Categorical lattice-valued topology
Lecture 3: lattice-valued preordered sets

Sergejs Solovjovs
Department of Mathematics and Statistics, Faculty of Science, Masaryk University
Kotlářská 2, 611 37 Brno, Czech Republic

Abstract
As an application of the theory of categorical lattice-valued topology, this lecture provides lattice-valued analogues of the well-known results that the construct Prost of preordered sets, firstly, is concretely isomorphic to a full concretely coreflective subcategory of the category Top of topological spaces, and, secondly, is (non-concretely) isomorphic to a full coreflective subcategory of the category TopSys of topological systems.

1. Specialization preorder and Alexandroff topology

Remark 1. There is a convenient correspondence between topological spaces and preordered sets [5, 7].

Definition 2. A preordered set is a pair \((X, \leq)\), which contains a set \(X\) and a binary relation \(\leq \subseteq X \times X\) (preorder on \(X\)), which is reflexive (\(x \leq x\) for every \(x \in X\)) and transitive (\(x \leq y\) and \(y \leq z\) implies \(x \leq z\), for every \(x, y, z \in X\)). Prost is the construct (i.e., a category, which is concrete over the category Set of sets and maps) of preordered sets and preorder-preserving maps.

Definition 3. A partially ordered set (poset) is a preordered set \((X, \leq)\), in which the relation \(\leq\) is, additionally, antisymmetric (\(x \leq y\) and \(y \leq x\) imply \(x = y\), for every \(x, y \in X\)), and therefore, is called a partial order on \(X\). Pos is the full subcategory of Prost of posets and order-preserving maps.

Definition 4. Top\(_0\) is the full subcategory of the category Top, comprising \(T_0\) topological spaces.

Theorem 5.
(1) There exists a concrete functor \(\text{Top} \xrightarrow{\text{Spec}} \text{Prost}\), which is defined on objects by \(\text{Spec}(X, \tau) = (X, \leq_\tau)\), where \(x \leq_\tau y\) iff \(y \in \text{cl}\{x\}\).

(2) There exists a full concrete functor \(\text{Prost} \xrightarrow{\text{At}} \text{Top}\), which is defined on objects by \(\text{At}(X, \leq) = (X, \tau_\leq)\), where \(\tau_\leq = \{U \subseteq X \mid U = \downarrow U\}\) with \(\downarrow U = \{x \in X \mid x \leq y\text{ for some } y \in U\}\).

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Email address: solovjovs@math.muni.cz (Sergejs Solovjovs)
URL: http://www.math.muni.cz/~solovjovs (Sergejs Solovjovs)

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(3) $At$ is a left-adjoint-right-inverse to $Spec$, the adjoint situation being concrete (the (co-)unit is given by the identity maps).

**Remark 6.** The functor $Spec$ uses the dual of the well-known specialization preorder [7], hence the notation. The full embedding $At$ employs the dual of the Alexandroff topology [7], which has the property that arbitrary intersections of open sets are open (every point then has a smallest neighborhood), and hence the notation.

**Theorem 7.** The concrete adjunction $At \dashv Spec : Top \to Prost$ restricts to the constructs $Pos$ and $Top_0$.

**Theorem 8.** The category $Prost$ is isomorphic to a full coreflective subcategory of $TopSys$.

**Proof.** Combine the adjunctions $\begin{array}{c}
Prost \\
\downarrow At \\
\downarrow \downarrow Spec \\
\downarrow E \\
\downarrow \downarrow TopSys
\end{array}$ (recall Lecture 2).

**Remark 9.** Theorem 8 opens the way to do the theory of domains [4] inside that of topological systems.

**Remark 10.** This lecture shows a lattice-valued analogue of Theorem 8 over the category $Loc$ of locales.

2. Lattice-valued preordered sets versus lattice-valued topological spaces

**Remark 11.** This section shows a lattice-valued analogue of the adjunction between $Prost$ and $Top$.

**Definition 12.** Given a subcategory $S$ of $Loc$, $S-Prost$ is the category, concrete over the product category $Set \times S$, whose objects (localic preordered sets) are triples $(X, L, P)$, where $X$ is a set, $L$ is an $S$-object, and $X \times X \xrightarrow{L} L$ is a map (localic preorder on $(X, L)$, with “$P$” standing for “preorder”) such that
(1) $P(x, x) = \top_L$ for every $x \in X$ (reflexivity);
(2) $P(y, y) \land P(z, y) \leq P(x, y)$ for every $x, y, z \in X$ (transitivity).

Morphisms (localic monotone maps) $(X_1, L_1, P_1) \xrightarrow{(f, \varphi)} (X_2, L_2, P_2)$ are $Set \times S$-morphisms $(X_1, L_1) \xrightarrow{(f, \varphi)} (X_2, L_2)$ such that $P_1(x, y) \leq (\varphi^0 \circ P_2)(f(x), f(y))$ for every $x, y \in X_1$.

**Remark 13.** The notion of lattice-valued preorder has already been much studied in the literature (see, e.g., [2, 3, 6, 8, 9] for its various representations).

**Remark 14.** The case of $S_L$ is denoted $L-Prost$. In particular, $2-Prost$ is isomorphic to $Prost$.

**Definition 15.** Recall from Lecture 1 that given a subcategory $S$ of $Loc$, $S-Top$ is the category, concrete over the product category $Set \times S$, whose objects (localic topological spaces) are triples $(X, L, \tau)$, where $X$ is a set, $L$ is an $S$-object, and $\tau$ (localic topology on $X$) is a subframe of the power frame $L^X$. Morphisms (localic continuous maps) $(X_1, L_1, \tau_1) \xrightarrow{(f, \varphi)} (X_2, L_2, \tau_2)$ are $Set \times S$-morphisms $(X_1, L_1) \xrightarrow{(f, \varphi)} (X_2, L_2)$ such that $(f, \varphi)^\tau_1(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$ (written more concisely as $((f, \varphi)^\tau_2)^\tau_1(\tau_2) \subseteq \tau_1$).

**Remark 16.** The next result lifts the classical functor $Prost \xrightarrow{At} Top$ to the variable-basis localic setting.

**Theorem 17.** There exists a full concrete embedding $S-Prost \xrightarrow{At} S-Top$ defined by $At(X, L, P) = (X, L, \tau_P)$, where $\tau_P = \{ \alpha \in L^X \mid P(x, y) \land \alpha(y) \leq \alpha(x) \text{ for every } x, y \in X \}$.

**Proof.** To show that $At$ is correct on objects, notice that given $\{ \alpha_s \mid s \in S \} \subseteq \tau$ and $x, y \in X$, $P(x, y) \land (\bigvee_{s \in S} \alpha_s(y)) \leq \bigvee_{s \in S}(P(x, y) \land \alpha_s(y)) \leq \bigvee_{s \in S} \alpha_s(x)$. Moreover, $P(x, y) \land (\bigwedge_{s \in S} \alpha_s(y)) = \bigwedge_{s \in S}(P(x, y) \land \alpha_s(y)) \leq \bigwedge_{s \in S} \alpha_s(x)$, i.e., similar to the classical case, the topology $\tau_P$ is closed under arbitrary meets.
To show that the functor is correct on morphisms, notice that given some localic monotone map \((X_1, L_1, P_1)^{(f, \varphi)} \to (X_2, L_2, P_2)\) and \(\alpha \in \tau_{P_2}\), \(P_1(x, y) \wedge (f, \varphi)(\alpha (y)) \leq (f, \varphi)(P_2(f(x), y)) \leq \varphi^\op(\alpha(f(y))) \leq (f, \varphi)(\alpha \circ f)(y)\).

To verify that the functor is full, notice first that given a localic preordered set \((X, L, P)\), for every \(z \in X\), it follows that \(\alpha_z = P(-, z) \in \tau_P\). Indeed, \(P(x, y) \wedge \alpha_z = P(x, y) \wedge \alpha_z \leq P(x, z) = \alpha_z(x)\) by the transitivity property of localic preorder.

Given a now localic continuous map \(\hat{At}(X_1, L_1, P_1)^{(f, \varphi)} \to At(X_2, L_2, P_2)\) and \(x, y \in X_1\), it follows that \(\alpha_{f(y)} \in \tau_{P_1}\), and therefore, \((f, \varphi)(\alpha_{f(y)}) \in \tau_{P_1}\), which yields \(P_1(x, y) \wedge (f, \varphi^\op(\alpha_{f(y)}))(y) \leq (f, \varphi)(\alpha_{f(y)})(x)\). As a consequence, we get that \(P_1(x, y) = P_1(x, y) \wedge \varphi^\op(\top_{L_1}) = P_1(x, y) \wedge \varphi^\op(P_2(f(y), y)) \leq (f, \varphi^\op \circ P_2)(f(x), f(y))\), where \(\top\) uses the reflexivity property of localic preorders. Thus, \((X_1, L_1, P_1)^{(f, \varphi)} \to (X_2, L_2, P_2)\) is localic monotone.

To check that \(At\) is injective on objects, suppose we have localic preordered sets \((X, L, P_1), (X, L, P_2)\) such that \(\tau_{P_1} = \tau_{P_2}\). Given \(x, y \in X\), it follows that \(\alpha^x_1 = P_1(-, y) \in \tau_{P_2}\), and therefore, \(P_2(x, y) \wedge \alpha^y_1 \leq \alpha^x_2\), which yields then \(P_2(x, y) \leq P_1(x, y)\) by the reflexivity property of localic preorders. In a similar manner, one gets that \(P_1(x, y) \leq P_2(x, y)\), which implies then \(P_1(x, y) = P_2(x, y)\).

\[\square\]

Remark 18. To define a localic analogue of the functor \(\text{Top} \xrightarrow{Spec} \text{Prost}\), one needs additional notions.

Remark 19. Every locale \(L\) has a binary operation \(L \times L \xrightarrow{op} L\) given by \(a \to b = \bigvee \{c \in L \mid a \wedge c \leq b\}\) \([7]\). This operation satisfies the following properties for every \(a, b, c \in L\):

1. \(c \wedge a \leq b\) if \(c \leq a \to b\);
2. \((a \to b) \wedge a \leq b\);
3. \((a \to b) \wedge (b \to c) \leq a \to c\);
4. \(a \to a = \top_L\);
5. \(\top_L \to a = a\).

Definition 20. \(\text{Loc}^*\) is the subcategory of \(\text{Loc}\), which has the same objects, and whose morphisms \(\varphi\) have the property that \(\varphi^\op\) preserves \(\wedge\) and \(\to\). Subcategories of \(\text{Loc}^*\) will be denoted \(\text{S}^*\).

Lemma 21. Given a localic homomorphism \(L_1 \xrightarrow{\varphi} L_2\), equivalent are:

1. \(\varphi^\op\) preserves \(\wedge\) and \(\to\);
2. \(\bigwedge_{s \in S}(\varphi^\op(a_s) \to \varphi^\op(b_s)) \leq \varphi^\op(\bigwedge_{s \in S}(a_s \to b_s))\) for every \(\{a_s \mid s \in S\}, \{b_s \mid s \in S\} \subseteq L_2\).

Proof. (1) \(\implies\) (2) is straightforward. To show (2) \(\implies\) (1), notice that given \(\bigwedge_{s \in S} \varphi^\op(s) = \bigwedge_{s \in S}(\varphi^\op(\bigvee_{L_1} s) \to \varphi^\op\bigvee_{L_2} s) \leq \varphi^\op(\bigwedge_{s \in S}(\bigvee_{L_1} s) \to s) = \varphi^\op(\bigwedge_{s \in S} s)\), which yields, \(\varphi^\op(\bigwedge_{s \in S} s) = \bigwedge_{s \in S} \varphi^\op(s)\), since \(\varphi^\op\) is monotone. Given now \(a, b \in L_2\), it follows that \(\varphi^\op(a \to b) \leq \varphi^\op(a) \wedge \varphi^\op(b)\), and therefore, \(\varphi^\op(a \to b) \leq \varphi^\op(a) \to \varphi^\op(b)\), which then gives rise (by (2)) to the desired equality \(\varphi^\op(a \to b) = \varphi^\op(a) \to \varphi^\op(b)\).

\[\square\]

Remark 22. There exists the restriction \(\text{At}^*\) of the functor \(\text{At}\) to the categories \(\text{S}^*\text{-Top}\) and \(\text{S}^*\text{-Prost}\).

Theorem 23. There exists a concrete functor \(\text{S}^*\text{-Top} \xrightarrow{Spec} \text{S}^*\text{-Prost}\) given by \(\text{Spec}(X, L, \tau) = (X, L, P_\tau)\), where \(P_\tau(x, y) = \bigwedge_{\alpha \in \tau}(\alpha(y) \to \alpha(x))\), which is a right-adjoint-left-inverse to \(\text{At}^*\). Additionally, the adjoint situation in question is concrete.

Proof. To show that the functor is correct on objects, notice, firstly, that \(P_\tau(x, y) = \bigwedge_{\alpha \in \tau}(\alpha(x) \to \alpha(y))\) \(\top_L\); and, secondly, that \(P_\tau(x, y) \wedge P_\tau(y, z) \leq (\bigwedge_{\alpha \in \tau}(\alpha(y) \to \alpha(x))) \wedge (\bigwedge_{\alpha \in \tau}(\alpha(z) \to \alpha(y))) = \bigwedge_{\alpha \in \tau}(\alpha(y) \to \alpha(x)) \wedge (\alpha(z) \to \alpha(y)) \leq \bigwedge_{\alpha \in \tau}(\alpha(x) \to \alpha(z)) = P_\tau(z, x)\).

To show that \(\text{Spec}\) is correct on morphisms, notice that given a localic continuous map \((X_1, L_1, \tau_1)^{(f, \varphi)} \to (X_2, L_2, \tau_2)\) and \(x, y \in X_1\), it follows that \(P_\tau(x, y) = \bigwedge_{\alpha \in \tau_1}(\alpha(y) \to \alpha(x)) \leq \bigwedge_{\beta \in \tau_2}(\alpha(y) \to \alpha(x))(y)\).
Lemma 21 ((f, φ)\textsuperscript{−}(β)(x)) = \bigwedge_{y \in \tau}((\varphi P \circ \beta \circ f)(y) \rightarrow (\varphi P \circ \beta \circ f)(x)) \subseteq \varphi P(\bigwedge_{y \in \tau}((β(y) \rightarrow β(x)))) = (\varphi P \circ P_\tau)(f(x), f(y)).

To verify that Spec is a right adjoint to At\textsuperscript{∗} (as well as the concreteness of the adjunction in question), we show that given a locale preorder set (X, L, P), (X, L, P) \xrightarrow{\text{Spec} \circ At\textsuperscript{∗}(X, L, P)} Spec \circ At\textsuperscript{∗}(X, L, P) is a Spec-universal arrow for (X, L, P). To check that (1_X, 1_L) is locale monotone, notice that given α ∈ τ and x, y ∈ X, P(x, y) ∧ α(y) ≤ α(x) implies P(x, y) ≤ α(y) → α(x). As a consequence, we obtain that P(x, y) ≤ \bigwedge_{y \in \tau}(α(y) → α(x)) = P_{\tau P}(x, y). Given a locale monotone map (X, L, P) \xrightarrow{(f, ϕ)} Spec(X', L', τ'), to see that At\textsuperscript{∗}(X, L, P) \xrightarrow{(f, ϕ)} (X', L', τ') is locale continuous, notice that given α ∈ τ' and x, y ∈ X, it follows that P(x, y) ∧ ((f, ϕ)\textsuperscript{−}(α))(y) = P(x, y) ∧ (ϕ \circ α \circ f)(y) ≤ (ϕ P \circ ϕ P)(f(x), f(y)) ∧ (ϕ P \circ α \circ f)(y) = ϕ P(P_{\tau}(f(x), f(y)) ∧ (α \circ f)(y)) ≤ ϕ P(((α(f(y)) → α(f(x)))) ∧ (α(f(y))) ≤ ϕ P(α(f(x))) = ((f, ϕ)\textsuperscript{−}(α))(x).

To check that Spec is a left inverse to At\textsuperscript{∗}, notice that given a locale preorder set (X, L, P), Spec \circ At\textsuperscript{∗}(X, L, P) = (X, L, P_P). For every x, y ∈ X, it follows then that (recall the third paragraph of the proof of Theorem 17) P_{\tau P}(x, y) ≤ α_P(y) → α_P(x) = P(y, y) → P_{\tau P}(x, y) = P(x, y). The converse inequality has been just shown in the previous paragraph, which yields the desired P_{\tau P}(x, y) = P(x, y).

□

Remark 24. Theorems 17, 23 provide a lattice-valued analogue of the Prost – Top correspondence.

Corollary 25. The category S\textsuperscript{∗}-Prost is concretely isomorphic to a full concretely coreflective subcategory of the category S\textsuperscript{∗}-Top.

Remark 26. Corollary 25 is applicable to every categories of the form L-Prost and L-Top. In particular, one gets back the classical crisp correspondence through the categories 2-Prost and 2-Top. The shift from fixed-basis to variable-basis distorts, however, the classical links between continuity and monotonicity.

2.1. Lattice-valued partially ordered sets

Definition 27. S-Top\textsubscript{0} is the full subcategory of S-Top, the objects of which (locale T\textsubscript{0} topological spaces) are locale topological spaces (X, L, τ), with the property that for every distinct x, y ∈ X, there exists α ∈ τ such that α(x) ≠ α(y) (locale T\textsubscript{0} separation axiom).

Definition 28 (9). S-Pos is the full subcategory of S-Prost, the objects of which (locale posets) are locale preorder sets (X, L, P), with the property that for every x, y ∈ X,

1. P(x, y) ∧ P(y, x) = τ_L implies x = y (antisymmetry).
3. Lattice-valued preordered sets versus lattice-valued topological systems

Definition 31. Recall from Lecture 2 that given a subcategory $S$ of $\text{Loc}$, $S$-$\text{TopSys}$ is the category, concrete over the product category $\text{Set} \times S \times \text{Loc}$, whose objects (localic topological systems) are tuples $(X, L, A, \satisfies)$, where $X$ is a set, $L$ is an $S$-object, $A$ is a locale, and $\satisfies$ is a map $X \times A \to L$ (localic satisfaction relation on $(X, L, A)$) such that $L \xrightarrow{\satisfies(x, -)^{op}} A$ is a localic homomorphism for every $x \in X$. Morphisms (localic continuous maps) $(X_1, L_1, A_1, \satisfies_1) \xrightarrow{(f, \phi)} (X_2, L_2, A_2, \satisfies_2)$ are $\text{Set} \times S \times \text{Loc}$-morphisms $(X_1, L_1, A_1) \xrightarrow{(f, \phi)} (X_2, L_2, A_2)$ such that $\forall x \in X_1, \exists a \in A_2$.

Remark 32. Similar to the case of the category $S'$-$\text{Top}$, we have to restrict ourselves to the category $S'$-$\text{TopSys}$, in order to deal with the setting of localic preorders.

Theorem 33. (1) There exists a full embedding $\text{S'}$-$\text{Top} \xleftarrow{E} \text{S'}$-$\text{TopSys}$, which is defined by $E((X_1, L_1, \tau_1) \xrightarrow{(f, \phi)} (X_2, L_2, \tau_2)) = (X_1, L_1, \tau_1, \satisfies_1) \xrightarrow{(f, \phi, (\tau_1)^{op})} (X_2, L_2, \tau_2, \satisfies_2)$, where $\satisfies_1(x, \alpha) = \alpha(x)$.

(2) There exists a functor $\text{S'}$-$\text{TopSys} \xrightarrow{\text{Spat}} \text{S'}$-$\text{Top}$, which is defined by $\text{Spat}(X_1, L_1, A_1, \satisfies_1) \xrightarrow{(f, \phi)} (X_2, L_2, \tau_2, \satisfies_2)$, where $\tau_2 = \{ \satisfies_1(x, -) \mid a \in A_1 \}$.

(3) The functor $\text{Spat}$ is a right-adjoint-left-inverse to the embedding $E$.

Proof. The definitions of both functors clearly fit our current localic setting. Moreover, given a localic topological system $(X, L, A, \satisfies)$, its respective $E$-co-universal arrow is provided by $\text{Spat}(X, L, A, \satisfies) \xrightarrow{(1_x, 1_L, \psi^{op})} (X, L, A, \satisfies)$. Where $A \xrightarrow{\psi} \{ \satisfies(-, a) \mid a \in A \}$, $\psi$ is a $\mathcal{A}$-map $\satisfies$. 

Corollary 34. The category $\text{S'}$-$\text{Top}$ is isomorphic to a full reflective subcategory of $\text{S'}$-$\text{TopSys}$.

Theorem 35. The category $\text{S'}$-$\text{PrTop}$ is isomorphic to a full reflective subcategory of $\text{S'}$-$\text{TopSys}$.

Proof. Combine the adjoint situations of Theorems 23 and 33.

Remark 36. Theorem 35 is applicable to the categories $\text{L-PrTop}$ and $\text{L-TopSys}$ for every locale $L$, e.g., the categories $\text{2-PrTop}$ and $\text{2-TopSys}$ give back the classical crisp $\text{PrTop} \to \text{TopSys}$ correspondence.

4. Lattice-valued preordered topological spaces and topological systems

Remark 37. The claim of Theorem 35 has an important deficiency: a restriction to a particular subcategory of $\text{Loc}$. While fixed-basis approach is not influenced by this restriction, variable-basis approach is indeed. This section restores the classical setting in case of variable-basis, namely, puts a localic preorder on both localic topological spaces and systems, to incorporate monotonicity in the very definition of continuity.

4.1. Lattice-valued preordered topological spaces

Definition 38 ([3]). Given a subcategory $S$ of $\text{Loc}$, $S$-$\text{PrTop}$ is the category, concrete over $\text{Set} \times S$, whose objects (localic preordered topological spaces) are tuples $(X, L, \tau, P)$, where $(X, L, \tau)$ is a localic topological space, $(X, L, P)$ is a localic preordered set, and, moreover, $P(x, y) \land \alpha(y) \leq \alpha(x)$ for every $x, y \in X$ and every $\alpha \in \tau$. Morphisms (localic monotone continuous maps) $(X_1, L_1, \tau_1, P_1) \xrightarrow{(f, \phi)} (X_2, L_2, \tau_2, P_2)$ are $\text{Set} \times S$-morphisms $(X_1, L_1) \xrightarrow{(f, \phi)} (X_2, L_2)$, which are both localic monotone and continuous.

Example 39. The category $\text{2-PrTop}$ is isomorphic to the category $\text{PrTop}$ of preordered topological spaces (which are triples $(X, \tau, \leq)$, containing both a topology $\tau$ and a preorder $\leq$, and satisfying additionally the condition $U \subseteq U$ for every $U \in \tau$) and continuous monotone maps.
Theorem 40.

(1) There exists a full concrete embedding \( \textbf{S-PrTop} \hookrightarrow \textbf{S-PrTop} \), which is given by \( \text{PrAt}(X, L, P) \rightarrow (X, L, \tau_P, P) \), where \( \tau_P = \{ \alpha \in L^X \mid P(x, y) \land \alpha(y) \leq \alpha(x) \text{ for every } x, y \in X \} \).

(2) There exists a concrete functor \( \textbf{S-PrTop} \rightarrow \textbf{S-PrTop} \), which is defined by \( U(X, L, \tau, P) = (X, L, P) \).

(3) \( U \) is a right-adjoint-left-inverse to \( \text{PrAt} \). Moreover, the adjunction in question is concrete.

Proof. Item (1) is a direct consequence of Theorem 17, and item (2) is easy. To check item (3), given a localic monotone map \( (X, L, P) \rightarrow (X', L', \tau', P') \), every \( x, y \in X \) and every \( \alpha \in \tau' \), it follows that \( P(x, y) \land (\varphi')^{-1}(\alpha)(y) = P(x, y) \land (\varphi' \circ \alpha \circ f)(y) \leq (\varphi' \circ P')(f(x), f(y)) \land (\varphi' \circ \alpha \circ f)(y) = \varphi'(P'(f(x), f(y)) \land \alpha(f(y))) = \varphi'(\alpha(f(x))) = ((\varphi')^{-1}(\alpha))(x) \), which yields then the desired localic continuity of \( \text{PrAt}(X, L, P) \rightarrow (X', L', \tau', P') \).

Corollary 41. The category \( \textbf{S-PrTop} \) is concretely isomorphic to a full concretely coreflective subcategory of the category \( \textbf{S-PrTop} \).

Remark 42. Corollary 41 provides an analogue of Corollary 25, which is still variable-basis, and does not impose any restriction on the category \( \textbf{Loc} \). One has though to pay a price for the possibility of dealing with the whole category \( \textbf{Loc} \), i.e., the functor \( \text{Spec} \) of Theorem 23 is now the forgetful functor \( U \) of Theorem 40.

4.2. Lattice-valued preordered topological systems

Definition 43 ([3]). Given a subcategory \( \textbf{S} \) of \( \textbf{Loc}, \textbf{S-PrTopSys} \) is the category, concrete over the product category \( \textbf{Set} \times \textbf{S} \times \textbf{Loc} \), whose objects (localic preordered topological systems) are quintuples \( (X, L, A, \equiv, P) \), where \( (X, L, A, \equiv) \) is a localic topological system, \( (X, L, P) \) is a localic preordered set, and, additionally, \( P(x, y) \equiv (y, a) \leq \equiv (x, a) \) for every \( x, y \in X \) and every \( a \in A \). Morphisms (localic monotone continuous maps) \( (X_1, L_1, A_1, \equiv_1, P_1) \rightarrow (X_2, L_2, A_2, \equiv_2, P_2) \) are \( \text{Set} \times \textbf{S} \times \textbf{Loc} \)-morphisms \( (X_1, L_1, A_1) \rightarrow (X_2, L_2, A_2) \), which are both monotonic and continuous.

Example 44. The category \( 2 \text{-} \textbf{PrTopSys} \) is isomorphic to the category \( \textbf{PrTopSys} \) of preordered topological systems (which are tuples \( (X, A, \equiv, \leq) \), containing both a satisfaction relation \( \equiv \) and a preorder \( \leq \) such that \( \text{ext}(a) = \downarrow \text{ext}(a) \) for every \( a \in A \), where \( \text{ext}(a) = \{ x \in X \mid x \equiv a \} \) and monotone continuous maps.

Theorem 45.

(1) There exists a full embedding \( \textbf{S-PrTop} \subset \textbf{S-PrTop} \) defined by \( \text{PrE}((X_1, L_1, \tau_1, P_1)) \rightarrow (X_2, L_2, \tau_2, P_2) = (X_1, L_1, \tau_1, P_1) \rightarrow (X_2, L_2, \tau_2, P_2, f, \varphi) \), where \( \varphi = \{ \varphi(a, \equiv) \mid a \in A \} \).

(2) There exists a functor \( \textbf{S-PrTop} \rightarrow \textbf{S-PrTop} \) defined by \( \text{SpAt}((X_1, L_1, A_1, \equiv, P_1)) \rightarrow (X_2, L_2, \tau_2, P_2, \varphi) \), where \( \varphi = \{ \varphi(a, \equiv) \mid a \in A \} \).

(3) The functor \( \text{PrSpAt} \) is a right-adjoint-left-inverse to the embedding \( \text{PrE} \).

Proof. Definitions 38, 43 ensure that the functors \( \text{PrE} \) and \( \text{PrSpAt} \) are correctly defined. The rest of the proof is similar to that of Theorem 33.

Corollary 46. The category \( \textbf{S-PrTop} \) is isomorphic to a full coreflective subcategory of \( \textbf{S-PrTopSys} \).

Theorem 47. The category \( \textbf{S-PrTop} \) is isomorphic to a full coreflective subcategory of \( \textbf{S-PrTopSys} \).

Proof. Combine the adjunctions of Theorems 40 and 45.

Remark 48. Theorem 47 provides a lattice-valued variable-basis analogue of the classical \( \textbf{Prost} \rightarrow \textbf{TopSys} \) correspondence. One has to pay a price for the use of the whole category \( \textbf{Loc} \), namely, to put a localic preorder on localic topological systems.
5. On the nature of the categories S-Prost and S-PrTop

Proposition 49. The category Loc-Prost is not topological over its ground category Set \times Loc.

Proof. It will be enough to construct a \( | - | \)-structured source, which has no \( | - | \)-initial lift in the category Loc-Prost. Let \( X \) be the two-element set \( \{ x, y \} \), let \( L_1 \) be the unit interval \([0, 1]\) (with its standard lattice-theoretic structure), let \( L_2 \) be the three-element chain \( \{ \bot, a, \top \} \), and let \( I \) be the set \([0, 1]\). Consider a \( | - | \)-structured source \( ((X, L_1), \varphi_1) \Rightarrow ((X, L_2, P)) \) for every \( i \in I \), where \( P(x, y) = P(y, x) = a \) and \( \varphi_i^{op}(a) = i \). It is easy to see that, firstly, \((X, L_2, P)\) is a localic preordered set, and, secondly, \( \varphi_i^{op}(a) = 1 \) is the only localic preorder on \((X, L_1)\), which makes \((X, L_1) \xrightarrow{(1_X, \varphi_1)} (X, L_2)\) localic monotone for every \( i \in I \). We show that the source \( ((X, L_1, \hat{P}) \xrightarrow{(1_X, \varphi_1)} (X, L_2, P)) \) is not \( | - | \)-initial.

Consider a source \( ((X, L_1, P'), (1_X, \varphi')) \Rightarrow ((X, L_2, P)) \) in Loc-Prost, where \( P'(x, y) = P'(y, x) = \frac{1}{2} \), and \( \varphi_i^{op}(a) = 1 \) for every \( i \in I \). Additionally, consider a \( \text{Set} \times \text{Loc} \)-morphism \((X, L_1) \xrightarrow{(1_X, \varphi)} (X, L_1)\) with

\[
\varphi^{op}(b) = \begin{cases} 
0, & b = 0 \\
1, & \text{otherwise.}
\end{cases}
\]

It is easy to see that the diagram

\[
\begin{array}{ccc}
(X, L_1, P') & \xrightarrow{(1_X, \varphi')} & (X, L_1, \hat{P}) \\
\downarrow & & \downarrow \\
(|X, L_1, P|) & \xrightarrow{(1_X, \varphi)} & (|X, L_1, \hat{P}|)
\end{array}
\]

commutes for every \( i \in I \). Since \( P'(x, y) = \frac{1}{2} > 0 = (\varphi^{op} \circ \hat{P})(x, y) \), it follows immediately that the map \(|(X, L_1, P')| \xrightarrow{1_{X, \varphi}} |(X, L_1, \hat{P})|\) is not localic monotone.

Remark 50. Proposition 49 is in a striking difference with the properties of the category Prost, which provides a topological construct.

Remark 51. The map \( L_1 \xrightarrow{\varphi^{op}} L_1 \) in Proposition 49 preserves finite meets, but not arbitrary ones. This issue plays a significant role, motivating a particular subcategory of the category Loc.

Definition 52. Loc\(^{\land}\) is the subcategory of the category Loc with the same objects, and whose morphisms \( \varphi \) have the property that \( \varphi^{op} \) is \( \land \)-preserving. Subcategories of the category Loc\(^{\land}\) will be denoted S\(^{\land}\).

Definition 53. A concrete category \((A, | - |)\) over \( X \) is called amnestic provided that given \( A \)-morphisms \( A \xrightarrow{f} B \) and \( B \xrightarrow{g} A \) such that \(|A| = |B| = X \) and \(|f| = |g| = 1_X \), it follows that \( A = B \).

Lemma 54. The concrete category S\(^{\land}\)-Prost is amnestic over its ground category Set \times S\(^{\land}\).

Proof. Suppose that both \((X, L, P_1) \xrightarrow{(1_X, 1_L)} (X, L, P_2)\) and \((X, L, P_2) \xrightarrow{(1_X, 1_L)} (X, L, P_1)\) are localic monotone. Given \( x, y \in X \), it follows then that \( P_1(x, y) \leq P_2(x, y) \) and \( P_2(x, y) \leq P_1(x, y) \). As a consequence, localic preorders \( P_1 \) and \( P_2 \) coincide, which is the desired result.

Theorem 55. The concrete category S\(^{\land}\)-Prost is topological over its ground category Set \times S\(^{\land}\).
Proof. Following the above lemma and Proposition 21.5 of [1], it will be enough to show that every $\emptyset$-initial lift. Given a $\emptyset$-structured source $((X, L) \xrightarrow{(f, \varphi)} ([X, L_i, P_i])_{i \in I})$, define a map $X \times X \xrightarrow{P} L$ by $P(x, y) = \bigwedge_{i \in I}(\varphi_i \circ P_i)(f_i(x), f_i(y))$.

To show that $P$ is a localic preorder on $(X, L)$, notice, firstly, that $P(x, x) = \bigwedge_{i \in I}(\varphi_i \circ P_i)(f_i(x), f_i(x)) = \bigwedge_{i \in I} \varphi_i \circ P_i(T_{L_i}L_i) = \bigwedge_{i \in I} \top_L = \top_L$; and, secondly,

$$P(x, y) \land P(y, z) = \bigwedge_{i \in I}(\varphi_i \circ P_i)(f_i(x), f_i(y)) \land (\bigwedge_{i \in I}(\varphi_i \circ P_i)(f_i(y), f_i(z))) = \bigwedge_{i \in I}((\varphi_i \circ P_i)(f_i(x), f_i(y)) \land (\varphi_i \circ P_i)(f_i(y), f_i(z))) = \bigwedge_{i \in I} \varphi_i \circ P_i(f_i(x), f_i(y)) \land P_i(f_i(y), f_i(z)) \leq \bigwedge_{i \in I} \varphi_i \circ P_i(f_i(x), f_i(z)) = P(f(x), f(z)).$$

The definition of the map $P$ implies that $((X, L, P) \xrightarrow{(f, \varphi)} ([X, L_i, P_i])_{i \in I})$ is a source in $S^\top$-Prost. Suppose we are given another source $((X', L', P') \xrightarrow{(f', \varphi')} ([X', L_i, P_i])_{i \in I})$ in $S^\top$-Prost and some $\text{Set} \times S^\top$-morphism $(X', L') \xrightarrow{(f', \varphi')}$ $(X, L)$ such that the triangle

$$\begin{array}{ccc}
\square ABC & \xrightarrow{(f, \varphi)} & \square A'B'C' \\
\xrightarrow{[X, L, P]} & & \xrightarrow{[X', L, P']}
\end{array}$$

commutes for every $i \in I$. Given $x, y \in X'$, it follows that $P'(x, y) \leq (\varphi_i \circ P_i)(f'_i(x), f'_i(y)) = (\varphi_i \circ \varphi_i \circ P_i)((f_i \circ f_i)(x), (f_i \circ f_i)(y))$ for every $i \in I$, and therefore, $P'(x, y) \leq \bigwedge_{i \in I}(\varphi_i \circ \varphi_i \circ P_i)((f_i \circ f_i)(x), (f_i \circ f_i)(y)) \overset{(1)}{=} \varphi_i \circ P_i(f_i \circ f_i)(x), (f_i \circ f_i)(y)) = (\varphi_i \circ P_i)(f(x), f(y))$, where (1) relies on the definition of the category $\text{Loc}^\top$. One gets a localic monotone map $(X', L', P') \xrightarrow{(f', \varphi')}$ $(X, L, P)$, which is the desired result. □

Remark 56. The case of the category 2-Prost in Theorem 55 gives back the classical result that the category Prost is a topological construct. □

Proposition 57. The category $\text{Loc-PrTop}$ is not topological over its ground category $\text{Set} \times \text{Loc}$. □

Proof. Suppose that $\text{Loc-PrTop}$ is topological over $\text{Set} \times \text{Loc}$. By Corollary 41, we know that Loc-Prost is concretely isomorphic to a full concretely beneath category of Loc-PrTop. By the dual of Propositions 21.30, 21.31 in [1], it follows then that $\text{Loc-Prost}$ is topological over $\text{Set} \times \text{Loc}$, which clearly contradicts Proposition 49 above. □

Remark 58. Proposition 57 shows a difference in the nature of the categories Loc-Top and Loc-PrTop, the former being topological, and the latter not. It appears though that changing the ground category of Loc-PrTop makes a difference. More precisely, in view of item (2) of Theorem 40, in the following, we consider S-PrTop as a concrete category over the category S-Prost. □

Lemma 59. The concrete category $\text{S-PrTop}$ is amnestic over the category $\text{S-Prost}$. □

Proof. Suppose that both $(X, L, \tau_1, P) \xrightarrow{([1 \times 1]_L)} (X, L, \tau_2, P)$ and $(X, L, \tau_2, P) \xrightarrow{([1 \times 1]_L)} (X, L, \tau_1, P)$ are localic monotone. It follows then that $\tau_2 \subseteq \tau_1$ and $\tau_2 \subseteq \tau_1$, respectively. As a consequence, localic topologies $\tau_1$ and $\tau_2$ coincide, which is the desired result. □

Theorem 60. The category $\text{S-PrTop}$ is topological over the category $\text{S-Prost}$. □
PROOF. Following the above lemma and Proposition 21.5 of [1], it is enough to show that every \(| - |\)-structured source has a \(| - |\)-initial lift. Given a \(| - |\)-structured source \(\{(X, L, P, f, \varphi_i) \mid (X_i, L_i, \tau_i, P_i)\}_{i \in I}\), let \(T = \bigcup_{i \in I} \{(f_i, \varphi_i)^{\circ \alpha}(\alpha) \mid \alpha \in \tau_i\}\), let \(\tau = \langle T \rangle\) (i.e., the subframe of \(L^X\), which is generated by \(T\), which is then a localic topology on \((X, L)\), and let \(\tau = \{\alpha \in \tau \mid P(x, y) \wedge \alpha(y) \leq \alpha(x) \text{ for every } x, y \in X\}\). Similiar to the first paragraph of the proof of Theorem 17, it follows that \(\tau\) is a localic topology on \((X, L)\), and therefore, by its very definition, one gets a localic preordered topological space \((X, L, \tau, P)\).

Given \(i \in I\), to show that the map \(\{(X, L, \tau, P) \mid \langle f_i, \varphi_i \rangle \mid (X_i, L_i, \tau_i, P_i)\}\) is localic continuous, we notice that given \(\alpha \in \tau_i\) and \(x, y \in X\), \(P(x, y) \wedge ((f_i, \varphi_i)^{\circ \alpha}(\alpha))(y) = P(x, y) \wedge (\varphi_i^{\circ \alpha} \circ f_i \circ \alpha)(y) \leq (\varphi_i^{\circ \alpha} \circ f_i)(x, y) \wedge (\varphi_i^{\circ \alpha} \circ f_i \circ \alpha)(y) = \varphi_i^{\circ \alpha}(P(f_i(x), f_i(y)) \wedge \alpha(f_i(y))) \leq \varphi_i^{\circ \alpha}(\alpha(f_i(x))) = ((f_i, \varphi_i)^{\circ \alpha})(\alpha)(x)\).

Suppose we are given another source \(\{(X', L', \tau', P') \mid \langle f'_i, \varphi'_i \rangle \mid (X_i, L_i, \tau_i, P_i)\}_{i \in I}\) in \(\textbf{S-PrTop}\) and some \(\textbf{Set} \times \textbf{S}\)-morphism \((X', L') \xrightarrow{\langle f', \varphi' \rangle} (X, L)\) such that the triangle

\[
\begin{array}{ccc}
\left|\left(X', L', \tau', P'\right)\right| & \xrightarrow{\langle f', \varphi' \rangle} & \left|\left(X, L, \tau, P\right)\right| \\
\left|\left(X, L, \tau, P\right)\right| & \xrightarrow{\langle f, \varphi \rangle} & \left|\left(X_i, L_i, \tau_i, P_i\right)\right|
\end{array}
\]

commutes for every \(i \in I\). Given \(i \in I\) and \(\alpha \in \tau_i\), it follows that \(\tau' \ni (f'_i, \varphi'_i)^{\circ \alpha}(\alpha) = \varphi_i^{\circ \alpha} \circ f_i = \varphi_i^{\circ \alpha} \circ \varphi_i^{\circ \alpha} \circ f_i \circ f_i = ((f, \varphi)^{\circ \alpha})(\alpha)\). As a consequence, we obtain that \(\langle (f, \varphi)^{\circ \alpha} \rangle \mapsto (\tau) \subseteq \tau'\). Since \((f, \varphi)^{\circ \alpha}\) is a frame homomorphism, it follows that \(\langle (f, \varphi)^{\circ \alpha} \rangle \mapsto (\tau) \subseteq \tau'\) (recall Lemma 56 from Lecture 2), and therefore, \(\langle (f, \varphi)^{\circ \alpha} \rangle \mapsto (\tau) \subsetneq \tau'\). As a consequence, one obtains a localic continuous map \(\langle X', L', \tau', P' \rangle \xrightarrow{\langle f, \varphi \rangle} (X, L, \tau, P)\), which provides the result in question. \[\square\]

Corollary 61. The category \(\textbf{S}^\perp\cdot\textbf{PrTop}\) is topological over \(\textbf{Set} \times \textbf{S}^\perp\).

PROOF. Follows from Theorems 55 and 60. \[\square\]

Remark 62. Consider the following commutative triangle

\[
\begin{array}{ccc}
\text{Loc-PrTop} & \xrightarrow{|-|_2} & \text{Loc-Prost} \\
\downarrow |\_{-1} & & \downarrow |\_{-3} \\
\text{Set} \times \text{Loc} & \xrightarrow{|-|_4} & \text{Loc-Prost}
\end{array}
\]

While the functor \(|-|_4\) is topological, \(|-|_2\) and \(|-|_3\) are not, which is different from the crisp case or, more generally, fixed-basis approach over a locale \(L\), where all three functors are topological. Speaking meta-mathematically (and recalling previous results of this lecture), variable-basis approach to preorder lacks some of its standard and well used to properties. \[\blacksquare\]

6. Quantale-valued preordered sets

Remark 63. Some of the results of this lecture could be obtained, replacing \(\textbf{Loc}\) with the category \(\textbf{UQuant}\) of unital quantales. In case of Theorem 17, given a unital quantale \(Q\) and a quantale preordered set \((X, Q, P)\) (replace \(\mathbb{T}_L\) with the quantale unit \(1_Q\), and \(\land\) with the quantale multiplication \(\otimes\) in Definition 12), the set \(\tau_P = \{\alpha \in Q^X \mid P(x, y) \otimes \alpha(y) \leq \alpha(x) \text{ for every } x, y \in X\}\) need not be a quantale topology on \((X, Q)\), i.e., a unital subquantale of \(Q^X\). While the set \(\tau_P\) is closed under \(\lor\), it may not contain the unit, and may be not closed under the multiplication. A solution is to take the topology on \((X, Q)\), generated by the set \(\tau_P\) (denoted \(\langle \tau_P \rangle\)). Using the subbasic continuity technique of Lecture 1, one gets an analogue of Theorem 17. \[\blacksquare\]
Theorem 64. There exists a full concrete embedding $\mathbf{S}^\ast\mathbf{-Prost} \subseteq^{\mathcal{A}_U} \mathbf{S}^\ast\mathbf{-Top}$ defined by $\mathcal{A}(X, Q, P) = (X, Q, \tau_P)$, where $\tau_P = \{\alpha \in Q^X \mid P(x, y) \otimes \alpha(y) \leq \alpha(x) \text{ for every } x, y \in X\}$.

Remark 65. In case of Theorem 29, given a quantic topological space $(X, Q, \tau)$, one defines a map $X \times X \xrightarrow{P_\tau} Q$ by $P_\tau(x, y) = \land_{\alpha \in \tau}(\alpha(y) \rightarrow_\tau \alpha(x))$, where $a \rightarrow_\tau b = \bigvee\{c \in Q \mid c \otimes a \leq b\}$. This map satisfies the transitivity property of quantic preorders (notice that $(b \rightarrow_\tau a) \otimes (c \rightarrow_\tau b) \leq c \rightarrow_\tau a$ for every $a, b, c \in Q$), but fails to be reflexive. Modifying though the reflexivity axiom of quantic preorders to run as follows:

$$1_Q \leq P(x, x) \text{ for every } x \in X \text{ (reflexivity),}$$

one gets a quantic preorder $P_\tau$ on $(X, Q)$, which fits the setting of Theorem 64. Easy calculations (the category $\mathbf{UQuant}^{op*}$ employs the operation $\rightarrow_\tau$ instead of $\rightarrow$) give the following analogue of Theorem 23. ■

Theorem 66. There exists a concrete functor $\mathbf{S}^\ast\mathbf{-Top} \xrightarrow{\mathcal{Spec}} \mathbf{S}^\ast\mathbf{-Prost}$, which is defined by $\mathcal{Spec}(X, Q, \tau) = (X, Q, P_\tau)$, where $P_\tau(x, y) = \land_{\alpha \in \tau}(\alpha(y) \rightarrow_\tau \alpha(x))$, and which is a right-adjoint-left-inverse to $\mathcal{A}^*$. Additionally, the adjoint situation in question is concrete.

Corollary 67. The category $\mathbf{S}^\ast\mathbf{-Prost}$ is concretely isomorphic to a full concretely coreflective subcategory of the category $\mathbf{S}^\ast\mathbf{-Top}$.

Proof. Follows from Theorems 64 and 66. □

Remark 68. The quantic analogue of Theorem 29 requires an additional stipulation on the quantale multiplication, to obtain a convenient formulation of the antisymmetry axiom. ■

Definition 69. $\mathbf{UQuant}^{op_1}$ is the full subcategory of $\mathbf{UQuant}^{op}$, whose objects $Q$ have the property that

$$1_Q \leq a \otimes b \text{ implies } 1_Q \leq a \land b \text{ for every } a, b \in Q. \tag{Q}$$

Remark 70. In the framework of the category $\mathbf{UQuant}^{op_1}$, define the axiom in question as

$$1_Q \leq P(x, y) \otimes P(y, x) \text{ implies } x = y \text{ (antisymmetry).}$$

With the above axiom, one can provide the quantic analogue of Theorem 29 (use now subcategories $\mathbf{C}^{*+}$ of the intersection of the subcategories $\mathbf{UQuant}^{op*}$ and $\mathbf{UQuant}^{op_1}$ of $\mathbf{UQuant}^{op}$; also notice that the lifting of item (1) of Theorem 29 does not require assumption (Q), but the lifting of item (2) does indeed). ■

References