Algebraic categorification and its applications, I

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“Definition”

Roughly speaking, categorification means an “upgrade” from set theory to category theory, in particular:

sets are upgraded to categories

functions are upgraded to functors

equalities are upgraded to isomorphisms
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Categorification in short

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Motivation

Question: Why do we need categorification?

Answer: Categories have more structure than sets.

This can be used to get new useful information about objects we study.
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Example: Khovanov homology — links and crossings

\( L \) — diagram of an oriented link

\( n_+ \) — number of right crossings

\( n_- \) — number of left crossings

right crossing

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Definition. The Kauffman bracket $\{L\} \in \mathbb{Z}[v, v^{-1}]$ of $L$ is defined via the following rule:

$$
\begin{cases}
\times \times \times = \{ \equiv \} - v \{ \} \\
\end{cases}
$$

together with $\{ \bigcirc L \} = (v + v^{-1}) \{L\}$

and normalized by the conditions $\{\emptyset\} = 1$. 
Definition. The Kauffman bracket \( \{L\} \in \mathbb{Z}[v, v^{-1}] \) of \( L \) is defined via the following rule:

\[
\begin{cases}
\{ \begin{array}{c}
\times
\end{array} \} = \{ \begin{array}{c}
\circlearrowleft
\end{array} \} - v \{ \begin{array}{c}
\circlearrowright
\end{array} \}
\end{cases}
\]

together with \( \{ \bigcirc L \} = (v + v^{-1}) \{L\} \)

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$$\begin{align*}
\{ \begin{array}{c}
\times \\
\end{array} \} &= \{ \begin{array}{c}
\equiv \\
\end{array} \} - v \{ \begin{array}{c}
\bigcirc \\
\end{array} \} \\
\{ \begin{array}{c}
\bigcirc \\
\end{array} \} &= (v + v^{-1})\{L\}
\end{align*}$$

The Kauffman bracket together with the normalization $\{\emptyset\} = 1$.
Definition. The Kauffman bracket \( \{L\} \in \mathbb{Z}[\nu, \nu^{-1}] \) of \( L \) is defined via the following rule:

\[
\begin{align*}
\{ \begin{array}{c}
\times \\
\hline
\end{array} \} &= \{ \begin{array}{c}
\equiv \\
\hline
\end{array} \} - \nu \{ \begin{array}{c}
\circ \\
\hline
\end{array} \} \\
\end{align*}
\]

Together with \( \{ \bigcirc L \} = (\nu + \nu^{-1})\{L\} \) and normalized by the conditions \( \{\emptyset\} = 1 \).
Definition. The Kauffman bracket $\{L\} \in \mathbb{Z}[v, v^{-1}]$ of $L$ is defined via the following rule:

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and normalized by the conditions $\{\emptyset\} = 1$. 
Definition. The unnormalized Jones polynomial $\hat{J}(L)$ of $L$ is defined by

$$\hat{J}(L) := (-1)^{n-} v^{n+2n} \{L\} \in \mathbb{Z}[v, v^{-1}]$$

Definition. The (usual) Jones polynomial $J(L)$ is defined via

$$(v + v^{-1})J(L) = \hat{J}(L).$$

Theorem. [Jones] $J(L)$ is an invariant of an oriented link.

Example. For the Hopf link

$$H := \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}$$

we have $\hat{J} = (v + v^{-1})(v + v^5)$ and $J(H) = v + v^5$. 
Example: Khovanov homology — Jones polynomial

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Example. For the Hopf link

$$H := \begin{tikzpicture}
\draw (0,0) circle (0.5 cm);
\draw (1,0) circle (0.5 cm);
\draw (0,0) -- (1,0);
\end{tikzpicture}$$

we have $\hat{J} = (v + v^{-1})(v + v^5)$ and $J(H) = v + v^5$. 

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Example. For the Hopf link

$$H := \begin{tikzpicture} [baseline=-.5ex]
\node (a) at (0,0) {O};
\node (b) at (1,0) {O};
\draw (a) to[out=90, in=180, looseness=1.5] (b);
\draw (b) to[out=90, in=0, looseness=1.5] (a);
\end{tikzpicture}$$

we have $\hat{J} = (v + v^{-1})(v + v^5)$ and $J(H) = v + v^5$. 
Example: Khovanov homology — Jones polynomial

Definition. The unnormalized Jones polynomial $\hat{J}(L)$ of $L$ is defined by

$$\hat{J}(L) := (-1)^n - \nu^{n+2n} \{L\} \in \mathbb{Z}[\nu, \nu^{-1}]$$

Definition. The (usual) Jones polynomial $J(L)$ is defined via

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Theorem. [Jones] $J(L)$ is an invariant of an oriented link.

Example. For the Hopf link

$$H := \begin{tikzpicture}
\draw (0,0) circle (1);
\draw (1,0) circle (1);
\end{tikzpicture}$$

we have $\hat{J} = (\nu + \nu^{-1})(\nu + \nu^5)$ and $J(H) = \nu + \nu^5$. 
Definition. The unnormalized Jones polynomial $\hat{J}(L)$ of $L$ is defined by

$$\hat{J}(L) := (-1)^n v^{n+2n-}\{L\} \in \mathbb{Z}[v, v^{-1}]$$

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Example. For the Hopf link

$$H := \begin{tikzpicture}
    \draw (0,0) circle (0.5);
    \draw (0,0) circle (0.5);
    \draw (0.5,0) to (0,0.5);
    \draw (0,0.5) to (0.5,0);
\end{tikzpicture}$$

we have $\hat{J} = (v + v^{-1})(v + v^5)$ and $J(H) = v + v^5$. 
Definition. The unnormalized Jones polynomial \( \hat{J}(L) \) of \( L \) is defined by

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\hat{J}(L) := (-1)^{n-} v^{n+2n-} \{L\} \in \mathbb{Z}[v, v^{-1}]
\]

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(v + v^{-1})J(L) = \hat{J}(L).
\]

Theorem. [Jones] \( J(L) \) is an invariant of an oriented link.

Example. For the Hopf link

\[
H := \begin{array}{c}
\includegraphics[width=0.1\textwidth]{hopf_links.png}
\end{array}
\]

we have \( \hat{J} = (v + v^{-1})(v + v^5) \) and \( J(H) = v + v^5 \).
Example: Khovanov homology — Jones polynomial

Definition. The unnormalized Jones polynomial $\hat{J}(L)$ of $L$ is defined by

$$\hat{J}(L) := (-1)^{n-1} v^{n_+ - 2n_-} \{L\} \in \mathbb{Z}[v, v^{-1}]$$

Definition. The (usual) Jones polynomial $J(L)$ is defined via

$$(v + v^{-1}) J(L) = \hat{J}(L).$$

Theorem. [Jones] $J(L)$ is an invariant of an oriented link.

Example. For the Hopf link

$$H := \begin{array}{c}
\includegraphics[width=0.2\textwidth]{hopf_link.png}
\end{array}$$

we have

$$\hat{J} = (v + v^{-1})(v + v^5)$$

and

$$J(H) = v + v^5.$$
Example: Khovanov homology — characterization of $J$

**Theorem.** The Jones polynomial is uniquely determined by the property $J(\bigcirc) = 1$

and the **skein relation**

\[
v^2 J \begin{array}{cc}
\begin{array}{c}
\nearrow \\
\nearrow
\end{array} &
\begin{array}{c}
\searrow \\
\searrow
\end{array}
\end{array} - v^{-2} J \begin{array}{cc}
\begin{array}{c}
\swarrow \\
\swarrow
\end{array} &
\begin{array}{c}
\nwsearrow \\
\nwsearrow
\end{array}
\end{array} = (v - v^{-1}) J \begin{array}{cc}
\begin{array}{c}
\nwsearrow \\
\nwsearrow
\end{array} &
\begin{array}{c}
\nearrow \\
\nearrow
\end{array}
\end{array} \]

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Example: Khovanov homology — characterization of $J$

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\[ v^2 J \begin{array}{c} \rightarrow \leftarrow \end{array} - v^{-2} J \begin{array}{c} \rightarrow \leftarrow \end{array} = (v - v^{-1}) J \begin{array}{c} \nearrow \searrow \end{array} \]
**Theorem.** The Jones polynomial is uniquely determined by the property $J(\bigcirc) = 1$

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\[
v^2 J \left( \begin{array}{c} \uparrow \downarrow \\ \downarrow \bigcirc \uparrow \end{array} \right) - v^{-2} J \left( \begin{array}{c} \uparrow \downarrow \\ \downarrow \bigcirc \uparrow \end{array} \right) = (v - v^{-1}) J \left( \begin{array}{c} \uparrow \\ \bigcirc \end{array} \right)
\]
Theorem. The Jones polynomial is uniquely determined by the property $J(\bigcirc) = 1$

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$$v^2 J \begin{pmatrix} \begin{array}{cc} \rightarrow & \rightarrow \\ \rightarrow & \rightarrow \end{array} \end{pmatrix} - v^{-2} J \begin{pmatrix} \begin{array}{cc} \rightarrow & \rightarrow \\ \rightarrow & \rightarrow \end{array} \end{pmatrix} = (v - v^{-1}) J \begin{pmatrix} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \end{pmatrix}$$
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$$v^2 J \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \end{pmatrix} - v^{-2} J \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \end{pmatrix} = (v - v^{-1}) J \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \end{array} \end{pmatrix}$$
Main idea: [Khovanov] Upgrade Kauffman bracket to a new bracket $[\cdot]$.

- $\mathbb{C}$-mod — category of finite dimensional $\mathbb{C}$-vector spaces.
- $\mathbb{C}$-gmod — category of finite dimensional graded $\mathbb{C}$-vector spaces.
- $\text{Com}^b(\mathbb{C}$-gmod) — category of finite complexes over $\mathbb{C}$-gmod.

$[\cdot]$ takes values in $\text{Com}^b(\mathbb{C}$-mod).

$V = \mathbb{C}$ in degree 1 $\oplus$ $\mathbb{C}$ in degree $-1$. 
Example: Khovanov homology — idea and ingredients

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\([\cdot, \cdot]\) takes values in \(\text{Com}^b(\mathbb{C}\text{-mod})\)

\(V \rightarrow \mathbb{C}\) in degree 1 \(\oplus \mathbb{C}\) in degree \(-1\)
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Main idea: [Khovanov] Upgrade Kauffman bracket to a new bracket $\langle \cdot, \cdot \rangle$

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Example: Khovanov homology — categorification

Categorification of normalization conditions:

\[ [\emptyset] = 0 \rightarrow C \rightarrow 0 \]

\[ [\bigcirc L] = V \otimes [L] \]

Categorification of the Kauffman bracket:

\[
\begin{bmatrix}
\begin{array}{c}
\times
\end{array}
\end{bmatrix} = \text{Total} \left( 0 \rightarrow \begin{bmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{bmatrix} \rightarrow \begin{bmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{bmatrix} \rightarrow \begin{bmatrix}
\begin{array}{c}
\bigcirc
\end{array}
\end{bmatrix} \right) \langle -1 \rangle \rightarrow 0
\]

Main difficulty: Definition of \( d \).
Example: Khovanov homology — categorification

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\[
\begin{array}{c}
\begin{array}{c}
\times \\
\end{array}
\end{array}
= \text{Total} \left( 0 \to \begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\end{array}
\end{array} \xrightarrow{d} \begin{array}{c}
\begin{array}{c}
\circlearrowright \\
\end{array}
\end{array} \right) \left( \langle -1 \rangle \to 0 \right)
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\[ \begin{array}{ccc}
\times & \rightarrow & \text{Total} \\
\downarrow & & \downarrow \quad \rightarrow \\
0 & \rightarrow & \begin{bmatrix}
\quad & \quad & d \\
\quad & \rightarrow & \end{bmatrix} \\
\end{array} \]

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\bigcirc
\end{array}
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\begin{bmatrix}
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= \text{Total}
\begin{bmatrix}
\begin{array}{c}
0 \rightarrow \\
\bigotimes
\end{array}
\end{bmatrix}
\xrightarrow{d}
\begin{bmatrix}
\begin{array}{c}
-1 \rightarrow
\end{array}
\end{bmatrix}
\]

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Example: Khovanov homology — categorification

Categorification of normalization conditions:

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\]

\[
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\]

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\[
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\begin{array}{c}
\end{array}
\end{bmatrix} \stackrel{d}{\rightarrow} \begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix} \langle -1 \rangle \rightarrow 0 \right)
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Example: Khovanov homology — the result

\[\cdot\] — shift in homological position

\langle \cdot \rangle — shift in grading

Theorem. [Khovanov]
Homology of \([\cdot][n_-]\langle n_+ - 2n_- \rangle\) is an invariant of an oriented link.

Note: \([\cdot][n_-]\langle n_+ - 2n_- \rangle\) is not an invariant of an oriented link.

Decategorification theorem. [Khovanov]
Graded Euler characteristic of \([L][n_-]\langle n_+ - 2n_- \rangle\) equals \(\hat{J}(L)\).

Benefit: Khovanov homology is a strictly stronger invariant than Jones polynomial. For example:

Theorem. [Kronheimer-Mrowka] Khovanov homology detects the unknot.
Example: Khovanov homology — the result

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Homology of $[[n]][n_−]⟨n_+ − 2n_−⟩$ is an invariant of an oriented link.

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**Benefit:** Khovanov homology is a strictly stronger invariant than Jones polynomial. For example:

**Theorem.** [Kronheimer-Mrowka] Khovanov homology detects the unknot.
Example: Khovanov homology — the result

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\langle \cdot \rangle \quad \text{— shift in grading}
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Elementary diagrams:

the cup diagram
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right crossing
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\( \mathfrak{g} \) — simple finite dimensional Lie algebra

\( U(\mathfrak{g}) \) — the universal enveloping algebra of \( \mathfrak{g} \)

Fact. \( U(\mathfrak{g}) \) is a cocommutative Hopf algebra.

Consequence. The isomorphism \( V \otimes W \cong W \otimes V \) is involutive.

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Alternative approach — tangles

**Tang** — the category of oriented tangles

**Objects:** Non-negative integers

Informally: $n \in \{0, 1, 2, \ldots \}$ should be thought of as a collection of $n$ points.

**Morphisms:** Oriented diagrams generated by (oriented) elementary diagrams (up to isotopy), connecting the corresponding points, read from bottom to top.

**Composition:** Concatenation

**Example 1:** An oriented cup diagram is a morphism from 0 to 2.

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Idea of quantum knot invariants. [Reshetikhin-Turaev]

Consider some $U_v(g)$.

$V$ — the “natural” $U_v(g)$-module

Define a functor $F : \text{Tang} \rightarrow U_v(g)\text{-mod}$

$F(n) := V^\otimes n$, where $F(0) := \mathbb{C}(\nu)$

$F(\text{elementary diagram}) := \text{certain explicit homomorphisms of } U_v(g)\text{-modules}$

oriented link $L \rightarrow$ tangle $T_L \rightarrow \text{endom. } F(T_L)$ of $\mathbb{C}(\nu)$

Consequence: $F(T_L)(1)$ is an invariant of $L$. 
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Quantum invariants — $U_q(\mathfrak{sl}_2)$

**Definition:** $U_q(\mathfrak{sl}_2)$ has generators $E, F, K, K^{-1}$ and relations

\[
KE = v^2 EK, \quad KF = v^{-2} FK, \quad KK^{-1} = K^{-1} K = 1,
\]

\[
EF - FE = \frac{K - K^{-1}}{v - v^{-1}}.
\]

**Hopf structure:**

\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}.
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Quantum invariants — natural $U_v(\mathfrak{sl}_2)$-module

Quantum numbers: $[a] := \frac{v^a - v^{-a}}{v - v^{-1}}$, $a \in \mathbb{Z}$

$V$ — the “natural” $U_v(\mathfrak{sl}_2)$-module

Basis: $w_0$ and $w_1$

Action:

$E w_k = [k + 1] w_{k+1}$, $F w_k = -[n - k + 1] w_{k-1}$,

$K^{\pm 1} w_k = -v^{\pm (2k-n)} w_k$

Notation: $w_{i_1} \otimes w_{i_2} \otimes \cdots \otimes w_{i_k}$ denoted by $i_1 i_2 \ldots i_k$

Consequence: Basis in $V^\otimes n$ consists of 0-1-sequences of length $n$. 
Quantum invariants — natural $U_v(\mathfrak{sl}_2)$-module

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$V$ — the “natural” $U_v(\mathfrak{sl}_2)$-module

Basis: $w_0$ and $w_1$

Action:

$Ew_k = [k + 1]w_{k+1}$, \hspace{1cm} $Fw_k = -[n - k + 1]w_{k-1}$,

$K^{\pm 1}w_k = -v^{\pm(2k-n)}w_k$

Notation: $w_{i_1} \otimes w_{i_2} \otimes \cdots \otimes w_{i_k}$ denoted by $i_1i_2\ldots i_k$

Consequence: Basis in $V \otimes^n$ consists of 0-1–sequences of length $n$. 
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Quantum invariants — action of tangles

Definition. The functor $F : \text{Tang} \to U_V(\mathfrak{sl}_2)\text{-mod}$ is given by:

$\cup : \mathbb{C}(v) \to \hat{\mathcal{Y}}_1^\otimes 2$ is given by:

$$1 \mapsto 01 + v10.$$ 

$\cap : \hat{\mathcal{Y}}_1^\otimes 2 \to \mathbb{C}(v)$ is given by:

$$00 \mapsto 0, \quad 11 \mapsto 0, \quad 01 \mapsto v^{-1}, \quad 10 \mapsto 1.$$ 

right crossing: $\hat{\mathcal{Y}}_1^\otimes 2 \to \hat{\mathcal{Y}}_1^\otimes 2$ is given by:

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Let $L$ be an oriented link. Then

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Categorification of quantum $U_v(\mathfrak{sl}_2)$-invariants — the idea

$\text{Cat}$ — category of categories

Idea: Construct a functor from $\text{Tang}$ to $\text{Cat}$?

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Approach via category $\mathcal{O}$

$\mathfrak{gl}_n$ — reductive Lie algebra over $\mathbb{C}$

$\mathfrak{gl}_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — standard triangular decomposition

$\mathcal{O}$ — BGG category $\mathcal{O}$

$S_n$ — the Weyl group of $\mathfrak{gl}_n$

Fact: $S_n$ acts on $\mathfrak{h}^*$ in the natural way

$M(\lambda)$ — Verma module with highest weight $\lambda - \rho$
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Blocks in $\mathcal{O}$

$\mathcal{O}_0$ — the principal block of $\mathcal{O}$

$k \in \{0, 1, 2, \ldots, n\}$

$S_k \times S_{n-k} \subset S_N$ — maximal Young subgroup

$p_k$ — corresponding parabolic subalgebra

$\mathcal{O}^{(k,n-k)}_0$ — parabolic subcategory of locally $p_k$-finite modules

Definition: $\mathcal{C}_n := \bigoplus_{k=0}^{n} \mathcal{O}^{(k,n-k)}_0$

Fact: $\mathcal{C}_n$ has $2^n$ simple objects up to isomorphism.

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Categorification of $V^\otimes n$ for $v = 1$

**Observation:** $\dim V^\otimes n = \text{rank}(\text{Gr}(C_n))$

$p \subset q$ — parabolic subalgebras

$I_{(p,q)} : \mathcal{O}^q \subset \mathcal{O}^p$ — natural inclusion

$Z_{(p,q)} : \mathcal{O}^p \subset \mathcal{O}^q$ — adjoint Zuckerman functors

**Note:** $Z_{(p,q)}$ is only right exact

**Action:** $E : D^b(\mathcal{O}(k,n-k)) \xrightarrow{1} D^b(\mathcal{O}(k,1,n-k-1)) \xrightarrow{\mathcal{L}Z} D^b(\mathcal{O}(k+1,n-k-1))$

**Action:** $F$ — adjoint to $E$

**Theorem.**[Bernstein-Frenkel-Khovanov] This categorifies $V^\otimes n$ for $v = 1$.

**Meaning:** Taking the Grothendieck group results in $V^\otimes n$.
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Volodymyr Mazorchuk

Algebraic categorification and its applications, I

23/29
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**Question:** Where can we find $\nu$?

**Answer:** Introduce grading.

Theorem. [Soergel] Each block of (parabolic) $\mathcal{O}$ is equivalent to the category of (ungraded) modules over a finite dimensional positively graded and even Koszul algebra.

$\tilde{C}_n$ — graded version of $C_n$

Theorem. [Stroppel] The action of graded Zuckerman functors on $D^b(\tilde{C}_n)$ categorifies $V^\otimes n$

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Fact. Projective functors commute with Zuckerman functors.

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Categorification of quantum $U_v(\mathfrak{sl}_2)$-invariants — shuffling functors

$s \in S_n$ — simple reflection

$\theta_s$ — wall-crossing functor

Fact. There are adjunctions $\theta_s \to \text{Id}$ and $\text{Id} \to \theta_s$

Definition. [Carlin] Shuffling functor $C_s := \text{Coker}(\text{Id} \to \theta_s)$ (adjoint: coshuffling)

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Cup diagram: Translation out of a wall.

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