, b_1)$. Therefore, by (4.55) of Theorem 4.33, for a matrix $M \in D(\lambda, b_2)$ there are (unique) matrices $V_1, V_2 \in \mathcal{V}$ such that

$$M = P(\lambda, b_j) + R(\lambda, b_j)\, V_j\, R(\bar\lambda, b_j), \quad j \in \{1,2\}. \tag{4.62}$$

Upon subtracting the two equations in (4.62) we get

$$P(\lambda, b_2) - P(\lambda, b_1) + R(\lambda, b_2)\, V_2\, R(\bar\lambda, b_2) = R(\lambda, b_1)\, V_1\, R(\bar\lambda, b_1).$$

This equation, when solved for V_1 in terms of V_2, has the form

$$V_1 = R^{-1}(\lambda, b_1) \left[P(\lambda, b_2) - P(\lambda, b_1) + R(\lambda, b_2)\, V_2\, R(\bar\lambda, b_2)\right] R^{-1}(\bar\lambda, b_1) =: T(V_2),$$

which defines a continuous mapping $T : \mathcal{V} \to \mathcal{V}$, $T(V_2) = V_1$. Since $\mathcal{V}$ is compact, it follows that the mapping T has a fixed point in $\mathcal{V}$, i.e., $T(V) = V$ for some matrix $V \in \mathcal{V}$. Equation $T(V) = V$ implies that

$$\begin{aligned} P(\lambda, b_2) - P(\lambda, b_1) &= R(\lambda, b_1)\, V R(\bar\lambda, b_1) - R(\lambda, b_2)\, V R(\bar\lambda, b_2) \\ &= [R(\lambda, b_1) - R(\lambda, b_2)]\, V R(\bar\lambda, b_1) - R(\lambda, b_2)\, V\, [R(\bar\lambda, b_1) - R(\bar\lambda, b_2)]. \end{aligned}$$

Hence, by $\|V\| \le 1$, we have

$$\begin{aligned} &\|P(\lambda, b_2) - P(\lambda, b_1)\| \\ &\quad \le \|R(\lambda, b_1) - R(\lambda, b_2)\| \left\|R(\bar\lambda, b_1)\right\| + \|R(\lambda, b_2)\| \left\|R(\bar\lambda, b_1) - R(\bar\lambda, b_2)\right\|. \end{aligned} \tag{4.63}$$

Since the functions $R(\lambda,\cdot)$ and $R(\bar\lambda,\cdot)$ are monotone nonincreasing, they are bounded. That is, for some $K > 0$ we have $\|R(\lambda, b)\| \le K$ and $\left\|R(\bar\lambda, b)\right\| \le K$ for all $b \in [b_0,\infty)_{\mathbb{T}}$.

Let $\varepsilon > 0$ be arbitrary. The convergence of $R(\lambda, b)$ and $R(\bar\lambda, b)$ as $b \to \infty$ yields the existence of $b_3 \in [b_0,\infty)_{\mathbb{T}}$ such that for every $b_1, b_2 \in [b_3,\infty)_{\mathbb{T}}$ with $b_1 < b_2$ we have

$$\|R(\nu, b_1) - R(\nu, b_2)\| \le \varepsilon/(2K), \quad \nu \in \{\lambda, \bar\lambda\}. \tag{4.64}$$

Using estimate (4.64) in inequality (4.63) we obtain for $b_2 > b_1 \ge b_3$

$$\|P(\lambda, b_2) - P(\lambda, b_1)\| < \varepsilon/(2K) \cdot K + \varepsilon/(2K) \cdot K = \varepsilon.$$

This means that the limit $P_+(\lambda) \in \mathbb{C}^{n\times n}$ in (4.61) exists, which completes the proof. ■

By Theorems 4.28 and 4.33 we know that the Weyl disks $D(\lambda, b)$ are closed, convex, and nested as $b \to \infty$. Therefore the limit of $D(\lambda, b)$ as $b \to \infty$ is a closed, convex, and nonempty set. This motivates the following definition, which can be found in the special cases of system $(\mathcal{S}_\lambda)$ in [102, Theorem 3.3], [34, Theorem 3.6], [45, Definition 4.7], [150, Theorem 4.12].

Definition 4.36 (Limiting Weyl disk). Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the set

$$D_+(\lambda) := \bigcap_{b\in[a,\infty)_{\mathbb{T}}} D(\lambda, b)$$

is called the *limiting Weyl disk*. The matrix $P_+(\lambda)$ from Theorem 4.35 is called the *center* of $D_+(\lambda)$ and the matrices $R_+(\lambda)$ and $R_+(\bar\lambda)$ from (4.60) its *matrix radii*.

As a consequence of Theorem 4.33 we obtain the following characterization of the limiting Weyl disk.

Corollary 4.37. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 4.29, we have*

$$D_+(\lambda) = P_+(\lambda) + R_+(\lambda)\, \mathcal{V}\, R_+(\bar{\lambda}),$$

where $\mathcal{V}$ is the set of all contractive matrices defined in (4.54).

From now on we assume that Hypothesis 4.29 holds, so that the limiting center $P_+(\lambda)$ and the limiting matrix radii $R_+(\lambda)$ and $R_+(\bar{\lambda})$ of $D_+(\lambda)$ are well defined.

Remark 4.38. By means of the nesting property of the disks (Theorem 4.28) and Theorems 4.22 and 4.24, it follows that the elements of the limiting Weyl disk $D_+(\lambda)$ are of the form

$$M_+(\lambda) \in D_+(\lambda), \quad M_+(\lambda) = \lim_{b \to \infty} M(\lambda, b, \alpha, \beta(b)), \tag{4.65}$$

where $\beta(b) \in \mathbb{C}^{n \times 2n}$ satisfies $\beta(b)\, \beta^*(b) = \mathcal{I}$ and $i\, \delta(\lambda)\, \beta(b)\, \mathcal{J} \beta^*(b) \geq 0$ for all $b \in [a, \infty)$. Moreover, from Lemma 4.18 we conclude that

$$M_+^*(\lambda) = M_+(\bar{\lambda}). \tag{4.66}$$

A matrix $M_+(\lambda)$ from (4.65) is called a *half-line Weyl–Titchmarsh $M(\lambda)$-function*. Also, as noted in [45, Section 4], see also [43, Theorem 2.18], the function $M_+(\lambda)$ is a Herglotz function with rank n and has a certain integral representation (which will not be needed in this chapter).

Our next result shows another characterization of the elements of $D_+(\lambda)$ in terms of the Weyl solution $\mathcal{X}(\cdot, \alpha, \lambda, M)$ defined in (4.33). This is a generalization of [112, p. 671], [102, Equation (3.2)], [34, Theorem 3.8(i)], [45, Theorem 4.8], [150, Theorem 4.15].

Theorem 4.39. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n \times n}$. The matrix M belongs to the limiting Weyl disk $D_+(\lambda)$ if and only if*

$$\int_a^\infty \mathcal{X}^{\sigma *}(t, \lambda, \alpha, M)\, \widetilde{\mathcal{W}}(t)\, \mathcal{X}^{\sigma}(t, \lambda, \alpha, M)\, \Delta t \leq \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}. \tag{4.67}$$

Proof. By Definition 4.36, we have $M \in D_+(\lambda)$ if and only if $M \in D(\lambda, b)$, i.e., $\mathcal{E}(M, b) \leq 0$, for all $b \in [a, \infty)_{\mathbb{T}}$. Therefore, by formula (4.34), we get

$$\int_a^b \mathcal{X}^{\sigma *}(t, \lambda, \alpha, M)\, \widetilde{\mathcal{W}}(t)\, \mathcal{X}^{\sigma}(t, \lambda, \alpha, M)\, \Delta t = \frac{\mathcal{E}(M, b)}{2\, |\operatorname{Im}(\lambda)|} + \frac{\delta(\lambda)\, \operatorname{Im}(M)}{|\operatorname{Im}(\lambda)|} \leq \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}$$

for every $b \in [a, \infty)_{\mathbb{T}}$, which is equivalent to inequality (4.67). ∎

Remark 4.40. In [34, Definition 3.4], the notion of a boundary of the limiting Weyl disk $D_+(\lambda)$ is discussed. This would be a "limiting Weyl circle" according to Definitions 4.21 and 4.36. The description of matrices $M \in \mathbb{C}^{n \times n}$ laying on this boundary follows from Theorems 4.39 and 4.22, giving for such matrices M the equality

$$\int_a^\infty \mathcal{X}^{\sigma *}(t, \lambda, \alpha, M)\, \widetilde{\mathcal{W}}(t)\, \mathcal{X}^{\sigma}(t, \lambda, \alpha, M)\, \Delta t = \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}. \tag{4.68}$$

Condition (4.68) is also equivalent to

$$\lim_{t\to\infty} \mathcal{X}^*(t,\lambda,\alpha,M)\,\mathcal{J}\mathcal{X}(t,\lambda,\alpha,M) = 0. \tag{4.69}$$

This is because, by (4.35) and the Lagrange identity (Corollary 4.6),

$$\begin{aligned}&\mathcal{X}^*(t,\lambda,\alpha,M)\,\mathcal{J}\mathcal{X}(t,\lambda,\alpha,M)\\ &\qquad = 2i\,\operatorname{Im}(\lambda)\left[\frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)} - \int_a^t \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\,\widetilde{\mathcal{W}}(s)\,\mathcal{X}^{\sigma}(s,\lambda,\alpha,M)\,\Delta s\right]\end{aligned}$$

for every $t \in [a,\infty)_{\mathbb{T}}$. From this we can see that the integral on the right-hand side above converges for $t \to \infty$ and (4.68) holds if and only if condition (4.69) is satisfied. Characterizations (4.68) and (4.69) of the matrices M on the boundary of the limiting Weyl disk $D_+(\lambda)$ generalize the corresponding results in [34, Theorems 3.8(ii) and 3.9], see also [142, Theorem 6.3].

Consider the linear space of square-integrable $\mathrm{C}^1_{\mathrm{prd}}$ functions

$$L^2_{\mathcal{W}} = L^2_{\mathcal{W}}[a,\infty)_{\mathbb{T}} := \left\{z : [a,\infty)_{\mathbb{T}} \to \mathbb{C}^{2n},\ z \in \mathrm{C}^1_{\mathrm{prd}},\ \|z(\cdot)\|_{\mathcal{W}} < \infty\right\},$$

where we define

$$\|z(\cdot)\|_{\mathcal{W}} := \sqrt{\langle z(\cdot), z(\cdot)\rangle_{\mathcal{W}}}, \quad \langle z(\cdot), \tilde{z}(\cdot)\rangle_{\mathcal{W}} := \int_a^\infty z^{\sigma *}(t)\,\widetilde{\mathcal{W}}(t)\,\tilde{z}^{\sigma}(t)\,\Delta t.$$

In the following result we prove that the space $L^2_{\mathcal{W}}$ contains the columns of the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$ when M belongs to the limiting Weyl disk $D_+(\lambda)$. This implies that there are at least n linearly independent solutions of system $(\mathcal{S}_\lambda)$ in $L^2_{\mathcal{W}}$. This is a generalization of [112, Theorem 5.1], [142, Theorem 4.1], [45, Theorem 4.10], [6, p. 716].

Theorem 4.41. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $M \in D_+(\lambda)$. The columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a linearly independent system of solutions of system $(\mathcal{S}_\lambda)$, each of which belongs to $L^2_{\mathcal{W}}$.*

Proof. Let us denote by $z_j(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M)\,e_j$ for $j \in \{1,\dots,n\}$ the columns of the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$, where e_j is the j-th unit vector. We prove that the functions $z_1(\cdot),\dots,z_n(\cdot)$ are linearly independent. Assume that $\sum_{j=1}^n c_j z_j(\cdot) = 0$ on $[a,\infty)_{\mathbb{T}}$ for some $c_1,\dots,c_n \in \mathbb{C}$. Then $\mathcal{X}(\cdot,\lambda,\alpha,M)\,c = 0$, where $c := (c_1^*,\dots,c_n^*)^* \in \mathbb{C}^n$. It follows by (4.35) that

$$2i\,c^*\operatorname{Im}(M)\,c = c^*\mathcal{X}^*(a,\lambda,\alpha,M)\,\mathcal{J}\mathcal{X}(a,\lambda,\alpha,M)\,c = 0,$$

which implies the equality $c^*\,\delta(\lambda)\,\operatorname{Im}(M)\,c = 0$. Using that $M \in D_+(\lambda) \subseteq D(\lambda,b)$ for some $b \in [b_0,\infty)_{\mathbb{T}}$, we obtain from Theorem 4.25 that the matrix $\delta(\lambda)\,\operatorname{Im}(M)$ is positive definite. Hence, $c = 0$ so that the functions $z_1(\cdot),\dots,z_n(\cdot)$ are linearly independent. Finally, for every $j \in \{1,\dots,n\}$ we get from Theorem 4.39 the inequality

$$\|z_j(\cdot)\|^2_{\mathcal{W}} = \int_a^\infty z_j^{\sigma *}(t)\,\widetilde{\mathcal{W}}(t)\,z_j^{\sigma}(t)\,\Delta t \overset{(4.67)}{\leq} e_j^*\,\frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}\,e_j \leq \frac{\|\,\delta(\lambda)\,\operatorname{Im}(M)\,\|}{|\operatorname{Im}(\lambda)|} < \infty.$$

Thus, $z_j(\cdot) \in L^2_{\mathcal{W}}$ for every $j \in \{1,\dots,n\}$ and the proof is complete. ∎

Denote by $\mathcal{N}(\lambda)$ the linear space of all square-integrable solutions of system ($\mathcal{S}_\lambda$), i.e.,

$$\mathcal{N}(\lambda) := \{\, z(\cdot) \in L^2_{\mathcal{W}},\ z(\cdot) \text{ solves } (\mathcal{S}_\lambda) \,\}.$$

Then as a consequence of Theorem 4.41 we obtain the estimate

$$\dim \mathcal{N}(\lambda) \geq n \quad \text{for each } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Next we discuss the situation when $\dim \mathcal{N}(\lambda) = n$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 4.42. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $\dim \mathcal{N}(\lambda) = n$. Then the matrix radii of the limiting Weyl disk $D_+(\lambda)$ satisfy $R_+(\lambda) = 0 = R_+(\bar{\lambda})$. Consequently, the set $D_+(\lambda)$ consists of the single matrix $M = P_+(\lambda)$, i.e., the center of $D_+(\lambda)$, which is given by formula (4.61) of Theorem 4.35.*

Proof. With the matrix radii $R_+(\lambda)$ and $R_+(\bar{\lambda})$ of $D_+(\lambda)$ defined in (4.60) and with the Weyl solution $\mathcal{X}(\cdot, \lambda, \alpha, M)$ given by a matrix $M \in D_+(\lambda)$ we observe that the columns of $\mathcal{X}(\cdot, \lambda, \alpha, M)$ form a basis of the space $\mathcal{N}(\lambda)$. Since the columns of the fundamental matrix $\Psi(\cdot, \lambda, \alpha) = \begin{pmatrix} Z(\cdot, \lambda, \alpha) & \widetilde{Z}(\cdot, \lambda, \alpha) \end{pmatrix}$ span all solutions of system ($\mathcal{S}_\lambda$), the definition of $\mathcal{X}(\cdot, \lambda, \alpha, M) = Z(\cdot, \lambda, \alpha) + \widetilde{Z}(\cdot, \lambda, \alpha)\, M$ yields that the columns of $\widetilde{Z}(\cdot, \lambda, \alpha)$ together with the columns of $\mathcal{X}(\cdot, \lambda, \alpha, M)$ form a basis of all solutions of system ($\mathcal{S}_\lambda$). Hence, from $\dim \mathcal{N}(\lambda) = n$ and Theorem 4.41 we get that the columns of $\widetilde{Z}(\cdot, \lambda, \alpha)$ do not belong to $L^2_{\mathcal{W}}$. Consequently, by formula (4.46), the Hermitian matrix functions $\mathcal{H}(\cdot, \lambda, \alpha)$ and $\mathcal{H}(\cdot, \bar{\lambda}, \alpha)$ defined in (4.45) are monotone nondecreasing on $[a, \infty)_{\mathbb{T}}$ without any upper bound, i.e., their eigenvalues – being real – tend to ∞. Therefore, the functions $R(\lambda, \cdot)$ and $R(\bar{\lambda}, \cdot)$ as defined in (4.57) have limits at ∞ equal to zero. That is, $R_+(\lambda) = 0$ and $R_+(\bar{\lambda}) = 0$. The fact that the set $D_+(\lambda) = \{P_+(\lambda)\}$ then follows from the characterization of $D_+(\lambda)$ in Corollary 4.37. ■

In the final result of this section we establish another characterization of the matrices M from the limiting Weyl disk $D_+(\lambda)$. In comparison with Theorem 4.39 we now use a similar condition to the one in Theorem 4.24 for the regular spectral problem. However, a stronger assumption than Hypothesis 4.29 is now required for this result to hold, compare with [44, Lemma 2.21] and [45, Theorem 4.16].

Hypothesis 4.43. *For every $a_0, b_0 \in (a, \infty)_{\mathbb{T}}$ with $a_0 < b_0$ and for every $\lambda \in \mathbb{C}$ we have*

$$\int_{a_0}^{b_0} \Psi^{\sigma *}(t, \lambda, \alpha)\, \widetilde{\mathcal{W}}(t)\, \Psi^{\sigma}(t, \lambda, \alpha)\, \Delta t > 0.$$

Under Hypothesis 4.43, the Weyl disks $D(\lambda, b)$ converge to the limiting disk "monotonically" as $b \to \infty$, i.e., the limiting Weyl disk $D_+(\lambda)$ is "open" in the sense that all its element lie inside $D_+(\lambda)$. This can be interpreted in view of Theorem 4.24 as $\mathcal{E}(M, t) < 0$ for all $t \in [a, \infty)_{\mathbb{T}}$.

Theorem 4.44. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n \times n}$. Under Hypothesis 4.43, the matrix $M \in D_+(\lambda)$ if and only if*

$$\mathcal{E}(M, t) < 0 \quad \text{for all } t \in [a, \infty)_{\mathbb{T}}. \tag{4.70}$$

Proof. If condition (4.70) holds, then $M \in D_+(\lambda)$ follows from the definition of $D_+(\lambda)$. Conversely, suppose that $M \in D_+(\lambda)$ and let $t \in [a, \infty)_{\mathbb{T}}$ be given. Then for any $b \in (t, \infty)_{\mathbb{T}}$ we have by formula (4.34) that

$$\begin{aligned}\mathcal{E}(M, t) &= -2\,\delta(\lambda)\ \mathrm{Im}(M) + 2\ |\mathrm{Im}(\lambda)| \int_a^t \mathcal{X}^{\sigma*}(s, \lambda, \alpha, M)\, \widetilde{\mathcal{W}}(s)\, \mathcal{X}^{\sigma}(s, \lambda, \alpha, M)\, \Delta s \\ &= \mathcal{E}(M, b) - 2\ |\mathrm{Im}(\lambda)| \int_t^b \mathcal{X}^{\sigma*}(s, \lambda, \alpha, M)\, \widetilde{\mathcal{W}}(s)\, \mathcal{X}^{\sigma}(s, \lambda, \alpha, M)\, \Delta s, \qquad (4.71)\end{aligned}$$

where we used identity (2.12). Since $M \in D_+(\lambda)$ is assumed, we have $M \in D(\lambda, b)$, i.e., $\mathcal{E}(M, b) \le 0$, while Hypothesis 4.43 implies the positivity of the integral over $[t, b]_{\mathbb{T}}$ in (4.71). Consequently, equation (4.71) yields that $\mathcal{E}(M, t) < 0$. ∎

Remark 4.45. If we partition the Weyl solution $\mathcal{X}(\cdot, \lambda) := \mathcal{X}(\cdot, \lambda, \alpha, M)$ into two $n \times n$ blocks $\mathcal{X}_1(\cdot, \lambda)$ and $\mathcal{X}_2(\cdot, \lambda)$ as in (4.39), then condition (4.70) can be written as

$$\delta(\lambda)\, \mathrm{Im}\, (\mathcal{X}_1^*(t, \lambda)\, \mathcal{X}_2(t, \lambda)) > 0 \quad \text{for all } t \in [a, \infty)_{\mathbb{T}}.$$

Therefore, by Remark 4.23(ii), the matrices $\mathcal{X}_1(t, \lambda)$ and $\mathcal{X}_2(t, \lambda)$ are invertible for all $t \in [a, \infty)_{\mathbb{T}}$. A standard argument then yields that the quotient $Q(\cdot, \lambda) := \mathcal{X}_2(\cdot, \lambda)\, \mathcal{X}_1^{-1}(\cdot, \lambda)$ satisfies the *Riccati matrix equation* (suppressing the argument t in the coefficients)

$$Q^{\Delta} - (\mathcal{C} + \mathcal{D}Q) + Q^{\sigma}(\mathcal{A} + \mathcal{B}Q) + \lambda \mathcal{W}[\mathcal{I} + \mu(\mathcal{A} + \mathcal{B}Q)] = 0, \quad t \in [a, \infty)_{\mathbb{T}},$$

see [58, Theorem 3], [94, Section 6], and [95].

4.5 Limit point and limit circle criteria

Throughout this section we assume that Hypothesis 4.29 is satisfied. The results from Theorem 4.41 and Lemma 4.42 motivate the following terminology, compare with [166, p. 75], [159, Definition 1.2] in the time scales scalar case $n = 1$, with [43, p. 3486], [133, p. 1668], [107, p. 274], [141, Definition 3.1], [137, Definition 1], [167, p. 2826] in the continuous case, and with [142, Definition 5.1], [45, Definition 4.12] in the discrete case.

Definition 4.46 (Limit point and limit circle case for system ($\mathcal{S}_\lambda$)). System ($\mathcal{S}_\lambda$) is said to be in the *limit point case* at ∞ (or of the *limit point type*) if

$$\dim \mathcal{N}(\lambda) = n \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

System ($\mathcal{S}_\lambda$) is said to be in the *limit circle case* at ∞ (or of the *limit circle type*) if

$$\dim \mathcal{N}(\lambda) = 2n \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Remark 4.47. According to Remark 4.38 (in which $\beta(b) \equiv \beta$), the center $P_+(\lambda)$ of the limiting Weyl disk $D_+(\lambda)$ can be expressed in the limit point case as

$$P_+(\lambda) = M_+(\lambda) = \lim_{b \to \infty} M(\lambda, b, \alpha, \beta),$$

where $\beta \in \Gamma$ is arbitrary but fixed.

Next we establish the first result of this section. Its continuous time version can be found in [107, Theorem 2.1], [112, Theorem 8.5] and the discrete time version in [44, Lemma 3.2], [45, Theorem 4.13].

Theorem 4.48. *Let system* ($\mathcal{S}_\lambda$) *be in the limit point or limit circle case, fix* $\alpha \in \Gamma$, *and let* $\lambda, \nu \in \mathbb{C} \setminus \mathbb{R}$. *Then*

$$\lim_{t\to\infty} \mathcal{X}_+^*(t,\lambda,\alpha,M_+(\lambda))\,\mathcal{J}\mathcal{X}_+(t,\nu,\alpha,M_+(\nu)) = 0, \tag{4.72}$$

where $\mathcal{X}_+(\cdot,\lambda,\alpha,M_+(\lambda))$ *and* $\mathcal{X}_+(\cdot,\nu,\alpha,M_+(\nu))$ *are the Weyl solutions of* ($\mathcal{S}_\lambda$) *and* ($\mathcal{S}_\nu$), *respectively, defined by* (4.33) *through the matrices* $M_+(\lambda)$ *and* $M_+(\nu)$, *which are determined by the limit in* (4.65).

Proof. For every $t \in [a,\infty)_{\mathbb{T}}$ and matrices $\beta(t) \in \mathbb{C}^{n\times 2n}$ such that the identities

$$\beta(t)\,\beta^*(t) = \mathcal{I} \quad \text{and} \quad i\,\delta(\lambda)\,\beta(t)\,\mathcal{J}\beta^*(t) \ge 0$$

holds true, and for $\kappa \in \{\lambda,\nu\}$ we define the matrix (compare with Definition 4.17)

$$M(\kappa,t,\alpha,\beta(t)) := -[\beta(t)\,\widetilde{Z}(t,\kappa,\alpha)]^{-1}\beta(t)\,Z(t,\kappa,\alpha).$$

Then, by Theorems 4.22 and 4.24, we have $M(\kappa,t,\alpha,\beta(t)) \in D(\kappa,t)$. Following the notation in (4.33), we consider the Weyl solutions $\mathcal{X}(\cdot,\kappa) := \mathcal{X}(\cdot,\kappa,\alpha,M(\kappa,t,\alpha,\beta(\cdot)))$. Similarly, let $\mathcal{X}_+(\cdot,\kappa) := \mathcal{X}(\cdot,\kappa,\alpha,M_+(\kappa))$ be the Weyl solutions corresponding to the matrices $M_+(\kappa) \in D_+(\kappa)$ from the statement of this theorem.

First assume that system ($\mathcal{S}_\lambda$) is of the limit point type. In this case, by Remark 4.47, we may take $\beta(t) \in \Gamma$ for all $t \in [a,\infty)_{\mathbb{T}}$. Hence, from Theorem 4.22 we get that $\beta(\cdot)\,\mathcal{X}(\cdot,\kappa) = 0$ on $[a,\infty)_{\mathbb{T}}$. By (4.23), for each $t \in [a,\infty)_{\mathbb{T}}$ and $\kappa \in \{\lambda,\nu\}$ there is a matrix $Q_\kappa(t) \in \mathbb{C}^{n\times n}$ such that $\mathcal{X}(\cdot,\kappa) = \mathcal{J}\beta^*(\cdot)\,Q_\kappa(\cdot)$ on $[a,\infty)_{\mathbb{T}}$. Hence, we have on $[a,\infty)_{\mathbb{T}}$

$$\begin{aligned} &\mathcal{X}_+^*(t,\lambda)\,\mathcal{J}\mathcal{X}_+(t,\nu) + F(t,\lambda,\nu,\beta(t)) + G(t,\lambda,\nu,\beta(t)) \\ &\qquad = \mathcal{X}^*(t,\lambda)\,\mathcal{J}\mathcal{X}(t,\nu) = Q_\lambda^*(t)\,\beta(t)\,\mathcal{J}\beta^*(t)\,Q_\nu(t) = 0, \end{aligned} \tag{4.73}$$

where we define

$$\begin{aligned} F(t,\lambda,\nu,\beta(t)) &:= \mathcal{X}_+^*(t,\lambda)\,\mathcal{J}\widetilde{Z}(t,\nu,\alpha)\,[M(\nu,t,\alpha,\beta(t)) - M_+(\nu)], \\ G(t,\lambda,\nu,\beta(t)) &:= [M^*(\lambda,t,\alpha,\beta(t)) - M_+^*(\lambda)]\,\widetilde{Z}^*(t,\lambda,\alpha)\,\mathcal{J}\mathcal{X}(t,\nu). \end{aligned}$$

If we show that

$$\lim_{t\to\infty} F(t,\lambda,\nu,\beta(t)) = 0, \quad \lim_{t\to\infty} G(t,\lambda,\nu,\beta(t)) = 0, \tag{4.74}$$

then equation (4.73) implies the result claimed in (4.72). First we prove the limit (4.74)(ii). Pick any $t \in [b_0,\infty)_{\mathbb{T}}$. By Theorem 4.33, Corollary 4.37, and $D_+(\lambda) \subseteq D(\lambda,t)$ we have

$$M(\lambda,t,\alpha,\beta(t)) = P(\lambda,t) + R(\lambda,t)\,U(t)\,R(\bar\lambda,t), \quad M_+(\lambda) = P(\lambda,t) + R(\lambda,t)\,V(t)\,R(\bar\lambda,t),$$

where $U(t) \in \mathcal{U}$ and $V(t) \in \mathcal{V}$. Therefore,

$$M(\lambda,t,\alpha,\beta(t)) - M_+(\lambda) = R(\lambda,t)\,[U(t) - V(t)]\,R(\bar\lambda,t).$$

Since $\widetilde{Z}(\cdot,\lambda,\alpha)$ and $\mathcal{X}(\cdot,\nu)$ are respectively solutions of systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) which satisfy $\widetilde{Z}^*(a,\lambda,\alpha)\,\mathcal{J}\mathcal{X}(a,\nu) = -\mathcal{I}$, it follows from Corollary 4.6 that

$$\widetilde{Z}^*(t,\lambda,\alpha)\,\mathcal{J}\mathcal{X}(t,\nu) = -\mathcal{I} + (\bar\lambda - \nu)\int_a^t \widetilde{Z}^{\sigma*}(s,\lambda,\alpha)\,\widetilde{\mathcal{W}}(s)\,\mathcal{X}^\sigma(s,\nu)\,\Delta s.$$

Hence, we can write

$$G(t,\lambda,\nu,\beta(t)) = R(\bar\lambda,t)\,[U^*(t)-V^*(t)]\,R(\lambda,t)\Big[\,(\bar\lambda-\nu)\int_a^t \tilde Z^{\sigma*}(s,\lambda,\alpha)\,\widetilde{\mathcal W}(s)\,\mathcal X^{\sigma}(s,\nu)\,\Delta s-\mathcal I\,\Big],$$

where we used the Hermitian property of $R(\lambda,t)$ and $R(\bar\lambda,t)$. Since we now assume that system ($\mathcal S_\lambda$) is in the limit point case, we know from Lemma 4.42 that $\lim_{t\to\infty} R(\lambda,t)=0$ and $\lim_{t\to\infty} R(\bar\lambda,t)=0$. Therefore, in order to establish (4.74)(ii) it is sufficient to show that

$$R(\lambda,t)\int_a^t \tilde Z^{\sigma*}(s,\lambda,\alpha)\,\widetilde{\mathcal W}(s)\,\mathcal X^{\sigma}(s,\nu)\,\Delta s$$

is bounded for $t\in[b_0,\infty)_{\mathbb T}$. Let $\eta\in\mathbb C^n$ be a unit vector and denote by $\mathcal X_j(\cdot,\nu):=\mathcal X(\cdot,\nu)\,e_j$ the j-th column of $\mathcal X(\cdot,\nu)$ for $j\in\{1,\dots,n\}$. With the definition of $R(\lambda,\cdot)$ in (4.57) we have

$$\begin{aligned}
&\Big|\int_a^t \eta^* R(\lambda,s)\,\tilde Z^{\sigma*}(s,\lambda,\alpha)\,\widetilde{\mathcal W}(s)\,\mathcal X_j^{\sigma}(s,\nu)\,\Delta s\Big|\\
&\quad\le \int_a^t \Big|\widetilde{\mathcal W}^{1/2}(s)\,\tilde Z^{\sigma*}(s,\lambda,\alpha)\,R(\lambda,s)\,\eta\Big|\cdot\Big|\widetilde{\mathcal W}^{1/2}(s)\,\mathcal X_j^{\sigma}(s,\nu)\Big|\ \Delta s\\
&\quad\overset{(2.17)}{\le}\Big(\int_a^t \eta^* R(\lambda,s)\,\tilde Z^{\sigma*}(s,\lambda,\alpha)\widetilde{\mathcal W}(s)\,\tilde Z^{\sigma}(s,\lambda,\alpha)\,R(\lambda,s)\,\eta\,\Delta s\Big)^{1/2}\times\\
&\qquad\qquad\times\Big(\int_a^t \mathcal X_j^{\sigma*}(s,\nu)\,\widetilde{\mathcal W}(s)\,\mathcal X_j^{\sigma}(s,\nu)\,\Delta s\Big)^{1/2},
\end{aligned}\tag{4.75}$$

where the last step follows from the Cauchy–Schwarz inequality (2.17) on time scales. From equation (4.46) we obtain

$$\mathcal H^{-1/2}(t,\lambda,\alpha)\int_a^t \tilde Z^{\sigma*}(s,\lambda,\alpha)\widetilde{\mathcal W}(s)\,\tilde Z^{\sigma}(s,\lambda,\alpha)\,\Delta s\,\mathcal H^{-1/2}(t,\lambda,\alpha)=\frac{1}{2\,|\mathrm{Im}(\lambda)|}\,\mathcal I,$$

so that the first term in the product in (4.75) is bounded by $1/\sqrt{2\,|\mathrm{Im}(\lambda)|}$. Moreover, from formula (4.34) we get that the second term in the product in (4.75) is bounded by the number $[e_j^*\,\mathrm{Im}(M(\nu,t,\alpha,\beta(t)))\,e_j]/\,\mathrm{Im}(\nu)$. Hence, upon recalling the limit in (4.65), we conclude that the product in (4.75) is bounded by

$$\frac{1}{2\,|\mathrm{Im}(\lambda)|}\cdot\frac{e_j^*\,\mathrm{Im}(M_+(\nu))\,e_j}{\mathrm{Im}(\nu)},$$

which is independent of t. Consequently, the limit (4.74)(ii) is established. The limit (4.74)(i) is then proven in a similar manner. The proof for the limit point case is finished.

If system ($\mathcal S_\lambda$) is in the limit circle case, then for $\kappa\in\{\lambda,\nu\}$ the columns of $\tilde Z(\cdot,\kappa,\alpha)$ and $\mathcal X_+(\cdot,\kappa)$ belong to $L^2_{\mathcal W}$, hence they are bounded in the $L^2_{\mathcal W}$ norm. In this case the limits in (4.74) easily follow from the limit (4.65) for $M_+(\kappa)$, $\kappa\in\{\lambda,\nu\}$. ∎

In the next result we provide a characterization of the system ($\mathcal S_\lambda$) being of the limit point type. Special cases of this statement can be found, e.g., in [142, Theorem 6.12] and [45, Theorem 4.14].

Theorem 4.49. *Let $\alpha\in\Gamma$. The system ($\mathcal S_\lambda$) is in the limit point case if and only if for every $\lambda\in\mathbb C\backslash\mathbb R$ and every square-integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\bar\lambda)$ of ($\mathcal S_\lambda$) and ($\mathcal S_{\bar\lambda}$), respectively, we have*

$$z_1^*(t,\lambda)\,\mathcal J z_2(t,\bar\lambda)=0\quad\text{for all } t\in[b_0,\infty)_{\mathbb T}.\tag{4.76}$$

Proof. Let ($\mathcal{S}_\lambda$) be in the limit point case. Fix any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and suppose that $z_1(\cdot,\lambda)$ and $z_2(\cdot,\bar\lambda)$ are solutions of ($\mathcal{S}_\lambda$) and ($\mathcal{S}_{\bar\lambda}$), respectively. Then, by Theorem 4.41 and Remark 4.38, there are vectors $\xi_1,\xi_2 \in \mathbb{C}^n$ such that $z_1(\cdot,\lambda) = \mathcal{X}_+(\cdot,\lambda)\,\xi_1$ and $z_2(\cdot,\bar\lambda) = \mathcal{X}_+(\cdot,\bar\lambda)\,\xi_2$ on $[a,\infty)_\mathbb{T}$, where $\mathcal{X}_+(\cdot,\kappa) := \mathcal{X}_+(\cdot,\kappa,\alpha,M_+(\kappa))$ are the Weyl solutions corresponding to some matrices $M_+(\kappa) \in D_+(\kappa)$ for $\kappa \in \{\lambda,\bar\lambda\}$. In fact, by Lemma 4.42 the matrix $M_+(\kappa)$ is equal to the center of the disk $D_+(\kappa)$. It follows that for any $t \in [b_0,\infty)_\mathbb{T}$ equality

$$\begin{aligned}\mathcal{X}_+^*(t,\lambda)\,\mathcal{J}\mathcal{X}_+(t,\bar\lambda) &\overset{(4.33)}{=} \big(\mathcal{I} \quad M_+^*(\lambda)\big)\,\Psi^*(t,\lambda,\alpha)\,\mathcal{J}\,\Psi(t,\bar\lambda,\alpha)\,\big(\mathcal{I} \quad M_+^*(\bar\lambda)\big)^* \\ &\overset{(4.17)(i)}{=} M_+^*(\bar\lambda) - M_+^*(\lambda) \overset{(4.66)}{=} 0\end{aligned}$$

holds, so that equation (4.76) is established.

Conversely, let $\nu \in \mathbb{C}\setminus\mathbb{R}$ be arbitrary but fixed, set $\lambda := \bar\nu$, and suppose that for every square-integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) condition (4.76) is satisfied. From Theorem 4.41 we know that for $M_+(\kappa) \in D_+(\kappa)$ the columns $\mathcal{X}_+^{[j]}(\cdot,\kappa)$, $j \in \{1,\dots,n\}$, of the Weyl solution $\mathcal{X}_+(\cdot,\kappa)$ are linearly independent square-integrable solutions of ($\mathcal{S}_\kappa$), $\kappa \in \{\lambda,\nu\}$. Therefore, $\dim\mathcal{N}(\lambda) \geq n$ and $\dim\mathcal{N}(\nu) \geq n$. Moreover, by identity (4.17)(i) we have

$$\mathcal{X}_+^*(t,\lambda)\,\mathcal{J}\mathcal{X}_+^{[j]}(t,\nu) = 0 \quad \text{for all } t \in [b_0,\infty)_\mathbb{T} \text{ and } j \in \{1,\dots,n\}. \tag{4.77}$$

Let $z(\cdot,\nu)$ be any square-integrable solution of system ($\mathcal{S}_\nu$). Then, by our assumption (4.76),

$$\mathcal{X}_+^*(t,\lambda)\,\mathcal{J}z(t,\nu) = 0 \quad \text{for all } t \in [b_0,\infty)_\mathbb{T}. \tag{4.78}$$

From (4.77) and (4.78) it follows that the vectors $\mathcal{X}_+^{[j]}(a,\nu)$, $j \in \{1,\dots,n\}$, and $z(a,\nu)$ are solutions of the linear homogeneous system

$$\mathcal{X}_+^*(a,\lambda)\,\mathcal{J}\eta = 0. \tag{4.79}$$

Since, by Theorem 4.41, the vectors $\mathcal{X}_+^{[j]}(a,\nu)$ for $j \in \{1,\dots,n\}$ represent a basis of the solution space of system (4.79), there exists a vector $\xi \in \mathbb{C}^n$ such that $z(a,\nu) = \mathcal{X}_+(a,\nu)\,\xi$. By the uniqueness of solutions of system ($\mathcal{S}_\nu$) we then get $z(\cdot,\nu) = \mathcal{X}_+(\cdot,\nu)\,\xi$ on $[a,\infty)_\mathbb{T}$. Hence, the solution $z(\cdot,\nu)$ is square-integrable and $\dim\mathcal{N}(\nu) = n$. Since $\nu \in \mathbb{C}\setminus\mathbb{R}$ was arbitrary, it follows that system ($\mathcal{S}_\lambda$) is in the limit point case. ■

As a consequence of the above result we obtain a characterization of the limit point case in terms of a condition similar to (4.76), but using a limit. This statement is a generalization of [107, Corollary 2.3], [44, Corollary 3.3], [142, Theorem 6.14], [45, Corollary 4.15], [34, Theorem 3.9], [150, Theorem 4.16].

Corollary 4.50. *Let $\alpha \in \Gamma$. System ($\mathcal{S}_\lambda$) is in the limit point case if and only if for every $\lambda,\nu \in \mathbb{C}\setminus\mathbb{R}$ and every square-integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$), respectively, we have*

$$\lim_{t\to\infty} z_1^*(t,\lambda)\,\mathcal{J}z_2(t,\nu) = 0. \tag{4.80}$$

Proof. The necessity follows directly from Theorem 4.48. Conversely, assume that condition (4.80) holds for every $\lambda,\nu \in \mathbb{C}\setminus\mathbb{R}$ and every square-integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$). Fix $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and set $\nu := \bar\lambda$. By Corollary 4.7 we know that $z_1^*(\cdot,\lambda)\,\mathcal{J}z_2(\cdot,\nu)$ is constant on $[a,\infty)_\mathbb{T}$. Therefore, by using condition (4.80) we can see that identity (4.76) must be satisfied, which yields by Theorem 4.49 that system ($\mathcal{S}_\lambda$) is of the limit point type. ■

4.6 Nonhomogeneous time scale symplectic systems

In this section we consider the nonhomogeneous time scale symplectic system

$$z^{\Delta}(t,\lambda) = \mathcal{S}(t,\lambda)\, z(t,\lambda) - \mathcal{J}\, \widetilde{\mathcal{W}}(t)\, f^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.81}$$

where the matrix function $\mathcal{S}(\cdot,\lambda)$ and $\widetilde{\mathcal{W}}(\cdot)$ are defined in (4.6) and (4.4), $f \in L^2_{\mathcal{W}}$, and where the associated homogeneous system ($\mathcal{S}_\lambda$) is either of the limit point or limit circle type at ∞. Together with system (4.81) we consider a second system of the same form but with a different spectral parameter and a different nonhomogeneous term

$$y^{\Delta}(t,\nu) = \mathcal{S}(t,\nu)\, y(t,\nu) - \mathcal{J}\, \widetilde{\mathcal{W}}(t)\, g^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.82}$$

with $g \in L^2_{\mathcal{W}}$. The following is a generalization of Theorem 4.5 to nonhomogeneous systems.

Theorem 4.51 (Lagrange identity). *Let $\lambda,\nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $y(\cdot,\nu)$ are $2n \times m$ solutions of systems* (4.81) *and* (4.82)*, respectively, then*

$$\begin{aligned}[z^*(t,\lambda)\, \mathcal{J} y(t,\nu)]^{\Delta} &= (\bar\lambda - \nu)\, z^{\sigma *}(t,\lambda)\, \widetilde{\mathcal{W}}(t)\, y^{\sigma}(t,\nu) \\ &\quad - f^{\sigma *}(t)\, \widetilde{\mathcal{W}}(t)\, y^{\sigma}(t,\nu) + z^{\sigma *}(t,\lambda)\, \widetilde{\mathcal{W}}(t)\, g^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}. \end{aligned} \tag{4.83}$$

Proof. Formula (4.83) follows by the product rule (2.3) with the aid of the relation

$$z^{\sigma}(t,\lambda) = [\mathcal{I} + \mu(t)\, \mathcal{S}(t,\lambda)]\, z(t,\lambda) + \mu(t)\, \widetilde{\mathcal{W}}(t)\, f^{\sigma}(t)$$

and identity (4.8). ∎

For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $t,s \in [a,\infty)_{\mathbb{T}}$ we define the function

$$G(t,s,\lambda,\alpha) := \begin{cases} \widetilde{Z}(t,\lambda,\alpha)\, \mathcal{X}^*_+(s,\bar\lambda,\alpha), & \text{for } t \in [a,s)_{\mathbb{T}}, \\ \mathcal{X}_+(t,\lambda,\alpha)\, \widetilde{Z}^*(s,\bar\lambda,\alpha), & \text{for } t \in [s,\infty)_{\mathbb{T}}, \end{cases} \tag{4.84}$$

where $\widetilde{Z}(\cdot,\lambda,\alpha)$ denotes the solution of system ($\mathcal{S}_\lambda$) given in (4.29), i.e., $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^*$, and $\mathcal{X}_+(\cdot,\lambda,\alpha) := \mathcal{X}(\cdot,\lambda,\alpha,M_+(\lambda))$ is the Weyl solution of ($\mathcal{S}_\lambda$) as in (4.33) determined by a matrix $M_+(\lambda) \in D_+(\lambda)$. This matrix $M_+(\lambda) \in D_+(\lambda)$ is arbitrary but fixed throughout this section. By interchanging the order of the arguments t and s we have

$$G(t,s,\lambda,\alpha) = \begin{cases} \mathcal{X}_+(t,\lambda,\alpha)\, \widetilde{Z}^*(s,\bar\lambda,\alpha), & \text{for } s \in [a,t]_{\mathbb{T}}, \\ \widetilde{Z}(t,\lambda,\alpha)\, \mathcal{X}^*_+(s,\bar\lambda,\alpha), & \text{for } s \in (t,\infty)_{\mathbb{T}}. \end{cases} \tag{4.85}$$

In the literature the function $G(\cdot,\cdot,\lambda,\alpha)$ is called a resolvent kernel, compare with [107, p. 283], [111, p. 15], [45, Equation (5.4)], and in this section it will play a role of the *Green function*.

Lemma 4.52. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathcal{X}_+(t,\lambda,\alpha)\, \widetilde{Z}^*(t,\bar\lambda,\alpha) - \widetilde{Z}(t,\lambda,\alpha)\, \mathcal{X}^*_+(t,\bar\lambda,\alpha) = \mathcal{J} \quad \text{for all } t \in [a,\infty)_{\mathbb{T}}. \tag{4.86}$$

Proof. Identity (4.86) follows by a direct calculation from the definition of $\mathcal{X}_+(\cdot,\lambda,\alpha)$ via (4.33) with a matrix $M_+(\lambda) \in D_+(\lambda)$ by using formulas (4.19) and (4.66). ∎

In the next lemma we summarize the properties of the function $G(\cdot,\cdot,\lambda,\alpha)$, which together with Proposition 4.54 and Theorem 4.55 justifies the terminology "Green function" of the system (4.81), compare with [14, Section 4]. A discrete version of the result below can be found in [45, Lemma 5.1].

Lemma 4.53. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The function $G(\cdot,\cdot,\lambda,\alpha)$ has the following properties:*

(i) $G^*(t,s,\lambda,\alpha) = G(s,t,\bar{\lambda},\alpha)$ *for every* $t,s \in [a,\infty)_{\mathbb{T}}$, $t \neq s$,

(ii) $G^*(t,t,\lambda,\alpha) = G(t,t,\bar{\lambda},\alpha) - \mathcal{J}$ *for every* $t \in [a,\infty)_{\mathbb{T}}$,

(iii) *for every right-scattered point* $t \in [a,\infty)_{\mathbb{T}}$ *it holds*

$$G(\sigma(t),\sigma(t),\lambda,\alpha) = [\mathcal{I} + \mu(t)\,\mathcal{S}(t,\lambda)]\, G(t,\sigma(t),\lambda,\alpha) + \mathcal{J},$$

(iv) *for every* $t,s \in [a,\infty)_{\mathbb{T}}$ *such that* $t \notin \mathcal{T}(s)$ *the function* $G(\cdot,s,\lambda,\alpha)$ *solves homogeneous system* ($\mathcal{S}_\lambda$) *on the set* $\mathcal{T}(s)$, *where*

$$\mathcal{T}(s) := \{\, \tau \in [a,\infty)_{\mathbb{T}},\ \ \tau \neq \rho(s) \text{ if } s \text{ is left-scattered} \,\},$$

(v) *the columns of* $G(\cdot,s,\lambda,\alpha)$ *belong to* $L^2_{\mathcal{W}}$ *for every* $s \in [a,\infty)_{\mathbb{T}}$, *and the columns of* $G(t,\cdot,\lambda,\alpha)$ *belong to* $L^2_{\mathcal{W}}$ *for every* $t \in [a,\infty)_{\mathbb{T}}$.

Proof. Condition (i) follows from the definition of $G(\cdot,s,\lambda,\alpha)$ in (4.84). Condition (ii) is a consequence of Lemma 4.52. Condition (iii) is proven from the definition of the function $G(\sigma(t),\sigma(t),\lambda,\alpha)$ in (4.84) by using Lemma 4.52 and

$$\widetilde{Z}(t,\lambda,\alpha) = \widetilde{Z}^{\sigma}(t,\lambda,\alpha) - \mu(t)\,\mathcal{S}(t,\lambda)\,\widetilde{Z}(t,\lambda,\alpha).$$

Concerning condition (iv), the function $G(\cdot,s,\lambda,\alpha)$ solves system ($\mathcal{S}_\lambda$) on $[s,\infty)_{\mathbb{T}}$ because $\mathcal{X}_+(\cdot,\lambda,\alpha)$ solves this system on $[s,\infty)_{\mathbb{T}}$. If $s \in (a,\infty)_{\mathbb{T}}$ is left-dense, then $G(\cdot,s,\lambda,\alpha)$ solves ($\mathcal{S}_\lambda$) on $[a,s)_{\mathbb{T}}$, since $\widetilde{Z}(\cdot,\lambda,\alpha)$ solves this system on $[a,s)_{\mathbb{T}}$. For the same reason $G(\cdot,s,\lambda,\alpha)$ solves ($\mathcal{S}_\lambda$) on $[a,\rho(s))_{\mathbb{T}}$ if $s \in (a,\infty)_{\mathbb{T}}$ is left-scattered. Condition (v) follows from the definition of $G(\cdot,s,\lambda,\alpha)$ in (4.84) used with $t \geq s$ and from the fact that the columns of $\mathcal{X}_+(\cdot,\lambda,\alpha)$ belong to $L^2_{\mathcal{W}}$, by Theorem 4.41. The columns of $G(t,\cdot,\lambda,\alpha)$ then belong to $L^2_{\mathcal{W}}$ by part (i) of this lemma. ∎

Since by Lemma 4.53(v) the columns of $G(t,\cdot,\lambda,\alpha)$ belong to $L^2_{\mathcal{W}}$, the function

$$\hat{z}(t,\lambda,\alpha) := -\int_a^\infty G(t,\sigma(s),\lambda,\alpha)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.87}$$

is well defined whenever $f \in L^2_{\mathcal{W}}$. Moreover, by using (4.85) we can write $\hat{z}(t,\lambda,\alpha)$ as

$$\begin{aligned}\hat{z}(t,\lambda,\alpha) = -\mathcal{X}_+(t,\lambda,\alpha) \int_a^t \widetilde{Z}^{\sigma*}(s,\bar{\lambda},\alpha)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s \\ - \widetilde{Z}(t,\lambda,\alpha)\int_t^\infty \mathcal{X}_+^{\sigma*}(s,\bar{\lambda},\alpha)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.\end{aligned} \tag{4.88}$$

Proposition 4.54. *For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $f \in L^2_{\mathcal{W}}$ the function $\hat{z}(\cdot,\lambda,\alpha)$ defined in (4.87) solves the nonhomogeneous system (4.81) with the initial condition $\alpha\,\hat{z}(a,\lambda,\alpha) = 0$.*

Proof. By the time scales product rule (2.3) when Δ-differentiating expression (4.88) we have for every $t \in [a,\infty)_{\mathbb{T}}$ (suppressing the dependence on α in the calculation below)

$$\begin{aligned}\hat{z}^{\Delta}(t,\lambda) &= -\mathcal{X}_{+}^{\Delta}(t,\lambda)\int_a^t \widetilde{Z}^{\sigma *}(s,\bar{\lambda})\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s - \mathcal{X}_{+}^{\sigma}(t,\lambda)\,\widetilde{Z}^{\sigma *}(t,\bar{\lambda})\,\widetilde{\mathcal{W}}(t)\,f^{\sigma}(t)\\ &\quad - \widetilde{Z}^{\Delta}(t,\lambda)\int_t^{\infty} \mathcal{X}_{+}^{\sigma *}(s,\bar{\lambda})\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s + \widetilde{Z}^{\sigma}(t,\lambda)\,\mathcal{X}_{+}^{\sigma *}(t,\bar{\lambda})\,\widetilde{\mathcal{W}}(t)\,f^{\sigma}(t)\\ &= \mathcal{S}(t,\lambda)\,\hat{z}(t,\lambda) - [\mathcal{X}_{+}^{\sigma}(t,\lambda)\,\widetilde{Z}^{\sigma *}(t,\bar{\lambda}) - \widetilde{Z}^{\sigma}(t,\lambda)\,\mathcal{X}_{+}^{\sigma *}(t,\bar{\lambda})]\,\widetilde{\mathcal{W}}(t)\,f^{\sigma}(t)\\ &\overset{(4.86)}{=} \mathcal{S}(t,\lambda)\,\hat{z}(t,\lambda) - \mathcal{J}\,\widetilde{\mathcal{W}}(t)\,f^{\sigma}(t).\end{aligned}$$

This shows that $\hat{z}(\cdot,\lambda,\alpha)$ is a solution of system (4.81). From equation (4.88) with $t=a$ we get

$$\alpha\,\hat{z}(a,\lambda,\alpha) = -\alpha\,\widetilde{Z}(a,\lambda,\alpha)\int_a^{\infty} \mathcal{X}_{+}^{\sigma *}(s,\bar{\lambda},\alpha)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s = 0,$$

where we used the initial condition $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^*$ and the fact $\alpha\,\mathcal{J}\alpha^* = 0$ coming from $\alpha \in \Gamma$. ■

The following theorem provides further properties of the solution $\hat{z}(\cdot,\lambda,\alpha)$ of system (4.81). It is a generalization of [106, Lemma 4.2], [112, Theorem 7.5], [45, Theorem 5.2] to time scales.

Theorem 4.55. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $f \in L^2_{\mathcal{W}}$. Suppose that system ($\mathcal{S}_\lambda$) is in the limit point or limit circle case. Then the solution $\hat{z}(\cdot,\lambda,\alpha)$ of system (4.81) defined in (4.87) belongs to $L^2_{\mathcal{W}}$ and satisfies*

$$\|\,\hat{z}(\cdot,\lambda,\alpha)\,\|_{\mathcal{W}} \leq \frac{1}{|\mathrm{Im}(\lambda)|}\,\|f\|_{\mathcal{W}}, \tag{4.89}$$

$$\lim_{t\to\infty} \mathcal{X}_{+}^{*}(t,\nu,\alpha)\,\mathcal{J}\hat{z}(t,\lambda,\alpha) = 0 \quad \text{for every } \nu \in \mathbb{C}\setminus\mathbb{R}. \tag{4.90}$$

Proof. To shorten the notation we suppress the dependence on α in all quantities appearing in this proof. Assume first that system ($\mathcal{S}_\lambda$) is in the limit point case. For every $r \in [a,\infty)_{\mathbb{T}}$ we define the function $f_r(\cdot) := f(\cdot)$ on $[a,r]_{\mathbb{T}}$ and $f_r(\cdot) := 0$ on $(r,\infty)_{\mathbb{T}}$, and the function

$$\hat{z}_r(t,\lambda) := -\int_a^{\infty} G(t,\sigma(s),\lambda)\,\widetilde{\mathcal{W}}(s)\,f_r^{\sigma}(s)\,\Delta s = -\int_a^{r} G(t,\sigma(s),\lambda)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s.$$

For every $t \in [r,\infty)_{\mathbb{T}}$ we have as in (4.88) that

$$\hat{z}_r(t,\lambda) = -\mathcal{X}_{+}(t,\lambda)\,g(r,\lambda), \qquad g(r,\lambda) := \int_a^r \widetilde{Z}^{\sigma *}(s,\bar{\lambda})\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s. \tag{4.91}$$

Since by Theorem 4.41 the solution $\mathcal{X}_{+}(\cdot,\lambda) \in L^2_{\mathcal{W}}$, equation (4.91) shows that $\hat{z}_r(\cdot,\lambda)$, being a multiple of $\mathcal{X}_{+}(\cdot,\lambda)$, also belongs to $L^2_{\mathcal{W}}$. Moreover, by Theorem 4.48,

$$\lim_{t\to\infty} \hat{z}_r^{*}(t,\lambda)\,\mathcal{J}\hat{z}_r(t,\lambda) \overset{(4.91)}{=} g^{*}(r,\lambda)\,\lim_{t\to\infty} \mathcal{X}_{+}^{*}(t,\lambda)\,\mathcal{J}\mathcal{X}_{+}(t,\lambda)\,g(r,\lambda) \overset{(4.72)}{=} 0. \tag{4.92}$$

On the other hand, $\hat{z}_r^{*}(a,\lambda)\,\mathcal{J}\hat{z}_r(a,\lambda) = 0$ and for any $t \in [a,\infty)_{\mathbb{T}}$ identity (4.83) implies

$$\begin{aligned}\hat{z}_r^{*}(t,\lambda)\,\mathcal{J}\hat{z}_r(t,\lambda) &= -2i\,\mathrm{Im}(\lambda)\int_a^t \hat{z}_r^{\sigma *}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,\hat{z}_r^{\sigma}(s,\lambda)\,\Delta s\\ &\quad + 2i\,\mathrm{Im}\left(\int_a^t \hat{z}_r^{\sigma *}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,f_r^{\sigma}(s)\,\Delta s\right).\end{aligned} \tag{4.93}$$

Combining equation (4.93) where $t \to \infty$, formula (4.92), and the definition on $f_r(\cdot)$ yields

$$\|\,\hat{z}_r(\cdot,\lambda)\,\|_{\mathcal{W}}^2 = \int_a^\infty \hat{z}_r^{\sigma*}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,\hat{z}_r^{\sigma}(s,\lambda)\,\Delta s = \frac{1}{\operatorname{Im}(\lambda)}\,\operatorname{Im}\left(\int_a^r \hat{z}_r^{\sigma*}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s\right).$$

By using the Cauchy–Schwarz inequality (2.17) on time scales and $\widetilde{\mathcal{W}}(\cdot) \geq 0$ we then have

$$\begin{aligned}
\|\,\hat{z}_r(\cdot,\lambda)\,\|_{\mathcal{W}}^2 &= \frac{1}{2i\operatorname{Im}(\lambda)}\left[\int_a^r \hat{z}_r^{\sigma*}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s - \int_a^r f^{\sigma*}(s)\,\widetilde{\mathcal{W}}(s)\,\hat{z}_r^{\sigma}(s,\lambda)\,\Delta s\right] \\
&\leq \frac{1}{|\operatorname{Im}(\lambda)|}\,\left|\int_a^r \hat{z}_r^{\sigma*}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s\right| \\
&\overset{(2.17)}{\leq} \frac{1}{|\operatorname{Im}(\lambda)|}\,\left(\int_a^r \hat{z}_r^{\sigma*}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,\hat{z}_r^{\sigma}(s,\lambda)\,\Delta s\right)^{1/2}\left(\int_a^r f^{\sigma*}(s)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s\right)^{1/2} \\
&\leq \frac{1}{|\operatorname{Im}(\lambda)|}\,\|\,\hat{z}_r(\cdot,\lambda)\,\|_{\mathcal{W}}\,\|f\|_{\mathcal{W}}\,.
\end{aligned}$$

Since $\|\,\hat{z}_r(\cdot,\lambda)\,\|_{\mathcal{W}}$ is finite by $\hat{z}_r(\cdot,\lambda) \in L^2_{\mathcal{W}}$, we get from the above calculation that

$$\|\,\hat{z}_r(\cdot,\lambda)\,\|_{\mathcal{W}} \leq \frac{1}{|\operatorname{Im}(\lambda)|}\,\|f\|_{\mathcal{W}}\,. \tag{4.94}$$

We will prove that (4.94) implies estimate (4.89) by the convergence argument. For any $t,r \in [a,\infty)_{\mathbb{T}}$ we observe that

$$\hat{z}(t,\lambda) - \hat{z}_r(t,\lambda) = -\int_r^\infty G(t,\sigma(s),\lambda)\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s.$$

Now we fix $q \in [a,r)_{\mathbb{T}}$. By the definition of $G(\cdot,\cdot,\lambda)$ in (4.84) we have for every $t \in [a,q]_{\mathbb{T}}$

$$\hat{z}(t,\lambda) - \hat{z}_r(t,\lambda) = -\widetilde{Z}(t,\lambda)\int_r^\infty \mathcal{X}_+^*(\sigma(s),\bar{\lambda})\,\widetilde{\mathcal{W}}(s)\,f^{\sigma}(s)\,\Delta s. \tag{4.95}$$

Since the functions $\mathcal{X}_+(\cdot,\bar{\lambda})$ and $f(\cdot)$ belong to $L^2_{\mathcal{W}}$, it follows that the right-hand side of (4.95) converges to zero as $r \to \infty$ for every $t \in [a,q]_{\mathbb{T}}$. Hence, $\hat{z}_r(\cdot,\lambda)$ converges to the function $\hat{z}(\cdot,\lambda)$ uniformly on $[a,q]_{\mathbb{T}}$. Since $\hat{z}(\cdot,\lambda) = \hat{z}_r(\cdot,\lambda)$ on $[a,q]_{\mathbb{T}}$, we have by $\widetilde{\mathcal{W}}(\cdot) \geq 0$ and (4.94) that

$$\int_a^q \hat{z}^{\sigma*}(s,\lambda)\,\widetilde{\mathcal{W}}(s)\,\hat{z}^{\sigma}(s,\lambda)\,\Delta s \leq \|\,\hat{z}_r(\cdot,\lambda)\,\|_{\mathcal{W}}^2 \overset{(4.94)}{\leq} \frac{1}{|\operatorname{Im}(\lambda)|^2}\,\|f\|_{\mathcal{W}}^2\,. \tag{4.96}$$

Since $q \in [a,\infty)_{\mathbb{T}}$ was arbitrary, inequality (4.96) implies the result in (4.89). In the limit circle case inequality (4.89) follows by the same argument by using the fact that all solutions of system ($\mathcal{S}_\lambda$) belong to $L^2_{\mathcal{W}}$.

Now we prove the existence of the limit (4.90). Assume that the system ($\mathcal{S}_\lambda$) is in the limit point case and let $\nu \in \mathbb{C} \setminus \mathbb{R}$ be arbitrary. Following the argument in the proof of [107, Lemma 4.1] and [45, Theorem 5.2], we have from identity (4.83) that for any $r,t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned}
\mathcal{X}_+^*(t,\nu)\,\mathcal{J}\hat{z}_r(t,\lambda) = \mathcal{X}_+^*(a,\nu)\,\mathcal{J}\hat{z}_r(a,\lambda) + (\bar{\nu}-\lambda)\int_a^t \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,\hat{z}_r^{\sigma}(s,\lambda)\,\Delta s \\
+ \int_a^t \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,f_r^{\sigma}(s)\,\Delta s.
\end{aligned} \tag{4.97}$$

Since for $t \in [r,\infty)_{\mathbb{T}}$ equality (4.91) holds, it follows that

$$\lim_{t\to\infty} \mathcal{X}_+^*(t,\nu)\,\mathcal{J}\hat{z}_r(t,\lambda) = -\lim_{t\to\infty} \mathcal{X}_+^*(t,\nu)\,\mathcal{J}\mathcal{X}_+(t,\lambda)\,g(r,\lambda) \overset{(4.72)}{=} 0.$$

Hence, by (4.97),

$$\mathcal{X}_+^*(a,\nu)\,\mathcal{J}\hat{z}_r(a,\lambda) = (\lambda-\bar{\nu})\int_a^\infty \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,\hat{z}_r^\sigma(s,\lambda)\,\Delta s \\ - \int_a^r \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,f^\sigma(s)\,\Delta s. \tag{4.98}$$

By the uniform convergence of $\hat{z}_r(\cdot,\lambda)$ to $\hat{z}(\cdot,\lambda)$ on compact intervals, we get from (4.98) with $r\to\infty$ the equality

$$\mathcal{X}_+^*(a,\nu)\,\mathcal{J}\hat{z}(a,\lambda) = (\lambda-\bar{\nu})\int_a^\infty \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,\hat{z}^\sigma(s,\lambda)\,\Delta s \\ - \int_a^\infty \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,f^\sigma(s)\,\Delta s. \tag{4.99}$$

On the other hand, by (4.83) we obtain for every $t \in [a,\infty)_{\mathbb{T}}$

$$\mathcal{X}_+^*(t,\nu)\,\mathcal{J}\hat{z}(t,\lambda) = \mathcal{X}_+^*(a,\nu)\,\mathcal{J}\hat{z}(a,\lambda) + (\bar{\nu}-\lambda)\int_a^t \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,\hat{z}^\sigma(s,\lambda)\,\Delta s \\ + \int_a^t \mathcal{X}_+^{\sigma*}(s,\nu)\,\widetilde{\mathcal{W}}(s)\,f^\sigma(s)\,\Delta s. \tag{4.100}$$

Upon taking the limit in (4.100) as $t\to\infty$ and using equality (4.99) we conclude that the limit in (4.90) holds true.

In the limit circle case the limit in (4.90) can be proved similarly as above, because all the solutions of ($\mathcal{S}_\lambda$) now belong to $L^2_{\mathcal{W}}$. However, in this case we can apply a direct argument to show that (4.90) holds. By formula (4.88) we get for every $t \in [a,\infty)_{\mathbb{T}}$

$$\mathcal{X}_+^*(t,\nu)\,\mathcal{J}\hat{z}(t,\lambda) = -\mathcal{X}_+^*(t,\nu)\,\mathcal{J}\mathcal{X}_+(t,\lambda)\int_a^t \widetilde{Z}^{\sigma*}(s,\bar{\lambda})\,\widetilde{\mathcal{W}}(s)\,f^\sigma(s)\,\Delta s \\ - \mathcal{X}_+^*(t,\nu)\,\mathcal{J}\widetilde{Z}(t,\lambda)\int_t^\infty \mathcal{X}_+^{\sigma*}(s,\bar{\lambda})\,\widetilde{\mathcal{W}}(s)\,f^\sigma(s)\,\Delta s. \tag{4.101}$$

The limit of the first term in (4.101) is zero because $\mathcal{X}_+^*(t,\nu)\,\mathcal{J}\mathcal{X}_+(t,\lambda)$ tends to zero for $t\to\infty$ by (4.72), and it is multiplied by a convergent integral as $t\to\infty$. Since the columns of $\widetilde{Z}(\cdot,\lambda)$ belong to $L^2_{\mathcal{W}}$, the function $\mathcal{X}_+^*(\cdot,\nu)\,\mathcal{J}\widetilde{Z}(\cdot,\lambda)$ is bounded on $[a,\infty)_{\mathbb{T}}$ and it is multiplied by an integral converging to zero as $t\to\infty$. Therefore, formula (4.90) follows. ∎

In the last result of this chapter we construct another solution of the nonhomogeneous system (4.81) satisfying condition (4.90) and such that it starts with a possibly nonzero initial condition at $t=a$. It can be considered as an extension of Theorem 4.55.

Corollary 4.56. *Let $\alpha\in\Gamma$ and $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Assume that ($\mathcal{S}_\lambda$) is in the limit point or limit circle case. For $f\in L^2_{\mathcal{W}}$ and $v\in\mathbb{C}^n$ we define*

$$\breve{z}(t,\lambda,\alpha) := \mathcal{X}_+(t,\lambda,\alpha)\,v + \hat{z}(t,\lambda,\alpha) \quad \text{for all } t\in[a,\infty)_{\mathbb{T}},$$

where $\hat{z}(\cdot,\lambda,\alpha)$ is given in (4.87). Then $\tilde{z}(\cdot,\lambda,\alpha)$ solves system (4.81) with $\alpha\,\tilde{z}(a,\lambda,\alpha)=v$,

$$\|\,\tilde{z}(\cdot,\lambda,\alpha)\,\|_{\mathcal{W}} \leq \frac{1}{|\mathrm{Im}(\lambda)|}\;\|f\|_{\mathcal{W}} + \|\,\mathcal{X}_{+}(\cdot,\lambda,\alpha)\,v\,\|_{\mathcal{W}}\,, \tag{4.102}$$

$$\lim_{t\to\infty} \mathcal{X}_{+}^{*}(t,\nu,\alpha)\,\mathcal{J}\,\tilde{z}(t,\lambda,\alpha)=0 \quad \text{for every } \nu\in\mathbb{C}\setminus\mathbb{R}. \tag{4.103}$$

In addition, if system $(\mathcal{S}_{\lambda})$ is in the limit point case, then $\tilde{z}(\cdot,\lambda,\alpha)$ is the only $L^2_{\mathcal{W}}$ solution of (4.81) satisfying $\alpha\,\tilde{z}(a,\lambda,\alpha)=v$.

Proof. As in the previous proof we suppress the dependence on α. Since the function $\mathcal{X}_{+}(\cdot,\lambda)\,v$ solves $(\mathcal{S}_{\lambda})$, it follows from Proposition 4.54 that $\tilde{z}(\cdot,\lambda,\alpha)$ solves the system (4.81) and $\alpha\,\tilde{z}(a,\lambda)=\alpha\,\mathcal{X}_{+}(a,\lambda)\,v=v$. Next, $\tilde{z}(\cdot,\lambda)\in L^2_{\mathcal{W}}$ as a sum of two $L^2_{\mathcal{W}}$ functions. The limit in (4.103) follows from the limit (4.90) of Theorem 4.55 and from identity (4.72), because

$$\lim_{t\to\infty} \mathcal{X}_{+}^{*}(t,\nu)\,\mathcal{J}\,\tilde{z}(t,\lambda)=\lim_{t\to\infty}\big\{\mathcal{X}_{+}^{*}(t,\nu)\,\mathcal{J}\,\mathcal{X}_{+}(t,\lambda)\,v+\mathcal{X}_{+}^{*}(t,\nu)\,\mathcal{J}\,\hat{z}(t,\lambda)\big\}=0.$$

Inequality (4.102) is obtained from estimate (4.89) by the triangle inequality.

Now we prove the uniqueness of $\tilde{z}(\cdot,\lambda)$ in the case of $(\mathcal{S}_{\lambda})$ being of the limit point type. If $z_1(\cdot,\lambda)$ and $z_2(\cdot,\lambda)$ are two $L^2_{\mathcal{W}}$ solutions of (4.81) satisfying $\alpha\,z_1(a,\lambda)=v=\alpha\,z_2(a,\lambda)$, then their difference $z(\cdot,\lambda):=z_1(\cdot,\lambda)-z_2(\cdot,\lambda)$ also belongs to $L^2_{\mathcal{W}}$ and solves system $(\mathcal{S}_{\lambda})$ with $\alpha\,z(\cdot,\lambda)=0$. Since $z(\cdot,\lambda)=\Psi(\cdot,\lambda)\,c$ for some $c\in\mathbb{C}^{2n}$, the initial condition $\alpha\,z(\cdot,\lambda)=0$ implies through (4.26) that $z(\cdot,\lambda)=\widetilde{Z}(\cdot,\lambda)\,d$ for some $d\in\mathbb{C}^{n}$. If $d\neq 0$, then $z(\cdot,\lambda)\notin L^2_{\mathcal{W}}$, because in the limit point case the columns of $\widetilde{Z}(\cdot,\lambda)$ do not belong to $L^2_{\mathcal{W}}$, which is a contradiction. Therefore, $d=0$ and the uniqueness of $\tilde{z}(\cdot,\lambda)$ is established. ■

4.7 Bibliographical notes

More details about the Hermitian components of a given matrix M can be found [75, pp. 268–269], see also [23, Fact 3.5.24]. For the proof of Proposition 4.15 we refer to [116, Section 5], [31, Corollary 1], [150, Theorem 3.6]. Moreover, since it is more appropriate and better-arranged, the references for special cases of the statements established in this chapter are given through the text.

We recall the fact that the results from this chapter were obtained by R. Šimon Hilscher and the author in [145] as a generalization of the Weyl–Titchmarsh theory developed for the discrete symplectic systems by S. L. Clark and the author in [45].

Mathematics is not a deductive science – that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.

Paul R. Halmos, see [81]

Chapter 5

Second order Sturm–Liouville equations on time scales

In this chapter we are interested in the second order Sturm–Liouville equations on time scales which are the most special, the simplest, and the most illustrative case of symplectic dynamic systems. In the continuous time case, Sturm–Liouville differential equations of the second order have the form

$$\big(p(t)\,y'\big)' + q(t)\,y = 0, \tag{5.1}$$

while in the discrete time case, the second order Sturm–Liouville difference equations are known in the form

$$\Delta\,(p_k\,\Delta y_k) + q_k\,y_{k+1} = 0. \tag{5.2}$$

There is not a unique analogue of these equations in the time scale theory. Basic properties were developed for the following dynamic equations

$$\Big(p(t)\,y^{\Delta}\Big)^{\nabla} + q(t)\,y = 0, \quad \Big(p(t)\,y^{\nabla}\Big)^{\Delta} + q(t)\,y = 0, \quad \text{and} \quad \Big(p(t)\,y^{\Delta}\Big)^{\Delta} + q(t)\,y^{\sigma} = 0, \tag{5.3}$$

which reduce to (5.1) and (5.2), if $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. Although different equations from (5.3) provide some advantage over the others in some calculations, we will see in this chapter that all dynamic equations displayed in (5.3) are justifiable in the theory of dynamic equations. In addition, equations (5.3)(i) and (5.3)(ii) are essentially equivalent and, moreover, it is becoming apparent that these equations are more convenient for the investigation, because they are self-adjoint – in the functional-analytic sense – whereas equation (5.3)(iii) is not, see [47, (c) p. 5] and also [46].

In each section of this chapter we present new results achieved for different forms of Sturm–Liouville dynamic equations in (5.3). More specifically, in the following section we characterize the domains of the Krein–von Neumann and Friedrichs extensions of operators associated with equation (5.3)(i). In Section 5.2 we introduce the concept of critical, subcritical, and supercritical operators in connection with equation (5.3)(ii).

Finally, in the last section we present the Weyl–Titchmarsh theory for dynamic equations of the second order with the left-hand side in the same form as in (5.3)(iii).

5.1 Krein–von Neumann and Friedrichs extensions for second order operators on time scales

The Friedrichs extension of linear differential operators was investigated in the literature intensively, especially for operators associated with the Sturm–Liouville equations. This theory was initiated in [73], where Friedrichs showed that for a symmetric, densely defined, linear operator $\mathcal{L}$ which is bounded below in a Hilbert space H, there exists a self-adjoint extension of $\mathcal{L}$ with the same lower bound. Such an extension was called "ausgezeichnet" (=excellent) and is known as the Friedrichs extension of $\mathcal{L}$. In this context the Friedrichs extension and its domain are denoted by $\mathcal{L}_F$ and $\mathcal{D}_\mathrm{F}$, respectively. The outstanding role of this extension in physics was illustrated in [161, Figure 5.1, p. 255].

The first characterization of the domain of the Friedrichs extension of the operator defined by the second order Sturm–Liouville equation on a finite interval was given in [74], where it is shown that it can be determined by the Dirichlet boundary conditions. About the same time Freudenthal presented in [72] the following description of the domain of $\mathcal{L}_F$:

$$\left.\begin{aligned} \mathcal{D}_\mathrm{F} = \{y \in \mathcal{D}_{\max} :\ \exists\, y_s \in \mathcal{D}_{\min} \text{ with } y_s \to y \text{ in } H \text{ as } s \to \infty \text{ and } \\ \langle \mathcal{L}[y_s - y_r],\, y_s - y_r \rangle \to 0 \text{ as } s, r \to \infty\}, \end{aligned}\right\} \tag{5.4}$$

which will be used in the proof of our main result about the Friedrichs extension, i.e., of Theorem 5.5. Moreover, this characterization is regarded as the definition of the Friedrichs extension (as an operator with the given domain) in some publications, see, e.g., [165, Definition 10.5.1] or [126, Definition 3.1]. Rellich observed in [136] that functions in the domain of $\mathcal{L}_\mathrm{F}$ behave near $\pm\infty$ like the recessive (or principal) solution of a certain disconjugate equation associated with $\mathcal{L}$. Other investigations of $\mathcal{L}_\mathrm{F}$ which extend the Friedrichs' result from 1936 can the reader find, e.g., in [110, 125, 139]. The question of the uniqueness of the Friedrichs extension was answered by Krein in [117]. Friedrichs extension of a singular differential operator on a finite and an infinite interval by using the recessive solution was given in [120]. An overview of the theory of the Friedrichs extension for operators associated with the second order Sturm–Liouville equation can be found in [165, Section 10.5].

The Friedrichs extension of linear difference operators connected with the second order Sturm–Liouville difference equations was considered in [20, 35, 128, 143]. The main motivation for this section lies in the results from [35]. In this reference, the authors deal with a positive linear operator defined by a three terms difference equation and characterize the domains of its Krein–von Neumann and Friedrichs extensions via the corresponding recessive solution. This result was extended to operators associated with the $2n$-th order Sturm–Liouville difference equations in [55].

The Krein–von Neumann (called also only Krein or only von Neumann) extension was developed as an analogue of the Friedrichs extension. In the continuous case, it was shown in [117, Theorem 2, p. 492] that the Friedrichs and the Krein–von Neumann extensions play extremal roles in the spectral theory of the given symmetric, densely defined, linear operator which is bounded below. Indeed, the Friedrichs extension is the greatest self-adjoint extension of the operator (Krein used the terminology the "hard" extension) and the Krein–von Neumann extension is the smallest self-adjoint extension of the operator (Krein called it as the “soft” extension). Moreover, the domain of the Krein-von Neumann extension can be characterized similarly to (5.4), see [11, Corrolary 2].

5.1.1 Preliminaries on equation (5.3)(i)

In this section, we deal with operator $\mathcal{L}[x](t)$ associated with the second order dynamic equation on a time scale unbounded above, i.e.,

$$\mathcal{L}[x](t) = 0, \quad \text{where } \mathcal{L}[x](t) := \left(p(t)\, x^{\Delta}\right)^{\nabla} + q(t)\, x, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{5.5}$$

where $p(\cdot)$ is continuous on $[a,\infty)_{\mathbb{T}}$, $q \in \mathrm{C}_{\mathrm{pld}}$ on $[a,\infty)_{\mathbb{T}}$, and p, q are real-valued functions such that

$$\inf_{t \in [a,b]_{\mathbb{T}}} |p(t)| > 0 \quad \text{for all} \quad b \in (a,\infty)_{\mathbb{T}}$$

holds. Additionally, the following equations

$$\left.\begin{array}{r} x^{\Delta\nabla} + a_1(t)\, x^{\nabla} + a_2(t)\, x = 0, \\ x^{\Delta\nabla} + b_1(t)\, x^{\Delta} + b_2(t)\, x = 0, \\ x^{\Delta\nabla} + c_1(t)\, x^{\nabla} + a_2(t)\, x^{\rho} = 0 \end{array}\right\} \tag{5.6}$$

can be given in the form (5.5) under some additional conditions regarding the smoothness and the regressivity of the coefficients. Now, we denote the set

$$\mathbb{D}_{\Delta\nabla} := \left\{x : [a,\infty)_{\mathbb{T}} \to \mathbb{C},\ x \in \mathrm{C}^1_{\mathrm{prd}} \text{ and } \left(p\, x^{\Delta}\right)(\cdot) \in \mathrm{C}^1_{\mathrm{pld}}\right\}$$

and by a *solution* of equation (5.5) we mean a function $x \in \mathbb{D}_{\Delta\nabla}$ satisfying this equation for all $t \in [\sigma(a),\infty)_{\mathbb{T}}$. For any $t_0 \in [a,\infty)_{\mathbb{T}}$ and any given constants $x_0, x_1 \in \mathbb{C}$ the initial value problem

$$\mathcal{L}[x](t) = 0, \quad x(t_0) = x_0, \quad x^{\Delta}(t_0) = x_1 \tag{5.7}$$

possesses a unique solution on $[a,\infty)_{\mathbb{T}}$.

In connection with equation (5.5), we define the *inner product* of functions $x(\cdot)$ and $y(\cdot)$ by

$$\langle x,\, y\rangle := \int_a^{\infty} \overline{x}(t)\, y(t) \nabla t.$$

We consider the Hilbert space $L^2 := L^2\left([a,\infty)_{\mathbb{T}}\right)$. Moreover, we suppose that equation (5.5) is in the *limit circle case*, i.e., with respect to Definition 4.46, there are two linearly independent solutions of (5.5) in L^2.

If $x, y : \mathbb{T} \to \mathbb{C}$ are Δ-differentiable functions on $\mathbb{T}^{\kappa}$, then we define their *Wronskian* by

$$W[x,y](t) := \overline{x}(t) y^{\Delta}(t) - \overline{x}^{\Delta}(t) y(t) \quad \text{for all } t \in \mathbb{T}^{\kappa}$$

and the *Lagrange bracket* by

$$[x,y](t) := p(t) W[x,y](t) \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{5.8}$$

It is known that for any $x, y : [a,\infty)_{\mathbb{T}} \to \mathbb{C}$ such that $x, y \in \mathbb{D}_{\Delta\nabla}$ the so-called *Lagrange identity* (or *Green's formula*) holds, i.e.,

$$\langle x,\, \mathcal{L}[y]\rangle - \langle \mathcal{L}[x],\, y\rangle = \left[p(t) W\, [x,y]\, (t)\right]_a^{\infty}, \tag{5.9}$$

where $[p(t)W[x,y](t)]^{\infty} := \lim_{t\to\infty} p(t)W[x,y](t)$. The calculation is based on formulas (2.16) and (2.15). It is natural fact that the Lagrange bracket of any two solutions of (5.5) is independent of t. Moreover, two solutions x, y of (5.5) are *linearly independent* on $[a,\infty)_{\mathbb{T}}$ if and only if $W[x,y](t) \neq 0$ for some (and hence for any) $t \in [a,\infty)_{\mathbb{T}}$.

By a straightforward calculation and using identities (2.14) and (2.10) we obtain for any function $x \in \mathbb{D}_{\Delta\nabla} \cap L^2$ that

$$\begin{aligned} \langle x, \mathcal{L}[x]\rangle &= \mathcal{F}_{\Delta\nabla}(x) - \Big[p(t)\overline{x}(t)x^{\Delta}(t)\Big]_a^{\infty}, \\ &\text{where } \mathcal{F}_{\Delta\nabla}(x) := \int_a^{\infty}\Big\{q(t)\,|x(t)|^2 - p^{\rho}(t)\left|x^{\nabla}(t)\right|^2\Big\}\nabla t \end{aligned} \tag{5.10}$$

is the so-called *quadratic functional* associated with equation (5.5).

A solution $x_{\mathrm{P}} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ of equation (5.5), which is given by the initial conditions

$$x_{\mathrm{P}}(a) = 0, \quad x_{\mathrm{P}}^{\Delta}(a) = \frac{1}{p(a)}, \tag{5.11}$$

is called the *principal solution* and a solution $x_{\mathrm{A}} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ of equation (5.5), which satisfies

$$x_{\mathrm{A}}(a) = -1, \quad x_{\mathrm{A}}^{\Delta}(a) = 0, \tag{5.12}$$

is called the *associated solution*. From initial conditions (5.11), (5.12) and from the constancy of the Lagrange bracket it follows that $x_{\mathrm{P}}(t)$ and $x_{\mathrm{A}}(t)$ are linearly independent with $W[x_{\mathrm{P}}, x_{\mathrm{A}}](a) = \frac{1}{p(a)}$. Thus, $[x_{\mathrm{P}}, x_{\mathrm{A}}](\cdot) \equiv 1$. Since equation (5.5) is supposed to be in the limit circle case, we have $x_{\mathrm{P}}, x_{\mathrm{A}} \in L^2$. Moreover, it can be verified by a direct computation that for any ∇-differentiable functions $x, y : [a,\infty)_{\mathbb{T}} \to \mathbb{C}$ the so-called *bracket decomposition*

$$[x,y](t) = [x,x_{\mathrm{P}}](t)\,[x_{\mathrm{A}},y](t) - [x,x_{\mathrm{A}}](t)\,[x_{\mathrm{P}},y](t) \tag{5.13}$$

holds true. A solution $u(\cdot)$ of equation (5.5) such that

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0$$

is satisfied for any solution $x(\cdot)$ of (5.5) which is linearly independent with $u(\cdot)$ and eventually $x(t) \neq 0$, is called the *recessive solution* at $+\infty$. The existence of the recessive solution is implied by the nonoscillation of equation (5.5) or by the positivity of the quadratic functional $\mathcal{F}(x)$, where $x \in \mathrm{C}^1_{\mathrm{pld}}$, $x(a) = 0$, and eventually $x(t) \equiv 0$.

Finally, the following lemma is a very useful tool for the proof of the main result of this section.

Lemma 5.1. *For $s \in (a,\infty)_{\mathbb{T}}$, let $x_s : [a,\infty)_{\mathbb{T}} \to \mathbb{C}$ be the solution of (5.5) such that $x_s(a) = 1$ and $x_s(s) = 0$. Then*

$$x_s(t) = -x_{\mathrm{A}}(t) + \frac{x_{\mathrm{A}}(s)}{x_{\mathrm{P}}(s)}\,x_{\mathrm{P}}(t) \quad \text{and} \quad \lim_{s\to\infty} x_s(t) = -x_{\mathrm{A}}(t) + \alpha\,x_{\mathrm{P}}(t) = u(t), \tag{5.14}$$

where $\alpha := \lim_{s\to\infty} \frac{x_{\mathrm{A}}(s)}{x_{\mathrm{P}}(s)}$ and $u(\cdot)$ is the recessive solution of (5.5) at $+\infty$.

5.1.2 Main results

In this subsection we establish our main results about the Krein–von Neumann and Friedrichs extensions of the operator $\mathcal{L}$ defined by the dynamic equation (5.5). Let us denote

$$\mathcal{D}_{\max} := \{x \in \mathbb{D}_{\Delta\nabla} \cap L^2 : \ \mathcal{L}[x] \in L^2\},$$
$$\mathcal{D}_0' := \{x \in \mathbb{D}_{\Delta\nabla} \cap L^2 : \ \operatorname{supp} x \subseteq (a,\infty)_{\mathbb{T}}\}.$$

It can be verified that the set $\mathcal{D}_0'$ is dense in L^2 and as a consequence of the Lagrange identity (5.9) we have that $\mathcal{L}$ defines a symmetric operator on $\mathcal{D}_0'$. The *maximal operator* $\mathcal{L}_{\max}$ generated by $\mathcal{L}$ is defined as $\mathcal{L}_{\max} : \mathcal{D}_{\max} \to L^2$, $\mathcal{L}_{\max}[x](t) := \mathcal{L}[x](t)$, and then the *minimal operator* is the closure of the restriction of the maximal operator to the set $\mathcal{D}_0'$. The domain of this minimal operator is denoted by $\mathcal{D}_{\min}$. We note that an operator generated by $\mathcal{L}$ with the domain $\mathcal{D}_0'$ is said to be the *pre-minimal operator*. It follows from the definition $\mathcal{D}_0'$ that $x(a) = 0$ for any $x \in \mathcal{D}_{\min}$ and $\mathcal{L}_{\min} = \mathcal{L}_{\max}^*$ (the adjoint operator in L^2), i.e.,

$$\big\langle \mathcal{L}[x],\, y \big\rangle = \big\langle x,\, \mathcal{L}[y] \big\rangle \quad \text{for all } x \in \mathcal{D}_{\min} \text{ and } y \in \mathcal{D}_{\max}. \tag{5.15}$$

For any $x, y \in \mathcal{D}_{\max}$ both inner products in the left-hand side of equation (5.9) are finite (due to the fact that all the elements are in L^2), so the limit

$$[x,y]^\infty := \lim_{t\to\infty} [x,y](t) = \big\langle x,\, \mathcal{L}[y] \big\rangle - \big\langle \mathcal{L}[x],\, y \big\rangle + [x,y](a) \tag{5.16}$$

exists and is finite. By using that $[x,y](a) = 0$ for any $x \in \mathcal{D}_{\min}$, $y \in \mathcal{D}_{\max}$, identities (5.15) and (5.16) yield that it is possible to describe the domain of the minimal operator explicitly as

$$\mathcal{D}_{\min} = \{x \in \mathcal{D}_{\max} : \ x(a) = 0, \ [x,y]^\infty = 0 \text{ for every } y \in \mathcal{D}_{\max}\},$$

which corresponds to the continuous time result in [144, p. 6] and to discrete time result in [35, p. 183]. Moreover, by using identity (5.13) we get

$$\mathcal{D}_{\min} = \{x \in \mathcal{D}_{\max} : \ x(a) = 0, \ [x, x_{\mathrm{A}}]^\infty = 0 = [x, x_{\mathrm{P}}]^\infty\}.$$

For any $h \in \mathbb{R} \cup \{\infty\}$ we denote by $\mathcal{L}_h$ the extension of $\mathcal{L}_{\min}$ with the domain

$$\mathcal{D}_h := \mathcal{D}(\mathcal{L}_h) = \{x \in \mathcal{D}_{\max} : \ x(a) = 0, \ [x, x_{\mathrm{A}}]^\infty - h\,[x, x_{\mathrm{P}}]^\infty = 0\}, \quad h \in \mathbb{R},$$
$$\mathcal{D}_\infty := \mathcal{D}(\mathcal{L}_\infty) = \{x \in \mathcal{D}_{\max} : \ x(a) = 0, \ [x, x_{\mathrm{P}}]^\infty = 0\}, \quad h = \infty.$$

Remark 5.2. Note that the value of $x(a)$ for $x \in \mathcal{D}_h$ or $x \in \mathcal{D}_\infty$ is irrelevant as in [35]. Hence, without loss of generality we can take it to be zero. Indeed, if $x(a) \neq 0$ and the point a is right-dense, then it is possible to modify the function $x(\cdot)$ by a suitable function with a compact support to obtain the desired value $x(a) = 0$, see [120, pp. 415–418]. If $x(a) \neq 0$ and the point a is right-scattered, then we can extend the time scale interval $[a,\infty)_{\mathbb{T}}$ by a right-scattered point $a_0 := a - 1$ in order to get $x(a_0) = 0$ without any change of the value of the corresponding quadratic functional. Now, if the function $x(t)$ has to be a solution of equation (5.5) on the interval $[a_0,\infty)_{\mathbb{T}}$, we define

$$p(a_0) := \Big[p(a)x^\Delta(a) + q(a)x(a)\Big] \big/ x(a)$$

while the value of $q(a_0)$ can be chosen arbitrary.

By simple calculations and with respect to Remark 5.2, it can be shown that $x_A - h\,x_P \in \mathcal{D}_h$ (when $h \in \mathbb{R}$) or $x_P \in \mathcal{D}_\infty$ (when $h = \infty$). Moreover, for $x, y \in \mathcal{D}_h$ the bracket decomposition (5.13) implies that $[x,y]^\infty = 0$. Hence, the Lagrange identity (5.9) yields that $\mathcal{L}_h$ is symmetric, i.e., $\langle \mathcal{L}_h[x], y\rangle = \langle x, \mathcal{L}_h[y]\rangle$ for every $x, y \in \mathcal{D}_h$. If $x \in \mathcal{D}(\mathcal{L}_h^*)$ we have

$$\langle \mathcal{L}_h[y], x\rangle = \langle y, \mathcal{L}_h[x]\rangle \quad \text{for all } y \in \mathcal{D}_h,$$

i.e., $\mathcal{L}_h$ is a self-adjoint operator. In particular,

$$[x, x_A]^\infty - h\,[x, x_P]^\infty = \langle x, \mathcal{L}_h[x_A - hx_P]\rangle - \langle \mathcal{L}_{\max}[x], x_A - hx_P\rangle = 0$$

and consequently $x \in \mathcal{D}_h$. It is easily seen that $\mathcal{D}(\mathcal{L}_h) \neq \mathcal{D}(\mathcal{L}_{h'})$ when $h \neq h'$. Moreover, in accordance with the continuous and discrete time theory we can say that every self-adjoint extension of $\mathcal{L}_{\min}$ has the form $\mathcal{L}_h$ for some $h \in \mathbb{R} \cup \{\infty\}$, see, e.g., [151, Section 2.6, Lemma 2.20].

We now define the *Krein–von Neumann extension* $\mathcal{L}_K$ to be a self-adjoint extension of $\mathcal{L}_{\min}$ with the domain

$$\mathcal{D}(\mathcal{L}_K) := \mathcal{D}_{\min} + \mathcal{N},$$

where $\mathcal{N}$ is the null space of $\mathcal{L}_{\max}$. The following result is a consequence of the previous discussion.

Theorem 5.3. *The domain of the Krein–von Neumann extension of $\mathcal{L}_{\min}$ corresponds to $h = 0$, i.e.,*

$$\mathcal{D}(\mathcal{L}_K) = \{x \in \mathcal{D}_{\max} :\ x(a) = 0,\ [x, x_A]^\infty = 0\}.$$

Proof. From the previous discussion it follows that $\mathcal{D}(\mathcal{L}_0)$ is the domain of a self-adjoint extension of $\mathcal{L}_{\min}$. We have to show that $\mathcal{D}(\mathcal{L}_K) \subseteq \mathcal{D}(\mathcal{L}_0)$. Obviously, $x_A \in \mathcal{N} \subset \mathcal{D}(\mathcal{L}_K)$, so that for any $x \in \mathcal{D}(\mathcal{L}_K)$ we have

$$[x, x_A]^\infty = \langle x, \mathcal{L}_{\max}[x_A]\rangle - \langle \mathcal{L}_{\max}[x], x_A\rangle = \langle x, \mathcal{L}_K[x_A]\rangle - \langle \mathcal{L}_K[x], x_A\rangle = 0,$$

where we apply the self-adjointness of the operator $\mathcal{L}_K$ in the last equality. ∎

Now, we assume that the minimal operator is bounded below (positive), i.e., there exists $\varepsilon > 0$ such that

$$\langle \mathcal{L}[x], x\rangle \geq \varepsilon \langle x, x\rangle \quad \text{for all } x \in \mathcal{D}_{\min}. \tag{5.17}$$

This assumption is not really restrictive, since, e.g., the nonoscillation of equation (5.5) implies that the operator $\mathcal{L}_{\min}$ is bounded below, see (5.10), i.e., $\langle \mathcal{L}[x], x\rangle \geq \gamma \langle x, x\rangle$ holds for some constant $\gamma \in \mathbb{R}$. The construction of the Friedrichs extension then applies to the operator $\mathcal{L}_{\min} - (\gamma - \varepsilon)\mathcal{I}$ for a suitable $\varepsilon > 0$, where $\mathcal{I}$ denotes the identity operator.

Remark 5.4. We have the following inclusions

$$\mathcal{D}_0' \subseteq \mathcal{D}_{\min} \subseteq \mathcal{D}_F \subseteq \mathcal{D}_{\max}.$$

Moreover, let $q = q_\pm := \dim \operatorname{Ker}(\mathcal{L}_{\min} \pm i\mathcal{I})$ be the deficiency indices of the minimal operator $\mathcal{L}_{\min}$. If $q = 0$, then the operator $\mathcal{L}_{\min}$ is self-adjoint and $\mathcal{D}_{\min} = \mathcal{D}_F$.

Theorem 5.5. *Assume that* (5.17) *holds true and that equation* (5.5) *possesses the recessive solution $u(\cdot)$. Then $u(\cdot)$ belongs to the domain of the Friedrichs extension of the operator $\mathcal{L}_{\min}$.*

Proof. Let the assumptions of the theorem be satisfied. For $s \in [a,\infty)_{\mathbb{T}}$ we consider the function $y_s(\cdot)$ defined by

$$y_s(t) := \begin{cases} x_s(t), & \text{for } t \in [a,s]_{\mathbb{T}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $x_s(t)$ is defined in Lemma 5.1. With respect to Remark 5.2, we conclude $y_s \in \mathcal{D}_{\min}$.

From Lemma 5.1 we know that $y_s(t) \to u(t)$ as $s \to \infty$. Hence, for each $t \in [a,\infty)_{\mathbb{T}}$, $r > s$, $r \in \mathbb{T}$, we obtain

$$\big\langle \mathcal{L}[y_r - y_s],\, y_r - y_s \big\rangle = \int_a^\infty \Big\{ y_s\, \overline{\mathcal{L}[y_s]} - y_r\, \overline{\mathcal{L}[y_s]} - y_s\, \overline{\mathcal{L}[y_r]} + y_r\, \overline{\mathcal{L}[y_r]} \Big\} \nabla t.$$

Now we distinguish the character of the point $s \in [a,\infty)_{\mathbb{T}}$. First, let s be a left-scattered point. Then we have

$$\begin{aligned}
\big\langle \mathcal{L}[y_r - y_s],\, y_r - y_s \big\rangle &= \int_a^{\rho(s)} \Big\{ y_s\, \overline{\mathcal{L}[y_s]} - y_r\, \overline{\mathcal{L}[y_s]} - y_s\, \overline{\mathcal{L}[y_r]} \Big\} \nabla t \\
&\quad + \int_{\rho(s)}^{s} \Big\{ y_s\, \overline{\mathcal{L}[y_s]} - y_r\, \overline{\mathcal{L}[y_s]} - y_s\, \overline{\mathcal{L}[y_r]} \Big\} \nabla t \\
&\quad + \int_{s}^{\infty} \Big\{ y_s\, \overline{\mathcal{L}[y_s]} - y_r\, \overline{\mathcal{L}[y_s]} - y_s\, \overline{\mathcal{L}[y_r]} \Big\} \nabla t \\
&\quad + \int_a^r y_r\, \overline{\mathcal{L}[y_r]}\, \nabla t + \int_r^\infty y_r\, \overline{\mathcal{L}[y_r]}\, \nabla t \\
&\overset{(2.18)}{=} -\nu(s)\, y_r(s)\, \overline{\mathcal{L}[y_s]}(s) \\
&= -\nu(s)\, x_r(s)\, \overline{\Big[\big(p(s)\, y_s^{\Delta}(s) \big)^{\nabla} + q(s) y_s(s) \Big]} \\
&\overset{(2.6),(5.14)}{=} x_r(s)\, p^{\rho}(s)\, \overline{\left(-x_{\mathrm{A}}^{\Delta}(\rho(s)) + \frac{x_{\mathrm{A}}(s)}{x_{\mathrm{P}}(s)} x_{\mathrm{P}}^{\Delta}(\rho(s)) \right)} \\
&= -\frac{x_r(s)}{x_{\mathrm{P}}(s)},
\end{aligned}$$

where we used the fact that $x_r(t)$ and $x_s(t)$ are solutions of equation (5.5), the constancy of the Lagrange bracket, and the following computation

$$\begin{aligned}
1 = p(s) W[x_{\mathrm{P}}, x_{\mathrm{A}}](s) &= p^{\rho}(s)\, W[x_{\mathrm{P}}, x_{\mathrm{A}}]\,(\rho(s)) \\
&\overset{(2.10)}{=} p^{\rho}(s) \Big\{ x_{\mathrm{P}}(s)\, x_{\mathrm{A}}^{\Delta}\,(\rho(s)) - x_{\mathrm{A}}(s)\, x_{\mathrm{P}}^{\Delta}\,(\rho(s)) \Big\}.
\end{aligned}$$

Moreover, the first identity in (5.14) yields

$$-\frac{x_r(s)}{x_{\mathrm{P}}(s)} = -\frac{-x_{\mathrm{A}}(s) + \frac{x_{\mathrm{A}}(r)}{x_{\mathrm{P}}(r)}\, x_{\mathrm{P}}(s)}{x_{\mathrm{P}}(s)} = \frac{x_{\mathrm{A}}(s)}{x_{\mathrm{P}}(s)} - \frac{x_{\mathrm{A}}(r)}{x_{\mathrm{P}}(r)} \to 0 \quad \text{for } r,s \to \infty.$$

Next, let the point s be left-dense and consider a left-sequence $\{s_k\}_{k=1}^{\infty}$ such that $s_k \nearrow s$.

Then we calculate

$$\begin{aligned}
&\langle \mathcal{L}[y_r - y_s], y_r - y_s \rangle \\
&\quad = \int_a^s \left\{ y_s \overline{\mathcal{L}[y_s]} - y_r \overline{\mathcal{L}[y_s]} - y_s \overline{\mathcal{L}[y_r]} \right\} \nabla t \\
&\qquad + \int_s^\infty \left\{ y_s \overline{\mathcal{L}[y_s]} - y_r \overline{\mathcal{L}[y_s]} - y_s \overline{\mathcal{L}[y_r]} \right\} \nabla t \\
&\qquad + \int_a^r y_r \overline{\mathcal{L}[y_r]} \Delta t + \int_r^\infty y_r \overline{\mathcal{L}[y_r]} \nabla t \\
&\quad = \lim_{k\to\infty} \int_a^{s_k} \left\{ y_s \overline{\mathcal{L}[y_s]} - y_r \overline{\mathcal{L}[y_s]} - y_s \overline{\mathcal{L}[y_r]} \right\} \nabla t = 0.
\end{aligned}$$

Hence, with respect to the Freudenthal's characterization displayed in (5.4) we proved that the recessive solution $u(\cdot)$ belongs to the domain of the Friedrichs extension of the minimal operator, i.e., $u \in \mathcal{D}(\mathcal{L}_{\mathrm{F}})$. ■

As a direct consequence of the previous theorem we obtain the following.

Theorem 5.6. *Assume that* (5.17) *holds true and that equation* (5.5) *possesses the recessive solution* $u(\cdot)$. *Then the domain of the Friedrichs extension of the minimal operator* $\mathcal{L}_{\min}$ *corresponds to* $h = \alpha$, *where* α *is from Lemma 5.1, i.e.,*

$$\mathcal{D}(\mathcal{L}_{\mathrm{F}}) = \{x \in \mathcal{D}_{\max} : \; x(a) = 0 \textit{ and } [x, u]^\infty = 0\}.$$

Proof. The statement follows directly from Theorem 5.5 and from the Lagrange identity in (5.9). ■

5.2 Critical second order operators on time scales

In this section we extend to time scales the discrete time result given in [77], where the theory of critical operators was investigated and were also discussed the connection between the Friedrichs extension and this theory. The concept of critical operators was recently studied in the continuous time case, e.g., in [76, 123, 129]. Let us recall the definition of critical operators as introduced in [77] and the main result of that paper which deals with the second order Sturm–Liouville difference equation

$$-\Delta(a_k \Delta y_{k-1}) + c_k y_k = 0, \quad a_k > 0, k \in \mathbb{Z}, \tag{5.18}$$

and the corresponding three-term symmetric recurrence relation

$$\tau(y) = 0, \quad \text{where } \tau(y) := -a_{k+1} y_{k+1} + b_k y_k - a_k y_{k-1} \text{ and } b_k = a_{k+1} + a_k + c_k. \tag{5.19}$$

Definition 5.7. Let (5.18) be disconjugate on $\mathbb{Z}$, i.e., $\tau \geq 0$. We say that the operator τ is *critical* if the recessive solutions at ∞ and $-\infty$ are linearly dependent, in the opposite case τ is said to be *subcritical*. If $\tau \not\geq 0$, i.e., (5.18) is not disconjugate, τ is said to be *supercritical*.

Theorem 5.8. *The following statements are equivalent.*

(i) *The operator* τ *is critical on* $\mathbb{Z}$.

(ii) *For any $\varepsilon > 0$ and $m \in \mathbb{Z}$, the operator $\tilde{\tau}$ which we get from τ by replacing a_m by $a_m - \varepsilon$, is supercritical on $\mathbb{Z}$, i.e., $\tilde{\tau} \ngeq 0$.*

(iii) *For any $\varepsilon > 0$ and $m \in \mathbb{Z}$, the operator $\hat{\tau}$ which we get from τ by replacing b_m by $b_m + \varepsilon$, is supercritical on $\mathbb{Z}$, i.e., $\hat{\tau} \ngeq 0$.*

In [77], the authors used a matrix operator associated with (5.19) and proved Theorem 5.8 by a construction of solutions of (5.18). More recently in [56], this result was extended to $2n$-order Sturm–Liouville difference operators, where the variational approach was used. We use a similar technique to prove the main result of this section.

5.2.1 Preliminaries on equation (5.3)(ii)

In this section we consider an unbounded time scale, i.e., $(-\infty, \infty)_{\mathbb{T}}$, and the operator $\mathcal{M}[x](t)$ associated with the equation given by

$$-\left(p(t)\,x^{\nabla}\right)^{\Delta} + q(t)\,x = 0, \quad \text{i.e.,} \quad \mathcal{M}[x](t) := -\left(p(t)\,x^{\nabla}\right)^{\Delta} + q(t)\,x \tag{5.20}$$

where p is continuous and positive, q is rd-continuous, and p, q are real-valued functions on $(-\infty, \infty)_{\mathbb{T}}$. Let us introduce a set

$$\mathbb{D}_{\nabla\Delta} := \left\{x : (-\infty, \infty)_{\mathbb{T}} \to \mathbb{R},\ x \in C^1_{\mathrm{pld}},\ \left(p\,x^{\nabla}\right)(\cdot) \in C^1_{\mathrm{prd}}\right\}.$$

Then $x \in \mathbb{D}_{\nabla\Delta}$ is a *solution* of the equation $\mathcal{M}[x](t) = 0$ if it is satisfied for all $t \in (-\infty, \infty)_{\mathbb{T}}$. Moreover, for any $t_0 \in (-\infty, \infty)_{\mathbb{T}}$ and any given constants x_0, x_1 the initial value problem

$$\mathcal{M}[x](t) = 0, \quad x(t_0) = x_0, \quad x^{\nabla}(t_0) = x_1 \tag{5.21}$$

possesses a unique solution on $(-\infty, \infty)_{\mathbb{T}}$. Similarly to (5.6), we can show that the following equations

$$\left.\begin{aligned} x^{\nabla\Delta} + a_1(t)\,x^{\nabla} + a_2(t)\,x &= 0,\\ x^{\nabla\Delta} + b_1(t)\,x^{\Delta} + b_2(t)\,x &= 0,\\ x^{\nabla\Delta} + c_1(t)\,x^{\Delta} + a_2(t)\,x^{\sigma} &= 0 \end{aligned}\right\} \tag{5.22}$$

can be written in the form (5.20).

Now, let $x \in \mathbb{D}_{\nabla\Delta}$ be a solution of (5.20) with a bounded support. Then we have

$$\begin{aligned} -\int_{-\infty}^{\infty} &\left[x\left(p(t)\,x^{\nabla}\right)^{\Delta} + q(t)\,x^2\right]\Delta t\\ &\overset{(2.13)}{=} \left[-p(t)\,x x^{\nabla}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left[x^{\Delta} p^{\sigma}(t) x^{\nabla}(\sigma(t)) + q(t)\,x^2\right]\Delta t\\ &= \int_{-\infty}^{\infty}\left[p^{\sigma}(t)\left(x^{\Delta}\right)^2 + q(t)\,x^2\right]\Delta t =: \mathcal{F}_{\nabla\Delta}(x). \end{aligned}$$

We recall the definition of generalized zeros and disconjugacy of equation (5.20).

Definition 5.9. We say that a function $x(\cdot)$ has a *generalized zero* in $(\rho(t_0), t_0]_{\mathbb{T}}$ if $x(t_0) = 0$ or $x^{\rho}(t_0)\,x(t_0) < 0$. Equation (5.20) is said to be *disconjugate* on $\mathbb{T}$ provided there is no nontrivial real solution of this equation with two (or more) generalized zeros in $\mathbb{T}$.

Finally, the following theorem concerning properties about the recessive solution for equation (5.20) is an important tool in the proof of the main result in this section.

Theorem 5.10 (Recessive and dominant solutions). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If (5.20) is nonoscillatory on $[a, \omega)_{\mathbb{T}}$, i.e., there is a nontrivial solution having only finitely many generalized zeros in $[a, \omega)_{\mathbb{T}}$, then there is a solution $u(\cdot)$ called a recessive solution at ω, such that $u(\cdot)$ is positive on $[t_0, \omega)_{\mathbb{T}}$ for some $t_0 \in \mathbb{T}$, and if $v(\cdot)$ is any second, linearly independent solution, called a dominant solution at ω, the following hold:*

(i) $\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0$,

(ii) $\int_{t_0}^{\omega} \frac{1}{p(t)u(t)u^{\rho}(t)} \nabla t = \infty$,

(iii) $\int_{b}^{\omega} \frac{1}{p(t)v(t)v^{\rho}(t)} \nabla t < \infty$ *for* $b < \omega$ *sufficiently close,*

(iv) $\frac{p(t)v^{\nabla}(t)}{v(t)} > \frac{p(t)u^{\nabla}(t)}{u(t)}$ *for* $t < \omega$ *sufficiently close.*

The recessive solution $u(\cdot)$ is unique up to multiplication by a nonzero constant.

The statement for recessive and dominant solutions at $\inf \mathbb{T}$ is analogical.

5.2.2 Main results

First of all, we extend Definition 5.7 to the operator $\mathcal{M}$.

Definition 5.11. Let (5.20) be disconjugate on $\mathbb{T}$. We say that the operator $\mathcal{M}$ is *critical* if the recessive solutions of (5.20) at ∞ and $-\infty$ are linearly dependent. In the opposite case $\mathcal{M}$ is said to be *subcritical*. If (5.20) is not disconjugate, operator $\mathcal{M}$ is said to be *supercritical*.

The main result of this section read as follows.

Theorem 5.12. *Let the operator $\mathcal{M}$ be critical on $\mathbb{T}$, let $I = [a, b]_{\mathbb{T}}$ be an arbitrary time scale interval, such that $\rho(b) > a$, and let $\varepsilon > 0$ be arbitrary. Then the operators*

$$\widetilde{\mathcal{M}}^{p} = -\left(\tilde{p}(t)\, x^{\nabla}\right)^{\Delta} + q(t)\, x,$$
$$\widetilde{\mathcal{M}}^{q} = -\left(p(t)\, x^{\nabla}\right)^{\Delta} + \tilde{q}(t)\, x,$$

where

$$\tilde{p}(t) = \begin{cases} p(t) - \varepsilon, & \text{for } t \in I, \\ p(t), & \text{otherwise}, \end{cases} \qquad \tilde{q}(t) = \begin{cases} q(t) - \varepsilon, & \text{for } t \in I, \\ q(t), & \text{otherwise}, \end{cases}$$

are supercritical on $\mathbb{T}$.

Proof. Let $h(\cdot)$ be a recessive solution of (5.20) both at ∞ and $-\infty$. Now, let us introduce a solution $x(\cdot)$ of equation (5.20) in the form

$$x(t) := \begin{cases} 0, & t \in (-\infty, K)_{\mathbb{T}}, \\ f(t), & t \in [K, L)_{\mathbb{T}}, \\ h(t), & t \in [L, M)_{\mathbb{T}}, \\ g(t), & t \in [M, N)_{\mathbb{T}}, \\ 0, & t \in [N, \infty)_{\mathbb{T}}, \end{cases} \tag{5.23}$$

where

$$f(t) = A\,h(t)\int_K^t \frac{1}{p(s)\,h(s)\,h^\rho(s)}\nabla s \quad \text{with} \quad A := \left(\int_K^L \frac{1}{p(\tau)\,h(\tau)\,h^\rho(\tau)}\nabla\tau\right)^{-1},$$

$$g(t) = B\,h(t)\int_t^N \frac{1}{p(s)\,h(s)\,h^\rho(s)}\nabla s \quad \text{with} \quad B := \left(\int_M^N \frac{1}{p(\tau)\,h(\tau)\,h^\rho(\tau)}\nabla\tau\right)^{-1},$$

and moreover $K < L < M < N$, $K, L, M, N \in \mathbb{T}$, are chosen such that $I \subset [M,N]_{\mathbb{T}}$ in case of the operator $\widetilde{\mathcal{M}}^p$ or $I \subset [L,M]_{\mathbb{T}}$ if we deal with $\widetilde{\mathcal{M}}^q$. Because

$$f(K) = 0,\ \ f(L) = h(L),\ \ h(M) = g(M),\ \text{ and } g(N) = 0, \tag{5.24}$$

x is continuous. Now, we calculate the functional $\mathcal{F}$ at x given by (5.23), i.e.,

$$\begin{aligned}
\mathcal{F}_{\nabla\Delta}(x) &= \int_{-\infty}^{\infty}\left[p^\sigma(x^\Delta)^2 + qx^2\right]\Delta t\\
&= \int_K^L\left[p^\sigma(f^\Delta)^2 + qf^2\right]\Delta t + \int_L^M\left[p^\sigma(h^\Delta)^2 + qh^2\right]\Delta t + \int_M^N\left[p^\sigma(g^\Delta)^2 + qg^2\right]\Delta t\\
&= \left[pff^\nabla\right]_K^L + \left[phh^\nabla\right]_L^M + \left[pgg^\nabla\right]_M^N\\
&= h(L)\left[p(L)\,f^\nabla(L) - p(L)\,h^\nabla(L)\right] + h(M)\left[p(M)\,h^\nabla(M) - p(M)\,g^\nabla(M)\right]\\
&= A + B.
\end{aligned}$$

Here we used at first the fact, that for $[\alpha,\beta]_{\mathbb{T}} \subseteq \operatorname{supp} x$ we have

$$\begin{aligned}
\mathcal{F}_{\nabla\Delta}(x) &= \int_\alpha^\beta\left[p^\sigma(x^\Delta)^2 + qx^2\right]\Delta t \overset{(2.13)}{=} \int_\alpha^\beta\left[(pxx^\nabla)^\Delta - (px^\nabla)^\Delta x + qx^2\right]\Delta t\\
&= \int_\alpha^\beta\left\{(pxx^\nabla)^\Delta + x\left[-(px^\nabla)^\Delta + qx\right]\right\}\Delta t = \left[pxx^\nabla\right]_\alpha^\beta,
\end{aligned}$$

identity (5.24), and the calculations

$$p(L)\,f^\nabla(L) = p(L)\,h^\nabla(L) + \frac{A}{h(L)}, \qquad p(M)\,g^\nabla(M) = p(M)\,h^\nabla(M) - \frac{B}{h(M)}.$$

Next, Theorem 5.10(ii) implies that A and B tend to 0 as $K \to -\infty$ and $N \to \infty$, respectively. Therefore we obtain for any $\delta > 0$ that

$$\mathcal{F}_{\nabla\Delta}(x) \le \frac{\delta}{2}$$

for K sufficiently negative and N sufficiently positive. Finally, we calculate the values of the functionals $\mathcal{F}^p$ and $\mathcal{F}^q$ corresponding to the operators $\widetilde{\mathcal{M}}^p$ and $\widetilde{\mathcal{M}}^q$, respectively. We obtain (see the choice of K, L, M, N above) the following inequalities

$$\mathcal{F}^p(x) = \mathcal{F}_{\nabla\Delta}(x) - \underbrace{\varepsilon\int_I (x^\Delta(t))^2\Delta t}_{=:\delta} \le \frac{\delta}{2} - \delta = -\frac{\delta}{2} < 0,$$

$$\mathcal{F}^q(x) = \mathcal{F}_{\nabla\Delta}(x) - \underbrace{\varepsilon\int_I (x(t))^2\Delta t}_{=:\delta} \le \frac{\delta}{2} - \delta = -\frac{\delta}{2} < 0.$$

Therefore $\widetilde{\mathcal{M}}^p$ and $\widetilde{\mathcal{M}}^q$ are supercritical and the theorem is proven. ■

We end this subsection by a direct consequence of Theorem 5.12.

Corollary 5.13. *Let the operator $\mathcal{M}$ be critical on $\mathbb{T}$, let $I = [a, b]_{\mathbb{T}}$ be a time scale interval, such that $\rho(b) > a$, and let $\varepsilon > 0$ be arbitrary. Then the operators*

$$\widehat{\mathcal{M}}^p = -\left(\hat{p}(t)\, y^\nabla\right)^\Delta + q(t)\, y,$$
$$\widehat{\mathcal{M}}^q = -\left(p(t)\, y^\nabla\right)^\Delta + \hat{q}(t)\, y,$$

where

$$\hat{p}(t) = \begin{cases} p(t) + \varepsilon, & \text{for } t \in I, \\ p(t), & \text{otherwise}, \end{cases} \qquad \hat{q}(t) = \begin{cases} q(t) + \varepsilon, & \text{for } t \in I, \\ q(t), & \text{otherwise}, \end{cases}$$

are subcritical on $\mathbb{T}$.

Proof. Suppose that $\widehat{\mathcal{M}}^p$ is critical. We perturb this operator in the sense of Theorem 5.12 (we replace $\hat{p}$ with $\tilde{\hat{p}}$). The resulting operator should be, according to Theorem 5.12, supercritical, which is a contradiction. The result concerning $\widehat{\mathcal{M}}^q$ is proven similarly. ∎

5.2.3 One term operator

In this subsection we give, as an example, a criterion of criticality of the following one term operator

$$\wp[x](t) := -\left(p(t)\, x^\nabla\right)^\Delta. \tag{5.25}$$

Theorem 5.14. *Operator $\wp$ is critical if and only if*

$$\int_0^\infty \frac{1}{p(t)} \nabla t = \infty = \int_{-\infty}^0 \frac{1}{p(t)} \nabla t. \tag{5.26}$$

Proof. At first we find the recessive solution of (5.25) at ∞. We use the following linearly independent solutions of (5.25) given by

$$x_1(t) = 1 \quad \text{and} \quad x_2(t) = \int_0^t \frac{1}{p(s)} \nabla s.$$

By a direct computation (compare with Theorem 5.10(i)) we obtain

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = \lim_{t\to\infty} \int_0^t \frac{1}{p(s)} \nabla s = \infty,$$

i.e., $x_1(t) = 1$ is the recessive solution at ∞. At $-\infty$ we use the solutions

$$x_1(t) = 1 \quad \text{and} \quad x_3(t) = \int_t^0 \frac{1}{p(s)} \nabla s.$$

Since

$$\lim_{t\to-\infty} \frac{x_3(t)}{x_1(t)} = \lim_{t\to-\infty} \int_t^0 \frac{1}{p(s)} \nabla s = \infty$$

holds, the function $x_1(\cdot)$ is the recessive solution both at ∞ and $-\infty$. Hence, the operator $\wp$ is critical.

On the other hand to verify the second implication, it suffices to show that if at least one of the integrals in (5.26) is convergent, the operator $\wp$ is not critical. Let us assume that $\int_0^\infty \frac{1}{p(t)} \nabla t < \infty$ and let us introduce the solutions

$$x_1(t) = 1 \quad \text{and} \quad x_4(t) = \int_0^\infty \frac{1}{p(s)} \nabla s - \int_0^t \frac{1}{p(s)} \nabla s = \int_t^\infty \frac{1}{p(s)} \nabla s.$$

We have

$$\lim_{t \to \infty} \frac{x_4(t)}{x_1(t)} = \lim_{t \to \infty} \int_t^\infty \frac{1}{p(s)} \nabla s = 0,$$

therefore $x_4(\cdot)$ is the recessive solution of (5.25) at ∞. This solution is linearly independent with the recessive solution at $-\infty$, which is $x_1(t) = 1$ (if the second integral is divergent) or (which we can verify analogously) it is $x_5(t) = \int_{-\infty}^t \frac{1}{p(s)} \nabla s$ (if the second integral is convergent). Altogether, the statement is proved. ∎

5.3 Weyl–Titchmarsh theory for second order equations on time scales

The purpose of this section is to provide an overview of cornerstones of the Weyl–Titchmarsh theory for the second order Sturm–Liouville dynamic equation in the form

$$-\left(p(t)\, x^\Delta\right)^\Delta + q(t)\, x^\sigma = \lambda\, w(t)\, x^\sigma, \quad t \in [a, \infty)_\mathbb{T} \tag{5.27}$$

as the special case of the Weyl–Titchmarsh theory given for symplectic dynamic systems in Chapter 4.

When studying differential equation (4.1), i.e.,

$$-\left(p(t)\, x'\right)' + q(t)\, x = \lambda\, w(t)\, x, \quad t \in [a, \infty),$$

a crucial role is played by the solutions $\theta(\cdot), \phi(\cdot) : [a, \infty) \to \mathbb{C}$ satisfying the following boundary conditions

$$\theta(a) = \sin\varphi, \quad p(a)\, \theta'(a) = \cos\varphi, \qquad \phi(a) = -\cos\varphi, \quad p(a)\, \phi'(a) = \sin\varphi, \tag{5.28}$$

where $\varphi \in [0, \pi)$. Weyl considered in his paper [160] equation (4.1) with continuous $p(\cdot)$ and $q(\cdot)$, $p(\cdot) > 0$, $w(\cdot) \equiv 1$ on $[a, \infty)$, and boundary conditions (5.28) in which $\varphi = \frac{\pi}{2}$. On the other hand, Titchmarsh studied in his book [156] equation (4.1) with $p(\cdot) \equiv 1$, continuous $q(\cdot)$, and $w(\cdot) \equiv 1$ on $[a, \infty)$, but with general boundary conditions in the form (5.28) having $\varphi \in [0, \pi)$.

Following the results in [160], the elements of the Weyl–Titchmarsh theory for equation (5.27) with $p(\cdot) = w(\cdot) \equiv 1$ and the time scale analogue of boundary conditions (5.28) with $\varphi = \frac{\pi}{2}$, see also (5.33), are considered in [166]. Similarly, when $p(\cdot)$ is piecewise continuously nabla-differentiable and $w(\cdot) \equiv 1$, the results in [160] are generalized in [109] to equation (5.27). A classification of the limit point and limit circle cases for the second order dynamic equations with mixed time scale delta and nabla derivatives and nonzero continuous coefficients is given in [159].

We present in this section an overview of the Weyl–Titchmarsh theory for equation (5.27). In particular, an important role in this theory is played by the $m(\lambda)$-*function*, whose natural properties

$$m(\bar{\lambda}) = \overline{m(\lambda)} \quad \text{and} \quad \operatorname{Im}(\lambda) \cdot \operatorname{Im}(m(\lambda)) > 0 \quad \text{for } \lambda \notin \mathbb{R}$$

remain true on general time scales. We construct the so-called *Weyl solution* and *Weyl disk*. We justify the terminology "disk" by its geometric properties, show explicitly the coordinates of the center of the disk, and calculate its radius. We show that the dichotomy mentioned above works in the same way (especially, that the Weyl solution is square-integrable) and present a necessary and sufficient criterion for the limit point case. Finally, we consider a nonhomogeneous problem associated with equation (5.27) and we express its solution explicitly. These results complete the study in [166], in which the square-integrable Weyl solution and the center and radius of the Weyl disk were obtained for the special case of (5.27) as we discuss above.

Our method is based on the transformation of equation (5.27) into a 2×2 dynamic system, see Lemma 5.15, which allows us to apply the results from Chapter 4. Also, it needs to be mentioned that when the coefficients of equation (5.27) satisfy the assumption that

$$p(\cdot), q(\cdot), w(\cdot) \text{ are rd-continuous and (5.30) holds,} \tag{5.29}$$

then some of our results, but not all of them, follow from [150], where equation (5.27) is considered as the special case of linear Hamiltonian system. This section provides a straightforward unification and extension of the Weyl–Titchmarsh theory for the second order Sturm–Liouville differential and difference equations.

5.3.1 Preliminaries on equation (5.3)(iii)

In the last section of this chapter we deal with equation (5.27) on $[a,\infty)_{\mathbb{T}}$, where the coefficients $p(\cdot), q(\cdot), w(\cdot)$ are real piecewise rd-continuous functions on $[a,\infty)_{\mathbb{T}}$ such that

$$\inf_{t\in[a,b]_{\mathbb{T}}} |p(t)| > 0 \quad \text{for all } b \in (a,\infty)_{\mathbb{T}} \quad \text{and} \quad w(t) > 0 \quad \text{for all } t \in [a,\infty)_{\mathbb{T}}, \tag{5.30}$$

and $\lambda \in \mathbb{C}$ plays a role of a spectral parameter. The function $p(\cdot)$ is allowed to change its sign.

For a given $\lambda \in \mathbb{C}$, we introduce the following set

$$\mathbb{D}_{\Delta\Delta}(\lambda) := \left\{ x(\cdot,\lambda) : [a,\infty)_{\mathbb{T}} \to \mathbb{C},\ x(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}} \text{ and } \left(p\, x^{\Delta}\right)(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}} \right\}.$$

A function $x(\cdot,\lambda) : [a,\infty)_{\mathbb{T}} \to \mathbb{C}$ is said to be a solution of equation (5.27) if $x(\cdot,\lambda) \in \mathbb{D}_{\Delta\Delta}(\lambda)$ and equation (5.27) is satisfied for all $t \in [a,\infty)_{\mathbb{T}}$. The existence and uniqueness of solutions of (5.27) together with the initial conditions $x(t_0) = x_0$ and $x^{\Delta}(t_0) = x_1$, where $x_0, x_1 \in \mathbb{C}$ are given constants, follows through the next lemma by the corresponding result for the symplectic dynamic system systems introduced in Section 4.1.

Lemma 5.15. *Equation (5.27) can be equivalently written as the time scale symplectic system, i.e., ($\mathcal{S}_\lambda$), of the form*

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\Delta} = \begin{pmatrix} 0 & 1/p(t) \\ q(t) & \mu(t)\, q(t)/p(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda\, w(t)\, x^{\sigma} \end{pmatrix},$$

where $u(t) := p(t)\, x^{\Delta}(t)$, or with the linear Hamiltonian dynamic system, i.e., (4.3), of the form

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\Delta} = \begin{pmatrix} 0 & 1/p(t) \\ q(t) - \lambda\, w(t) & 0 \end{pmatrix} \begin{pmatrix} x^{\sigma} \\ u \end{pmatrix}.$$

Proof. The statements follow by straightforward calculations. ■

For any two functions $x(\cdot,\lambda), y(\cdot,\nu) \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,\infty)_{\mathbb{T}}$, where $\lambda,\nu \in \mathbb{C}$, we define their *Wronskian* by

$$\widetilde{W}[x(t,\lambda), y(t,\nu)] = x(t,\lambda)\, y^{\Delta}(t,\nu) - x^{\Delta}(t,\lambda)\, y(t,\nu) \quad \text{for all } t \in [a,\infty)_{\mathbb{T}}, \tag{5.31}$$

which will be used in the formulation of our results similarly to [156]. This form does not correspond with the Wronskian introduced in (5.8) because solutions of (5.27) are real with respect to t and complex only with respect to the spectral parameter λ. Moreover, this Wronskian is constant with respect to t and consequently two solutions of (5.27) are linearly independent if and only if $\widetilde{W}[x,y](t) \not\equiv 0$.

5.3.2 Main results

Throughout this section we assume that $\beta_1, \beta_2 \in \mathbb{C}$ are given numbers such that

$$\text{(i)} \quad \beta_1\overline{\beta}_1 + \beta_2\overline{\beta}_2 = 1, \quad \text{(ii)} \quad \overline{\beta}_1\beta_2 - \beta_1\overline{\beta}_2 = 0. \tag{5.32}$$

Following (5.28), we denote by $\theta(\cdot,\lambda,\varphi)$ and $\phi(\cdot,\lambda,\varphi)$ the solutions of (5.27) satisfying the initial conditions

$$\left.\begin{aligned} \theta(a,\lambda,\varphi) &= \sin\varphi, & p(a)\,\theta^{\Delta}(a,\lambda,\varphi) &= \cos\varphi, \\ \phi(a,\lambda,\varphi) &= -\cos\varphi, & p(a)\,\phi^{\Delta}(a,\lambda,\varphi) &= \sin\varphi, \end{aligned}\right\} \tag{5.33}$$

where $\lambda \in \mathbb{C}$ and $\varphi \in [0,\pi)$. Our first result is associated with the regular spectral problem.

Theorem 5.16. *Consider the boundary value problem*

$$(5.27) \text{ with} \quad x(a)\,\sin\varphi + p(a)\,x^{\Delta}(a)\,\cos\varphi = 0, \quad \beta_1\,x(b) + \beta_2\,p(b)\,x^{\Delta}(b) = 0, \tag{5.34}$$

where $b \in [a,\infty)_{\mathbb{T}}$ is fixed. Then

(i) *a number $\lambda \in \mathbb{C}$ is an eigenvalue of (5.34) if and only if*

$$\beta_1\,\phi(b,\lambda,\varphi) + \beta_2\,p(b)\,\phi^{\Delta}(b,\lambda,\varphi) = 0,$$

(ii) *the eigenvalues of (5.34) are real and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product*

$$\langle x(\cdot), y(\cdot)\rangle_{w,b} := \int_a^b w(t)\,\overline{x}^{\sigma}(t)\,y^{\sigma}(t)\,\Delta t. \tag{5.35}$$

The following definition corresponds in the continuous case to [156, identity (2.1.5)].

Definition 5.17 ($m(\lambda)$-function)**.** For any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $b \in [a,\infty)_{\mathbb{T}}$ we define the $m(\lambda)$-*function*

$$m(b,\lambda,\varphi) := -\frac{\beta_1\,\theta(b,\lambda,\varphi) + \beta_2\,p(b)\,\theta^{\Delta}(b,\lambda,\varphi)}{\beta_1\,\phi(b,\lambda,\varphi) + \beta_2\,p(b)\,\phi^{\Delta}(b,\lambda,\varphi)}.$$

In the following theorem we present the fundamental properties of the $m(\lambda)$-function.

Theorem 5.18. *The $m(\lambda)$-function is an entire function in λ and it satisfies*

$$\overline{m}(b,\lambda,\varphi) = m(b,\overline{\lambda},\varphi) \quad \text{for every } \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Now, we define the so-called *Weyl solution*, compare with [156, p. 25].

Definition 5.19 (Weyl solution)**.** For any $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $m \in \mathbb{C}$ we define the *Weyl solution*

$$\mathcal{X}(\cdot,\lambda,\varphi,m) := \theta(\cdot,\lambda,\varphi) + m\,\phi(\cdot,\lambda,\varphi). \tag{5.36}$$

In the following theorem, we describe how the $m(\lambda)$-function depends on the value of φ used in the initial conditions (5.33).

Theorem 5.20. *For any $\varphi,\psi \in [0,\pi)$ we have*

$$m(b,\lambda,\varphi) = \frac{\sin(\varphi-\psi) + \cos(\varphi-\psi)\,m(b,\lambda,\psi)}{\cos(\varphi-\psi) + \sin(\varphi-\psi)\,m(b,\lambda,\psi)}.$$

Now, we define the corresponding Weyl disk. The justification of this terminology follows from Theorem 5.23 bellow.

Definition 5.21 (Weyl disk)**.** By using the function

$$\mathcal{E}(m) := i\,\delta(\lambda)\,p(b)\,W\left[\mathcal{X}(b,\overline{\lambda},\varphi,m),\mathcal{X}(b,\lambda,\varphi,m)\right]$$

with fixed $b \in [a,\infty)_{\mathbb{T}}$ we construct the Weyl disk to be the set

$$D(b,\lambda) := \{m \in \mathbb{C} : \ \mathcal{E}(m) \le 0\}.$$

A characterization of elements of the Weyl disk is formulated in the following theorem. In this result the numbers $\beta_1,\beta_2 \in \mathbb{C}$ satisfy only the first identity in (5.32), while the second identity from (5.32) is replaced by an inequality.

Theorem 5.22. *The number $m \in \mathbb{C}$ belongs to the Weyl disk $D(b,\lambda)$ if and only if there exist $\beta_1,\beta_2 \in \mathbb{C}$ such that (5.32)(i) holds, $\overline{\beta}_1\beta_2 - \beta_1\overline{\beta}_2 \ge 0$, and*

$$\beta_1\,\mathcal{X}(b,\lambda,\varphi,m) + \beta_2\,p(b)\,\mathcal{X}^{\Delta}(b,\lambda,\varphi,m) = 0.$$

Moreover, in this case we have $m = m(b,\lambda,\varphi)$.

The following properties represent the time scales analogies of the geometric characterization of the Weyl disk obtained for the continuous time case in [156, p. 24].

Theorem 5.23. *For $b \in [a,\infty)_{\mathbb{T}}$, the Weyl disk $D(b,\lambda)$ has the form*

$$D(b,\lambda) = \{c(b,\lambda) + r(b,\lambda)\,v,\ \ v \in \mathbb{C} \text{ with } \|v\| \le 1\},$$

where the center $c(b,\lambda) \in \mathbb{C}$ is the point

$$c(b,\lambda) = -W[\phi(b,\overline{\lambda}),\theta(b,\lambda)]\,/\,W[\phi(b,\overline{\lambda}),\phi(b,\lambda)],$$

and the radius $r(b,\lambda) \in \mathbb{R}$ is given by

$$1/r(b,\lambda) = \left|p(b)\,W[\phi(b,\overline{\lambda}),\phi(b,\lambda)]\right|.$$

From now on, we consider the corresponding singular spectral problem. For this case we use the semi-inner product which is the limit of $\langle\cdot,\cdot\rangle_{w,b}$ as $b\to\infty$, i.e.,

$$\langle x(\cdot), y(\cdot)\rangle_w := \int_a^\infty w(t)\,\overline{x}^\sigma(t)\,y^\sigma(t)\,\Delta t.$$

First of all we use the nesting property of the Weyl disks and define the limiting Weyl disk.

Theorem 5.24. *The Weyl disks $D(b,\lambda)$ are closed, convex, and nested with respect to $b\to\infty$. Hence, the so-called limiting Weyl disk*

$$D_+(\lambda) := \bigcap_{b\in[a,\infty)_{\mathbb{T}}} D(b,\lambda)$$

is closed, convex, and nonempty. Moreover, the limits $\lim_{b\to\infty} c(b,\lambda)$ and $\lim_{b\to\infty} r(b,\lambda)$ exist and define the center $c_+(\lambda)$ and the radius $r_+(\lambda)$ of the limiting Weyl disk $D_+(\lambda)$, i.e.,

$$c_+(\lambda) = \lim_{b\to\infty} c(b,\lambda), \quad r_+(\lambda) = \lim_{b\to\infty} r(b,\lambda) \geq 0.$$

Next, we introduce the linear space of complex square-integrable $\mathrm{C}^1_{\mathrm{prd}}$ functions on the interval $[a,\infty)_{\mathbb{T}}$, i.e.,

$$L^2_w := \left\{ x\in \mathrm{C}^1_{\mathrm{prd}},\ \|x(\cdot)\|^2_w := \int_a^\infty w(t)\ |x^\sigma(t)|^2\ \Delta t < \infty \right\}.$$

The following theorem says that the Weyl solution is square-integrable, which corresponds to [156, inequality (2.1.9)] in the continuous time case.

Theorem 5.25. *For any $m\in D_+(\lambda)$ we have*

$$\|\mathcal{X}(\cdot,\lambda,\varphi,m)\|^2_w \leq (\mathrm{Im}\,(m)) \,/\, \mathrm{Im}\,(\lambda)\,.$$

From Theorem 5.25 it follows that there is always at least one square-integrable solution. Hence, it is natural to classify the second order Sturm–Liouville equations on time scales of the form (5.27) depending on their number of linearly independent square-integrable solutions.

Definition 5.26. Equation (5.27) is said to be in the *limit point case* if, for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$, there is exactly one (up to a multiplicative constant) square-integrable solution on $[a,\infty)_{\mathbb{T}}$, while it is said to be in the *limit circle case* if there are two linearly independent square-integrable solutions on $[a,\infty)_{\mathbb{T}}$.

The limit point case is characterized in the next theorem.

Theorem 5.27. *The following statements are equivalent.*

(i) *Equation (5.27) is in the limit point case.*

(ii) *For every $\lambda\in\mathbb{C}\setminus\mathbb{R}$ we have $r_+(\lambda)=0$, and consequently $D_+(\lambda)=\{c_+(\lambda)\}$.*

(iii) *For every $\lambda,\nu\in\mathbb{C}\setminus\mathbb{R}$ and every square-integrable solutions $x_1(\cdot,\lambda)$ and $x_2(\cdot,\nu)$ of (5.27) with the spectral parameter equal to λ and ν, respectively, we have*

$$\lim_{t\to\infty} p(t)\, W[x_1(t,\overline{\lambda}), x_2(t,\nu)] = 0.$$

Now, for $f(\cdot) \in L^2_w$ we consider the nonhomogeneous equation

$$\left(p(t)\, x^{\Delta}\right)^{\Delta} + q(t) x^{\sigma} = \lambda w(t)\, x^{\sigma} + w(t)\, f^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{5.37}$$

The form of the solution of the nonhomogeneous problem given in the following theorem corresponds to the continuous time case in [156, identity (2.6.1)]. For $m \in D_+(\lambda)$ we define the *Green function*

$$G(t,s,\lambda) := \begin{cases} \left[\theta(t,\lambda) + m\, \phi(t,\lambda)\right] \phi(s,\lambda), & s \in [a,t]_{\mathbb{T}}, \\ \phi(t,\lambda) \left[\theta(s,\lambda) + m\, \phi(s,\lambda)\right], & s \in [t,\infty)_{\mathbb{T}}. \end{cases}$$

Theorem 5.28. *The function*

$$\hat{x}(t,\lambda) = -\int_a^{\infty} G(t,\sigma(s),\lambda)\, w(s) f^{\sigma}(s)\, \Delta s, \quad t \in [a,\infty)_{\mathbb{T}},$$

solves equation (5.37) and satisfies the boundary conditions

$$\hat{x}(a,\lambda) \sin\varphi + p(a)\, \hat{x}^{\Delta}(a,\lambda) \cos\varphi = 0 \quad \text{and} \quad \lim_{t\to\infty} p(t)\, W[\mathcal{X}(t,\bar{\nu},\varphi,m), \hat{x}(t,\lambda)] = 0$$

for every $\nu \in \mathbb{C} \setminus \mathbb{R}$. Moreover, we have the estimate

$$\|\hat{x}(\cdot,\lambda)\|_w \le \|f(\cdot)\|_w \,/ |\mathrm{Im}\,(\lambda)| \,.$$

5.4 Bibliographical notes

Equation (5.5) was studied, e.g., in [10,14,121] and in [32, Section 8.4]. The equivalency of (5.5) and (5.6) follows from [33, Theorem 4.23 – Corollary 4.25]. The existence of a unique solution of the initial value problem (5.7) was shown in [14, Theorem 3.1]. The Hilbert space $L^2([a,\infty)_{\mathbb{T}})$ was introduced in [47], see also [140]. The constancy of the Lagrange bracket was proven in [14, Theorem 3.2]. The existence of the recessive solution of equation (5.5) follows from [33, Theorem 4.55]. The content of Lemma 5.1 can be proved by using the same arguments as in [35, Lemma 2]. The density of the set $\mathcal{D}_0'$ in the Hilbert space L^2 follows from [47, Theorem 3.12]. In Section 5.1 we suppose that equation (5.5) is in the limit circle case. In the opposite case, i.e., in the limit point case, there is a unique (up to a multiplicative constant) solution in L^2 and the domains of the Krein–von Neumann and Friedrichs extensions have to be characterized by using an another construction, compare with [167].

Equation (5.20) was investigated in [7–9,80]. The existence of a unique solution of the initial problem (5.21) was proven in [7, Theorem 3.6]. The definition of a generalized zero and the concept of the disconjugacy of equation (5.20), i.e., the content of Definition 5.9, are introduced in an analogy with [121, Definition 4.43 and Definition 4.44], see also [2,62]. The content of Theorem 5.10 can be found in [32, Theorem 4.55]. For the continuous and discrete analogies of the one term operator $\wp[x]$ given in (5.25) we refer to [56,59,82].

The existence of a unique solution of an initial problem associated with (5.27) follows from [57, Corollary 7.12] via the transformation given in Lemma 5.15. Moreover, basic theory for equation (5.27) with $\lambda = 0$ was established in [32, Chapter 4].

The main results presented in Section 5.1 were published by the author in [164]. Another generalization of the Friedrichs extension (for an operator associated with a linear Hamiltonian differential system) was developed by R. Šimon Hilscher and the author in [144]. The results concerning critical operators from Section 5.2 were given by P. Hasil and the author in [83]. The results from Section 5.3 are derived as a special case of the time scale symplectic system from [145] presented in Chapter 4, their particular forms for the second order dynamic equation are from paper [146] by R. Šimon Hilscher and the author. They follow from the corresponding results in [145] or in Chapter 4, and some of them under assumption (5.29) from [150]. More precisely, Theorems 5.16, 5.22–5.25 and Definition 5.21 have their counterparts both in [145, 150]. The remaining results in Section 5.3 follow from [145] or Chapter 4 alone.

Bibliography

[1] R. P. Agarwal, M. Bohner, and A. C. Peterson, *Inequalities on time scales: A survey*, Math. Inequal. Appl. **4** (2001), no. 4, 535–557.

[2] R. P. Agarwal, M. Bohner, and P. J. Y. Wong, *Sturm–Liouville eigenvalue problems on time scales*, Appl. Math. Comput. **99** (1999), no. 2-3, 153–166.

[3] C. D. Ahlbrandt, *Equivalence of discrete Euler equations and discrete Hamiltonian systems*, J. Math. Anal. Appl. **180** (1993), no. 2, 498–517.

[4] C. D. Ahlbrandt, M. Bohner, and J. Ridenhour, *Hamiltonian systems on time scales*, J. Math. Anal. Appl. **250** (2000), no. 2, 561–578.

[5] D. R. Anderson, *Discrete trigonometric matrix functions*, Panamer. Math. J. **7** (1997), no. 1, 39–54.

[6] D. R. Anderson, *Titchmarsh–Sims–Weyl theory for complex Hamiltonian systems on Sturmian time scales*, J. Math. Anal. Appl. **373** (2010), no. 2, 709–725.

[7] D. R. Anderson, G. S. Guseinov, and J. Hoffacker, *Higher-order self-adjoint boundary-value problems on time scales*, J. Comput. Appl. Math. **194** (2006), no. 2, 309–342.

[8] D. R. Anderson and J. Hoffacker, *Green's function for an even order mixed derivative problem on time scales*, in "Dynamic Equations on Time Scales" (M. Bohner and B. Kaymakçalan, editors), Dynam. Systems Appl. **12** (2003), no. 1-2, 9–22.

[9] D. R. Anderson and J. Hoffacker, *A stacked delta-nabla self-adjoint problem of even order*, Math. Comput. Modelling **38** (2003), no. 5-6, 481–494.

[10] D. R. Anderson and J. Hoffacker, *Even order self adjoint time scale problems*, Electron. J. Differential Equations (2005), no. 24, 9 pp. (electronic).

[11] T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, Tôhoku Math. J. (2) **22** (1970), 65–75.

[12] T. Asahi, *Spectral theory of the difference equations*, Progr. Theoret. Phys. No. **36** (1966), 55–96.

[13] F. M. Atici, D. C. Biles, and A. Lebedinsky, *An application of time scales to economics*, Math. Comput. Modelling **43** (2006), no. 7-8, 718–726.

[14] F. M. Atici and G. S. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, in "Dynamic Equations on Time Scales" (R. P. Agarwal, M. Bohner, and D. O'Regan, editors), J. Comput. Appl. Math. **141** (2002), no. 1-2, 75–99.

[15] F. M. Atici and F. Uysal, *A production-inventory model of HMMS on time scales*, Appl. Math. Lett. **21** (2008), no. 3, 236–243.

[16] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mathematics in Science and Engineering, Vol. 8, Academic Press, New York, 1964.

[17] B. Aulbach and L. Neidhart, *Integration on measure chains*, in "New Progress in Difference Equations" (Proceedings of the Sixth International Conference on Difference Equations, Augsburg, 2001), B. Aulbach, S. Elaydi, and G. Ladas (editors), pp. 239–252, CRC, Boca Raton, 2004.

[18] J. H. Barrett, *A Prüfer transformation for matrix differential equations*, Proc. Amer. Math. Soc. **8** (1957), 510–518.

[19] E. T. Bell, *Men of Mathematics, Volumes I and II*, Pinguin Books, Melbourne–London–Baltimore, 1953.

[20] M. Benammar and W. D. Evans, *On the Friedrichs extension of semi-bounded difference operators*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 1, 167–177.

[21] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and Its Applications, Vol. 2, Hindawi Publishing Corporation, New York, 2006.

[22] C. Bennewitz and W. N. Everitt, *The Titchmarsh–Weyl eigenfunction expansion theorem for Sturm–Liouville differential equations*, in "Sturm–Liouville theory: Past and Present" (Proceedings of the International Colloquium, Geneva, 2003), W. O. Amrein, A. M. Hinz, and D. B. Pearson (editors), pp. 137–171, Birkhäuser Verlag, Basel, 2005.

[23] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory*, Princeton University Press, Princeton, 2005.

[24] P. Bohl, *Über eine Differentialgleichung der Störungstheorie* (in German), J. Reine Angew. Math. **131** (1906), no. 1-2, 268–321.

[25] M. Bohner, S. Clark, and J. Ridenhour, *Lyapunov inequalities for time scales*, J. Inequal. Appl. **7** (2002), no. 1, 61–77.

[26] M. Bohner and O. Došlý, *Trigonometric transformations of symplectic difference systems*, J. Differential Equations **163** (2000), no. 1, 113–129.

[27] M. Bohner and O. Došlý, *The discrete Prüfer transformation*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2715–2726 (electronic).

[28] M. Bohner and O. Došlý, *Trigonometric systems in oscillation theory of difference equations*, in "Dynamic Systems and Aplications, Vol. 3." (Proceedings of the Third International Conference on Dynamic Systems and Applications, Atlanta, GA, 1999), pp. 99–104, Dynamic, Atlanta, 2001.

[29] M. Bohner, G. Gelles, and J. Heim, *Multiplier-accelerator models on time scales*, Int. J. Stat. Econ. **4** (2010), no. S10, 1–12.

[30] M. Bohner and G. Guseinov, *Riemann and Lebesgue integration*, in "Advances in Dynamic Equations on Time Scales", M. Bohner and A. C. Peterson (editors), pp. 117–163, Birkhäuser Verlag, Boston, 2003.

[31] M. Bohner and R. Hilscher, *An eigenvalue problem for linear Hamiltonian dynamic systems*, Fasc. Math. (2005), no. 35, 35–49.

[32] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser Verlag, Boston, 2001.

[33] M. Bohner and A. C. Peterson (editors), *Advances in Dynamic Equations on Time Scales*, Birkhäuser Verlag, Boston, 2003.

[34] M. Bohner and S. Sun, *Weyl–Titchmarsh theory for symplectic difference systems*, Appl. Math. Comput. **216** (2010), no. 10, 2855–2864.

[35] B. M. Brown and J. S. Christiansen, *On the Krein and Friedrichs extensions of a positive Jacobi operator*, Expo. Math. **23** (2005), no. 2, 179–186.

[36] B. M. Brown and W. D. Evans, *On an extension of Copson's inequality for infinite series*, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), no. 1-2, 169–183.

[37] B. M. Brown, W. D. Evans, and L. L. Littlejohn, *Orthogonal polynomials and extensions of Copson's inequality*, in "Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA, Granada, 1991)", J. Comput. Appl. Math. **48** (1993), no. 1-2, 33–48.

[38] B. M. Brown, W. D. Evans, and M. Plum, *Titchmarsh–Sims–Weyl theory for complex Hamiltonian systems*, Proc. London Math. Soc. (3) **87** (2003), no. 2, 419–450.

[39] M. C. Caputo, *Time scales: From nabla calculus to delta calculus and vice versa via duality*, Int. J. Difference Equ. **5** (2010), no. 1, 25–40.

[40] D. Chelkak and E. Korotyaev, *Weyl–Titchmarsh functions of vector-valued Sturm–Liouville operators on the unit interval*, J. Funct. Anal. **257** (2009), no. 5, 1546–1588.

[41] S. L. Clark, *A spectral analysis for self-adjoint operators generated by a class of second order difference equations*, J. Math. Anal. Appl. **197** (1996), no. 1, 267–285.

[42] S. L. Clark and F. Gesztesy, *Weyl–Titchmarsh M-function asymptotics for matrix-valued Schrödinger operators*, Proc. London Math. Soc. (3) **82** (2001), no. 3, 701–724.

[43] S. L. Clark and F. Gesztesy, *Weyl–Titchmarsh M-function asymptotics, local uniqueness results, trace formulas, and Borg-type theorems for Dirac operators*, Trans. Amer. Math. Soc. **354** (2002), no. 9, 3475–3534 (electronic).

[44] S. L. Clark and F. Gesztesy, *On Weyl–Titchmarsh theory for singular finite difference Hamiltonian systems*, J. Comput. Appl. Math. **171** (2004), no. 1-2, 151–184.

[45] S. L. Clark and P. Zemánek, *On a Weyl–Titchmarsh theory for discrete symplectic systems on a half line*, Appl. Math. Comput. **217** (2010), no. 7, 2952–2976.

[46] F. A. Davidson and B. P. Rynne, *The formulation of second-order boundary value problems on time scales*, Adv. Difference Equ. (2006), Art. ID 31430, 10 pp. (electronic).

[47] F. A. Davidson and B. P. Rynne, *Self-adjoint boundary-value problems on time-scales*, Electron. J. Differential Equations (2007), no. 175, 10 pp. (electronic).

[48] P. J. Davis and R. Hersh, *The Mathematical Experience*, with an introduction by G.–C. Rota, Houghton Mifflin, Boston, 1982.

[49] Z. Došlá and D. Škrabáková, *Phases of linear difference equations and symplectic systems*, Math. Bohem. **128** (2003), no. 3, 293–308.

[50] O. Došlý, *On transformations of selfadjoint linear differential systems*, Arch. Math. (Brno) **21** (1985), no. 3, 159–170.

[51] O. Došlý, *Phase matrix of linear differential systems*, Časopis Pěst. Mat. **110** (1985), no. 2, 183–192, 208.

[52] O. Došlý, *On some properties of trigonometric matrices*, Časopis Pěst. Mat. **112** (1987), no. 2, 188–196.

[53] O. Došlý, *Riccati matrix differential equation and classification of disconjugate differential systems*, Arch. Math. (Brno) **23** (1987), no. 4, 231–241.

[54] O. Došlý, *The Bohl Transformation and its applications*, in "2004 – Dynamical Systems and Applications" (Proceedings of the International Conference, Antalya, 2004), pp. 371–385, GBS Publishers & Distributors, Delhi, 2005.

[55] O. Došlý and P. Hasil, *Friedrichs extension of operators defined by symmetric banded matrices*, Linear Algebra Appl. **430** (2009), no. 8-9, 1966–1975.

[56] O. Došlý and P. Hasil, *Critical higher order Sturm–Liouville difference operators*, J. Difference Equ. Appl., to appear.

[57] O. Došlý, S. Hilger, and R. Hilscher, *Symplectic dynamic systems*, in "Advances in Dynamic Equations on Time Scales", M. Bohner and A. C. Peterson (editors), pp. 293–334, Birkhäuser Verlag, Boston, 2003.

[58] O. Došlý and R. Hilscher, *Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales*, J. Differ. Equations Appl. **7** (2001), no. 2, 265–295.

[59] O. Došlý and J. Komenda, *Conjugacy criteria and principal solutions of self-adjoint differential equations*, Arch. Math. (Brno) **31** (1995), no. 3, 217–238.

[60] O. Došlý and Š. Pechancová, *Trigonometric recurrence relations and tridiagonal trigonometric matrices*, Int. J. Difference Equ. **1** (2006), no. 1, 19–29.

[61] O. Došlý and Z. Pospíšil, *Hyperbolic transformation and hyperbolic difference systems*, Fasc. Math. (2001), no. 32, 25–48.

[62] L. H. Erbe and S. Hilger, *Sturmian theory on measure chains*, Differential Equations Dynam. Systems **1** (1993), no. 3, 223–244.

[63] L. H. Erbe and P. X. Yan, *Disconjugacy for linear Hamiltonian difference systems*, J. Math. Anal. Appl. **167** (1992), no. 2, 355–367.

[64] L. H. Erbe and P. X. Yan, *Qualitative properties of Hamiltonian difference systems*, J. Math. Anal. Appl. **171** (1992), no. 2, 334–345.

[65] G. J. Etgen, *A note on trigonometric matrices*, Proc. Amer. Math. Soc. **17** (1966), 1226–1232.

[66] G. J. Etgen, *Oscillatory properties of certain nonlinear matrix differential systems of second order*, Trans. Amer. Math. Soc. **122** (1966), 289–310.

[67] W. N. Everitt, *A personal history of the m-coefficient*, J. Comput. Appl. Math. **171** (2004), no. 1-2, 185–197.

[68] W. N. Everitt, D. B. Hinton, and J. K. Shaw, *The asymptotic form of the Titchmarsh–Weyl coefficient for Dirac systems*, J. London Math. Soc. (2) **27** (1983), no. 3, 465–476.

[69] K. Feng, *The Hamiltonian way for computing Hamiltonian dynamics*, in "Applied and Industrial Mathematics" (Venice, 1989), Math. Appl., Vol. 56, pp. 17–35, Kluwer Acad. Publ., Dordrecht, 1991.

[70] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics I: Mainly mechanics, radiation, and heat*, Addison-Wesley Publishing, Reading, 1963.

[71] K. Filakovský, *Vlastnosti řešení hyperbolického systému* (in Czech, [Properties of Solutions of Hyperbolic System]), Master's Thesis, University of J. E. Purkyně, Brno, 1987.

[72] H. Freudenthal, *Über die Friedrichssche Fortsetzung halbbeschränkter Hermitescher Operatoren* (in German), Proc. Akad. Wet. Amsterdam **39** (1936), 832–833.

[73] K. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren, zweiter Teil* (in German), Math. Ann. **109** (1934), no. 1, 685–713.

[74] K. Friedrichs, *Über die ausgezeichnete Randbedingung in der Spektraltheorie der halbbeschränkten gewöhnlichen Differentialoperatoren zweiter Ordnung* (in German), Math. Ann. **112** (1936), no. 1, 1–23.

[75] F. R. Gantmacher, *The Theory of Matrices, Volumes I and II*, translated from the Russian by K. A. Hirsch, reprint of the 1959 translations, AMS Chelsea Publishing, Providence, 1998.

[76] F. Gesztesy and Z. Zhao, *On critical and subcritical Sturm-Liouville operators*, J. Funct. Anal. **98** (1991), no. 2, 311–345.

[77] F. Gesztesy and Z. Zhao, *Critical and subcritical Jacobi operators defined as Friedrichs extensions*, J. Differential Equations **103** (1993), no. 1, 68–93.

[78] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid dynamical systems: Robust stability and control for systems that combine continuous-time and discrete-time dynamics*, IEEE Control Syst. Mag. **29** (2009), no. 2, 28–93.

[79] G. S. Guseinov, *Integration on time scales*, J. Math. Anal. Appl. **285** (2003), no. 1, 107–127.

[80] G. S. Guseinov, *Self-adjoint boundary value problems on time scales and symmetric Green's functions*, Turkish J. Math. **29** (2005), no. 4, 365–380.

[81] P. R. Halmos, *I Want To Be a Mathematician: An Automathography*, Springer-Verlag, New York, 1985.

[82] P. Hasil, *Criterion of p-criticality for one term* $2n$*-order difference operators*, Arch. Math. (Brno), to appear.

[83] P. Hasil and P. Zemánek, *Critical second order operators on time scales*, in "Proceedings of the 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications" (Dresden, 2010), W. Feng, Z. Feng, M. Grasselli, X. Lu, S. Siegmund, and J. Voigt (editors), AIMS Proceedings, American Institute of Mathematical Sciences (AIMS), Springfield, to appear.

[84] E. Hellinger, *Zur Stieltjesschen Kettenbruchtheorie* (in German), Math. Ann. **86** (1922), no. 1-2, 18–29.

[85] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten* (in German, [Calculus on Measure Chains with Application to Central Manifolds]), Ph.D. Dissertation, University of Würzburg, Würzburg, 1988.

[86] S. Hilger, *Analysis on measure chains — a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1-2, 18–56.

[87] S. Hilger, *Matrix Lie theory and measure chains*, in "Dynamic Equations on Time Scales" (R. P. Agarwal, M. Bohner, and D. O'Regan, editors), J. Comput. Appl. Math. **141** (2002), no. 1-2, 197–217.

[88] R. Hilscher, *Linear Hamiltonian systems on time scales: Transformations*, in "Discrete and Continuous Hamiltonian Systems" (R. P. Agarwal and M. Bohner, editors), Dynam. Systems Appl. **8** (1999), no. 3-4, 489–501.

[89] R. Hilscher, *Linear Hamiltonian systems on time scales: Positivity of quadratic functionals*, in "Boundary Value Problems and Related Topics" (L. H. Erbe and A. C. Peterson, editors), Math. Comput. Modelling **32** (2000), no. 5-6, 507–527.

[90] R. Hilscher, *Positivity of quadratic functionals on time scales: Necessity*, Math. Nachr. **226** (2001), 85–98.

[91] R. Hilscher and P. Řehák, *Riccati inequality, disconjugacy, and reciprocity principle for linear Hamiltonian dynamic systems*, in "Dynamic Equations on Time Scales" (M. Bohner and B. Kaymakçalan, editors), Dynam. Systems Appl. **12** (2003), no. 1-2, 171–189.

[92] R. Hilscher and V. Zeidan, *Calculus of variations on time scales: Weak local piecewise C^1_{rd} solutions with variable endpoints*, J. Math. Anal. Appl. **289** (2004), no. 1, 143–166.

[93] R. Hilscher and V. Zeidan, *Time scale symplectic systems without normality*, J. Differential Equations **230** (2006), no. 1, 140–173.

[94] R. Hilscher and V. Zeidan, *Applications of time scale symplectic systems without normality*, J. Math. Anal. Appl. **340** (2008), no. 1, 451–465.

[95] R. Hilscher and V. Zeidan, *Riccati equations for abnormal time scale quadratic functionals*, J. Differential Equations **244** (2008), no. 6, 1410–1447.

[96] R. Hilscher and V. Zeidan, *Time scale embedding theorem and coercivity of quadratic functionals*, Analysis (Munich) **28** (2008), no. 1, 1–28.

[97] R. Hilscher and V. Zeidan, *Nabla time scale symplectic systems and related quadratic functionals*, Differ. Equ. Dyn. Syst. **18** (2010), no. 1-2, 163–198.

[98] R. Hilscher and V. Zeidan, *Reid roundabout theorems for time scale symplectic systems*, in "Discrete Dynamics and Difference Equations" (Proceedings of the Twelfth International Conference on Difference Equations and Applications , Lisbon, 2007), S. Elaydi, H. Oliveira, J. M. Ferreira, and J. F. Alves (editors), World Scientific Publishing, London, 2010.

[99] R. Hilscher, V. Zeidan, and W. Kratz, *Differentiation of solutions of dynamic equations on time scales with respect to parameters*, Adv. Dyn. Syst. Appl. **4** (2009), no. 1, 35–54.

[100] R. Hilscher and P. Zemánek, *Trigonometric and hyperbolic systems on time scales*, Dynam. Systems Appl. **18** (2009), no. 3-4, 483–505.

[101] D. B. Hinton and A. Schneider, *On the Titchmarsh–Weyl coefficients for singular S-Hermitian systems. I*, Math. Nachr. **163** (1993), 323–342.

[102] D. B. Hinton and A. Schneider, *On the Titchmarsh–Weyl coefficients for singular S-Hermitian systems. II*, Math. Nachr. **185** (1997), 67–84.

[103] D. B. Hinton and J. K. Shaw, *On Titchmarsh–Weyl $M(\lambda)$-functions for linear Hamiltonian systems*, J. Differential Equations **40** (1981), no. 3, 316–342.

[104] D. B. Hinton and J. K. Shaw, *On the spectrum of a singular Hamiltonian system*, Quaestiones Math. **5** (1982/83), no. 1, 29–81.

[105] D. B. Hinton and J. K. Shaw, *Parameterization of the $M(\lambda)$ function for a Hamiltonian system of limit circle type*, Proc. Roy. Soc. Edinburgh Sect. A **93** (1982/83), no. 3-4, 349–360.

[106] D. B. Hinton and J. K. Shaw, *Hamiltonian systems of limit point or limit circle type with both endpoints singular*, J. Differential Equations **50** (1983), no. 3, 444–464.

[107] D. B. Hinton and J. K. Shaw, *On boundary value problems for Hamiltonian systems with two singular points*, SIAM J. Math. Anal. **15** (1984), no. 2, 272–286.

[108] D. B. Hinton and J. K. Shaw, *On the spectrum of a singular Hamiltonian system. II*, Quaestiones Math. **10** (1986), no. 1, 1–48.

[109] A. Huseynov, *Limit point and limit circle cases for dynamic equations on time scales*, Hacet. J. Math. Stat. **39** (2010), no. 3, 379–392.

[110] H. Kalf, *A characterization of the Friedrichs extension of Sturm–Liouville operators*, J. London Math. Soc. (2) **17** (1978), no. 3, 511–521.

[111] V. I. Kogan and F. S. Rofe-Beketov, *On square-integrable solutions of symmetric systems of differential equations of arbitrary order*, Proc. Roy. Soc. Edinburgh Sect. A **74** (1974/75), 5–40 (1976).

[112] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with one singular point*, SIAM J. Math. Anal. **20** (1989), no. 3, 664–700.

[113] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with two singular points*, SIAM J. Math. Anal. **20** (1989), no. 3, 701–715.

[114] A. M. Krall, *A limit-point criterion for linear Hamiltonian systems*, Appl. Anal. **61** (1996), no. 1-2, 115–119.

[115] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Mathematical Topics, Vol. 6, Akademie Verlag, Berlin, 1995.

[116] W. Kratz, R. Šimon Hilscher, and V. Zeidan, *Eigenvalue and oscillation theorems for time scale symplectic systems*, in "Dynamic Equations on Time Scales and Applications" (L. H. Erbe and A. C. Peterson, editors), Int. J. Dyn. Syst. Differ. Equ. **3** (2011), no. 1–2, 84–131.

[117] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications I* (in Russian with English summary), Rec. Math. [Mat. Sbornik] N.S. **20(62)** (1947), 431–495.

[118] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, Vol. 6, World Scientific Publishing, Teaneck, 1989.

[119] M. Lesch and M. M. Malamud, *On the deficiency indices and self-adjointness of symmetric Hamiltonian systems*, J. Differential Equations **189** (2003), no. 2, 556–615.

[120] M. Marletta and A. Zettl, *The Friedrichs extension of singular differential operators*, J. Differential Equations **160** (2000), no. 2, 404–421.

[121] K. Messer, *Second order self-adjoint equations with mixed derivatives*, in "Advances in Dynamic Equations on Time Scales", M. Bohner and A. C. Peterson (editors), pp. 85–116, Birkhäuser Verlag, Boston, 2003.

[122] S. J. Monaquel and K. M. Schmidt, *On M-functions and operator theory for non-self-adjoint discrete Hamiltonian systems*, in "Special Issue: 65th birthday of Prof. Desmond Evans", J. Comput. Appl. Math. **208** (2007), no. 1, 82–101.

[123] M. Murata, *Positive solutions and large time behaviors of Schrödinger semigroups, Simon's problem*, J. Funct. Anal. **56** (1984), no. 3, 300–310.

[124] R. Nevanlinna, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem* (in German), Ann. Acad. Sci. Fenn. A **18** (1922), no. 5, 53 pp.

[125] H.-D. Niessen and A. Zettl, *The Friedrichs extension of regular ordinary differential operators*, Proc. Roy. Soc. Edinburgh Sect. A **114** (1990), no. 3-4, 229–236.

[126] H.-D. Niessen and A. Zettl, *Singular Sturm–Liouville problems: The Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc. (3) **64** (1992), no. 3, 545–578.

[127] S. Orlov, *Nested matrix disks analytically depending on a parameter, and theorems on the invariance of ranks of radii of limiting disks*, Math. USSR Izv. **10** (1976), 565–613 (1977).

[128] H. L. Pedersen, *Stieltjes moment problems and the Friedrichs extension of a positive definite operator*, J. Approx. Theory **83** (1995), no. 3, 289–307.

[129] Y. Pinchover, *A Liouville-type theorem for Schrödinger operators*, Comm. Math. Phys. **272** (2007), no. 1, 75–84.

[130] Z. Pospíšil, *Hyperbolic sine and cosine functions on measure chains*, in "Proceedings of the Third World Congress of Nonlinear Analysts, Part 2 (Catania, 2000)", Nonlinear Anal. **47** (2001), no. 2, 861–872.

[131] Z. Pospíšil, *Hyperbolic matrix systems, functions and transformations on time scales*, author preprint, October 2001.

[132] Z. Pospíšil, *An inverse problem for matrix trigonometric and hyperbolic functions on measure chains*, in "Colloquium on Differential and Difference Equations — CDDE 2002" (Brno, 2002), Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math., Vol. 13, pp. 205–211, Masaryk Univ., Brno, 2003.

[133] J. Qi and S. Chen, *Strong limit-point classification of singular Hamiltonian expressions*, Proc. Amer. Math. Soc. **132** (2004), no. 6, 1667–1674 (electronic).

[134] W. T. Reid, *A Prüfer transformation for differential systems*, Pacific J. Math. **8** (1958), no. 3, 575–584.

[135] W. T. Reid, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1971.

[136] F. Rellich, *Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung* (in German), Math. Ann. **122** (1951), no. 5, 343–368.

[137] C. Remling, *Geometric characterization of singular self-adjoint boundary conditions for Hamiltonian systems*, Appl. Anal. **60** (1996), no. 1-2, 49–61.

[138] G. Ren and Y. Shi, *The defect index of singular symmetric linear difference equations with real coefficients*, Proc. Amer. Math. Soc. **138** (2010), no. 7, 2463–2475.

[139] R. Rosenberger, *A new characterization of the Friedrichs extension of semibounded Sturm–Liouville operators*, J. London Math. Soc. (2) **31** (1985), no. 3, 501–510.

[140] B. P. Rynne, *L^2 spaces and boundary value problems on time-scales*, J. Math. Anal. Appl. **328** (2007), no. 2, 1217–1236.

[141] Y. Shi, *On the rank of the matrix radius of the limiting set for a singular linear Hamiltonian system*, Linear Algebra Appl. **376** (2004), 109–123.

[142] Y. Shi, *Weyl–Titchmarsh theory for a class of discrete linear Hamiltonian systems*, Linear Algebra Appl. **416** (2006), no. 2-3, 452–519.

[143] B. Simon, *Spectral analysis of rank one perturbations and applications*, in "Mathematical Quantum Theory II: Schrödinger Operators" (Vancouver, 1993), CRM Proc. Lecture Notes, Vol. 8, pp. 109–149, Amer. Math. Soc., Providence, 1995.

[144] R. Šimon Hilscher and P. Zemánek, *Friedrichs extension of operators defined by linear Hamiltonian systems on unbounded interval*, in "Equadiff 12", Proceedings of the Conference on Differential Equations and their Applications (Brno, 2009), J. Diblík, O. Došlý, P. Drábek, and E. Feistauer, editors, Math. Bohem. **135** (2010), no. 2, 209–222.

[145] R. Šimon Hilscher and P. Zemánek, *Weyl–Titchmarsh theory for time scale symplectic systems on half line*, in "Recent Progress in Differential and Difference Equations", Proceedings of the Conference on Differential and Difference Equations and Applications (Rajecké Teplice, 2010), J. Diblík, E. Braverman, Y. Rogovchenko, and M. Růžičková, editors, Abstr. Appl. Anal. **2011** (2011), Art. ID 738520, 41 pp. (electronic).

[146] R. Šimon Hilscher and P. Zemánek, *Overview of Weyl–Titchmarsh theory for second order Sturm–Liouville equations on time scales*, submitted, February 2011.

[147] A. R. Sims, *Secondary conditions for linear differential operators of the second order*, J. Math. Mech. **6** (1957), 247–285.

[148] H. Sun and Y. Shi, *Limit-point and limit-circle criteria for singular second-order linear difference equations with complex coefficients*, Comput. Math. Appl. **52** (2006), no. 3-4, 539–554.

[149] H. Sun and Y. Shi, *Strong limit point criteria for a class of singular discrete linear Hamiltonian systems*, J. Math. Anal. Appl. **336** (2007), no. 1, 224–242.

[150] S. Sun, M. Bohner, and S. Chen, *Weyl–Titchmarsh theory for Hamiltonian dynamic systems*, Abstr. Appl. Anal. **2010** (2010), Art. ID 514760, 18 pp. (electronic).

[151] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs, Vol. 72, American Mathematical Society, Providence, RI, 2000.

[152] C. C. Tisdell and A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal. **68** (2008), no. 11, 3504–3524.

[153] E. C. Titchmarsh, *On expansions in eigenfunctions IV*, Quart. J. Math., Oxford Ser. **12** (1941), 33–50.

[154] E. C. Titchmarsh, *On expansions in eigenfunctions V*, Quart. J. Math., Oxford Ser. **12** (1941), 89–107.

[155] E. C. Titchmarsh, *On expansions in eigenfunctions VI*, Quart. J. Math., Oxford Ser. **12** (1941), 154–166.

[156] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations, Part I*, second edition, Clarendon Press, Oxford, 1962.

[157] Y. Wang, Y. Shi, and G. Ren, *Transformations for complex discrete linear Hamiltonian and symplectic systems*, Bull. Austral. Math. Soc. **75** (2007), no. 2, 179–191.

[158] J. Weidmann, *Spectral theory of Sturm-Liouville operators approximation by regular problems*, in "Sturm–Liouville theory: Past and Present" (Proceedings of the International Colloquium, Geneva, 2003), W. O. Amrein, A. M. Hinz, and D. B. Pearson (editors), pp. 75–98, Birkhäuser Verlag, Basel, 2005.

[159] J. Weiss, *Limit-point criteria for a second order dynamic equation on time scales*, Nonlinear Dyn. Syst. Theory **9** (2009), no. 1, 99–108.

[160] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen* (in German), Math. Ann. **68** (1910), no. 2, 220–269.

[161] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Applied Mathematical Sciences, Vol. 108, Springer-Verlag, New York, 1995.

[162] P. Zemánek, *Symplektické diferenční systémy* (in Czech, [Symplectic Difference Systems]), Master's Thesis, Masaryk University, Brno, 2007.

[163] P. Zemánek, *Discrete trigonometric and hyperbolic systems: An overview*, Ulmer Seminare über Funktionalanalysis und Differentialgleichungen **14** (2009), 345–359.

[164] P. Zemánek, *Krein-von Neumann and Friedrichs extensions for second order operators on time scales*, in "Dynamic Equations on Time Scales and Applications" (L. H. Erbe and A. C. Peterson, editors), Int. J. Dyn. Syst. Differ. Equ. **3** (2011), no. 1–2, 132–144.

[165] A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, Vol. 121, American Mathematical Society, Providence, 2005.

[166] C. Zhang and L. Zhang, *Classification for a class of second-order singular equations on time scales*, in "Proceedings of the Eighth ACIS International Conference on Software Engineering, Artificial Intelligence, Networking, and Parallel/Distributed Computing" (Qingdao, 2007), W. Feng and F. Gao (editors), pp. 73–76, IEEE Computer Society, Washington, 2007.

[167] Z. Zheng, J. Qi, and S. Chen, *Eigenvalues below the lower bound of minimal operators of singular Hamiltonian expressions*, Comput. Math. Appl. **56** (2008), no. 11, 2825–2833.

Appendix

Further research

The topics presented in this dissertation can be extended in various ways. We sketch some of the related problems in this section.

Although solutions of scalar trigonometric systems are periodic in the continuous and discrete time cases, this property was not studied for a general trigonometric system on time scales in Chapter 3. The coefficients determining the hyperbolic system on any time scale are given in Section 3.4. The problem of finding similar coefficients for trigonometric systems is unsolved.

The Weyl–Titchmarsh theory is intensively studied in the literature, especially in the continuous time. In Chapter 4, fundamental results which enable us to generalize the works of Clark, Everitt, Gesztesy, Hinton, Krall, and Shaw are given. Moreover, these results will be new also in the discrete time case.

The results given in Sections 5.1 and 5.2 are established in the the simplest case of symplectic systems on time scales, i.e., for the second order Sturm–Liouville equations on time scales. They can be generalized to higher order Sturm–Liouville equations as it was done recently in the continuous and discrete time cases.

List of author's publications (as of July 14, 2011)

The publications cited in the Web of Science (WOS) database are denoted by these letters and completed with the corresponding impact factor (IF).

Published:

[A1] S. L. Clark and P. Zemánek, *On a Weyl–Titchmarsh theory for discrete symplectic systems on a half line*, Appl. Math. Comput. **217** (2010), no. 7, 2952–2976. WOS, IF 2009: 1.124.

[A2] R. Hilscher and P. Zemánek, *Trigonometric and hyperbolic systems on time scales*, Dynam. Systems Appl. **18** (2009), no. 3-4, 483–505. WOS, IF 2009: 0.500.

[A3] R. Šimon Hilscher and P. Zemánek, *Definiteness of quadratic functionals for Hamiltonian and symplectic systems: a survey*, Int. J. Difference Equ. **4** (2009), no. 1, 49–67.

[A4] R. Šimon Hilscher and P. Zemánek, *Friedrichs extension of operators defined by linear Hamiltonian systems on unbounded interval*, in: "Equadiff 12", Proceedings of the Conference on Differential Equations and their Applications (Brno, 2009), J. Diblík, O. Došlý, P. Drábek, and E. Feistauer, editors, Math. Bohem. **135** (2010), no. 2, 209–222.

[A5] R. Šimon Hilscher and P. Zemánek, *Weyl–Titchmarsh theory for time scale symplectic systems on half line*, in: "Recent Progress in Differential and Difference Equations", Proceedings of the Conference on Differential and Difference Equations and Applications (Rajecké Teplice, 2010), J. Diblík, E. Braverman, Y. Rogovchenko, and M. Růžičková, editors, Abstr. Appl. Anal. **2011** (2011), Art. ID 738520, 41 pp. (electronic). WOS, IF 2009: 2.221.

[A6] P. Zemánek, *Krein-von Neumann and Friedrichs extensions for second order operators on time scales*, in "Dynamic Equations on Time Scales and Applications" (L. H. Erbe and A. C. Peterson, editors), Int. J. Dyn. Syst. Differ. Equ. **3** (2011), no. 1–2, 132–144.

[A7] P. Zemánek, *Discrete trigonometric and hyperbolic systems: An overview*, Ulmer Seminare über Funktionalanalysis und Differentialgleichungen **14** (2009), 345–359.

Accepted:

[A8] P. Hasil and P. Zemánek, *Critical second order operators on time scales*, in: "Proceedings of the 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications" (Dresden, 2010), W. Feng, Z. Feng, M. Grasselli, X. Lu, S. Siegmund, and J. Voigt, editors, AIMS Proceedings, American Institute of Mathematical Sciences (AIMS), Springfield, to appear.

Submitted:

[A9] R. Šimon Hilscher and P. Zemánek, *Overview of Weyl–Titchmarsh theory for second order Sturm–Liouville equations on time scales*, submitted, February 2011.

Author's curriculum vitae

Personal data:	Petr Zemánek, born on April 21, 1983, in Znojmo (Czech Republic).	
University education:	*Since 2007*	the doctoral student of the degree program "Mathematics" with the field of the study "Mathematical analysis" (Faculty of Science, Masaryk University),

	June 13, 2007	MSc. degree in the study program "Mathematics" in the field of study "Mathematics" with the specialization in mathematical analysis (Faculty of Science, Masaryk University) and with the master's thesis "Symplectic Difference Systems" (in Czech, supervisor Assoc. Prof. Roman Šimon Hilscher, DSc.),
	June 30, 2005	BSc. degree in the study program "Mathematics" with the field of study "Mathematics" (Faculty of Science, Masaryk University) and bachelor's thesis "Linear Differential Equations of the n-th Order with Constant Coefficients and Their Applications" (in Czech, supervisor prof. RNDr. Miroslav Bartušek, DrSc.).

Pedagogical activities: Seminars (at Faculty of Science and Faculty of Informatics, Masaryk University; University Ulm; Mendel University) with

- mathematical analysis in $\mathbb{R}$,
- mathematical analysis in $\mathbb{C}$,
- linear algebra,
- differential equations,
- elementary probability.

Academical Stays:

- *March–April 2008*, University Ulm (Ulm, Germany), within ERASMUS scholarship,
- *April 2009*, Missouri University of Science and Technology (Rolla, MO, USA).

Appreciation of Science Community: The Prize of the Dean of the Faculty of Science (Masaryk University) for excellent study results and success in the research (in 2010).

International conferences with active participation:

- The 14th International Conference on Difference Equations and Applications (Istanbul, Turkey, July 21–25, 2008), title of the talk: *Trigonometric and hyperbolic systems on time scales*,
- Equadiff 12 (Brno, Czech Republic, July 20–24, 2009), title of the talk: *Friedrichs extension of operators defined by linear Hamiltonian systems on infinite interval*,
- The 15th International Conference on Difference Equations and Applications (Estoril, Portugal, October 19–23, 2009), title of the talk: *Friedrichs extension of operators defined by linear Hamiltonian systems on infinite interval*,

- The 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications (Dresden, Germany, May 25–28, 2010), title of the talk: *Krein-von Neumann and Friedrichs extensions for second order operators on time scales*,
- Conference on Differential and Difference Equations and Applications 2010 (Rajecké Teplice, Slovakia, June 21–25, 2010), title of the talk: *On Weyl-Titchmarsh theory for dynamic symplectic systems on time scales*,
- The 16th International Conference on Difference Equations and Applications (Riga, Latvia, July 18–24, 2010), title of the talk: *On Weyl-Titchmarsh theory for dynamic symplectic systems on time scales*,
- Colloquium on Differential Equations and Integration Theory, Dedicated to the memory of Stefan Schwabik (Křtiny, Czech Republic, October 14–17, 2010), title of the poster: *Overview of Weyl Titchmarsh theory for second order Sturm-Liouville equations on time scales*.

New Results in Theory of Symplectic Systems on Time Scales
MSc. Petr Zemánek

Published by Masaryk University
Printed by Tribun EU s.r.o., Cejl 32, Brno 602 00

Typeset in $\LaTeX 2_\varepsilon$
2011, 1st edition
45 copies

ISBN 978-80-210-5515-5

ISBN 978-80-210-5515-5

muni PRESS

9 788021 055155 >