MASARYK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

Exercises in Global Analysis (with solutions)

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BRNO 2017

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Introduction

Dear reader,

the textbook of exercises which you are about to read is a complementary study material for the course Introduction to global analysis, instructed on the Department of Mathematics and Statistics of Masaryk University. This is a project under MU Development Fund for the year 2016, and the authors are very grateful for being supported by university. We also wish to express our gratitude to the guarantee of the project, profesor J. Slovak, for all the help during the realization of this textbook.

Our aim is to facilitate the understanding of extensive theory of the course. We approach this by giving detailed solution of numerous exercises. In the beginning of each chapter, definitions, theorems and lemmas are stated, most of them in unchanged version and without proofs. The original form of these, together with proofs of theorems and lemmas, can be found in book of I. Kolar [1]. We have humbly enriched some parts of the text with commentaries, mostly those sections we considered to be less comprehensible at first glance, or the parts that seemed to hold the possibility of gently enlarge the student's horizon. We will be happy if you find this textbook comprehensible and uplifting, nevertheless, there is a certain willingness required on the part of a student due to the level we intended to create. For the most comfortable reading we encourage you to study the above mentioned literature, also, as they are the primary reference for the course, as well as for this study material.

In case you come across some mistakes, we would be grateful for letting us know on our university mail. We wish you a pleasant reading.

Smooth mappings of real spaces

Definition 1.1.

Let $U \subset \mathbb{R}^n$ be open set and $f\colon U \to \mathbb{R}$ a function. We say that f is ***r*-times differentiable** or that f is of **class C^r** if it has all continuous partial derivatives $f^{(1)}, \dots, f^{(r)}$, up to the order r, at all points in U.

Remark. Function of class C^0 means continuous function. Function of class C^∞ is called **infinitely differentiable** or **smooth**. **Analytical** functions, i.e. those which can be expanded into power series, are said to be of class C^ω. The following implication holds

$$f \in C^\omega \Rightarrow f \in C^\infty .$$

Theorem 1.2.

Function $\lambda : \mathbb{R} \to \mathbb{R}$ given by

$$\lambda(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t \geq 0 \end{cases}$$

is smooth.

Theorem 1.3.

For arbitrary real constant $c > 0$ the function $\chi_c : \mathbb{R} \to \mathbb{R}$ given by

$$\chi_c(t) = \frac{\lambda(t)}{\lambda(t) + \lambda(c-t)}$$

is smooth.

Remark. Note that function from theorem 1.3 take only value 0 or 1

$$\chi_c(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t \geq c \end{cases}$$

and is nondecreasing.

Definition 1.4.

The Euclidean norm of a vector $x \in \mathbb{R}^n$ is given by

$$||x|| = \sqrt{(x^1)^2 + \cdots + (x^n)^2} \,. \tag{1.5}$$

Using norm we can define n-dimensional ball with center in a and radius r

$$B(a,r) = \{x \in \mathbb{R}^n, ||x-a|| < r\} \ . \tag{1.6}$$

Topological closure is denoted by $\overline{B}(a,r)$.

Theorem 1.7.

Consider function $\mu\colon \mathbb{R}^n \to \mathbb{R}$ dependent on three parameters $a \in \mathbb{R}^n,\ r > 0,\ c > 0$, given by

$$\mu(x) = 1 - \chi_c(||x-a|| - r) \ . \tag{1.8}$$

Then following holds

- function μ is smooth,
- $0 \leq \mu(x)$ and $\mu(x) = 0$ if and only if $x \notin B(a, r+c)$
- $\mu(x) \leq 1\ \forall x \in \mathbb{R}^n$ and $\mu(x) = 1$ if and only if $x \in \overline{B}(a,r)$.

Definition 1.9.

By **the support of a function** $f\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open, is understood the set of points in which f has non-zero value.

Remark. It holds for the function μ that it is constant in a certain neighbourhood of a point a and it's support is compact set $\overline{B}(a, r+c)$.

The following statement is called **Whitney's theorem** and describes important characteristic feature of smooth functions.

Theorem 1.10.

Every closed subset $S \subset \mathbb{R}^n$ is a set of zero points of some non-negative smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$.

Definition 1.11.

Let I be arbitrary index set. Open cover $(V_\alpha),\ \alpha \in I$, of open set $U \subset \mathbb{R}^n$ is called **locally finite** if for every point $x \in U$ there is a neighbourhood of x having non-empty intersection with only finitely many elements of the covering (V_α).

The following statement is one of the many versions of so called **partition of unity** theorems. It can be used to globalize certain local constructions.

Theorem 1.12.

Let $(V_\alpha),\ \alpha \in I$ be locally finite open cover of an open set $U \subset \mathbb{R}^n$. There is a system of non-negative smooth functions $(f_\alpha),\ \alpha \in I$ on U such that the following holds

a) $f_\alpha(x) \neq 0$ if and only if $x \in V_\alpha$,

b) $\sum_{\alpha\in} f_\alpha(x) = 1 \ \forall x \in U$.

Remark. From now on we will use the notation $f\colon U \to V$ solely in the case of a mapping between open sets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^k$, unless not stated otherwise.

Definition 1.13.

Consider a mapping $f\colon U \to V$ given by k-tuple of functions

$$y^1 = f^1\left(x^1,\ldots,x^n\right),\ldots,y^k = f^k\left(x^1,\ldots,x^n\right) \ , \tag{1.14}$$

called **components of mapping** f. We also write

$$y^p = f^p\left(x^i\right),\ i = 1,\ldots,n,\ p = 1,\ldots,k \tag{1.15}$$

or shortly just $y = f(x)$. We define f to be **differentiable mapping of class** C^r if all the components of f are of class C^r, $r = 1,\ldots,\infty,\omega$. **Smooth mapping** is a mapping of class C^∞.

Theorem 1.16.

Consider open sets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^k, W \subset \mathbb{R}^m$ and a pair of mappings $f\colon U \to V$, $g\colon V \to W$ of class C^r. The composed mapping $g \circ f\colon U \to W$ is also of class C^r.

Remark. For the clarity of computation let us fix the following range of index symbols

$$i,j = 1,\ldots,n, \qquad p,q = 1,\ldots,k, \qquad s,t = 1,\ldots m\ . \tag{1.17}$$

What we mean is that, for example, $\left(x^i\right)$ represents n-tuple of coordinates in $\mathbb{R}^n$. Analogously $\left(f^p\left(x^i\right)\right)$ is a k-tuple of mapping components of $f\colon U \to V$ expressed in coordinates $\left(x^i\right)$. Similarly for the rest of indeces.

Definition 1.18.

Matrix $\left(\frac{\partial f^p(a)}{\partial x^i}\right)$ is called the **Jacobian matrix** of f at a point $a \in U$. Rank of this matrix is denoted $Rk_a f$ and called **rank of a mapping** f at a. In the special case $k = n$ we can consider the determinant $\det\left(\frac{\partial f^p(a)}{\partial x^i}\right)$ which is called the **Jacobian** of mapping f at a.

Theorem 1.19.

The Jacobi matrix of composed mapping $g \circ f$ at a is product of Jacobian matrices $\left(\frac{\partial g^s(f(a))}{\partial y^p}\right)$ and $\left(\frac{\partial f^p(a)}{\partial x^i}\right)$.

Definition 1.20.

Let $U, V \subset \mathbb{R}^n$ be open sets. Bijective mapping $f\colon U \to V$ is called **diffeomorphism of class** C^r if f and inverse $f^{-1}\colon V \to U$ are both of class C^r, $r \geq 1$.

Remark. Diffeomorphism between open sets can be understood as a system of curvilinear coordinates.

Theorem 1.21.

Let $f\colon U \to V$ be diffeomorphism. For the corresponding Jacobian the following is true

$$\det\left(\frac{\partial f^p(a)}{\partial x^i}\right) \neq 0 \ \ \forall a \in U \ . \tag{1.22}$$

We consider to be well known (from elementary calculus) that the circle cannot be described as a graph of a single function. Also we know that given the equation of circle

$$(x-x_0)^2+(y-y_0)-r^2=0$$

we can express the y variable as a positive and negative square root. Then, we can describe the circle piece-wisely as a graph of these roots. Generalizing this idea leads to question whether it is possible to localy describe geometrical object, given by family of equations, as a graph of some mapping. Such function can be implicitely contained in the equations, hence the name of the following statement: **implicit mapping theorem**.

Theorem 1.23.

Let $G^p\left(x^1,\ldots,x^n,y^1,\ldots,y^k\right)$, $p=1,\ldots,k$, are function of class C^r, $r\geq 1$ defined on a neighbourhood W of a point $\left(a^1,\ldots,a^n,b^1,\ldots,b^k\right)\in\mathbb{R}^{n+k}$ satisfying

$$G^p\left(a^1,\ldots,a^n,b^1,\ldots,b^k\right)=0 \tag{1.24}$$

$$\det\left(\frac{\partial G^p\left(a^1,\ldots,a^n,b^1,\ldots,b^k\right)}{\partial y^q}\right)\neq 0\ . \tag{1.25}$$

Then there exists a neighbourhood U of $\left(a^1,\ldots,a^n\right)\in\mathbb{R}^n$ and neighbourhoood V of $\left(b^1,\ldots,b^k\right)\in\mathbb{R}^k$ such that $U\times V\subset W$ and for every point $\left(x^1,\ldots,x^n\right)\in U$ corresponds precisely one point $\left(y^1,\ldots,y^k\right)\in V$ satisfying

$$G^p\left(x^1,\ldots,x^n,y^1,\ldots,y^k\right)=0\ . \tag{1.26}$$

Functions $y^p=f^p\left(x^1,\ldots,x^n\right)$ given in this way are also of class C^r.

Lemma 1.27.

Let $f\colon U\to\mathbb{R}^n$ be a mapping of class C^r, $U\subset\mathbb{R}^n$. If the Jacobian of f is non-zero in every point then $f(U)$ is open set.

Theorem 1.28.

Let $f\colon U\to\mathbb{R}^n$ be a mapping of class C^r, $U\subset\mathbb{R}^n$. If there is a point $a\in U$ in which the Jacobian of f is non-zero then there is a neighbourhood $\tilde{U}\subset U$ of the point a and a neighbourhood $V\subset\mathbb{R}^n$ of $f(a)$ such that the restricted mapping $f|_{\tilde{U}}\colon\tilde{U}\to V$ is diffeomorphism.

Definition 1.29.

Mapping $f\colon U \to \mathbb{R}^k$, $U \subset \mathbb{R}^n$ is called **immersion** if $Rk_a f = n$ for all $a \in U$.

Remark. Rank of the Jacobi matrix $\left(\frac{\partial f^p(a)}{\partial x^i}\right)$ is less or equal to the minimum of dimensions n and k. Therefore, immersion implies $n \leq k$.

Theorem 1.30.

If $f\colon U \to \mathbb{R}^k$ is an immersion then for every $a \in U$ there are neighbourhoods $\tilde{U}$ of a, neighbourhood V of $f(a)$ and curvilinear system of coordinates $\tilde{y}^p$ on V such that the restricted mapping $f|_{\tilde{U}}$ is of the form

$$\tilde{y}^1 = x^1, \ldots, \tilde{y}^n = x^n, \tilde{y}^{n+1} = 0, \ldots \tilde{y}^k = 0\ . \tag{1.31}$$

Definition 1.32.

Mapping $f\colon U \to V$ between open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ is called **submersion** if $Rk_a f = k$ for all $a \in U$.

Remark. Because of the same reason as in the case of immersion, the dimensional condition must hold, i.e. $n \geq k$.

Theorem 1.33.

If $f\colon U \to V$ is a submersion then for all points $a \in U$ there is a neighbourhood $\tilde{U}$ of the point a and a curvilinear system of coordinates $\tilde{x}^i$ on $\tilde{U}$ such that the restricted mapping $f|_{\tilde{U}}$ is of the form

$$y^1 = \tilde{x}^1, \ldots, y^k = \tilde{x}^k\ . \tag{1.34}$$

Let us summarize the immersions and submersions in the following observation. Every immersion is localy inclusion $\mathbb{R}^n \to \mathbb{R}^{n+m}$, $\left(x^1, \ldots, x^n\right) \mapsto \left(x^1, \ldots, x^n, 0, \ldots, 0\right)$. Every submersion is localy projection $\mathbb{R}^{k+m} \to \mathbb{R}^k$, $\left(x^1, \ldots, x^k, x^{k+1}, \ldots, x^{k+m}\right) \mapsto \left(x^1, \ldots, x^k\right)$.

Exercise 1.35.

Give an example of a map which

1. a) is of the class C^0 but not of the class C^1 b) is of the class C^1 but not of the class C^2 c) is of the class C^2
2. is of the class C^r
3. is of the class C^{r+1} but not of the class C^r
4. is smooth and analytic
5. is smooth but not analytic
6. is smooth, invertible but not a diffeomorphism
7. is a diffeomorphism

Solution. 1. Example of class C^0 map is function absolute value function $f(x) = |x|$ which is everywhere continuous but does not have derivation at 0.
C^1 map is for example $f(x) = \left(\sin x, x^{\frac{3}{2}}\right)$. Eventhough the first component $f_1(x) = \sin x$ is smooth, the second $f_2(x) = x^{\frac{3}{2}}$ is of class C^1. Analogously, $f(x,y,z) = \left(x^2+y+z^4, x^{\frac{5}{2}}, e^x\right)$ is of class C^2. Generaly: let i be index of class to which the component of a given map belongs. The class of a map is according to 1.13 given by the class with the smallest i.

2. Function $f(x) = x^{\frac{2r+1}{2}}$ has r continuous derivatives and the r^{th} derivative is

$$f^{(r)}(x) = (2r+1)!! \cdot x^{\frac{1}{2}} ,$$

where $n!! = n \cdot (n-2) \cdot \dots \cdot 1$ means double factorial. Simultaneously, $f^{(r+1)}(x) = \frac{1}{2}(2r+1)!! \cdot x^{\frac{-1}{2}}$ is not defined at 0, hence, not differentiable.

3. The case is not possible (trivially). Let every component of f be continuously differentiable up to order $r+1$ including. Then every component of f must be r-times continuously differentiable, i.e. f belongs to C^r.

4. Function given by polynom $f(x) = a_1x^1 + a_2x^2 + \dots + a_nx^n$ is smooth. Arbitrary mapping with components given by polynomial functions is smooth mapping which is also analytic. Power series expansion of a polynomial is the polynomial itself. Another example is the exponential function $f(x) = e^x$ which is smooth and analytic function having the Taylor expansion of the form

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

5. Smooth mapping is, by definition, differentiable up to arbitrary order. Therefore, we can compute the Taylor series. The question is whether the corresponding power series is (pointwise) convergent to the original function. We met example of function satisfying our conditions in theorem 1.2

$$\lambda(t) = \begin{cases} 0 & t \le 0 \\ e^{-\frac{1}{t}} & t \ge 0 \end{cases}$$

The function is smooth but not analytic and it requires non-trivial proof to show this fact. We provide this example without proof as an illustration of the set inequality $C^\infty \neq C^\omega$. Let us remark that non-analytical smooth function of complex variable does not exist.

6. Function $f(x) = x^n$ considered on $[0,\infty)$ is smooth and invertible. The inverse $f^{-1}(x) = x^{\frac{1}{n}}$ is not differentiable at 0 (derivation of f^{-1} approaches infinity from the right). Since inverse of f is not differentiable, f cannot be diffeomorphism.

7. Exponential function $f(x) = e^x$ with inverse $f^{-1} = \ln x$ is an example of smooth diffeomorphism of interval $(0,\infty)$. Similarly for mapping $f(x,y) = (f_1(x,y), f_2(x,y)) = (x^2+y^2+1, e^{xy})$ which has the matrix of partial derivatives in the following form
$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ ye^{xy} & xe^{xy} \end{pmatrix}$$
being non-zero on $\mathbb{R}^2 \setminus \{(0,0)\}$. Theorem 1.28 says that for each point in $\mathbb{R}^2$ there is a neighbourhood on which f can be restricted to diffeomorphism.

$\diamond$

Exercise 1.36.

Show that both translation and linear isomorphisms of $\mathbb{R}^n$ are diffeomorphisms.

Solution. Translation is given by a shifting vector. Let us denote it $U = (u_1, \ldots, u_n)$. By adding U to original coordinates we get new coordinates
$$\left(x^1, \ldots, x^n\right) \mapsto \left(y^1, \ldots, y^n\right) = \left(x^1+u^1, \ldots, x^n+u^n\right)$$
Since new coordinates y^i are fucntions of the original ones, i.e. $y^i = y^i\left(x^1, \ldots, x^n\right)$, we can compute the matrix of partial derivatives $\left(\frac{\partial y^i}{\partial x^j}\right)$ which will be equal to the identity matrix because u_i are constant values
$$\frac{\partial y^i}{\partial x^j} = \frac{\partial x^i + u^i}{\partial x^j} = \delta^i_j\,,$$
where δ^i_j is Kronecker delta. The determinant is equal to one for arbitrary point of $\mathbb{R}^n$, thus, theorem 1.28 yields the result.
We recall well known fact from linear algebra, that is, every linear isomorphism of $\mathbb{R}^n$ (in fact every linear map between real vector spaces) can be described by some $n \times n$ real matrix with non-zero determinant
$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$$
and the image under the map is given by matrix multiplication
$$\left(x^1, \ldots, x^n\right) \mapsto \left(a_{11}x^1 + \cdots + a_{1n}x^n,\ \ldots\ , a_{n1}x^1 + \cdots + a_{nn}x^n\right)\ .$$
We can see that the matrix of partial derivatives of new variables $(y^i) = (a_{i1}x^1 + \cdots + a_{in}x^n)$ with respect to the original coordinates (x^j) is equal to A. The deteminant of A is non-zero by assumption, hence, every linear isomorphism is diffeomorphism. $\diamond$

Exercise 1.37.

Determine the domains of the following transformation of coordinates on which they are diffeomorphisms. The transformations are from standard coordinates to

1. polar coordinates,
2. cylindrical coordinates,
3. spherical coordinates,
4. hyperbolic coordinates.

Solution.

1. Polar coordinates describe the $\mathbb{R}^2$ plane

$$x(r,\varphi) = r\cos\varphi \,,$$
$$y(r,\varphi) = r\sin\varphi \,,$$

where r is the distance of a given point from the origin and φ is angle between line through the point and the origin and positive part of x axis. Let us compute the matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

The determinant is

$$J = \cos\varphi \cdot r\cos\varphi - (-\sin\varphi \cdot \sin\varphi) = r\left(\cos^2\varphi + \sin^2\varphi\right) = r$$

For Jacobian to be non-zero we muset satisfie $r > 0$ and φ must be from $[0,2\pi)$ for coordinates to be bijective. Theorem 1.28 says that polar coordinates are diffeomorphism on $(0,\infty)\times[0,2\pi)$.

2. Cylindrical coordinates describes points of $\mathbb{R}^3$

$$x(r,\varphi,z) = r\cos\varphi \,,$$
$$y(r,\varphi,z) = r\sin\varphi \,,$$
$$z(r,\varphi,z) = z$$

where $r > 0$ is the distance of a given point from the origin and $\varphi \in (0,2\pi)$ is angle between the line through the point and the origin and positive part of x axis (measured for $z = 0$ or, equivalently, we can consider the projection of the xy plane). Jacobi matrix of transformation is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using Laplace expansion along the third row we see that the value of Jacobian is the same as for polar coordinates

$$J = r\,.$$

and due to 1.28 cylindrical coordinates are diffeomorphism on $(0,\infty)\times(0,2\pi)$.

3. Spherical coordinates describes $\mathbb{R}^3$

$$x(r,\varphi,\theta) = r\sin\theta\cos\varphi$$
$$y(r,\varphi,\theta) = r\sin\theta\sin\varphi$$
$$z(r,\varphi,\theta) = r\cos\theta \,,$$

where $r > 0$ is the distance of a given point from the origin, $\theta \in (0,\pi)$ is angle between the line through the point and the origin with positive part of z axis, and φ $\varphi \in (0,2\pi)$ is angle between the line and positive part of x axis (measured for $z = 0$). Jacobi matrix of transformation is

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

Determinant computed with Laplace expansion along the third row is

$$\begin{aligned} \det J &= r^2\cos\theta \cdot \det\begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \end{pmatrix} \\ &+ r^2\sin\theta \cdot \det\begin{pmatrix} \sin\theta\cos\varphi & -\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi \end{pmatrix} \\ &= r^2\cos\theta\left(\sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi\right) \\ &+ r^2\sin\theta\left(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi\right) \\ &= r^2\sin\theta\cos\theta + r^2\sin\theta\sin^2\theta \\ &= r^2\sin\theta\ . \end{aligned}$$

We get non-zero value for $r > 0,\ \ \theta \in (0,\pi)$. For cooridnates to be diffeomorphism we need not only $J \neq 0$ but also $\varphi \in [0,2\pi)$ (because of bijection condition). Theorem 1.28 yields that on $(0,\infty) \times (0,\pi) \times [0,2\pi)$ the spherical coordinates are diffeomorphism.

4. Hyperbolic coordinates are of the form

$$\begin{aligned} x(u,v) &= ve^u \\ y(u,v) &= ve^{-u} \end{aligned}$$

and diffeomorphically maps $(\mathbb{R} \setminus \{(0,0)\}) \times (0,\infty)$ on $(0,\infty) \times (0,\infty)$ which is in accordance with theorem 1.28, given the Jacobian is

$$J = \det\begin{pmatrix} uve^u & e^u \\ -uve^{-u} & e^{-u} \end{pmatrix} = uv\left(e^{u-u} + e^{u-u}\right) = uv$$

$\diamond$

Exercise 1.38.

Give an example of a map which is

1. immersion,
2. submersion.

Solution. Every diffeomorphism is immersion. Parametrization of line in $\mathbb{R}^n$ $p\colon \mathbb{R} \to \mathbb{R}^n$, $t \mapsto \big(p^1(t), \dots, p^n(t)\big)$ is immersion. Analogously, a plane given by the graph of a function $q\colon \mathbb{R}^2 \to \mathbb{R}^3$, $(x,y) \mapsto (x,y,f(x,y))$ is immersion of $\mathbb{R}^2$ (or some subset of $\mathbb{R}^2$) into $\mathbb{R}^3$.
Every diffeomorphism is also submersion. Basic example of submersion is projection onto the first k-components $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$, $(x^1, \dots, x^n) \mapsto (x^1, \dots, x^k)$, $k \leq n$. Map $f\colon \mathbb{R}^2 \setminus (0,0) \to S^1$, $(x,y) \mapsto \frac{(x,y)}{\sqrt{x^2+y^2}}$ mapping $\mathbb{R}^2$ without the origin on the unit circle with centered in origin is submersion. ◇

Exercise 1.39.

Find diffeomorphism of the open ball : $B(a,r)$ with $\mathbb{R}^n$.

Solution. Firstly we use the consequence of theorem **??**, that is, composition of diffeomorphisms is diffeomorphism. Using this fact, we are looking for convenient maps such that their composition will be the searched diffeomorphism. Consider the mapping

$$\varphi\colon B(0,1) \to \mathbb{R}^n \ ,$$

defined on the open ball centered at the origin, $B(0,1) = \{x \in \mathbb{R}^n, ||x|| < 1\}$, given by

$$x \mapsto \frac{x}{\sqrt{1-||x||^2}} \ ,$$

where $||-||$ is the euclidean norm 1.5. The inverse is

$$\begin{aligned} \varphi^{-1}\colon \mathbb{R}^n &\to B(0,1) \ , \\ y &\mapsto \frac{y}{\sqrt{1+||y||^2}} \ . \end{aligned}$$

Let us check the map is diffeomprhism. Both φ and φ^{-1} are C^∞ maps. Differentiability can fail in denominator which is smooth on suitable domains. For φ holds $1-||x||^2 > 0$ (the ball is open with unit radius, i.e. $||x||^2 < 1$) and for φ^{-1} holds $1+||y||^2 > 0$. Only trouble is the euclidean norm which is not differentiable at zero. We can evade this problem by altering the norm slightly. In sufficiently small neighbourhood of zero, let us change the square root function (occuring in $||-||$) by convenient increasing function. Using partition of unity we can ”glue” such function with the square root on the rest of domain. We will avoid further details in this rather intuitive explanation for the sake of not loosing the original idea. Hence, symboll $||-||$ will mean the altered norm and above defined φ will be understood with respect to this change. We check that composition of f and f^{-1} yields identity.

$$
\begin{aligned}
\left(f^{-1}\circ f\right)(x) &= f^{-1}\left(\frac{x}{(1-||x||^2)^{\frac{1}{2}}}\right) \\
&= \frac{\frac{x}{(1-||x||^2)^{\frac{1}{2}}}}{\left(1+||\frac{x}{(1-||x||^2)^{\frac{1}{2}}}||^2\right)^{\frac{1}{2}}} \\
&= \frac{\frac{x}{(1-||x||^2)^{\frac{1}{2}}}}{\left(1+\frac{||x||^2}{1-||x||^2}\right)^{\frac{1}{2}}} \\
&= x
\end{aligned}
\qquad
\begin{aligned}
\left(f\circ f^{-1}\right)(y) &= f\left(\frac{y}{(1+||y||^2)^{\frac{1}{2}}}\right) \\
&= \frac{\frac{y}{(1+||y||^2)^{\frac{1}{2}}}}{\left(1-||\frac{y}{(1+||y||^2)^{\frac{1}{2}}}||^2\right)^{\frac{1}{2}}} \\
&= \frac{\frac{y}{(1+||y||^2)^{\frac{1}{2}}}}{\left(1-\frac{||y||^2}{1+||y||^2}\right)^{\frac{1}{2}}} \\
&= y
\end{aligned}
$$

In the left column we used $1-||x||^2>0$ and similarly we used $1+||y||^2>0$ in the right column. So far we have diffeomorphism between $B(0,1)$ and $\mathbb{R}^n$. Example 1.36 shows that linear isomorphisms and translations are diffeomorphisms, thus, we can find a suitable shift and scaling to produce a diffeomorphism ϕ between the open ball centered at zero with radius r and the unit open ball $B(0,1)$, i.e.

$$\phi:\, B(a,r)\to B(0,1)$$

The desired diffeomorphism is $\varphi\circ\phi$

$$
\begin{tikzcd}
B(a,r) \arrow[rr, dashed, "\varphi\circ\phi"] \arrow[dr, "\phi"'] & & \mathbb{R}^n \\
& B(0,1) \arrow[ur, "\varphi"'] &
\end{tikzcd}
$$

◇

Submanifolds of real spaces

Natural generalization of curves or surfaces in $\mathbb{R}^n$ is the m-dimensional submanifold of $\mathbb{R}^n$.

Definition 2.1.

Subset $M \subset \mathbb{R}^n$ is called m-**dimensional submanifold of class** $C^r, m \leq n$, if for every point $x \in M$ there is a neighbourhood W of x together with diffeomorphism $\psi \colon W \to V \subset \mathbb{R}^n$ of class C^r which maps $W \cap M$ on open subset $U \subset V$ given by the equations

$$x^{m+1} = 0, \ldots, x^n = 0 \ . \tag{2.2}$$

Definition 2.3.

Restriction of diffeomorphism ψ from the definition of submanifold induces a mapping $\phi \colon W \cap M \to U$ called **local coordinate system** on submanifold M. Inverse of ϕ, considered as mapping $\phi^{-1} \colon U \to \mathbb{R}^n$, is called **local parametric description of submanifold** M.

- The previous definition implies that m-dimensional submanifold can be locally viewed as a piece of m-dimensional linear subspace in $\mathbb{R}^n$ which has been curved. The local information of a specific distortion is encoded in diffeomprhism ψ.

- Diffeomorphism ψ is telling us only locally how to ”flatten”, so to speak, the curved submanifold to linear subspace (and vice versa). This means that ”straightening” the whole submanifold might not be possible (i.e. from the definition of submanifold does not follow the existence of global diffeomorphism between the submanifold and a linear subspace. Examples of such cases would be cone, Möebius strip, sphere, Klein bottle, rotational paraboloid or a torus.

The following theorem enables us to describe submanifold using system of equations.

Theorem 2.4.

Let $f \colon U \to \mathbb{R}^{n-m}$ be a mapping of class $C^r, U \subset \mathbb{R}^n$ and $b \in \mathbb{R}^{n-m}$ a point. If f has rank $n-m$ in every point of set $f^{-1}(b)$ then $f^{-1}(b)$ is m-dimensional submanifold of class C^r.

Definition 2.5.

Let ϕ_1 and ϕ_2 be local coordinate systems on a submanifold M (given by restriction of diffeomorphisms as in 2.3) with domains W_1, W_2 respectively and let U_{12}, U_{21} be open subsets in $\mathbb{R}^m$ given by the image of intersection $W_1 \cap W_2 \cap M$ under ϕ_1, ϕ_2, i.e.

$$U_{12} = \phi_1 \left(W_1 \cap W_2 \cap M \right), \quad U_{21} = \phi_2 \left(W_1 \cap W_2 \cap M \right) \ .$$

The composed mapping $\phi_{12} := \phi_2 \circ \phi_1^{-1} \colon U_{12} \to U_{21}$ is called **transition map** between the pair of local coordinate systems (ϕ_1, ϕ_2).

Theorem 2.6.

Every transition map ϕ_{12} of a given pair of local coordinates (ϕ_1, ϕ_2) is diffeomprhism of class C^r between open subsets $U_{12} \subset \mathbb{R}^m$ and $U_{21} \subset \mathbb{R}^m$.

In the next steps we will extend the notion of differentiability of mapping between real spaces on the case of submanifolds. In order to do that we firstly define continuity.

Remark. Let us recall that $\mathbb{R}^n$ is a topological space. The topology is generated by open balls. It follows that a submanifold $M \subset \mathbb{R}^n$ is also a topological space with respect to the subspace topology.

Definition 2.7.

A mapping $f \colon M \to N$ between submanifolds is called **continuous** if it is a continuous mapping in the topological sense.

Definition 2.8.

Let $M \subset \mathbb{R}^n$, $N \subset \mathbb{R}^k$ be submanifolds of class C^r. We call $f \colon M \to N$ a **mapping of class** C^s, $s \leq r$ if for every $a \in M$ there is a neighbourhood $U \subset M$ of a, a neighbourhood $V \subset N$ of $f(a)$ satisfying $f(U) \subset V$ and local coordinate systems

$$\phi \colon U \to W_1 \subset \mathbb{R}^m, \ \psi \colon V \to W_2 \subset \mathbb{R}^l$$

such that the composition

$$\psi \circ f \circ \phi^{-1} \colon W_1 \to W_2$$

is of class C^s.

Remark. Definition of mapping of class C^s between submanifolds of class C^r, where necessarily $s \leq r$, is independent of the choice of local coordinates.

Theorem 2.9.

Let $M \subset \mathbb{R}^n$, $N \subset \mathbb{R}^k$, $P \subset \mathbb{R}^l$ be submanifolds of class C^r and let

$$f \colon M \to N, \quad g \colon N \to P$$

be mappings of class C^s, $s \leq r$. The composed mapping

$$g \circ f \colon M \to P$$

is also of class C^s.

We use letters x, y, z to denote variables in spaces of dimension less then 4.

Exercise 2.10.

Decide whether the following subsets are submanifolds of euclidean space. Determine their dimension and class.

1. Bernoulli lemniscate.
2. General vector space and affine subspace in $\mathbb{R}^n$.
3. Solution of $x^n = \sin\left(x^1 x^2 \dots x^n\right)$ and graph of arbitrary smooth function.
4. Subset in $\mathbb{R}^3$ given by $x^2 + y^2 + z^2 = r^2$, $x - y = 0$, $r > 0$.

Solution. Majority of solutions will be consequence of theorem 2.4, hence, let us describe how the theorem can be used. Mapping $f\colon U \to \mathbb{R}^{n-m}$, $U \subset \mathbb{R}^n$ from the theorem is given by $(n-m)$-tuple of functions

$$f^s\left(x^1, \dots, x^n\right), \quad s = 1, \dots, n-m$$

and a point $b \in \mathbb{R}^{n-m}$, i.e. $b = \left(b^1, \dots, b^{n-m}\right)$ is $(n-m)$-tuple of numbers. Therefore, the set $f^{-1}(b)$ is given by a system of $(n-m)$ equations

$$f^s\left(x^1, \dots, x^n\right) = b^s, \quad s = 1, \dots, n-m \; .$$

Next information we can read from the statement is that on a subset $f^{-1}(b)$ is the rank of f maximal. We defined rank of a mapping, $Rk_a f$, as a rank of the corresponding Jacobi matrix $\left(\frac{\partial f^s}{\partial x^i}\right)$. Due to dimension we know that the rank must be $n-m$.

1. Bernoulli lemniscate is a plane curve given by two points A_1, A_2 on the x axis in distance a from the origin. Point P of the lemniscate must satisfie $||PA_1|| \cdot ||PA_2|| = a^2$.
   Since the curve intersects itself at $(0,0)$ there cannot exists a neighbourhood $W \subset \mathbb{R}^2$ of the point such that it's intersection with lemniscate would be diffeomorphic with open line segment. The reason is that every diffeomorphism is a continuous bijection and the self intersecting curve cannot be image of an injective mapping. By theorem 2.4 there must be a point in which the rank of f will not be maximal. We will find this point. In cartesian coordinates, lemniscate can be described as

   $$\left(x^2 + y^2\right)^2 = 2a^2\left(x^2 - y^2\right), \quad a > 0$$

   and $a^2 > x^2$ is satisfied. Let us rewrite the equation to the form $f(x,y) = 0$

   $$2a^2x^2 - 2a^2y^2 - x^4 - 2x^2y^2 - y^4 = 0$$

   The Jacobian will have only one row since f has only one component.

   $$\left(\frac{\partial f^s}{\partial x^i}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(4a^2x - 4x^3 - 4xy^2, -4a^2y - 4x^2y - 4y^3\right) \; .$$

At least one column must be non-zero for the matrix to have maximal rank (i.e. 1) at every point of $\mathbb{R}^2$. So, we examine the conditions on variable implied by setting columns to zero. We begin with the second

$$0 = -4a^2y - 4x^2y - 4y^3 = -4y\left(a^2 + x^2 + y^2\right) \ .$$

Since we assume $a > 0$, the bracketed expression must be greater or equal to zero. Hence, $y = 0$, which we substitute into the first column

$$0 = 4a^2x - 4x^3 = 4x\left(a^2 - x^2\right)$$

and $a^2 > x^2$ implies $x = 0$. We conclude that f her zero rank at $(0,0)$.

2. From the definition of submanifold follows that for arbitrary n, $\mathbb{R}^n$ is submanifold of dimension n. It is thanks to existence of global standard coordinates $\left(x^1,\dots,x^n\right)$. In other words, to find diffeomorphism from definition 2.1, it is enough to take the identity map on the whole space. Let us proceed with case of general vector space V of dimension n. It is well known from course of linear algebra that two vector spaces of same dimension are isomorphic. Choice of base on V yields coordinates on V and uniquely determines a diffeomorphism with $\mathbb{R}^n$ (linear isomorphism given by transformation matrix from the given basis to the standard one). Therefore, vector space of dimension n is n-dimensional submanifold.
We can solve the case of affine subspace by determining how we can obtain vector subspace from affine one. A choice of origin in affine subspace gives a unique shifting vector as a difference between the chosen origin and origin of ambient vector space. Since every shift is diffeomorphism and composition of diffeomorphisms is difeomorphism, we conclude that every affine subspace is submanifold with dimension equal to dimension of it's underlying vector space. Moreover, it is smooth submanifold because we can describe it as a solution of linear equation $a_1x^1 + a_2x^2 + \dots a_nx^n + c = 0$, i.e. equation of the form $f\left(x^1,\dots,x^n\right) = 0$, where f is smooth.

3. We rewrite the equation $x^n = \sin\left(x^1x^2\dots x^n\right)$ to the form $f\left(x^1,\dots,x^n\right) = 0$

$$\sin\left(x^1x^2\dots x^n\right) - x^n = 0 \ .$$

For the sake of clarity, use the following substituion $\mathbf{z} = x^1x^2\cdot\dots\cdot x^n$ with partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x^1} &= x^2\dots x^n\cos\left(\mathbf{z}\right)\\
\frac{\partial f}{\partial x^2} &= x^1x^3\dots x^n\cos\left(\mathbf{z}\right)\\
&\vdots\\
\frac{\partial f}{\partial x^1} &= x^1\dots x^{n-1}\cos\left(\mathbf{z}\right) - 1
\end{aligned}$$

Jacobi matrix consisting of only one row is

$$\left(x^2 \dots x^n \cos(\mathbf{z}), x^1 x^3 \dots x^n \cos(\mathbf{z}), \dots\dots, x^1 x^2 \dots x^{n-1} \cos(\mathbf{z}) - 1\right) .$$

We will show that at least one column will be always non-zero. Let all but the last column be zero. Then either $x^1 = x^2 = \cdots = x^n = 0$ holds or $\cos(\mathbf{z}) = 0$ is true. Both cases yields value of -1 in the last column. Therefore, the given object is $(n-1)$-dimensional submanifold and is smooth, since $f\left(x^1, \dots, x^n\right) = \sin\left(x^1 x^2 \dots x^n\right) - x^n$ is smooth function.
For analogous reasons, graph of arbitrary function $f \in C^r$

$$\left\{\left(x^1, \dots, x^n, f\left(x^1, \dots, x^n\right)\right)\right\} \subset \mathbb{R}^{n+1}$$

is n-dimensional submanifold of class C^r. Equation of graph is

$$f\left(x^1, \dots, x^n\right) = x^{n+1} \Leftrightarrow F\left(x^1, \dots, x^{n+1}\right) = f - x^{n+1} = 0$$

and corresponding (one row) Jacobi matrix of F has partial derivatives in the first n rows and -1 in the last one. The very same argument we can use for $f \in C^\infty$, thus, graph of smooth function is smooth manifold of dimension n.

4. We will give two solutions to this problem. Eventhough the first one uses resuluts of later chapter we will describe the idea. Equation $x^2 + y^2 + z^2 = r^2$ describes 2-dimensional sphere centered at origin. Equation $x - y = 0$ describes plane containing line $y = x$ and the z axis. Set given as a solution of both equations is intersection of the plane and the sphere, circle of radius r centered at origin. In a proceeding chapter we will see that any circle is a smooth submanifold of dimension 1.
The second approach uses theorem 2.4. Function at hand is of the form

$$f(x,y,z) = \left(f_1(x,y,z), f_2(x,y,z)\right) = \left(x^2 + y^2 + z^2 - r^2, x - y\right)$$

and has 2×3 Jacobi matrix

$$\left(\frac{\partial f^s}{\partial x^i}\right) = \begin{pmatrix} 2x & 2y & 2z \\ 1 & -1 & 0 \end{pmatrix}$$

The second row is always non-zero and the first one is zero only at the origin which is, nevertheless, not contained in the sphere. Therefore, Jacobi matrix will have maximal rank on the given subset. Due to theorem 2.4, it is a smooth 2-dimensional submanifold. ◇

Exercise 2.11.

Show that if $M \subset \mathbb{R}^n$ is an open subset then M is a smooth submanifold of dimension n.

Solution. We have shown in exercise 2.10 that $\mathbb{R}^n$ is a smooth submanifold of dimension n. Let M be open subset in $\mathbb{R}^n$. According to 2.1 we need to find a neighbourhood around every $a \in M$ which is diffeomorphic with open subset in $\mathbb{R}^n$. But this is trivial, since M itself is given by points with inherited coordinate description from ambient $\mathbb{R}^n$. Thus, the neighbourhood is M itself and diffeomorphism is the identity map. ◇

Exercise 2.12.

Prove that the set of linear isomorphisms of $\mathbb{R}^n$ is smooth submanifold. Determine it's dimension.

Solution. Firstly we realize that linear isomorphisms of $\mathbb{R}^n$ corresponds to real matrices $n \times n$ with non-zero determinant. The space of all such matrices is denoted $\mathrm{GL}(n,\mathbb{R})$. Every $A \in \mathrm{GL}(n,\mathbb{R})$, $A = (a_{ij})$ consits of n^2 components a_{ij}. Hence, we have injective mapping

$$\mathrm{GL}(n,\mathbb{R}) \hookrightarrow \mathbb{R}^{n^2} ,$$

given by identifying A with n^2-tuple of real numbers

$$(a_{ij}) \mapsto (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots\ldots, a_{n1}, a_{n2}, \ldots, a_{nn}) . \tag{2.13}$$

Determinant, then, is a smooth function

$$\det\colon \mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n^2} \to \mathbb{R} .$$

Smoothness follows from formula for determinant known from linear algebra

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} ,$$

where S_n is the set of all permutations on n elements. The right-hand side is a polynomial in variables a_{ij} which is a smooth function. Thus, we can describe the set of linear isomorphisms as an inverse image of an open subset along a continous function

$$\mathrm{GL}(n,\mathbb{R}) = \det{}^{-1}(\mathbb{R} \setminus \{0\}) .$$

Therefore, $\mathrm{GL}(n,\mathbb{R})$ is open subset in $\mathbb{R}^{n^2}$ because inverse image of open subset along continuous fuction is open subset. According to 2.11 is the set of all linear isomorphism of $\mathbb{R}^n$ smooth submanifold of dimension n^2. ◇

Exercise 2.14.

Show that the set of all real matrices with determinant equal to 1 is smooth submanifold.

Solution. We will use similar idea as in exercise 2.12. We already know that the space of all $n \times n$ matricies can be injectively mapped to $\mathbb{R}^{n^2}$ using 2.13. Also, the determinant is smooth mapping from the space of matrices to real numbers. The set of all matrices with determinant 1, denoted $\mathrm{SL}(n,\mathbb{R})$, is inverse image of determinant map

$$\det\colon \mathrm{SL}(n,\mathbb{R}) \subset \mathbb{R}^{n^2} \to \mathbb{R} ,$$
$$\mathrm{SL}(n,\mathbb{R}) = \det{}^{-1}(1) .$$

We will show that Jacobi matrix of det has maximal rank on subset $\mathrm{SL}(n,\mathbb{R})$. It is enough to show that given $A \in \mathrm{SL}(n,\mathbb{R})$ there is a pair i, j satisfying

$$\frac{\partial \det(A)}{\partial a_{ij}} \neq 0 \ ,$$

i.e. Jacobi matrix will have at least one non-zero value and, therefore, maximal rank. Applying Laplace expansion on determinant of $A = (a_{ij})$ along i-th row is

$$\det(A) = (-1)^{i+1} a_{i1} M_{i1} + (-1)^{i+2} a_{i2} M_{i2} + \cdots + (-1)^{i+n} a_{in} M_{in}$$

where M_{ij} is the minor (or cofactor) correspoind to element a_{ij}. We proceed by contradiction. Suppose that the following holds

$$\frac{\partial \det(A)}{\partial a_{ij}} = (-1)^{i+j} M_{ij} = 0$$

for fixed i and arbitrary j, leading to

$$\frac{\partial \det(A)}{\partial a_{i1}} = M_{i1} = \cdots = \frac{\partial \det(A)}{\partial a_{in}} = M_{in} = 0 \ .$$

Having all minors for a given row equal to zero implies Laplace expansion along this row vanishes, implying $\det(A) = 0$. Nevertheless, we assumed $A \in \mathrm{SL}(n,\mathbb{R})$, hence the contradiction. We have shown that Jacobi matrix has maximal rank at arbitrary point A of subset $\mathrm{SL}(n,\mathbb{R})$. Therefore, by theorem 2.4, it is a smooth submanifold of dimension $n^2 - 1$. ◇

Exercise 2.15.

Decide whether a general regular convex polygon with n vertices, $n \geq 3$, is a submanifold of real plane.

Solution. We will solve this problem by showing that arbitrary part of polygon containing some vertex is not a submanifold, implying that the whole polygon cannot be a submanifold. So, consider some part of polygon containing vertex V and two adjacent sides h_1, h_2 with open ends. Recalling results from previous section, we know that rotation and shift applied on some object does not influence whether the object is submanifold or not. Thus, without loss of generality, let us suppose that the part h_1Vh_2 lies in $\mathbb{R}^2$ in a position in which the vertex V is identified with origin and h_1 lying on the x axis. Then, we can consider angle α from h_2 to h_1 to be from interval $(0,\pi) \setminus \{\frac{\pi}{2}\}$ (case $\alpha = \frac{\pi}{2}$ can be turned to the previous case using rotation). Polyline h_1Vh_2 is not a submanifold beacuse it is a graph of a function which cannot be differentiable at the origin, since the following reason. Prescription of this function, let us denote it f, would be defined picese-wisely by two linear functions with the break point at the origin: descripition of the first part would be dependent on α and the second one would be constantly zero. Precise form of this piece-wise linear f we do not need. What is important is the break point in which the function cannot be continuously differentiable, because α is, by assumption, non-zero (i.e. derivative from the left would differ from derivative from the right at $(0,0)$). If we would like to describe $h_1 V h_2 \subset \mathbb{R}^2$ as a submanifold with accordance to definition 2.1, we would come to the following contradiction. Two different desripitons of some neighbourhood of V, the first one given by function f and the second one arbitrary, let us denote it g, must differ by a smooth transition map ψ such that the composition $f \circ \psi \circ g^{-1}$ is smooth. Nevertheless, composition preserves the least of orders of differentiability which, because of f, would be zero. Thus, $h_1 V h_2$ cannot be submanifold and nor can be true for the general polygon containing it. ◇

Exercise 2.16.

Showh that given a map $f\colon M \to N$ of class C^r between submanifold of class C^s implies $r \leq s$.

Solution. Let $f\colon M \to N$ be a C^r map. For f to be of class C^r, due to **??**, there must be for each point $a \in M$ a neighbourhood $U \subset M$, a neihbourhood $V \subset N$ for $f(a)$ satisfying $f(U) \subset V$, and local coordinates ϕ on U and ψ on V such that the composition $\psi \circ f \circ \phi^{-1}$ is a C^r map. The situation can be depicted in the following diagram

$$\begin{tikzcd} U \arrow[r, "f"] & V \arrow[d, "\psi"] \\ \phi(U) \arrow[u, "\phi^{-1}"] \arrow[r, "\psi\circ f\circ\phi^{-1}"'] & \psi(V) \end{tikzcd}$$

The order of differentiability of composed map is equal to the least order of composed maps. Both ψ and ϕ in $\psi \circ f \circ \phi^{-1}$ are, by assumption, of class C^s due to 2.1. Therefore, $r \leq s$ holds. ◇

Exercise 2.17.

Decide whether a subset of $\mathbb{R}^2$, given by two circles (with non-zero radius) with one point intersection, is a submanifold.

Solution. We can solve the problem using topological argument. From the definiton of submanifold it follows that for each point of submanifold there is a suitable vicinity of the point together with a diffeomorphism f on simply connected subset of $\mathbb{R}^n$. It is a basic fact in topology that simply connected subset of $\mathbb{R}^n$ can be homeomorphicaly mapped on the whole $\mathbb{R}^n$. Since every diffeomorphism is homeomorphism, every submanifold of euclidean space must be locally homeomorphic to $\mathbb{R}^n$. Let M be the set given by two circles intersecting at a single point x. Suppose that M is a submanifold of $\mathbb{R}^2$ and consider an arbitrary neighbourhood U of x. From topology we know that the space given by $\mathbb{R}^2$ from which we remove a point has only one connected component. Removing the point x from U gives four connected components. Hence the contradiction, since $\mathbb{R}^2$ without (arbitrary) point should be homeomorphic to U without x. At the same time, the number of components is invariant under homeomorphisms, thus, the situation cannot occure and M cannot be submanifold of $\mathbb{R}^2$. ◇

Smooth manifolds and smooth mappings

Definition 3.1.

n-dimensional **topological manifold** is a separable topological space M with countable basis, which is locally homeomorphic to $\mathbb{R}^n$. This means that $\forall x \in M$ there exists a neighborhood U, open set $V \subset \mathbb{R}^n$ and homeomorhphism $\varphi : U \to V$.

Definition 3.2.

Every homeomorphism $\varphi : U \to V$, where $U \in M$ and $V \in \mathbb{R}^n$ are open sets is called **local chart** on M. Coordinates $\varphi(a)$, where $a \in U$ are called coordinates of a in φ If $0 \in V$, then $\varphi^{-1}(0)$ is called a **center of the chart** φ.

Definition 3.3.

Let us assume M and two charts φ_1 and φ_2 on sets U_1 and U_2. Induced mapping $\varphi_{12} = \varphi_2 \circ \left(\varphi_1^{-1}\big|_{V_{12}} \right) : V_{12} \to V_{21}$ is called **transition mapping** of φ_1 and φ_2. This is sometimes referred to as a **change of coordinates** on the overlap.

Definition 3.4.

Two maps $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ on a topological manifold M are **$\mathbf{C^r}$-related** if transition mapping φ_{12} is C^r-diffeomorphism.

Definition 3.5.

C^r **atlas** $\mathscr{A}$ on a topological manifold M is a set of C^r-related charts $\varphi_\alpha : U_\alpha \to V_\alpha$, such that their domains U_α cover M. This atlas is also called **differential structure**.

Remark. We can consider an atlas on $\mathbb{R}^n$ where $\mathrm{id}_{\mathbb{R}^n}$ is not compatible with it. Such an atlas is called a **non-trivial differential structure**.

Definition 3.6.

Map $\phi_0 : U_0 \to V_0$ is **compatible** with a C^r atlas $\mathscr{A}$ if every transition map $\varphi_{0\alpha} : V_{0\alpha} \to V_{\alpha 0}$ is C^r-diffeomorphism.

Definition 3.7.

Atlas $\mathscr{A}$ on a topological manifold M is called **complete** if it contains all compatible charts.

Theorem 3.8.

Let $\mathscr{A}$ be an arbitrary C^r atlas on M. If we add all compatible charts, we obtain a complete atlas.

Definition 3.9.

Continuous mapping $f : M \to N$ is of a class C^s, if for every mapping $\phi : U \to V$ compatible with $\mathscr{A}$ and for every chart $\psi : W \to Z$ compatible with $\mathscr{B}$, such that $f(U) \in W$, the induced mapping $\psi \circ \varphi^{-1} : V \to Z$ is of class C^s

Definition 3.10.

Differentiable manifold of class C^r is a topological manifold with C^r atlas $\mathscr{A}$.

Theorem 3.11.

Continuous mapping $f : M \to N$ between two C^r manifolds is of a class C^s $((s \leq r)$ if for every $x \in M$ there exist local charts φ on M and ψ on N such that the induced mapping $\varphi^{-1} \circ f \circ \psi$ is of class C^s in a neighborhood of x.

Theorem 3.12.

Let M, N and P be C^r manifolds and $f : M \to N$, $g : N \to P$ are C^s mappings. Then the composition $g \circ f : M \to P$ is also C^s.

Definition 3.13.

C^s **diffeomorphism** of two $n-$dimensional C^r manifolds M_1 and M_2 is a bijective C^s mapping $f : M_1 \to M_2$, such that inverse mapping $f^{-1} : M_2 \to M_1$ is also C^s.

Definition 3.14.

Product of C^r manifolds M and N given by atlases $\mathscr{A}$ and $\mathscr{B}$ is a C^r manifold on direct product of topological manifolds $M \times N$ with differential structure $\mathscr{A} \times \mathscr{B}$.

Remark. Even if $\mathscr{A}$ and $\mathscr{B}$ are complete, atlas $\mathscr{A} \times \mathscr{B}$ is not.

Definition 3.15.

Let us assume $n-$dimensional C^r manifold M. We say that subset $N \subset M$ is $k-$dimensional C^s **submanifold** if N with atlas obtained by a restriction from atlas of M is $k-$dimensional C^s manifold.

Remark. We say that manifold or mapping is smooth if it's C^∞. From now on, we will omit the word smooth.

Exercise 3.16.

Are following mappings local charts?

1. $f : \mathbb{R} \to \mathbb{R}$, $f(t) = t^{2k}$, $k \in \mathbb{N}$
2. $f : \mathbb{R} \to \mathbb{R}$, $f(t) = t^{2k-1}$, $k \in \mathbb{N}$
3. $f : (-a,a) \to \mathbb{R}$, $f(t) = tan(\frac{\pi t}{2a})$, $a \in \mathbb{R}$

Solution.

1. First we can note that this mapping is not injective. The image of this mapping is an interval $[0,\infty)$. This set is not even open in $\mathbb{R}$! Furthermore, images of open sets containing 0 are not open, $f((-a,a)) = \left[0, a^{2k}\right)$
2. These mappings are local charts. They are injective and every open set is mapped into another open set.
3. This example show us that $\mathbb{R}$ is $C^\infty-$diffeomorphic to an open interval $(-a,a)$. Mapping is injective and image of any open set is open set again.

$\diamond$

Exercise 3.17.

Can one construct an atlas on the sphere S^n with only one chart?

Solution. If one could construct such a chart $\varphi : S^n \to \mathbb{R}^n$ it would have to be a homeomorphism onto an open subset of $\mathbb{R}^n$. Since S^n is compact, $\varphi(S^n)$ would be a closed, as well as open set of $\mathbb{R}^n$, hence it would be $\mathbb{R}^n$. This is absurd, because $\mathbb{R}^n$ is not compact. Therefore one can not construct an atlas on S^n with only one chart. $\diamond$

Exercise 3.18.

Consider open subsets U and V of the unit circle S^1 in $\mathbb{R}^2$ given by

$$U = \{(\cos\alpha, \sin\alpha) : \alpha \in (0, 2\pi)\}$$

$$V = \{(\cos\alpha, \sin\alpha) : \alpha \in (-\pi, \pi)\}$$

Prove that $\mathscr{A} = \{(U,\varphi), (V,\psi)\}$, where

$$\varphi : U \to \mathbb{R},\ \varphi(\cos\alpha, \sin\alpha) = \alpha$$

$$\psi : V \to \mathbb{R},\ \psi(\cos\alpha, \sin\alpha) = \alpha$$

is an atlas on S^1.

Solution. One has $U \cup V = S^1$, as one can see from the figure. The maps φ and ψ are homeomorphisms onto $(0, 2\pi)$ and $(-\pi, \pi)$ respectively, hence (U, φ) and (V, ψ) are local charts on S^1. The transition map $\psi \circ \varphi^{-1}$ is given by

$$\alpha \to (\cos\alpha, \sin\alpha) \to \begin{cases} \alpha, & \alpha \in (0, \pi) \\ \alpha - 2\pi, & \alpha \in (\pi, 2\pi) \end{cases}$$

This is a diffeomorphism on $U \cap V$, thus $\mathscr{A}$ is an atlas on S^1.

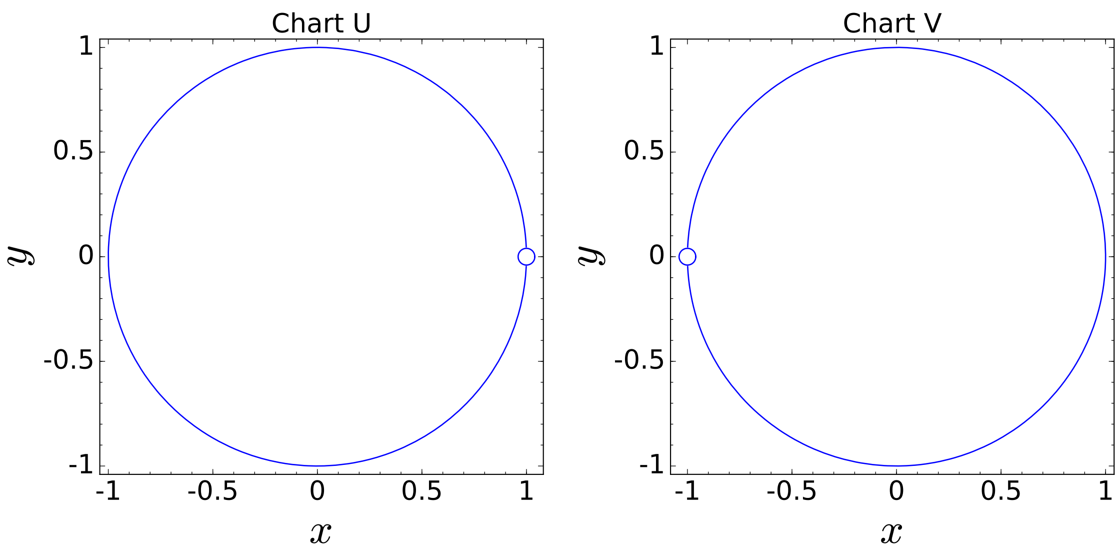

Figure 3.1: Charts U and V on S^1

◇

Exercise 3.19.

Define an atlas on the cylindrical surface

$$M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2, 0 < z < h\right\}$$

where $h, r \in \mathbb{R}^+$

Solution. We only need to endow the circles $S^1_r = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\right\}$ with an atlas. However we've done this in a previous exercise. Then $U \times (0, h)$, $V \times (0, h)$ are open subsets in M and we define an atlas on M by

$$\mathscr{A} = \{(U \times (0, h)\,; \varphi \times \mathrm{id})\,, (V \times (0, h)\,; \psi \times \mathrm{id})\}$$

The transition map between the charts is

$$(\psi \times \mathrm{id}) \circ (\varphi \times \mathrm{id})^{-1} = \left(\psi \circ \varphi^{-1}\right) \times \mathrm{id}$$

which is $C^\infty-$diffeomorphism and therefore $\mathscr{A}$ is an atlas on M. ◇

Remark. This procedure can be extended to a product of arbitrary manifolds $M = M_1 \times M_2$.

Exercise 3.20.

For each positive real number r, consider the map $\phi_r : \mathbb{R} \to \mathbb{R}$, where $\phi_r(t) = t$ if $t \leq 0$ and $\phi_r(t) = rt$ if $t > 0$. Prove that the atlases $\{(\mathbb{R}, \phi_r)\}$ define an uncountable set of differential structures on $\mathbb{R}$. Are the respective differentiable manifolds diffeomorphic?

Solution. ϕ_r is a homeomorphism for every positive real r, therefore it's an atlas. To check whether differential structure are equivalent, we have to verify for which r and s are mappings ϕ_r and ϕ_s C^r-related. To do this, we have to calculate $\phi_r \circ \phi_s^{-1}$, which is

$$\left(\phi_r \circ \phi_s^{-1}\right)(t) = \begin{cases} t, & t \leq 0 \\ \frac{r}{s}t, & t > 0 \end{cases}$$

These functions are discontinuous if $r \neq s$, therefore the differential structures are different. However, let us assume a mapping

$$\phi : \mathbb{R}_r \to \mathbb{R}_s = \begin{cases} t \to t, & t \leq 0 \\ t \to \frac{r}{s}t, & t > 0 \end{cases}$$

This mapping is a diffeomorphism, because $\phi_s \circ \phi \phi_r^{-1}$ is an identity map, which is C^∞. $\diamond$

Tangent bundles, tangent mappings

Definition 4.1.

Path on M is smooth mapping $f : I \to M$, where $I \in \mathbb{R}$ is an open interval.

Remark. Path is sometimes called **smooth motion**, because it does not contain only a "trajectory", but also "motion" that created it. If $f(I) \in M$ is a curve and f is it's parametrization, we have **parametrized curve** on M.

Remark. If M is an open subset $U \subset \mathbb{R}^n$, then path $f(t) : I \to U$ has tangent vector $\frac{\mathrm{d}f^i(t_0))}{\mathrm{dt}}$ at every $t_0 \in I$. This vector can be understood as a vector fixed at a point $f(t_0)$.

For simplicity, from now on we assume that $0 \in I$.

Definition 4.2.

Choose a point $a \in U$ and assume a path $f_b = a + tb$, which for sufficiently small t lies in U. Then $\frac{\mathrm{d}f(0)}{\mathrm{d}t} = b$. The pair (a, b) is called a **tangent vector** to U at a. Set of all such vectors is called a **tangent space** of U at a and is denoted $T_aU \cong \mathbb{R}^n$. Union of tangent spaces at all points $TU = \bigcup_{a \in U} T_aU \cong U \times \mathbb{R}^n$ is called **tangent bundle**.

Remark. We will also use f' to denote $\frac{\mathrm{d}f}{\mathrm{d}t}$.

Remark. Definition of tangent space and tangent bundle is more abstract in case of general topological manifold M.

Remark. Function $h : I \to \mathbb{R}$ has first order zero at $t_0 \in I$ if

$$h(t_0) = \frac{\mathrm{d}h(t_0)}{\mathrm{d}t} = 0$$

Definition 4.3.

Two paths $f, g : I \to M$ are in **contact** at t_0 if $f(t_0) = g(t_0)$ and for every smooth function $\varphi : M \to \mathbb{R}$, induced function $\varphi \circ f - \varphi \circ g : I \to \mathbb{R}$ has first order zero at t_0.

Theorem 4.4.

Two paths $f, g : I \to M$ are in contact at 0 if $f(0) = g(0) = a$ and there exists coordinate system x^i at some neighborhood of a, such that

$$\frac{\mathrm{d}f^i(0)}{\mathrm{d}t} = \frac{\mathrm{d}g^i(0)}{\mathrm{dt}}$$

This allows us to define tangent vectors as equivalence class of paths which are in contact.

Definition 4.5.

Equivalence class A of paths $f(t)$ on M with $f(0) = a$ with a first order contact at 0 is called a **tangent vector** of M at a. We denote it $A = \frac{\mathrm{d}f(0)}{\mathrm{d}t}$. Set of all such equivalence classes is **tangent space** T_aM.

Definition 4.6.

Derivative of function φ in direction of A is given by

$$A\varphi = \frac{\mathrm{d}\,(\varphi \circ f)\,(0)}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial \varphi(a)}{\partial x^i} \frac{\mathrm{d}f^i(0)}{\mathrm{d}t}$$

Definition 4.7.

Let us assume functions $\varphi, \psi : M \to \mathbb{R}$, for which $\varphi(a) = \psi(a)$ and $A\varphi = A\psi$ for every $A \in T_aM$.Then these functions has coinciding **differentials**. We will denote this equivalence class as $\mathrm{d}_a\varphi$.

Remark. If two functions have coinciding differentials, following equalities hold

- $\varphi(a) = \psi(a)$
- $\frac{\partial \varphi(a)}{\partial x^i} = \frac{\partial \psi(a)}{\partial x^i}$

Theorem 4.8.

Space of all differentials T_a^*M is $n-$dimensional vector space, because following identities hold

- $(\mathrm{d}_a\varphi) + (\mathrm{d}_a\psi) = \mathrm{d}_a\,(\varphi + \psi)$
- k $(\mathrm{d}_a\varphi) = \mathrm{d}_a\,(k\varphi)$

This space is called **cotangent space** of M at a. Elements of this space are called **covectors**.

Theorem 4.9.

Tangent vectors at a coincide with linear maps $T_a^*M \to \mathbb{R}$.

Definition 4.10.

Let f be a map to a different manifold N and $A = \frac{\mathrm{d}h(0)}{\mathrm{d}t} \in T_aM$, where h is some path at M. Then $f \circ h : I \to N$ is a path at N and tangent vector $\frac{\mathrm{d}f \circ h(t_0)}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial f(a)}{\partial x^i} \frac{\mathrm{d}h^i(0)}{\mathrm{d}t}$ depends only on A. Therefore the map $T_af : T_aM \to T_{f(a)}N$ is linear. This map is called a **tangent map** f at a. Map $Tf : TM \to TN$, with $Tf = \bigcup_{a \in M} T_af$ is called tangent map f. Tangent map is also denoted f_*.

Theorem 4.11.

For composition of maps, following identity holds

$$T(g \circ f) = (Tg) \circ (Tf)$$

Theorem 4.12.

If $f : M \to N$ is smooth map, so is $Tf : TM \to TN$.

Theorem 4.13.

If M is $m-$dimensional submanifold in $\mathbb{R}^n$, then TM is $2m-$dimensional submanifold in $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

Definition 4.14.

Rank of map f at a point a, denoted by $Rk_a f$ is a rank of linear map $T_a f$.

Exercise 4.15.

Consider the map $f : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \to x^3 + xy + y^3 + 1$

1. Compute the tangent map $f_* : T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$
2. At which points is f_* injective?
3. At which points is f_* surjective?

Solution.

1. First we evaluate the tangent map of the basis vectors

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_p\right) = \frac{\partial f(p)}{\partial x}\left.\frac{\partial}{\partial t}\right|_{f(p)} = \left(3x^2 + y\right)\left.\frac{\partial}{\partial t}\right|_{f(p)}$$
$$f_*\left(\left.\frac{\partial}{\partial y}\right|_p\right) = \frac{\partial f(p)}{\partial y}\left.\frac{\partial}{\partial t}\right|_{f(p)} = \left(3y^2 + x\right)\left.\frac{\partial}{\partial t}\right|_{f(p)}$$

   Because $f*$ is linear map, this suffices to determine its action on arbitrary $v \in T_p\mathbb{R}^2$. The image of a vector $a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} \in T_pM$ is

$$f_*\left(a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y}\right) = \left[a_x\left(3x^2 + y\right) + a_y\left(3y^2 + x\right)\right]\left.\frac{\partial}{\partial t}\right|_{f(p)}$$

2. f_* cannot be injective since $\dim\mathbb{R}^2 > \dim\mathbb{R}$.

3. If f_* is not surjective at a point p, image of any vector will be mapped to zero. Set of such a points is given by

$$P = \left\{(x,y) \in \mathbb{R}^2 : x + 3y^2 = 0, 3x^2 + y = 0\right\}$$

After solving this set of equations we get $P = \left\{\left(-\frac{1}{3}, -\frac{1}{3}\right), (0,0)\right\}$. Therefore, f_* is surjective on $\mathbb{R}^2/P$

◇

Exercise 4.16.

Let $g : \mathbb{R}^2 \to \mathbb{R}^3$, $(x,y) \to \left(x^2y + y^2, x - 2y^3, ye^x\right)$

1. Compute $g_{*(x,y)}$.
2. Find $g_*\left(4\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)_{(0,1)}$.
3. Find conditions that α_x, α_y, α_z must satisfy for the vector $\left(\alpha_x\frac{\partial}{\partial x} + \alpha_y\frac{\partial}{\partial y} + \alpha_z\frac{\partial}{\partial z}\right)_{g(0,0)}$ to be the image of some vector by g_*.

Solution. 1. As in previous exercise, we evaluate the tangent map on basis vectors, which allows us to find the matrix of linear mapping. In this case the matrix is

$$g_{*(x,y)} = \begin{pmatrix} 2xy & x^2 + 2y \\ 1 & -6y^2 \\ ye^x & e^x \end{pmatrix}$$

2. Let us now evaluate the tangent mapping on $4\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$

$$g_*\left(4\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)_{(x,y)} = \begin{pmatrix} 2xy & x^2 + 2y \\ 1 & -6y^2 \\ ye^x & e^x \end{pmatrix}_{g(x,y)} \begin{pmatrix} 4 \\ -1 \end{pmatrix}_{(x,y)} = \begin{pmatrix} 8xy - x^2 - 2y \\ 4 + 6y^2 \\ (4y-1)\,e^x \end{pmatrix}_{g(x,y)}$$

3. The matrix of a tangent map at $(0,0)$ is

$$g_{*(0,0)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence we obtain condition $\alpha_x = 0$.

◇

Exercise 4.17.

Consider the curve σ in $\mathbb{R}^2$ defined by $x = \cos t$, $y = \sin t$, $t \in (0,\pi)$ and the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 2x + y^3$. Find the vector v tangent to σ at $\pi/4$ and calculate vf.

Solution. We have $\sigma'(t) = (-\sin t, \cos t)$, thus

$$\sigma'(\pi/4) = \left(-\frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\right)_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)}$$

Hence

$$\sigma'(\pi/4) f = \left(-\frac{\sqrt{2}}{2}\frac{\partial f}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial f}{\partial y}\right)_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)} = -\frac{\sqrt{2}}{4}$$

◇

Exercise 4.18.

Consider the path in $\mathbb{R}^2$ given by $\sigma(t) = (x(t), y(t)) = \left(t^2 - 1, t^3 - t\right)$. Calculate $\sigma(t)$ and $\sigma'(t)$ for $t = \pm 1$. Compare the values.

Solution. We have

$$\sigma(1) = \sigma(-1) = (0,0)$$

For the derivative we have $\sigma'(t) = \left(2t, 3t^2 - 1\right)$. This gives us

$$\sigma'(1) = (2,2)$$
$$\sigma'(-1) = (-2,2)$$

The setting can be seen on a following figure.

◇

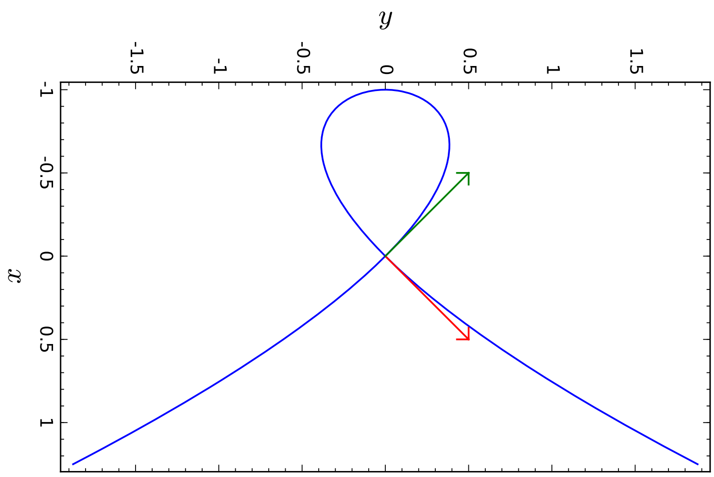

Figure 4.1: Path $\sigma(t)$ together with tangent vectors at $t = \pm 1$.

Vector fields

Definition 5.1.

Vector field on M is a smooth mapping $X : M \to TM$, such that $p \circ X = \mathrm{id}_M$

Definition 5.2.

Derivative in direction of a vector field X of a smooth function $f : M \to \mathbb{R}$ is $Xf : M \to \mathbb{R}$ given by $(Xa)(a) = X(a) f$, where $X(a)$ is a derivation in direction of a vector $X(a)$. In local coordinates we have

$$Xf = \sum_{i=1}^{n} X^i(a) \frac{\partial f}{\partial x^i}$$

Theorem 5.3.

For every pair of vector fields X, Y on M exists one and only one vector field $[X,Y]$ on M, such that for every function f on M, following identity holds

$$[X,Y] f = X(Yf) - Y(Xf)$$

Such vector field is called **Lie bracket** of X, Y. This vector field has following coordinate expression

$$[X,Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

Theorem 5.4.

For every $k, l \in \mathbb{R}$ and every vector fields X, Y, Z and for every function f on M, following identities hold

$$[kX + lY, Z] = k[X,Z] + l[Y,Z]$$

$$[X,Y] = -[Y,X]$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

$$[fX,Y] = f[X,Y] - (Yf)X$$

Definition 5.5.

Vector fields $X : M \to TM$ and $Y : N \to TN$ are $f-$**related** if $(Tf) \circ X = Y \circ f$.

Lemma 5.6.

Vector fields X and Y are $f-$related if and only if for every function $h : N \to \mathbb{R}$, following identity holds

$$X(h \circ f) = (Yh) \circ f$$

Theorem 5.7.

If X_1, X_2 are vector fields on M and Y_1, Y_2 are vector fields on N such that X_1, Y_1 and X_2, Y_2 are $f-$related, brackets $[X_1, X_2]$ and $[Y_1, Y_2]$ are $f-$related as well.

Definition 5.8.

We say that vector field X is **tangent to submanifold** N if $X(x) \in T_x N$ for every $x \in N$.

Theorem 5.9.

If vector fields X, Y are tangent to N, so is $[X, Y]$.

Definition 5.10.

Path $f : I \to M$ is **integral curve** of a vector field X iff

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = X(f(t))$$

Remark. In local coordinates, we have $X = \sum_{i=1}^{n} X^i(x) \frac{\partial}{\partial x^i}$, therefore every integral curve of X is solution of a system of differential equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i\left(x^1, \ldots, x^n\right)$$

Note that right-hand side is independent of t.

Definition 5.11.

For every $x \in M$ exists one and only one **maximal integral curve** $f_x : \mathbb{R} \supset I_x \to M$ such that $f_x(0) = x$. Maximal means that I_x can't be extended anymore. Due to existence theorem for differential equations we know that set

$$\mathbb{R} \times M \supset \mathscr{D}X := \bigcup_{x \in M} I_x \times \{x\}$$

is open and we can define the **flow of a vector field** X as a smooth map

$$Fl^X : \mathscr{D}X \to M, \quad Fl^X(t, x) = Fl_t^X(x) = f_x(t)\ .$$

Definition 5.12.

If vector field X is **complete**, following identity holds for every $t, x \in \mathbb{R}$

$$Fl^X_{t+s} = Fl^X_t \circ Fl^X_s$$

Remark. If X is not complete, previous theorem still holds for small enough $t, s \in \mathbb{R}$.

Definition 5.13.

Support of a vector field X is closure of a set of points where X is nonzero.

Theorem 5.14.

Every vector field X with compact support K is complete on any manifold M.

Definition 5.15.

$k-$dimensional **distribution** S on M is an assignment of a $k-$dimensional linear subspace $S(x) \subset T_xM$ at every $x \in M$.

Remark. Vector field $X \in S$ if $X(x) \in S(x)$ for every $x \in M$.

Definition 5.16.

Distribution S is **smooth** if for every $x \in M$ exists a neighborhood U together with k smooth vector fields $X_1, ..., X_k$, such that vectors $X_1(x), ..., X_k(x)$ form a basis of $S(x)$ for every $x \in U$.

Remark. From now on, we will assume smooth distributions.

Definition 5.17.

$k-$dimensional submanifold $N \subset M$ is called an **integral manifold** of distribution S if $T_xN = S(x)$ for every $x \in N$.

Definition 5.18.

Distribution S is **integrable** if for every $x \in M$ there exist an integral submanifold of S containing x.

Definition 5.19.

Distribution S is **involutive** if for every X_1, X_2 defined on $U \subset M$ that belong to S, also their Lie bracket $[X_1, X_2]$ belongs to S.

Theorem 5.20.

Distribution is integrable if and only if it is involutive.

Theorem 5.21.

Let S be an involutive distribution. Then for every $x \in M$ exists local coordinate system $y^1, \dots, y^n$ in some neighborhood U, such that $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}$ form a basis of S on U.

Exercise 5.22.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the C^∞ function defined by $f(x,y,z) = x^2 + y^2 - 1$, which defines a differentiable structure on $C = f^{-1}(0)$. Consider the vector fields on $\mathbb{R}^3$

1. $X = \left(x^2 - 1\right) \frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$
2. $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2xz^2\frac{\partial}{\partial z}$

Are they tangent to C?

Solution. All we need to do is calculate the derivative of a function f in the direction of a given vector field in every point of C. If the derivative is zero, the value of function does not change in the direction of X and therefore the implicit equation $f(x,y,z) = 0$ will still hold.

1.
$$\begin{aligned} Xf &= \left(x^2 - 1\right)\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + xz\frac{\partial f}{\partial z} = \\ &= 2x\left(x^2 + y^2 - 1\right) \end{aligned}$$

   We can clearly see that Xf is zero when restricted to C, therefore X is tangent to C.

2.
$$\begin{aligned} Yf &= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2xz^2\frac{\partial f}{\partial z} = \\ &= 2\left(x^2 + y^2\right) \end{aligned}$$

   This vector field is not tangent, because $Yf|_C = 2$.

◇

Exercise 5.23.

Consider the vector fields

$$X = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}, Y = y\frac{\partial}{\partial y}$$

on $\mathbb{R}^3$ and the map $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2y$. Calculate:

1. Is this distribution involutive?
2. $[X,Y]_{(1,1,0)}$
3. $(fX)_{(1,1,0)}$
4. $(Xf)(1,1,0)$
5. $f_*\left(X_{(1,1,0)}\right)$

Solution. 1. First, we calculate the Lie bracket of vector fields. In practice, it's very useful to calculate the action of a Lie bracket on some test function f using following identity

$$[X,Y]f = X(Yf) - Y(Xf)$$

In our case, we get

$$\begin{aligned}
[X,Y]f &= \left(xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial y}\right) - \left(y\frac{\partial}{\partial y}\right)\left(xy\frac{\partial f}{\partial x} + x^2\frac{\partial f}{\partial z}\right) = \\
&= xy^2\frac{\partial^2 f}{\partial x\partial y} + x^2y\frac{\partial^2 f}{\partial z\partial y} - xy\frac{\partial f}{\partial x} - xy^2\frac{\partial^2 f}{\partial x\partial y} - x^2y\frac{\partial^2 f}{\partial z\partial y} = -xy\frac{\partial f}{\partial x}
\end{aligned}$$

We can clearly see that this distribution is not involutive. We can construct basis

$$([X,Y], Y, [X,Y]+X) = \left(-xy\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial z}\right)$$

in which all the vectors are linearly independent.

2. To calculate the value of a Lie bracket, we just evaluate it in the point $(1,1,0)$, which gives us

$$[X,Y]_{(1,1,0)} = -\left.\frac{\partial}{\partial x}\right|_{(1,1,0)}$$

3. $(fX)_{(1,1,0)} = f(1,1,0)X_{(1,1,0)} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)_{(1,1,0)}$

4. $(Xf)(1,1,0) = \left(xy\frac{\partial f}{\partial x} + x^2\frac{\partial f}{\partial z}\right)_{(1,1,0)} = \left(\frac{\partial f}{\partial x}\right)(1,1,0) = 2$

5. $f_*\left(X_{(1,1,0)}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\begin{pmatrix}1\\0\\1\end{pmatrix} = 2\left.\frac{\partial}{\partial t}\right|_1$

where t denotes the canonical coordinate on $\mathbb{R}$.

◇

Exercise 5.24.

Let us assume a vector field $X = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}$. How would this vector field look like at $\mathbb{R}^3$ with

1. cylindrical coordinates (r, ϕ, z) given by

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}$$

2. spherical coordinates (r, ϕ, θ) given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Solution.

1. First, we have to calculate the Jacobian of the transformation

$$J = \begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The field in cylindrical coordinates is given by

$$X = f_1(r, \phi, z)\frac{\partial}{\partial r} + f_2(r, \phi, z)\frac{\partial}{\partial \phi} + f_3(r, \phi, z)\frac{\partial}{\partial z}$$

Therefore

$$\begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

This gives us system of equations for components f_i

$$\begin{aligned} f_1\cos\phi - f_2 r\sin\phi &= 2 \\ f_1\sin\phi + f_2 r\cos\phi &= -1 \\ f_3 &= 3 \end{aligned}$$

This system can be easily solved and the solution gives us following expression for X in cylindrical coordinates

$$X = (2\cos\phi - \sin\phi)\frac{\partial}{\partial r} - \frac{2\sin\phi + \cos\phi}{r}\frac{\partial}{\partial \phi} + 3\frac{\partial}{\partial z}$$

2. Just as in previous exercise, we start with calculating Jacobian

$$J = \begin{pmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\cos\phi \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix}$$

In spherical coordinates is our vector field given by

$$X = f_1(r,\phi,\theta)\frac{\partial}{\partial r} + f_2(r,\phi,\theta)\frac{\partial}{\partial \phi} + f_3(r,\phi,\theta)\frac{\partial}{\partial \theta}$$

and thus we have to solve following system of equations

$$\begin{pmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\cos\phi \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

After solving the system, we find that the components of a vector field are

$$\begin{aligned} f_1 &= (2\cos\phi - \sin\phi)\sin\theta + 3\cos\theta \\ f_2 &= \frac{(2\cos\phi - \sin\phi)\cos\theta - 3\sin\theta}{r} \\ f_3 &= \frac{2(\sin\phi - \cos\phi)}{r\sin\theta} \end{aligned}$$

◇

Exercise 5.25.

For each vector field find its integral curves and tell if it's complete or not

1. $X = \frac{\partial}{\partial y} + e^x \frac{\partial}{\partial z}$
2. $X = e^{-x} \frac{\partial}{\partial x}$

Solution. 1. To find integral curves, we have to solve following set of differential equations

$$\begin{aligned} x'(t) &= 0 \\ y'(t) &= 1 \\ z'(t) &= e^{x(t)} \end{aligned}$$

The solution passing through the point (x_0, y_0, z_0) is

$$\begin{aligned} x'(t) &= x_0 \\ y'(t) &= y_0 + t \\ z'(t) &= e^{x_0}t + z_0 \end{aligned}$$

This is defined for every $t \in \mathbb{R}$ and thus is complete

2. Differential equation for an integral curve of this vector field is

$$\mathrm{e}^{x(t)}x'(t) = 1$$

therefore

$$\mathrm{e}^{x(t)} = t + C$$

The integral curve passing through x_0 is

$$x(t) = \log\left(t + e^{x_0}\right)$$

This vector field is not complete, it's defined only for $t > -\mathrm{e}^{x_0}$.

◇

Exercise 5.26.

Consider the distribution $\mathscr{D}$ on $\mathbb{R}^3$ determined by

$$X = \frac{\partial}{\partial x} + \frac{2xz}{1+x^2+y^2}\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y} + \frac{2yz}{1+x^2+y^2}\frac{\partial}{\partial z}$$

1. Is $\mathscr{D}$ involutive?
2. Calculate the local flows of X and Y.
3. If $\mathscr{D}$ is involutive, find its integral surface.

Solution. 1. To find whether the distribution is involutive, we have to calculate Lie brackets between basis vector fields. In our case we have $[X,Y] = 0 \in \mathrm{span}(X,Y)$ and therefore $\mathscr{D}$ is involutive.

2. First we calculate the local flow of X. We immediately have $x = x_0 + t$ and $y = y_0$, all that remains is z, for which we have

$$\frac{z'}{z} = \frac{2(x_0+t)}{1+(x_0+t)^2+y_0^2}$$

The solution to this equation passing through (x_0, y_0, z_0) is

$$z = z_0\frac{1+(x+t)^2+y^2}{1+x_0^2+y_0^2}$$

This gives us local flow

$$\phi_t(x,y,z) = \left(x+t, y, z\frac{1+(x+t)^2+y^2}{1+x^2+y^2}\right)$$

Local flow for Y is obtained in similar way, the most helpful is to note that if we switch x and y we get the same equations. The flow therefore is

$$\psi_s(x,y,z) = \left(x, y+s, z\frac{1+x^2+(y+s)^2}{1+x^2+y^2}\right)$$

3. There are two ways to obtain integral manifold of a distribution. First way is to simply compose flows, which will give us $\psi(t,s) = (\psi_s \circ \phi_t)(x_0, y_0, z_0)$. This result is, however, not very interesting. Better way is to look for a covector that annihilates $\mathscr{D}$. This covector would belong to a conormal bundle of the integral submanifold and therefore it must be zero when restricted to a cotangent bundle of the integral submanifold. That is, for example

$$\alpha = 2xz\mathrm{d}x + 2yz\mathrm{d}y - \left(1 + x^2 + y^2\right)\mathrm{d}z$$

This covector can be further simplified

$$\begin{aligned}\alpha &= z\mathrm{d}\left(1 + x^2 + y^2\right) - \left(1 + x^2 + y^2\right)\mathrm{d}z = \\ &= -\left(1 + x^2 + y^2\right)^2 \mathrm{d}\left(\frac{z}{1 + x^2 + y^2}\right)\end{aligned}$$

This covector will be zero if and only if we are differentiating a constant, therefore $\frac{z}{1+x^2+y^2} = const.$

◇

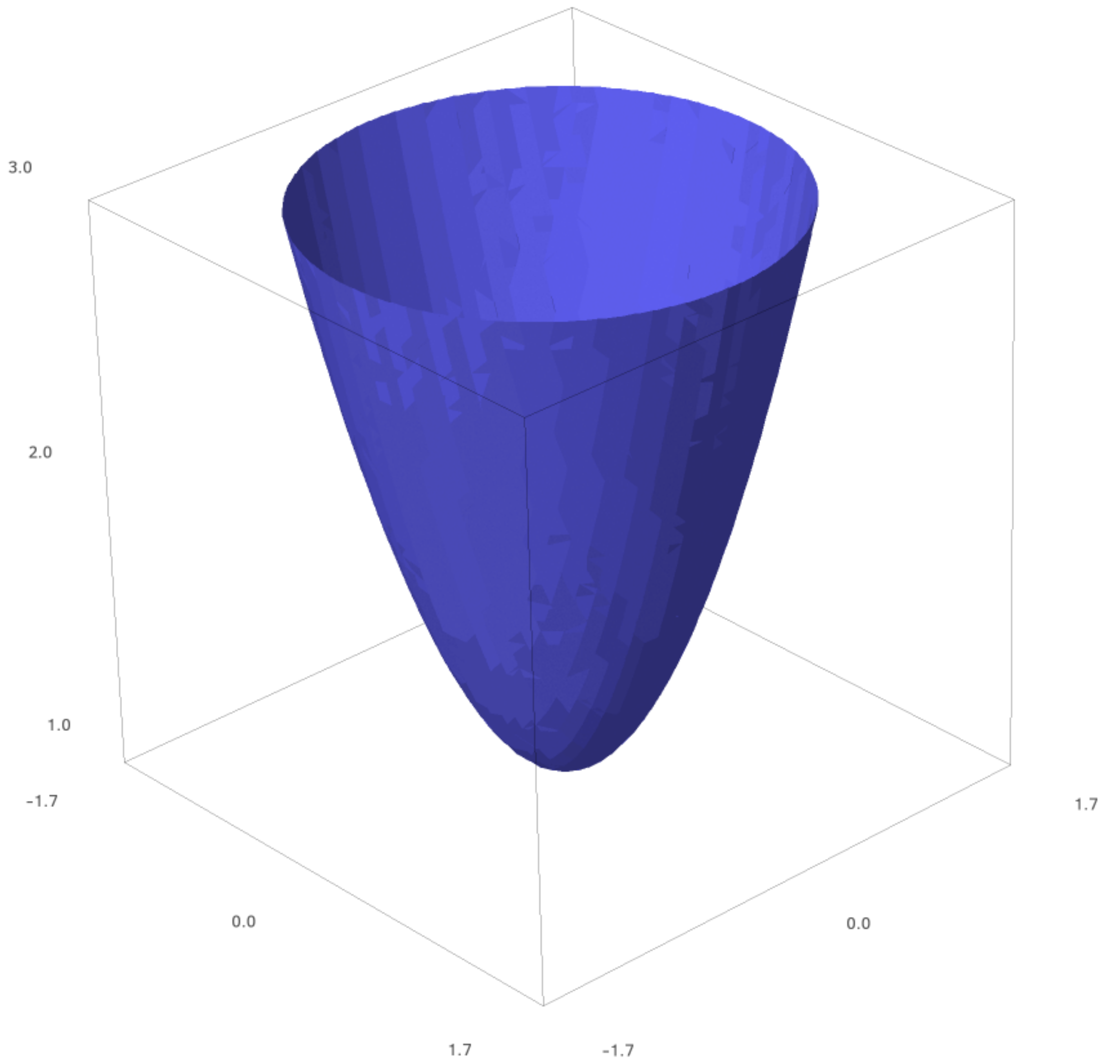

Figure 5.1: Integral submanifold and distribution $\mathfrak{D}$.

Tensors and tensor fields

Tensors

In this part we will study algebraic properties of multilinear maps. All vector spaces will be finitely dimensional.

Definition 6.1.

Consider $r+1$ vector spaces $V_1, \dots, V_r, W$. A map

$$f\colon V_1 \times \dots \times V_r \to W$$

is called **multilinear**, if it is linear in every component, i.e. for each $i \in I = \{1, \dots, r\}$ and every set of vectors $v_1 \in V_1, \dots, v_{i-1} \in V_{i-1}, v_{i+1} \in V_{i+1}, \dots, v_r \in V_r$ we get a multilinear map

$$f(v_1, \dots, v_{i-1}, -, v_{i+1}, \dots, v_r) : V_i \to W \ .$$

The space of all such multilinear maps is denoted $L(V_1, \dots, V_r; W)$. In the special case $V_1 = \dots = V_r = V$, we call $L\Bigl(\underbrace{V, \dots, V}_{r\text{-times}}; W\Bigr)$ the space of all **r-linear maps** from V to W.

We can define vector space structure on $L(V_1, \dots, V_r; W)$.

Proposition 6.2.

For arbitrary multilinear maps $f, g \in L(V_1, \dots, V_r; W)$ we define addition

$$(f+g)(v_1, \dots, v_r) = f(v_1, \dots, v_r) + g(v_1, \dots, v_r) \tag{6.3}$$

and for $k \in \mathbb{R}$ we define scalar multiplication

$$(kf)(v_1, \dots, v_r) = k(f(v_1, \dots, v_r)) \ . \tag{6.4}$$

$L(V_1, \dots, V_r; W)$ is a vector space with respect to this operations.

Special case of the previous definition is $r = 1,\ W = \mathbb{R}$ considered in the following definition.

Definition 6.5.

Space $L(V, \mathbb{R})$ of all linear functions on V is called **dual space** of V, denoted V^*. Elements of dual space are called **linear 1-forms** or **covectors**. Space $(V^*)^*$ is called **second dual** and is denoted V^{**}.

Proposition 6.6.

For every finite dimensional vector space V (non-canonically) holds $V \cong V^*$ and (canonically, using evaluation map) $V \cong V^{**}$.

Remark. The notion of a canonical isomoprhism in the previous proposition means that given all the isomoprhisms of a finite dimensional vector space there is a unique one which is independent of a chosen description of the abstract vector space.

Definition 6.7.

We define for a vector space V it's **r-th tensor power**, $\bigotimes^r V$, as a space of all r-linear maps from V^* to $\mathbb{R}$, i.e.

$$\bigotimes^r V := L(V^*, \ldots, V^*; \mathbb{R}) \ .$$

Elements of $\bigotimes^r V$ are called **tensors of order** r, **r-th order** tensors or simply r-tensors.

Picking a basis $\{e_1, \ldots, e_n\}$ in V determines coordinate descripition of a vector v with respect to this basis, $v = \sum_{i=1}^n v^i e_i$ (note the position of coordinate indices) where n is dimension of V.

Definition 6.8.

We define **dual basis** $\{d^1, \ldots, d^n\}$ in V^* by equations

$$d^i(e_j) = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ . \tag{6.9}$$

1-form $u \in V^*$ can be described in dual basis coordinates followingly

$$u = \sum_{i=1}^n u_i d^i \tag{6.10}$$

(note the position of coordinate indices) and for arbitrary $v \in V$ we have

$$u(v) = \sum_{i=1}^n u_i v^i \ . \tag{6.11}$$

Let us recall that every vector $v \in V$ can be understood as a 1-form on V^*

$$v \colon V^* \to \mathbb{R}$$

Definition 6.12.

We define tensor product $v_1 \otimes \cdots \otimes v_r$ of a set of vectors $v_1, \ldots, v_r$

$$v_1 \otimes \cdots \otimes v_r \in \bigotimes^r V$$

by prescription of a value on r-tuple of 1-forms

$$(v_1 \otimes \cdots \otimes v_r)(u_1, \ldots, u_r) = u_1(v_1) \cdot \cdots \cdot u_r(v_r) \in \mathbb{R} \ , \tag{6.13}$$

where $u_i(v_i)$ is value of vector v_i on 1-form u_i.

we proceed by introducing tensor product of arbitrary r-tensor and s-tensor over V.

Definition 6.14.

For arbitrary $A \in \bigotimes^r V$ and $B \in \bigotimes^s V$ we define tensor product $A \otimes B \in \bigotimes^{r+s} V$ by prescripiton of value on $r+s$ covectors

$$(A \otimes B)(u_1, \ldots, u_{r+s}) = A(u_1, \ldots, u_r) \cdot B(u_1, \ldots, u_s) \ . \tag{6.15}$$

Definition 6.16.

r-linear map A is called **symmetric** if the following hold

$$A(v_1, \ldots, v_r) = A\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) \tag{6.17}$$

for arbitrary permutation $\sigma \in \mathrm{P}_r$, where P_r is the group of all permutation of some r-element set.

Definition 6.18.

Subset $S^r V \subset \bigotimes^s V$ of all symmetric r-linear maps from V^* to $\mathbb{R}$ is called **r-th symmetric tensor power of** V.

Proposition 6.19.

r-th symmetric tensor power $S^r V$ is a linear subspace in $\bigotimes^r V$.

Definition 6.20.

We define **symmetrization** of a r-linear map $A\colon V^* \times \cdots \times V^* \to \mathbb{R}$

$$\mathrm{Sym}\colon \bigotimes^r V \to S^r V$$

by the equation

$$\mathrm{Sym}(A)(u_1, \ldots, u_r) = \frac{1}{r!} \sum_{\sigma \in P_r} A\left(u_{\sigma(1)}, \ldots, u_{\sigma(r)}\right) \ . \tag{6.21}$$

Definition 6.22.

r-linear map A is called **antisymmetric** or **alternating** if for any given permutation $\sigma \in \mathrm{P}_r$ the following holds

$$A(v_1, \ldots, v_r) = \mathrm{sgn}(\sigma) A\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) \ , \tag{6.23}$$

where $\mathrm{sgn}(\sigma)$ is the sing of permutation.

Remark. Symmetric/antisymmetric tensor can be recognized by the following property. Interchanging two arguments of this multilinear maps change/reverse the sign after evaluation (with respect to the original value).

Definition 6.24.

Subset $\Lambda^r V \subset \bigotimes^r V$ of all antisymmetric r-linear maps from V^* to $\mathbb{R}$ is called **r-th exterior tensor power of V**.

Proposition 6.25.

---

r-th exterior tensor power $\Lambda^r V$ is a linear subspace in $\bigotimes^r V$.

Definition 6.26.

We define **antisymmetrization** of a r-linear map $A\colon V^* \times \cdots \times V^* \to \mathbb{R}$

$$\mathrm{Alt}\colon \bigotimes^r V \to \Lambda^r V$$

by prescription

$$\mathrm{Alt}\,(A)\,(u_1,\ldots,u_r) = \frac{1}{r!}\sum_{\sigma\in P_r} \mathrm{sgn}(\sigma) A\left(u_{\sigma(1)},\ldots,u_{\sigma(r)}\right) \ . \tag{6.27}$$

Remark. For $A \in \bigotimes^2 V$ we have $\mathrm{Sym}\left(A^{ij}\right) = \frac{1}{2}\left(A^{ij}+A^{ji}\right)$ and $\mathrm{Alt}\left(A^{ij}\right) = \frac{1}{2}\left(A^{ij}-A^{ji}\right)$. Consequence of this in the special case $r=2$ is decomposition of $A = \left(A^{ij}\right)$ in terms of symmetric and antisymmetric part

$$A = \mathrm{Sym}A + \mathrm{Alt}A \tag{6.28}$$

$$\left(A^{ij}\right) = \frac{1}{2}\left(A^{ij}+A^{ji}\right) + \frac{1}{2}\left(A^{ij}-A^{ji}\right) \tag{6.29}$$

Definition 6.30.

Given $v_1,\ldots,v_r \in V$, we define **exterior product of vectors** by

$$v_1\wedge\cdots\wedge v_r = \mathrm{Alt}(v_1\otimes\cdots\otimes v_r) \in \Lambda^r V \ . \tag{6.31}$$

Theorem 6.32.

For each $\sigma \in \mathrm{P}_r$ the following holds

$$v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(r)} = \mathrm{sgn}(\sigma)\cdot v_1\wedge\cdots\wedge v_r \ . \tag{6.33}$$

Definition 6.34.

We define **exterior product of tensors** $A\wedge B \in \Lambda^{r+s}$ for arbitrary $A \in \Lambda^r V,\ B \in \Lambda^s V$ by prescription

$$A\wedge B = \mathrm{Alt}\,(A\otimes B) \ . \tag{6.35}$$

Theorem 6.36.

For $A \in \Lambda^r V,\ B \in \Lambda^s V$ the following holds

$$A \wedge B = (-1)^{rs} B \wedge A \ . \tag{6.37}$$

Remark. Consider a vector space V with a chosen basis $\alpha = \{v_1, \ldots, v_n\}$ and a vector space W with a chosen basis β. Let us recall that linear map $f\colon V \to W$ can be reprezented, with respect to these basis, as a matrix (a^i_j) (where i is row index and j column idex). Value a^i_j is coordinate descripiton (with respect to α, β) of f and is given by the i-th coordinate(in β) of the image of j-th basis vector of α, i.e. $\big(f(v_j)\big)^i$.

Definition 6.38.

Dual map of a linear map $f\colon V \to W$

$$f^*\colon W^* \to V^*$$

is defined by equation

$$f^*(u)(v) = u\,(f(v)) \ , \tag{6.39}$$

where $v \in V$ a $u \in W^*$.

Definition 6.40.

We define ***r*-th tensor power of a map** $f\colon V \to W$

$$\otimes^r f\colon \bigotimes^r V \to \bigotimes^r W$$

by a prescription on $A \in \bigotimes^r V$

$$(A\colon V^* \times \cdots \times V^* \to \mathbb{R}) \longmapsto \big(A \circ (f^* \times \cdots \times f^*) : W^* \times \cdots \times W^* \to \mathbb{R}\big) \ .$$

Theorem 6.41.

The following holds: $(\otimes^r f)\,(S^r V) \subset S^r V$ and $(\otimes^r f)\,(\Lambda^r V) \subset \Lambda^r V$.

Remark. Notice that $\Lambda^n \mathbb{R}^n$ is a space of dimension 1 which can be deduced from constructing a basis and using elementary combinatorics to compute number of vectors in it. Particularly, every $A \in \Lambda^n \mathbb{R}^n$ is defined by only one coordinate $A^{1\ldots n}$.

Theorem 6.42.

Let $f = \left(a^i_j\right) : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and $A \in \Lambda^n \mathbb{R}^n$. For image $B = \otimes^n f(A)$ of induced map $\otimes^r f\colon \Lambda^n \mathbb{R}^n \to \Lambda^n \mathbb{R}^n$ the following holds

$$B^{1\ldots n} = \det\left(a^i_j\right) A^{1\ldots n} \tag{6.43}$$

Definition 6.44.

Tensor power of type (r,s) of a vector space V is defined as a space of all multilinear maps

$$L\left(\underbrace{V^*,\dots,V^*}_{r\text{-krát}},\underbrace{V,\dots,V}_{s\text{-krát}};\mathbb{R}\right)$$

and is denoted $\bigotimes^r_s V$ or $\bigotimes^r V \otimes \bigotimes^s V^*$. Elements of this space are called **mixed tensors**.

Remark. Two special cases of the previous definitions are $r = 0$ and $s = 0$ which corresponds to $\bigotimes^r V$ and $\bigotimes^s V^*$. Elements of these spaces are called **pure** or unmixed tensors.

Analogously as in the previous part, we can define tensor product of two vector spaces using multilinear maps.

Definition 6.45.

Tensor product $V \otimes W$ **of vector spaces** V, W is given by

$$V \otimes W = L(V^*, W^*; \mathbb{R}) \ . \tag{6.46}$$

Remark. Special case $W = V^*$ of the last definition yields $\bigotimes^1_1 V$.

Lemma 6.47.

There is a unique linear map

$$C\colon \bigotimes{}^1_1 V \to \mathbb{R}$$

such that for every element of the form $v \otimes u$, $v \in V$, $u \in V^*$ the following holds

$$C(v \otimes u) = u(v) \ . \tag{6.48}$$

Using map C from lemma 6.47, we can define a generalization of the matrix trace, called contraction.

Definition 6.49.

For $A = (A^i_j) \in \bigotimes^1_1 V$ we define **contraction of** $(1,1)$ **tensor** as a value given by

$$C(A) = \sum_{i=1}^{n} A^i_i \ . \tag{6.50}$$

Remark. If we understand A as a matrix then $C(A)$ is the matrix trace.

To define contraction on general (r,s) tensors we use the following idea. Given arbitrary $A \in \bigotimes_1^1 V$, we can fix one of arguments of this bilinear map, thus, getting a linear map

$$\begin{tikzcd} & A(-,v) : V^* \to \mathbb{R} \arrow[r, rightsquigarrow] & A(-,v) \in V \\ A\colon V^* \times V \to \mathbb{R} \arrow[ur, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \\ & A(u,-) : V \to \mathbb{R} \arrow[r, rightsquigarrow] & A(u,-) \in V^* \end{tikzcd}$$

In this manner, using $v \in V$ or $u \in V^*$, we can restrict domain of the tensor, i.e. $A \mapsto A(-,v) \in \bigotimes_0^1 V$ or $A \mapsto A(u,-) \in \bigotimes_1^0 V$, hence, lowering the tensor order. Similarly, for tensor $A \in \bigotimes_s^r V$ we can fix all of it's arguments but those on positions k, l, where $1 \le k \le r$ and $1 \le l \le s$, and get a new $(1,1)$ tensor $\tilde{A}$

$$\tilde{A}(-,-) = A\left(u_1, \ldots, u_k, -, u_{k+1}, \ldots, u_r, v_1, \ldots, v_l, -, v_{l+1}, \ldots, v_s\right) \in \bigotimes_1^1 V \ .$$

Then, we can apply contraction defined in 6.49 on $\tilde{A}$ leading to the definition below.

Definition 6.51.

For $A \in \bigotimes_s^r V$ the **contraction $C_l^k(A)$ of tensor A to k-th upper index and l-th lower index** by prescription

$$C_l^k(A) = C\left(\tilde{A}\right) \ . \tag{6.52}$$

Remark. Contraction to k-th upper and l-th lower index can be seen as the trace of mapping $\tilde{A}$. For $A = \left(A^{i_1 \ldots i_r}_{j_1 \ldots j_r}\right)$ we can write contraction in coordinates as follows

$$C_l^k(A) = \sum_{h=1}^{n} A^{i_1 \ldots h \ldots i_r}_{j_1 \ldots h \ldots j_r} \ , \tag{6.53}$$

where the summation index h is on the k-th position of upper and l-th position of the lower multiindex. The operation of contraction can be iterated whenever condition $r,\ s \ge 2$ is satisfied.

Tensor fields

Firstly we define cotangent bundle T^*M which is a necessary step to define general tensor fields, since, tensor powers of tangent and cotangent bundle is ambient space of tensor fields. We have met tensor bundle TM already in the fourth chapter of this text. As a set, TM is given as a union of tangent vector spaces, i.e. $TM = \bigcup_{x \in M} T_x M$. Structure of a smooth manifold is inherited from the underlying base manifold M. Analogously for $T*M$. As a set, it is a union of cotangent spaces

$$T^*M = \bigcup_{x \in M} T_x^* M \ , \tag{6.54}$$

equipped with projection on the base manifold

$$\pi\colon T^*M \to M\ , T_x^*M \mapsto x\ . \tag{6.55}$$

For open set $V \subset \mathbb{R}^n$ it holds that $T^*V = V \times (\mathbb{R}^n)^*$. Given local chart $\varphi\colon U \to V$ on M, we have a tangent map $T_x\varphi\colon T_xM \to T_{\varphi(x)}V = \{\varphi(x) \times \mathbb{R}^n\}$, $x \in U$ and a dual map $T_x^*\varphi\colon \{\varphi(x) \times (\mathbb{R}^n)^* \to T_x^*M$. Mapping $T_x\varphi$ is a linear isomorphism, thus, the dual is invertible with inverse $(T_x^*)^{-1}\varphi\colon T_x^*M \to \{\varphi(x) \times (\mathbb{R}^n)^*$. Taking union over all points $x \in U$ yields bijection $(T_x^*)^{-1}\varphi\colon \pi^{-1}(U) \to V \times (\mathbb{R}^n)^*$ which can be used to lift open subset from $V \times (\mathbb{R}^n)^*$ up to $\pi^{-1}(U)$ and get a basis of topology on $T*M$. Therefore, we see how to introduce structure of topological space on T^*M. Similarly, we introduce smooth structure using transition maps $\varphi_{12}\colon V_{12} \to V_{21}$ of the atlas on M by point-wisely considering inverse of tangent map dual, i.e. $T_x^*\varphi_{12}\colon V_{12} \times (\mathbb{R}^n)^* \to V_{21} \times (\mathbb{R}^n)^*$. Transition maps $T_x^*\varphi_{12}$ are of class C^∞, due to φ_{12} are C^∞ maps. Thereby, T^*M is a smooth manifold.

Definition 6.56.

Manifold T^*M is called **cotangent bundle** of a manifold M.

Remark. Given a smooth map $f\colon M \to N$ between manifolds there is a tangent map at $x \in M$. In case of function $f\colon M \to \mathbb{R}$ on a manifold, the tangent map is $T_xf\colon T_xM \to T_x\mathbb{R} = \mathbb{R}$. Using dual map $(T_xf)^*$ we can define assigement $x \mapsto (\mathrm{d}f)(x) := (T_xf)^*$, i.e.

$$\begin{aligned} \mathrm{d}f\colon M &\to T^*M\ ,\\ x &\mapsto (\mathrm{d}f)(x)\ . \end{aligned}$$

Map $\mathrm{d}f$ is called the **differential of a function** f. Value of differential at x is the dual to tangent map of f at a point x.

The next step on our way to defin tensor fields is bundle tensor product of tensor power TM a T^*M. Similarly as in case of cotangent bundle we can show that this space can be equipped by a smooth manifold structure using base manifold M.

In preceding section we have seen how to construct tensor powers of a given vector space V. Thus, let r, s be arbitrary and define $V = T_xM$, $x \in M$. Having $\bigotimes_s^r T_xM = \bigotimes^r T_xM \otimes \bigotimes^s T_x^*M$ we can define the following set

$$\bigotimes_s^r TM = \bigcup_{x\in M} \bigotimes_s^r T_xM \tag{6.57}$$

which is equipped with a projection map $p\colon \bigotimes_s^r TM \to M$ by the virtue of $A \in \bigotimes_s^r TM$ being defined over some subset $S \subset M$, i.e. $p(A) = S$, which is true for arbitrary A. Also, for a given $V \subset \mathbb{R}^n$ we have

$$\bigotimes_s^r TV = V \times \bigotimes_s^r \mathbb{R}^n\ .$$

Thus, for a local chart $\varphi\colon U \to V$ we can construct point-wisely induced bijective map

$$\left(\otimes^r (T\varphi)\right) \otimes \left(\otimes^s (T^*\varphi)^{-1}\right)\colon p^{-1} \to V \times \bigotimes_s^r \mathbb{R}^n\ . \tag{6.58}$$

This map is a local chart and induces topology on $\bigotimes_s^r TM$. Basis of topology is given by inverse image $p^{-1}(U)$ of open subsets $U \subset V \times \bigotimes_s^r \mathbb{R}^n$. Finally, a smooth transition map φ_{12} between two overlaping charts on M induces smooth transition map $\left(\otimes^r (T\varphi)_{12}\right) \otimes \left(\otimes^s (T^*\varphi_{12})^{-1}\right)$ between charts 6.58, hence, obtaining smooth manifold structure.

Definition 6.59.

Manifold $\bigotimes_s^r TM$ is called **tensor bundle of type** (r,s) of M.

Now we can easily define the notion of a tensor field on manifold as a smooth *section* of a tensor bundle.

Definition 6.60.

Smooth map $A\colon M \to \bigotimes_s^r TM$ satisfying $p \circ A = id_M$, where p is the projection from general tensor bundle on the underlying manifold M, $p\colon \bigotimes_s^r TM \to M$, is called **tensor field of type** (r,s) on M.

Remark. Tensor field A on M of type (r,s) can be described in local coordinates $\left(x^i\right)$ by smooth functions $A^{i_1\dots i_r}_{j_1\dots j_s}(x)$. Value of tensor field at a given point $x \in M$ is a tensor. Field contains the information of smooth change of the tensor when moving from point to point.

Definition 6.61.

Let $f\colon M \to N$ be a smooth map. A map between tensor bundles is defined point-wisely as

$$\otimes^r Tf := \bigcup_{x\in M} \otimes^r T_x f\colon \bigotimes^r TM \to \bigotimes^r TN\,, \tag{6.62}$$

where$\otimes^r T_x f\colon \bigotimes^r T_xM \to \bigotimes^r T_{f(x)}N$ is tensor power of (linear) tangent map $T_x f$ in the sense of 6.40.

If we wish to define induced mapping in the opposite direction, i.e. between tensor powers of cotangent bundles we must realize that the tangent maps are not bijection in general. Hence, let us proceed as follows. Consider surjective smooth map $f\colon M \to N$ and tensor field A of type $(0,s)$ on N, i.e. $A\colon N \to \bigotimes_s TN$. By surjectivity of f there is for each $y \in N$ the inverse image $x \in M$, $f(x) = y$ and a tangent map at x, $T_x f\colon T_xM \to T_yN$. A is a multilinear map $A(y)\colon \underbrace{T_yN \times \cdots \times T_yN}_{s\text{-krát}} \to \mathbb{R}$ t y and we can compose it with $(T_x f)^s$ which is defined standardly

$$(T_x f)^s := \left(\underbrace{T_x f \times \cdots \times T_x f}_{s\text{-krát}}\right) : \underbrace{T_xM \times \cdots \times T_xM}_{s\text{-krát}} \longrightarrow \underbrace{T_yN \times \cdots \times T_yN}_{s\text{-krát}}\,.$$

We get commutative diagram

$$\begin{tikzcd}
T_xM \times \cdots \times T_xM \arrow[rr, "(T_x f)^s"] \arrow[dr, dashed, "A(y)\circ(T_x f)^s"'] & & T_yN \times \cdots \times T_yN \arrow[dl, "A(y)"] \\
& \mathbb{R} &
\end{tikzcd}$$

in which we see that a field A on N induces along f a new filed on M, leading to the following definition.

Definition 6.63.

Consider $A \in \bigotimes_s TN$ and a smooth surjective map $f\colon M \to N$. **Pullback of** A **of type** $(0,s)$ **along** f is defined point-wisely as

$$f^*A(x) = A\left(f(x)\right) \circ \left(T_x f\right)^s \ . \tag{6.64}$$

New field of the same type $(0,s)$ is denoted $f^*(A)$ and at a point $x \in M$ defines a tensor $f^*(A)(x)$. Especially, for a tensor field of type $(0,0)$, i.e. a function $g\colon N \to \mathbb{R}$ we define $f^*g = g \circ f\colon M \to \mathbb{R}$.

Lemma 6.65.

Let $g\colon Q \to M$ and $f\colon M \to N$ be smooth maps. For pullback of a tensor field $A \in \bigotimes_s TN$ the following holds

$$(f \circ g)^* A = g^*\left(f^*A\right) \ . \tag{6.66}$$

Definition 6.67.

Differential k**-form on** M or **exterior** k**-form** is a antisymetric tensor field of type $(0,k)$ on M, i.e. a tensor field $A\colon M \to \Lambda^k T^*M$ which is at every point of M antisymetric tensor.

Definition 6.68.

For a differential k-form $A\colon M \to \Lambda^k T^*M$ and l-form $B\colon M \to \Lambda^l T^*M$ we define point-wisely their **exterior product** $A \wedge B\colon M \to \Lambda^{k+l}T^*M$

$$(A \wedge B)(x) := A(x) \wedge B(x) \ . \tag{6.69}$$

Remark. In the special case of f being 0-form on M (i.e. a function) equation $f \wedge A = fA$ holds. Put differently, exterior product of a function with a form is given by multiplying the function with the form.

Theorem 6.70.

For a smooth map $f\colon M \to N$ and differential forms A,B on N the following holds

$$f^*(A \wedge B) = (f^*A) \wedge (f^*B) \ . \tag{6.71}$$

Let us denote the space of all differential k-forms on M by symbol $\Omega^k M$. Especially for $k=0$, $\Omega^0 M = C^\infty(M,\mathbb{R})$ is the space of all smooth functions on M.

Exercises

Exercise 6.72.

Prove statement 6.6 and describe canonical isomorphism between vector space V and the second dual space V^{**}.

Solution. Let us recall the definition of dual space V^*. It is a vector space consisting of linear 1-forms on V, i.e. linear maps from V to the underlying field of scalars. In our case (we work only with real space) $V^* = L(V;\mathbb{R}) = \{u\colon V \to \mathbb{R}|\ u \text{ je lineární}\}$. The dualization proces in case of finitely dimensional spaces preserves the dimension. This can be deduced by constructing basis on V^*. On a dual space, there is a **dual basis** defined with respect to basis $\{e_1,\dots,e_n\}$ on V by requiring $f^i(e_j) = \delta^i_j$ (where δ^i_j is Kronecker delta). Dual basis $\{f^1,\dots,f^n\}$ has the same number of elements as the original basis on V, thus, $n = \dim V = \dim V^*$. Using dualization again yields $V^{**} = \{\alpha\colon V^* \to \mathbb{R}|\alpha \text{ je lineární}\}$ having the same dimension n. Hence we know that $V \cong V^{**}$ and it remains to describe the canonical isomorphism. Every (immutably chosen) vector $v \in V$ can be understood as a linear 1-form on V^* because for any $u \in V^*$ we have the following prescription called evaluation

$$\mathrm{Ev}_v(u) = u(v)\ , \tag{6.73}$$

sending $u\colon V \to \mathbb{R}$ on the corresponding value at v. Linearity of Ev_v follows from linearity of 1-form u, thus, $\mathrm{Ev}_v \in V^{**}$. So, we can assign element of second dual space to each $v \in V$

$$\mathrm{Ev}\colon V \to V^{**}\ , \tag{6.74}$$

$$v \mapsto \mathrm{Ev}_v \tag{6.75}$$

and this assignement is unambiguous - injectivity comes directly from 6.73: $\mathrm{Ev}_v \equiv 0 \Rightarrow u(v) = 0\ \forall u \in V^* \Rightarrow v = 0$. Therefore, Ev is isomorphism as it is injective linear map between vector spaces of the same dimension. Moreover, evaluation map is independent of a choice of basis on V and V^{**}, thus canonical. ◇

Remark. Dualization $(-)^*$ can be seen as a map from the class of all vector spaces to itself which sends arbitrary vector space to it's dual. Moreover, this map satisfies definition categorical notion of *functor*, conrcetely *endofunctor* of the category Vect of all vector spaces over a given field of scalars. We can, then, describe the fact that Ev is canonical using notion of natural isomorphism of second dual functor, $(-)^{**}\colon \mathit{Vect} \to \mathit{Vect}$, with identical functor on *Vect*.

Exercise 6.76.

Using tensor powers of real vector space V describe the following spaces.

1. Space of scalars $\mathbb{R}$.

2. Space of vectors V.

3. Space of 1-forms V^*.

4. Space of bilinear forms on V.

5. Space of inner products on V.

Solution. 1. Space of scalars $\mathbb{R}$ can be described as a 0-th tensor power. According to definition 6.7 we have

$$\bigotimes^0 V = L\left(\underbrace{V^*,\dots,V^*}_{\text{0-krát}};\mathbb{R}\right) .$$

Let us recall that from definition of multilinearity follows that with respect to every set indexing the domain product space there are certain conditons that must be satisfied (put differently: certain axioms must be satisfied in every component of the product). Our index set is empty, hence, no condition will be violated by any map. Therefore, we can say that every map with domain being empty product is linear. Furthermore, from set theory we know that product over empty index set corresponds to one element set

$$\underbrace{V^*\times\cdots\times V^*}_{\text{0-krát}} \cong \{\star\}$$

and the set of all maps from one element set to $\mathbb{R}$ is $\mathbb{R}$ itself. So, we have

$$\bigotimes^0 V = \mathbb{R} .$$

2. Vector space V is the first tensor power. Due to indentification $V \cong V^{**}$ stated in 6.6 we can directly, with respect to definition 6.7, write

$$\bigotimes^1 V = L(V^*;\mathbb{R}) = V^{**} \cong V .$$

3. Space of 1-forms on V^* can be describd as the firs tensor power ovec the dual space V^* which can be shown by comparing dimensions. In previous part we got $\bigotimes^1 V \cong V$, thus $\bigotimes^1 V$ has the same dimension as V. Here V is an abstract vector space, so, we can choose arbitrary vector space and the equality of dimensions will still hold. Aplying this idea on the dual space V^*, the following must hold $\dim\left(\bigotimes^1 V^*\right) = \dim(V^*)$. Since we know that finitely dimensional vector space having the same dimension are isomoprhic, we conclude

$$\bigotimes_1 V = \bigotimes^1 V^* \cong V^* .$$

It is likely that it came up to mind of the reader to solve this exercise similarly as the previous part, directly from definition 6.7, i.e.

$$\bigotimes^1 V^* = L(V^{**};\mathbb{R}) \cong L(V;\mathbb{R}) = V^* .$$

Being our wish to be precise we ought to examine the isomorphism $L((V^{**};\mathbb{R}) \cong L(V;\mathbb{R})$ which is not so clear to be true at the first glance eventhough, it is expectable. To show this we can use *the universal property* of tensor product which is part of categorical definition of tensor product.

4. Space of bilinear forms on V, i.e. bilinear maps from $V \times V$ to $\mathbb{R}$, can be seen as the second tensor power, which follows straightly from definition 6.7.

$$\mathrm{Bilin}\,(V;\mathbb{R}) = \{\beta \colon V \times V \to \mathbb{R} |\ \beta \text{ je bilinární}\} \cong \bigotimes\nolimits_2 V \ .$$

5. Space of inner products on V can be described as a factor space of the second tensor power by a suitable subspace (or generating relations). Inner product is a positive definite symetric bilinear form (we can meet definitions that does not require pozitivity which we would called *pseudo*-inner product). Let $N =< u \otimes v - v \otimes u | u, v \in V^* >$, $N \subset \bigotimes_2 V$ be linear subspace. The space of all inner product on V can be written as $(\bigotimes_2 V)/N$.

◇

Exercise 6.77.

Prove the formula 6.11.

Solution. Let $\{e_1, \dots, e_n\}$ be basis in V and $\{d_1, \dots, d_n\}$ dual basis in V^*. With respect to these basis, we can write $v \in V$ in coordinates as $v = \sum_{i=1}^n v^i e_i$ and $u \in V^*$ has coordinate descripiton $\sum_{j=1}^n u_j d^j$. Given linearity of 1-forms, in coordinates we have

$$u(v) = \left(\sum_{j=1}^n u_j d^j\right)\left(\sum_{i=1}^n v^i e_i\right) = \sum_{i,j=1}^n u_j v^i d^j(e_i) = \sum_{i,j=1}^n u_j v^i \delta_i^j = \sum_{i=1}^n u_i v^i \ .$$

which was intended to be shown. ◇

Exercise 6.78.

Show that for r-tensor α of the form $\alpha = v_1 \otimes \cdots \otimes (a v_i + b\tilde{v}_i) \otimes \cdots \otimes v_r$, where a, b are scalars, the following holds

$$\alpha = a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_r) + b(v_1 \otimes \cdots \otimes \tilde{v}_i \otimes \cdots \otimes v_r) \ . \tag{6.79}$$

Solution. Inserting arbitrary argument $(u_1, \dots, u_r)$ in definition 6.13 we get by simple computation

$$\begin{aligned}
\alpha\,(u_1, \dots, u_r) &= v_1(u_1) \dots (a v_i + b\tilde{v}_i)(u_i) \dots v_r(u_r) \\
&= v_1(u_1) \dots (a v_i(u_i) + b\tilde{v}_i(u_i)) \dots v_r(u_r) \\
&= a v_1(u_1) \dots v_i(u_i) \dots v_r(u_r) + b v_1(u_1) \dots \tilde{v}_i(u_i) \dots v_r(u_r) \\
&= a v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_r\,(u_1, \dots, u_r) + b v_1 \otimes \cdots \otimes \tilde{v}_i \otimes \cdots \otimes v_r\,(u_1, \dots, u_r)
\end{aligned}$$

This reslut is true for arbitrary $(u_1, \dots, u_r)$, thus, 6.79 holds. ◇

Exercise 6.80.

Give a coordinate description of tensor product of two vectors $v, w \in V$, where $\dim V = 2$. Let A be a matrix 2×2. Determine the space to which element of the form $A \otimes v$ belongs.

Solution. 1. Let $\{e_1, e_2\}$ be basis in V, with respect to which we write $v = v^1 e_1 + v^2 e_2$, $w = w^1 e_1 + w^2 e_2$. Product $v \otimes w$ is element of $\bigotimes^2 V$. This space can be described using basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$. Then, with accordance to 6.79 the following is true

$$\begin{aligned} v \otimes w &= (v^1 e_1 + v^2 e_2) \otimes (w^1 e_1 + w^2 e_2) \\ &= v^1 w^1 e_1 \otimes e_1 + v^1 w^2 e_1 \otimes e_2 + v^2 w^1 e_2 \otimes e_1 + v^2 w^2 e_2 \otimes e_2 \end{aligned}$$

In coordinates we have $v \otimes w = \left(v^1 w^1, v^1 w^2, v^2 w^1, v^2 w^2\right)$.

2. Matrix 2×2 can be seen as a tensor of type $(1,1)$, i.e. $A \in \bigotimes_1^1 V$. In second part of previous exercise 6.76 we have seen that element of vector space is also element of the first tensor power: $v \in \bigotimes^1 V$. We conclude that $A \otimes v \in \bigotimes_1^2 V$.

◇

Exercise 6.81.

Using basis $\varepsilon = \{e_1, \dots, e_n\}$ in V construct basis in r-th tensor power $\bigotimes^r V$ and determine dimension of this space. Also, consider element $A = v_1 \otimes \cdots \otimes v_r$, where all v_i belongs to V and describe it in coordinates. Also, find coordinate description of $B \in \bigotimes^r V$ and $C \in \bigotimes_s V$.

Solution. We have seen in exercise 6.80 basis for $\bigotimes^2 V$. Analogously we can proceed in case of $\bigotimes^r V$. Basis $\alpha = \{e_{i_1} \otimes \cdots \otimes e_{i_r} | e_{i_j} \in V,\ 1 \le i_1 \le n, \dots, 1 \le i_r \le n\}$ consits of tensor products of r basis vectors in V. On the i-th position of product we can choose arbitrary basis vector and the choice may be repeated on more positions at once. Thus, there are n posibilities of how to choose e_i on each of r positions and dimension of $\bigotimes^r V$ (i.e. number of elements in α) is r^n.
In the next step we describe vectors defining A with respect to ε. For arbitrary v_i we have $v_i = \sum_{j=1}^n v_i^j e_j$. Inserting this expression in the product we get

$$\left(\sum_{j=1}^n v_1^j e_j\right) \otimes \cdots \otimes \left(\sum_{j=1}^n v_r^j e_j\right) = \sum_{1 \le i_1 \le n, \dots, 1 \le i_r \le n} v_1^{i_1} \dots v_r^{i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}\,, \tag{6.82}$$

which shows us that coordinate of A with respect to α, on position $i_1 \dots i_r$ is $v_1^{i_1} \dots v_r^{i_r}$. Coordinates of general tensor B, which can be given as a sum of simple tensors (e.g. $B = (v_1 \otimes \cdots \otimes v_r) + (\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_r)$ can be computed (viz 6.79) by adding coordinates of simple tensors in the sum. Therefore, B has on position $i_1 \dots i_r$ coordinate $B^{i_1 \dots i_r} = v_1^{i_1} \dots v_r^{i_r} + \tilde{v}_1^{i_1} \dots \tilde{v}_r^{i_r}$ Analogously we write, in the same manner as in case of vectors or matrices, coordinate description of tensor in round brackets $B = (B^{i_1 \dots i_r})$, i.e.

$$B = \sum_{1 \le i_1 \le n, \dots, 1 \le i_r \le n} B^{i_1 \dots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}\,. \tag{6.83}$$

Similarly for $C \in \bigotimes_s V$. Basis $\tilde{\alpha}$ on $\bigotimes_s V$ can be constructed from dual basis $\{d^1, \dots, d^n\}$, $\tilde{\alpha} = \{d^{i_1} \otimes \cdots \otimes d^{i_s} |\ d^{i_j} \in V^*,\ 1 \le i_1 \le n, \dots, 1 \le i_r \le n\}$. With respect to $\tilde{\alpha}$ we have

$C = (C_{i_1 \ldots i_s})$, i.e.

$$C = \sum_{1 \le i_1 \le n, \ldots, 1 \le i_r \le n} C_{i_1 \ldots i_s} d^{i_1} \otimes \cdots \otimes d^{i_s} \; . \tag{6.84}$$

◇

Exercise 6.85.

Given $A \in \bigotimes^r V$ and $B \in \bigotimes^s V$ arbitrary, find coordinates of $r+s$ type tensor $A \otimes B$.

Solution. Coordinates of A with respect to $\alpha = \{e_{i_1} \otimes \cdots \otimes e_{i_r} | \; 1 \le i_1 \le n, \ldots, 1 \le i_r \le n\}$ are $A = (A^{i_1 \cdots i_r})$ and coordinates of B with respect to $\beta = \{e_{j_1} \otimes \cdots \otimes e_{j_s} | \; 1 \le i_1 \le n, \ldots, 1 \le i_s \le n\}$ are $B = (B^{j_1 \cdots j_s})$. Coordinates of product $A \otimes B$ are $(A^{i_1 \cdots i_r} B^{j_1 \cdots j_s})$ which follows directly from 6.79. ◇

Exercise 6.86.

Given $A \in \bigotimes^r V$ determine value of $A(u_1, \ldots, u_r)$, i.e. evaluate A in arbitrary argument $(u_1, \ldots, u_r)$, where (necesarily due to definition of A) $u_i \in V^*$ for all i.

Solution. We already know that w e can choose basis $\alpha = \{e_{i_1} \otimes \cdots \otimes e_{i_r} | \; 1 \le i_1 \le n, \ldots, 1 \le i_r \le n\}$ on $\bigotimes^r V$ and with respect to this basis write $A = \sum_{1 \le i_1 \le \cdots \le i_r \le n} A^{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}$. Applying the defining equality 6.13 for each summand of the sum yields

$$\begin{aligned}
A(u_1, \ldots, u_r) &= \Big(\sum_{1 \le i_1 \le n, \ldots, 1 \le i_r \le n} A^{i_1 \ldots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \Big)(u_1, \ldots, u_r) \\
&= \sum_{1 \le i_1 \le n, \ldots, 1 \le i_r \le n} \big(A^{i_1 \ldots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}(u_1, \ldots, u_r) \big) \\
&= \sum_{1 \le i_1 \le n, \ldots, 1 \le i_r \le n} \big(A^{i_1 \ldots i_r} e_{i_1}(u_1) \ldots e_{i_r}(u_r) \big) \\
&= \sum_{1 \le i_1 \le n, \ldots, 1 \le i_r \le n} \big(A^{i_1 \ldots i_r} u_1^{i_1} \ldots u_r^{i_r} \big)
\end{aligned}$$

where $u_k^{i_j}$ is coordinate of vector u_k on i_j position. ◇

Exercise 6.87.

Describe in coordinates arbitrary antisymetric tensor $B \in \Lambda^r V$. Prove the following

$$B^{i_1 \ldots i_r} = \operatorname{sgn}(\sigma) B^{i_{\sigma(1)} \cdots i_{\sigma(r)}} \; , \tag{6.88}$$

where σ is arbitrary permutation of set containing r-elements.

Solution. In exercise 6.81 we described a basis on r-th exterior power of V, specifically $\alpha = \{e_{i_1} \wedge \cdots \wedge e_{i_r} | 1 \le i_1 < \cdots < i_r \le n\}$. Coordinate description of $B \in \Lambda^r V$ with respect to α are $B = \sum_{1 \le i_1 < \cdots < i_r \le n} B^{i_1 \ldots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}$. Consider new set of generators $\beta = \{e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_r)} | 1 \le i_1 < \cdots < i_r \le n\}$ where $\sigma \in \mathrm{P}_r$ is a chosen permutation. This set is again a basis in $\Lambda^r V$ and due to theorem 6.32 is the difference between $e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_r)}$ and $e_{i_1} \wedge \cdots \wedge e_{i_r}$ given only by sign. Then, B can ber written with respect to this new basis as $B = \sum_{1 \le i_1 < \cdots < i_r \le n} B^{\sigma(i_1)\ldots\sigma(i_r)} e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_r)}$. Since every tensor is, in the same manner as vector, a geometrical object, it is independent of our choice of decripition (basis in which is described), thus, the following equation holds

$$\sum_{1 \le i_1 < \cdots < i_r \le n} B^{i_1 \ldots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} = B = \sum_{1 \le i_1 < \cdots < i_r \le n} B^{\sigma(i_1)\ldots\sigma(i_r)} e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_r)} \, .$$

Right-hand side of the equation can be rewritten using 6.32

$$\sum_{1 \le i_1 < \cdots < i_r \le n} B^{i_1 \ldots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} = B = \sum_{1 \le i_1 < \cdots < i_r \le n} B^{\sigma(i_1)\ldots\sigma(i_r)} \operatorname{sgn}(\sigma) e_{i_1} \wedge \cdots \wedge e_{i_r} \, .$$

Simple comparison of coefficients yields the result $B^{i_1 \ldots i_r} = \operatorname{sgn}(\sigma) B^{\sigma(i_1)\ldots\sigma(i_r)}$. ◇

Exercise 6.89.

Determine pullback of the following forms on $\mathbb{R}^3$: $\omega = xyz\mathrm{d}y$, $\theta = \omega \wedge (e^x \mathrm{d}x + e^y \mathrm{d}y)$ and $\mu = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ along map $f \colon \mathbb{R}^2 \to \mathbb{R}^3$, $(u, v) \mapsto (u^2, 2uv, e^{uv})$.

Solution. Consider coordinates x, y, z on $\mathbb{R}^3$ and corresponding dual coordinates $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ on cotangent bundle $T^*\mathbb{R}^3$ with respect to which we have written ω, θ, μ. Note that unlike the case of general manifold, these coordinates are defined globally and description of above forms holds on the whole $\mathbb{R}^3$. To compute pullbacks we shall use 6.70. Also, we will need the commutativity of d with pullbacks, i.e. $f^*(\mathrm{d}\alpha) = \mathrm{d}(f^*\alpha)$. This property is discussed in the next chapter and holds true for arbitrary forn α. Also note that f^* is linear which is a direct consequence of definition of pullback 6.63. Let us compute

$$\begin{aligned} f^*\omega &= f^*(xyz\mathrm{d}y) \\ &= u^2 2uve^{uv} (\mathrm{d}(2uv)) \\ &= 2u^3ve^{uv} (2v\mathrm{d}u + 2u\mathrm{d}v) \\ &= 4u^3v^2e^{uv}\mathrm{d}u + 4u^4ve^{uv}\mathrm{d}v \, . \end{aligned}$$

Further, given $f^*\theta$ we have

$$f^*\theta = f^*(\omega \wedge (e^x\mathrm{d}x + e^y\mathrm{d}y)) = f^*\omega \wedge f^*(e^x\mathrm{d}x + e^y\mathrm{d}y) \, .$$

Since we have already computed $f^*\omega$, it is enough to compute the second factor

$$\begin{aligned} f^*(e^x\mathrm{d}x + e^y\mathrm{d}y) &= e^{u^2} 2u\mathrm{d}u + e^{2uv}(2v\mathrm{d}u + 2u\mathrm{d}v) \\ &= 2\left(e^{u^2}u + e^{2uv}v\right)\mathrm{d}u + 2ue^{2uv}\mathrm{d}v \, . \end{aligned}$$

Substitute this result together with result of $f^*\omega$ in $f^*\theta$ gives

$$f^*\theta = (4u^3v^2e^{uv}\mathrm{d}u + 4u^4ve^{uv}\mathrm{d}v) \wedge \left(2\left(e^{u^2}u + e^{2uv}v\right)\mathrm{d}u + 2ue^{2uv}\mathrm{d}v\right) \, .$$

Multiply bracketed terms and omit terms with $\mathrm{d}\alpha \wedge \mathrm{d}\alpha$ since 1-forms anticommutes $\mathrm{d}u \wedge \mathrm{d}v = -\mathrm{d}v \wedge \mathrm{d}u$

$$\begin{aligned} f^*\theta &= 8u^4v^2e^{3uv}\mathrm{d}u \wedge \mathrm{d}v + 8u^4ve^{uv}\left(e^{u^2}u + e^{2uv}v\right) \\ &= -8u^5ve^{u(u+v)}\mathrm{d}u \wedge \mathrm{d}v \; . \end{aligned}$$

Before proceeding to the last part of computation, let us estimate the outcome. Form $\mu = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is a 3-form on $\mathbb{R}^3$. Because every $(n+1)$-form on n-dimensional manifold vanishes and pullback preserves order of form we expect the 3-form $f^*\mu$ on $\mathbb{R}^2$ to be zero. Indeed

$$\begin{aligned} f^*\mu &= f^*\left(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\right) \\ &= \mathrm{d}(u^2) \wedge \mathrm{d}(2uv) \wedge \mathrm{d}\mathrm{d}(e^{uv}) \\ &= 2u\mathrm{d}u \wedge 2\underbrace{(v\mathrm{d}u + u\mathrm{d}v)}_{\alpha} \wedge e^{uv}\underbrace{(v\mathrm{d}u + u\mathrm{d}v)}_{\alpha} \\ &= 4ue^{uv}\mathrm{d}u \wedge \underbrace{\alpha \wedge \alpha}_{0} = 0 \; . \end{aligned}$$

$\diamond$

Integration of exterior forms

Exterior derivative

In the first part of this chapter we introduce important operation on the space of differential k-forms (shortly k-forms). We recall that a function $f\colon M \to \mathbb{R}$ can be seen as a 0-form and differential $\mathrm{d}f$ is a 1-form. We will use notion of $\mathbb{R}$**-lineárního zobrazení** as a map which is linear with respect to real numbers (as we did implicitly so far).

Theorem 7.1.

There is a unique $\mathbb{R}$-linear map $\mathrm{d}\colon \Omega^k M \to \Omega^{k+1} M,\ k = 0, 1, \ldots, n$ such that the following conditions hold

1. given a function $f \in \Omega^0 M$, $\mathrm{d}f$ is differential of the function
2. $\mathrm{d}(\omega \wedge \varphi) = (\mathrm{d}\omega \wedge \varphi) + (-1)^k (\omega \wedge \mathrm{d}\varphi)$ for $\omega \in \Omega^k M,\ \varphi \in \Omega^l M$,
3. for each function $f \in \Omega^0 M$ holds $\mathrm{d}(\mathrm{d}f) = 0$.

Definition 7.2.

Map from theorem 7.1 is called **exterior derivative**.

Remark. Exterior derivative is operation which asigns to each k-form a $(k+1)$-form. Note that for a function f we have $f \wedge \omega = f\omega$ which, aplied on the second condition in theorem 7.1, gives $\mathrm{d}(f\omega) = \mathrm{d}f \wedge \omega + f\mathrm{d}\omega$. Next important observation here is the fact that all three properties of the above theorem holds also for restriction of forms on arbitrary open subset $U \subset M$ which can be efficiently used to prove the theorem in local coordinates.

Theorem 7.3.

For each k-form ω the following holds

$$\mathrm{d}(\mathrm{d}\omega) = 0\,. \tag{7.4}$$

Theorem 7.5.

For every smooth map $f\colon M \to N$ and every l-form ω on N the following holds

$$\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)\,. \tag{7.6}$$

Definition 7.7.

k-form ω on M is called closed if $\mathrm{d}\omega = 0$ is satisfied. We call ω exact if there is a $(k-1)$-form φ such that $\mathrm{d}\varphi = \omega$.

According to theorem 7.3 we can immediately see that each exact form is closed. Theorem below is about the opposite implication and is, due to historical reasons, often stated as lemma. It is an important result describing relation between exact and closed form.

Theorem 7.8.

(Poincaré Lemma). Let $\omega \in \Omega^k\mathbb{R}^n$ be such that $\mathrm{d}\omega = 0$ is satisfied. Then there is a $\varphi \in \Omega^{k-1}\mathbb{R}^n$ such that $\omega = \mathrm{d}\varphi$.

Remark. The theorem above states: every closed k-form on $\mathbb{R}^n$ is exact. Let us note that Poincaré lemma is often formulated in more general form, using topological notion of simple conectendess.

Theorem 7.9.

Let ω be closd 1-form on $\mathbb{R}^n$ such that for functions f, g on $\mathbb{R}^n$, $\mathrm{d}f = \omega = \mathrm{d}g$ holds. Then $g = f + c$, $c \in \mathbb{R}$.

Proposition 7.10.

Let Z^kM be space of all closed k-forms on M and B^kM space of all exact k-forms on M. Then the following holds

1. Both Z^kM and B^kM are vector space
2. B^kM is a vector subspace in Z^kM.

Definition 7.11.

Vector factorspace $H^kM = Z^kM/B^kM$ is called **k-th de-Rham space** (or **k-th de-Rham cohomology group**) **of manifold** M. For $k = 0$ we set $H^0M = Z^0M$. Element of H^kM is denoted $[\omega]$.

Remark. Representants of class $[\omega] \in H^kM$ are of the form $\omega + \mathrm{d}\varphi$, where ω is closed k-form and φ is $(k-1)$-form.

Theorem 7.12.

Let $f\colon M \to N$ be smooth mapping, $\omega \in Z^kM$. The prescripiton

$$f^{\#}[\omega] = [f^*\omega] \tag{7.13}$$

defines linear map $f^{\#}\colon H^kN \to H^kM$.

Theorem 7.14.

Let $f\colon M \to N,\ g\colon N \to P$ be smooth maps. The following holds

$$(g \circ f)^{\#} = f^{\#} \circ g^{\#} \ . \tag{7.15}$$

The next theorem shows how we can use de-Rham cohomology to distinguish two manifolds.

Theorem 7.16.

Let $f\colon M \to N$ be diffeomorphism. Then $f^{\#}\colon H^kN \to H^kM$ is a linear isomorphism.

In **??** we introduced an algebraic object (vector space H^kM) on a manifold M to obtain new information about the manifold. Subsequent lemma and corresponding theorem show that algebraic structure defined on M can be further enriched with a certain type of multiplication derived from exterior product.

Lemma 7.17.

Consider $\omega \in \Omega^kM$ and $\varphi \in \Omega^lM$.

1. If ω and φ are closed then $\omega \wedge \varphi$ is closed.
2. If ω is exact and φ is closed then $\omega \wedge \varphi$ is exact.

Theorem 7.18.

Let $\omega \in H^kM, \varphi \in H^lM$. The rule $[\omega] \wedge [\varphi] = [\omega \wedge \varphi]$ defines a bilinear map $H^kM \times H^lM \to H^{k+l}M$.

Definition 7.19.

Let $\dim M = n$ and consider the direct product of vector spaces $HM = H^0M \otimes \cdots \otimes H^nM$. Take elements $\omega, \varphi \in HM$ which can be written in the form of a formal sum $\omega = \omega_1 + \cdots + \omega_n,\ \varphi = \varphi_1 + \cdots + \varphi_n$, where $\omega_i, \varphi_i \in H^iM$ for all i. We define the product on HM

$$\omega \wedge \varphi = (\omega_1 + \cdots + \omega_n) \wedge (\varphi_1 + \cdots + \varphi_n) \ . \tag{7.20}$$

We call HM with respect to this product the **de-Rham ring of manifold** M.

Remark. In product $(\omega_1 + \cdots + \omega_n) \wedge (\varphi_1 + \cdots + \varphi_n)$ it can happen that for $\omega_{i_1} \wedge \varphi_{j_1}$ and $\omega_{i_2} \wedge \varphi_{j_2}$ equality $i_1 + j_1 = i_2 + j_2$ holds. In such case both element are to be summed. Note that if inequality $i + j > n$ holds for $omega_i + \varphi_j$ then the result of product vanishes.

Theorem 7.21.

Let X, Y be vector fields on M, $\omega \in \Omega^1M$. Then the following holds

$$\mathrm{d}\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) \ . \tag{7.22}$$

Remark. Let us investigate the formula 7.22. ω being 1-form means that we can evaluate it on a vector field to get a function, which further on can be differentiated in direction of a vector field. This is the meaning of $X\omega(Y)$. The symbol $[X,Y]$ is a value of the Lie bracket on two vector fields which is again a vector field and, thus, can be evaluated on a 1-form, which is denoted by $\omega([X,Y])$. Finally let us point out that $\mathrm{d}\omega$ is a 2-form and has two vector fields as it's arguments for which we write $\mathrm{d}\omega(X,Y)$.

Integration of differential forms

Integration of differential forms is closely related with notion of orientation of a manifold. In case of vector spaces, orientation is defined by a choice of base. We say then, that this basis is positive (or negative) and every other basis related to the chosen one by a transformation matrix with positive determinant is a basis with the same orientation. Every other basis (related with the chosen one by transformation with negative determinant) is said to by negative. This definition can be extendend on manifolds using tangent spaces.

Definition 7.23.

By **orientation of manifold** M we understand such a choice of orientation of all tangent spaces that given $a \in M$ there is a local chart $\varphi\colon U \to V$, $a \in U$ such that $\frac{\partial}{\partial x^1}|_x, \ldots, \frac{\partial}{\partial x^n}|_x$ is a positive basis in T_xM for all $x \in U$.

Theorem 7.24.

There are at most two orientations on a connected manifold.

Remark. Unlike the case of vector spaces, neither definition 7.23 nor theorem 7.24 ensure that there is a global orientation on a manifold. Thus, we define the notion of orientability.

Definition 7.25.

Manifol on which there is an orietation is called **orientable**. In other case we call manifold non-orientable. Orientable manifol with chosen orientation is called **oriented**.

Definition 7.26.

By **standard n-dimensional simplex** (or just n-simplex) $\Delta_n \subset \mathbb{R}^n$ we understand subset given as a solution of rozumíme podmnožinu zadanou rovnicí $x^1 + \cdots + x^n \leq 1$, $x^i \geq 0\ \forall i = 1,\ldots,n$. Subset $s_i \subset \Delta_i$ satisfying $x^i = 0$ is called **i-dimensional face of Δ_n** (or shortly **i-th face of Δ_n**) and subset satisfying $x^1 + \cdots + x^n = 1$ is called **0-th face**.

To be able to view simpleces as geometrial objects which does not necessarily lie in the first hyperoctant of coordinate system we introduce more general definition, encapsulating the same idea.

Definition 7.27.

Subset $D_k \subset \mathbb{R}^n$ is called **k-dimensional siplex**, $k \le n$ if there is such a affine coordinate system $y^1, \dots, y^n$ in $\mathbb{R}^n$ in which D_k is given by $y^{k+1} = 0, \dots, y^n = 0,\ y^1 + \cdots + y^k \le 1,\ y^i \ge 0\ \forall i = 1, \dots, k.$

Remark. D_k is standard k-dimensional simplex in affine subspace given by equations $y^{k+1} = 0, \dots, y^n = 0$ which can be identified with $\mathbb{R}^k$ via coordinates $y^1, \dots, y^k$ from definition 7.27. Also note that i-th face if Δ_n is $(n-1)$-dimensional simplex. Similarly for D_k, i-th face is $(k-1)$-dimensional simplex. In this manner we can proceed up to case of 1-dimensional faces which are (quite naturally) called **edges of simplex**, and of 0-dimensional faces which are called **vertices of simplex**. k-dimensonal face of k-simplex is the simplex itself.

Definition 7.28.

Boundary of n-simplex is the union of all $(n-1)$-dimensional faces of the simplex, denoted by $\partial\Delta_n$. Complement of boundary is called **interior of n-simplex** and is denoted $\Delta_n^0 = \Delta_n \setminus \partial\Delta_n$. **Orientation of a simplex** is defined to be orientation of the ambient space $\mathbb{R}^n$ in which the simplex is defined. **Orientation of $(n-1)$-face of oriented n-simplex** is defined using outer normal principle (orientation of $(n-1)$-face is inherited from orientation of affine subspace in which the face lies; orientation of this subspace is such that by complementing the basis of the underlying vector space, using outer normal, outer with respect to the simplex, results in new basis oriented as the ambient $\mathbb{R}^n$).

Now we want to define simplex on the smooth manifold. Then, we will use these objects to define the notion of integral of differential form.

Definition 7.29.

Let M be n-dimensional manifol. Subset $\sigma_k \subset M$ is called **curvilinear k-simplex** (or shortly k-simplex if the notion is unambigous in a given context) if there is such a neighbourhood U of σ_k and such a local chart $\varphi\colon U \to V$ that $\varphi(\sigma_k)$ is a standard k-simplex in subspace given by equations $x^{k+1} = 0, \dots, x^n = 0$.

Remark. For a standard simplex we have notions of faces, boundary, interior and orientation. Using local chart from definition 7.29, that identify simplex on manifold with the standard simplex, we can introduce notions of faces s_i, boundary $\partial\sigma_k$, interior σ_k^0 and orietation on a curvilinear simlex. Orientation of n-dimensional manifold defines orientation of each n-simplex σ_n. Observe that σ_k^0 is a k-dimensional submanifold in M.

Before we proceed with definition of integration, let us mention the following preparatory idea. Consider n-form ω on n-dimensonal manifold M and oriented n-simplex $\sigma_n \subset M$. Let $\varphi\colon U \to V$ be local chart which maps σ_n to Δ_n and preserves the orientation. Then, pullback of restricted form ω along φ is n-form $\omega_\varphi := \left(\varphi^{-1}\right)^* (\omega|_U)$ on $V \subset \mathbb{R}^n$. Because $\omega_\varphi \in \Lambda^n T_x^* \mathbb{R}^n$ and $\dim(\Lambda^n T_x^* \mathbb{R}^n) = 1$, the form can be written as $\omega_\varphi = a_\varphi \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, where $a_\varphi = a_\varphi\left(x^1, \dots, x^n\right)$ is function on V.

Definition 7.30.

Integral of n-form ω over oriented n-simplex $\sigma_n \subset M$ is defined as

$$\int\limits_{\sigma_n} \omega = \int \cdots \int\limits_{\Delta_n} a_\varphi \mathrm{d}x^1 \ldots \mathrm{d}x^n \,, \tag{7.31}$$

where the right-hand side of the equation is (classical) multiple integral in $\mathbb{R}^n$.

Theorem 7.32.

Definition 7.30 does not depend on the choice of local chart φ.

Subsequent lemma describes how transformation of multiple integrals works and is used to prove the above theorem.

Lemma 7.33.

Let $W, \bar{W}$ be two neighbourhoods Δ_n and $f \colon W \to \bar{W}$, $y^i = f^i(x)$ be a diffeomorphism such that $f(\Delta_n) = \Delta_n$. For each smooth function $b(y) = b(y^1, \ldots, y^n)$ on $\bar{W}$ the following holds

$$\int \cdots \int\limits_{\Delta_n} b(y) \mathrm{d}y^1 \ldots \mathrm{d}y^n = \int \cdots \int\limits_{\Delta_n} b(f(x)) \left| \det \left(\frac{\partial f^i}{\partial x^j} \right) \right| \mathrm{d}x^1 \ldots \mathrm{d}x^n \,. \tag{7.34}$$

Proposition 7.35.

Let $\omega, \varphi \in \Omega^n M$, $c, d \in \mathbb{R}$ and let σ_n be curvilinear oriented simplex on M. Denote $-\sigma_n$ the simplex with opposite orientation. The following holds

$$\int\limits_{\sigma_n} (c\omega + d\varphi) = c \int\limits_{\sigma_n} \omega + d \int\limits_{\sigma_n} \varphi \tag{7.36}$$

$$\int\limits_{-\sigma_n} \omega = - \int\limits_{\sigma_n} \omega \,. \tag{7.37}$$

Similarly as in the case of itegration of n-forms on an n-dimensional manifold, we shall use some preparatory idea to define integration of k-forms, $k \leq n$. Let M be n-dimensinal manifold, $\sigma_k \subset M$ oriented k-simplex and $\varphi \colon U \to V, \sigma_k \subset U$ local chart from definition 7.29 identifying σ_k with standard Δ_k. Then $\varphi^{-1}\left(V \cap \mathbb{R}^k\right)$, where $\mathbb{R}^k \subset \mathbb{R}^n$ is subspace given by equations $x^{k+1} = 0, \ldots, x^n = 0$, is a k-dimensional submanifold in M and σ_k is oriented k-simplex on this submanifold, which we denote N. The following definition uses pullback of ω along embedding of the submanifold $i_N \colon N \hookrightarrow M$, which is a k-form $i_N^* \omega$ on N.

Definition 7.38.

We define the **integral of a k-form ω** over a oriented k-simplex $\sigma_k \subset N \subset M$ as

$$\int\limits_{\sigma_k} \omega = \int\limits_{\sigma_k} i_N^* \omega \,. \tag{7.39}$$

Proposition 7.40.

Definition 7.38 does not depend on a submanifold N.

Proposition 7.41.

Let $\omega, \varphi \in \Omega^k M$, $\sigma_k \subset M$ be an oriented k-simplex, $-\sigma_k$ siplex with opposite orientation, where $k \leq n = \dim(M)$ and c, d are real constants. The following holds

$$\int\limits_{\sigma_k} (c\omega + d\varphi) = c \int\limits_{\sigma_k} \omega + d \int\limits_{\sigma_k} \varphi \tag{7.42}$$

$$\int\limits_{-\sigma_k} \omega = - \int\limits_{\sigma_k} \omega \,. \tag{7.43}$$

Definition 7.44.

k**-dimensional polyhedron** P (or shortly k-polyhedron) on a manifold M is a finite subset $\{\sigma_k^1, \dots, \sigma_k^m\}$ of k-simplices on M such that the union of arbitrary pair from this subset is either shared face of dimension smaller then k, or an empty set. We say that P is **oriented** if all k-simplices of which P is made, are oriented. Then we use notation $P = \sigma_k^1 + \dots + \sigma_k^m$.

Remark. Boundary $\partial\sigma_k$ of an oriented k-simplex is sum of it's oriented $(k-1)$-faces s_i, i.e. $\partial\sigma_k = s_0 + s_1 + \dots + s_n$. Analogously, we denote boundary of a polyhedron by $\partial P = \partial\sigma_k^1 + \dots + \partial\sigma_k^m$.

Definition 7.45.

Integral of a k-form ω over an oriented k-polyhedron P is defined by

$$\int\limits_{P} \omega = \int\limits_{\sigma_k^1} \omega + \dots + \int\limits_{\sigma_k^m} \omega \,. \tag{7.46}$$

Theorem 7.47.

(General Stokes' theorem). Let P be an oriented k-polyhedron on a manifold M, ∂P it's boundary oriented by the principle of outer normal and ω be a $(k-1)$-forma on M. The the following holds

$$\int\limits_{\partial P} \omega = \int\limits_{P} \mathrm{d}\omega \,. \tag{7.48}$$

Exercise 7.49.

Let ω be k-form on a manifold M of dimension n. Calculate coordinate description of exterior derivative $\mathrm{d}\omega$.

Solution. Let us describe ω in local coordinates (x^i)

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \, .$$

We split the exercise in three cases: $k = 0$, $0 < k < n$ and $k = n$. The first one, $k = 0$, means that the considered form is a smooth function on M, i.e. $\omega = \omega\left(x^1, \ldots, x^n\right)$. The first condition in theorem 7.1 then asserts that the exterior derivative is differential, hence

$$\mathrm{d}\omega = \sum_{i=1}^{n} \frac{\partial \omega}{\partial x^i} \mathrm{d}x^i \, .$$

We solve the case $0 < k < n$ using all three conditions stated in 7.1. Also, we will use tha fact that the exterior derivative is linear, which yields

$$\mathrm{d}\omega = \mathrm{d}\left(\sum \omega_{i_1\ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) = \sum \mathrm{d}\left(\omega_{i_1\ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)$$

and we can colve for each summand separately. Let us rewrite the multiplication by smooth function via exterior product with 0-form $\omega_{i_1\ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \omega_{i_1\ldots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$. Now we can use condition 2 from theorem 7.1

$$\begin{aligned} \mathrm{d}\left(\omega_{i_1\ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) &= \mathrm{d}(\omega_{i_1\ldots i_k}) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &\quad + (-1)^0 (\omega_{i_1\ldots i_k}) \wedge \mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \, . \end{aligned}$$

Note that $\omega_{i_1\ldots i_k}$ is a smooth function, i.e.

$$\mathrm{d}\omega_{i_1\ldots i_k} = \sum_{i=1}^{n} \frac{\partial \omega_{i_1\ldots i_k}}{\partial x^i} \mathrm{d}x^i \, .$$

Differentiate bracketed terms in $\mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right)$

$$\mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) = \sum \left(\mathrm{d}(\mathrm{d}x^{i_1}) \wedge \cdots \wedge \mathrm{d}x^{i_k} + \cdots + (-1)^{k-1} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}(\mathrm{d}x^{i_k})\right)$$

and use 3 to get

$$\mathrm{d}\left(\omega_{i_1\ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) = \left(\sum_{j=1}^{n} \frac{\partial \omega_{i_1\ldots i_k}}{\partial x^j} \mathrm{d}x^j\right) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \, .$$

Altogether we have

$$\mathrm{d}\omega = \sum_{1\le i_1<\cdots<i_k\le n} \left(\sum_{j=1}^{n} \frac{\partial \omega_{i_1\ldots i_k}}{\partial x^j} \mathrm{d}x^j\right) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \, . \tag{7.50}$$

Case $k = n$ means that considered form can be written as $\omega = \omega(x^1, \ldots, x^n) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, where $\omega(x^1, \ldots, x^n)$ is a smooth function on M. Differentiating as in the previous case yields $\mathrm{d}\omega = 0$. ◇

Exercise 7.51.

Consider an n-dimensional euclidena space E_n which is equipped with the standard inner product $< -,- >$. Each vector space X on E_n uniquely defines a 1-form ω_X by inserting X as the first argument of the product

$$\omega_X =< X,- > \ . \tag{7.52}$$

for arbitrary vector field Y, then, we have $\omega_X(Y) =< X,Y >$. Further, X defines $(n-1)$-form ω^X by

$$\omega^X(Y_1,\dots,Y_{n-1}) = X \wedge Y_1 \wedge \dots \wedge Y_{n-1} \ , \tag{7.53}$$

where Y_i, $i = 1,\dots,n-1$ are vector field on E_n. Find the coordinate descripiton of both ω_X and ω^X in dimension 3. For vector field $X = \left(x, x+y, z^2\right)$ find ω_X, ω^X and the value of ω_X in vector field $Y = \left(z^2, \sin x, y^2\right)$ at point $\left(\frac{\pi}{2}, 2, 3\right)$. In the same point determine the value of ω^X with respect to Y and $Z = (x,y,z)$.

Remark. Before we start our calculation let us note that the forms ω_X and ω^X can be defined on more general spaces than euclidean ones, namely Riemannian manifolds which are manifold equipped with the notion of inner product. We will be dealing with these kinds of manifolds in later chapters.

Solution. Consider such coordinates (x^i) on E_n that yields coordinates $\left(\frac{\partial}{\partial x^i}\right)$ on T^*E_n which, at each point x, defines orthogonal basis of the tangent space T_xE_n. We also have dual coordinates $\left(\mathrm{d}x^i\right)$ on T^*E_n. Let us first compute general description of both forms. In coordinates we write $X = \sum\limits_{i=1}^{3} X^i \frac{\partial}{\partial x^i}$, $Y = \sum\limits_{j=1}^{3} Y^j \frac{\partial}{\partial x^j}$. Compute

$$\omega_X(Y) =< X,Y > =< \sum_{i=1}^{3} X^i \frac{\partial}{\partial x^i}, \sum_{j=1}^{3} Y^j \frac{\partial}{\partial x^j} > \tag{7.54}$$

$$= \sum_{i,j=1}^{3} X^i Y^j < \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} > \qquad \text{bilinearita} \tag{7.55}$$

$$= \sum_{i,j=1}^{3} X^i Y^j \delta^i_j \qquad \text{ortogonalita} \tag{7.56}$$

$$= \sum_{i=1}^{3} X^i Y^i \tag{7.57}$$

at the same time we have

$$\omega_X =< X,- >= \sum_{i=1}^{3} X^i < \frac{\partial}{\partial x^i}, - > \ .$$

By comparing that with the result 7.57 we see that $< \frac{\partial}{\partial x^i}, - >$ must be a 1-form with value with respect to Y being i-th coordinate Y^i. This corresponds to element of dual basis, $\mathrm{d}x^i$, thus

$$\omega_X = \sum_{i=1}^{3} X^i \mathrm{d}x^i \ . \tag{7.58}$$

Consider further $Z = \sum_{k=1}^{3} Z^k \frac{\partial}{\partial x^k}$. Then $\omega^X(Y,Z)$ is a function with the value at x corresponding to the volume of an oriented paralellpiped given by vectors $X(x), Y(x), Z(x)$. Therefore, we can write

$$\omega^X(Y,Z) = \det \begin{pmatrix} X^1 & Y^1 & Z^1 \\ X^2 & Y^2 & Z^2 \\ X^3 & Y^3 & Z^3 \end{pmatrix} \ .$$

Using the Laplace expansion along the first column we get

$$\omega^X(Y,Z) = X^1\left(Y^2Z^3 - Y^3Z^2\right) - X^2\left(Y^1Z^3 - Y^3Z^1\right) + X^3\left(Y^1Z^2 - Y^2Z^1\right) \ . \tag{7.59}$$

Generic 2-form in dimension 3 is of the form

$$f_1 \mathrm{d}x^1 \wedge \mathrm{d}x^2 + f_2 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + f_3 \mathrm{d}x^3 \wedge \mathrm{d}x^1 \ .$$

Inserting fields Y and Z gives (directly from the definition of exterior product)

$$f_1\left(Y^1Z^2 - Y^2Z^1\right) + f_2\left(Y^2Z^3 - Y^3Z^2\right) + f_3\left(Y^3Z^1 - Y^1Z^3\right) \ . \tag{7.60}$$

Comparing this with 7.59 we see that $f_1 = X^3,\ f_2 = X^1,\ f_3 = X^2$, hence, we can write ω^X in the following form

$$\omega^X = X^3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 + X^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + X^2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 \ . \tag{7.61}$$

Before inserting specific values, let us use classical notation in dimension three $x^1 = x,\ x^2 = y,\ x^3 = z$. Given vector field $X = \left(x, x+y, z^2\right)$, the 1-form ω_X can be written as

$$\omega_X = x\mathrm{d}x + (x+y)\mathrm{d}y + z^2\mathrm{d}z \ .$$

Applying vector field $Y = \left(z^2, \sin x, y^2\right)$ we get

$$\omega_X(Y) = \omega_X(z^2, \sin x, y^2) = xz^2 + (x+y)\sin x + z^2y^2 \ ,$$

which at point $x = \left(\frac{\pi}{2}, 2, 3\right)$ yields the value of $\omega_X(Y)(x) = 5\pi + 38$. Finally, the 2-form ω^X is

$$\omega^X = z^2 \mathrm{d}x \wedge \mathrm{d}y + x\mathrm{d}y \wedge \mathrm{d}z + (x+y)\mathrm{d}z \wedge \mathrm{d}x \ .$$

Insert Y and $Z = (x,y,z)$ and rearrange

$$\begin{aligned} \omega^X(Y,Z) &= \omega^X\left((z^2, \sin x, y^2), (x,y,z)\right) \\ &= z^2\left(z^2y - x\sin x\right) + x\left(z\sin x - y^3\right) + (x+y)\left(xy^2 - z^3\right) \ . \end{aligned}$$

Valute at x is $\omega^X(Y,Z)(x) = 9\left(18 - \frac{pi}{2}\right) + \frac{pi}{2}(-5) + (\frac{pi}{2} + 2)(2\pi - 27) = \pi^2 - \frac{33\pi}{2} + 108$. ◇

Remark 1. Differential forms shown in exercise 7.51 have interesting interpretation in physics: ω_X can be used to compute the work done on a particle traveling along a curve inside a force field X and ω^X can be used to compute the cflow of X over an oriented surface. We will see concrete examples of this in the exercise below.

Exercise 7.62.

Compute the flow of field $X = \left(x^2+y^2, 1, z\right)$ over the surface $M = \{(x,y,z) \in \mathbb{R}^3 | \, x^2+y^2 = z, \; z \in < 0,2)\}$. Determine the work done by X along curve γ which is part of helix in the first octant of standard coordinates, having z-axis as the center of rotation.

Solution. Equation $x^2+y^2 = z$ consider in $\mathbb{R}^3$ describes rotational paraboloid symetric with respect to z-axis with vertex at $(0,0,0)$. Condition $z \in < 0,2)$ means that paraboloid is bounded from above. It is an oriented surface over which we can integrate in accordance to theory of this chapter. Taking into account the specific form of X and boundedness of M, we expect the result in the form of a finite number. Due to 1, the flow of X over M is equal to integral of the differential form $\omega^X \in \Omega^2\mathbb{R}^3$ given by 7.61, where we substitute X from the exercise. So, we can compute the flow as

$$\int_M \omega^X = \int_M (x^2+y^2)\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z + z\mathrm{d}z \wedge \mathrm{d}x \, .$$

Both the field X and the surface M exhibits rotational symetries of a circle which is a hint to use polar coordinates on M, $p \colon \mathbb{R}^2 \to M \subset \mathbb{R}^3$, in which the paraboloid can be parametrized as follows

$$\begin{aligned} x(r,\varphi) &= r\cos\varphi \\ y(r,\varphi) &= r\sin\varphi \\ z(r,\varphi) &= r^2 \, . \end{aligned} \tag{7.63}$$

where $(r,\varphi) \in N =< 0, \sqrt{2}) \times < 0, 2\pi >$. The original integral is transformed in accordance to 7.38

$$\int_M \omega^X = \int_N p^*\omega^X \, ,$$

where p^* is the pullback along composed map $p \colon \underset{r,\varphi}{\mathbb{R}^2} \to \underset{x(r,\varphi),y(r,\varphi)}{\mathbb{R}^2} \to \underset{x,y,x^2+y^2}{\mathbb{R}^3}$ (this is to be viewed as composing the transformation into polar coordintes with parametrization of paraboloid in the standard coordinates). To compute $p^*\omega^X$, we firstly use linearity of p^* and the fact that the pullback of a 0-form is given by precomposition

$$p^*\omega^X = p^*\left((x^2+y^2)\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z + z\mathrm{d}z \wedge \mathrm{d}x\right) \tag{7.64}$$

$$= r^2 p^*(\mathrm{d}x \wedge \mathrm{d}y) + p^*(\mathrm{d}y \wedge \mathrm{d}z) + r^2 p^*(\mathrm{d}z \wedge \mathrm{d}x) \, . \tag{7.65}$$

In the next step we use commutativity of pullback with exterior derivative, stated in theorem 7.5, and compatibility with exterior product $p^*(\mathrm{d}\alpha \wedge \mathrm{d}\beta) = \mathrm{d}p^*\alpha \wedge \mathrm{d}p^*\beta$. Hence, we can firstly compute the differentials of transformed coordinates 7.63, then obtain pullbacks of basis forms by exterior multiplication. For differentials we have

$$\begin{aligned}
\mathrm{d}x &= \cos\varphi \mathrm{d}r - r\sin\varphi \mathrm{d}\varphi \\
\mathrm{d}y &= \sin\varphi \mathrm{d}r + r\cos\varphi \mathrm{d}\varphi \\
\mathrm{d}z &= 2r\mathrm{d}r \ .
\end{aligned}$$

Because terms of the form $\mathrm{d}\alpha \wedge \mathrm{d}\alpha$ vanishes, exterior products are of the form

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= r\cos^2\varphi \mathrm{d}r \wedge \mathrm{d}\varphi - r\sin^2\varphi \mathrm{d}\varphi \wedge \mathrm{d}r = r\mathrm{d}r \wedge \mathrm{d}\varphi \\
\mathrm{d}y \wedge \mathrm{d}z &= -2r^2\cos\varphi \mathrm{d}r \wedge \mathrm{d}\varphi \\
\mathrm{d}z \wedge \mathrm{d}x &= 2r^2\sin\varphi \mathrm{d}r \wedge \mathrm{d}\varphi
\end{aligned}$$

which is to be substitued in 7.65

$$\begin{aligned}
p^*\omega^X &= r^3\mathrm{d}r \wedge \mathrm{d}\varphi - 2r^2\cos\varphi \mathrm{d}r \wedge \mathrm{d}\varphi + 2r^4\sin\varphi \mathrm{d}r \wedge \mathrm{d}\varphi \\
&= \left(r^3 - 2r^2\cos\varphi + 2r^4\sin\varphi\right) \mathrm{d}r \wedge \mathrm{d}\varphi \ .
\end{aligned}$$

Finally, we can apply defintion 7.30 to compute the flow

$$\begin{aligned}
\int_N p^*\omega^X &= \int_N \left(r^3 - 2r^2\cos\varphi + 2r^4\sin\varphi\right) \mathrm{d}r \wedge \mathrm{d}\varphi \\
&= \int_0^{\sqrt{2}} \int_0^{2\pi} \left(r^3 - 2r^2\cos\varphi + 2r^4\sin\varphi\right) \mathrm{d}r\mathrm{d}\varphi \\
&= 2\pi \int_0^{\sqrt{2}} r^3\mathrm{d}r = 2\pi \ .
\end{aligned}$$

The flow of X over the surface M is 2π (of suitable units). Let us proceed to compute the work donne by X along the helix $\gamma \subset \mathbb{R}^3$. This curve can be parametrized as

$$\begin{aligned}
x(\varphi) &= a\cos\varphi \\
y(\varphi) &= a\sin\varphi \\
z(\varphi) &= R\varphi \ ,
\end{aligned} \tag{7.66}$$

where a, R are non-negative real constants. Sice we consider only the first octant part of the helix, φ is taken from $(0, \frac{\pi}{2})$. Work done by the field X can be computed using 1-form ω_X. We computed general expression of this form in dimension 3 in exercise 7.51. Thus, substituing for X in 7.58 and applying d yields

$$\begin{aligned}
\omega_X &= X^1\mathrm{d}x + X^2\mathrm{d}y + X^3\mathrm{d}z \\
&= (x^2 + y^2)\mathrm{d}x + \mathrm{d}y + z\mathrm{d}z \\
&= (a - a\sin\varphi + a\cos\varphi + R)\mathrm{d}\varphi \ .
\end{aligned}$$

Integral is of the form

$$\int_{\gamma}\omega_X = \int_{0}^{\frac{\pi}{2}}(a - a\sin\varphi + a\cos\varphi + R)\mathrm{d}\varphi = \frac{\pi}{2}(a+R)$$

Therefore, the work done by X along γ amounts to $\frac{\pi}{2}(a+R)$ (suitable units). $\diamond$

Exercise 7.67.

Integrate the form $\omega \in \Omega^2\mathbb{R}^4$, $\omega = \mathrm{d}x^3 \wedge \mathrm{d}x^4 + x^1x^3\mathrm{d}x^2 \wedge \mathrm{d}x^4$ on a subset $M \subset \mathbb{R}^4$ given by equations $(x^1)^2 + (x^2)^2 = 1,\ (x^3)^2 + (x^4)^2 = 1$.

Solution. Firstly we choose more convenient description of M. Observe that both defining equations $(x^1)^2 + (x^2)^2 = 1,\ (x^3)^2 + (x^4)^2 = 1$ describes a unit circle. In other words, projection of M on x^1x^2 plane gives a unit circle, and for each point of this circle we have another one, given by condition on x^3, x^4 coordinates. Thus, we are dealing with the product of two unit circles$S^1 \times S^1$ which is a 2-dimensional plane called torus. Because we consider this plane to be embedded in $\mathbb{R}^4$ we can use simple parametrization $p\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by polar coodrinates, one for each circle.

$$\begin{aligned} x^1(\varphi,\theta) &= \cos\varphi & \qquad x^3(\varphi,\theta) &= \cos\theta \\ x^2(\varphi,\theta) &= \sin\varphi & \qquad x^4(\varphi,\theta) &= \sin\theta\ , \end{aligned}$$

where $(\varphi,\theta) \in N =< 0, 2\pi > \times < 0, 2\pi >$. Corresponding differential are

$$\begin{aligned} \mathrm{d}x^1 &= -\sin\mathrm{d}\varphi & \qquad \mathrm{d}x^3 &= -\sin\mathrm{d}\theta \\ \mathrm{d}x^2 &= \cos\mathrm{d}\varphi & \qquad \mathrm{d}x^4 &= \cos\mathrm{d}\theta\ . \end{aligned}$$

Applying exterior product yields

$$\begin{aligned} \mathrm{d}x^3 \wedge \mathrm{d}x^4 &= -\sin\theta\cos\theta\mathrm{d}\theta \wedge \mathrm{d}\theta = 0 \\ \mathrm{d}x^2 \wedge \mathrm{d}x^4 &= \cos\varphi\cos\theta\mathrm{d}\varphi \wedge \mathrm{d}\theta\ , \end{aligned}$$

hence, the pullback of ω along parametrization p is of the form

$$p^*\omega = \cos^2\varphi\cos^2\theta\mathrm{d}\varphi \wedge \mathrm{d}\theta\ .$$

Using definitions 7.38 and 7.30, we are now ready to determine the value of $\int_M \omega$. Let us compute

$$\begin{aligned}
\int\limits_N p^*\omega &= \int\limits_N \cos^2\varphi\cos^2\theta \mathrm{d}\varphi\wedge\mathrm{d}\theta \\
&= \int\limits_0^{2\pi}\int\limits_0^{2\pi} \cos^2\varphi\cos^2\theta \mathrm{d}\varphi\wedge\mathrm{d}\theta \\
&= \int\limits_0^{2\pi}\int\limits_0^{2\pi} \left(\frac{1+\cos 2\varphi}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)\mathrm{d}\varphi\mathrm{d}\theta \\
&= \frac{1}{4}\int\limits_0^{2\pi}\int\limits_0^{2\pi} \mathrm{d}\varphi\mathrm{d}\theta = \pi^2 \,.
\end{aligned}$$

We conclude $\int\limits_M \omega = \pi^2$. ◇

Exercise 7.68.

Compute $\int\limits_M \omega$, where $\omega = xz\mathrm{d}x\wedge\mathrm{d}y + xy\mathrm{d}y\wedge\mathrm{d}z + 2yz\mathrm{d}z\wedge\mathrm{d}x$ and M is boundary of the standard 3-simplex in $\mathbb{R}^3$.

Solution. Exercise can be easily solved using Stokes' theorem 7.47. Sine M is boundary of 3-simplex Δ_3, we can write

$$\int\limits_M \omega = \int\limits_{\Delta_3} \mathrm{d}\omega \,.$$

Let us make use of results from exercise 7.49, where we have computed coordinate description of $\mathrm{d}\omega$ for arbitrary ω. In our case, due to 7.50, and applying condition 2 stated in theorem 7.1 it holds

$$\mathrm{d}\omega = y\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + 2z\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + x\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = (x+y+2z)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \,.$$

Subsequent computation follows from definition 7.30

$$
\begin{aligned}
\int\limits_{\Delta_3} \mathrm{d}\omega &= \int\limits_{\Delta_3} (x+y+2yz)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&= \int\limits_0^1 \int\limits_0^{1-x} \int\limits_0^{1-x-y} (x+y+2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int\limits_0^1 \int\limits_0^{1-x} [xz+yz+z^2]_0^{1-x-y}\mathrm{d}x\mathrm{d}y \\
&= \int\limits_0^1 \int\limits_0^{1-x} \big(x(1-x-y)+y(1-x-y)+(1-x-y)^2\big) \\
&= \int\limits_0^1 \int\limits_0^{1-x} (1-x-y)\,\mathrm{d}x\mathrm{d}y \\
&= \int\limits_0^1 \big((1-x)-x(1-x)-\frac{1}{2}(1-x)^2\big)\mathrm{d}x \\
&= \int\limits_0^1 (-\frac{1}{2}x^2-x+\frac{1}{2})\mathrm{d}x = -\frac{1}{6}
\end{aligned}
$$

Altogether we have $\int\limits_M \omega = -\frac{1}{6}$. ◇

Exercise 7.69.

Give some examples of non-orientable manifolds.

Solution. Amongs the most classical examples of non-orientable manifold is the Möbius strip, given by parametrization

$$
\begin{aligned}
x(\varphi,r) &= \left(1+\frac{r}{2}\cos\frac{\varphi}{2}\right)\cos\varphi \\
y(\varphi,r) &= \left(1+\frac{r}{2}\cos\frac{\varphi}{2}\right)\sin\varphi \\
z(\varphi,r) &= \frac{r}{2}\sin\varphi\ ,
\end{aligned}
$$

where $\varphi\in[0,2\pi]$ a $r\in(-1,1)$ (r controls the width of the strip). Topologicaly, we can construct this space by gluing two opposite sites of a rectangle along the opposite direction (gluing in the same direction leads to cylinder). Another example of non-oriented manifold is the Klein bottle. Parametrization of this object is quite complicated and we will not describe it, yet, we shall give the topological construction. By gluing both pairs of opposite sites of a rectangle, first pair gluing along the same direction, the second pair gluing along the opposite direction, results in Klein bottle. Also, we can proceed by gluing two Möbius strips along their bounderies in the same direction. Note that, unlike the case of the Möbius strip, the Klein bottle cannot be embedded in $\mathbb{R}^3$. The least dimension for the bottle to be embedded in real space is four. $\diamond$

Submanifolds of Euclidean space

Definition 8.1.

Assume local parametrization f of an $n-$dimensional submanifold M in $m-$dimensional Euclidean space E_m

$$x^p = f^p\left(u^1, \ldots, u^n\right),\ p = 1, \ldots, m$$

First fundamental form g_{ij} is induced from an Euclidean scalar product $(-,-)$.

$$g_{ij}(u) = \left(\frac{\partial f(u)}{\partial u^i}, \frac{\partial f(u)}{\partial u^j}\right) = g_{ji}(u)$$

Remark. **Scalar product** of tangent vectors $A, B \in T_u M$ is given by

$$(A,B) = \sum_{i,j=1}^{n} g_{ij}(u) a^i b^j$$

Remark. **Length of a curve** C at M is given by

$$s = \int_I \sqrt{\sum_{i,j=1}^{n} g_{ij}(u(t)) \frac{\mathrm{d}u^i}{\mathrm{d}t} \frac{\mathrm{d}u^j}{\mathrm{d}t}} \mathrm{d}t$$

Definition 8.2.

Diffeomorphism $f : M \to \bar{M}$ is called **isometry** if scalar product is invariant for every $x \in M$.

Definition 8.3.

Intrinsic geometry of a submanifold M are properties of M that remain invariant under isometries. These properties are derived from first fundamental form.

Remark. Other properties are called **extrinsic**.

Definition 8.4.

Normal space $N_x M$ of a submanifold $M \subset E_m$ at a point x is set of all vectors orthogonal to its tangent space $T_x M$.

Remark. For every $x \in M$ we have $T_x E_m = T_x M + N_x M$.

Definition 8.5.

We say that vectors $v(t) \in T_{p(t)}M$ are **parallely transported** along a path $p(t)$ on M if $\frac{\mathrm{d}v(t)}{\mathrm{d}t} \in N_{p(t)}$ for every $t \in I$.

Definition 8.6.

In local coordinates, Christoffel symbols can be expressed as

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} \tilde{g}^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

with $\sum_{l=1}^{n} \tilde{g}^{kl} g_{lm} = \delta^k_m$.

Theorem 8.7.

Christoffel symbols Γ^k_{ij} belong to intrinsic geometry.

Theorem 8.8.

Parallel transport condition is

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{jk}(p(t))\, v^j \frac{\mathrm{d}p^k}{\mathrm{d}t} = 0$$

Exercise 8.9.

Find all isometries of $\mathbb{R}^n$ with Euclidean metric.

Solution. We are looking for a transformation

$$y^i = f^i(x)$$

Jacobian of such a transformation is

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial f^i}{\partial x^j}$$

Metric is $(0,2)$-tensor and it transforms as

$$g'_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} g_{kl} = \frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j} g_{kl}$$

Since we want metric to remain invariant, we have

$$g_{ij} = \frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j} g_{kl}$$

Now we can make use of a fact that g_{ij} is constant and we take a derivative

$$0 = \frac{\partial^2 f^k}{\partial x^i \partial x^m} \frac{\partial f^l}{\partial x^j} g_{kl} + \frac{\partial f^k}{\partial x^i} \frac{\partial^2 f^l}{\partial x^j \partial x^m} g_{kl}$$

Now i and j are free indices so we can switch them in a first term. We also make use of symmetry of g and switch k and l as well. We get

$$0 = \frac{\partial^2 f^l}{\partial x^j \partial x^m} \frac{\partial f^k}{\partial x^i} g_{kl} + \frac{\partial f^k}{\partial x^i} \frac{\partial^2 f^l}{\partial x^j \partial x^m} g_{kl}$$

We finally get

$$0 = \frac{\partial f^k}{\partial x^i} \left(\frac{\partial^2 f^l}{\partial x^j \partial x^m} + \frac{\partial^2 f^l}{\partial x^j \partial x^m} \right) g_{kl}$$

and thus

$$\frac{\partial^2 f^l}{\partial x^j \partial x^m} = 0$$

We will be looking for global affine isomorphisms that leave metric invariant. Affine isomorphisms are of form

$$y^i = A^i_j x^j + y^i_0$$

Jacobian of such a transformation is

$$\frac{\partial y^i}{\partial x^j} = A^i_k \frac{\partial x^k}{\partial x^j} = A^i_k \delta^k_j = A^i_j$$

where δ^k_j is a Kronecker delta. From now on, it will be useful to use matrix notation instead of indices. In matrix notation, this transformation rule is

$$\mathbf{g}' = \mathbf{A}^\intercal \mathbf{g} \mathbf{A}$$

In case of isometries of Euclidean metric, matrix of linear transformation has to satisfy

$$\mathbf{E} = \mathbf{A}^\intercal \mathbf{E} \mathbf{A}$$

where $\mathbf{E}$ is $n-$dimensional identity matrix. What can we say about these matrices? First, let's take det of this equation. We get

$$\det \mathbf{E} = \det \mathbf{E} \ \det \mathbf{A}^2$$

since $\det \mathbf{A} = \det \mathbf{A}^\intercal$ This tells us that

$$\det \mathbf{A} = \pm 1$$

and all these transformations are invertible. How do we compose the affine transformations however? We can't do it by simple matrix multiplication. The most convenient way is to look at these transformations as on linear transformations $\mathbb{R}^{n+1}$ with $x_{n+1} = 1$. Such a transformation is

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \mathbf{y_0} \\ 0 & 1 \end{pmatrix}$$

and the action is

$$\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \begin{pmatrix} \mathbf{A} & \mathbf{y_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{Ax} + \mathbf{y_0} \\ 1 \end{pmatrix}$$

Operation is usual matrix multiplication. This gives us associativity. The unit element is a unit matrix. To show that these transformations form a group, all we have to do is check if they are closed under the composition. This is pretty straightforward

$$\begin{pmatrix} \mathbf{A} & \mathbf{y_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{z_0} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{AB} & \mathbf{Az_0} + \mathbf{y_0} \\ 0 & 1 \end{pmatrix}$$

Thus we can talk about isometry group of $\mathbb{R}^n$, which will be denoted by $ISO(n)$. How many parameters does $ISO(n)$ have? First, we check the parameters of $\mathbf{A}$.To do this, we can use Taylor expansion. Let's assume that

$$\mathbf{A} = \mathbf{E} + \varepsilon\mathbf{X} + O\left(\varepsilon^2\right)$$

The equation, up to second order in ε is

$$\mathbf{E} = \mathbf{E} + \varepsilon\left(\mathbf{X}^\mathsf{T} + \mathbf{X}\right) + O\left(\varepsilon^2\right)$$

The infinitesimal transformation X is antisymmetric, which gives us $\frac{n(n-1)}{2}$ parameters. This is in fast a subgroup of $ISO(n)$, denoted by $O(n)$. This group consist of all rotations and reflections of Euclidean space. If we add n parameters of y_0, we get

$$\dim\, ISO(n) = \frac{n(n+1)}{2}$$

which is a dimension of $O(n+1)$. This is not a coincidence, since $\mathbb{R}^n$ can be thought of as a limiting case of a sphere $S^n_R \subset \mathbb{R}^{n+1}$ whose radius R is taken to the infinity. If we take a look at action of $O(n+1)$ around the point $(0,...,1)$, we recover the action $ISO(n)$ up to first order. This is a special case of the procedure called group contraction. ◇

Exercise 8.10.

Show that second fundamental form $\mathbb{I}$ does not belong to an intrinsic geometry of a submanifold.

Solution. To illustrate this example, I will examine the second fundamental form of a surface embedded in $\mathbb{R}^3$. First, let's assume that our surface is a parametric surface

$$z = z(x,y)$$

and that plane $z = 0$ is tangent at the origin. From Taylor expansion we get

$$z = \frac{\partial^2 z}{\partial x^2}\frac{x^2}{2} + \frac{\partial^2 z}{\partial x \partial y} xy + \frac{\partial^2 z}{\partial y^2}\frac{y^2}{2} + \text{higher order}$$

The second fundamental form is a quadratic form

$$\mathbb{I} = \frac{\partial^2 z}{\partial x^2}\mathrm{d}x^2 + 2\frac{\partial^2 z}{\partial x \partial y}\mathrm{d}x\mathrm{d}y + \frac{\partial^2 z}{\partial y^2}\mathrm{d}y^2$$

Let us denote $L = \frac{\partial^2 z}{\partial x^2}$, $M = \frac{\partial^2 z}{\partial x \partial y}$ and $N = \frac{\partial^2 z}{\partial y^2}$.

In general, surface in $\mathbb{R}^3$ is parametrized by a smooth, vector valued function $\mathbf{r}(u,v)$. For a convenience, derivatives of r will be denoted by lower indices. In this notation, we can write

$$\mathbb{I} = b_{ij}\mathrm{d}u^i\mathrm{d}u^j$$

Since parametrization is regular, $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent in the domain of $\mathbf{r}$. This allows us to calculate a unique unit normal vector field, given by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r_u} \times \mathbf{r_v}|}$$

The coefficients b_{ij} are given by a projections of partial derivatives into a tangent plane. This can be written as

$$b_{ij} = r^k_{ij} n_k$$

where n_k are components of a normal covector. We can see that the components of second fundamental form depend on the embedding of the submanifold via the normal vector field. Because of this, second fundamental form is not a part of the intrinsic geometry of a submanifold. $\diamond$

Exercise 8.11.

Find normal spaces for following submanifolds at every point $p \in M$

1. $S^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$
2. $H^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\right\}$

Solution. These submanifolds are of a codimension 1, their normal space will therefore be one dimensional. We can use the result of a previous exercise, namely

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r_u} \times \mathbf{r_v}|}$$

1. First, we parametrize our sphere by spherical coordinates
$$
\begin{aligned}
x &= \cos\varphi\sin\theta,\\
y &= \sin\varphi\sin\theta,\\
z &= \cos\theta.
\end{aligned}
$$
with $\varphi \in (0,2\pi)$ and $\theta \in (0,\pi)$. To calculate our normal vector field, we have to calculate derivatives of a parametrization with respect to intrinsic coordinates
$$
\mathbf{r}_\varphi = \begin{pmatrix} -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta \\ 0 \end{pmatrix}, \qquad \mathbf{r}_\theta = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix}
$$
Now, we have to calculate their cross product
$$
\mathbf{r}_\varphi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\varphi\sin\theta & \cos\varphi\sin\theta & 0 \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \end{vmatrix} = -\sin\theta\mathbf{r}
$$
the unit normal vector therefore is
$$
\mathbf{n} = \frac{\mathbf{r}_\varphi \times \mathbf{r}_\theta}{\left|\mathbf{r}_\varphi \times \mathbf{r}_\theta\right|} = \frac{-\sin\theta\mathbf{r}}{\sin\theta\,|\mathbf{r}|} = -\hat{\mathbf{r}}
$$
where $\hat{\mathbf{r}}$ is a unit position vector.

2. In this case, parametrization is
$$
\begin{aligned}
x &= \cos\varphi\cosh\theta,\\
y &= \sin\varphi\cosh\theta,\\
z &= \sinh\theta.
\end{aligned}
$$
with $\varphi \in (0,2\pi)$ and $\theta \in \mathbb{R}$. We calculate derivatives of a parametrization again
$$
\mathbf{r}_\varphi = \begin{pmatrix} -\sin\varphi\cosh\theta \\ \cos\varphi\cosh\theta \\ 0 \end{pmatrix}, \qquad \mathbf{r}_\theta = \begin{pmatrix} \cos\varphi\sinh\theta \\ \sin\varphi\sinh\theta \\ \cosh\theta \end{pmatrix}
$$
The cross product is
$$
\mathbf{r}_\varphi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\varphi\cosh\theta & \cos\varphi\cosh\theta & 0 \\ \cos\varphi\sinh\theta & \sin\varphi\sinh\theta & -\cosh\theta \end{vmatrix} = \cosh\theta \begin{pmatrix} x \\ y \\ -z \end{pmatrix} = \cosh\theta\tilde{\mathbf{r}}
$$
where $\tilde{\mathbf{r}} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$. In this case, normal vector field is
$$
\mathbf{n} = \frac{\cosh\theta\,\tilde{\mathbf{r}}}{\cosh\theta\,|\tilde{\mathbf{r}}|} = \hat{\tilde{\mathbf{r}}}
$$

◇

Riemann space

Definition 9.1.

Riemann metric on a manifold M is a smooth mapping $g\colon M \to S^2_+T^*M$ such that $p \circ g = \mathrm{id}_M$, with p the projection on a tensor bundle $S^2_+T^*M$, i.e. $p\colon S^2_+T^*M \to M$, and S^2_+ denote space of all positive definite quadratic forms. Pair (M,g) is called a Riemann space or Riemann manifold.

Remark. In local coordinates x^i na M has g a coordinate expression $g = \left(g_{ij}(x)\right)$, $g_{ij} = g_{ji}$. We can also write it as

$$g = \sum_{i,j=1}^{n} g_{ij}(x)\,\mathrm{d}x^i \mathrm{d}x^j$$

or

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}(x)\,\mathrm{d}x^i \mathrm{d}x^j.$$

$\mathrm{d}s$ is a length element on M.

Definition 9.2.

Length of a curve C, with parametric expression $f(t)$, $t \in [a,b]$, from point $f(a)$ to $f(b)$ is a number

$$s = \int_a^b \sqrt{g\left(\frac{\mathrm{d}f}{\mathrm{d}t}, \frac{\mathrm{d}f}{\mathrm{d}t}\right)}\,\mathrm{d}t.$$

If the coordinate expression for f is $x^i = x^i(t)$, then

$$s = \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}(x(t))\,\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}}\,\mathrm{d}t. \tag{9.3}$$

It is clear that length of a curve is independent on its parametrization.

Definition 9.4.

Let (M,g) and $(\bar{M},\bar{g})$ be two Riemann spaces and let $f\colon M \to \bar{M}$ be a smooth mapping. if $g(v,w) = \bar{g}(T_xf(v), T_xf(w))$ for every $x \in M$ a $v,w \in T_xM$, then f is called isometry.

Theorem 9.5.

If (M,g) is a Riemann manifold and $f\colon N \to M$ is immersion, then f^*g is Riemann metric on N. If f has a coordinate expression $x^i = f^i(y^p)$ and $k = \dim N$, then

$$f^*g = \sum_{p,q}^{k} \sum_{i,j=1}^{n} g_{ij}(f(y)) \frac{\partial f^i}{\partial y^p} \frac{\partial f^j}{\partial y^q} \mathrm{d}y^p \mathrm{d}y^q.$$

Theorem 9.6.

On every topological manifold M exists a Riemann metric.

Theorem 9.7.

For a positive definit quadratic form is $g\colon V \to V^*$ linear isomorphism.

Definition 9.8.

Let (M,g) be a Riemann space and $f\colon M \to \mathbb{R}$ a function. Vector field $\tilde{g}(\mathrm{d}f) := \mathrm{grad} f$, where $\tilde{g}$ is a matrix inverse to a matrix of metric is called a gradient of f. Gradient has following coordinate expression

$$(\mathrm{grad} f)^i = \sum_{j=1}^{n} \tilde{g}^{ij} \frac{\partial f}{\partial x^j}.$$

Definition 9.9.

Let (M,g) be a oriented Riemann space, i,e. manifold M is orientable. Then the every tangent space T_xM is oriented vector space with scalar product and for every n-touple of vectors $v_1, \ldots, v_n \in T_xM$ we can define exterior product

$$[v_1, \ldots, v_n]_x \in \mathbb{R}.$$

This exterior product defines n-form $\mathrm{vol}(g) : M \to \bigwedge^n T^*M$, called volume n-form of an oriented Riemann space (M,g).

Theorem 9.10.

In coordinate system $\left(x^1, \ldots, x^n\right)$ with orientation matching (M,g) holds

$$\mathrm{vol}(g) = \sqrt{\det\left(g_{ij}\right)}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Theorem 9.11.

Number $\mathrm{Vol}\sigma_n = \int_{\sigma_n} \mathrm{vol}(g)$ is called a volume of a simplex $\sigma_n \subset (M,g)$ and following formula holds

$$\mathrm{Vol}(\sigma_n) = \underset{\Delta_n}{\int \cdots \int} \sqrt{\det\left(g_{ij}\right)}\, \mathrm{d}x^1 \ldots \mathrm{d}x^n.$$

Definition 9.12.

Set $\mathbb{H}^2 := \left\{(x,y) \in \mathbb{R}^2 \mid y > 0\right\}$ with a metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$ is called a upper half plane.

Exercise 9.13.

Let (M,g) be $n-$dimensional Riemann space, $x \in M$ is a point on a manifold, $\{e_i\}_{i=1}^n$ is orthonormal basis T_xM a $\left\{f^i\right\}_{i=1}^n$ jis a basis of dual space such that $f^j(e_i) = \delta_i^j$. For $X \in T_xM$ and $\alpha \in T_x^*M$ we define musical isomorphisms

$$\begin{aligned} \flat\colon T_xM \to T_x^*M, \qquad & X^\flat = g(X,-), \\ \sharp\colon T_x^*M \to T_xM, \qquad & \alpha^\sharp = \tilde{g}(\alpha,-). \end{aligned}$$

Prove that following formulas hold

$$g\left(X,\alpha^\sharp\right) = \alpha(X) = \tilde{g}\left(X^\flat\alpha\right).$$

Solution. First, let us remind how the evaluation of forms on vectors works:

$$\alpha(X) = \alpha_i f^i\left(X^j e_j\right) = \alpha_i X^j f^j(e_i) = \alpha_i X^j \delta_i^j = \alpha_i X^i.$$

For coordinate expressions of the isomorphisms we have

$$\begin{aligned} \left(X^\flat\right)_j &= \left(X^\flat\right)_i f^i\left(e_j\right) = X^\flat\left(e_j\right) = g\left(X,e_j\right) = X^i g\left(e_i,e_j\right) = X^i g_{ij}, \\ \left(\alpha^\sharp\right)^j &= \left(\alpha^\sharp\right)^i f^j(e_i) = f^j\left(\alpha^\sharp\right) = \tilde{g}\left(\alpha,f^j\right) = \alpha_i \tilde{g}\left(f^i,f^j\right) = \alpha_i \tilde{g}^{ij}. \end{aligned}$$

Furthermore

$$\begin{aligned} g\left(X,\alpha^\sharp\right) &= X^i g_{ij}\tilde{g}^{jk}\alpha_k = X^i\delta_i^k\alpha_k = X^i\alpha_i = \alpha(X), \\ \tilde{g}\left(X^\flat,\alpha\right) &= \tilde{g}^{ij}g_{jk}X^k\alpha_j = \delta_k^j X^k\alpha_j = X^j\alpha_j = \alpha(X). \end{aligned}$$

◇

Exercise 9.14.

Let us assume the upper half plane $\mathbb{H}^2$, as a subset of $\mathbb{C}$instead of a subset of $\mathbb{R}^2$ with the identification $z = x + \mathrm{i}y$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a square matrix with unit determinant. Show that these matrices form a group. This group is called a special linear group $\mathrm{SL}(2,\mathbb{R})$. Now let's assume a mapping

$$\begin{aligned} \psi\colon \mathrm{SL}(2,\mathbb{R}) \times \mathbb{H}^2 &\to \mathbb{H}^2, \\ (A,z) &\mapsto Az = \frac{az+b}{cz+d}. \end{aligned}$$

Show that this map is a left action of a group $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$.In the end show that $\mathrm{SL}(2,\mathbb{R})$ is an isometry group of $\mathbb{H}^2$, i.e. that every element of $\mathrm{SL}(2,\mathbb{R})$ leaves the form of metric invariant:

$$\mathrm{d}s^2 = \frac{(\mathrm{d}\Re z)^2 + (\mathrm{d}\Im z)^2}{(\Im z)^2}.$$

Solution. Operation in $\mathrm{SL}(2,\mathbb{R})$ is a matrix multiplication, which is associative. Unit element if an identity matrix. Inverse element exists because the determinant is nonzero. Set of these matrices is also closed under the operation becuase determinant of product of matrices will also be 1.

Next we will show that the image of ψ is $\mathbb{H}^2$. To do this we will have to calculate an imaginary part of Az,

$$\begin{aligned}\Im Az &= \Im\left(\frac{az+b}{cz+d}\right) = \Im\left(\frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)}\right) = \Im\left(\frac{(ax+\mathrm{i}ay+b)(cx+d-\mathrm{i}cy)}{|cz+d|^2}\right) = \\ &= \frac{1}{|cz+d|^2}\Im\left((x^2+y^2)ac+(ad+bc)x+\mathrm{i}(ad-bc)y+bd\right) = \frac{y}{|cz+d|^2} > 0.\end{aligned}$$

For an action $\cdot$ of a group G on a set X following identities hold

$$\begin{aligned} e\cdot x &= x, && \forall x\in X,\\ (gh)\cdot x &= g\cdot(h\cdot x), && \forall x\in X,\ g,h\in G,\end{aligned}$$

where e is a unit element of a group. For a unit matrix E we have $Ez = \frac{z+0}{0+1} = z$. Now let $B = \begin{pmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & \tilde{d}\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, then

$$B(Az) = B\frac{az+b}{cz+d} = \frac{\tilde{a}\frac{az+b}{cz+d}+\tilde{b}}{\tilde{c}\frac{az+b}{cz+d}+\tilde{d}} = \frac{\left(\tilde{a}a+\tilde{b}c\right)z+\tilde{a}b+\tilde{b}d}{\left(\tilde{c}a+\tilde{d}c\right)z+\tilde{c}b+\tilde{d}d} = (BA)z,$$

and therefore ψ is really an action of a group on a set.

Metric can be rewritten as

$$\mathrm{d}s^2 = \frac{(\mathrm{d}\Re z)^2+(\mathrm{d}\Im z)^2}{(\Im z)^2} = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(\Im z)^2}$$

and let us denote $w = Az$. From previous calculations we know that

$$\Im w = \frac{\Im z}{|cz+d|^2}.$$

This gives us

$$\begin{aligned}\mathrm{d}w &= \mathrm{d}\left(\frac{az+b}{cz+d}\right) = \frac{a\mathrm{d}z\,(cz+d)-c\mathrm{d}z\,(az+b)}{(cz+d)^2} = \frac{\mathrm{d}z}{(cz+d)^2},\\ \mathrm{d}\bar{w} &= \mathrm{d}\left(\frac{a\bar{z}+b}{c\bar{z}+d}\right) = \frac{a\mathrm{d}\bar{z}\,(c\bar{z}+d)-c\mathrm{d}\bar{z}\,(a\bar{z}+b)}{(c\bar{z}+d)^2} = \frac{\mathrm{d}\bar{z}}{(c\bar{z}+d)^2}.\end{aligned}$$

Therefore

$$\frac{\mathrm{d}w\mathrm{d}\bar{w}}{(\Im w)^2} = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(cz+d)^2(c\bar{z}+d)^2}\frac{|cz+d|^4}{(\Im z)^2} = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(\Im z)^2}.$$

◇

Exercise 9.15.

Calculate the metric induced from $\mathbb{R}^3$ with Euclidean scalar product on a

1. sphere S^2,
2. torus T^2.

Solution. 1. Sphere is usually parametrized by $\varphi \in [0,2\pi)$ a $\theta \in [0,\pi]$, via

$$\begin{aligned} x &= \cos\varphi\sin\theta, \\ y &= \sin\varphi\sin\theta, \\ z &= \cos\theta. \end{aligned}$$

Differentials are

$$\begin{aligned} \mathrm{d}x &= -\sin\varphi\sin\theta\,\mathrm{d}\varphi + \cos\varphi\cos\theta\,\mathrm{d}\theta, \\ \mathrm{d}y &= \cos\varphi\sin\theta\,\mathrm{d}\varphi + \sin\varphi\cos\theta\,\mathrm{d}\theta, \\ \mathrm{d}z &= -\sin\theta\,\mathrm{d}\theta. \end{aligned}$$

We can insert them into the expression for length element of $\mathbb{R}^3$, tj. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, and we get length element of a sphere

$$\mathrm{d}s^2 = \sin^2\theta\,\mathrm{d}\varphi^2 + \mathrm{d}\theta^2.$$

2. Torus is parametrized by $\varphi \in [0,2\pi)$ a $\theta \in [0,2\pi)$, via

$$\begin{aligned} x &= (R + r\cos\theta)\cos\varphi, \\ y &= (R + r\cos\theta)\sin\varphi, \\ z &= r\sin\theta, \end{aligned}$$

where $R > 0$ is the distance of the center of a cylinder and the origin of coordinate system and $r > 0$ is a radius of a cylinder. Differentials are

$$\begin{aligned} \mathrm{d}x &= -(R + r\cos\theta)\sin\varphi\,\mathrm{d}\varphi - r\sin\varphi\cos\theta\,\mathrm{d}\theta, \\ \mathrm{d}y &= (R + r\cos\theta)\cos\varphi\,\mathrm{d}\varphi - r\sin\theta\sin\varphi\,\mathrm{d}\theta, \\ \mathrm{d}z &= r\cos\theta\,\mathrm{d}\theta. \end{aligned}$$

Inserting the differentials into the length element of $\mathbb{R}^3$ gives us

$$\mathrm{d}s^2 = (R + r\cos\theta)^2\,\mathrm{d}\varphi^2 + r^2\,\mathrm{d}\theta^2.$$

◇

Exercise 9.16.

Calculate the metric of S^2 in a stereographic projection.

Solution. Stereographic projection on plane with Cartesian coordinates (u,v) is given by

$$x = \frac{2u}{u^2+v^2+1}, \quad y = \frac{2v}{u^2+v^2+1}, \quad z = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

The differentials are

$$\begin{aligned}
\mathrm{d}x &= 2\frac{\left(u^2+v^2+1\right)-2u^2}{\left(u^2+v^2+1\right)^2}\mathrm{d}u - \frac{4uv}{\left(u^2+v^2+1\right)^2}\mathrm{d}v,\\
\mathrm{d}y &= -\frac{4uv}{\left(u^2+v^2+1\right)^2}\mathrm{d}u + 2\frac{\left(u^2+v^2+1\right)-2v^2}{\left(u^2+v^2+1\right)^2}\mathrm{d}v,\\
\mathrm{d}z &= \frac{4u\,\mathrm{d}u+4v\,\mathrm{d}v}{\left(u^2+v^2+1\right)^2}.
\end{aligned}$$

Inserting the differentials into the length element of $\mathbb{R}^3$ gives us

$$\mathrm{d}s^2 = \frac{4}{(u^2+v^2+1)}\left(\mathrm{d}u^2+\mathrm{d}v^2\right).$$

◇

Exercise 9.17.

Calculate the length of a circle

$$(x-a)^2+(y-b)^2 = r^2,$$

where $a,b \in \mathbb{R}$ abd $r>0$ in

1. $\mathbb{R}^2$ with a metric
$$\mathrm{d}s^2 = \frac{4}{(x^2+y^2+1)}\left(\mathrm{d}x^2+\mathrm{d}y^2\right).$$
2. upper half plane with a restriction $b>r$.

Solution. In both cases we arrive to a similar integral. Let us now calculate this integral generally. For $\alpha,\beta,\gamma \in \mathbb{R}$ we have

$$I := \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\alpha t^2+\beta t+\gamma} = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\left(\sqrt{\alpha}t+\frac{\beta}{2\sqrt{\alpha}}\right)^2+\delta},$$

where $\delta = \gamma - \frac{\beta^2}{4\alpha}$. Now

$$I = \frac{1}{\delta}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\left(\sqrt{\frac{\alpha}{\delta}}t+\frac{\beta}{2\sqrt{\alpha\delta}}\right)^2+1} = \left|\begin{array}{c} \sqrt{\frac{\alpha}{\delta}}t+\frac{\beta}{2\sqrt{\alpha\delta}} = s \\ \sqrt{\frac{\alpha}{\delta}}\mathrm{d}t = \mathrm{d}s \end{array}\right| = \frac{1}{\sqrt{\alpha\delta}}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s^2+1} =$$
$$= \frac{\pi}{\sqrt{\alpha\gamma-\frac{\beta^2}{4}}}.$$

1. We parametrize the circle via angle $\varphi \in [0, 2\pi)$ as

$$x = r\cos\varphi + a$$
$$y = r\sin\varphi + b.$$

Now we make a use of the expression for a length of a curve

$$s = \int_0^{2\pi} \sqrt{\frac{4r^2\left(\sin^2\varphi + \cos^2\varphi\right)}{\left(1 + (r\cos\varphi + a)^2 + (r\sin\varphi + b)^2\right)^2}}\,\mathrm{d}\varphi.$$

This integral can be solved via substitution $t = \tan\frac{\varphi}{2}$, that gives us

$$\begin{aligned} s &= 4r\int_{-\infty}^{\infty} \frac{1}{(1 + r^2 + a^2 + b^2) + 2r\left[a\left(\frac{1-t^2}{t^2+1}\right) + 2b\frac{t}{t^2+1}\right]}\frac{\mathrm{d}t}{t^2+1} = \\ &= 4r\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(1 + r^2 + a^2 + b^2 - 2ra)\cdot t^2 + 4rb\cdot t + (1 + r^2 + a^2 + b^2 + 2ra)}, \end{aligned}$$

This is an integral I for $\alpha = 1 + r^2 + a^2 + b^2 - 2ra$, $\beta = 4rb$ a $\gamma = 1 + r^2 + a^2 + b^2 + 2ra$, and the length of a circle is

$$s = \frac{4r\pi}{\sqrt{(1 + r^2 + a^2 + b^2)^2 - 4a^2r^2 - 4b^2r^2}}.$$

2. We will use the same parametrization as in previous case. The length of a circle is

$$s = r\int_0^{2\pi} \frac{\mathrm{d}\varphi}{r\sin\varphi + b}.$$

We use the same substitution as well and the integral I. We get

$$s = 2r\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{bt^2 + 2rt + b} = \frac{2r\pi}{\sqrt{b^2 - r^2}}.$$

◇

Exercise 9.18.

Calculate the volume and the surface of a torus.

Solution. We use the result of the Exercise 9.15. Volume element is

$$\mathrm{vol}\,(g) = r\,(R + r\cos\theta)\,\mathrm{d}\varphi \wedge \mathrm{d}\theta.$$

Integration over $\varphi \in [0,2\pi)$ and $\theta \in [0,2\pi)$ gives us

$$S = \int_0^{2\pi} \int_0^{2\pi} r\,(R + r\cos\theta)\,\mathrm{d}\varphi \mathrm{d}\theta = 4\pi^2 rR.$$

Volume can be calculated in two ways. First way is tu assume the radius of a cylinder r as addition coordinate, induce the metric on a full torus and integrate the induced volume form. The other way is to simply integrate the formula for the surface over $r' \in (0,r]$.In both cases we get the integral

$$V = 4\pi^2 R \int_0^r r' \mathrm{d}r' = 2\pi^2 r^2 R.$$

◇

Exercise 9.19.

Show that $\mathbb{R}^n$ for $n \geq 2$ has in hyperspherical coordinates $(r, \phi_1, \ldots, \phi_{n-1})$ given by

$$\begin{aligned}
x_1 &= r\cos\phi_1,\\
x_2 &= r\sin\phi_1\cos\phi_2,\\
&\vdots\\
x_{n-1} &= r\sin\phi_1 \ldots \sin\phi_{n-2}\cos\phi_{n-1},\\
x_n &= r\sin\phi_1 \ldots \sin\phi_{n-2}\sin\phi_{n-1},
\end{aligned}$$

where $r > 0$, $\phi_1, \ldots, \phi_{n-2} \in [0,\pi)$ a $\phi_{n-1} \in [0,2\pi)$, metric of the form

$$\mathrm{d}s_n^2 = \mathrm{d}r^2 + r^2 \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i-1} \sin^2\phi_j \right) \mathrm{d}\phi_i^2.$$

Solution. We can use the induciton. For $n = 2$ we have polar coordinates

$$\mathrm{d}s_2^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\phi_1^2.$$

Assume that the metric in hyperspherical coordinates $(r, \phi_1, \ldots, \phi_{n-2})$ on $\mathbb{R}^{n-1}$ is of the form

$$\mathrm{d}s_{n-1}^n = \mathrm{d}r^2 + r^2 \sum_{i=1}^{n-2} \left(\prod_{j=1}^{i-1} \sin^2\phi_j \right) \mathrm{d}\phi_i^2,$$

on $\mathbb{R}^n$ we have coordinates $(\tilde{x}_1, \ldots, \tilde{x}_n)$, given by

$$\begin{aligned}
\tilde{x}_1 &= x_1,\\
&\vdots\\
\tilde{x}_{n-2} &= x_{n-2},\\
\tilde{x}_{n-1} &= x_{n-1}\cos\phi_{n-1},\\
\tilde{x}_n &= x_{n-1}\sin\phi_{n-1},
\end{aligned}$$

where $(x_1,\dots,x_{n-1})$ are Cartesian coordinates on $\mathbb{R}^{n-1}$ and ϕ_{n-1} is additional hyperspherical coordinate. Length element on $\mathbb{R}^n$ is

$$\mathrm{d}s_n^2 = \mathrm{d}\tilde{x}_1^2 + \dots + \tilde{x}_n^2.$$

After subsituting the differentials we get

$$\begin{aligned}
\mathrm{d}s_n^2 &= \mathrm{d}x_1^2 + \dots + \mathrm{d}x_{n-2}^2 + \mathrm{d}x_{n-1}^2 \cos^2\phi_{n-1} + x_{n-1}^2 \sin^2\phi_{n-1}\,\mathrm{d}\phi_{n-1}^2 - \\
&\quad - 2x_{n-1}\cos\phi_{n-1}\sin\phi_{n-1}\,\mathrm{d}x_{n-1}\,\mathrm{d}\phi_{n-1} + \sin^2\phi_{n-1}\,\mathrm{d}x_{n-1}^2 + x_{n-1}^2\cos^2\phi_{n-1}\,\mathrm{d}\phi_{n-1}^2 + \\
&\quad + 2x_{n-1}\cos\phi_{n-1}\sin\phi_{n-1}\,\mathrm{d}x_{n-1}\,\mathrm{d}\phi_{n-1} = \\
&= \mathrm{d}s_{n-1}^2 + x_{n-1}^2\,\mathrm{d}\phi_{n-1}^2 = \mathrm{d}r^2 + r^2 \sum_{i=1}^{n-2}\left(\prod_{j=1}^{i-1}\sin^2\phi_j\right)\mathrm{d}\phi_i^2 + r^2\prod_{j=1}^{n-2}\sin^2\phi_j\,\mathrm{d}\phi_{n-1}^2 = \\
&= \mathrm{d}r^2 + r^2\sum_{i=1}^{n-1}\left(\prod_{j=1}^{i-1}\sin^2\phi_j\right)\mathrm{d}\phi_i^2.
\end{aligned}$$

◇

Exercise 9.20.

Assume Riemann space $(\mathbb{R}^3, g)$, where g is the metric from Exercise 9.19. How does the gradient of f looks like in this space?

Solution. Let us denote $\phi_0 := r$. Metric is diagonal, inverse metric can be obtained very easily

$$\tilde{g}^{00} = 1, \quad \tilde{g}^{11} = \frac{1}{r^2}, \quad \tilde{g}^{22} = \frac{1}{r^2\sin^2\phi_1}.$$

Components of a gradient are

$$(\operatorname{grad} f)^0 = \frac{\partial f}{\partial r}, \quad (\operatorname{grad} f)^1 = \frac{1}{r^2}\frac{\partial f}{\partial \phi^1}, \quad (\operatorname{grad} f)^2 = \frac{1}{r^2\sin^2\phi_1}\frac{\partial f}{\partial \phi_2}.$$

◇

Exercise 9.21.

Calculate ”surface” and ”volume” of an n-dimensional ball of a radius R.

Solution. First we will calculate a ”surface” of an n-dimensional ball of a radius R, i.e. we will calculate the volume of its boundary, $n-1$-dimensional sphere of a radius R. We will use the result of the Exercise 9.19. Metric induces od a sphere from $\mathbb{R}^n$ is

$$\mathrm{d}s_{S^{n-1}}^2 = R^2\sum_{i=1}^{n-1}\left(\prod_{j=1}^{i-1}\sin^2\phi_j\right)\mathrm{d}\phi_i^2.$$

Volume is then given by an integral of the volume form over whole sphere

$$R^{n-1}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\dots\sin\phi_{n-2}\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\dots\mathrm{d}\phi_{n-1} =: S_R(n-1).$$

Integration of this $(n-1)$-dimensional integral can be avoided by using the integral of a Gauss function. We can use

$$I := \int_{\mathbb{R}^n} \mathrm{e}^{-\sum_{i=1}^n x_i^2}\,\mathrm{d}x_1\ldots\mathrm{d}x_n = \pi^{n/2}.$$

We can recalculate this integral in the hyperspherical coordinates. Hyperspherical coordinates however do not cover entire $\mathbb{R}^n$. However, the noncovered part of $\mathbb{R}^n$ has zero measure and can be ignored in this calculation.

$$I = \int_0^\infty\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi \mathrm{e}^{-r^2} r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\ldots\sin\phi_{n-2}\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\ldots\mathrm{d}\phi_{n-1}\,\mathrm{d}r =$$
$$= S_1(n-1)\int_0^\infty \mathrm{e}^{-r^2} r^{n-1}\mathrm{d}r.$$

Integral over r can be simplified by a substitution

$$\int_0^\infty \mathrm{e}^{-r^2} r^{n-1}\mathrm{d}r = \left|\begin{array}{c} r^2 = t \\ 2r\mathrm{d}r = \mathrm{d}t \end{array}\right| = \frac{1}{2}\int_0^\infty \mathrm{e}^{-t}\,t^{\frac{n}{2}-1}\mathrm{d}t = \frac{1}{2}\Gamma\left(\frac{n}{2}\right),$$

where $\Gamma(n)$ is a Gamma function. We know that both values of I has to be the same and the volume of an $(n-1)$-dimensional sphere is

$$S_1(n-1) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

After rescaling be a constant R the volume form rescales with R^{n-1} and the volume of $(n-1)$-dimensional sphere with radius R is

$$S_R(n-1) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}R^{n-1}.$$

Volume of $n-$dimensional ball can be obtained by integrating of $S_R(n-1)$ over the radius, because in the metric of $n-$dimensional ball is new dependence on r, i.e.

$$V_R(n) = \int_0^R S_r(n-1)\,\mathrm{d}r = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}R^n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}R^n.$$

◇

Exercise 9.22.

Assume a sphere S^2 with metric induced from $\mathbb{R}^3$. Find curves that have the same angle with every meridian given by

$$\begin{aligned} p:\varphi &= \varphi_0, & \varphi_0 &= \text{const} \in [0,2\pi),\\ \theta &= t, & t &\in (0,\pi), \end{aligned}$$

These curves are called loxodromes and were of great importance for a medieval maritime navigation.

Solution. Let us denote such a curve by $l(t) = (\varphi(t), \theta(t))$. Tangent vector of meridian is $\dot{p} = (0,1)$. Let $\beta = \text{const} \in (0,\pi)$ be an angle between curves $l(t)$ a $p(t)$. Then

$$\cos\beta = \frac{g\left(\dot{p}, \dot{l}\right)}{\sqrt{g(\dot{p},\dot{p}) \cdot g(\dot{l},\dot{l})}},$$

where g is a metric. We substitute tangent vectors to curves and get

$$\cos\beta = \frac{\dot{\theta}}{\sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2}}. \tag{9.23}$$

From the denominator we get a condition

$$\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2 \neq 0. \tag{9.24}$$

Equation (9.23) can be rewritten as

$$\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2 = \frac{\dot{\theta}^2}{\cos^2\beta}, \tag{9.25}$$

if $\beta \neq \frac{\pi}{2}$. For $\beta = \frac{\pi}{2}$ we get from (9.23) $\theta = \text{const}$ and the restriction (9.24) tells us that $\varphi = At + B$, where $A, B \in \mathbb{R}$. These curves are circles of latitude. We can rewrite this equation to (9.25)

$$\begin{aligned}
\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2 &= \frac{\dot{\theta}^2\left(\sin^2\beta + \cos^2\beta\right)}{\cos^2\beta}, \\
\sin^2\theta\dot{\varphi}^2 &= \dot{\theta}^2\tan^2\beta, \\
\frac{\dot{\varphi}^2}{\dot{\theta}^2} &= \frac{\tan^2\beta}{\sin^2\theta}.
\end{aligned}$$

We eliminate t, take a square root and get differential equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\theta} = \pm\frac{\tan\beta}{\sin\theta},$$

with a solution

$$\pm\text{cotan}\,\beta\,(\varphi + \alpha) = \ln\left(\tan\frac{\theta}{2}\right),$$

where $\alpha \in \mathbb{R}$ is integration constant. ◇

Parallel transport of vector fields

Definition 10.1.

Let (M,g) be n-dimensional Riemann manifold with local coordinates x^i, $i = 1,\ldots,n$. Objects

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} \tilde{g}^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right),$$

where $\tilde{g}$ is matrix inverse to the matrix of a metric g, are called Christoffel symbols of a metric g.

Definition 10.2.

We say that vector field $v(t) = \left(v^i(t)\right)$ along a path $p(t) = \left(p^i(t)\right)$ is being transported iff

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{jk}\left(p(t)\right) v^j \frac{\mathrm{d}p^k}{\mathrm{d}t} = 0.$$

Definition 10.3.

For a set of $v(t)$ tangent vectors along $p(t)$ on Riemann manifold (M,g) we define a covariant derivative $\frac{\nabla v(t)}{\mathrm{d}t} = \left(\frac{\nabla v^i(t)}{\mathrm{d}t}\right)$ via coordinate expression

$$\frac{\nabla v^i}{\mathrm{d}t} = \frac{\mathrm{d}v^i}{\mathrm{d}t} + \sum_{j,k=1}^{n} \Gamma^i_{jk}\left(p(t)\right) v^j \frac{\mathrm{d}p^k}{\mathrm{d}t}.$$

Definition 10.4.

Covariant derivative of a vector field Y in direction of a vector field X on a Riemann manifold (M,g) is a vector field $\nabla_X Y$ with a coordinate expression

$$(\nabla_X Y)^i = \sum_{j,k=1}^{n}\left(\frac{\partial Y^j(x)}{\partial x^j} + \Gamma^i_{kj}(x)Y^k(x)\right)X^j(x),$$

where $x \in M$.

Theorem 10.5.

For arbitrary vector fields X,Y na M and function $f\colon M \to \mathbb{R}$ holds

1. $\nabla_X\left(Y_1 + Y_2\right) = \nabla_X Y_1 + \nabla_X Y_2$,

2. $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$,
3. $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$,
4. $\nabla_{fX} Y = f\nabla_X Y$.

Definition 10.6.

Mapping $\nabla\colon \mathscr{X}M \times \mathscr{X}M \to \mathscr{X}M$, $(X,Y) \mapsto \nabla_X Y$, which fulfills 1. - 4. of Theorem 10.5, is called a linear connection on manifold M,

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Definition 10.7.

Linear connection of Definition 10.4 is called Levi-Civita connection of metric g on manifold M.

Definition 10.8.

Tensor field ∇Y of type $(1,1)$ is called a covariant differential of vector field Y. Coordinate expression is

$$(\nabla Y)^i_j = \frac{\partial Y^i}{\partial x^j} + \sum_{k=1}^{n} \Gamma^i_{kj} Y^k.$$

Definition 10.9.

Let (M,g) be a Riemann manifold and $f\colon M \to \mathbb{R}$ a function. Mapping $f \mapsto \Delta f$ given in local coordinates by

$$\Delta f = \sum_{i=1}^{n} (\nabla \mathrm{grad} f)^i_i$$

is called a Laplace operator.

Definition 10.10.

Path $\gamma\colon I \to M$, where $I \subseteq \mathbb{R}$, is a geodesic path of a linear connection ∇, if its tangent vector $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$ is paralelly transported along γ, i.e.

$$\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0.$$

Theorem 10.11.

Geodesic path $\gamma(t) = (x^i(t))$ is given by a system of second order ordinary differential equations

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk}(x) \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \quad i = 1, \dots, n.$$

Theorem 10.12.

If $\gamma(t)$ is geodesic path on M, the path $\gamma(at+b)$ is also geodesic form every $a,b \in \mathbb{R}$.

Definition 10.13.

Curve $C \subset M$ is a geodesic curve of a linear connection ∇ (or just geodesic), if there exists a parametrization $\gamma(t)$ that is geodesic path.

Exercise 10.14.

Calculate Christoffel symbols of

1. $\mathbb{R}^3$ in spherical coordinates and S^2 with metric induced from $\mathbb{R}^3$,
2. a torus T^2 with metric induced from $\mathbb{R}^3$,
3. upper half plane, i.e. $\mathbb{H}^2 = \left\{(x^1,x^2) \in \mathbb{R}^2 \,\middle|\, x^2 > 0\right\}$, with metric $\mathrm{d}s^2 = \left(x^2\right)^{-2}\left(\left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2\right)$.

Solution. 1. Spherical coordinates are given by

$$\begin{aligned} x &= r\cos\varphi\sin\theta, \\ y &= r\sin\varphi\sin\theta, \\ z &= r\cos\theta. \end{aligned}$$

Our notation is $r := \varphi^0$, $\varphi^1 := \varphi$ a $\varphi^2 := \theta$. Nonzero components of metric are $g_{00} = 1$, $g_{11} = r^2\sin^2\theta$ and $g_{22} = r^2$, nonzero components of an inverse metric are $\tilde{g}^{00} = 1$, $\tilde{g}^{11} = \frac{1}{r^2\sin^2\theta}$ a $\tilde{g}^{22} = r^{-2}$. Christoffel symbols are calculated from the definition

$$\Gamma^0_{00} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{0i}\left(\frac{\partial g_{0i}}{\partial\varphi^0} + \frac{\partial g_{i0}}{\partial\varphi^0} - \frac{\partial g_{00}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{00}\left(\frac{\partial g_{00}}{\partial\varphi^0} + \frac{\partial g_{00}}{\partial\varphi^0} - \frac{\partial g_{00}}{\partial\varphi^0}\right) = 0,$$

$$\Gamma^0_{01} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{0i}\left(\frac{\partial g_{0i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^0} - \frac{\partial g_{01}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{00}\left(\frac{\partial g_{00}}{\partial\varphi^1} + \frac{\partial g_{01}}{\partial\varphi^0} - \frac{\partial g_{01}}{\partial\varphi^0}\right) = 0,$$

$$\Gamma^0_{02} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{0i}\left(\frac{\partial g_{0i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^0} - \frac{\partial g_{02}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{00}\left(\frac{\partial g_{00}}{\partial\varphi^2} + \frac{\partial g_{02}}{\partial\varphi^0} - \frac{\partial g_{02}}{\partial\varphi^0}\right) = 0,$$

$$\Gamma^0_{11} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{0i}\left(\frac{\partial g_{1i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{00}\left(\frac{\partial g_{10}}{\partial\varphi^1} + \frac{\partial g_{01}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^0}\right) = -r\sin^2\theta,$$

$$\Gamma^0_{12} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{0i}\left(\frac{\partial g_{1i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{00}\left(\frac{\partial g_{10}}{\partial\varphi^2} + \frac{\partial g_{02}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^0}\right) = 0,$$

$$\Gamma^0_{22} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{0i}\left(\frac{\partial g_{2i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^2} - \frac{\partial g_{22}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{00}\left(\frac{\partial g_{20}}{\partial\varphi^2} + \frac{\partial g_{02}}{\partial\varphi^2} - \frac{\partial g_{22}}{\partial\varphi^0}\right) = -r,$$

$$\Gamma^1_{00} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{1i}\left(\frac{\partial g_{0i}}{\partial\varphi^0} + \frac{\partial g_{i0}}{\partial\varphi^0} - \frac{\partial g_{00}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{01}}{\partial\varphi^0} + \frac{\partial g_{10}}{\partial\varphi^0} - \frac{\partial g_{00}}{\partial\varphi^1}\right) = 0,$$

$$\Gamma^1_{01} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{1i}\left(\frac{\partial g_{0i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^0} - \frac{\partial g_{01}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{01}}{\partial\varphi^1} + \frac{\partial g_{11}}{\partial\varphi^0} - \frac{\partial g_{01}}{\partial\varphi^1}\right) =$$

$$= \frac{1}{2}\frac{1}{r^2\sin^2\theta}2r\sin^2\theta = \frac{1}{r},$$

$$\Gamma^1_{02} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{1i}\left(\frac{\partial g_{0i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^0} - \frac{\partial g_{02}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{01}}{\partial\varphi^2} + \frac{\partial g_{12}}{\partial\varphi^0} - \frac{\partial g_{02}}{\partial\varphi^1}\right) = 0,$$

$$\Gamma^1_{11} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{1i}\left(\frac{\partial g_{1i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{11}}{\partial\varphi^1} + \frac{\partial g_{11}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^1}\right) = 0,$$

$$\Gamma^1_{12} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{1i}\left(\frac{\partial g_{1i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{11}}{\partial\varphi^2} + \frac{\partial g_{12}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^1}\right) =$$

$$= \frac{1}{2}\frac{1}{r^2\sin^2\theta}2r^2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta},$$

$$\Gamma^1_{22} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{1i}\left(\frac{\partial g_{2i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^2} - \frac{\partial g_{22}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{21}}{\partial\varphi^2} + \frac{\partial g_{12}}{\partial\varphi^2} - \frac{\partial g_{22}}{\partial\varphi^1}\right) = 0,$$

$$\Gamma^2_{00} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{2i}\left(\frac{\partial g_{0i}}{\partial\varphi^0} + \frac{\partial g_{i0}}{\partial\varphi^0} - \frac{\partial g_{00}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{02}}{\partial\varphi^0} + \frac{\partial g_{20}}{\partial\varphi^0} - \frac{\partial g_{00}}{\partial\varphi^2}\right) = 0,$$

$$\Gamma^2_{01} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{2i}\left(\frac{\partial g_{0i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^0} - \frac{\partial g_{01}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{02}}{\partial\varphi^1} + \frac{\partial g_{21}}{\partial\varphi^0} - \frac{\partial g_{01}}{\partial\varphi^2}\right) = 0,$$

$$\Gamma^2_{02} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{2i}\left(\frac{\partial g_{0i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^0} - \frac{\partial g_{02}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{02}}{\partial\varphi^2} + \frac{\partial g_{22}}{\partial\varphi^0} - \frac{\partial g_{02}}{\partial\varphi^2}\right) = \frac{1}{2}\frac{1}{r^2}2r =$$

$$= \frac{1}{r},$$

$$\Gamma^2_{11} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{2i}\left(\frac{\partial g_{1i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{12}}{\partial\varphi^1} + \frac{\partial g_{21}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^2}\right) =$$

$$= -\frac{1}{2}\frac{1}{r^2}2r^2\sin\theta\cos\theta = -\sin\theta\cos\theta,$$

$$\Gamma^2_{12} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{2i}\left(\frac{\partial g_{1i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{12}}{\partial\varphi^2} + \frac{\partial g_{22}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^2}\right) = 0,$$

$$\Gamma^2_{22} = \frac{1}{2}\sum_{i=0}^{2}\tilde{g}^{2i}\left(\frac{\partial g_{2i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^2} - \frac{\partial g_{22}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{22}}{\partial\varphi^2} + \frac{\partial g_{22}}{\partial\varphi^2} - \frac{\partial g_{22}}{\partial\varphi^2}\right) = 0.$$

We got six nonzero symbols. Christoffel symbols on a sphere S^2 are obtained by setting $r = 1$, i.e.

$$\Gamma^1_{12} = \frac{\cos\theta}{\sin\theta}, \quad \Gamma^2_{11} = -\sin\theta\cos\theta.$$

2. We again use notation $\varphi^1 := \varphi$, $\varphi^2 := \theta$. Nonzero components of metric and inverse metric are $g_{11} = (R + r\cos\theta)^2$, $g_{22} = r^2$, $\tilde{g}^{11} = \frac{1}{(R+r\cos\theta)^2}$ a $g^{22} = \frac{1}{r^2}$. As in previous case $g_{11} = g_{11}(\theta)$ a $g_{22} = \text{const}$, there will be only two nonzero Christoffel symbols

$$\begin{aligned}
\Gamma^1_{12} &= \frac{1}{2}\tilde{g}^{1i}\left(\frac{\partial g_{1i}}{\partial\varphi^2} + \frac{\partial g_{i2}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{11}\left(\frac{\partial g_{11}}{\partial\varphi^2} + \frac{\partial g_{12}}{\partial\varphi^1} - \frac{\partial g_{12}}{\partial\varphi^1}\right) = \\
&= \frac{1}{2}\frac{2(R+r\cos\theta)(-r\sin\theta)}{(R+r\cos\theta)^2} = -\frac{r\sin\theta}{R+r\cos\theta}, \\
\Gamma^2_{11} &= \frac{1}{2}\tilde{g}^{2i}\left(\frac{\partial g_{1i}}{\partial\varphi^1} + \frac{\partial g_{i1}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^i}\right) = \frac{1}{2}\tilde{g}^{22}\left(\frac{\partial g_{12}}{\partial\varphi^1} + \frac{\partial g_{21}}{\partial\varphi^1} - \frac{\partial g_{11}}{\partial\varphi^2}\right) = \\
&= -\frac{1}{2}\frac{2(R+r\cos\theta)(-r\sin\theta)}{r^2} = \frac{1}{r}(R+r\cos\theta)\sin\theta.
\end{aligned}$$

3.Nonzero components of a metric and inverse metric are $g_{11} = g_{22} = \left(x^2\right)^{-2}$ a $\tilde{g}^{11} = \tilde{g}^{22} = \left(x^2\right)^2$. Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}\tilde{g}^{1i}\left(\frac{\partial g_{1i}}{\partial x^1} + \frac{\partial g_{i1}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^i}\right) = 0, \\
\Gamma^1_{12} &= \frac{1}{2}\tilde{g}^{1i}\left(\frac{\partial g_{1i}}{\partial x^2} + \frac{\partial g_{i2}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^i}\right) = \frac{1}{2}\tilde{g}^{11}\frac{\partial g_{11}}{\partial x^2} = \frac{1}{2}\left(x^2\right)^2\frac{-2}{\left(x^2\right)^3} = -\frac{1}{x^2}, \\
\Gamma^1_{22} &= \frac{1}{2}\tilde{g}^{1i}\left(\frac{\partial g_{2i}}{\partial x^2} + \frac{\partial g_{i2}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^i}\right) = 0, \\
\Gamma^2_{11} &= \frac{1}{2}\tilde{g}^{2i}\left(\frac{\partial g_{1i}}{\partial x^1} + \frac{\partial g_{i1}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^i}\right) = -\frac{1}{2}\tilde{g}^{22}\frac{\partial g_{11}}{\partial x^2} = -\frac{1}{2}\left(x^2\right)^2\frac{-2}{\left(x^2\right)^3} = \frac{1}{x^2}, \\
\Gamma^2_{12} &= \frac{1}{2}\tilde{g}^{2i}\left(\frac{\partial g_{1i}}{\partial x^2} + \frac{\partial g_{i2}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^i}\right) = 0, \\
\Gamma^2_{22} &= \frac{1}{2}\tilde{g}^{2i}\left(\frac{\partial g_{2i}}{\partial x^2} + \frac{\partial g_{i2}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^i}\right) = \frac{1}{2}\tilde{g}^{22}\frac{\partial g_{11}}{\partial x^2} = \frac{1}{2}\left(x^2\right)^2\frac{-2}{\left(x^2\right)^3} = -\frac{1}{x^2}.
\end{aligned}$$

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Exercise 10.15.

Consider $\mathbb{R}^2$ in cartesian and polar coordinates. Find vector field in these coordinate systems that is created by paralelly transporting tangent vector of a curve $\sigma(t) = (x(t), y(t)) = (\cos t, \sin t)$, $t \in [0, 2\pi)$, in the point $t = 0$, along the curve itself.

Solution. In polar coordinates (r, φ) the curve is parametrized by $\sigma_{\text{pol}}(t) = (1, t)$, $t \in [0, 2\pi)$. In Cartesian coordinates, the tangent vector at the origin is $\dot{\sigma}(0) = (0, 1)$ while for polar coordinates it is $\dot{\sigma}_{\text{pol}}(0) = (0, 1)$. Let X be our vector field in Cartesian coordinates.Since all Christoffel symbols are zero, parallel transport equations are simply

$$\dot{X}^i = 0.$$

Taking initial condition into account, we find that the vector field is constatnt along the path

$$X = \frac{\partial}{\partial y}.$$

In polar coordinates it will not be that easy, since two Christoffel symbols are nonzero: $\Gamma^r_{\varphi\varphi} = -r$ and $\Gamma^\varphi_{r\varphi} = r^{-1}$. Parallel transport equations for our vector field X_{pol} are

$$\begin{aligned} \dot{X}^r_{\text{pol}} + \Gamma^r_{\varphi\varphi}(\sigma(t))\,\dot{\sigma}^\varphi(t) X^\varphi_{\text{pol}} = 0 \quad &\Rightarrow \quad \dot{X}^r_{\text{pol}} - X^\varphi_{\text{pol}} = 0, \\ \dot{X}^\varphi_{\text{pol}} + \Gamma^\varphi_{r\varphi}(\sigma(t))\left(\dot{\sigma}^\varphi(t) X^r_{\text{pol}} + \dot{\sigma}^r(t) X^\varphi_{\text{pol}}\right) = 0 \quad &\Rightarrow \quad \dot{X}^\varphi_{\text{pol}} + X^r_{\text{pol}} = 0. \end{aligned}$$

We get vector field

$$X_{\text{pol}} = \sin t \frac{\partial}{\partial r} + \cos t \frac{\partial}{\partial \varphi}.$$

We can see that the vector field changes its direction along the path. These two vector fields coincide however. We can verify this by coordinate transformation

$$\begin{aligned} X^x &= \frac{\partial x}{\partial r}(\sigma_{\text{pol}}(t)) X^r_{\text{pol}} + \frac{\partial x}{\partial \varphi}(\sigma_{\text{pol}}(t)) X^\varphi_{\text{pol}} = \\ &= \cos\varphi(\sigma_{\text{pol}}(t)) \sin t + (-r\sin\varphi)\left(\sigma_{\text{pol}}(t)\right)\cos t = \cos t \sin t - \sin t \cos t = 0, \\ X^y &= \frac{\partial y}{\partial r}\left(\sigma_{\text{pol}}(t)\right) X^r_{\text{pol}} + \frac{\partial y}{\partial \varphi}\left(\sigma_{\text{pol}}(t)\right) X^\varphi_{\text{pol}} = \\ &= \sin\varphi\left(\sigma_{\text{pol}}(t)\right)\sin t + (r\cos\varphi)\left(\sigma_{\text{pol}}(t)\right)\cos t = \sin^2 t + \cos^2 t = 1. \end{aligned}$$

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Exercise 10.16.

Let us consider sphere S^2 with metric induced from $\mathbb{R}^3$. Solve the equation of parallel transport along circle of latitude.

$$\begin{aligned} \gamma\colon\ \varphi &= t, \quad t \in [0, 2\pi], \\ \theta &= \theta_0, \quad \theta_0 = \text{const} \in (0, \pi). \end{aligned}$$

For arbitrary $v(0) \in T_{\gamma(0)}S^2$ find a matrix A, such that

$$v(2\pi) = Av(0),$$

where $v(2\pi) \in T_{\gamma(2\pi)}S^2 = T_{\gamma(0)}S^2$ is an image of $v(0)$ after parallel transport and find an angle between them. Finally, show that the scalar product is invariant under the parallel transport.

Solution. We calculated Christoffel symbols in Exercise 10.14, parallel transport equations are

$$\begin{aligned} \frac{\mathrm{d}v^\varphi}{\mathrm{d}t} + \frac{\cos\theta_0}{\sin\theta_0} v^\theta &= 0, \\ \frac{\mathrm{d}v^\theta}{\mathrm{d}t} - \sin\theta_0 \cos\theta_0 v^\varphi &= 0. \end{aligned}$$

We differentiate first of the equations and insert $\dot{v}^\theta$ from the second, we get

$$\ddot{v}^\varphi + \cos^2\theta_0 v^\varphi = 0,$$

with a solution

$$v^\varphi = A\sin(t\cos\theta_0) + B\cos(t\cos\theta_0),$$

kde $A, B \in \mathbb{R}$. We can find second component of a vector v

$$v^\theta = -A\sin\theta_0\cos(t\cos\theta_0) + B\sin\theta_0\sin(t\cos\theta_0).$$

Now we have to find integration constants A, B. Let $v(0) = (a, b)$, where $a, b \in \mathbb{R}$, then

$$\begin{aligned} a &= v^\varphi(0) = B, \\ b &= v^\theta(0) = -A\sin\theta_0, \end{aligned}$$

and we get

$$\begin{aligned} v^\varphi &= -\frac{b}{\sin\theta_0}\sin(t\cos\theta_0) + a\cos(t\cos\theta_0), \\ v^\theta &= b\cos(t\cos\theta_0) + a\sin\theta_0\sin(t\cos\theta_0). \end{aligned}$$

These two equations (for $t = 2\pi$) can be written in matrix form. This gives us our matrix A,

$$v(2\pi) = \begin{pmatrix} v^\varphi(2\pi) \\ v^\theta(2\pi) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\cos\theta_0) & -\frac{\sin(2\pi\cos\theta_0)}{\sin\theta_0} \\ \sin\theta_0\sin(2\pi\cos\theta_0) & \cos(2\pi\cos\theta_0) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = Av(0).$$

Since space $T_{\gamma(0)}\mathrm{S}^2$ is a vector space with a scalar product given by matrix

$$g = \begin{pmatrix} \sin^2\theta_0 & 0 \\ 0 & 1 \end{pmatrix},$$

angle α between the vector and its image under parallel transport can be determined by

$$\cos\alpha = \frac{g(v(0), v(2\pi))}{\sqrt{g(v(0), (v(0)) \cdot g(v(2\pi), v(2\pi))}}.$$

After the matrix multiplication we find out that

$$\alpha = 2\pi\cos\theta_0,$$

which means that for the equator, image will coincide with the original vector. Invariance of a scalar product means that for every pair of vectors $u, w \in T_{\gamma(0)}\mathrm{S}^2$ following identity holds

$$g(Au, Aw) = g(u, w).$$

In local coordinates,

$$(Au)^T g(Aw) = u^T gw.$$

Since u, w are arbitrarty,

$$A^T gA = g,$$

must hold. This can be verified via matrix multiplication. $\diamond$

Exercise 10.17.

Let's assume linear connection ∇ na $\mathbb{R}^2$ with all Christoffel symbols zero, except of $\Gamma^1_{21} = 1$. Calculate parallel transport of tangent vector $\dot{\sigma}(0)$ to a curve

$$\sigma(t) = \left(-3\mathrm{e}^{-t} + 8, t-2\right)$$

along the curve. Is $\sigma(t)$ geodesic?

Solution. Tangent vector along $\sigma(t)$ is $\dot{\sigma}(t) = (3\mathrm{e}^{-t}, 1)$, for $t = 0$ we have $\dot{\sigma}(0) = (3, 1)$. Let $Y(t)$ be our vector field, parallel transport equations are

$$\dot{Y}^i + \sum_{j,k=1}^{2} \Gamma^i_{jk} Y^j \dot{\sigma}^k = 0.$$

We can see that Christoffel symbols are not symmetric in lower indices, we have to be careful with the ordering. We get

$$\begin{aligned} \dot{Y}^1 + 3Y^2\mathrm{e}^{-t} &= 0, \\ \dot{Y}^2 &= 0. \end{aligned}$$

Second equation can be integrated straight away, intial condition gives us $Y^2 = 1$. We insert this into first equation and use initial condition again. Parallely transported vector field is $\sigma(t)$ je tvaru

$$Y = \left(3\mathrm{e}^{-t}, 1\right),$$

which is the same as tangent vector field $\dot{\sigma}(t)$ to the curve $\sigma(t)$. $\sigma(t)$ is geodesic. ◇

Exercise 10.18.

Let f be a function on Riemann manifold $\left(\mathbb{R}^3, g\right)$, where g is a metric in spherical coordinates. Calculate Δf.

Solution. Our goal is to calculate

$$\Delta f = \sum_{i=1}^{n} (\nabla \operatorname{grad} f)^i_i = \sum_{i=1}^{n} \left(\frac{\partial \left(\operatorname{grad} f\right)^i}{\partial x^i} + \sum_{k=1}^{n} \Gamma^i_{ki} \left(\operatorname{grad} f\right)^k \right).$$

We calculated gradient in spherical coordinates in Exercise 9.20, Christoffel symbols were calculated in Exercise 10.14. We get

$$\begin{aligned} \Delta f &= \frac{\partial \left(\operatorname{grad} f\right)^r}{\partial r} + \frac{\partial \left(\operatorname{grad} f\right)^\varphi}{\partial \varphi} + \Gamma^\varphi_{\varphi r} \left(\operatorname{grad} f\right)^r + \Gamma^\varphi_{\varphi \theta} \left(\operatorname{grad} f\right)^\theta + \\ &\quad + \frac{\partial \left(\operatorname{grad} f\right)^\theta}{\partial \theta} + \Gamma^\theta_{\theta r} \left(\operatorname{grad} f\right)^r = \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\cos\theta}{r^2 \sin\theta} \frac{\partial f}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2}. \end{aligned}$$

◇

Exercise 10.19.

Find geodesics on

1. sphere S^2,

2. upper half plane, i.e. $\mathbb{H}^2 = \left\{(x^1,x^2)\in\mathbb{R}^2 \,\middle|\, x^2>0\right\}$, with metric $\mathrm{d}s^2 = \left(x^2\right)^{-2}\left(\left(\mathrm{d}x^1\right)^2+\left(\mathrm{d}x^2\right)^2\right)$.

Solution. 1. Nonzero Christoffel symbols are $\Gamma^{\varphi}_{\varphi\theta} = \operatorname{cotan}\theta$ a $\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$. We are looking for a geodesic $\gamma(t) = (\varphi(t),\theta(t))$. Geodesic equations are

$$\begin{aligned}
\ddot\varphi + 2\Gamma^{\varphi}_{\varphi\theta}\dot\varphi\dot\theta = 0 \quad &\Rightarrow \quad \ddot\varphi + 2\frac{\cos\theta}{\sin\theta}\dot\varphi\dot\theta = 0,\\
\ddot\theta + \Gamma^{\theta}_{\varphi\varphi}\dot\varphi^2 = 0 \quad &\Rightarrow \quad \ddot\theta - \sin\theta\cos\theta\dot\varphi^2.
\end{aligned}$$

We multiply first equation by $\sin^2\theta$ and get

$$\begin{aligned}
\ddot\varphi\sin^2\theta + 2\sin\theta\cos\theta\dot\varphi\dot\theta &= 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot\varphi\sin^2\theta\right) &= 0,\\
\dot\varphi\sin^2\theta &= C,
\end{aligned}$$

where $C\in\mathbb{R}$. We can express $\dot\varphi$ and insert into second geodesic equation,

$$\ddot\theta - C^2\frac{\cos\theta}{\sin^3\theta} = 0.$$

We multiply this equation by $2\dot\theta$ and rearrange as before

$$\begin{aligned}
2\ddot\theta\dot\theta - 2C^2\frac{\cos\theta}{\sin^3\theta}\dot\theta &= 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot\theta^2 + C^2\operatorname{cotan}^2\theta\right) &= 0,\\
\dot\theta^2 + C^2\operatorname{cotan}^2\theta &= K^2,
\end{aligned}$$

where $K\in\mathbb{R}$. Now we get rid of t and we will look for an implicit equation of a geodesic, i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{\dot\theta}{\dot\varphi} = \frac{\sin^2\theta}{C}\sqrt{K^2 - C^2\operatorname{cotan}^2\theta}. \tag{10.20}$$

For $C=0$ we get $\gamma(t) = (\mathrm{const}, Kt+\mathrm{const}')$. These curves are meridians. Now let's take a look at equation (10.20) again. One of the integrals is trivial, the other one is

$$\begin{aligned}
\int\frac{C\mathrm{d}\theta}{\sin^2\theta\sqrt{K^2-C^2\operatorname{cotan}^2\theta}} &= \int\frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{K^2C^{-2}-\operatorname{cotan}^2\theta}} = \left|\begin{array}{c}\operatorname{cotan}\theta = x\\ -\frac{\mathrm{d}\theta}{\sin^2\theta} = \mathrm{d}x\end{array}\right| =\\
&= -\int\frac{\mathrm{d}x}{\sqrt{K^2C^{-2}-x^2}} = -\arcsin\frac{x}{K^2C^{-2}} + \mathrm{const} =\\
&= -\arcsin\frac{\operatorname{cotan}\theta}{K^2C^{-2}} + \mathrm{const},
\end{aligned}$$

The integration constants of both integrals were combined. We have

$$-\arcsin\frac{\cot\theta}{K^2C^{-2}}=\varphi+B.$$

After few rearrangements we get the equation of a geodesic

$$K^2\cos B\sin\theta\sin\varphi+K^2\sin B\sin\theta\cos\varphi+C^2\cos\theta=0. \tag{10.21}$$

We can take a look at these curves in $\mathbb{R}^3$. These curves are subsets of planes

$$\alpha x+\beta y+\delta z=0,$$

where α, β a δ are respective constants, i.e. planes passing through an origin. Geodesics in $\mathbb{R}^3$ are arcs of intersetion of a sphere with planes passing through the origin. Intersection of a sphere with such a plane is called great circle.We can see that implicit equation (10.21) also contains case $C=0$, that describe meridians.
2. Nonzero Christoffel symbols are $\Gamma^1_{12}=-\left(x^2\right)^{-1}$, $\Gamma^2_{11}=\left(x^2\right)^{-1}$ a $\Gamma^2_{22}=-\left(x^2\right)^{-1}$. We are looking for a geodesic $\gamma(t)=\left(x^1(t),x^2(t)\right)$. Geodesic equations are

$$\begin{aligned}\ddot{x}^1+2\Gamma^1_{12}\dot{x}^1\dot{x}^2=0 \quad &\Rightarrow \quad \ddot{x}^1-2\frac{\dot{x}^1\dot{x}^2}{x^2}=0,\\ \ddot{x}^2+\Gamma^2_{11}\left(\dot{x}^1\right)^2+\Gamma^2_{22}\left(\dot{x}^2\right)^2=0 \quad &\Rightarrow \quad \ddot{x}^2+\frac{\left(\dot{x}^1\right)^2}{x^2}+\frac{\left(\dot{x}^2\right)^2}{x^2}=0.\end{aligned}$$

We divide first equation by $\left(x^2\right)^2$ and we get

$$\begin{aligned}\frac{\ddot{x}^1}{(x^2)^2}-2\frac{\dot{x}^1\dot{x}^2}{(x^2)^3}&=0,\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^1}{(x^2)^2}\right)&=0,\\ \frac{\dot{x}^1}{(x^2)^2}&=C,\end{aligned} \tag{10.22}$$

where $C\in\mathbb{R}$. Thus we can insert $\dot{x}^1$ into second equation that we multiply by $2\dot{x}^2\left(x^2\right)^{-2}$ and we get

$$\begin{aligned}2\frac{\dot{x}^2\ddot{x}^2}{(x^2)^2}+2C^2x^2\dot{x}^2-2\frac{\left(\dot{x}^2\right)^3}{(x^2)^3}&=0,\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\left(\dot{x}^2\right)^2}{(x^2)^2}+C^2\left(x^2\right)^2\right)&=0,\\ \frac{\left(\dot{x}^2\right)^2}{(x^2)^2}+C^2\left(x^2\right)^2&=D^2,\end{aligned} \tag{10.23}$$

where $D \in \mathbb{R}$. We can combine (10.22) and (10.23)to get rid of t and find implicit equation of geodesic. We get equations

$$\left(\frac{\mathrm{d}x^2}{\mathrm{d}x^1}\right)^2 = \left(\frac{\dot{x}^2}{\dot{x}^1}\right)^2 = \frac{D^2}{C^2\left(x^2\right)^2} - 1 \tag{10.24}$$

For $C = 0$ geodesics are $\gamma(t) = \left(\mathrm{const}, E\mathrm{e}^{Dt}\right)$, where $E > 0$, i.e. half lines in upper half plane perpendicular to x-axis. Let's take a look at equation (10.24) again. Integration of this equation gives us

$$-\sqrt{\frac{D^2}{C^2} - \left(x^2\right)^2} = x^1 - S,$$

where $S \in \mathbb{R}$ is integration constant. We rewrite this equation

$$\left(x^1 - S\right)^2 + \left(x^2\right)^2 = \left(\frac{D}{C}\right)^2.$$

Geodesics are circles with center in $[S, 0]$ and with radius DC^{-1}. ◇

Exercise 10.25.

Let's assume $\mathbb{R}^2$ with linear connection ∇ and curve $\sigma(t) = (\mathrm{const}, y(t))$, where $y(t)\colon I \to \mathbb{R}$ is smooth function. Find equations for ∇, so that every $\sigma(t)$ is geodesic.

Solution. We have to make sure that the geodesic equation, i.e.

$$\ddot{\sigma}^i + \sum_{j,k=1}^{2} \Gamma^i_{jk}\dot{\sigma}^j\dot{\sigma}^k = 0,$$

is fulfilled for every $y(t)$. Derivatives of a curve $\sigma(t)$ are $\dot{\sigma}(t) = (0, \dot{y}(t))$ a $\ddot{\sigma}(t) = (0, \ddot{y}(t))$. We insert these to geodesic equations and get

$$\Gamma^1_{22}\dot{y}^2 = 0,$$
$$\ddot{y} + \Gamma^2_{22}\dot{y} = 0.$$

The equation for ∇ is $\Gamma^1_{22} = 0$. ◇

Exercise 10.26.

Since geodesics are curves on Riemann manifold on dimension n between two points with minimal distance, they can be obtained by minimization of length of a curve (9.3). Since $x(t)$ is given by an integral, necessary condition for an extrem of an integral is that integrand $L(x(t), \dot{x}(t), t)$ fulfills Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^k} = \frac{\partial L}{\partial x^k}, \quad k = 1, \ldots, n.$$

Show that Euler-Lagrange equations for a curve $x(t)$ and integral (9.3) are geodesic equations.

Solution. We don't have to care about square root it's an increasing function on $(0,\infty)$, thus we have

$$L(x(t),\dot{x}(t),t) = \sum_{i,j=1}^{n} \dot{x}^i(t)\dot{x}^j(t)g_{ij}(x(t)).$$

Let's start with calculating a derivative $\frac{\partial L}{\partial \dot{x}^k}$. This derivative is nonzero iff $i = k$ or $j = k$. This can be written using Kronecker delta as

$$\frac{\partial L}{\partial \dot{x}^k} = \sum_{i,j=1}^{n} \left(\dot{x}^i g_{ij}\delta_k^j + \dot{x}^j g_{ij}\delta_k^i\right) = \sum_{i=1}^{n} \dot{x}^i g_{ik} + \sum_{j=1}^{n} \dot{x}^j g_{kj}.$$

We differentiate this with respect to t and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^k} = \sum_{i=1}^{n} \ddot{x}^i g_{ik} + \sum_{j=1}^{n} \ddot{x}^j g_{kj} + \sum_{i,j=1}^{n} \left(\dot{x}^i \frac{\partial g_{ik}}{\partial x^j}\dot{x}^j + \dot{x}^j \frac{\partial g_{jk}}{\partial x^i}\dot{x}^i\right).$$

Thanks to the symmetry of metric tensor we can rewrite two sums as one. Righthand side of Euler-Lagrange equations is simply

$$\frac{\partial L}{\partial x^k} = \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j.$$

We compare both sides and get

$$2\sum_{i=1}^{n} \ddot{x}^i g_{ik} + \sum_{i,j=1}^{n} \left(\dot{x}^i \frac{\partial g_{ik}}{\partial x^j}\dot{x}^j + \dot{x}^j \frac{\partial g_{jk}}{\partial x^i}\dot{x}^i\right) = \sum_{i,j=1}^{n} \frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j.$$

We can multiply this equation by $\sum_{k=1}^{n} \tilde{g}^{lk}$ and subtract righthand side. We get the geodesic equation

$$\ddot{x}^l + \sum_{i,j=1}^{n} \frac{1}{2}\sum_{k=1}^{n} \tilde{g}^{lk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j = 0.$$

$\diamond$

Torsion and curvature of a linear connection

Definition 11.1.

Mapping $T\colon \mathscr{X}M \times \mathscr{X}M \to \mathscr{X}M$ defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad X,Y \in \mathscr{X}M$$

is called a torsion of a linear connection ∇ on M. Coordinate expression of a vector field $T(X,Y)$ in local coordinates is

$$(T(X,Y))^i = \sum_{j,k=1}^{n} \left(\Gamma^i_{kj} - \Gamma^i_{jk}\right) X^j Y^j.$$

Therefore T is a $(1,2)$ tensor field witch coordinate expression

$$\Gamma^i_{kj} - \Gamma^i_{jk}.$$

Definition 11.2.

If $T = 0$, ∇ is torsionless linear connection.

Remark. Levi-Civita connection of Riemann space is torsionless.

Definition 11.3.

Mapping $R\colon \mathscr{X}M \times \mathscr{X}M \times \mathscr{X}M \to \mathscr{X}M$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X,Y,Z \in \mathscr{X}M$$

is a curvature of a linear connection ∇ on M.Curvature is $(1,3)$ tensor field $(1,3)$ with coordinate expression

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lk}}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} + \sum_{h=1}^{n} \left(\Gamma^i_{hj}\Gamma^h_{lk} - \Gamma^i_{hk}\Gamma^h_{lj}\right).$$

Theorem 11.4.

If ∇ is torsionless linear connection, then its curvature R fulfills first Bianchi identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

for every $X,Y,Z \in \mathscr{X}M$ Coordinate expression of this identity is

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0.$$

Definition 11.5.

Let (M,g) be a Riemann manifold and ∇ Levi-Civita connection. We can define a covariant form of a curvature tensor $R(X,Y,Z,U)$, where $X,Y,Z,U \in \mathscr{X}M$, as

$$R(X,Y,Z,U) = g\left(R(X,Y)Z,U\right).$$

$R(X,Y,Z,U)$ is $(0,4)$ tensor with a coordinate expression

$$R_{klhi} = \sum_{j=1}^{n} g_{ij} R^{j}_{klh}.$$

Remark. In Exercise 12.6 we found out that for covariant form of curvature tensor $R(X,Y,Z,U) = -R(X,Y,U,Z)$ holds. In coordinates this is

$$R_{ijkl} = -R_{ijlk}.$$

Definition 11.6.

Let (M,g) be a Riemann manifold with a Levi-Civita connection ∇. Ricci curvature tensor Ric is defined as a contraction of curvature tensor, i.e.

$$\mathrm{Ric}_{ij} = -\sum_{k=1}^{n} R^{k}_{ikj}.$$

Ricci scalar, or scalar curvature, Scal is defined as

$$\mathrm{Scal} = \sum_{i,j=1}^{n} \tilde{g}^{ij} \mathrm{Ric}_{ij},$$

where $\tilde{g}$ is matrix inverse to matrix of metric g.

Exercise 11.7.

On a smooth manifold M are two connections ∇ and $\widetilde{\nabla}$, whose Christoffel symbols are related by

$$\widetilde{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \delta^{i}_{j}\theta_k,$$

where $\theta \in \Omega^1(M)$ is differential 1-form. Calculate the difference of their curvatures, given by tensor $\widetilde{R}^{i}_{jkl} - R^{i}_{jkl}$.

Solution. The calculation is very straighforward. We'll perform it in local coordinates.

$$\begin{aligned}
\widetilde{R}^i_{jkl} &= \frac{\partial \widetilde{\Gamma}^i_{lk}}{\partial x^j} - \frac{\partial \widetilde{\Gamma}^i_{lj}}{\partial x^k} + \widetilde{\Gamma}^i_{hj}\widetilde{\Gamma}^h_{lk} - \widetilde{\Gamma}^i_{hk}\widetilde{\Gamma}^h_{lj} = \\
&= \frac{\partial \Gamma^i_{lk}}{\partial x^j} + \delta^i_l \frac{\partial \theta_k}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \delta^i_l \frac{\partial \theta_j}{\partial x^k} + \left(\Gamma^i_{hj} + \delta^i_h \theta_j\right)\left(\Gamma^h_{lk} + \delta^h_l \theta_k\right) - \\
&\quad - \left(\Gamma^i_{hk} + \delta^i_h \theta_k\right)\left(\Gamma^h_{lj} + \delta^h_l \theta_j\right) = \\
&= R^i_{jkl} + \delta^i_l \left(\frac{\partial \theta_k}{\partial x^j} - \frac{\partial \theta_j}{\partial x^k}\right) + \Gamma^i_{hj}\delta^h_l \theta_k + \delta^i_h \theta_j \Gamma^h_{lk} + \delta^i_h \theta_j \delta^h_l \theta_k - \Gamma^i_{hk}\delta^h_l \theta_j - \\
&\quad - \delta^i_h \theta_k \Gamma^h_{lj} - \delta^i_h \theta_k \delta^h_l \theta_j = \\
&= R^i_{jkl} + \delta^i_l \left(\frac{\partial \theta_k}{\partial x^j} - \frac{\partial \theta_j}{\partial x^k}\right) + \Gamma^i_{lj}\theta_k + \Gamma^i_{lk}\theta_j + \delta^i_l \theta_j \theta_k - \Gamma^i_{lk}\theta_j - \Gamma^i_{lj}\theta_k - \delta^i_l \theta_k \theta_j = \\
&= R^i_{jkl} + \delta^i_l \left(\frac{\partial \theta_k}{\partial x^j} - \frac{\partial \theta_j}{\partial x^k}\right).
\end{aligned}$$

◇

Exercise 11.8.

Let M and M' be two smooth manifolds with linear connections ∇, ∇' respectively. Connection is invariant under a diffeomorphism $\varphi\colon M \to M'$ if for every $x \in M$

$$T_x\varphi\left(\nabla_X Y\right) = \left(\nabla'_{X'} Y'\right)(\varphi(x)),$$

holds where $X \in \mathscr{X}(M)$ and $X' \in \mathscr{X}(M')$ are φ-related vector fields, or $Y \in \mathscr{X}(M)$ a $Y' \in \mathscr{X}(M')$ are φ-related. Show that

$$T\varphi\left(\mathscr{T}(X,Y)\right) = \mathscr{T}'\left(T\varphi X, T\varphi Y\right),$$

where $\mathscr{T}, \mathscr{T}'$ are torsion tensors on M and M' respectively.

Solution. Vector fields X a X' are φ-related, if for every $x \in M$

$$T_x\varphi X = X'(\varphi(x))$$

holds. If X,Y are φ-related to X',Y', so is the Lie bracket $[X,Y]$ φ-related to $[X',Y']$. For every $x \in M$ holds

$$\begin{aligned}
T_x\varphi\left(\mathscr{T}(X,Y)\right) &= T_x\varphi\left(\nabla_X Y - \nabla_Y X - [X,Y]\right) = \left(\nabla'_{X'}Y'\right)(\varphi(x)) - \left(\nabla'_{Y'}X'\right)(\varphi(x)) - \\
&\quad - [X',Y'](\varphi(x)) = \\
&= \mathscr{T}'\left(X',Y'\right)(\varphi(x)) = \mathscr{T}'\left(T_x\varphi X, T_x\varphi Y\right).
\end{aligned}$$

◇

Exercise 11.9.

Show that for a covariant form of curvature tensor $R(X,Y,Z,U)$ following identity holds

$$R_{jkli} = R_{lijk}.$$

Solution. Let us assume two instances of a first Bianchi identity

$$R_{jkli} + R_{klji} + R_{ljki} = 0,$$
$$R_{jkil} + R_{kijl} + R_{ijkl} = 0.$$

We can substract them and use the antisymmetry in a first pair of indices in R_{jkil}, which gives us

$$2R_{jkli} + R_{klji} + R_{ljki} - R_{kijl} - R_{ijkl} = 0. \tag{11.10}$$

We can do this again but this time we switch indices $(i \leftrightarrow k)$ and $(l \leftrightarrow j)$, leading us to

$$2R_{lijk} + R_{ijlk} + R_{jlik} - R_{iklj} - R_{klij} = 0.$$

Using the antisymmetry in first and second pair of indices

$$2R_{lijk} - R_{ijkl} + R_{ljki} - R_{kijl} + R_{klji} = 0,$$

and substract from the equation (11.10), which gives

$$R_{jkli} = R_{lijk}.$$

◇

Exercise 11.11.

Show that Ricci curvature tensor Ric is symmetric, i.e. $\mathrm{Ric}_{ij} = \mathrm{Ric}_{ji}$.

Solution. We start from the first Bianchi identity

$$R^k_{ilj} + R^k_{lji} + R^k_{jil} = 0,$$

in which we contract indices k and l, i.e.

$$\sum_{k=1}^{n} \left(R^k_{ikj} + R^k_{kji} + R^k_{jik} \right) = 0,$$
$$-\mathrm{Ric}_{ij} + \mathrm{Ric}_{ji} + \sum_{k=1}^{n} R^k_{jik} = 0,$$

where we used the antisymmetry of curvature in the first pair of indices. Now we show that $\sum_{k=1}^{n} R^k_{jik} = 0$. First we lower the index k using the inverse metric, then we use the symmetry of a metric and antisymmetry of a curvature in the second pair of indices, i.e.

$$\sum_{k=1}^{n} R^k_{jik} = \sum_{k,l=1}^{n} g^{lk} R_{jikl} = -\sum_{k,l=1}^{n} g^{kl} R_{jilk} = -\sum_{l=1}^{n} R^l_{jil} \quad \Rightarrow \quad \sum_{k=1}^{n} R^k_{jik} = 0.$$

◇

Exercise 11.12.

Find the components of a curvature tensor, Ricci tensor and Ricci scalar in following cases

1. sphere S^2 with metric induced from $\mathbb{R}^3$,
2. torus T^2 with metric induced from $\mathbb{R}^3$,
3. upper half plane, i.e. $\mathbb{H}^2 = \left\{(x^1,x^2) \in \mathbb{R}^2 \,\middle|\, x^2 > 0\right\}$, with metric $ds^2 = \left(x^2\right)^{-2} \left(\left(dx^1\right)^2 + \left(dx^2\right)^2\right)$.

Solution. Christoffel symbols are already calculated in the Exercise 10.14.

1. We calculate one of the components from the definition, rest can be deduced from the symmetries of the curvature tensor. We again use the notation $\varphi^1 = \varphi$ and $\varphi^2 = \theta$.

$$R^2_{211} = \frac{\partial \Gamma^2_{11}}{\partial \varphi^2} - \frac{\partial \Gamma^2_{12}}{\partial \varphi^1} + \sum_{i=1}^{2} \left(\Gamma^2_{2i}\Gamma^i_{11} - \Gamma^2_{1i}\Gamma^i_{21}\right) = \frac{\partial \Gamma^2_{11}}{\partial \varphi^2} - \Gamma^2_{11}\Gamma^1_{21} =$$
$$= -\cos^2\theta + \sin^2\theta - (-\sin\theta\cos\theta)\frac{\cos\theta}{\sin\theta} = \sin^2\theta.$$

Then $R^2_{121} = -\sin^2\theta$. From these components we calculate covariant component of curvature tensor R_{1212} by lowering the index,

$$R_{1212} = \sum_{i=1}^{2} R^i_{121} g_{i2} = R^2_{121} g_{22} = -\sin^2\theta,$$

where we used the fact that the metric is diagonal. Now we use the antisymmetry in second pair of indices,i.e. $R_{1221} = \sin^2\theta$, and we raise the last index

$$R^1_{122} = \sum_{i=1}^{2} \tilde{g}^{1i} R_{122i} = \tilde{g}^{11} R_{1221} = \frac{\sin^2\theta}{\sin^2\theta} = 1,$$

where we used the fact that R_{1222} is zero due to the antisymmetry. The last nonzero component is $R^1_{212} = -1$.Ricci tensor components are obtained as follows

$$\mathrm{Ric}_{11} = -\sum_{i=1}^{2} R^i_{1i1} = -R^2_{121} = \sin^2\theta,$$
$$\mathrm{Ric}_{22} = -\sum_{i=1}^{2} R^i_{2i2} = -R^1_{212} = 1,$$
$$\mathrm{Ric}_{12} = -\sum_{i=1}^{2} R^i_{1i2} = 0.$$

Then the Ricci scalar is

$$\mathrm{Scal} = \sum_{i,j=1}^{2} \tilde{g}^{ij}\mathrm{Ric}_{ij} = \tilde{g}^{11}\mathrm{Ric}_{11} + \tilde{g}^{22}\mathrm{Ric}_{22} = \frac{1}{\sin^2\theta}\sin^2\theta + 1\cdot 1 = 2.$$

2. The procedure is same as for sphere. We calculate one of the components

$$
\begin{aligned}
R^2_{211} &= \frac{\partial \Gamma^2_{11}}{\partial \varphi^2} - \frac{\partial \Gamma^2_{12}}{\partial \varphi^1} + \sum_{i=1}^{2} \left(\Gamma^2_{2i}\Gamma^i_{11} - \Gamma^2_{1i}\Gamma^i_{21} \right) = \frac{\partial \Gamma^2_{11}}{\partial \varphi^2} - \Gamma^2_{11}\Gamma^1_{21} = \\
&= \frac{\cos\theta}{r}(R + r\cos\theta) - \sin^2\theta - \frac{1}{r}(R + r\cos\theta)\sin\theta \left(-\frac{r\sin\theta}{R + r\cos\theta} \right) = \\
&= \frac{\cos\theta\,(R + r\cos\theta)}{r}.
\end{aligned}
$$

Now we lower the upper index

$$
R_{2112} = \sum_{i=1}^{2} g_{i2} R^i_{211} = r\cos\theta\,(R + r\cos\theta).
$$

We use the antisymmetry and a raise the last index

$$
R^1_{212} = \sum_{i=1}^{2} \tilde{g}^{i1} R_{212i} = -\frac{r\cos\theta}{R + r\cos\theta}.
$$

We get two more nonzero components from the antisymmetry in first pair of indices. Ricci tensor is

$$
\begin{aligned}
\mathrm{Ric}_{11} &= -R^2_{121} = \frac{\cos\theta\,(R + r\cos\theta)}{r}, \\
\mathrm{Ric}_{22} &= -R^1_{212} = \frac{r\cos\theta}{R + r\cos\theta}, \\
\mathrm{Ric}_{12} &= 0.
\end{aligned}
$$

Ricci scalar is

$$
\mathrm{Scal} = \sum_{i,j=1}^{2} \tilde{g}^{ij}\mathrm{Ric}_{ij} = \tilde{g}^{11}\mathrm{Ric}_{11} + \tilde{g}^{22}\mathrm{Ric}_{22} = \frac{2\cos\theta}{r(R + r\cos\theta)}.
$$

3. The procedure is same as in previous exercises, calculate one component of curvature tensor

$$
\begin{aligned}
R^1_{122} &= \frac{\partial \Gamma^1_{22}}{\partial x^1} - \frac{\partial \Gamma^1_{12}}{\partial x^2} + \sum_{i=1}^{2} \left(\Gamma^1_{1i}\Gamma^i_{22} - \Gamma^1_{2i}\Gamma^i_{12} \right) = -\frac{\partial \Gamma^2_{12}}{\partial x^2} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^1_{12} = \\
&= -\frac{1}{(x^2)^2} + \frac{1}{(x^2)^2} - \frac{1}{(x^2)^2} = -\frac{1}{(x^2)^2}.
\end{aligned}
$$

Lower the index $R_{1221} = -\left(x^2\right)^{-4}$. Use the antisymmetry and raise the index $R^2_{121} = \left(x^2\right)^{-2}$. Ricci tensor is

$$
\begin{aligned}
\mathrm{Ric}_{11} &= -R^2_{121} = -\frac{1}{(x^2)^2}, \\
\mathrm{Ric}_{22} &= -R^1_{212} = -\frac{1}{(x^2)^2}, \\
\mathrm{Ric}_{12} &= 0,
\end{aligned}
$$

and Ricci scalar is

$$\mathrm{Scal} = \sum_{i,j=1}^{2} \tilde{g}^{ij}\mathrm{Ric}_{ij} = \tilde{g}^{11}\mathrm{Ric}_{11} + \tilde{g}^{22}\mathrm{Ric}_{22} = -2.$$

◇

Covariant differentiation of tensor fields

Definition 12.1.

Let M be a smooth manifold with a linear connection ∇. Covarian differentiation of 1-form ω in direction of $X \in \mathscr{X}M$ is 1-form $\nabla_X\omega\colon M \to T^*M$, for which, for every $Y \in \mathscr{X}M$ fulfills

$$X\langle\omega,Y\rangle = \langle\nabla_X\omega,Y\rangle + \langle\omega,\nabla_X Y\rangle.$$

Coordinate expression of covariant differentiation of 1-form is

$$(\nabla_X\omega)_i = \sum_{j,k=1}^{n}\left(\frac{\partial\omega_i}{\partial x^j} - \Gamma^k_{ij}\omega_k\right)X^j.$$

Definition 12.2.

Let M be smooth manifold with linear connection ∇, A is (r,s) tensor, $X, Y_1,\dots,Y_s \in \mathscr{X}M$ are vector fields and $\omega_1,\dots,\omega_r \in \Omega(M)$ are 1-forms.

$$A\left(Y_1\dots,Y_s,\omega_1,\dots,\omega_r\right): M \to \mathbb{R}$$

is a function on M. Covariant differentiation of A in direction of X, i.e. $\nabla_X A$ is defined as

$$\begin{aligned} X(A(Y_1,\dots,Y_s,\omega_1,\dots,\omega_r)) = (\nabla_X A)\left(Y_1,\dots,Y_s,\omega_1,\dots,\omega_r\right) + \\ + A\left(\nabla_X Y_1,\dots,Y_s,\omega_1,\dots,\omega_r\right) + \dots + A\left(Y_1,\dots,\nabla_X Y_s,\omega_1,\dots,\omega_r\right) + \\ + A\left(Y_1,\dots,Y_s,\nabla_X\omega_1,\dots,\omega_r\right) + \dots + A\left(Y_1,\dots,Y_s,\omega_1,\dots,\nabla_X\omega_r\right). \end{aligned}$$

Coordinate expression of covariant differentiation of (r,s) tensor is

$$\frac{\partial A^{i_1\dots i_r}_{j_1\dots j_s}}{\partial x^k} + \sum_{l=1}^{n}\left(\Gamma^{i_1}_{lk}A^{l i_2\dots i_r}_{j_1\dots j_s} + \dots + \Gamma^{i_r}_{lk}A^{i_1\dots i_{r-1}l}_{j_1\dots j_s} - \Gamma^{l}_{j_1 k}A^{i_1\dots i_r}_{l j_2\dots j_s} - \dots - \Gamma^{l}_{j_s k}A^{i_1\dots i_r}_{j_1\dots j_{s-1}l}\right).$$

Remark. Covariante derivative of $(0,2)$ tensor field g on M in direction $X \in \mathscr{X}M$ is

$$Xg\left(Y,Z\right) = \left(\nabla_X g\right)\left(Y,Z\right) + g\left(\nabla_X Y,Z\right) + g\left(Y,\nabla_X Z\right).$$

Theorem 12.3.

Levi-Civita connection ∇ is unique torsionless linear connection on (M,g) such that $\nabla_X g = 0$ for every $X \in \mathscr{X}M$.

Theorem 12.4.

Let (M,g) be Riemann manifold with a Levi-Civita connection ∇. For every vector field $X,Y,Z \in \mathscr{X}M$ following identity holds

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Exercise 12.5.

Let (M,g) be Riemann manifold with a Levi-Civita connection ∇. Show that for every $X,Y,Z \in \mathscr{X}(M)$ Koszul formula holds.

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - \\ & - g([Y,Z],X) + g([Z,X],Y). \end{aligned}$$

Solution. Let us remind that for Levi-Civita connection we have

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ [X,Y] &= \nabla_X Y - \nabla_Y X. \end{aligned}$$

Let's calculate

$$\begin{aligned} Xg(Y,Z) + Yg(Z,X) - Zg(X,Y\rangle = {} & g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + \\ & + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y). \end{aligned}$$

We can use the linearity and symmetry of scalar product

$$\begin{aligned} Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) = {} & g(\nabla_X Y, Z) + g(Z, \nabla_Y X) + \\ & + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X). \end{aligned}$$

Now we use torsionless of Levi-Civita connection and linearity of scalar product again

$$\begin{aligned} Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) = {} & g(\nabla_X Y, Z) + g(Z, \nabla_X Y) - g(Z, [X,Y]) + \\ & + g([X,Z],Y) + g([Y,Z],X). \end{aligned}$$

Symmetry of scalar product and antisymmetry of Lie brackets give us Koszul formula. ◇

Exercise 12.6.

Let (M,g) be Riemann manifold with a Levi-Civita connection ∇. Show that for every vector field $X,Y,U,Z \in \mathscr{X}M$ we have

$$g(R(X,Y)Z,U) = -g(R(X,Y)U,Z),$$

where R is curvature of Levi-Civita connection.

Solution. Theorem 12.4 gives us

$$[X,Y]\,g(U,Z) = g\left(\nabla_{[X,Y]}U,Z\right) + g\left(U,\nabla_{[X,Y]}Z\right).$$

We apply this twice and get

$$\begin{aligned} XY g(U,Z) &= X\left(g\left(\nabla_Y U,Z\right) + g\left(U,\nabla_Y Z\right)\right) = \\ &= g\left(\nabla_X\nabla_Y U,Z\right) + g\left(\nabla_Y U,\nabla_X Z\right) + g\left(\nabla_X U,\nabla_Y Z\right) + g\left(U,\nabla_X\nabla_Y Z\right). \end{aligned}$$

Let's assume the expression $(XY - YX - [X,Y])\,g(U,Z)$. $XY - YX$ is a Lie bracket, so

$$(XY - YX - [X,Y])\,g(U,Z) = 0.$$

However, if we apply vector fields on scalar $g(U,Z)$ we get

$$\begin{aligned} 0 &= g\left(\nabla_X\nabla_Y U,Z\right) + g\left(\nabla_Y U,\nabla_X Z\right) + g\left(\nabla_X U,\nabla_Y Z\right) + g\left(U,\nabla_X\nabla_Y Z\right) - \\ &\quad - g\left(\nabla_Y\nabla_X U,Z\right) - g\left(\nabla_X U,\nabla_Y Z\right) - g\left(\nabla_Y U,\nabla_X Z\right) - g\left(U,\nabla_Y\nabla_X Z\right) - \\ &\quad - g\left(\nabla_{[X,Y]}U,Z\right) - g\left(U,\nabla_{[X,Y]}Z\right) = \\ &= g\left(\nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U,Z\right) + g\left(U,\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right) = \\ &= g\left(R(X,Y)U,Z\right) + g\left(U,R(X,Y)Z\right), \end{aligned}$$

which gives us the identity we wanted to show. ◇

Exercise 12.7.

Let (M,g) be Riemann manifold with a Levi-Civita connection ∇. Show that for every $X,Y,Z,U \in \mathscr{X}M$ second Bianchi identity holds

$$\left(\nabla_X R\right)(Y,Z,U) + \left(\nabla_Y R\right)(Z,X,U) + \left(\nabla_Z R\right)(X,Y,U) = 0.$$

Solution. First let's examine action of X on scalar $\langle R(Y,Z)U,\omega\rangle$, where $\omega \in \Omega(M)$. $\langle R(Y,Z)U,\omega\rangle$ is evaluation of a vector field $R(Y,Z)U$ on 1-form ω, then

$$X\langle R(Y,Z)U,\omega\rangle = \langle \nabla_X(R(Y,Z)U),\omega\rangle + \langle R(Y,Z)U,\nabla_X\omega\rangle.$$

Or $\langle R(Y,Z)U,\omega\rangle$ can be viewed as evaluation of curvature on three vector fields and one 1-form. We have

$$\begin{aligned} X\langle R(Y,Z)U,\omega\rangle &= X(R(Y,Z,U,\omega)) = \left(\nabla_X R\right)(Y,Z,U,\omega) + R\left(\nabla_X Y,Z,U,\omega\right) + \\ &\quad + R\left(Y,\nabla_X Z,U,\omega\right) + R\left(Y,Z,\nabla_X U,\omega\right) + R\left(Y,Z,U,\nabla_X\omega\right) = \\ &= \langle\left(\nabla_X R\right)(Y,Z,U),\omega\rangle + \langle R\left(\nabla_X Y,Z\right)U,\omega\rangle + \langle R\left(Y,\nabla_X Z\right)U,\omega\rangle + \\ &\quad + \langle R(Y,Z)\nabla_X U,\omega\rangle + \langle R(Y,Z)U,\nabla_X\omega\rangle. \end{aligned}$$

We can see that

$$\left(\nabla_X R\right)(Y,Z,U) = \nabla_X\left(R(Y,Z)U\right) - R\left(\nabla_X Y,Z\right)U - R\left(Y,\nabla_X Z\right)U - R(Y,Z)\nabla_X U.$$

Our goal is to show that the expression

$$\begin{aligned}(\nabla_X R)(Y,Z,U)+(\nabla_Y R)(Z,X,U)+(\nabla_Z R)(X,Y,U)= \qquad (12.8)\\ =\nabla_X(R(Y,Z)U)-R(\nabla_X Y,Z)U-R(Y,\nabla_X Z)U-R(Y,Z)\nabla_X U+\\ +\nabla_Y(R(Z,X)U)-R(\nabla_Y Z,X)U-R(Z,\nabla_Y X)U-R(Z,X)\nabla_Y U+\\ +\nabla_Z(R(X,Y)U)-R(\nabla_Z X,Y)U-R(X,\nabla_Z Y)U-R(X,Y)\nabla_Z U\end{aligned}$$

is identically equal to zero. First we use the antisymmetry of curvature in first entries and the fact that Levi-Civita connection is torsionless to rewrite the terms in (12.8) with covariant derivative in first two entries

$$\begin{aligned}-R(\nabla_X Y,Z)U-R(Y,\nabla_X Z)U-R(\nabla_Y Z,X)U-R(Z,\nabla_Y X)U-\\ -R(\nabla_Z X,Y)U-R(X,\nabla_Z Y)U=\\ =R(Z,[X,Y])U+R(X,[Y,Z]U)+R(Y,[Z,X])U.\end{aligned}$$

We combine this with terms from (12.8) containing covariant derivative of U. Definition of curvature gives us

$$\begin{aligned}R(Z,[X,Y])U+R(X,[Y,Z]U)+R(Y,[Z,X])U-R(Y,Z)\nabla_X U-\\ -R(Z,X)\nabla_Y U-R(X,Y)\nabla_Z U=\\ =\nabla_Z\nabla_{[X,Y]}U-\nabla_{[X,Y]}\nabla_Z U-\nabla_{[Z,[X,Y]]}U+\\ +\nabla_X\nabla_{[Y,Z]}U-\nabla_{[Y,Z]}\nabla_X U-\nabla_{[X,[Y,Z]]}U+\\ +\nabla_Y\nabla_{[Z,X]}U-\nabla_{[Z,X]}\nabla_Y U-\nabla_{[Y,[Z,X]]}U-\\ -\nabla_Y\nabla_Z\nabla_X U+\nabla_Z\nabla_Y\nabla_X U+\nabla_{[Y,Z]}\nabla_X U-\\ -\nabla_Z\nabla_X\nabla_Y U+\nabla_X\nabla_Z\nabla_Y U+\nabla_{[Z,X]}\nabla_Y U-\\ -\nabla_X\nabla_Y\nabla_Z U+\nabla_Y\nabla_X\nabla_Z U+\nabla_{[X,Y]}\nabla_Z U=\\ =\nabla_X\left(\nabla_{[Y,Z]}U+\nabla_Z\nabla_Y U-\nabla_Y\nabla_Z U\right)+\\ +\nabla_Y\left(\nabla_{[Z,X]}U+\nabla_X\nabla_Z U-\nabla_Z\nabla_X U\right)+\\ +\nabla_Z\left(\nabla_{[X,Y]}U+\nabla_Y\nabla_X U-\nabla_X\nabla_Y U\right)-\nabla_{[Z,[X,Y]]+[X,[Y,Z]]+[Y,[Z,X]]}U=\\ =-\nabla_X(R(Y,Z)U)-\nabla_Y(R(Z,X)U)-\nabla_Z(R(X,Y)U),\end{aligned}$$

where we used the Jacobi identity for vector fields. The resulting terms cancel remaining terms in (12.8). ◇

Exercise 12.9.

Find coordinate expression of second Bianchi identity.

Solution. Components of curvature tensor are R^i_{jkl}. We apply covariant derivative in direction of $X=\sum_{h=1}^n X^h\frac{\partial}{\partial x^h}$ leading us to

$$\nabla_X R^i_{jkl}=\sum_{h=1}^n X^h\nabla_{\frac{\partial}{\partial x^h}}R^i_{jkl}.$$

Let us denote covariant derivative in direction of $\frac{\partial}{\partial x^h}$ as ∇_h. Second Bianchi identity is

$$0 = \sum_{h,j,k,l=1}^{n} \left(\left(X^h \nabla_h R^i_{jkl} \right) Y^j Z^k U^l + \left(Y^h \nabla_h R^i_{jkl} \right) Z^j X^k U^l + \left(Z^h \nabla_h R^i_{jkl} \right) X^j Y^k U^l \right)$$

We relabel contracting indices in second term $(h \to j \to k \to h)$ and in third term $(h \to k \to j \to h)$ which gives us

$$0 = \sum_{h,j,k,l=1}^{n} \left(\nabla_h R^i_{jkl} + \nabla_j R^i_{khl} + \nabla_k R^i_{hjl} \right) X^h Y^j Z^k U^l,$$

The coordinate expression for second Bianchi identity is

$$\nabla_h R^i_{jkl} + \nabla_j R^i_{khl} + \nabla_k R^i_{hjl} = 0.$$

$\diamond$

Exercise 12.10.

Show that following identity holds

$$\sum_{h,k=1}^{n} \tilde{g}^{hk} \nabla_h \mathrm{Ric}_{kj} = \frac{1}{2} \nabla_j \mathrm{Scal},$$

Solution. First, let us contract second Bianch identity and use the antisymmetry in first two indices of curvature tensor

$$\sum_{k=1}^{n} \left(\nabla_h R^k_{jkl} + \nabla_j R^k_{khl} + \nabla_k R^k_{hjl} \right) = 0,$$

$$-\nabla_h \mathrm{Ric}_{jl} + \nabla_j \mathrm{Ric}_{hl} + \sum_{k=1}^{n} \nabla_k R^k_{hjl} = 0.$$

We can multiply this by $\sum_{h,l=1}^{n} \tilde{g}^{hl}$ and use the fact that the metric is covariantly constant, i.e. $\nabla g = 0$

$$\sum_{h,l=1}^{n} \tilde{g}^{hl} \left(-\nabla_h \mathrm{Ric}_{jl} + \nabla_j \mathrm{Ric}_{hl} + \sum_{k=1}^{n} \nabla_k R^k_{hjl} \right) = 0,$$

$$-\sum_{h,l=1}^{n} \tilde{g}^{hl} \nabla_h \mathrm{Ric}_{jl} + \nabla_j \mathrm{Scal} + \sum_{k,h,l=1}^{n} \tilde{g}^{hl} \nabla_k R^k_{hjl} = 0.$$

Last term can be manipulated using symmetries of curvature

$$\sum_{k,h,l=1}^{n} \tilde{g}^{hl} \nabla_k R^k_{hjl} = \sum_{k,h,l,m=1}^{n} \tilde{g}^{hl} \nabla_k \tilde{g}^{mk} R_{hjlm} = \sum_{k,h,l,m=1}^{n} \tilde{g}^{hl} \nabla_k \tilde{g}^{mk} R_{jhml} =$$

$$= \sum_{k,h,m=1}^{n} \nabla_k \tilde{g}^{mk} R^h_{jhm} = -\sum_{k,m=1}^{n} \tilde{g}^{mk} \nabla_k \mathrm{Ric}_{jm}.$$

After relabeling $(k \to h)$ a $(m \to l)$ and plugging into last equation we get the identity. $\diamond$

Exercise 12.11.

Let (M,g) be connected $n-$dimensional Riemann manifold with a Levi-Civita connection ∇ and Ricci tensor is of form

$$R_{ij} = \mu(x^k) g_{ij},$$

where $\mu(x^k)$ is a function of coordinates. Show that for $n > 2$ the function $\mu(x^k)$ has to be constant.

Solution. First let's take a look at the case $n = 1$. Curvature, Ricci tensor and scalar curvature are identically zero and $\mu(x^k)$ is an arbitrary function. For $n > 1$ we calculate Ricci scalar first. The expression $\sum_{i=1}^n \delta_i^i$ is just a trace of $n-$dimensional unit matrix. Therefore, the Ricci scalar is

$$R = \sum_{i,j=1}^n \tilde{g}^{ij} R_{ij} = \sum_{i,j=1}^n \tilde{g}^{ij} \mu g_{ij} = \mu \sum_{i=1}^n \delta_i^i = n\mu.$$

We also apply the expression for a covariant derivative of Ricci scalar from previous exercise, i.e.

$$\sum_{h,k=1}^n \tilde{g}^{hk} \nabla_h \mathrm{Ric}_{kj} = \frac{1}{2} \nabla_j \mathrm{Scal},$$
$$\sum_{h,k=1}^n \tilde{g}^{hk} \nabla_h \left(\mu g_{kj}\right) = \frac{1}{2} \nabla_j \left(n\mu\right).$$

μ is a scalar function and covariant derivative is just ordinary derivative. Because the metric is covariantly constant, we get

$$\sum_{h,k=1}^n \tilde{g}^{hk} g_{kj} \frac{\partial \mu}{\partial x^h} = \frac{n}{2} \frac{\partial \mu}{\partial x^j},$$
$$\sum_{h=1}^n \delta_j^h \frac{\partial \mu}{\partial x^h} = \frac{n}{2} \frac{\partial \mu}{\partial x^j},$$
$$\frac{\partial \mu}{\partial x^j} = \frac{n}{2} \frac{\partial \mu}{\partial x^j}.$$

This condition is fulfilled if and only if $n = 2$. For $n > 2$ we have

$$\frac{\partial \mu}{\partial x^j} = 0,$$

and μ is a constant. $\diamond$

Exercise 12.12.

Let ∇ be torsionless connection on a smooth manifold M. Show that for vector fields $X, Y \in \mathscr{X} M$, the exteriror derivative of 1-form $\omega \in \Omega(M)$ is

$$\mathrm{d}\omega(X,Y) = \langle \nabla_X \omega, Y \rangle - \langle \nabla_Y \omega, X \rangle.$$

Solution. Let us remind that

$$d\omega(X,Y) = X\langle\omega,Y\rangle - Y\langle\omega,X\rangle - \langle\omega,[X,Y]\rangle\,.$$

In first two terms we use definition of a covariant derivative, in last term we use that the connection has no torsion. We get

$$d\omega(X,Y) = \langle\nabla_X\omega,Y\rangle + \langle\omega,\nabla_X Y\rangle - \langle\nabla_Y\omega,X\rangle - \langle\omega,\nabla_Y X\rangle - \langle\omega,\nabla_X Y - \nabla_Y X\rangle\,.$$

The identity is obtained by linearity. $\diamond$

All definitions, theorems, lemmas and remarks has been taken from I. Kolar textbook which serves as a main study material for the course. Eventhough we made few alterations of the original formulation, we also aimed to preserve the idea behind them. During the process of writing this text, we have been inspired by various literature, amongst which was collection of exercises with solutions, written by associate professor Anton Galaev, Dr. rer. nat.

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