Naposledy upraveno: 22. února 2020.

Úkol 1

Příklad 1.1. Derivujte parciálně podle všech proměnných: $f_1 = c + x + y^2 + z^3$, $f_2 = ct\sin(xyz)$, $f_3 = \frac{\cos(ct)}{r^2}$, kde $r^2 = x^2 + y^2 + z^2$.

$$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial t} = c\sin(xyz) \qquad \frac{\partial f_3}{\partial t} = -c\frac{\sin(ct)}{r^2}$$

$$\frac{\partial f_1}{\partial y} = 2y \qquad \frac{\partial f_2}{\partial x} = ct\cos(xyz)\cdot yz \qquad \frac{\partial f_3}{\partial x} = -\frac{2x\cos ct}{(x^2+y^2+z^2)^2} = -2x\frac{\cos(ct)}{r^4}$$

$$\frac{\partial f_1}{\partial z} = 3z^2 \qquad \frac{\partial f_2}{\partial y} = ct\cos(xyz)\cdot xz \qquad \frac{\partial f_3}{\partial y} = -\frac{2y\cos ct}{(x^2+y^2+z^2)^2} = -2y\frac{\cos(ct)}{r^4}$$

$$\frac{\partial f_1}{\partial t} = 0 \qquad \frac{\partial f_2}{\partial z} = ct\cos(xyz)\cdot xy \qquad \frac{\partial f_3}{\partial z} = -\frac{2z\cos ct}{(x^2+y^2+z^2)^2} = -2z\frac{\cos(ct)}{r^4}$$

Příklad 1.2. Najděte divergenci a rotaci vektorového pole $\vec{V} = \omega(-y, x, 0)$.

$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial(\omega(-y))}{\partial x} + \frac{\partial(\omega x)}{\partial y} + \frac{\partial(\omega\cdot 0)}{\partial z}\right) = \vec{0}$$

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial 0}{\partial y} - \frac{\partial(\omega y)}{\partial z}, \frac{\partial(-\omega y)}{\partial z} - \frac{\partial 0}{\partial x}, \frac{\partial(\omega x)}{\partial z} - \frac{\partial(-\omega y)}{\partial x}\right) = (0, 0, 2\omega)$$

Příklad 1.3. Dokažte identitu: $\vec{\nabla}\times(\vec{V}\times\vec{r}) - 2\vec{V} = \vec{0}$, kde $\vec{r} = (x, y, z)$ a $\vec{V}$ je konstantní vektor.

Pro lehkou úsporu místa budu dokazovat pouze, že $\vec{\nabla}\times(\vec{V}\times\vec{r}) = 2\vec{V}$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times((V_x, V_y, V_z)\times(x, y, z)) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times((V_y\cdot z - V_z\cdot y), (V_z\cdot x - V_x\cdot z), (V_x\cdot y - V_y\cdot x)) =$$
$$= \left(\frac{\partial}{\partial y}(V_x\cdot y - V_y\cdot x) - \frac{\partial}{\partial z}(V_z\cdot x - V_x\cdot z), \frac{\partial}{\partial z}(V_y\cdot z - V_z\cdot y) - \frac{\partial}{\partial x}(V_x\cdot y - V_y\cdot x),\right.$$
$$\left.\frac{\partial}{\partial x}(V_z\cdot x - V_x\cdot z) - \frac{\partial}{\partial y}(V_y\cdot z - V_z\cdot y)\right) = ((V_x - (-V_x)), (V_y - (-V_y)), (V_z - (-V_z))) = (2V_x, 2V_y, 2V_z) = 2\vec{V}$$

Příklad 1.4. Dokažte identity $\vec{\nabla}\times(\vec{V}F) = F\vec{\nabla}\times\vec{V} + (\vec{\nabla}F)\times\vec{V}$, $\vec{\nabla}\cdot(\vec{V}F) = F\vec{\nabla}\cdot\vec{V} + (\vec{\nabla}F)\cdot\vec{V}$, kde $\vec{V}$ představuje obecné trojrozměrné vektorové pole a F obecnou funkci.

Identity můžeme dokázat tak, že rozepíšeme operátory a vektory do složek a ty upravíme do požadovaného tvaru.

$$= \nabla\times(\vec{V}F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times((V_x, V_y, V_z)\cdot F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times((V_xF, V_yF, V_zF)) =$$
$$= \left(\frac{\partial V_zF}{\partial y} - \frac{\partial V_yF}{\partial z}, \frac{\partial V_xF}{\partial z} - \frac{\partial V_zF}{\partial x}, \frac{\partial V_yF}{\partial x} - \frac{\partial V_xF}{\partial y}\right) =$$
$$= \left(F\frac{\partial V_z}{\partial y} + V_z\frac{\partial F}{\partial y} - F\frac{\partial V_y}{\partial z} - V_y\frac{\partial F}{\partial z}, F\frac{\partial V_x}{\partial z} + V_x\frac{\partial F}{\partial z} - F\frac{\partial V_z}{\partial x} - V_z\frac{\partial F}{\partial x}, F\frac{\partial V_y}{\partial x} + V_y\frac{\partial F}{\partial x} - F\frac{\partial V_x}{\partial y} - V_x\frac{\partial F}{\partial y}\right) =$$
$$= F\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) + \left(V_z\frac{\partial F}{\partial y} - V_y\frac{\partial F}{\partial z}, V_x\frac{\partial F}{\partial z}V_z\frac{\partial F}{\partial x}, V_y\frac{\partial F}{\partial x} - V_x\frac{\partial F}{\partial y}\right) =$$
$$= F(\nabla\times\vec{V}) + (\nabla F)\times\vec{V}$$

$$\nabla \cdot (\vec{V}F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \cdot ((V_x, V_y, V_z)F) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)(V_xF, V_yF, V_yF) =$$
$$= \left(\frac{\partial}{\partial x}(V_xF) + \frac{\partial}{\partial y}(V_yF) + \frac{\partial}{\partial z}(V_zF)\right) = \left(F\frac{\partial V_x}{\partial x} + V_x\frac{\partial F}{\partial x} + F\frac{\partial V_y}{\partial y} + V_y\frac{\partial F}{\partial y} + F\frac{\partial V_z}{\partial z} + V_z\frac{\partial F}{\partial z}\right) =$$
$$= F\left(\nabla \cdot \vec{V}\right) + (\nabla F) \cdot \vec{V}$$

Nebo můžeme použít definici vektorového součinu

$$\vec{A} \times \vec{B} = \varepsilon_{klm} A_l B_m \tag{1}$$

a Einsteinovu sumační konvenci

$$\sum_{i=1}^{n} a_i b_i = a_i b_i \tag{2}$$

a zjednodušený zápis derivace podle k-té složky jako

$$\frac{\partial}{\partial k} = \partial_k \tag{3}$$

Důkazy budou následně vypadat takto:

$$\left[\nabla \times (\vec{V}F)\right]_k = \varepsilon_{klm}\partial_l(\vec{V}F)_m = \varepsilon_{klm}\partial_l(V_mF) = \varepsilon_{klm}(\partial_l V_m)F + \varepsilon_{klm}(\partial_l F)V_m = \left[F(\nabla \times \vec{V})\right]_k + \left[(\nabla F) \times \vec{V}\right]_k$$

Protože jde o obecnou složku k bude výsledný výraz platit i pro jakoukoli jinou složku. Můžeme tedy psát

$$\nabla \times \left(\vec{V}F\right) = F\left(\nabla \times \vec{V}\right) + (\nabla F)\,\vec{V}$$

$$\nabla \cdot (\vec{V}F) = \partial_k(V_kF) = (\partial_k V_k)F + (\partial_k F)V_k = (\nabla\vec{V})F + (\nabla F)\vec{V}$$

Příklad 1.5. Vypočtěte integrál po třech různých libovolných křivkách z bodu $\vec{r} = (0,0,0)$ do bodu $\vec{r'} = (1,1,1)$. Vektorové pole je $\vec{V} = (y, -x, z)$.

První trajektorie:

$$\begin{aligned} &x = t, y = t, z = t \qquad \text{kde } t \in [0,1] \\ &\vec{V} = (t, -t, t) \\ &\mathrm{d}\vec{r} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t = (1,1,1)\,\mathrm{d}t \end{aligned}$$

$$\int_{\mathcal{C}} \vec{V}\mathrm{d}\vec{r} = \int_0^1 (t, -t, t) \cdot (1,1,1)\mathrm{d}t = \int_0^1 t - t + t\mathrm{d}t = \int_0^1 t\mathrm{d}t = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

Druhá trajektorie:

$$\begin{aligned} &x = t^3, y = t^5, z = t^2 \qquad \text{kde } t \in [0,1] \\ &\vec{V} = (t^5, -t^3, t^2) \\ &\mathrm{d}\vec{r} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t = \left(3t^2, 5t^4, 2t\right)\mathrm{d}t \end{aligned}$$

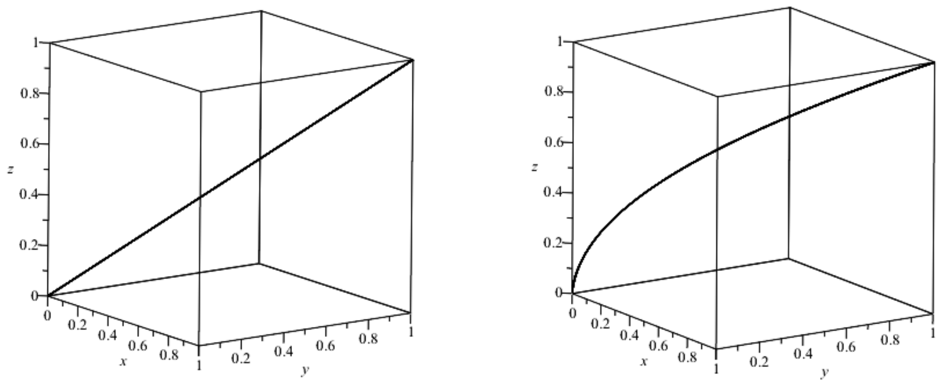
$$\int_{\mathcal{C}} \vec{V}\mathrm{d}\vec{r} = \int_0^1 (t^5, -t^3, t^2) \cdot (3t^2, 5t^4, 2t)\mathrm{d}t = \int_0^1 3t^7 - 5t^7 + 2t^3\mathrm{d}t =$$
$$= \int_0^1 (-2t^7 + 2t^3)\mathrm{d}t = \left[-\frac{2t^8}{8} + \frac{2t^4}{4}\right]_0^1 = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$$

Třetí trajektorie:

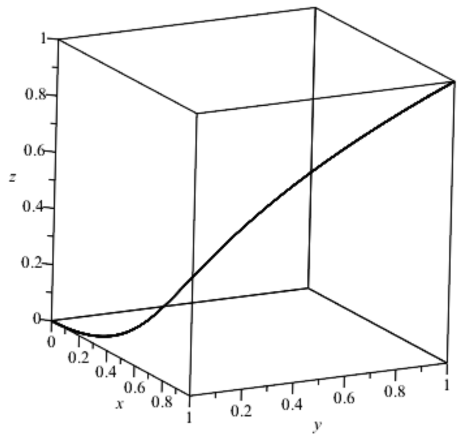
$$
\begin{aligned}
&x = 2t - t^2, y = t^5, z = t^2 \qquad \text{kde } t \in [0,1]\\
&\vec{V} = (t^5, t^2 - 2t, t^2)\\
&\mathrm{d}\vec{r} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t = \big(2 - 2t, 5t^4, 2t\big)\,\mathrm{d}t
\end{aligned}
$$

$$
\int_{\mathcal{C}} \vec{V}\,\mathrm{d}\vec{r} = \int_0^1 (t^5, t^2 - 2t, t^2)\cdot(2 - 2t, 5t^4, 2t)\;\mathrm{d}t = \int_0^1 (2t^5 - 2t^6 + 5t^6 - 10t^5 + 2t^3)\mathrm{d}t =
$$

$$
= \int_0^1 (3t^6 - 8t^5 + 2t^3)\mathrm{d}t = \left[\frac{3t^7}{7} - \frac{8t^6}{6} + \frac{2t^4}{4}\right]_0^1 = \frac{3}{7} - \frac{8}{6} + \frac{1}{2} = \frac{18 - 56 + 21}{42} = \frac{-17}{42}
$$



Obrázek 1: Vlevo křivka pro první parametrizaci, což je přímka. Vpravo křivka pro druhou parametrizaci.



Obrázek 2: Křivka pro třetí parametrizaci

Ověření výpočtu křivkových integrálů v Maplu.

```
with(VectorCalculus):
SetCoordinates(cartesian[x,y,z]);
LineInt(VectorField(<y,-x,z>),Path(<t,t,t>,t=0..1));
LineInt(VectorField(<y,-x,z>),Path(<t^3,t^5,t^2>,t=0..1));
LineInt(VectorField(<y,-x,z>),Path(<2*t-t^2,t^5,t^2>,t=0..1));
```