Categorical lattice-valued topology
Lecture 2: lattice-valued topological systems

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Abstract
This lecture introduces lattice-valued topological systems as models of topological theories, and shows lattice-valued analogues of the system spatialization (topological spaces from topological systems) and localication (locales from topological systems) procedures. As an application of the theory of topological systems, we obtain the equivalence between the categories of state property systems of D. Aerts and closure spaces.

1. Basics on topological systems

1.1. Topological systems and continuous maps

**Remark 1.** Topological systems were introduced by S. Vickers in 1989 in [8] as a common framework for topological spaces and the underlying algebraic structures of their topologies – locales, in order to provide a convenient way to switch between spatial (pointed) and localic (pointfree) topological settings.

**Definition 2.** A frame is a complete lattice $A$ such that $a \land (\bigvee S) = \bigvee_{s \in S} (a \land s)$ for every $a \in A$ and every $S \subseteq A$. A frame homomorphism $A_1 \xrightarrow{\varphi} A_2$ is a map, which preserves finite (including the empty) meets and arbitrary joins. Frm is the category (variety) of frames and frame homomorphisms. Loc is the dual category of Frm, whose objects (resp. morphisms) are called locales (resp. localic homomorphisms).

**Definition 3.** A topological system is a triple $(X, A, \models)$, where $X$ is a set, $A$ is a locale and $\models \subseteq X \times A$ is a binary relation (satisfaction relation on $(X, A)$) such that for every $S \subseteq A$ and every $x \in X$

1. $x \models S$ if $x \models s$ for some $s \in S$;
2. if $S$ is finite, then $x \models \bigwedge S$ if $x \models s$ for every $s \in S$.

A topological system morphism $(X_1, A_1, \models_1) \xrightarrow{(f, \varphi)} (X_2, A_2, \models_2)$ (called continuous map) contains a map $f : X_1 \xrightarrow{f} X_2$ and a localic homomorphism $A_1 \xrightarrow{\varphi} A_2$ such that for every $x \in X_1$ and every $a \in A_2$, $x \models_1 \varphi^\varphi(a)$ iff $f(x) \models_2 a$. TopSys is the category of topological systems and continuous maps, which is concrete over the product category Set $\times$ Loc.

**Remark 4.** The main examples of topological systems are given by topological spaces and locales.

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1.2. Topological spaces as topological systems

**Theorem 5.** There exists a full embedding $\operatorname{Top} \xrightarrow{E} \operatorname{TopSys}$ defined by $E((X, \tau_1) \xrightarrow{f} (X, \tau_2)) = (X_1, \tau_1, \vdash_1) \xrightarrow{f_{\tau}^{-\circ}} (X_2, \tau_2, \vdash_2)$, where $x \vdash_1 U$ if $x \in U$.

**Proof.** To show that the functor is correct on morphisms, notice that given $x \in X_1$ and $V \in \tau_2$, it follows that $x \in f_{\tau}^{-\circ}(V)$ if $f(x) \in V$. To show that the functor is full, notice that given a continuous map $(X_1, \tau_1, \vdash_1) \xrightarrow{f_{\tau}} (X_2, \tau_2, \vdash_2)$, for every $V \in \tau_2$, it follows that $x \in f_{\tau}^{-\circ}(V)$ if $f(x) \in V$.

**Theorem 6.** The embedding $\operatorname{Spat} \xrightarrow{E} \operatorname{TopSys}$ has a right adjoint $\operatorname{TopSys} \xleftarrow{\operatorname{Spat}} \operatorname{Top}$, which is defined by $\operatorname{Spat}((X_1, A_1, \vdash_1) \xrightarrow{f_{\vdash}} (X_2, A_2, \vdash_2)) = (X_1, \{\text{ext}(a) | a \in A_1\}) \xrightarrow{f_{\vdash}} (X_2, \{\text{ext}(b) | b \in A_2\})$, where $\text{ext}(c) = \{x \in X_1 | x \Vdash c\}$.

**Proof.** To show that the functor is correct on objects, notice, e.g., that given $a, b \in A_1$, it follows that $\text{ext}(a) \cap \text{ext}(b) = \{x \in X_1 | x \Vdash a \text{ and } x \Vdash b\} = \text{ext}(a \land b)$. To show that the functor is correct on morphisms, notice that given $a \in A_2$, it follows that $x \in f_{\vdash}(\text{ext}(a))$ iff $f(x) \in \text{ext}(a)$ iff $f(x) \Vdash_2 a$ iff $x \Vdash \varphi_{\text{op}}(a)$ iff $x \in \text{ext}(\varphi_{\text{op}}(a))$, and therefore, $f_{\vdash}(\text{ext}(a)) = \text{ext}(\varphi_{\text{op}}(a))$.

**Corollary 7.** $\operatorname{Top}$ is isomorphic to a full reflective subcategory of the category $\operatorname{TopSys}$.

**Remark 8.** The functor of Theorem 6 is called the system spatialization procedure.

1.3. Locales as topological systems

**Definition 9.** Given a frame $A$, $\operatorname{Pt}(A)$ stands for the set $\operatorname{ Frm}(A, 2)$ of all frame homomorphisms $A \rightarrow 2$, whose elements are called the points of $A$ and are denoted $p$.

**Theorem 10.** There exists a full embedding $\operatorname{Loc} \xrightarrow{E} \operatorname{TopSys}$, which is defined by $E(A_1 \xrightarrow{\varphi} A_2) = (\operatorname{Pt}(A_1), A_1, \vdash_1) \xrightarrow{(\varphi_{\text{op}})^{-\circ}} (\operatorname{Pt}(A_2), A_2, \vdash_2)$, where $p \vdash_1 c$ iff $p(c) = \top$, and $(\varphi_{\text{op}})^{\top}_2(p) = p \circ \varphi_{\text{op}}$.

**Proof.** To show that the functor is correct on objects, notice that given $p \in \operatorname{Pt}(A_1)$ and $S \subseteq A_1$, it follows that $p \vdash_1 S$ iff $p(\bigvee S) = \top$ if $\bigvee_{a \in S} p(s) = \top$ if $p(s) = \top$ for some $s \in S$ if $p \vdash_1 s$ for some $s \in S$. To show that the functor is correct on morphisms, notice that given $p \in \operatorname{Pt}(A_1)$ and $a \in A_2$, it follows that $p \vdash_1 \varphi_{\text{op}}(a)$ iff $p \circ \varphi_{\text{op}}(a) = \top$ iff $((\varphi_{\text{op}})^{\top}_2(p))(a) = \top$ iff $((\varphi_{\text{op}})^{\top}_2(p))(a) \Vdash_2 a$. To show that the functor is full, notice that given a continuous map $(\operatorname{Pt}(A_1), A_1, \vdash_1) \xrightarrow{f_{\vdash}} (\operatorname{Pt}(A_2), A_2, \vdash_2)$, for every $p \in \operatorname{Pt}(A_1)$, it follows that $(f(p))(a) = \top$ iff $(f(p)) \Vdash_2 a$ iff $p \vdash_1 \varphi_{\text{op}}(a)$ iff $p(\varphi_{\text{op}}(a)) = \top$.
Definition 15 (Fixed-basis approach). Given a locale \(L\), an \(L\)-topological system is a triple \((X, A, \VDash)\), where \(X\) is a set, \(A\) is a locale, and \(X \times A \xrightarrow{\sigma} L\) is a map (\(L\)-satisfaction relation on \((X, A)\)) such that \(A \xrightarrow{\mathsf{ht}(x, -)} L\) is a frame homomorphism for every \(x \in X\). An \(L\)-topological system morphism \((X_1, A_1, \vdash_1) \xrightarrow{(f, \varphi)} (X_2, A_2, \vdash_2)\) (called \(L\)-continuous map) contains a map \(X_1 \xrightarrow{f} X_2\) and a localic homomorphism \(A_1 \xrightarrow{\varphi} A_2\) such that \(\vdash_1(x, \varphi\circ\mathsf{ht}(f(x), a)) = \vdash_2(f(x), a)\) for every \(x \in X_1\) and every \(a \in A_2\). \(L\text{-TopSys}\) is the category of \(L\)-topological systems and \(L\)-continuous maps.

Remark 16. The category \(2\text{-TopSys}\) is isomorphic to the category \(\text{TopSys}\) of S. Vickers.

Definition 17 (Variable-basis approach). Given a subcategory \(C\) of \(\text{Loc}\), a \(C\)-\text{topological system} is a tuple \((X, L, A, \vdash)\), which is an \(L\)-topological system for some \(C\)-object \(L\). A \(C\)-\text{topological system morphism} \((X_1, L_1, A_1, \vdash_1) \xrightarrow{(f, \varphi)} (X_2, L_2, A_2, \vdash_2)\) (called \(C\)-continuous map) contains a map \(X_1 \xrightarrow{f} X_2\) and localic homomorphisms \(L_1 \xrightarrow{\varphi} L_2\) such that \(\vdash_1(x, f^\varphi(b)) = \vdash_2(f(x), a)\) for every \(x \in X_1\) and every \(a \in A_2\). \(C\text{-TopSys}\) is the category of \(C\)-topological systems and \(C\)-continuous maps.

Remark 18. In [8], S. Vickers provided an alternative definition of topological systems, which was extended to lattice-valued case by J. T. Denniston, A. Melton and S. E. Rodabaugh in [4].

Proposition 19. Given sets \(X, Y, Z\), where exists a bijection \(\text{Set}(X \times Y, Z) \xrightarrow{h} \text{Set}(Y, Z^X)\) given by \(((h(f))(y)) = f(x, y)\). Its inverse map \(\text{Set}(Y, Z^X) \xrightarrow{g} \text{Set}(X \times Y, Z)\) is given by \(g(f)(x, y) = (f(y))(x)\).

Definition 20. Given a subcategory \(C\) of \(\text{Loc}\), a \(C\)-\text{topological system} is a tuple \((X, L, A, \kappa)\), where \((X, L, A)\) is a \(\text{Set} \times C \times \text{Loc}\)-object and \(A \xrightarrow{\kappa} L^X\) is a frame homomorphism. A \(C\)-\text{topological system morphism} \((X_1, L_1, A_1, \kappa_1) \xrightarrow{(f, \varphi)\kappa} (X_2, L_2, A_2, \kappa_2)\) is a \(\text{Set} \times C \times \text{Loc}\)-morphism such that the diagram

\[
\begin{array}{ccc}
A_2 & \xrightarrow{\varphi^\kappa} & A_1 \\
\kappa_2 \downarrow & & \kappa_1 \downarrow \\
L_2^X & \xrightarrow{(f, \varphi)} & L_1^X
\end{array}
\]

commutes. \(C\text{-TopSys}_a\) is the category of \(C\)-topological systems and \(C\)-continuous maps.

Remark 21. Proposition 19 provides an isomorphism between the categories \(C\text{-TopSys}\) and \(C\text{-TopSys}_a\).

3. Categorical lattice-valued topological systems

Remark 22. Definition 20 suggests a categorical approach to lattice-valued topological systems, which is based in powerset operators and topological theories.

Definition 23. Let \(X \xrightarrow{T} B^{op}\) be a \(t\)-theory. \(\text{TopSys}(T)\) is the comma category \((T \downarrow 1_{B^{op}})\), concrete over the product category \(\text{Set} \times B^{op}\), whose objects \((T\text{-systems})\) are triples \((X, \kappa, B)\), which are made by \(B^{op}\)-morphisms \(TX \xrightarrow{\mathsf{ht}} B\) (\(T\)-satisfaction relation on \((X, B)\)), and whose morphisms \((T\text{-continuous morphisms})\) \((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)\) are \(\text{Set} \times B^{op}\)-morphisms \((X_1, B_1) \xrightarrow{(f, \varphi)} (X_2, B_2)\) such that the diagram

\[
\begin{array}{ccc}
TX_1 & \xrightarrow{Tf} & TX_2 \\
\kappa_1 \downarrow & & \kappa_2 \downarrow \\
B_1 & \xrightarrow{\varphi} & B_2
\end{array}
\]

commutes.
Example 24. The case of the ground category $X = \text{Set} \times S$, where $S$ is a subcategory of $A^{\text{op}}$ for some variety $A$, is called variety-based approach. In particular, $\text{TopSys}((\mathcal{V}_A, B))$ provides the category $A_n \text{-TopSys}$, which is the framework for fixed-basis variety-based topological systems, whereas $\text{TopSys}((\mathcal{V}, B))$ gives the category $(S, B) \text{-TopSys}$ (the case $A = B$ is shortened to $S \text{-TopSys}$), which is the framework for variable-basis variety-based topological systems. More specific,

1. $\text{TopSys}((\mathcal{P}, \text{ Frm}))$ is isomorphic to the category $\text{TopSys}$ of topological systems of S. Vickers [8];
2. $\text{TopSys}((\mathcal{P}, \text{ Set}))$ is isomorphic to the category $\text{IntSys}$ of interchange systems of J. T. Denniston, A. Melton and S. E. Rodabaugh [5, 6];
3. $\text{TopSys}((\mathcal{R}_A, \text{ Frm}))$, where $S = \text{Loc}$, is isomorphic to the category $\text{Loc-TopSys}$ of lattice-valued topological systems of J. T. Denniston et al. [4].

3.1. Spatialization of lattice-valued topological systems

Remark 25. One of the main results of the theory of lattice-valued topological systems is the possibility of representing the category $\text{Top}(T)$ as a full subcategory (with convenient properties) of the category $\text{TopSys}(T)$. The embedding though is not concrete, since the ground categories in question are different.

Theorem 26.

1. There exists a full embedding $\text{Top}(T) \xrightarrow{E} \text{TopSys}(T)$, which is given by $E((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, e_{\tau_1}^{\text{op}}, \tau_1) \xrightarrow{(f, \varphi)} (X_2, e_{\tau_2}^{\text{op}}, \tau_2)$, where $e_{\tau_i}$ denotes the inclusion $\tau_i \xrightarrow{\iota_i} TX_i$, and $\varphi^{\text{op}}$ stands for the restriction $\tau_2 \xrightarrow{(Tf)^{\text{op}} \iota_2} \tau_1$.
2. There exists a functor $\text{TopSys}(T) \xrightarrow{\text{Spat}} \text{Top}(T)$ given by $\text{Spat}((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = (X_1, (\kappa_1^{\text{op}})^{\to}(B_1)) \xrightarrow{f} (X_2, (\kappa_2^{\text{op}})^{\to}(B_2))$.
3. $\text{Spat}$ is a right-adjoint-left-inverse to $E$.

Proof.

Ad (1). To show that the functor is full, notice that given a $T$-continuous morphism $(X_1, e_{\tau_1}^{\text{op}}, \tau_1) \xrightarrow{(f, \varphi)} (X_2, e_{\tau_2}^{\text{op}}, \tau_2)$, commutativity of the diagram

\[
\begin{array}{ccc}
\tau_2 & \xrightarrow{\varphi^{\text{op}}} & \tau_1 \\
\downarrow{e_{\tau_2}} & & \downarrow{e_{\tau_1}} \\
TX_2 & \xrightarrow{(Tf)^{\text{op}}} & TX_1
\end{array}
\]

implies that $((Tf)^{\text{op}})^{\to}(\tau_2) \subseteq \tau_1$, and therefore, $(X_1, \tau_1) \xrightarrow{(f, \varphi)} (X_2, \tau_2)$ is $T$-continuous with $Ef = (f, \varphi^{\text{op}})$.

Ad (2). To show that $\text{Spat}(f, \varphi)$ is $T$-continuous, notice that given $b \in B_2$, it follows that $(Tf)^{\text{op}} \circ \kappa_2^{\text{op}}(b) = \kappa_1^{\text{op}} \circ \varphi^{\text{op}}(b) \in (\kappa_1^{\text{op}})^{\to}(B_1)$.

Ad (3). Straightforward computations show that $\text{Spat}E = 1_{\text{Top}(T)}$. For the first claim, it will be enough to show that every system $(X, \kappa, B)$ has an $E$-co-universal arrow, i.e., a $\text{TopSys}(T)$-morphism $E\text{Spat}(X, \kappa, B) \xrightarrow{\varepsilon} (X, \kappa, B)$ such that for every $\text{TopSys}(T)$-morphism $E(X', \tau') \xrightarrow{(f, \varphi)} (X, \kappa, B)$, there exists a unique $\text{Top}(T)$-morphism $(X', \tau') \xrightarrow{\varepsilon} \text{Spat}(X, \kappa, B)$ such that the following triangle commutes

\[
\begin{array}{ccc}
E(X', \tau') & \xrightarrow{(f, \varphi)} & (X, \kappa, B) \\
\downarrow{E\varepsilon} & & \downarrow{\varepsilon} \\
E\text{Spat}(X, \kappa, B) & \xrightarrow{\varepsilon} & (X, \kappa, B)
\end{array}
\]
There is a $T$-continuous morphism $(E\text{Spat}(X,\kappa,B) = (X,e_{X,B}^{\text{op}},\kappa_{X,B})) \rightarrow (X,\kappa,B)$. Given a $\text{TopSys}(T)$-morphism $E(X',\tau') \rightarrow (X,\kappa,B)$, the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\kappa_{X,B}} & TX \\
\downarrow \phi^{\text{op}} & & \downarrow \epsilon_{\tau'} \\
\tau' & \xrightarrow{(f,\varphi)} & TX'
\end{array}
\]

commutes, and therefore, $(X',\tau') \xrightarrow{f} (\text{Spat}(X,\kappa,B) = (X,(\kappa_{X,B})^{\text{op}}(B)))$ is a $\text{Top}(T)$-morphism, which makes the above-mentioned triangle commute, and whose uniqueness is clear.

**Corollary 27.** $\text{Top}(T)$ is isomorphic to a full (regular mono)-coreflective subcategory of $\text{TopSys}(T)$.

**Proof.** In view of Theorem 26, it will be enough to show that given a $T$-system $(X,\kappa,B)$, the map $B \xrightarrow{\kappa_{X,B}} (\kappa_{X,B})^{\text{op}}(B)$ is a regular epimorphism in $B$. Define $C = \{(b_1,b_2) \in B \times B | \kappa_{X,B}(b_1) = \kappa_{X,B}(b_2)\}$ (the kernel of $\kappa_{X,B}$), and let $C \xrightarrow{\pi_2} B$ be given by $\pi_i(b_1,b_2) = b_i$ for $i \in \{1,2\}$. It is easy to see that $(\kappa_{X,B},(\kappa_{X,B})^{\text{op}}(B))$ is a coequalizer of $(\pi_1,\pi_2)$, which proves the claim.

**Remark 28.** The functor $\text{Spat}$ of Theorem 26 extends the system spatialization procedure of S. Vickers.

### 3.2. Localization of lattice-valued topological systems

**Remark 29.** The second important result of the theory of lattice-valued topological systems is the possibility of representing the category $B^{\text{op}}$ as a full subcategory (with convenient properties) of the category $\text{TopSys}(T)$. Unlike $\text{Top}(T)$, however, the embedding of $B^{\text{op}}$ into $\text{TopSys}(T)$ is not always possible.

**Proposition 30.** There exists a functor $\text{TopSys}(T) \xrightarrow{\text{Loc}} B^{\text{op}}$, which is defined by $\text{Loc}((X_1,\kappa_1,B_1) \xrightarrow{(f,\varphi)} (X_2,\kappa_2,B_2)) = B_1 \xrightarrow{\varphi} B_2$.

**Remark 31.** The functor $\text{Loc}$ of Proposition 30 extends the system localization procedure of S. Vickers.

**Theorem 32.** Given a $t$-theory $X \xrightarrow{T} B^{\text{op}}$, the following are equivalent:

1. there exists an adjoint situation $(\eta,\varepsilon) : T \dashv \text{Pt} : B^{\text{op}} \rightarrow X$;
2. there exists a full embedding $B^{\text{op}} \xrightarrow{E} \text{TopSys}(T)$ such that $\text{Loc}$ is a left-adjoint-left-inverse to $E$.

$B^{\text{op}}$ then is isomorphic to a full (in general, neither mono- nor epi-) reflective subcategory of $\text{TopSys}(T)$.

**Proof.**

Ad (1) $\Rightarrow$ (2). Define a functor $B^{\text{op}} \xrightarrow{E} \text{TopSys}(T)$ by $E(B_1,\varphi) B_2) = (\text{Pt}B_1,\varepsilon_{B_1},B_1) \xrightarrow{(f,\varphi)} (\text{Pt}B_2,\varepsilon_{B_2},B_2)$. Correctness of $E$ on morphisms follows from commutativity of the diagram

\[
\begin{array}{ccc}
TPtB_1 & \xrightarrow{Tf} & TPtB_2 \\
\downarrow \varepsilon_{B_1} & & \downarrow \varepsilon_{B_2} \\
B_1 & \xrightarrow{\varphi} & B_2
\end{array}
\]

Moreover, $E$ is clearly an embedding. To verify that $E$ is full, notice that given a $T$-continuous morphism $(\text{Pt}B_1,\varepsilon_{B_1},B_1) \xrightarrow{(f,\varphi)} (\text{Pt}B_2,\varepsilon_{B_2},B_2)$, commutativity of the diagram

\[
\begin{array}{ccc}
TPtB_1 & \xrightarrow{\varepsilon_{B_1}} & B_1 \\
\downarrow T\varphi & & \downarrow \varphi \\
TPtB_2 & \xrightarrow{\varepsilon_{B_2}} & B_2
\end{array}
\]
implies that \( \varepsilon_{B_2} \circ TP \varphi = \varepsilon_{B_2} \circ T f \), and therefore, \( PT \varphi = f \). Given a \( T \)-system \( (X, \kappa, B) \), straightforward calculations show that \( (X, \kappa, B) \left( f = \text{Pred}_X \cdot 1_B \right) \) \( (PTB, \varepsilon_B, B) = E_{\text{Loc}}(X, \kappa, B) \) provides an \( E \)-universal arrow for \( (X, \kappa, B) \). It is also easy to see that \( \text{Loc}E = 1_{B^p} \).

Ad (2) \( \Rightarrow \) (1). Suppose we have an adjunction \( \text{Loc} \dashv E : B^p \rightarrow \text{TopSys}(T) \). There clearly exists a functor \( \text{TopSys}(T) \xrightarrow{\text{Gr}} \text{X} \), which is defined by \( \text{Gr}((X_1, \kappa_1, B_1) \left( f, \varphi \right) (X_2, \kappa_2, B_2)) = X_1 \xrightarrow{f} X_2 \) ("Gr" is an abbreviation for "ground category"). Moreover, there exists a full embedding \( \text{X} \xrightarrow{M} \text{TopSys}(T) \), which is given by \( M(X_1 \xrightarrow{f} X_2) = (X_1, 1_{TX_1}, TX_1) \left( f, \text{Gr} \right) (X_2, 1_{TX_2}, TX_2) \). Straightforward calculations show that \( M \) is a left-adjoint-right-inverse to \( \text{Gr} \), e.g., given a \( T \)-system \( (X, \kappa, B) \), it follows that \( (M \text{Gr}(X, \kappa, B) = (X, 1_{TX}, TX)) \left( f, \text{Gr} \right) (X, \kappa, B) \) provides an \( M \)-co-universal arrow for \( (X, \kappa, B) \). The two adjoint situations

\[
\text{X} \xrightarrow{M} \text{TopSys}(T) \quad \xrightarrow{\text{Loc}} \quad \text{TopSys}(T) \xrightarrow{E} B^p
\]

give rise to the required one

\[
\text{X} \xrightarrow{T_{\text{Loc}}M} \text{TopSys}(T) \xrightarrow{\text{Gr}E} B^p.
\]

Theorem 33. Every topological theory \( \text{Set} \xrightarrow{T_A} B^p \), which is induced by the \( \text{bp} \)-theory \( \text{Set} \xrightarrow{V_A=(-)^\chi} A^p \), has a right adjoint.

Proof. Define a functor \( B^p \xrightarrow{PT_A} \text{Set} \), by \( PT_A(B_1 \left( \varepsilon \right) B_2) = PT_A B_1 \left( \varepsilon \right) PT_A B_2 \), where \( PT_A B_1 = B(B_1, ||A||) \) and \( \left( PT_A \varphi \right)(p) = \varphi \circ p \). Straightforward calculations show that given a \( B \)-algebra \( B \), the map \( B \xrightarrow{\varepsilon} \left( T_A PT_A B = ||A||B(||A||)\right) \), defined by \( \left( \varepsilon_B(b)(p) \right) = p(b) \), provides a \( T_A \)-co-universal arrow for \( B \).

Remark 34. Theorem 33 cannot be restated for variable-basis topological theories.

Proposition 35. Consider the \( t \)-theory \( \text{Set} \times A^p \xrightarrow{T_A} A^p \) given by \( \left( X_1, A_1 \right) \left( \chi \right) (X_2, A_2) \rightarrow A_1^X \left( (f, \chi)^\sim \right) A_2^X \), where \( (f, \chi)^\sim (\alpha) = \chi \circ \alpha \circ f \), and, moreover, assume that there exists an \( A \)-algebra \( A \), whose underlying set is finite, e.g., has the cardinality \( n \). Then \( T \) has no right adjoint.

Proof. If \( T \) has a right adjoint, then \( T \) preserves colimits, and thus, coproducts. Given a singleton set 1, \( T((1, A) \coprod (1, A)) = T(1 \coprod A) = (1 \coprod A) \cong A^1 \times A^1 \). If \( T((1, A) \coprod (1, A)) \cong T((1, A) \times T(1, A)) \), then \( n^2 = \text{Card}((1 \times A) \coprod (1 \times A)) = \text{Card}(A^1 \times A^1) = n^2 \), which is a contradiction.

3.3. Sobriety-spatiality equivalence revisited

Remark 36. This subsection provides a more general analogue of the sobriety-spatiality equivalence, which is considered at the end of Lecture 1.

Remark 37. Let \( X \xrightarrow{T} B^p \) be a \( t \)-theory, which has a right adjoint. One has the adjoint situations

\[
\text{Top}(T) \xrightarrow{\epsilon} \text{TopSys}(T) \xrightarrow{\text{Loc}} B^p,
\]

which give rise to the adjunction \( \text{Top}(T) \xrightarrow{\eta=\text{Loc}E} B^p \), or, more precisely, \( (\eta, \epsilon) : O \dashv PT : B^p \rightarrow \text{Top}(T) \).

Definition 38. \( \text{Spat} \) is the full subcategory of \( B^p \), which contains \( B \)-algebras \( B \) such that \( OPTB \xrightarrow{\varepsilon_B} B \) is an isomorphism.

Definition 39. \( \text{Sob} \) is the full subcategory of \( \text{Top}(T) \), which contains \( T \)-spaces \( (X, \tau) \) such that \( (X, \tau) \xrightarrow{\eta(X, \tau)} PTO(X, \tau) \) is an isomorphism.

Proposition 40. The adjunction \( \text{Top}(T) \xrightarrow{\eta} B^p \) restricts to an equivalence \( \text{Sob} \xrightarrow{\pi} \text{Spat} \).
Definition 41. A $T$-space $(X, \tau)$ is called separated provided that $(X, \tau) \nrightarrow PTO(X, \tau)$ is a monomorphism. $\Top_s(T)$ stands for the full subcategory of $\Top(T)$ of separated $T$-spaces.

Example 42. Separated $T$-spaces in the category $\Top((P, \Frm))$, which is isomorphic to $\Top$ (recall Lecture 1), are precisely the crisp $T_0$ topological spaces.

Theorem 43. Let $T$ be a t-theory in $X$. If $X$ is a $(\text{Retr, Mono})$-category, where $\text{Retr}$ (resp. $\text{Mono}$) is the class of retractions (resp. monomorphisms) in $X$, then $\Top_s(T)$ is a reflective subcategory of $\Top(T)$.

Example 44. The category $\Set$ is a $(\text{Retr, Mono})$-category, and therefore, Theorem 43 is applicable to every t-theory in $\Set$, which is generated by, e.g., a bp-theory of the form $\Set \nrightarrow A^{\op}$.

3.4. $\TopSys(T)$ is an essentially algebraic category

Remark 45. The category $\Top(T)$ is topological over its ground category $X$ (see Lecture 1). The category $\TopSys(T)$, however, is essentially algebraic over its ground category $X \times B^{\op}$.

Theorem 46 ([11]). A concrete category $(C, |−|)$ over $X$ is essentially algebraic iff the following conditions are satisfied:

1. $|−|$ creates isomorphisms,
2. $|−|$ has a left adjoint,
3. $C$ is $(\text{Epi, Mono-Source})$-factorizable.

Proposition 47. The functor $\TopSys(T) \nrightarrow X \times B^{\op}$ creates isomorphisms.

Proof. Given an $X \times B^{\op}$-isomorphism $(X_1, B_1) \nrightarrow [(X_2, \kappa_2, B_2)]$, the unique $T$-satisfaction relation on $(X_1, B_1)$, which makes $(f, \varphi)$ an isomorphism in $\TopSys(T)$, can be defined by $\kappa_1 = \varphi^{-1} \circ \kappa_2 \circ T_f$.

Proposition 48. The functor $\TopSys(T) \nrightarrow X \times B^{\op}$ has a left adjoint.

Proof. It is enough to show that every $X \times B^{\op}$-object $(X, A)$ has a $|−|$-universal arrow, i.e., an $X \times B^{\op}$-morphism $(X_1, B_1) \nrightarrow [(X_2, \kappa_2, B_2)]$ such that for every $X \times B^{\op}$-morphism $(X_1, B_1) \nrightarrow [(X_3, \kappa_3, B_3)]$, there exists a unique $T$-continuous morphism $(X_2, \kappa_2, B_2) \nrightarrow [(X_3, \kappa_3, B_3)]$ such that the triangle

\[
\begin{array}{ccc}
(X_1, B_1) & \nrightarrow & [(X_2, \kappa_2, B_2)] \\
(f, \varphi) & \downarrow & \psi \\
(X_2, \kappa_2, B_2) & \nrightarrow & [(X_3, \kappa_3, B_3)]
\end{array}
\]

commutes.

There exists an $X \times B^{\op}$-morphism $(X_1, B_1) \nrightarrow [(X_3, \kappa_3, B_3)]$ such that there exists a unique $B$-homomorphism $B_3 \nrightarrow TX_1 \times B_1$, which is defined by commutativity of the following diagram:

\[
\begin{array}{ccc}
TX_3 & \nrightarrow & B_3 \\
\kappa_3^{op} & \downarrow & \psi \\
TX_1 \times B_1 & \nrightarrow & B_1
\end{array}
\]
The left-hand side of the diagram implies that \((X_1, \pi^\text{op}_{TX_1}, TX_1 \times B_1) \xrightarrow{(f, \varphi)} (X_3, \kappa_3, B_3)\) is a TopSys\((T)\)-morphism, and the right-hand side of the diagram gives commutativity of the above-mentioned triangle.

Given another TopSys\((T)\)-morphism \((X_1, \pi^\text{op}_{TX_1}, TX_1 \times B_1) \xrightarrow{(f, \varphi')} (X_3, \kappa_3, B_3)\) with the same property, it follows that \((f, \varphi) = [(f', \varphi') \circ \eta] = [(f', \varphi') \circ (X_1, \pi^\text{op}_{B_1}) = (f', \varphi' \circ \pi^\text{op}_B)\), and thus, \(f' = f\) and \(\pi_B \circ \varphi^\text{op} = \varphi^\text{op}\). Moreover, \((Tf)^\text{op} \circ \kappa^\text{op}_3 = (Tf')^\text{op} \circ \kappa^\text{op}_3 = \pi^\text{op}_{TX_1} \circ \psi^\text{op}\) by the condition of \(T\)-continuous morphisms, and therefore, \(\psi = \psi^\text{op}\) by the universal property of products. 

\[\square\]

**Proposition 49.** If every source \(S\) in \(X\) has an (Epi, Mono-Source)-factorization \(S = \mathcal{M} \circ \circ\) such that \((Te)^\text{op}\) is an injective map, then the category TopSys\((T)\) is (Epi, Mono-Source)-factorizable.

**Proof.** Let \(S = ((X, \kappa, B) \xrightarrow{(f, \varphi)} (X_i, \kappa_i, B_i))_{i \in I}\) be a source in TopSys\((T)\). By the assumption, there exists an (Epi, Mono-Source)-factorization \(X \xrightarrow{\lambda} X_i \xrightarrow{\widetilde{\nu}} X_i\) of the source \(K = (X \xrightarrow{\lambda} X_i)_{i \in I}\) with the above-mentioned property. Define \(\overline{B} = \bigcup_{i \in I}(\varphi^\text{op}_i)^{-1}(B_i)\) (recall that \((S)\) is the smallest \(B\)-subalgebra containing \(S\)), let \(\overline{B} \xrightarrow{\psi} B\) be the inclusion, and let \(B_i \xrightarrow{\psi_i} \overline{B} \xrightarrow{\psi} B\) be an (Epi-Sink, Mono) factorization of the sink \(T = (B_i \xrightarrow{\varphi^\text{op}_i} B)_{i \in I}\) in \(B\). Moreover, easy considerations show that \(\overline{B} = \omega^B(c_j)|c_j \in K\) with \(K = \bigcup_{i \in I}(\varphi^\text{op}_i)^{-1}(B_i)\), where \(\omega^B((c_j))\) is a word (in the algebraic sense), consisting of operations on \(B\) and taking elements of \(K\) as arguments (notice that we deliberately omit the set, \(j\) ranges over, since, in general, it consists of the union of some \(n_\lambda\)s; also bear in mind that the category TopSys\((T)\) contains no operations at all, being just an element of \(K\)). In view of this characterization, define a map \(\overline{B} \xrightarrow{\pi^\text{op}} TX\) by \(\pi^\text{op}(\omega^B((c_j))) = \omega^TX((Tm_j)^\text{op} \circ \kappa_j^\text{op}(c_j))\) with \(c_j = \varphi_j(b_j)\). To verify correctness of the definition, suppose that \(\omega^B((c_j)) = \omega^B((c'_{j}))\), and get then \((Te)^\text{op}((Tm_j)^\text{op} \circ \kappa_j^\text{op}(b_j))) = \omega^TX((Tf_j)^\text{op} \circ \kappa_j^\text{op}(b_j)) = \omega^TX((Tm_j)^\text{op} \circ \kappa_j^\text{op}(b_j)) = \omega^TX((Tf_j)^\text{op} \circ \kappa_j^\text{op}(b_j)) = \kappa^\text{op}\circ \omega^B((c_j)) = \kappa^\text{op}\circ \omega^B((c'_{j})) = \omega^TX(((Tm_j)^\text{op} \circ \kappa_j^\text{op}(b_j)))\), and therefore, \(\omega^TX((Tm_j)^\text{op} \circ \kappa_j^\text{op}(b_j)) = \omega^TX((Tm_j)^\text{op} \circ \kappa_j^\text{op}(b'_j)))\) by the assumption of the proposition. To prove that \(\pi^\text{op}\) is a homomorphism, take \(\lambda \in \Lambda\) and \(d_j \in \overline{B}\), getting \(\pi^\text{op}(\omega^B((d_j))) = \pi^\text{op}(\omega^B((c_j))) = \omega^TX((Tm_j)^\text{op} \circ \kappa_j^\text{op}(b_j)))\), which proves that \((\overline{X}, \pi, \overline{B})\) is a \(T\)-system and, at the same time, obtaining the following diagram

![Diagram](https://via.placeholder.com/150)

It shows to come commutativity of both its left and right rectangles. For the former, notice that given \(b_i \in B_i, (Te)^\text{op} \circ \pi^\text{op} \circ \psi_i(b_i) = (Te)^\text{op} \circ \pi^\text{op} \circ \varphi_i^\text{op}(b_i) = (Te)^\text{op} \circ \pi^\text{op} \circ \varphi_i^\text{op}(b_i)\) by the definition of \(\pi^\text{op}\), and therefore, \((Te)^\text{op} \circ \pi^\text{op} \circ \psi_i = (Te)^\text{op} \circ \pi^\text{op} \circ \kappa_i^\text{op}\), which implies \(\pi^\text{op} \circ \psi_i = (Tm_i)^\text{op} \circ \kappa_i^\text{op}\). For the latter, use the fact that \((Te)^\text{op} \circ \pi^\text{op} \circ \psi_i = (Te)^\text{op} \circ \pi^\text{op} \circ \kappa_i^\text{op}\) and \(\kappa_i^\text{op} = \kappa^\text{op} \circ \psi_i = \kappa^\text{op} \circ \psi_i\), for every \(i \in I\), implies \((Te)^\text{op} \circ \pi^\text{op} = \kappa^\text{op} \circ \psi_i\). As a result, \((X, \kappa, B) \xrightarrow{(f, \varphi)} (X_i, \kappa_i, B_i) = (X, \kappa, B) \xrightarrow{(e, \psi^\text{op})} (X, \pi, \overline{B}) \xrightarrow{(m_i, \psi_i^\text{op})} (X, \kappa, B)\) is the required (Epi, Mono-Source)-factorization of \(S\). 

\[\square\]

8
Theorem 50. Given a $t$-theory $X \xrightarrow{T} B^{\text{op}}$, with the property that every source $S$ in $X$ has an $(Epi, Mono)$-factorization $S = M \circ e$ such that $(Te)^{\text{op}}$ has an injective underlying map, the category $\text{TopSys}(T)$ is essentially algebraic over $X \times B^{\text{op}}$.

Proof. Follows from Propositions 47, 48, 49. □

Remark 51. By Theorem 50, all the categories of the form $\text{TopSys}((\mathcal{V}, B))$ (e.g., the category $\text{TopSys}$ of S. Vickers) are essentially algebraic.

4. An application of the theory of lattice-valued topological systems

4.1. State property systems of D. Aerts and closure spaces

Remark 52. State property systems were introduced by D. Aerts in 1999 in [2] as the basic mathematical structure in the Geneva-Brussels approach to the foundation of physics. Moreover, the category of state property systems is equivalent to the category of closure spaces [3].

Definition 53. A closure space is a pair $(X, F)$, where $X$ is a set, and $F$ is a family of subsets of $X$, which satisfies the following properties:

1. $\emptyset \in F$;
2. if $F_i \in F$ for $i \in I$, then $\bigcap_{i \in I} F_i \in F$.

A map $X_1 \xrightarrow{f} X_2$ between closure spaces $(X_1, F_1)$ and $(X_2, F_2)$ is said to be continuous provided that $(f^{-1})(F_2) \subseteq F_1$. $\text{Cls}$ is the category of closure spaces and continuous maps.

Definition 54. $CSL$ is the variety of closure semilattices ($c$-semilattices), i.e., $\mathcal{A}$-semilattices, with the singled out bottom element $\perp$.

Remark 55. $\text{Top}((\mathcal{P}, CSL))$ is isomorphic to the category $\text{Cls}$.

Definition 56. A state property system is a triple $(X, A, \kappa)$, where $X$ is a set, $A$ is a $c$-semilattice, and $A \xrightarrow{\kappa} \mathcal{P}X$ is an injective $c$-semilattice homomorphism (Cartan map). A state property system morphism $(X_1, A_1, \kappa_1) \xrightarrow{(f, \varphi)} (X_2, A_2, \kappa_2)$ is a Set $\times CSL^{\text{op}}$-morphism $(X_1, A_1) \xrightarrow{(f, \varphi)} (X_2, A_2)$ such that the diagram

\[
\begin{array}{ccc}
A_2 & \xrightarrow{\varphi} & A_1 \\
\kappa_2 \downarrow & & \kappa_1 \\
\mathcal{P}X_2 & \xrightarrow{f} & \mathcal{P}X_1
\end{array}
\]

commutes. $\text{SP}$ is the category of state property systems and state property system morphisms.

Theorem 57 ([3]). The categories $\text{SP}$ and $\text{Cls}$ are equivalent.

4.2. Lattice-valued state property systems and closure spaces

Definition 58. A $T$-system $(X, \kappa, B)$ is called separated provided that the map $B \xrightarrow{\kappa^{\text{op}}} TX$ is injective. $\text{TopSys}_s(T)$ stands for the full subcategory of $\text{TopSys}(T)$ of separated $T$-systems.

Example 59. $\text{TopSys}_s((\mathcal{P}, CSL))$ is isomorphic to the category $\text{SP}$ of D. Aerts.

Proposition 60. There exist the restrictions $\text{Top}(T) \xrightarrow{E} \text{TopSys}_s(T)$ and $\text{TopSys}_s(T) \xrightarrow{\text{Spat}} \text{Top}(T)$ of the functors $E$ and Spat of Theorem 26, respectively.
Proof. It will be enough to show that given a $T$-space $(X, \tau)$, $E(X, \tau)$ is a separated $T$-system, which follows immediately from Theorem 26 (1).

Theorem 61. The functors $\text{Top}(T) \xleftarrow{\text{E}} \text{TopSys}_s(T)$ and $\text{TopSys}_s(T) \xrightarrow{\text{Spat}} \text{Top}(T)$ provide an equivalence between the categories $\text{Top}(T)$ and $\text{TopSys}_s(T)$.

Proof. By Theorem 26 (3), $\text{Spat}$ is a right-adjoint-left-inverse to $E$. To prove the theorem, it is enough to show that for every separated $T$-system $(X, \kappa, B)$, the $E$-co-universal arrow $E\text{Spat}(X, \kappa, B) \xrightarrow{\varepsilon = (1_X, \kappa)} (X, \kappa, B)$ from the proof of Theorem 26 (3) is an isomorphism. The claim follows from the definition of $\varepsilon$, since $B \xrightarrow{\kappa}\text{op} (\kappa) \xrightarrow{\text{op}} (B)$ is always surjective, and it is injective by the property of separated $T$-systems. □

Corollary 62. The category $\text{TopSys}_s(T)$ is a full (regular mono)-coreflective subcategory of $\text{TopSys}(T)$.


Remark 63. The relevance of Theorem 61 is slightly undermined by the fact that $\text{Top}(T)$ and $\text{TopSys}_s(T)$ have different ground categories, i.e., $X$ and $X \times B^{\text{op}}$, respectively. In particular, notwithstanding the result that $\text{Top}(T)$ is topological over its ground category, $\text{TopSys}_s(T)$ never needs to have the same property. On the other hand, being topological, $\text{Top}(T)$ is (co)complete provided that its ground category is (co)complete, and that can be transferred immediately to the category $\text{TopSys}_s(T)$.

References


