Parabolic Geometries

Andreas Čap

Jan Slovák

Author address:

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A–1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A–1090 Wien, Austria
E-mail address: `Andreas.Cap@esi.ac.at`

Department of Algebra and Geometry, Masaryk University in Brno, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
E-mail address: `slovak@math.muni.cz`

Abstract.

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Background

CHAPTER 1

Cartan Geometries

Cartan geometries, or generalized spaces in the original terminology, were introduced by E. Cartan, see e.g. [**Cartan, 1952**], associating a geometric structure to any homogeneous space G/H. The geometric structure corresponding to G/H is formed by smooth manifolds having the same dimension as G/H together with a smooth principal H–bundle endowed with what is nowadays called a Cartan connection, possibly satisfying some normalization condition. On one hand, this led to a conceptual explanation of the point of view that Riemannian geometry studies "curved analogs" of Euclidean space. On the other hand, this allows to consider analogous generalizations for more general homogeneous spaces, which define geometries in the sense of Klein's Erlangen program. The book [**Sharpe, 1997**] explains this point of view and studies several basic examples of this situation.

For structures like Riemannian metrics, the point of view of Cartan connections, while conceptually nice, is not really necessary, since due to the simplicity of the situation, the normalized Cartan connection is essentially equivalent to a principal connection which corresponds to the Levi–Civita connection on the tangent bundle. Probably because of this reason and the enormous success of principal connections in the subsequent period, Cartan's ideas did not receive as much attention as they deserve in our opinion. Moreover, if one studies more complicated structures then the point of view of Cartan connections is really a strong simplification.

The aim of this introductory chapter is twofold. On one hand, we want to discuss the fundamentals of Cartan geometries as a basis for the further developments in more specific situations. Most of the theory developed in this chapter works in the full generality of arbitrary Cartan connections, without any restrictions to special groups. On the other hand, we want to discuss how rather well known structures like Riemannian metrics can be understood in the picture of Cartan geometries.

1.1. Prolog — a few examples of homogeneous spaces

We start by considering several examples of viewing spheres as homogeneous spaces of various Lie groups. In any of these examples, there is a geometric structure which is easily seen to be invariant under the action of the group in question. Much more subtle arguments show that in fact the group can in all examples be characterized as the group of those diffeomorphisms which preserve that geometric structure.

The structures showing up in this way can then be defined on (almost) arbitrary smooth manifolds. While they are very different, they have some common features:

- They are interesting from a local point of view, i.e. there are nontrivial local invariants.
- Any automorphism of such a structure is determined already by a finite jet (i.e. its value and the value of finitely many derivatives) in a single point.

- The automorphism group of such a structure on any smooth manifold is finite dimensional (with the dimension being bounded from above by the dimension of the group showing up in our examples).
- In general, such structures are very far from being homogeneous. Indeed, generically they do not even admit local automorphisms.

As we shall see later on, all these properties (together with many others) are consequences of the fact that the structures can be described by Cartan geometries, which means that in spite of the last point of this list, they admit a description which is quite similar to a homogeneous space.

The main purpose of these examples is on one hand to show the broadity of the concept of "geometric structure" that we use. Secondly, we want to show that fairly complicated geometric structures emerge naturally from rather simple group actions. Finally, many of the constructions that we will present in the book are easily visible in the homogeneous models, so we will regulary refer back to these examples. Since our purpose here is motivation, we will partly only outline arguments instead of giving full proofs.

At this place, we should tell that we restart our exposition from scratch in section 1.2, reviewing the basic elements of differential geometry. The current section plays indeed the role of a prolog and the reader might go ahead from the beginning and to browse in the sections 1.2, 1.3 below only if necessary, but he or she might also prefer to start in 1.2 and to come back to the individual examples in this sections later on.

As in most parts of the book, we shall mostly consider real Lie groups and so we often omit the letter $\mathbb{R}$ from our notation. Thus we write $GL(n)$, $O(n)$, $SL(n)$, etc. instead of the more complete notation $GL(n,\mathbb{R})$, $O(n,\mathbb{R})$, $SL(n,\mathbb{R})$, in this section.

1.1.1. The Riemannian round sphere. This is certainly among the best known examples of homogeneous spaces. We consider the sphere S^n as the unit sphere $\{x : |x| = 1\}$ in $\mathbb{R}^{n+1}$. The group $G = O(n+1)$ of orthogonal $(n+1)\times(n+1)$–matrices acts on $\mathbb{R}^{n+1}$ leaving the unit sphere invariant, so we get a smooth left action $O(n+1) \times S^n \to S^n$. Since any unit vector x in $\mathbb{R}^{n+1}$ can be extended to an orthonormal basis of $\mathbb{R}^{n+1}$, and the matrix having the elements of this bases as column vectors is orthognal and maps the first vector e_1 in the standard basis of $\mathbb{R}^{n+1}$ to x, this action is transitive. Moreover, the isotropy group of e_1 is immediately seen to be

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \right\} \subset G.$$

Any matrix H of the latter form is orthogonal if and only if the $n \times n$–matrix A is orthogonal. Thus $H \cong O(n)$ and our left action induces a diffeomorphism of the sphere S^n with the homogeneous space $O(n+1)/O(n)$.

To see the geometric structure invariant under this action, we just have to analyze the tangent spaces of S^n in this picture. By the description of S^n as the unit sphere (which uses the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^{n+1}$) we get for the tangent spaces $T_xS^n = \{y \in \mathbb{R}^{n+1} : \langle x, y\rangle = 0\}$. Thus the standard inner product on $\mathbb{R}^{n+1}$ induces a positive definit inner product on each tangent space of S^n, which obviously depends smoothly on the footpoint, so we get a *Riemannian metric* g on S^n, which is usually called the *standard metric* or the *round metric*.

For a matrix $A \in O(n+1)$, a point $x \in S^n$ and an element $y \in T_xS^n$, orthogonality of A immediately implies that $Ay \in T_{Ax}S^n$, and since $x \mapsto Ax$ is the restriction to S^n of a linear map, the tangent map $T_xS^n \to T_{Ax}S^n$ of the action by A is given by $y \mapsto Ay$. Visibly, this is an isometry between the tangent spaces, since the Riemannian metric is induced from the standard inner product on $\mathbb{R}^{n+1}$, so $O(n+1)$ is contained in the group $\mathrm{Isom}(S^n, g)$ of isometries of the round sphere.

There are at least two ways to see that actually these two groups coincide, i.e. any isometry of S^n is induced by an orthogonal transformation of $\mathbb{R}^{n+1}$. Since analogs of both these two lines of argument will be important for us in the sequel, we outline both of them.

The first line of argument is based on reducing to fixed jets. Let $\varphi \in \mathrm{Isom}(S^n, g)$ be any isometry, and consider $\varphi(e_1) \in S^n$. Since the action of $O(n+1)$ on S^n is transitive, we find a matrix $A \in O(n+1)$ such that $Ae_1 = \varphi(e_1)$, so $A^{-1} \circ \varphi$ lies in the subgroup $\mathrm{Isom}_{e_1}(S^n, g)$ of isometries fixing e_1. Clearly, $\varphi \in O(n+1)$ if and only if $A^{-1} \circ \varphi$ lies in the isotropy subgroup $H = O_{e_1}(n+1)$. Hence, it suffices to show that any isometry whose zero–jet in e_1 coincides with the zero–jet of the identity, actually lies in $O(n+1)$.

But if φ is such an isometry, its tangent map in e_1 is an orthogonal automorphism of $T_{e_1}S^n$, and thus we can find a (actually unique) matrix A in the isotropy subgroup H which induces the same orthogonal map on $T_{e_1}S^n$. As above, this reduces the problem of showing that $\mathrm{Isom}(S^n, g) = O(n+1)$ to the case of those isometries, whose one–jet in e_1 coincides with the one–jet of the identity. Hence suppose that $\varphi \in \mathrm{Isom}(S^n, g)$ is an isometry which maps e_1 to itself and whose tangent map $T_{e_1}\varphi$ is the identity. Then in the Riemannian case it can be seen from elementary arguments that φ actually must be the identity as follows:

Since φ is an isometry, it must map geodesics to geodesics. More precisely, if $I \subset \mathbb{R}$ is an interval containing zero and $c : I \to S^n$ is a geodesic starting at $c(0)$ in direction $c'(0)$ then $\varphi \circ c$ is the geodesic with starting point $\varphi(c(0))$ and initial direction $T_{c(0)}\varphi \cdot c'(0)$. Since any point x_0 has an open neighborhood whose points can be reached by geodesics starting at x_0, the set on which the one–jet of φ coincides with the one–jet of the identity is open, and clearly its complement is open as well. Since S^n is connected, we conclude that φ coincides with the identity everywhere.

The second line of argument is based on the analysis of the Maurer–Cartan form on G. It is less elementary, but easier to generalize to the other cases of interest. We have already observed above that $A \mapsto Ae_1$ induces a diffeomorphism $O(n+1)/H \to S^n$, where $H \cong O(n)$ is the isotropy subgroup of e_1. In particular, the resulting projection $p : O(n+1) \to S^n$ is a principal fiber bundle with structure group $O(n)$. Let $\mathfrak{g}$ be the Lie algebra of $G = O(n+1)$, i.e. the Lie algebra of all skew symmetric $(n+1) \times (n+1)$–Matrices. Splitting matrices into blocks of sizes one and n, we see that

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & -v^t \\ v & A \end{pmatrix} : v \in \mathbb{R}^n, A \in \mathfrak{o}(n) \right\}.$$

The latter description of the subgroup H implies immediately that the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponding to H is given by those matrices for which $v = 0$. From the block form above, it follows immediately that the vector subspace $\mathfrak{n}$ consisting of thoses matrices, for which $A = 0$ is complementary to $\mathfrak{h}$, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ as a vector space, and $\mathfrak{n} \cong \mathbb{R}^n$.

The adjoint action restricted to the subgroup H makes $\mathfrak{g}$ into a representation of the group $H \cong O(n)$, and since the adjoint action for matrix groups is given by conjugation, one imeediately sees that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ is H–invariant, and $\mathfrak{n}$ is (isomorphic to) the standard representation of $O(n)$ on $\mathbb{R}^n$.

Now consider the Maurer–Cartan form $\omega \in \Omega^1(G; \mathfrak{g})$ on the group G, i.e. the trivialization of the tangent bundle of G by left translations, viewed as a differential form, cf. 1.2.4. Then ω is G–invariant and thus in particular H–invariant. Moreover, the H–invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ splits ω into a sum $\omega = \omega_{\mathfrak{h}} \oplus \omega_{\mathfrak{n}}$ of H–invariant forms with values in $\mathfrak{h} \cong \mathfrak{o}(n)$ and $\mathfrak{n} \cong \mathbb{R}^n$. For any point $A \in G$ the kernel of $\omega_{\mathfrak{n}}(A)$ is exactly the vertical tangent space $V_A G$ in A, so $\omega_{\mathfrak{n}}(A)$ induces a linear isomorphism $T_A G / V_A G \cong T_{p(A)} S^n \to \mathbb{R}^n$. Since in addition $\omega_{\mathfrak{n}}$ is H–invariant, we can view $p : G \to S^n$ as a reduction to the group $O(n)$ of the frame bundle of S^n and this is exactly the orthonormal frame bundle corresponding to the Riemann metric g. In particular, any isometry $\varphi : S^n \to S^n$ lifts to an automorphism $\tilde{\varphi}$ of the H–principal bundle $p : G \to S^n$, which in addition satisfies $\tilde{\varphi}^* \omega_{\mathfrak{n}} = \omega_{\mathfrak{n}}$.

On the other hand, since the component $\omega_{\mathfrak{h}} \in \Omega^1(G; \mathfrak{h})$ of ω is H–invariant, it follows immediately from the properties of the Maurer–Cartan form, that $\omega_{\mathfrak{h}}$ defines a principal connection on the principal bundle $p : G \to S^n$. This principal connection induces a linear connection on the tangent bundle of S^n which preserves the Riemannian metric g and which is torsion free by the Maurer–Cartan equation. Thus $\omega_{\mathfrak{h}}$ is the principal connection corresponding to the Levi–Civita connection on S^n. In particular, this implies that for any isometry φ of S^n, the induced diffeomorphism $\tilde{\varphi} : G \to G$ satisfies $\tilde{\varphi}^* \omega = \omega$. But now there is a rather simple general result, the uniqueness part in the “fundamental theorem of calculus”, cf. theorem 1.2.4 below, which says that any diffeomorphism with this property is the left multiplication by some element of G. In particular, this implies that $\mathrm{Isom}(S^n, g) = O(n+1)$.

The curved analogs of S^n viewed as a homogeneous space of $O(n+1)$ in the sense of E. Cartan's generalized spaces are exactly n–dimensional Riemannian manifolds. In fact, usually one would rather take the Euclidean space E^n, viewed as a homogeneous space of the group $\mathrm{Euc}(n)$ of euclidean motions, as the homogeneous model of Riemannian n–manifolds, but as we shall see in 1.1.2 below, this makes only a difference in the notion of curvature, but not in the notion of a curved analog itself. For Riemannian manifolds, the properties listed in the beginning of 1.1 are well known, and indeed can be proved without using Cartan connections, which however offer a very conceptual approach to these properties.

1.1.2. Some variations on the Riemannian sphere. There are several variations of the Riemannian sphere, in which the overall picture looks very similar. The first possible variation is to replace the group $O(n+1)$ by a group with the same Lie algebra, with $SO(n+1)$ and $Spin(n+1)$ being the most important choices. Clearly, $SO(n+1)$ acts on the sphere S^n by orientation preserving isometries and precisely as in 1.1.1 one shows that any orientation preserving isometry of S^n is obtained in that way. Moreover, the projection $SO(n+1) \to S^n$ is exactly the oriented orthonormal frame bundle of S^n with structure group $SO(n)$. The Cartan geometries corresponding to this homogeneous space are exactly oriented n–dimensional Riemannian manifolds. In the case of $Spin(n+1)$, we get isotropy subgroup $Spin(n)$ and the action of $Spin(n+1)$ on $\mathbb{R}^{n+1}$ and thus the action on S^n factors over the covering homomorphism $Spin(n+1) \to SO(n+1)$. This means that the resulting $Spin(n)$–principal bundle $Spin(n+1) \to S^n$ is an extension of the oriented orthonormal frame bundle of S^n and thus defines a *spin structure* on the oriented Riemannian manifold S^n. $Spin(n+1)$ is then exactly the group of all automorphisms of this spin strucutre, and the curved analogs of this homogeneous space are exactly all n–dimensional Riemannian spin–manifolds.

The second possible variation is to replace $O(n+1)$ by the orthogonal group $O(p+1,q)$ of an inner product or arbitrary signature $(p+1,q)$. Then one has to replace S^n by the (noncompact) unit sphere of that inner product. Choosing the inner product in such a way that the first vector in the standard basis of $\mathbb{R}^{p+1+q}$ has norm one, one proceeds similarly as in 1.1.1 to show that the action on the unit sphere is transitive and the isotropy subgroup of e_1 is isomorphic to $O(p,q)$. Visibly, $O(p+1,q)$ acts by isometries of the pseudo–Riemannian metric (with signature (p,q)) on the unit sphere induced from the inner product on $\mathbb{R}^{p+1+q}$, and similarly as in 1.1.1 one shows that these are all isometries. The curved analogs of this homogeneous space are exactly $p+q$–dimensional pseudo–Riemannian manifolds with metrics of signature (p,q). Clearly, one can pass to other groups with Lie algebra $\mathfrak{o}(p+1,q)$, in particular $SO(p+1,q)$, $SO^{\uparrow}(p+1,q)$ (the connected component of the identity in $SO(p+1,q)$), and $Spin(p+1,q)$, which leads to oriented pseudo–Riemannian manifolds, space– and time–oriented pseudo–Riemannian manifolds, and pseudo–Riemannian spin–manifolds of fixed dimension and signature, respectively.

The final variation we discuss is the passage to Euclidean space, which does not look like a variation at the first sight, since the group $\mathrm{Euc}(n)$ of Euclidean motions in n–dimensions is very different from $O(n+1)$. Nevertheless, we obtain the same notion of a curved analog as for the sphere. To get a nice presentation of the Euclidean group $\mathrm{Euc}(n)$, we consider the n–dimensional Euclidean space E^n as the affine hyperplane $\{x_{n+1}=1\}$ in $\mathbb{R}^{n+1}$. Then the usual Euclidean metric on E^n is simply induced from the standard inner product on $\mathbb{R}^{n+1}$. The group $\mathrm{Euc}(n)$ is then simply the group of all linear automorphisms of $\mathbb{R}^{n+1}$, which map the hyperplane $\{x_{n+1}=1\}$ to itself and restrict to an isometry of that hyperplane. A simple computation shows that

$$\mathrm{Euc}(n)=\left\{\begin{pmatrix} A & v\\ 0 & 1\end{pmatrix} : A\in O(n), v\in\mathbb{R}^n\right\}$$

and the isotropy group $H\cong O(n)$ of the last vector e_{n+1} in the standard basis of $\mathbb{R}^{n+1}$ consists exactly of those elements of $\mathrm{Euc}(n)$, for which $v=0$. On the level of Lie algebras, we have

$$\mathfrak{euc}(n)=\left\{\begin{pmatrix} A & v\\ 0 & 0\end{pmatrix} : A\in\mathfrak{o}(n), v\in\mathbb{R}^n\right\}$$

which again gives a decomposition $\mathfrak{euc}(n)=\mathfrak{h}\oplus\mathfrak{n}$ as a vector space. Using again the fact that the adjoint action on a matrix group is given by conjugation, we see that this actually is an H–invariant decomposition and $\mathfrak{n}$ is isomorphic to the standard representation $\mathbb{R}^n$ of $H\cong O(n)$.

This shows that while $\mathrm{Euc}(n)$ and $O(n+1)$ are very different Lie groups, they both contain $O(n)$ as a Lie subgroup and via the adjoint actions, the Lie algebras $\mathfrak{euc}(n)$ and $o(n+1)$ are isomorphic representations of $O(n)$. Since in the definition of a Cartan geometry corresponding to a homogeneous space G/H only the subgoup H and the H–module $\mathfrak{g}$ (and *not* the Lie algebra $\mathfrak{g}$) enters, the homogeneous spaces $E^n=\mathrm{Euc}(n)/O(n)$ and $S^n\cong O(n+1)/O(n)$ lead to the same generalized spaces, namely n–dimensional Riemannian manifolds. The only point in which the Lie algebra structure of $\mathfrak{g}$ enters is the definition of curvature, which differs in the two settings.

1.1.3. The projective sphere. The projective sphere is our first example of a parabolic geometry. In this example we view S^n as a homogeneous space of the group $SL(n+1,\mathbb{R})$, by considering it as the spaces of rays through the origin in $\mathbb{R}^{n+1}$. So consider the open subset $C := \mathbb{R}^{n+1} \setminus 0 \subset \mathbb{R}^{n+1}$ and the equivalence relation $x \sim y$ if and only if there exists a number $t \in \mathbb{R}$, $t > 0$ such that $y = tx$. Then obviously $C/\sim$ is is diffeomorphic to S^n. Moreover, the action of $\mathbb{R}^+$ on $\mathbb{R}^{n+1}$ by multiplication is clearly smooth, and restricts to a free action on C. Thus, the natural projection $\pi : C \to S^n$ is a principal bundle with structure group $\mathbb{R}^+$.

The action of $SL(n+1,\mathbb{R})$ restricts to an action on C, and since the action is linear, it induces an action on S^n, which obviously is transitive. (In fact, we could also start with $GL(n,\mathbb{R})$, but the induced action on S^n factors over $SL(n,\mathbb{R})$ anyhow.) The isotropy group P of the ray through the first vector in the standard basis of $\mathbb{R}^{n+1}$ consists of all matrices of the form

$$P = \left\{ \begin{pmatrix} \det(A)^{-1} & \varphi \\ 0 & A \end{pmatrix},\ A \in GL(n,\mathbb{R}),\ \varphi \in \mathbb{R}^{n*} \right\}.$$

This is a simple example of a *parabolic subgroup* of $SL(n+1,\mathbb{R})$. It is easy to verify that P is the semidirect product of $GL(n,\mathbb{R})$ and the vector group $\mathbb{R}^{n*}$.

To see the geometric structure on S^n which is invariant under the action of $SL(n+1,\mathbb{R})$, we start by observing that $GL(n+1,\mathbb{R})$ is precisely the group of those affine isomorphisms of $\mathbb{R}^{n+1}$ which leaves the subset C invariant. Affine isomorphisms are exactly those, which leave the canonical flat connection ∇ on $\mathbb{R}^{n+1}$ invariant. Hence any object on $S^n = C/\sim$ that can be constructed from the flat connection on $\mathbb{R}^{n+1}$ is invariant under the action of $SL(n+1,\mathbb{R})$. The natural idea is to try to construct a connection on S^n from the flat connection on $\mathbb{R}^{n+1}$ in a natural way. This is impossible, one has to make a choice to get such a connection, so what one naturally gets is an equivalence class of connections, which defines a *projective structure* on S^n.

A simple way to push down the flat connection from C to S^n, is to lift vector fields from S^n to C, and there is the obvious choice to lift them to $\mathbb{R}_+$–invariant fields. For $t \in \mathbb{R}_+$ let $m_t : C \to C$ be the multiplication by t, and let us denote by $\mathfrak{X}(C)^{\mathbb{R}_+}$ the space of those vector fields on C, for which $m_t^*\xi = \xi$, where m_t^* denotes the pullback by the diffeomorphism m_t. This means that $\xi \in \mathfrak{X}(C)^{\mathbb{R}_+}$ if and only if $\xi(tx) = T_x m_t \cdot \xi(x)$, or viewing vector fields on C just as $\mathbb{R}^{n+1}$–valued functions, if and only if $\xi(tx) = t\xi(x)$. If $\xi(tx) = T_x m_t \cdot \xi(x)$, then $T_{tx}\pi(\xi(tx)) = T_x\pi(\xi(x))$, so for $\xi \in \mathfrak{X}(C)^{\mathbb{R}_+}$ we find a unique vector field $\pi_*(\xi) \in \mathfrak{X}(S^n)$ such that $T\pi \circ \xi = \pi_*(\xi) \circ \pi$. Hence invariant vector fields are among the projectable fields $\mathfrak{X}_\pi(C)$, and we get a linear map $\pi_* : \mathfrak{X}(C)^{\mathbb{R}_+} \to \mathfrak{X}(S^n)$. It is well known that $\mathfrak{X}(C)^{\mathbb{R}_+}$ is a Lie subalgebra of $\mathfrak{X}(C)$ and that π_* is a homomorphism of Lie algebras. Moreover, for any smooth function $f : S^n \to \mathbb{R}$ and $\xi \in \mathfrak{X}_\pi(C)$ we have $(f \circ \pi)\xi \in \mathfrak{X}_\pi(C)$ and $\pi_*((f \circ \pi)\xi) = f\pi_*(\xi)$.

The mapping π_* is not injective, and indeed we can easily describe its kernel. Let E be the Euler vector field on C, i.e. the inclusion $C \hookrightarrow \mathbb{R}^{n+1}$, viewed as a vector field on C. Alternatively, E is the fundamental vector field on the principal $\mathbb{R}_+$–bundle $\pi : C \to S^n$ corresponding to the element 1 in the Lie algebra $\mathbb{R}$ of $\mathbb{R}_+$. Visibly, $E \in \mathfrak{X}(C)^{\mathbb{R}_+}$ and $\pi_*(E) = 0$. On the other hand, if $\xi \in \mathfrak{X}(C)^{\mathbb{R}_+}$ is such that

$\pi_*(\xi) = 0$, then for any $x \in C$, the tangent vector $\xi(x)$ must be in the kernel of $T_x\pi$, and thus must be a multiple of $E(x)$. Hence, we get a smooth function $f : C \to \mathbb{R}$, such that $\xi = fE$. Since both ξ and E are invariant, it follows immediately that $f(tx) = f(x)$, so f comes from a smooth function on S^n.

Any principal connection on the principal $\mathbb{R}^+$–bundle $\pi : C \to S^n$ provides us with a horizontal lift of tangent vectors, which induces a splitting $\sigma : \mathfrak{X}(S^n) \to \mathfrak{X}(C)^{\mathbb{R}_+}$ of the projection π_* which is linear over smooth funnctions on S^n. Moreover, these principal connections are well known to form an affine space modelled on the space $\Omega^1(S^n)$ of one–forms on S^n. In terms of the splittings, this means that if σ and $\hat{\sigma}$ are the splittings corresponding to two principal connections, then there is a unique one–form Υ on S^n such that $\hat{\sigma}(\xi) = \sigma(\xi) + (\Upsilon(\xi) \circ \pi)E$. Alternatively, one can easily construct one splitting σ as above, e.g. from the splitting of π defined by including S^n as the unit–sphere and then verify directly that the difference of two such splittings must be the product of E with a function coming from S^n.

Suppose that we have chosen a splitting $\sigma : \mathfrak{X}(S^n) \to \mathfrak{X}(C)^{\mathbb{R}_+}$ as above. Then for $\xi, \eta \in \mathfrak{X}(S^n)$ consider $\nabla_{\sigma(\xi)}\sigma(\eta)$. Since any multiplication m_t is an affine map, this is again an invariant vector field, so we may define $\nabla^\sigma_\xi \eta := \pi_*(\nabla_{\sigma(\xi)}\sigma(\eta)) \in \mathfrak{X}(S^n)$. Using the fact that σ and π_* are linear over smooth functions on $\mathbb{R}^n$ one immediately verifies that for $f \in C^\infty(S^n, \mathbb{R})$ we have $\nabla^\sigma_{f\xi}\eta = f\nabla^\sigma_\xi \eta$ and $\nabla^\sigma_\xi(f\eta) = (\xi \cdot f)\eta + f\nabla^\sigma_\xi \eta$, so ∇^σ is a linear connection on TS^n. Moreover, we can also easily compute how this connection depends on σ. If $\hat{\sigma}$ is another section, we get $\Upsilon \in \Omega^1(S^n)$ such that $\hat{\sigma}(\xi) = \sigma(\xi) + (\Upsilon(\xi) \circ \pi)E$. Since any $\sigma(\xi)$ viewed as a function on C is homogeneous of degree one, we get $\nabla_E\sigma(\xi) = \sigma(\xi)$ and from the definition of E it follows that $\nabla_{\sigma(\xi)}E = \sigma(\xi)$. Using these two facts, as well as $\pi_*(E) = 0$ and the Leibnitz rule for ∇, one concludes easily that $\nabla^{\hat{\sigma}}_\xi \eta = \nabla^\sigma_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. But this means exactly that $\nabla^{\hat{\sigma}}$ is *projectively equivalent* to ∇^σ, and conversely, any connection which is projectively equivalent to ∇^σ is of the form $\nabla^{\hat{\sigma}}$. If we use the inclusion of S^n as the unit sphere to construct a splitting σ as above, then the connection obtained is just the Levi–Civita connection of the round sphere from 1.1.1, so the geometric structure invariant under the action of $SL(n + 1, \mathbb{R})$ is the projective class of the Levi–Civita connection.

There is a well known geometric interpretation of projective equivalence of connections: Namely, two connections ∇ and $\hat{\nabla}$ are projectively equivalent if and only if they have the same geodesics up to reparametrizations. This leads to the point of view that a projective manifold is a manifold together with a family of unparametrized curves emanating from each point. The geodesics on the round sphere are exactly the great circles, i.e. the images of planes through zero under the map π. Visibly, this family of curves is invariant under the action of $SL(n + 1, \mathbb{R})$, while the parametrization of the curves is not, so also this point of view reflects nicely in our homogeneous picture.

The geodesic lines come with a class of parametrizations. How to see them nicely?

It can be shown that any diffeomorphism of S^n preserving this standard projective structure comes from the action of $SL(n+1, \mathbb{R})$ using both lines of argument outlined in 1.1.1, but in both versions the proof is more difficult than in the Riemannian case. Concerning the reduction to fixed jets, it is easy to show that any one jet of a diffeomorphism in a fixed point x_0 of S^n can be realized by the action of an element of $SL(n + 1, \mathbb{R})$, so one may restrict to those diffeomorphisms whose one jet at x_0 coincides with the one jet of the identity. Further, one proves that any projective diffeomorphism is determined by its two–jet in one point, and for diffeomorphisms whose one–jet in a point coincides with the one–jet of the identity, there is an n–dimensional family of possible two–jets. Finally, one proves that also the action of $SL(n+1,\mathbb{R})$ produces an n–dimensional family of possible two–jets which implies the result. It is also possible to use the second line of argument, i.e. to show that the Maurer–Cartan form on $SL(n+1,\mathbb{R})$ is determined by the projective structure on S^n. This will follow from our later results on the existence of canonical Cartan connections for projective manifolds.

The curved analogs of the homogeneous space $S^n \cong SL(n+1,\mathbb{R})/P$ are exactly the n–dimensional projective manifolds, i.e. n–manifolds endowed with a projective equivalence class of torsion free connections.

1.1.4. The projective contact sphere. The idea to get a transitive action on S^n by viewing it as the space of rays in $\mathbb{R}^{n+1}$ clearly generalizes to certain subgoups of $SL(n+1,\mathbb{R})$. If we take an orthogonal group, then we return to the setting of 1.1.1, since then the imbedding of S^n as the unit sphere is equivariant, so we can view it as a subspace as well. Hence in the realm of simple groups the main remaining possibility is to consider the group $Sp(2n,\mathbb{R})$ of all linear automorphisms of $\mathbb{R}^{2n}$ which preserve a non–degenerate skew symmetric bilinear form ω, which we at the same time view as the standard symplectic form on $\mathbb{R}^{2n}$. So as in 1.1.3 above consider $C = \mathbb{R}^{2n} \setminus 0$ and the principal $\mathbb{R}_+$ bundle $\pi : C \to S^{2n-1}$. The action of $Sp(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$ restricts to C and factors to a transitive action on S^{2n-1}, and the isotropy group P of the line through e_1 again turns out to be a parabolic subgroup of $Sp(2n,\mathbb{R})$.

The geometric structure corresponding to this transitive action is significantly more complicated than a projective strucure, since it involves a nontrivial filtration on the tangent bundle. More precisely, this is an example of a so–called parabolic contact structure, i.e. a parabolic geometry which has an underlying contact structure.

Note first that the $\mathbb{R}_+$–principal bundle gives rise to a number of real line bundles over S^{2n-1}. Namely, for any real number α we have the representation $t \mapsto t^\alpha$ of $\mathbb{R}_+$ on $\mathbb{R}$, and we denote by $\mathcal{E}[\alpha]$ the associated bundle to C corresponding to this representation. Via the usual correspondence between sections of an associated bundle and functions on the total space of a principal bundle, smooth sections of $\mathcal{E}[\alpha]$ are in bijective correspondence with smooth functions $f : C \to \mathbb{R}$ such that $f(tx) = t^{-\alpha} f(x)$.

As in the projective case, consider the Lie algebra $\mathfrak{X}(C)^{\mathbb{R}_+}$ of invariant vector fields on C, and the map $\pi_* : \mathfrak{X}(C)^{\mathbb{R}_+} \to \mathfrak{X}(S^{2n-1})$. Considering sections of this map linear over $C^\infty(S^{2n-1},\mathbb{R})$, we know that if σ and $\hat\sigma$ are two such sections, then there is a one–form $\Upsilon \in \Omega^1(S^{2n-1})$ such that $\hat\sigma(\xi) = \sigma(\xi) + (\Upsilon(\xi)\circ\pi)E$, where E denotes the Euler vector field. In particular, since ω is skew, $\omega(\sigma(\xi), E)$ is a smooth function on C which is independent of the choice of σ. Moreover, since both $\sigma(\xi)$ and E are invariant, one immediately concludes that this function is homogeneous of degree 2, and thus corresponds to a smooth section $q(\xi)$ of $\mathcal{E}[-2]$. By construction, the map $\xi \mapsto q(\xi)$ is linear over $C^\infty(S^{2n-1},\mathbb{R})$ so it comes from a homomorphism $q : TS^{2n-1} \to \mathcal{E}[-2]$ of vector bundles, which is easily seen to be nonzero and thus surjective on each fiber. In particular, this implies that $H := \ker(q)$ is a smooth subbundle of TS^{2n-1} of rank $2(n-1)$.

Next, for sections ξ, η of the subbundle H consider the section $q([\xi,\eta])$ of $\mathcal{E}[-2]$ obtained by projecting the Lie bracket. This is immediately seen to be bilinear over smooth functions, and thus is induced by a skew symmetric bundle map $\mathcal{L} : H \times H \to \mathcal{E}[-2]$. By definition, $\mathcal{L}(\xi, \eta)$ corresponds to the function $\omega(X, E)$, where $X \in \mathfrak{X}(C)^{\mathbb{R}_+}$ is any lift of $[\xi, \eta]$ to an invariant vector field on C. In particular, we may take $X = [\sigma(\xi), \sigma(\eta)]$ for some section σ as above. But now since $\sigma(\xi)$ is invariant and E generates the $\mathbb{R}_+$–action on C, we get $[\sigma(\xi), E] = 0$ and by assumption ξ is a section of H, so $\omega(\sigma(\xi), E) = 0$. Expanding the usual formula for the exterior derivative we then see that $0 = d\omega(\sigma(\xi), \sigma(\eta), E)$ implies $\omega([\sigma(\xi), \sigma(\eta)], E) = E \cdot \omega(\sigma(\xi), \sigma(\eta))$, which in turn equals $2\omega(\sigma(\xi), \sigma(\eta))$ by homogeneity. In particular, this implies that $\mathcal{L}$ is non–degenerate, and thus H defines a *contact structure* on S^{2n-1}.

From the construction it is clear that the contact structure H is invariant under the action of $Sp(2n, \mathbb{R})$. However, it is well known that the space of contact diffeomorphisms of a contact manifold is infinite dimensional, so this contact structure cannot be enough to characterize the diffeomorphisms coming from the action of $Sp(2n, \mathbb{R})$, but we need an additional input. A simple way to get this additional input is to just observe that $Sp(2n, \mathbb{R}) \subset SL(2n, \mathbb{R})$ so the projective structure from 1.1.3 on S^n is preserved by the action of $Sp(2n, \mathbb{R})$. It is then easy to verify that the diffeomorphisms coming from $Sp(2n, \mathbb{R})$ are exactly those, which preserve both the contact structure and the projective class of the Levi–Civita connection of the round metric. However, while this is a characterization, it is very misleading, in particular if one thinks about curved analogs of this homogeneous space. The point about this is that the contact structure and the projective class are not well compatible, for example none of the connections in the projective class preserves the subbundle H. To get a good description of the structure, we have to use similar arguments as in 1.1.3 to construct an analog of a projective strucutre which is compatible with the contact structure.

To do this, we first observe that any splitting $\sigma : \mathfrak{X}(S^n) \to \mathfrak{X}(C)^{\mathbb{R}^+}$ provides us also with a splitting of the sequence $0 \to H \to TS^{2n-1} \to \mathcal{E}[-2] \to 0$. To see this, consider a point $x \in C$, the tangent space T_xC, the value $E(x)$ of the Euler field, and the complement $E(x)^\perp = \{\xi \in T_xC : \omega(\xi, E(x)) = 0\}$. Having given an arbitrary element $\eta \in T_xC$, then non–degeneracy of ω implies that there exists an element $\tilde{\eta} \in E(x)^\perp$ such that $\omega(\xi, \tilde{\eta}) = \omega(\xi, \eta)$ for all $\xi \in E(x)^\perp$ and that this element is unique up to adding a multiple of $E(x)$. In particular, applying this construction (in each point) to a vector field of the form $\sigma(\eta)$ for $\eta \in \mathfrak{X}(S^{2n-1})$, we get a projectable vector field, whose projection $p_\sigma(\eta) \in \mathfrak{X}(S^{2n-1})$ is independent of the choices (apart from the choice of σ) and by construction $p_\sigma(\eta)$ is a section of the subbundle H. It is easy to verify that p_σ is linear over smooth functions, and thus a splitting $TS^{2n-1} \to H$ of the inclusion of H. Notice that by construction we have $\omega(\sigma(\xi), \sigma(p_\sigma(\eta))) = \omega(\sigma(\xi), \sigma(\eta))$ for all sections ξ of H and all $\eta \in \mathfrak{X}(S^{2n-1})$. Moreover, from the construction it follows easily that the projection p_σ depends only on the restriction of σ to the space $\Gamma(H)$ of sections of H.

Now we are ready to describe the main structure, namely a class of partial connections ∇^σ on H, which are an affine space modelled on the space of sections of the dual bundle $H^* = L(H, \mathbb{R})$. That is, any ∇^σ should be a bilinear operator $\Gamma(H) \times \Gamma(H) \to \Gamma(H)$, $(\xi, \eta) \mapsto \nabla^\sigma_\xi \eta$ which is linear over $C^\infty(S^{2n-1}, \mathbb{R})$ in the first variable and satisfies the Leibnitz rule in the second variable (which makes sense since we deal with a subbundle of the tangent bundle). So this is like a linear connection except for the fact that we may only differentiate in H–directions. For

$\xi, \eta \in \Gamma(H)$ we define $\nabla^\sigma_\xi \eta := p_\sigma(\pi_*(\nabla_{\sigma(\xi)}\sigma(\eta)))$, where ∇ is the flat connection on C. One immediatley verifies that this is a partial connection, and by construction it depends only on the restriction of σ to $\Gamma(H)$. Moreover, as in 1.1.3 for two such restrictions there is a section Υ of the bundle H^* such that $\hat{\sigma}(\xi) = \sigma(\xi) + (\Upsilon(\xi)\circ\pi)E$, which easily implies that $\nabla^{\hat{\sigma}}_\xi \eta = \nabla^\sigma_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. Thus, we have constructed a *projective class of partial connections* on S^{2n-1}.

One can verify by direct computations that this projective class has two further properties: On one hand, the induced partial connection on the second exterior power $\Lambda^2 H$ leaves the subbundle $\ker(\mathcal{L})$ invariant. (This essentially follows from the fact that ω is covariantly constant for the flat connection on C.) This also implies that the projection p_σ can be recovered from the partial connection ∇^σ, which we will discuss in more detail in **??**. On the other hand, the connections in the projective class are torsion free in the sense that $\nabla^\sigma_\xi \eta - \nabla^\sigma_\eta \xi = p_\sigma([\xi, \eta])$, which immediately follows from torsion–freeness of ∇.

It can then be verified that the diffeomorphisms coming from the action of $Sp(2n, \mathbb{R})$ are exactly those, which preserve the contact structure H as well as the projective class $[\nabla^\sigma]$ of partial connections on H. For us, this result will be a special case of the theorem on the existnce of a canonical Cartan connection for projective contact structures.

The curved analogs of S^{2n-1}, viewed as a homogeneous space of $Sp(2n, \mathbb{R})$ are the so–called projective contact structures on smooth manifolds of dimension $2n-1$. Such a structure on a manifold M consists of a smooth subbundle $H \subset TM$ of rank $2(n-1)$ such that the algebraic operator $\mathcal{L} : H \times H \to TM/H$ induced by the Lie bracket is non–degenerate and a projective class of partial connections ∇ on H. On this projective class, there are two requirements: On one hand, for any connection in the projective class the induced connection on $\Lambda^2 H$ has to preserve the subbundle $\ker(\mathcal{L})$. Using this, one shows that the connection gives rise to a splitting $TM \to H$ of the inclusion of H, which then allows us to require that the connections are torsion free in the sense stated above.

1.1.5. The conformal sphere. Projective structures and projective contact structures are rather untypical examples of parabolic geometries, since they involve the choice of a class of connections. In fact, essentially these are the only two examples in which such a choice is required, while all other parabolic geometries are determined by a first order G–structure or even a weakening of such a structure. Our next example is certainly the best studied example of a parabolic geometry, namely conformal structures. This example will be kept in mind as a guiding line throughout the book.

Consider $\mathbb{R}^{n+2}$ endowed with an inner product $\langle\ ,\ \rangle$ of signature $(n+1, 1)$, and the "light cone" C of nonzero null–vectors, i.e. $C = \{v \in \mathbb{R}^{n+2} : v \neq 0, \langle v, v\rangle = 0\}$. Taking the standard inner product of this signature, C contains of all nonzero solutions of the equation $x_{n+2}^2 = x_1^2 + \cdots + x_{n+1}^2$. In particular, for any element of C the last coordinate is nonzero. As in 1.1.3 consider the space of lines in C, i.e. the quotient of $C/\sim$ where two points in C are equivalent if and only if they are nonzero multiples of each other. Again, the natural projection $\pi : C \to C/\sim$ is a principal bundle with structure group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Similar as in 1.1.4 the representation $t \mapsto |t|^\alpha$ gives rise to a real line bundle $\mathcal{E}[\alpha] \to C/\sim$ for any real number α.

Viewing S^n as the unit sphere in the subspace $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$, the map $x \mapsto (x, 1)$ includes S^n into C and thus induces a smooth map to the quotient space $C/\sim$. Conversely, for any element of C the last coordinate is nonzero, and hence for $v = (v_1, \dots, v_{n+2}) \in C$, we may form $\frac{1}{v_{n+2}}(v_1, \dots, v_{n+1})$ which lies in $S^n \subset \mathbb{R}^{n+1}$ since $v \in C$. Thus we get a smooth map $C \to S^n$ which visibly factors to an inverse $C/\sim \to S^n$ of the map $S^n \to C/\sim$ from above, and so we may view S^n as the space of null lines in $(\mathbb{R}^{n+2}, \langle\ ,\ \rangle)$.

But this immediately suggests yet another way to make S^n into a homogeneous space. Namely, consider the orthogonal group $O(n+1,1)$ of the inner product $\langle\ ,\ \rangle$. This acts on $\mathbb{R}^{n+2}$ and by definitions leaves the null–cone C invariant. Since it acts linearly, this action factors to a smooth action on $S^n \cong C/\sim$. Elementary arguments show that this action is transitive and thus induces a diffeomorphism $S^n \cong O(n+1,1)/P$, where P is the isotropy subgroup of some point of S^n. The subgroup P is usually called the *Poincaré conformal group*. We will describe this group in more details later on, see **??**. For now let us just note that P is the semidirect product of the conformal group $CO(n)$ with an n–dimensional vector group and that P is a parabolic subgroup of $O(n+1,1)$.

To analyze the geometric meaning of this transitive action, let us determine the tangent spaces of S^n. Denote by $\pi : C \to S^n$ the natural projection. For an element $v \in C$, the tangent space to C at v clearly is given by $T_vC = \{w \in \mathbb{R}^{n+2} : \langle v, w\rangle = 0\}$. Since v by definition is a null vector the line $\ell_v = \{tv : t \in \mathbb{R}\}$ is contained in T_vC. The projection π induces a linear isomorphism $T_vC/\ker(T_v\pi) \to T_{\pi(v)}S^n$, and clearly the kernel $\ker(T_v\pi)$ is exactly $\ell_v \subset T_vC$. But this now means that for a point $x \in S^n$ corresponding to a line $\ell \setminus \{0\} \subset C$ choosing any element of $\ell \setminus \{0\}$ induces a linear isomorphism $T_xS^n \cong \ell^\perp/\ell$, where $\ell^\perp$ denotes the orthogonal complement of ℓ, which contains ℓ since ℓ is null. Again since ℓ is null, the inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^{n+2}$ induces a well defined positive definit inner product on $\ell^\perp/\ell$. But the dependence of the linear isomorphism $T_xS^n \cong \ell^\perp/\ell$ on the choice of the preimage of x in C is easy to see: From the fact that $\pi \circ m_\lambda = \pi$, where m_λ denotes the multiplication by $\lambda \in \mathbb{R}^*$ it follows immediately that the isomorphism $T_xS^n \cong \ell^\perp/\ell$ depends on the choice of the preimage only by a scalar multiple. In particular, this changes the induced inner product on T_xS^n by multiplication with the square of this number. Consequently, on each tangent space we get a positive definite inner product up to positive multiples, and thus a conformal structure on S^n, which visibly is invariant under the action of $O(n+1,1)$.

There is another quite usual description of a conformal structure which also can be seen nicely in this picture. As in 1.1.3 we may lift vector fields on S^n to $\mathbb{R}^*$–invariant vector fields on C and such a lift is unique up to adding a function times the restriction of the Euler vector field. But since C consists of null vectors and T_vC consists of vectors perpendicular to v, for the lifts $\sigma(\xi)$ and $\sigma(\eta)$ of two vector fields $\xi, \eta \in \mathfrak{X}(S^n)$ the function $\langle\sigma(\xi), \sigma(\eta)\rangle : C \to \mathbb{R}$ is idenpendent of the choice of the lifts and homogeneous of degree two, thus defining a section of the bundle $\mathcal{E}[-2]$. This means that we get a canonical metric with values in the line bundle $\mathcal{E}[-2]$, which is a well known description of a conformal structure.

Again, both lines of argument outlined in 1.1.1 can be used to show that a diffeomorphism preserving the standard conformal structure on S^n is given by the action of an element of $O(n+1,1)$, and we will prove this later as a consequence of the existence of a unique normal Cartan connection on a conformal manifold.

Clearly, one may also replace $O(n+1,1)$ by $SO(n+1,1)$ or $Spin(n+1,1)$ or the respective groups in signature $(p+1,q+1)$ to get similar variations as in 1.1.2. However, there is no analog to the flat Euclidean space, i.e. conformal geometry does not allow a homogeneous model isomorphic to $\mathbb{R}^n$, since on $\mathbb{R}^n$ there are local conformal transformations which do not globalize, and this is impossible for the flat model of a Cartan geometry, see **??**.

1.1.6. The CR–sphere. Our final example of viewing spheres as homogeneous spaces is the strictly pseudoconvex CR–structure on odd dimensional spheres, which certainly is the most complicated of the examples we discuss. The basis for this example is a Hermitian analog of the construction of the conformal sphere in 1.1.5 above. As the projective contact sphere from 1.1.4, the CR–sphere is an example of a parabolic contact structure.

Consider $\mathbb{C}^{n+2}$ with a Hermitian inner product $\langle\ ,\ \rangle$ of signature $(n+1,1)$. Let C be the cone of nonzero nullvectors, which is a submanifold of real codimension one, and consider the quotient $C/\sim$ by the action of $\mathbb{C}^* = \mathbb{C}\setminus 0$. Hence $\pi : C \to C/\sim$ is a principal bundle with structure group $\mathbb{C}^*$. There are various real and complex line bundles associated to this principal bundle, but we will only need the real line bundles $\mathcal{E}[\alpha,\alpha]$ for $\alpha\in\mathbb{R}$ corresponding to the representation $z\mapsto |z|^{2\alpha} = z^\alpha\bar{z}^\alpha$ of $\mathbb{C}^*$ on $\mathbb{R}$. Similarly as in 1.1.5 above one easily verifies directly that $C/\sim$ is diffeomorphic to the sphere S^{2n+1}. The standard action of $SU(n+1,1)$ on $\mathbb{C}^{n+2}$ then leaves the null cone C invariant and thus factors to an action on S^{2n+1} which can be easily shown directly to be transitive. The stabilizer of a point in S^{2n+1} is a parabolic subgroup P of $SU(n+1,1)$.

While the analysis of tangent spaces is quite similar to the conformal case, there are some significant differences. For $v\in C$, the tangent space T_vC is now given by $\{w\in\mathbb{C}^{n+2} : \mathrm{Re}(\langle v,w\rangle)=0\}$, so this is no more independent of the choice of a point v lying over a fixed point $x\in S^{2n+1}$. On the other hand, the projection $\pi : C\to S^{2n+1}$ again induces a linear isomorhism $T_{\pi(v)}S^{2n+1}\cong T_vC/\{zv : z\in\mathbb{C}\}$, so this time we clearly have to factor out the complex line through v. As before, we may lift vector fields on S^{2n+1} to $\mathbb{C}^*$–invariant vector fields on C and such lifts are unique up to multiples of the Euler vector field by a complex valued function coming from the sphere. Since we stay in the null–cone this means that the imaginary part $\mathrm{Im}(\langle v,\sigma(\xi)(v)\rangle)$ is a real valued function on C which is independent of the choice of the lift and homogeneous, so it corresponds to a section of $\mathcal{E}[-1,-1]$. Similarly as in 1.1.4 this gives us a surjective bundle map $TS^{2n+1}\to\mathcal{E}[-1,-1]$, whose kernel $H\subset TS^{2n+1}$ is a smooth subbundle of rank $2n$, which by construction is invariant under the action of $SU(n+1,1)$.

The subbundle H is easy to describe explicitly: For each $v\in C$ the fiber $H_{\pi(v)}$ of H in the point $\pi(v)$ is exactly the image under $T\pi$ of the subspace $\{w\in\mathbb{C}^{n+2} : \langle v,w\rangle = 0\}$. Since applying $T\pi$ amounts to factoring out a complex subspace, this endows $H_{\pi(v)}$ with a complex structure, which visibly is independent of the choice of v, so we get an almost complex structure J on the subbundle H, making it into a rank n–complex subbundle. This means that the subbundle H defines an *almost–CR structure* (of CR–dimension n and codimension one) on S^{2n+1}. $H = TS^{2n+1}\cap JTS^{2n+1}$ should also be remarked

The subbundle H gives rise to a skew symmetric bundle map $\mathcal{L} : H\times H\to\mathcal{E}[-1,-1]$ induced by the Lie bracket of vector fields. Up to a multiple, $\mathcal{L}$ is the imaginary part of the Levi–form of the almost–CR structure H. Similarly as in 1.1.4 one computes directly that $\mathcal{L}(\xi,\eta)$ is represented by the function $\mathrm{Im}(\langle\sigma(\xi),\sigma(\eta)\rangle)$

for any lifting map $\sigma : \mathfrak{X}(S^{2n+1}) \to \mathfrak{X}(C)^{\mathbb{C}^*}$ which is linear over smooth functions on S^{2n+1}. This is the imaginary part of a positive definit Hermitian form, so H defines a contact structure on S^{2n+1}, the almost–CR structure is strictly pseudoconvex and $\mathcal{L}$ is compatible with the almost complex structure on H in the sense that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$, which means that the almost–CR structure is partially integrable. In fact, the embedding of C into $\mathbb{C}^{n+2} \setminus 0$ induces an embedding of S^{2n+1} into the complex projective space $\mathbb{C}P^{n+1}$ under which the subspace H_x maps to the maximal complex subspace of $T_xS^{2n+1} \subset T_x\mathbb{C}P^{n+1}$. Thus, our almost–CR structure is integrable, so it is a CR structure, but this is not important for our purposes.

As in the examples before it follows from the existence of a canonical Cartan connection for CR structures that any CR–diffeomorphism, i.e. any contact diffeomorphism such that the restriction of its tangent map to the subbundle H is complex linear, is given by the action of an element of $SU(n+1, 1)$. The curved analogs of this homogeneous space are all strictly pseudoconvex partially integrable almost–CR manifolds of dimension $2n+1$, i.e. manifolds equipped with a rank n complex subbundle H in their tangent bundle such that the corresponding bundle map $\mathcal{L} : H \times H \to TM/H$ is the imainary part of a positive definit Hermitain form in appropriate coordinates and compatible with the almost complex structure on H in the sense stated above.

1.2. Some background from differential geometry

In this section, we review some basic facts on differential geometry and analysis on manifolds which will be necessary for the further development. Our main purpose here is to fix the notation and conventions used in the sequel, as well as to give a more detailed listing of prerequisites for the further text. The basic reference for this section is [**KMS**]. At the same time, we stress the basic concepts of frames, natural budles, and the role of the symmetry groups in the properties of geometric objects. This will remain one of the main features of our exposition in the entire book.

1.2.1. Smooth manifolds. Unless otherwise stated, all manifolds we consider are finite dimensional and second countable and we assume that all connected components have the same dimension. Any manifold comes equipped with a maximal *atlas*, i.e. a maximal collection of open subsets $U_\alpha \subset M$ together with homeomorphisms $u_\alpha : U_\alpha \to u_\alpha(U_\alpha)$ onto open subsets of $\mathbb{R}^n$, such that for all α, β with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the subset $u_\alpha(U_{\alpha\beta})$ is open and the composition $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1}; u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ is smooth (C^∞). A *chart* on M is any element (U_α, u_α) of this maximal atlas. Such a chart gives rise to local coordinates $u_\alpha^i : U_\alpha \to \mathbb{R}$ on M. A map $f : M \to N$ between smooth manifolds is smooth, if and only if its expression in one (or equivalently any) local coordinate system around any point in M is smooth. In particular, we may consider the space $C^\infty(M, \mathbb{R})$ of smooth real valued functions on M, which forms an algebra under the pointwise operations. For $f \in C^\infty(M, \mathbb{R})$ the *support* $\operatorname{supp}(f)$ of f is the closure of the set of all $x \in M$ such that $f(x) \neq 0$. The concept of support generalizes in an obvious way to smooth functions with values in a vector space and to smooth sections of vector bundles, see 1.2.5 below.

A fundamental result is that any open covering of a smooth manifold M admits a subordinate *partition of unity*. This means that if $\{V_i : i \in I\}$ is a family of open subsets of M such that $M = \cup_{i\in I} V_i$, then there exits a family f_α of smooth functions on M with values in $[0,1] \subset \mathbb{R}$ such that for any α there exists an $i \in I$ with $\operatorname{supp}(f_\alpha) \subset V_i$, the sets $\operatorname{supp}(f_\alpha)$ still cover M, any point $x \in M$ has a neighborhood which meets only finitely many of the sets $\operatorname{supp}(f_\alpha)$, and $\sum_\alpha f_\alpha(x) = 1$.

For a point $x \in M$, the *tangent space* T_xM to M in the point x is defined to be the space of all linear maps $\xi_x : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ which are derivations at x, i.e. which satisfy the Leibnitz rule $\xi_x(\varphi\psi) = \xi_x(\varphi)\psi(x) + \varphi(x)\xi_x(\psi)$. These derivations form a vector space whose dimension equals the dimension of the manifold. If $f : M \to N$ is a smooth map between smooth manifolds, then for $x \in M$ and $\xi_x \in T_xM$, we define $T_xf \cdot \xi : C^\infty(N,\mathbb{R}) \to \mathbb{R}$ by $(T_xf \cdot \xi_x)(\varphi) := \xi_x(\varphi \circ f)$. On immediately verifies that $T_xf \cdot \xi_x \in T_{f(x)}N$ and this defines a linear map $T_xf : T_xM \to T_{f(x)}N$, the *tangent map* of f at x. A simple way how to define a tangent vector in T_xM is to consider a parameterized curve $c(t)$ with $c(0) = x$ and to define the derivative ξ_x of functions f by the usual derivative of $f \circ c$ at 0. We write $\xi_x = \frac{\partial}{\partial t}_{|0} c$ and all tangent vectors are available in this way.

The union TM of all tangent spaces is called the *tangent bundle* of the manifold M and for each smooth map $f : M \to N$ we get the *tangent map* $Tf : TM \to TN$ of f by putting together the tangent maps at the individual points of M. The tangent bundle is endowed with the unique smooth structure such that the obvious projection $p : TM \to M$ and all tangent maps Tf become smooth maps. In this picture, the chain rule just states that T is a covariant functor on the category $\mathcal{M}$ of smooth manifolds, i.e. $T(g \circ f) = Tg \circ Tf$. The individual tangent spaces carry the natural vector space structure of the derivatives and the tangent bundles TM are locally isomorphic to $U \times \mathbb{R}^m$, $m = \dim M$, $U \subset M$, around each point in M. Thus, the values of the tangent functor are vector bundles $TM \to M$ and their linear morphism, see 1.2.6 below for the terminology for bundles.

A *vector field* is a section $\xi : M \to TM$ of the projection p, i.e. an assignment of a derivation $\xi(x) \in T_xM$ to each point $x \in M$. Thus ξ is a derivation $C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$. The space of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$. The commutator of two vector fields (derivations) ξ, $\eta \in \mathfrak{X}(M)$ is again a vector field called the *Lie bracket* $[\xi,\eta]$. The action of the functor T on invertible smooth maps $f : M \to N$ clearly induces the covariant action on vector fields which we denote by $\xi \mapsto f_*\xi$, $(f_*\xi)(y) = Tf(\xi(f^{-1}(y)))$. There is also the contravariant analog, $(f^*\xi)(x) = Tf^{-1}(\xi(f(x)))$.

For each given vector field ξ on M, we may ask for a one–parametric family of diffeomorphisms φ_t, $t \in \mathbb{R}$, such that $\varphi_0 = \mathrm{id}_M$ and $\frac{\partial}{\partial t}_{|0}\varphi_t(x) = \xi(x)$ for all x. In coordinates, this requirement represents a system of ordinary differential equations in $(\dim M + 1)$ variables x and t, with the fixed initial conditions $\varphi_0(x) = x$, and its (maximal) solution $(t,x) \mapsto \varphi_t(x)$ always exists. We call it the *flow* of the vector field ξ and we write Fl^ξ_t for φ_t. A vector field is called *complete* if its flow exists globally for all $t \in \mathbb{R}$. Let us notice the obvious relation between the action of T and the flows: $\mathrm{Fl}^{\varphi_*\xi}_t \circ\varphi = \varphi \circ \mathrm{Fl}^\xi_t$ for all diffeomorphisms φ.

The dual bundles to $TM \to M$ are the cotangent bundles $T^*M \to M$, i.e. their elements are the first differentials of smooth functions at the individual points of M. In contrast to the tangent functor, T^* is naturally a contravariant functor, but we may also view T^* as a covariant functor acting only on the (locally) invertible smooth maps $f : M \to N$, $T^*f(\alpha)(\xi) = \alpha(Tf^{-1}\xi)$ for all $\alpha \in T^*_xM$, $\xi \in T_{f(x)}N$.

The sections of the obvious projection $\pi : T^*M \to M$ are called *one–forms*. A simple way how to define a one–form α is to consider the differential df of a real function f at each point $x \in M$. We write $\Omega^1(M) = \Omega^1(M;\mathbb{R})$ for the space of all smooth real one–forms on M. If f has values in any other space V which allows to consider the differential (e.g. any finite dimensional vector space) then df is a V–valued one–form and we write $df \in \Omega^1(M;V)$. Of course, not all one–forms on M are achieved as differentials of functions. The latter ones generate them only locally.

The antisymmetric k–linear forms $\alpha : \Lambda^k T_xM \to \mathbb{R}$ are the elements of the kth exterior power $\Lambda^k T^*M$ of the cotangent bundle and the sections of the obvious projection are called *k–forms* on M. Similarly to one–forms, there is the contravariant action of all smooth mappings $f : M \to N$ on forms, $f^*\alpha(\xi_1,\dots,\xi_k) = \alpha(f_*\xi_1,\dots,f_*\xi_k)$. In particular, the 0–forms are just smooth functions h on M and f^*h is the composition $h\circ f$. The differential of functions, $d : C^\infty(M) \to \Omega^1(M)$, is a special case of the *exterior differential*. In general, the exterior differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is given for all $\xi_i \in \mathfrak{X}(M)$ by the formula

$$d\omega(\xi_0,\dots,\xi_k) = \sum_{i=0}^{k}(-1)^i\xi_i(\omega(\xi_0,\overset{i}{\dots},\xi_k)) + \sum_{i<j}^{k}(-1)^{i+j}\omega([\xi_i,\xi_j],\xi_0,\overset{i,j}{\dots},\xi_k)$$

where the symbols i or i,j over the dots mean skipping of the indicated arguments. The same formula applies for any V–valued form in $\Omega^k(M,V)$.

The exterior differential d is the only linear differential operator which is invariantly defined on all manifolds, see [**KMS**, Theorem 34.2]. Here the invariance means the commuting with the action of local diffeomorphisms, i.e. $\varphi^*(d\omega) = d(\varphi^*\omega)$. One of the aims of this book is to develop general tools for the study of such basic operators in the realm of more specific geometric structures on manifolds.

1.2.2. Distributions and foliations. A *distribution* $\mathcal{D}$ on a manifold M is a subset $\mathcal{D} \subset TM$ such that for each $x \in M$ the subset $\mathcal{D}_x = \mathcal{D} \cap T_xM$ is a vector subspace in T_xM. By elementary linear algebra, each distribution can be defined as the kernel of a one–form ω valued in a suitable vector space V. The dimension of V is at least $\dim M - \max_{x\in M}\{\dim \mathcal{D}_x\}$. The distribution $\mathcal{D}$ is said to be of *constant rank k* if $\dim \mathcal{D}_x = k$ is constant, and $\mathcal{D}$ is *smooth* if it is defined by a smooth form ω. A *regular distribution* is of constant rank and smooth. An *integral manifold N* of a distribution $\mathcal{D}$ is an immersed submanifold such that all tangent directions in $TN \subset TM$ are in $\mathcal{D}$. A *maximal integral submanifold* is an integral manifold whose dimension is equal to the rank of $\mathcal{D}$ at all its points. A distribution $\mathcal{D}$ is called *integrable* if there is a maximal integral submanifold through each point $x \in M$.

Each vector field ξ on M spans a one–dimensional distribution whose maximal integral submanifolds are the (unparametrized) flow lines $\mathrm{Fl}^\xi_t(x)$. In general, there are no maximal integral submanifolds and the obstruction to their existence is given by the Lie bracket of vector fields lying in $\mathcal{D}$. The distribution $\mathcal{D}$ is called *involutive* if for all vector fields ξ and $\eta \in \mathfrak{X}(M)$ with $\xi(x) \in \mathcal{D}$, $\eta(x) \in \mathcal{D}$ for all $x \in M$, also $[\xi,\eta](x) \in \mathcal{D}$ for all $x \in M$. The following proposition (called often Frobenius theorem) appears in all standard textbooks on differential geometry. For its general version on distributions of non–constant ranks see e.g. [**KMS**, section 3].

PROPOSITION. *A regular distribution $\mathcal{D} \subset TM$ is integrable if and only if it is involutive.*

An integrable regular distribution of rank k defines the *regular foliation*, i.e. the expression of M as the disjoint union of k–dimensional immersed submanifolds called *leaves* of the foliation. Of course, the leaves are just the (maximal) maximal integral submanifolds of the distributions. On the other hand, smooth foliations define the asociated distributions which are integrable.

An immediate and very useful consequence of the latter proposition and the above formula for the exterior differential reformulates the involutivity of $\mathcal{D}$ in terms of the defining one–form ω.

COROLLARY. *Let $\omega \in \Omega^1(M;V)$ be a smooth V–valued one–form and assume that the dimension of $\ker\omega(x)$ is constant for all $x \in M$. Then the distribution $\mathcal{D} = \ker\omega$ is integrable if and only if $d\omega(x)(X,Y) = 0$ for all X, $Y \in \mathcal{D}_x$.*

Equivalently, the condition from the corollary can be reformulated as follows: $\mathcal{D}$ is the intersection of the kernels of $\dim V$ forms ω_i and all $d\omega_i$ belong to the ideal generated by the forms ω_i in the entire algebra of exterior forms.

Locally, the leaves of each regular foliation of rank k in an n–dimensional manifold look like the k–planes $\mathbb{R}^k$ in $\mathbb{R}^{k+m}$, $k+m = n$. However, globally the topological structure of the leaves may be very complicated.

1.2.3. Lie groups and their Lie algebras. Groups will appear mostly as sets of actual or potential symmetries of the geometries we shall deal with. On the general manifolds, we have seen the groups of all diffeomorphisms $\varphi : M \to M$. These are examples of infinite dimensional Lie groups and the flow of a complete vector field is a one–dimensional subgroup. If a geometric structure is given on M, e.g. a fixed Riemannian metric, then the subgroup of all diffeomorphisms leaving this structure invariant is often a finite dimensional manifold. This will be the typical case in this book.

A *Lie group* G is a smooth manifold endowed with a smooth mapping $\mu : G \times G \to G$, the *multiplication*, which defines a group structure on G. Using the implicit function theorem one shows that then also the inversion mapping $\nu : G \to G$ is smooth. Given an element $g \in G$, we define the *left translation* $\lambda_g : G \to G$ by $\lambda_g(h) = \mu(g,h) = gh$, and the *right translation* $\rho^g : G \to G$ by $\rho^g(h) = hg$. Both λ_g and ρ^g are diffeomorphisms of G with inverses $\lambda_{g^{-1}}$ and $\rho^{g^{-1}}$, respectively. Further, one clearly has $\lambda_g \circ \lambda_h = \lambda_{gh}$ and $\rho^g \circ \rho^h = \rho^{hg}$, which also explains the use of sub– and superscripts.

Let G be a Lie group and let $\xi \in \mathfrak{X}(G)$ be a smooth vector field on G. Then ξ is called *left invariant*, if and only if $(\lambda_g)^*\xi = \xi$ for all $g \in G$, or equivalently $\xi(gh) = T_h\lambda_g \cdot \xi(h)$ for all $g,h \in G$. The last equation shows that any left invariant vector field ξ is uniquely determined by its value $\xi(e) \in T_eG$ in the unit element e of G. Conversely, it is easy to see that any $X \in T_eG$ extends to a left invariant vector field L_X on G. Consequently, there is a linear isomorphism between the space $\mathfrak{X}_L(G)$ of left invariant vector fields on G and the tangent space T_eG of G at the unit element. Since the pullback along a diffeomorphism is compatible with the Lie bracket of vector fields, the subspace $\mathfrak{X}_L(G)$ is a Lie subalgebra of the Lie algebra of all vector fields on G. Via the linear isomorphism, this gives rise to a Lie bracket on the tangent space T_eG, which is explicitly given by $L_{[X,Y]} = [L_X, L_Y]$ for $X,Y \in T_eG$. The space T_eG together with this Lie bracket is called the *Lie algebra* $\mathfrak{g}$ of the Lie group G.

Similarly as for left invariant vector fields, one has the Lie subalgebra $\mathfrak{X}_R(G)$ of *right invariant* vector fields on G. Again, any right invariant vector field on G is uniquely determined by its value in e, and any $X \in \mathfrak{g}$ extends uniquely to $R_X \in$

$\mathfrak{X}_R(G)$. It is easy to see that $R_X = \nu^*(L_{-X})$, where ν is the inversion on G, which in particular implies that $[R_X, R_Y] = R_{-[X,Y]}$. Another basic result is that right invariant vector fields commute with left invariant vector fields, i.e. $[L_X, R_Y] = 0$ for all $X, Y \in \mathfrak{g}$.

It follows from the construction of the Lie algebra of a Lie group that if G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and $\varphi : G \to H$ is a *homomorphism*, i.e. a smooth map compatible with the group structures, then the tangent map $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. For example, if G is a Lie group and $g \in G$ is any element, the conjugation $h \mapsto ghg^{-1}$ by g defines an automorphism of G, so the tangent map at zero defines an automorphism $\mathrm{Ad}(g)$ of the Lie algebra $\mathfrak{g}$, called the *adjoint action* of g on $\mathfrak{g}$.

The simplest example of a Lie group is the group $GL(n, \mathbb{K})$ of linear automorphisms of $\mathbb{K}^n$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, which may be also viewed as the group of invertible $n \times n$–matrices with entries from $\mathbb{K}$. This is a smooth manifold, since it is an open subset of the vector space $M_n(\mathbb{K})$ of all $n \times n$–matrices with entries from $\mathbb{K}$, and clearly matrix multiplication is smooth. From this it follows that the Lie algebra $\mathfrak{gl}(n, \mathbb{K})$ of $GL(n, \mathbb{K})$ equals $M_n(\mathbb{K})$, and one easily shows that the adjoint representation is given by the conjugation of matrices, while the Lie bracket is given by the commutator of matrices.

If G is an arbitrary Lie group, then a real or complex (finite dimensional) *representation* of G is a homomorphism φ from G to some $GL(n, \mathbb{R})$ respectively $GL(n, \mathbb{C})$. For such a representation φ the tangent map φ' at $e \in G$ is a homomorphism $\varphi' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{K})$ of Lie algebras, i.e. a representation of $\mathfrak{g}$, called the *infinitesimal representation* corresponding to φ. Equivalently, one may describe a representation of G as a smooth map $\hat{\varphi} : G \times \mathbb{K}^n \to \mathbb{K}^n$ which is linear in the second argument and has the property that $\hat{\varphi}(gh, v) = \hat{\varphi}(g, \hat{\varphi}(h, v))$ for all $g, h \in G$ and $v \in \mathbb{K}^n$. Up to a choice of a basis, this is equivalent to a representation $G \to GL(V)$ for any real or complex vector space V, and more generally V may be even infinite dimensional. Similarly, the infinitesimal representations then have values in the space of all linear mappings $L(V, V)$. In this picture, if there is no risk of confusion, we will often omit the name of the representation and simply use the notation $(g, v) \mapsto g \cdot v$ or gv. The adjoint action associates to any element $g \in G$ an automorphism $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$, which in particular is a linear isomorphism, so this defines a map $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Since mapping g to the conjugation by g is a homomorphism, Ad is a group homomorphism and it is easy to see that it is smooth, so it defines a representation of G, the *adjoint representation*. The infinitesimal representation $\mathrm{ad} = \mathrm{Ad}' : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ turns out to be given by $\mathrm{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

For each element X in the Lie algebra $\mathfrak{g}$ of a Lie group G, we have the corresponding left invariant vector field L_X on G. The invariance of L_X easily implies that the vector field L_X is complete, i.e. that its flow $\mathrm{Fl}_t^{L_X}$ is defined for all times t. In particular, we thus may define the *exponential mapping* $\exp : \mathfrak{g} \to G$ by mapping $X \in \mathfrak{g}$ to the flow of L_X up to time $t = 1$ through the unit element e of G. One then easily verifies that $\exp : \mathfrak{g} \to G$ is a smooth map, whose tangent map at $0 \in \mathfrak{g}$ is the identity, so exp restricts to a diffeomorphism from an open neighborhood of $0 \in \mathfrak{g}$ to an open neighborhood of $e \in G$. Further, the flows of left invariant vector fields are given by $\mathrm{Fl}_t^{L_X}(g) = g \exp(tX)$ and for right invariant vector fields one gets $\mathrm{Fl}_t^{R_X}(g) = \exp(tX)g$. In particular, the flows through e of L_X and R_X coincide, and by the flow property, this defines a *one–parameter subgroup* of G, i.e. a smooth homomorphism from the additive group $\mathbb{R}$ to G. Conversely, any such one–parameter subgroup of G is determined by its derivative at e, so $t \mapsto \exp(tX)$ is the unique curve $\alpha : \mathbb{R} \to G$ such that $\alpha(t+s) = \alpha(t)\alpha(s)$ and such that $\alpha'(0) = X$. For the group $GL(n, \mathbb{K})$ of invertible $n \times n$–matrices, the exponential mapping is given by the usual exponential of matrices, i.e. $\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

If $\varphi : G \to H$ is a homomorphism of Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and exponential mappings $\exp^G$ and $\exp^H$, then the description of the exponential map as the solution of an ordinary differential equation easily implies that $\varphi \circ \exp^G = \exp^H \circ \varphi'$. In particular, the values of φ on the image of $\exp^G$ are completely determined by the Lie algebra homomorphism φ'. From the fact that the image of $\exp^G$ contains an open neighborhood of the unit $e \in G$ one next concludes that the subgroup generated by this image is exactly the connected component G_0 of e in G, so the restriction of φ to G_0 is determined by φ'. In particular, if G is connected, then any homomorphism from G to some Lie group is determined by its tangent map at e. In particular, we may apply this to representations of G: For any representation $\varphi : G \to GL(n, \mathbb{K})$ and any $X \in \mathfrak{g}$, we get $\varphi(\exp(X)) = e^{\varphi'(X)} = \sum_{n=0}^{\infty} \frac{1}{n!} \varphi'(X)^n$. In particular, $\varphi'(X) = \frac{\partial}{\partial t}_{|0} \varphi(\exp(tX))$. Moreover, if G is connected, then any representation is determined by the corresponding infinitesimal representation.

1.2.4. Maurer–Cartan form. The left invariant vector fields lead to a trivialization of the tangent bundle of any Lie group. More precisely, the mapping $G \times \mathfrak{g} \to TG$ which is given by $(g, X) \mapsto L_X(g)$ is an isomorphism of vector bundles. The inverse of this isomorphism can be conveniently encoded as a one–form $\omega \in \Omega^1(G; \mathfrak{g})$ on G with values in the Lie algebra $\mathfrak{g}$, which is defined by $\omega(g)(\xi) := T_g\lambda_{g^{-1}} \cdot \xi$. This one–form is called the (left) *Maurer–Cartan form* on G. From the definition of ω it is obvious that $\omega(L_X) = X$, $\lambda_g^* \omega = \omega$, and for each $g \in G$ the map $\omega(g) : T_gG \to \mathfrak{g}$ is a linear isomorphism. Moreover by definition, $(\rho^g)^*\omega(h)(\xi) = \omega(hg)(T\rho^g \cdot \xi) = T\lambda_{g^{-1}h^{-1}} T\rho^g \cdot \xi$. Since left multiplications commute with right multiplications and $\lambda_{g^{-1}} \circ \rho^g$ is the conjugation with g^{-1}, we conclude that this equals $\mathrm{Ad}(g^{-1})(\omega(h)(\xi))$, and thus we get $(\rho^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$. Finally, consider the exterior derivative $d\omega$ of the Maurer–Cartan form. Since ω is constant on left invariant vector fields, the formula for the exterior derivative in 1.2.1 implies $d\omega(L_X, L_Y) = -\omega([L_X, L_Y])$ for $X, Y \in \mathfrak{g}$, which equals $-[X, Y]$ by definition of the Lie bracket on $\mathfrak{g}$. Since in each point the values of left invariant vector fields make up the whole tangent space, this implies for all $\xi, \eta \in TG$ the *Maurer–Cartan equation*

$$0 = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

The Maurer–Cartan form leads to a notion of differentiating functions with values in a Lie group. Indeed, if G is a Lie group with Lie algebra $\mathfrak{g}$, M is an arbitrary smooth manifold and $f : M \to G$ is an arbitrary smooth function, then we define the *left logarithmic derivative* $\delta f : TM \to \mathfrak{g}$ by $\delta f(\xi_x) = \omega(f(x))(T_x f \cdot \xi_x)$, so this is just obtained by composing the tangent map $TM \to TG$ with the trivialization of TG provided by ω and projecting out the second component. Another way to express this is $\delta f(x) = f^*\omega(x) = T\lambda_{f(x)^{-1}} \circ T_x f$ and, in particular, the Maurer–Cartan form itself equals $\omega = \delta\,\mathrm{id}_G$.

Let us look at some examples. If G is the additive real line $G = (\mathbb{R}, +)$ with standard coordinates x then $\omega_G = dx \in \Omega^1(G, \mathbb{R})$ and the left logarithmic derivative is just the usual differential. The isomorphic example with the multiplicative positive real line $G = (\mathbb{R}^+, \cdot)$ leads to $\omega = \frac{1}{x}dx$ and the usual logarithmic derivative of real functions, $f \mapsto (f/f')dx$. For the general linear group G we compute easily $\omega_G \in \Omega^1(GL(n;\mathbb{R}), \mathfrak{gl}(n, \mathbb{R}))$ in the usual matrix component coordinates $g = (x_{ij}) \in GL(n, \mathbb{R})$, $\omega_G(g) = g^{-1}(dx_{ij})$. For example in dimension two

$$\omega_{GL(2,R)} = \begin{pmatrix} x_{22}/\det g & -x_{12}/\det g \\ -x_{21}/\det g & x_{11}/\det g \end{pmatrix} \begin{pmatrix} dx_{11} & dx_{12} \\ dx_{21} & dx_{22} \end{pmatrix}.$$

The Maurer–Cartan form provides the infinitessimal information on the multiplication while the Maurer–Cartan equation gives the only local obstruction to its integrability. The explicit local formulation is in the following theorem, see [**Sharpe, 1997**, Chapter 3] for more details

THEOREM. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and the Maurer–Cartan form ω_G and M be a connected manifold. For any smooth mappings f_1, $f_2 : M \to G$ satisfying $\delta f_1 = \delta f_2$, there is a unique element $c \in G$ (integration constant) such that $f_2(x) = c \cdot f_1(x)$ for all $x \in M$. Moreover, let ω be a $\mathfrak{g}$–valued one–form on M satisfying the structure equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$. Then, for each point $x \in M$, there is a neighborhood U of x and a smooth map $f : U \to G$ such that $\omega_{|U} = \delta f = f^*\omega_G$. If such an f exists then it is unique up to a left translation by an element $g \in G$.*

PROOF. A straightforward computation establishes the formulae for the actions of the multiplication μ and inversion ν on the Maurer–Cartan form ω_G, cf. [**Sharpe, 1997**, page 113]

$$\begin{aligned} \mu^*\omega_G|_{T_{g,h}(G\times G)} &= \operatorname{Ad}^{-1}(h)(\omega_G \circ T\pi_1) + \omega_G \circ T\pi_2 \\ i^*\omega_G|_{T_g G} &= -\operatorname{Ad}(g)\omega_G \end{aligned}$$

where π_i are the obvious projections $G \times G \to G$. Let us consider two functions f_1, $f_2 : M \to G$ and define $h(x) = f_2(x) \cdot (\nu \circ f_1)(x)$. Our first goal is to prove $h^*\omega_G = 0$ since then $\omega_G \circ Th$ is identically zero and this means that h is constant on connected M. Thus we compute with the help of the latter equalities

$$\begin{aligned} h^*(\omega_G)|_{T_x G} &= (f_2^*, f_1^* \circ \nu^*) \circ \mu^*(\omega_G)(x) \\ &= \operatorname{Ad}(f_1(x))(\omega_G \circ Tf_2) + (-\operatorname{Ad}(f_1(x))(\omega_G \circ Tf_1)) \\ &= \operatorname{Ad} f_1(x)(f_2^*\omega_G - f_1^*\omega_G) = 0. \end{aligned}$$

In order to prove the existence of $f : M \to G$ such that $\delta f = \omega$ for a prescribed $\mathfrak{g}$–valued one form ω on M, let us discuss its graph in $M \times G$. Thus, we want to construct an integrable distribution of the right dimension with a non–degenerate projection π_M onto M. Let us write $\Omega = \pi_M^*\omega - \pi_G^*\omega_G$, where $\pi_G : M \times G \to G$ is the other projection. We claim that the desired distribution is the kernel of Ω. Clearly $T\pi_M(\xi, \eta) = 0$ with ξ, $\eta \in \ker \Omega$ implies $\xi = 0$, and thus $\omega_G(\eta) = 0$, and so the projection $T\pi_M$ resticts to isomorphisms on the individual fibers of $\ker \Omega$. Now, each maximal integral submanifold of $\ker \Omega$ yields the local inverse $F : U \subset M \to L \subset M \times G$ and $F(x) = (x, f(x))$ for suitable $f : U \to G$. Clearly $F^*\Omega = 0$ and

$$\omega - f^*\omega_G = (\pi_M \circ F)^*\omega - (\pi_G \circ F)^*\omega_G = F^*\Omega = 0.$$

This is just the required equality $\delta f = \omega$ in all points in U.

It remains to show that such maximal integral submanifolds exist. A straightforward computation using $\pi_M^*(\omega) = \pi_G^*(\omega_G) + \Omega$ and the structure equations shows

$$\begin{aligned} d\Omega &= \pi_M^*(d\omega) - \pi_G^*(d\omega_G) = -\frac{1}{2}(\pi_M^*[\omega,\omega] - \pi_G^*[\omega_G,\omega_G]) \\ &= -\frac{1}{2}([\pi_M^*\omega, \pi_M^*\omega] - [\pi_G^*\omega_G, \pi_G^*\omega_G]) = -\frac{1}{2}([\pi_G^*\omega_G, \Omega] + [\Omega, \pi_G^*\omega_G] + [\Omega,\Omega]) \end{aligned}$$

and so $d\Omega$ vanishes whenever both arguments are in $\ker\Omega$. This is the usual integrability condition on the distribution $\ker\Omega$, cf. corollary 1.2.2, and the proof is complete. $\square$

Let us notice that for one–dimensional $M \subset \mathbb{R}$, all 2–forms are zero, the Lie bracket $[\omega,\omega]$ must vanish as well since the image $\omega(T_xM)$ is always one–dimensional, and so the structure equations are satisfied automatically. Due to the uniqueness up to constants, we may always achieve a global existence of f in this case over each interval in $\mathbb{R}$. In particular, the theorem with the additive group $G = \mathbb{R}$ and $M = \mathbb{R}$ yields the standard theorem from Calculus on the unique existence of primitive functions, up to constants.

If we add the requirement that $\omega : T_xM \to \mathfrak{g}$ be a linear isomorphism, then the theorem implies that there is a unique group structure locally around $x \in M$ which is locally isomorphic to that of G via f, and has the Maurer–Cartan form $\omega = f^*\omega_G$. A global version of this theorem works for connected and simply connected manifolds M, or under suitable vanishing of the monodromy representation, see again [**Sharpe, 1997**, Chapter 3] for more details.

Remark. Let us point out that the treatment in [**Sharpe, 1997**] goes further. In constrast to our assumption of having the Lie group G given, there only the one–form ω with values in a fixed Lie algebra $\mathfrak{g}$ is considered. Then the whole group structure μ on the manifold M of the right dimension is the subject of the construction.

1.2.5. Lie subgroups and homogeneous spaces. An embedded *Lie subgroup* H of a Lie group G is a submanifold, which at the same time is a subgroup. We shall omit the adjective embedded in the sequel. A Lie subgroup is automatically a closed subset of G and, conversely, it can be shown (see [**KMS**, Theorem 5.5] that any closed subgroup of a Lie group G is an (embedded) Lie subgroup. For a Lie subgroup, the tangent space $\mathfrak{h} = T_eH \subset T_eG = \mathfrak{g}$ is a Lie subalgebra, i.e. the Lie bracket of two elements of $\mathfrak{h}$ again lies in $\mathfrak{h}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, there may exist Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ for which there is no Lie subgroup $H \subset G$ having $\mathfrak{h}$ as the tangent space at the identity. This can be seen from the case of the real torus $\mathbb{T}^2 = S^1 \times S^1$. The Lie algebra of this is $\mathbb{R}^2$ with the trivial Lie bracket, so a subalgebra in there is just a linear subspace. Now if one takes a line of irrational slope, then it is easy to see that if $H \subset G$ is a subgroup, containing a small submanifold which contains the unit element and has the given line as tangent space, then H must be dense in G, so it cannot be a Lie subgroup. Therefore, one defines a *virtual Lie subgroup* of a Lie group G to be a Lie group H together with a homomorphism $i : H \to G$ which is an injective immersion. The derivative $i' : \mathfrak{h} \to \mathfrak{g}$ is then the inclusion of a Lie subalgebra.

The above example generalizes to all one–parametric subgroups $\exp tX : \mathbb{R} \to G$, for $X \in \mathfrak{g}$. Either the image is topologically a circle or it is a line. A circle is a (embedded) subgroup, while a line is only immersed in general.

The Lie groups appear often as the symmetry groups on some manifolds, i.e. as groups of transformations on these manifolds. More explicitly, a left action of a Lie group G on a manifold M is a mapping $G \times M \to M$, $(g, x) \mapsto g \cdot x$ such that $h \cdot (g \cdot x) = (hg) \cdot x$. Similarly, right actions $M \times G \to M$ satisfy $x \cdot (gh) = (x \cdot g) \cdot h$. Only rarely such actions are effective in the sense that $g \cdot x = x$ for one element $x \in M$ implies $g = e$. So we associate the *isotropy subgroup* G_x to each element $x \in M$, $G_x = \{g \in G, g \cdot x = x\}$. By the definition, each isotropy subgroup is a closed subgroup in G.

Each (left) action ℓ of a Lie group G on a manifold M defines the so called *fundamental vector fields* by the formula $\zeta_X(x) = \frac{\partial}{\partial t}_{|0} \ell_{\exp tX}(x)$, for all $x \in M$ and X in the Lie algebra of G. Similarly we obtain the fundamental vector fields for the right actions. These fundamental vector fields provide infinitesimal versions of the Lie group actions. In particular, the left–invariant vector fields on the Lie group G iteself are obtained as the fundamental vector fields with respect to the left multiplication by elements in G. The fundamental field mapping to right actions yields a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, while $\zeta_{[X,Y]} = -[\zeta_X, \zeta_Y]$ for the left actions.

A simple computation yields $T_x \ell_g \cdot \zeta_X(x) = \zeta_{\mathrm{Ad}(g)X}(g \cdot x)$, i.e. $\ell_g^* \zeta_X = \zeta_{\mathrm{Ad}(g^{-1}) \cdot X}$, for left actions. For each point $x \in M$ we also define the isotropy subalgebra $\mathfrak{g}_x \subset \mathfrak{g}$ of elements X with $\zeta_X(x) = 0$.

For each subgroup $H \subset G$, there is the obvious left action of G on the space of right cosets G/H given by multiplication on representatives. If H is closed, there is a unique smooth structure on G/H such that the projection $G \to G/H$ is smooth and smooth functions on G/H are just the smooth functions on G constant on the cosets.

If M is a manifold with a *transitive action* of G, i.e. for any x, $y \in M$ there is $g \in G$ such that $g \cdot x = y$, then all isotropy subgroups G_x are uniquely isomorphic up to conjugation and M is uniquely identified with G/G_x, up to the choice of the *origin* $x \in M$. The manifolds $M \simeq G/G_x$, are called *homogeneous spaces* and they were at the very core of Felix Klein's approach to geometry in his famous Erlangen programme. Thus we also call them *Klein geometries* and it depends on the context whether the origin $o \in M$, and so the isomorphism $M \simeq G/G_o$, has been fixed. Usually we assume that M is connected.

In each closed $H \subset G$, there is the unique maximal normal subgroup $K \subset H$ of G. Clearly an action of an element $g \in G$ on G/H is trivial if and only if $g \in K$. We say that the Klein geometry is *effective* if $K = \{e\}$ is the trivial subgroup and *locally effective* if K is discrete. For practical reasons, it is not always reasonable to pass from a given homogeneous space G/H to its effective version $(G/K)/(H/K)$ and we shall meet many examples. On the other hand, mostly the Lie algebra $\mathfrak{k}$ is then trivial and so the geometry is locally effective. Another important case is the *splitted Klein geometry*, where a complementary Lie subalgebra $\mathfrak{g}_- \subset \mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$, is fixed. Finally, a *reductive Klein geometry* is a splitted geometry where $\mathfrak{g}_-$ is invariant with respect to the adjoint action of H.

A careful study of the Klein geometries is available in [**Sharpe, 1997**, Chapter 4] and we come back to this crucial part of our background in section 1.4 below.

EXAMPLE. The best known examples are the real affine and euclidean spaces. Let us consider the set of points in $A_n = \mathbb{R}^n$ as the domain of all affine transformations $x \mapsto Bx + b$ where B is any invertible matrix in $GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$ is any translation vector. In fact, we could have told this without any reference to an origin and coordinates, but our choice allows a nice description. Let us write

$$A(n, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ b & B \end{pmatrix},\ B \in GL(n, \mathbb{R}),\ b \in \mathbb{R}^n \right\} \subset GL(n+1, \mathbb{R})$$

for the general affine group acting on elements $(1, x^T) \in \mathbb{R}^n$. The action is obviously transitive and the isotropy subgroup of the origin $(0, \dots, 0) \in \mathbb{R}^n$ is just $GL(n, \mathbb{R})$, i.e. the subgroup of elements with $b = 0$. Thus $A_n \simeq A(n, \mathbb{R})/GL(n, \mathbb{R})$.

Most geometric properties of Klein geometries are always encoded in the algebraic properties of the isotropy subgroup. Let us look on them more closely. The most striking feature is the existence of the subgroup of translations in $A(n, \mathbb{R})$. The conjugation on a translation $b \in \mathbb{R}^m$ by an element $B \in GL(n, \mathbb{R})$ is $b \mapsto Bb$. Thus, the whole group $A(n, \mathbb{R})$ is the semidirect product $\mathbb{R}^n \rtimes GL(n, \mathbb{R})$ with the normal commutative factor $\mathbb{R}^n$ formed by translations and the direct factor $GL(n, \mathbb{R})$.

The corresponding Lie algebra $\mathfrak{a}(n, \mathbb{R}) \subset \mathfrak{gl}(n+1, \mathbb{R})$ consists of all matrices with first row zero and the Lie bracket is

$$\left[\begin{pmatrix} 0 & 0 \\ v & X \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ w & Y \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ Xw - Yv & XY - YX \end{pmatrix}.$$

The one–parametric subgroups corresponding to elements in $\mathbb{R}^n \subset \mathfrak{a}(n, \mathbb{R})$ are the straight lines through the origin with the usual parameterization and their actions on a fixed point $(1, x^T) \in \mathbb{R}^n$ yield all parameterized lines in the affine space:

$$\left(\exp t \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ tv & E \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix} = x + tv.$$

The euclidean space $\mathbb{R}^n$ is obtained in the same way if we consider the subgroup of euclidean motions which may be viewed as the subgroup

$$E(n, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ b & B \end{pmatrix}, B \in O(n, \mathbb{R}), b \in \mathbb{R}^n \right\} \subset A(n, \mathbb{R}).$$

See 1.1.2 for more comments on the geometric implications of this construction.

1.2.6. Principal and associated bundles. A *fibred manifold* is a surjective submersion $p : Y \to M$, a trivial fibred manifold with fiber S is $\pi_M : M \times S \to M$. *Fibred morphisms* $\varphi : Y \to Y'$ are smooth mappings between fibred manifolds such that they cover a smooth mapping $\varphi_0 : M \to M'$ over the base manifolds, i.e. $p' \circ \varphi = \varphi_0 \circ p$. A *fiber bundle* with standard fiber S is a fibred manifold $Y \to M$ which is locally isomorphic (via fibred morphisms) to the trivial bundle $\mathbb{R}^n \times S \to \mathbb{R}^n$ over open neighborhoods of all its points. A *vector bundle* is a fiber bundle with a fiberwise structure of vector space, whose standard fiber is a vector space isomorphic to all its fibers.

The projection $G \to G/H$ onto a homogeneous space clearly is a fiber bundle with standard fiber H. Moreover, the right multiplication on G defines an action of H on the fibers. A *principal bundle* $\mathcal{P}$ over M is a fiber bundle $\mathcal{P} \to M$ with a right action of its *structure group* H which is *free* and *transitive* on the fibers, i.e. if $u \cdot h = u$ for one element $u \in \mathcal{P}$ then h is the identity in H, and for any two elements u, u' in the same fiber of $\mathcal{P}$ there is $h \in H$ with $u' = u \cdot h$. The structure group is always assumed to be a Lie group and we also say $\mathcal{P}$ is a principal H–bundle. A *morphism of principal bundles* is a fibred morphism commuting with the principal actions, i.e. $\varphi(u \cdot h) = \varphi(u) \cdot h$. More generally, a morphism ψ of the structure group may be taken into account, i.e. $\varphi(u \cdot h) = \varphi(u) \cdot \psi(h)$. All Klein geometries (with a fixed origin) are principal bundles $G \to G/H$ with structure groups H.

The trivial principal bundles are of the form $\mathbb{R}^m \times H$ with the obvious right action. Local sections $\sigma : U \subset M \to \mathcal{P}$ define special morphisms of principal bundles, called *local trivializations*, $U \times H \to \mathcal{P}$, $(x, h) = \sigma(x) \cdot h$. The choice of a covering of the base manifold M by the open domains U_α of such local sections σ_α defines the so called *cocycle of transition functions* $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to H$ with the property

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot \tau_{\alpha\beta}(x).$$

Clearly $\tau_{\alpha\gamma}(x) = \tau_{\alpha\beta}(x) \cdot \tau_{\beta\gamma}(x)$ whenever both of the transition functions are defined. Equivalently, a cocycle of transition functions over an open covering of M defines the principal bundle $\mathcal{P} \to M$, up to isomorphisms. In particular, a principal bundle allows a global trivialization if and only if it admits a global section.

A *reduction of principal bundle* $\mathcal{P}$ to structure group $K \subset H$ is a principal bundle $\mathcal{R} \to M$ with structure group K over the same base, together with a principal bundle morphisms $\iota : \mathcal{R} \to \mathcal{P}$ with respect to the inclusion $i : K \to H$, covering the identity on M. A simple argument based on the cocycles of transition functions shows the following fact:

LEMMA. *There is the bijective correspondence between the set of reductions of a given principal bundle $\mathcal{P} \to M$ with structure group H to its subgroup K and the set of smooth global sections of the fiber bundle $\mathcal{P}/K \to M$.*

The simplest example of a principal bundle $\mathcal{P} \to$ pt (with a one–point base manifold) is the space of all basis of an m–dimensional vector space. Clearly $\mathcal{P}$ is isomorphic to $GL(m, \mathbb{R})$ once we fix one of the basis, but there is no canonical choice like that. Only a little bit more advanced example is the homogeneous space $\mathcal{P} = A(n, \mathbb{R}) \to A(n, \mathbb{R})/GL(n, \mathbb{R}) = \mathbb{R}^n$, i.e. the standard affine space. Again, the structure group is $GL(n, \mathbb{R})$ and we may view the individual fibers over the cosets $x \in \mathbb{R}^n$ as the spaces of basis of the modeling vector space $\mathbb{R}^n$ without any distinguished choice of one of them. In other words, $\mathcal{P}$ is the bundle of the basis of the individual tangent spaces at the points of the affine space.

A straightforward generalization of the latter example to general manifolds is the *linear frame bundle* $\mathcal{P}^1 M \to M$ whose fibers are formed by all basis of the individual tangent spaces. Again, its structure group is $GL(n, \mathbb{R})$ where n is the dimension of M. In analogy to this example, we shall most often call all elements in principal bundles *frames*. Another similar example is the general frame bundle $\mathcal{GL}(\mathcal{V})$ of a vector bundle $\mathcal{V} \to M$ with structure group $GL(V)$, where V is the standard fiber of the vector bundle $\mathcal{V}$.

The basic linear algebra teaches how a choice of a basis of a given vector space V leads to distinguished basis of all tensor products of this space. The crucial role is played there by the standard representation λ of the group $GL(n, \mathbb{R})$ on $\mathbb{R}^n \simeq V$ and its tensor products. More explicitly, any vector $v \in V$ is given by its coordinates with respect to any of the basis $u \in \mathcal{P} = GL(V)$, and so we may view v as a mapping $v : \mathcal{P} \to \mathbb{R}^n$ with the property $v(ug) = \lambda(g^{-1})(v)$, and all such mappings may be understood as a vector. This is exactly the way we shall treat most geometric objects in the realm of the so called associated bundles, see the proposition below.

An *associated bundle* to a principal bundle $\mathcal{P} \to M$ with standard fiber S, structure group H, and its left action $\lambda : H \times S \to S$, is the quotient $\mathcal{P} \times_\lambda S$ of the fiber bundle $\mathcal{P} \times S \to M$ with respect to the equivalence $(u, s) \equiv (u \cdot h, \lambda(h^{-1}, s))$. There is the obvious fiber bundle structure with standard fiber S on $\mathcal{P} \times_\lambda S$ and we shall write $\{u, s\}$ for the equivalence classes. If the left action is a linear representation of the structure group on a vector space V, then the associated bundle $\mathcal{P} \times_\lambda V$ is a vector bundle.

Each principal bundle morphism $\varphi : \mathcal{P} \to \mathcal{P}'$ with structure group G defines the fibred morphisms $\mathcal{P} \times_\lambda S \to \mathcal{P}' \times_\lambda S$, $\{u, s\} \mapsto \{\varphi(u), s\}$, for all left actions λ on the standard fiber S. Thus, the construction of the associated bundles from the fixed left actions λ is functorial. Of course, for linear actions this functorial construction has values in vector bundles and their linear morphisms. On the other hand, any smooth mapping $f : S \to S'$ commuting with given left actions defines fibred morphisms $\mathcal{P} \times_\lambda S \to \mathcal{P} \times_{\lambda'} S'$, $\{u, s\} \mapsto \{u, f(s)\}$ over the identity on the base manifolds M.

We have seen some examples already. The trivial representation $\mathbb{R}$ of $H = GL(n, \mathbb{R})$ provides the trivial associated bundles $\mathcal{P}^1 M \times_H \mathbb{R} = M \times \mathbb{R}$, $\dim M = n$, whose sections are the smooth functions, while the standard representation of H gives $\mathcal{P}^1 M \times_H \mathbb{R}^n$. We identify the latter space with TM viewing $\{u, X\}$ as the tangent vector whose coordinates with respect to u are X. Similarly, we relate the dual representation $\mathbb{R}^{n*}$ to the bundle T^*M, all tensor products of the latter two representations to tensor bundles, etc.

Proposition. *The space $C^\infty(\mathcal{P} \times_\lambda S)$ of all smooth sections s of the associated space with structure group H is in the natural bijective correspondence with the H–equivariant mappings $\tilde{s} \in C^\infty(\mathcal{P}, S)^H$. Using an arbitrary local section σ of $\mathcal{P}$, this correspondence is given as $s(x) = \{\sigma(x), \tilde{s}(\sigma(x))\}$.*

Proof. If $\tilde{s}$ is equivariant, i.e. $\tilde{s}(u \cdot h) = \lambda(h^{-1})(\tilde{s}(u))$, then the above formula defines a section. On the other hand, if σ is a fixed section defined at the point x in the underlying manifold M, then the above equality defines uniquely $\tilde{s}$ along the fiber over x. □

1.2.7. Natural bundles and jets. A *natural bundle* F on the category $\mathcal{M}_n$ of n–dimensional manifolds and locally invertible smooth mappings is a functor assigning to any n–manifold N a fiber bundle $p_N : F(N) \to N$ and to any local diffeomorphism $f : N_1 \to N_2$ a bundle map $F(f) : F(N_1) \to F(N_2)$ such that $p_{N_2} \circ F(f) = f \circ p_{N_1}$. Furthermore, F is regular, i.e. if P is any smooth manifold and $f : P \times N_1 \to N_2$ is smooth and such that for each $x \in P$ the map $f_x : N_1 \to N_2$ defined by $f_x(y) := f(x, y)$ is a local diffeomorphism, then we assume that the map $P \times F(N_1) \to F(N_2)$ defined by $(x, a) \mapsto F(f_x)(a)$ is smooth, too. Thus regularity means that smoothly parametrized families of local diffeomorphisms are transformed into smoothly parametrized families. Usually, one assumes in addition that F is local, i.e. that for any inclusion $i : U \hookrightarrow N$ of an open subset, $F(i)$ is the inclusion $p_N^{-1}(U) \hookrightarrow F(N)$.

The axiomatic study of natural bundles has been one of the main aims of [**KMS**]. In particular, the regularity follows from the functoriality and locality conditions, and an explicit description of all natural bundles in terms of associated spaces is deduced. In order to state the result, we have to recall another basic concept of differential geometry.

Let M, N be smooth manifolds, $x \in M$ a point. Two smooth mappings f, $g : M \to N$ are said to have the same *jet of order* r at x (briefly r–jet) if their partial derivatives at x up to order r in some local chart on M coincide (and then of course in all charts around x by the chain rule). We write then $j^r_x f = j^r_x g$. The point x is called the *source of the jet* while $f(x)$ is called the *target of the jet* $j^r_x f$. The space of all r–jets with source in M and target in N is denoted by $J^r(M, N)$. Similarly, we write $J^r_x(M, N)$, $J^r(M, N)_y$, or $J^r_x(M, N)_y$ if the source, target or both are fixed.

The chain rule implies immediately that the composition of jets in $J^r(M, N)_y$ and $J^r_y(N, Q)$ is well defined by the formula $(j^r_y g) \circ (j^r_x f) = j^r_x(g \circ f)$. The jet $j^r_x f$ is called invertible if there is $j^r_{f(x)} g$ such that $j^r_x(g \circ f) = j^r_x \operatorname{id}_M$.

Using the canonical coordinate charts on $\mathbb{R}^n$ and $\mathbb{R}^m$ and the translations, we obtain the identification $J^r(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times \mathbb{R}^n \times J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$ and the Taylor coefficients yield canonical coordinates on the r–jets keeping the origins. Now, let us notice that the construction of the jet spaces $J^r(M, N)$ is functorial in both arguments, and this functoriality induces coordinate charts on $J^r(M, N)$ from any charts on M and N. More explicitly, the composition of jets with jets of invertible smooth maps apply to the coordinate charts. Thus we have the smooth fibration $J^r(M, N) \to M \times N$ and also the obvious smooth projections $\pi^r_s : J^r(M, N) \to J^s(M, N)$, $r > s$. In particular π^r_0 is the product of the source and target projections.

For any fibred manifold $\pi : E \to M$ we write $J^r(E \to M)$, or rather briefly $J^r(E)$, for the submanifold in $J^r(M, E)$ of all jets of sections of π. Clearly, $J^r(\ )$ is a functor acting on locally invertible fibred morphisms, the *jet prolongation* functor. In this context, there is the rth order universal differential operator $j^r : C^\infty(E \to M) \to C^\infty(J^r E \to M)$, $s \mapsto (x \mapsto j^r_x s)$. Indeed, every rth order differential operator $D : C^\infty(E \to M) \to C^\infty(E' \to M)$ factors through a fibred morphism $\tilde{D} : J^r(E \to M) \to E'$.

Using jets, we can reconstruct many of the basic geometric objects. For example, the tangent bundles are identified with the first order jets of curves at $0 \in \mathbb{R}$, $TM = J^1_0(\mathbb{R}, M)$, the cotangent bundles are one–jets of functions, $T^*M = J^1(M, \mathbb{R})_0$. Let us notice that the vector bundle structure comes naturally on the cotangent spaces in this kinematic approach to the definition. The general linear group is (identified with) the subspace of all invertible jets $\operatorname{inv} J^1_0(\mathbb{R}^n, \mathbb{R}^n)_0$, the linear frame bundle $\mathcal{P}^1 M$ over an n–dimensional manifold is the principal bundle $\operatorname{inv} J^1_0(\mathbb{R}^n, M)$ with the principal action defined by the jet composition.

All these examples allow a simple higher order generalization. The rth frame bundle $\mathcal{P}^r M = \operatorname{inv} J^r_0(\mathbb{R}^n, M)$ consists of r–jets of local charts and carries the principal right action of the rth differential group $G^r_n = \operatorname{inv} J^r_0(\mathbb{R}, \mathbb{R})_0$ by the jet composition. $T^r_1 M = J^r_0(\mathbb{R}, M)$ is the associated bundle to $\mathcal{P}^r M$ with respect to the obvious left action of G^r_n on $S = J^r_0(\mathbb{R}, \mathbb{R}^n)$, etc.

THEOREM ([**KMS**, theorem 22.1]). *All local natural bundles on n–dimensional manifolds are obtained as associated spaces corresponding to left actions of the differential groups G^r_n on finite dimensional manifolds.*

The lowest possible r from the theorem is called the *order of the natural bundle*, which is subject of a strict estimate depending on the dimension of the bases and the dimension of the fibers. Let us notice that the composition of the jet prolongation functor with an kth order natural bundle F is the $(k + r)$th order natural bundle $J^r F$. The theorem was proved first by Palais and Terng with the regularity assumed, by Epstein and Thurston in the full strength, sharp estimates of the orders were obtained by Zajtz, see [**KMS**] for further development and more bibliographic details.

1.2.8. Lie derivatives. We shall recover the usual concept of Lie derivatives in the general functorial context of natural bundles. Consider any natural bundle F. For each section $s : N \to FN$ of its value, there is the contravariant action of any invertible smooth mapping $\varphi : M \to N$, $\varphi^* s = F\varphi^{-1} \circ s \circ \varphi$, the so called *pullback* of the section s. In particular, this can be applied to the individual diffeomorphisms in the flow of a fixed vector field $\xi \in \mathfrak{X}(M)$ and since the natural bundles are regular, we obtain again a one–parametric group $F(\mathrm{Fl}^{\xi}_t)$ on the total space FM. Let us write $\mathcal{F}\xi$ for the corresponding vector field on FM. Now, we define the *Lie derivative* $\mathcal{L}_\xi s$ of the section s in the direction of the vector field ξ by the formula

$$\mathcal{L}_\xi s = Ts \circ \xi - \mathcal{F}\xi \circ s$$

and a straightforward computation reveals that this is equivalent to

$$\mathcal{L}_\xi s = \tfrac{\partial}{\partial t}_{|0}(\mathrm{Fl}^{\xi}_t)^* s = \tfrac{\partial}{\partial t}_{|0} F(\mathrm{Fl}^{\xi}_{-t}) \circ s \circ \mathrm{Fl}^{\xi}_t\,.$$

In particular, the values of the Lie derivative are in the vertical tangent space $VFM \subset TFM$.

If F is a natural vector bundle, then VFM is naturally equivalent to $FM \oplus FM$ and, under this identificaton, the first component of $\mathcal{L}_\xi s$ is s and we consider the second component as the Lie derivative.

In the special case of the tangent bundle TM we obtain the standard Lie bracket, i.e. $\mathcal{L}_\xi \eta = [\xi, \eta]$ and for all tensor bundles the definition recovers the classical approach. In particular, we easily deduce the Leibnitz rule for general tensor products of natural vector bundles

$$\mathcal{L}_\xi (s_1 \otimes s_2) = (\mathcal{L}_\xi s_1) \otimes s_2 + s_1 \otimes (\mathcal{L}_\xi s_2)$$

and formulae for the evaluation of forms α on vector fields η

$$\mathcal{L}_\xi \langle \eta, \alpha \rangle = \langle \mathcal{L}_\xi \eta, \alpha \rangle + \langle \eta, \mathcal{L}_\xi \alpha \rangle.$$

More generally, for k–times contravariant tensor field τ we obtain

$$(\mathcal{L}_\xi \tau)(\eta_1, \dots, \eta_k) = \mathcal{L}_\xi(\tau(\eta_1, \dots, \eta_k)) - \sum_{i=1}^{k} \tau(\eta_1, \dots, \mathcal{L}_\xi(\eta_i), \dots, \eta_k).$$

In general, the Lie derivative always depends on all derivatives of the vector fields up to the order of the natural bundle. Thus the only cases where the Lie derivative is algebraic in the direction of the derivation are natural bundles of order zero, e.g. the smooth (vector valued) functions.

More generally, we introduce the Lie derivative of sections of any fibred manifold $Y \to M$ in the direction of a projectable vector field $\tilde{\xi} \in \mathfrak{X}(Y)$ over $\xi \in \mathfrak{X}(M)$ by the the same formulae as above, with $\mathcal{F}\xi$ replaced by $\tilde{\xi}$ and $F(\mathrm{Fl}^{\xi}_{-t})$ by $\mathrm{Fl}^{\tilde{\xi}}_{-t}$. Even more generally, an analogous formula leads to the Lie derivative of mappings $f : M \to N$ between manifolds with respect to two vector fields $\xi \in \mathfrak{X}(M)$, $\eta \in \mathfrak{X}(N)$, see [**KMS**, chapter XI] for details.

1.3. Survey on connections

In this section we first recall several approaches to the general notion of a connection and then we provide the survey of all basic features of general, principal, and affine connections. Our treatment prepares further steps towards a self–contained theory and calculus for Cartan connections worked out later.

1.3.1. Linear connections. The background of the general idea of a connection is extremely natural. We simply wish to differentiate given functions on manifolds and, more generally, sections of fibred manifolds in a given direction. We have seen two instances of similar operations already.

First, the absolute parallelism on a Lie group G defined by the Maurer–Cartan form allows to differentiate mappings $f : M \to G$ with results in the tangent space of one reference point, the unit. This is the left logarithmic derivative defined in 1.2.4. We shall see below how to link this derivative to the trivial principal connection on the trivial principal bundle $M \times G \to M$.

Second, on all natural bundles, there is the notion of Lie derivative of sections in the direction of vector fields, cf. 1.2.8. However, this depends on derivatives of the vector fields, not only on their values. So wee need another concept.

Let us consider an arbitrary vector bundle $\mathcal{V} \to M$ with standard fiber V. We wish to have derivatives of sections $s \in C^\infty(\mathcal{V})$ which are algebraic in the direction and satisfy the Leibnitz rule at least with respect to the multiplication by smooth functions, and such an operation is usualy understood as a 'connection'. It is well known (see e.g. [**KMS**, Lemma 7.3]) that smooth and algebraic evaluations on sections of vector bundles are just those which are linear over $C^\infty(M)$. Thus, a *linear connection* on a vector bundle $\mathcal{V} \to M$ is often defined as a family of first order linear differential operators ∇_ξ on the space $C^\infty(\mathcal{V})$ of smooth sections which depend smoothly on vector fields ξ on M and satisfy for all smooth functions f and vector fields ξ, η

$$\begin{aligned} \nabla_{\xi+\eta} s &= \nabla_\xi s + \nabla_\eta s \\ \nabla_\xi f s &= (\mathcal{L}_\xi f) s + f \nabla_\xi s \\ \nabla_{f\xi} s &= f \nabla_\xi s. \end{aligned}$$

In local trivializations $\mathbb{R}^n \times \mathbb{R}^k \to \mathcal{V}$, the coordinate–like expression of the section and vector field is $s = s^p$, $\xi = \xi^i \frac{\partial}{\partial x^i}$, and a straightforward application of the first and the third of the above equalities shows that each such connection is given by

$$\nabla_\xi s = A^{pi}_{qj} \tfrac{\partial s^q}{\partial x^i} \xi^j + B^p_{qi} s^q \xi^i$$

with appropriate smooth functions A^{pi}_{qj} and B^p_{qi} on $\mathbb{R}^n$. Of course, we use the standard summation conventions. Now, the second equality clearly implies $A^{pi}_{qj} = \delta^p_q \delta^i_j$. Thus, in coordinates, the differential operator ∇_ξ reads as

$$\nabla_\xi s(x) = \tfrac{\partial s^p}{\partial x^i}(x) \xi^i(x) + B^p_{qi}(x) s^q(x) \xi^i(x).$$

In particular, this formula defines uniquely the jets $j^1_x s$ for each fixed target $y = s(x)$, such that $\nabla_{\xi_x} s = 0$ for all $\xi_x \in T_x M$.

The latter observation is equivalent to the definition of the so called *horizontal distribution* $\mathcal{H} \subset T\mathcal{V}$, $\mathcal{H}_y = Ts(T_x M)$ for any representative of the above jet $j^1_x s$ with $s(x) = y$. Now, this horizontal distribution defines the unique lifts of vector fields $\xi \in \mathfrak{X}(M)$ to projectable horizontal fields $\gamma(\xi)$ on $\mathcal{V}$ over ξ, with values in $\mathcal{H}$.

Indeed, $\gamma(\xi)(y) = Ts_y \circ \xi$ where s_y is any representative of the jet $j^1_x s_y$ constructed above.

LEMMA. *Let ∇_ξ be a linear connection on a vector bundle $\mathcal{V} \to M$ and let $\gamma : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{V})$ be the induced horizontal lift of vector fields . Then ∇_ξ is the second component in the vertical tangent space $V\mathcal{V} = \mathcal{V} \oplus \mathcal{V}$ of the expression given by the formula*

$$\nabla_\xi s = Ts \circ \xi - \gamma(\xi) \circ s = \tfrac{\partial}{\partial t}_{|0} \operatorname{Fl}^{\gamma(\xi)}_{-t} \circ s \circ \operatorname{Fl}^\xi_t = \mathcal{L}_{\gamma(\xi)} s.$$

PROOF. The formula follows directly from the above coordinate formula and the definition of the horizontal lift γ, see also the discussion on Lie derivatives in 1.2.8. $\square$

In coordinates (x^i, y^p) on the local trivialization $\mathbb{R}^n \times \mathbb{R}^k$ of $\mathcal{V}$, we may write $\gamma(\xi)(x,y) = \xi^i(x)\frac{\partial}{\partial x^i} + \gamma^p_i(x,y)\xi^i(x)\frac{\partial}{\partial y^p}$. Now, the linearity of $\nabla_\xi s$ in both ξ and the section s implies immediately that γ is linear in the fiber coordinate too. Thus $\gamma^p_i(x,y) = \gamma^p_{qi}(x)y^q$ and the formula from the above lemma gets the most classical coordinate form

$$\nabla_\xi s = \frac{\partial s^p}{\partial x^i}\xi^i - \gamma^p_{qi}s^q\xi^i$$

with the usual summation conventions. In particular, if $\mathcal{V}$ is the usual tangent bundle $TM \to M$, this is the standard formula for a linear connection with Christoffel symbols γ^j_{ki}.

Let us notice, however, that the linearity of the connection was not essential in the formulation and proof of the above lemma and this leads to the following general definition.

1.3.2. General connections. A *general connection* on an arbitrary fibred manifold $p : Y \to M$ is a horizontal distribution $\mathcal{H} \subset TY$ which is complementary to the vertical tangent bundle VY. For each $y \in Y$, $\mathcal{H}(y) \subset T_yY$ is called the *horizontal subspace* of the connection.

There are three equivalent definitions: First, the specification of the horizontal subspaces is equivalent to the choice of one–jets of sections $j^1_x s_y$ with target $s_y(x) = y$, for each $y \in Y$, i.e. a connection is defined a the smooth section of the first jet prolongation $J^1Y \to Y$. Second, there is the induced horizontal lift of vector fields $\gamma : \mathfrak{X}(M) \to \mathfrak{X}_{\mathrm{hor}}(Y)$ with values in horizontal vector fields. Finally, we have $TY = VY \oplus \mathcal{H}$, which is equivalent to the *vertical projection* $\Psi : TY \to VY$ with kernel $\mathcal{H}$. Of course, the vertical projection defines also the *horizontal projection* $\chi = \mathrm{id}_{TY} - \Psi$. Let us notice that the latter definition even does not need the fibration, but we shall not meet this generality.

The *curvature* R is the obvious obstruction against the integrability of the horizontal distribution: take two vector fields ξ, $\eta \in \mathfrak{X}(M)$ and consider $R(\xi,\eta) = [\gamma(\xi), \gamma(\eta)] - \gamma([\xi,\eta])$. A direct computation shows that the values $R(\xi,\eta)$ are vertical vector fields in $C^\infty(VY)$ and, moreover, R is a bilinear operation with respect to the $C^\infty(M)$–module structure, thus algebraic in its arguments. Since $\gamma(\xi)$ is a projectable field over ξ determined by ξ uniquely, $R(\xi,\eta) = 0$ if and only if the horizontal distribution is involutive. Thus a connection $Y \to J^1Y$ has vanishing curvature if and only if it is localy given by local trivializations $s : \mathbb{R}^m \times \mathbb{R}^n \to Y$. A more sophisticated approach uses the vertical projection Ψ and defines the curvature as the Frölicher–Nijenhuis bracket $[\Psi, \Psi]$, see [**KMS**, section 9].

Since one–dimensional distributions are always involutive, a general connection defines the so called *parallel transport* on $p : Y \to M$: For each (parameterized) curve $c : (a, b) \subset \mathbb{R} \to M$, $c(0) = x$, and $y \in Y$, $p(y) = x$, there is the unique curve $\tilde{c} : (a', b') \subset (a, b) \to Y$ on a maximal interval (a', b') containing the origin, such that $\tilde{c}(0) = y$, $p \circ \tilde{c} = c$ on the domain, and $T\tilde{c}$ has values in the horizontal distribution.

The *absolute derivative* ∇_ξ defined by the general connection is given by the formula in lemma 1.3.1 and has values in the vertical tangent bundle to Y. Clearly, this formula may be also explained in the terms of the parallel transport: Take a section s and a direction $\xi_x \in T_xM$. Extend ξ_x into a vector field ξ, consider its flow through x and the corresponding parallel transport. Then the derivative of the pullback of the section by the parallel transport along the flow lines of ξ is exactly our formula. This is probably the first historical version of the differential with respect to a connection.

There is also another concept of derivative, the *exterior absolute differential* $d^\nabla : \Omega^k(Y; V) \to \Omega^{k+1}(Y; V)$ for any vector space V, defined by means of the pullback with respect to the horizontal projection, i.e. by the formula $d^\nabla = \chi^* d$. We shall see some explicit relation between the latter two concepts under specific geometric assumptions, but they are completely independent in our present general setting.

For fiber bundles, we may recover a non–linear version of the Christoffel symbols. Indeed, for each local trivialisation $\mathbb{R}^m \times S \to Y$, the horizontal lift of vector fields is described uniquely by mappings $\gamma_i : TM \to \mathfrak{X}(S)$, so that $\gamma(\xi^i \frac{\partial}{\partial x^i}) = \xi^i \frac{\partial}{\partial x^i} + \gamma_i \xi^i$ and similarly we obtain the formula for the absolute derivative.

In the special case of a vector bundle $Y \to M$, its first jet prolongation J^1Y carries a vector bundle structure too. As a simple exercise we obtain that a general connection $\sigma : Y \to J^1Y$ is linear if and only if the section σ is a linear morphism.

1.3.3. Principal connections. Let $p : \mathcal{P} \to M$ be a principal fiber bundle. A *principal connection* on $\mathcal{P}$ is a (general) connection whose horizontal distribution is invariant with respect to the principal action of the structure group G, i.e. $\mathcal{H}_{u\cdot g} = Tr^g(\mathcal{H}_u)$. Equivalently, we may express the same requirement by the equivalent conditions:

(1) the horizontal lifts of vector fields are right invariant vector fields on the principal bundle $\mathcal{P}$, i.e. $(r^g)^*\gamma(\xi) = \gamma(\xi)$,

(2) the corresponding section $\sigma : \mathcal{P} \to J^1\mathcal{P}$ is G–equivariant with respect to the obvious induced G–action on $J^1\mathcal{P}$,

(3) there is a $\mathfrak{g}$–valued one–form on $\mathcal{P}$, again called γ, which is G–equivariant with respect to the adjoint action, i.e. $(r^g)^*\gamma = \mathrm{Ad}_{g^{-1}} \circ \gamma$, such that $\mathcal{H}_u = \ker \gamma$ and $\gamma(\zeta_X(u)) = X$ where ζ_X is any *fundamental vector field*, i.e. $\zeta_X(u) = \frac{\partial}{\partial t}_{|0} u \cdot \exp tX$, $X \in \mathfrak{g}$.

We call the latter one–form γ the *principal connection form*, or briefly principal connection, on $\mathcal{P}$. The vertical projection is given in terms of the connection form γ by the formula $\Phi(\xi)(u) = \zeta_{\gamma(\xi)}(u)$. Let us also notice the immediate consequence of the first condition, namely the right invariance of the horizontal lifts ensures that also the parallel transport along curves is right invariant. This means that for any curve c, the corresponding parallel transport $\tilde{c}$ satisfies $\tilde{c}(t)(u \cdot g) = \tilde{c}(t)(u) \cdot g$ for all $u \in \mathcal{P}$ and $g \in G$.

Similarly, the definition of the curvature R of a principal connection by means of the horizontal lifts shows that R, viewed as an element in $\Omega^2(M, V\mathcal{P})$ is G–equivariant. But exactly as with the connection form itself, we may replace this $V\mathcal{P}$ valued curvature by the horizontal curvature form $\rho \in p^*\Omega^2(M, \mathfrak{g})$, $R(\xi,\eta)(u) = \zeta_{\rho(\xi,\eta)(u)}(u)$, which satisfies $(r^g)^*\rho = \mathrm{Ad}_{g^{-1}} \circ \rho$. The evaluation of the expression $d\gamma + \frac{1}{2}[\gamma,\gamma]$ on two commuting vector fields $\xi = \zeta_X$ and $\eta \in \mathfrak{X}(\mathcal{P})$ arbitrary, yields

$$d\gamma(\xi,\eta) + [\gamma(\xi),\gamma(\eta)] = \mathcal{L}_{\zeta_X}(\gamma(\eta)) - \mathcal{L}_\eta(\gamma(\zeta_X)) - \gamma([\zeta_X,\eta]) + [X,\gamma(\eta)]$$

where the bracket $[\zeta_X,\eta]$ and the derivative of the constant mapping $\gamma(\zeta_X)$ clearly vanish. Moreover, $\mathcal{L}_{\zeta_X}(\gamma(\eta)) = (\mathcal{L}_{\zeta_X}\gamma)(\xi)$ since the fields commute. The flow of ζ_X is $r^{\exp tX}$ and so the invariance of γ implies

$$d\gamma(\xi,\eta) + [\gamma(\xi),\gamma(\eta)] = -\,\mathrm{ad}(X)(\gamma(\eta)) + [X,\gamma(\eta)] = 0.$$

Thus the expression in question is strictly horizontal. Next, if both ξ and η are horizontal, we may choose them as horizontal lifts of fields on the base manifolds and so the connection form vanishes on both of ξ and η, and the whole expression coincides exactly with our original definition of the curvature form. So we have proved for all principal connections the standard formula

$$\rho = d\gamma + \tfrac{1}{2}[\gamma,\gamma].$$

The coordinate version of the principal connection form γ on a local trivialization $\mathbb{R}^m \times G \to \mathcal{P}$ is given nicely as

$$\gamma = \omega_G - \gamma_i dx^i.$$

Here ω_G is the Maurer–Cartan form of G and the *Christoffel symbols* γ_i are determined by the restriction to the distinguished section $(x,e) \subset \mathbb{R}^n \times G$.

In particular, for the principal connections on the linear frame bundle $\mathcal{P}^1M$ on a manifold M we obtain the usual Christoffels γ^k_{ji}. The choice of identically vanishing Christofells γ_i yields the so called trivial connection on the trivial principal bundle $M \times G$. Let us notice that then the absolute differential of sections $M \to M \times G$, viewed as functions $M \to G$, is exactly the left logarithmic derivative.

Obviously, the exterior differential of the connection form γ equals

$$d\gamma = d\omega_G - d\gamma_i \wedge dx^i$$

and the evaluation of $d\gamma + \frac{1}{2}[\gamma,\gamma]$ yields the usual coordinate expression for the curvature form. Indeed, if we write $\gamma^p_i Y_p$ for the expression of $\gamma_i(x,e)$ in the basis Y_p of $\mathfrak{g}$, $\rho = \rho^p Y_p$, and c^p_{qr} for the corresponding structure constants of $\mathfrak{g}$, i.e. $[Y_q, Y_r] = -c^p_{qr} Y_p$, we obtain the expression of ρ at the points (x,e) in $M \times G$

$$\rho^p = \left(\frac{\partial \gamma^p_i}{\partial x^j} - c^p_{qr}\gamma^q_i\gamma^r_j \right) dx^i \wedge dx^j.$$

Let us notice that the curvature form is also expressed as $\rho = d^\nabla\gamma$ by means of the exterior absolute derivative of the connection form itself.

1.3.4. Induced connections. As we saw in 1.2.6, the elements of principal fiber bundles $\mathcal{P}$ may be viewed as frames allowing coordinate–like description of elements of the associated bundles $\mathcal{S} = \mathcal{P} \times_G S$. Thus if we understand the horizontal distribution $\mathcal{H} \subset T\mathcal{P}$ as a way to define 'constant frame fields' up to order one, we have the same concept for all induced bundles $\mathcal{P} \times_G S$ as well. Simply we keep the coordinates constant and move the frames in the direction of the horizontal distributions.

In order make the definitions more explicit, we need to understand the tangent bundle of $\mathcal{S}$ as an associated bundle to $T\mathcal{P}$. First, notice that TG is again a Lie group with multiplication given by $j^1_0 c \cdot j^1_0 d = j^1_0(t \mapsto c(t) \cdot d(t))$ and TG acts on $T\mathcal{P}$ in the obvious way defining the principal fiber bundle structure on $T\mathcal{P} \to TM$. Now, each left action $\lambda : G \times S \to S$ gives rise to the left action $T\lambda : TG \times TS \to TS$, defined again by $T\lambda(j^1_0 g, j^1_0 s) = j^1_0(t \mapsto \lambda(g(t), s(t)))$. In particular, there is the embedding $G \subset TG$ and $S \subset TS$ by means of the zero sections and clearly the restriction of the above structure to the vertical bundle $V\mathcal{S}$ yields the identification $V\mathcal{S} = \mathcal{P} \times_G TS$.

Let $\lambda : G \times S \to S$ be a left action of the structure group G of the principal bundle $p : \mathcal{P} \to M$. Then each principal connection with the horizontal distribution $\mathcal{H}$ defines the horizontal distribution $\mathcal{H}^S \subset T\mathcal{S} = T\mathcal{P} \times_{T\lambda} TS$ on the associated bundle $p^S : \mathcal{S} = \mathcal{P} \times_\lambda S \to M$ by the formula

$$\mathcal{H}^S(\{u, s\}) = \{\{\xi(u), 0_s\} \in T\mathcal{P} \times_{T\lambda} TS, \xi(u) \in \mathcal{H}(u), u \in \mathcal{P}\}.$$

Indeed, for each tangent vector $\xi(u) \in \mathcal{H}(u)$, the projection $Tp^S(\{\xi(u), 0_s\})$ is just $Tp(\xi(u))$. Thus, any fixed principal connection γ on $\mathcal{P}$ yields on each associated bundle $\mathcal{S}$ the general connection γ^S. We call it the *induced connection* on $\mathcal{S}$.

These connections have remarkable properties. We shall formulate them explicitly for associated vector bundles, but general version is obtained analogously and we comment on that below.

Proposition. *Let γ^V be the connection induced on the associated vector bundle $\mathcal{V} = \mathcal{P} \times_G V$ by the principal connection with the connection form $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{g})$.*
(1) The induced connection is a linear connection on $\mathcal{V}$ with the horizontal lift of vector fields $\xi \mapsto \gamma^V(\xi) = \{\gamma(\xi), 0\} \in \mathfrak{X}(\mathcal{V})$.
(2) The frame form of the absolute derivative $\nabla_\xi \tilde{s}$ of a section with the frame form $s : \mathcal{P} \to V$ is the Lie derivative $\mathcal{L}_{\gamma(\xi)} s : \mathcal{P} \to TV = V \oplus V$ of the vector valued function s in the direction of the horizontal lift and its second component is again G–equivariant, thus a section of the same bundle.
(3) In local trivialization $\mathbb{R}^n \times V \to \mathcal{V}$, the absolute derivative is defined by

$$\nabla_\xi s = \frac{\partial s}{\partial x^i}\xi^i + \lambda'(\gamma_i \xi^i) \circ s$$

where $\lambda' : \mathfrak{g} \to \mathfrak{gl}(V)$ is the tangent (Lie algebra homomorphism) to $\lambda : G \to GL(V)$.
(4) The parallel transport c^V on $\mathcal{V}$ along the curve c on M with the initial condition $c^V(0) = \{\tilde{c}(0), v_0\}$ is given by $c^V(t) = \{\tilde{c}(t), v_0\}$, where $\tilde{c}(t)$ is the parallel transport on $\mathcal{P}$ along c.
(5) The curvature $R^V \in \Omega^2(M; V\mathcal{V})$ of the induced connection γ^V is given by $R^V(\xi, \eta)(\{u, v\}) = \{\zeta_{\rho(\xi,\eta)(u)}(u), 0_v\}$.
(6) Using the identification $V\mathcal{V} = \mathcal{V} \oplus \mathcal{V}$, $R^V(\xi, \eta) \in GL(E)$ for all $\xi, \eta \in \mathfrak{X}(M)$ and $R^V(\xi, \eta)(\chi) = \lambda'(\rho(\xi, \eta))(\chi)$.

Proof. We start with the claim (2): Let us recall that we use the same notation for the connection form γ as for the horizontal lift of the vector fields $\xi \mapsto \gamma(\xi)$ and let γ^V be the associated horizontal lift. Clearly $\gamma^V : \xi \mapsto \{\gamma(\xi)(u), 0_s\} \in T_{\{u,s\}}\mathcal{V}$. Let us choose some local section σ of $\mathcal{P}$, so that we may write $x \mapsto \{\sigma(x), s(\sigma(x))\}$ for the section of $\mathcal{S}$. The formula for the absolute derivative, see 1.3.1, then yields

$$\nabla_\xi\{\sigma, s \circ \sigma\} = T\{\sigma, s \circ \sigma\} \circ \xi - \{\gamma(\xi) \circ \sigma, 0\}.$$

Now, we may fix a frame $u \in \mathcal{P}$ over $x \in M$ and choose σ in such a way that $\gamma(\xi)(u) = T\sigma(\xi(x))$ for all ξ, i.e. a section σ which represents the defining jet of γ with target at u. This choice implies

$$\nabla_\xi\{\sigma, s \circ \sigma\} = \{\gamma(\xi)(u), Ts \circ \gamma(\xi)(u)\} - \{\gamma(\xi)(u), 0\}$$

which is exactly the vertical vector in $T\mathcal{V}$ which gives the value of $\mathcal{L}_{\gamma(\xi)}s$. Evaluating the same expression in $u \cdot g \in \mathcal{P}$, we obtain

$$\mathcal{L}_{\gamma(\xi)}(s)(u \cdot g) = Ts \circ \gamma(\xi) \circ r^g(u) = T(s \circ r^g) \circ \gamma(\xi)(u) = T\lambda_{g^{-1}} \cdot \mathcal{L}_{\gamma(\xi)}(u)$$

where we used the right invariance of $\gamma(\xi)$ and equivariance of s.

(1): In fact, the Lie derivative $\mathcal{L}_{\gamma(\xi)}s$ of the functions $s : \mathcal{P} \to S$ obviously has all properties required in 1.3.1 for the linear connections. In order to prove our claim, it suffices that the horizontal distribution defined by such a linear connection is exactly the distribution $\mathcal{H}^V$. But this is obvious from the construction.

(3): Follows from the general formula and the equivariance of s.

(4): The curve $c^V(t)$ obviously has tangent vectors in the horizontal distribution $\mathcal{H}^V$ and it has the right value $c^V(0)$. Since the parallel transport is given uniquely by its initial condition, this is sufficient.

(5): The formula will be clear once we understand the Lie bracket of vector fields on $\mathcal{V}$ in terms of the structure of the associated bundle. But the general formulae for the brackets in 1.2.8 shows that the fields $\tilde{\xi} = \{\xi, 0\}$ and $\tilde{\eta} = \{\eta, 0\}$ on $\mathcal{V}$ have the bracket

$$[\tilde{\xi}, \tilde{\eta}] = \{[\xi, \eta], 0\}.$$

Thus, the definition of the curvature R^V, evaluated on vector fields ξ and η in $\mathfrak{X}(M)$ yields

$$R^V(\xi, \eta)(\{u, v\}) = \{[\gamma(\xi), \gamma(\eta)] - \gamma([\xi, \eta]), 0\}$$

and this is equivalent to the required formula.

(6): This formula is an immediate consequence of the property (5) and the identification $V\mathcal{V} = \mathcal{V} \oplus \mathcal{V}$. □

The same reasoning applies to general associated bundles, except we have to be careful with the proper meaning of the formulae. In particular, they give results in the vertical tangent space to the fibred manifolds and so the splitting from (2) cannot be applied in general. Especially, dealing with affine bundles, we obtain the absolute derivatives as sections of the modelling associated vector bundles.

1.3.5. Affine connections on manifolds. The previous construction of induced connections can be often inverted. Each linear connection on a vector bundle $\mathcal{V}$ over M with standard fiber V is always induced by the unique principal connection on the frame bundle of $\mathcal{V}$, i.e. the principal bundle of all basis of the fibers of $\mathcal{V}$ with the structure group $GL(V)$.

In particular, the linear connections on the tangent bundle TM of a smooth manifold M are in one–to–one correspondence with the principal connections on the linear frame bundle $\mathcal{P}^1M$. The latter connections are briefly referred to as *linear connections on* M. Once we fix such a connection γ, then there are the induced connections on all tensor bundles (and more generally all associated bundles to $\mathcal{P}^1M$). The above coordinate formula 1.3.4(3) is just the classical formula for the *covariant derivative* with respect to a linear connection on a manifold.

The next ingredient which is specific for linear connections on manifolds is the existence of the *canonical form* $\theta \in \Omega^1(\mathcal{P}^1M; \mathbb{R}^m)$, $m = \dim M$. This is the form which takes the projection $\xi = Tp(\tilde{\xi})$ of a vector $\tilde{\xi} \in T_u\mathcal{P}^1M$ and yields the coordinates of ξ in the basis u. The names *solder form* or *soldering form* are often used for θ in the literature.

The canonical form is $GL(m,\mathbb{R})$–equivariant with respect to the standard action Ad of $GL(m,\mathbb{R})$ on $\mathbb{R}^m$ and it is strictly horizontal, i.e.
(1) $r_g^*\theta = \mathrm{Ad}_{g^{-1}} \circ \theta$
(2) $\theta(\tilde{\xi}) = 0$ if and only if $\tilde{\xi}$ is vertical.

The second claim is obvious from the definition of θ, the first one is just the transformation rule for coordinates under a change of the bases.

The choice of the notation Ad has a good reason. Let us recall the group $A(n,\mathbb{R})$ of all affine transformations of $\mathbb{R}^m$, see 1.2.5, and its abelian normal subgroup $\mathbb{R}^m$. The adjoint action on the corresponding subalgebra is just the standard action of the complementary subgroup $GL(m,\mathbb{R})$.

Now, given a linear connection $\gamma \in \Omega^1(\mathcal{P}^1M;\mathfrak{gl}(m,\mathbb{R}))$, we may consider the one–form $\omega = \theta + \gamma \in \Omega^1(\mathcal{P}^1M;\mathfrak{a}(m,\mathbb{R}))$. The following three properties of ω are obvious consequences of the definition and the already known facts:

Lemma. *The form $\omega = \theta + \gamma$ has the following properties*
(3) ω is $GL(m,\mathbb{R})$ equivariant, i.e. $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$,
(4) ω reproduces the fundamental fields, i.e. $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{gl}(m,\mathbb{R})$,
(5) ω is an absolute parallelism, i.e. $\omega_{|T_u\mathcal{P}^1M} : T_u\mathcal{P}^1M \to \mathfrak{a}(m,\mathbb{R})$ is a linear isomorphism for each frame u.

In fact, the canonical form θ is the specific ingredient which distinguishes the linear frame bundle from general principal bundles with structure group $GL(m,\mathbb{R})$. More explicitly, any principal $GL(m,\mathbb{R})$–bundle $\mathcal{P}$ over an m–dimensional manifold M equipped with a form θ with properties (1) and (2) is canonically isomorphic to $\mathcal{P}^1M$. In this isomorphism, a frame $u \in \mathcal{P}$ yields the basis of TM via the isomorphism $\theta(u) : T_{p(u)}M \to \mathbb{R}^m$.

We define an *affine connection* on an m–dimensional manifold M as a principal $GL(m,\mathbb{R})$–bundle $p : \mathcal{P} \to M$, together with a fixed form $\omega \in \Omega^1(\mathcal{P},\mathfrak{a}(m,\mathbb{R}))$ which has the properties (3), (4), (5) above. Let us summarize the basic properties of affine connections on manifolds. We write $\omega = \theta + \gamma$ for the decomposition into the $\mathbb{R}^m$–component and the $\mathfrak{gl}(m,\mathbb{R})$–component of ω.

Proposition. *Let $\omega \in \Omega^1(\mathcal{P},\mathfrak{a}(m,\mathbb{R}))$ be an affine connection.*
(6) The $\mathbb{R}^m$–component θ of ω has properties (1) and (2) of the canonical forms and there is the uniquely defined isomorphism of principal $GL(m,\mathbb{R})$–bundles $\varphi : \mathcal{P} \to \mathcal{P}^1M$ such that θ is the pullback of the canonical form. Moreover, the $\mathfrak{gl}(m,\mathbb{R})$–component γ is a principal connection on $\mathcal{P}$.
(7) For each $X \in \mathfrak{a}(m,\mathbb{R})$, there is the constant vector field $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ such that $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{P}$. Moreover,

$$\omega^{-1}(X)(u\cdot g) = Tr^g \cdot \omega^{-1}(\mathrm{Ad}_g \cdot X)(u)$$

for all $g \in GL(m,\mathbb{R})$, $u \in \mathcal{P}$, and ω^{-1} recovers the fundamental vector fields for all $X \in \mathfrak{gl}(m,\mathbb{R})$.
(8) The tangent bundle TM is associated to $\mathcal{P}$ via the identification

$$\{u,X\} \in \mathcal{P} \times_{GL(m,\mathbb{R})} \mathbb{R}^m \quad \mapsto \quad Tp(\omega^{-1}(X)(u)) = \xi \in T_{p(u)}M,$$

while the horizontal lift of ξ to $T_u\mathcal{P}$ is just $\omega^{-1}(X)(u)$.
(9) The absolute derivative (i.e. the covariant derivative) with respect to γ is given

for all sections $s : \mathcal{P} \to S$ of associated bundles $\mathcal{P} \times_{GL(m,\mathbb{R})} S$ by the formula

$$\nabla^\gamma_\xi s(u) = \mathcal{L}_{\omega^{-1}(X)} s(u)$$

where $\xi = \{u, X\}$.

(10) The formula $K = d\omega + \frac{1}{2}[\omega, \omega]$ defines a horizontal 2–form in $\Omega^2(\mathcal{P}, \mathbb{R}^m \oplus \mathfrak{gl}(m,\mathbb{R}))$. Its second component is the curvature of γ while the first component is the torsion of γ.

(11) For each $u \in \mathcal{P}$, the mapping $\varphi_u : \mathbb{R}^m \to M$, $\varphi_u(X) = p \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$ is a well defined diffeomorphism on a neighborhood of the origin in $\mathbb{R}^m$, the so called normal coordinates *of γ. These coordinates satisfy $\varphi_{u \cdot g} = \varphi_u \circ \mathrm{Ad}_g$.*

(12) The geodesics of γ are the curves $\alpha_u^X(t) = p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$, i.e. the images of straigt lines in the normal coordinates. They satisfy the equation $\nabla^\gamma_{\dot\alpha}\dot\alpha = 0$, $\alpha(0) = p(u)$, with the initial condition $\dot\alpha(0) = Tp(\omega^{-1}(X)(u))$. Each tangent vector in T_xM defines exactly one geodetic curve.

Proof. All claims are simple consequences of the definitions and previous considerations. Moreover, we shall reprove them all later in the framework of the general Cartan connections. □

Example. The Maurer–Cartan form on $A(m,\mathbb{R})$ defines the flat affine connection on the homogeneous space, the affine space $\mathbb{R}^m$, see 1.2.5. The normal coordinates are just the usual affine coordinates, the geodesics are just all straight lines, etc.

1.3.6. Connections on G–structures. Let $H \subset GL(m,\mathbb{R})$ be a closed subgroup, M an m–dimensional manifold, and let $B = \mathbb{R}^m \rtimes H \subset A(m,\mathbb{R})$ be the affine extension of H in the affine group $A(m,\mathbb{R})$. A *G–structure on M with structure group H* is a principal H–bundle $p : \mathcal{P} \to M$ together with a form $\theta \in \Omega^1(\mathcal{P}; \mathbb{R}^m)$ satisfying 1.3.5(1)–(2) with respect to elements $g \in H$. Then $\mathcal{P}$ is identified with a principal subbudle $\mathcal{P} \subset \mathcal{P}^1M$ by means of the form θ: each element u is identified with a frame in $\mathcal{P}^1M$ with coordinate mapping $\xi \mapsto \theta(u)(\tilde\xi)$ where $\tilde\xi$ is any lift of ξ to $T_u\mathcal{P}$. Equivalently, a G–structure with structure group H is a reduction $\iota : \mathcal{P} \to \mathcal{P}^1M$ to the structure group H. Then the pullback of the canonical form is the form θ required above.

This definition allows straightforward generalizations. Let us consider a group H such that its Lie algebra is a subalgebra in $\mathfrak{gl}(m,\mathbb{R})$ and let $\mathfrak{b} = \mathbb{R}^m \rtimes \mathfrak{h}$ be the affine extension of $\mathfrak{h}$. The principal H–bundles together with the forms θ as above are then identified with coverings of subbundles in $\mathcal{P}^1M$ and we shall often use the name (first order) *G–structure with structure group H* in this context too. Even more generally, we might consider a principal fiber bundle $\mathcal{P} \to M$ with structure group $\tilde H$ and a homomorphism of principal bundles $\mathcal{P} \to \mathcal{Q} \subset \mathcal{P}^1M$ over a group homomorphism $\tilde H \to H \subset GL(m,\mathbb{R})$. We talk than about the *generalized first order G–structures*. Obviously, the latter two concepts coincide if and only if the homomorphism $H \to H_1$ is injective on the level of the Lie algebras.

A *connection on the G-structure* $\mathcal{P}$ is an affine connection $\omega = \theta + \gamma \in \Omega^1(\mathcal{P}; \mathfrak{b})$, where θ is the defininig canonical form.

The G–structures as subbundles in $p : \mathcal{P}^1M \to M$ allow yet another simple description. Let $H \subset GL(m,\mathbb{R})$ be the structure group and consider the bundle $\mathcal{P}^1M/H$. Each section σ of this bundle defines a principal subbundle $Q \subset \mathcal{P}^1M$ of frames $u \in \mathcal{P}^1M$ with $[u] = \sigma(p(u)) \in \mathcal{P}^1M/H$. On the other hand, each reduction $Q \subset \mathcal{P}^1M$ defines the section $\sigma = \pi(Q)$ where $\pi : \mathcal{P}^1M \to \mathcal{P}^1M/H$ is the factor projection. Thus, there is the bijective correpsondence between smooth global sections of $\mathcal{P}^1M/H$ and the G–structures with structure group H. Notice $\mathcal{P}^1M/H \simeq Q \times_H (GL(m,\mathbb{R})/H)$ with $[u] = \{w,[g]\}$ where $u = wg$, $w \in Q$, and so $\sigma : Q \to GL(m,\mathbb{R})/H$ is given by the constant mapping $w \mapsto [e]$. In particular, the defining section σ is covariantly constant for all connections on the G–structure $Q \subset \mathcal{P}^1M$.

Example. There are many well known G–structures, let us name just a few:
(1) Riemannian structure is the reduction of $\mathcal{P}^1M$ to orthonormal frames, i.e. to $H = O(m,\mathbb{R}) \subset GL(m,\mathbb{R})$. The associated bundles $\mathcal{P}^1M/H$ are the bundles of non–denerate positive definite symmetric 2–tensors, the Riemannian metrics.
(2) An almost symplectic structure is a reduction of $\mathcal{P}^1M$ to the symplectic group $H = Sp(m,\mathbb{R})$. Here the bundle $\mathcal{P}^1M/H$ is the tensor bundle of non-degenerate antisymmetric 2–tensors and its sections are called sympletic forms.
(3) An absolute parallelism is the reduction to the trivial subgroup $\{1\} \subset GL(m,\mathbb{R})$. In this case, the G–structures are the global trivializations of $\mathcal{P}^1M$.
(4) A Riemannian spin structure is a two–fold covering of the first example.

Remark. Each principal connection on a reduction $\mathcal{Q}$ of a principal bundle $\mathcal{P}$ extends canonically to a principal connection on $\mathcal{P}$. Indeed, the form is given on the vertical vectors in all frames of P and we may simply extend the principal connection form by the right multiplication on all vectors in all frames. The other direction, i.e. the question whether a given connection on $\mathcal{P}$ comes from a connection on $\mathcal{Q}$ by this canonical extension is much harder. The answer is known under the name Ambrose–Singer Theorem and its version for general fiber bundles is given in [**KMS**, 9.11].

All properties of affine connections listed in proposition 1.3.5 are valid for connections on G–structures and they may easily be deduced by means of the latter remark, but they can be also proven directly. Again, they will follow from our general treatment of the Cartan connections below.

Let us pay a special attention to the $\mathbb{R}^m$–component K_- of the curvature K, the *torsion* T of a connection on the G-structure. More explicitly,

$$\begin{aligned} T(\xi,\eta)(u) &= K_-(\tilde{\xi},\tilde{\eta})(u) = d\theta(\tilde{\xi},\tilde{\eta})(u) + [\theta(\tilde{\xi}) + \gamma(\tilde{\xi}), \theta(\tilde{\eta}) + \gamma(\tilde{\eta})]_{\mathbb{R}^m} \\ &= d\theta(\tilde{\xi},\tilde{\eta})(u) \end{aligned}$$

where $u \in \mathcal{P}$, $\xi, \eta \in \mathfrak{X}(M)$, $\tilde{\xi}$ and $\tilde{\eta}$ are the horizontal lifts, and the $\mathbb{R}^m$–component of the bracket vanishes since $\mathbb{R}^m$ is abelian and the principal connection form γ vanishes on the horizontal subspaces. This formula also shows that the torsion is obtain equivalently as $T = d^\gamma\theta$ in terms of the exterior absolute derivative.

Now, let us notice that the tangent vector $\{u, T(\xi,\eta)(u)\} \in T_xM$ does not depend on the choice of the frame u. Indeed, the right invariance of the horizontal lift and the equivariance of θ imply exactly this behavior. Thus we may understand the torsion as a TM–valued form $T \in \Omega^2(M;TM)$. With this identification in mind, a simple computation using the definition of the covariant derivative ∇_ξ as the Lie derivative of the frame form $\theta(\tilde{\eta})$ of the vector field η in the direction of the horizontal lift $\tilde{\xi}$ of the vector field ξ reveals

$$\begin{aligned} T(\xi,\eta)(x) &= \{u, \big(d\theta(\tilde{\xi},\tilde{\eta})\big)(u)\} \\ &= \{u, \big(\mathcal{L}_{\tilde{\xi}}\theta(\tilde{\eta}) - \mathcal{L}_{\tilde{\eta}}\theta(\tilde{\xi}) - \theta([\tilde{\xi},\tilde{\eta}])\big)(u)\} \\ &= \big(\nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta]\big)(x) \end{aligned}$$

which is the more usual (defining) formula for the torsion of an affine connection. Moreover, in any local coordinates we obtain

$$\begin{aligned} T(\xi,\eta)(x) = &\left(\frac{\partial\eta^j}{\partial x^i}\xi^i - \gamma^j_{ik}\eta^i\xi^k - \frac{\partial\xi^j}{\partial x^i}\eta^i + \gamma^j_{ik}\xi^i\eta^k - [\xi,\eta]\right)\frac{\partial}{\partial x^j}(x) = \\ &= \left((\gamma^j_{ik} - \gamma^j_{ki})\xi^i\eta^k\right)\frac{\partial}{\partial x^j}(x), \end{aligned}$$

and thus the torsion is given by the antisymmetrization of the Christoffels,

$$T(\xi^i,\eta^k) = T^j_{ik}\xi^i\eta^k = \gamma^j_{[ik]}\xi^i\eta^k.$$

1.3.7. Partial connections. Our first attempt to the definition of a connection was the family of the first order linear differential operators ∇_ξ on sections of a given vector bundle $\mathcal{V} \to M$, defined for all vector fields $\xi \in \mathfrak{X}(M)$, satisfying a set of axioms, and in fact the values $\nabla_\xi s(x)$ depend on the value $\xi(x)$ only, see 1.3.1. In this context, a *partial connections* is simply a family of such operators with ξ restricted to a fixed distribution $\mathcal{D} \subset \mathfrak{X}(M)$.

If we follow the same track as above starting at 1.3.2, we can easily find the equivalent definition of the partial connection on $\mathcal{V}$ in the terms of partial horizontal distribution, the partial horizontal lift of vector fields, etc. We just copy the concepts from the beginning of this section restricted to vectors from a fixed distribution $\mathcal{D}$. Similarly, the right invariant partial horizontal distributions form the partial principal connections. The definition of the curvature does not make sense any more in general since the bracket of two fields from the distribution need not to be in the distribution again.

A partial linear connection on TM is called a *partial affine connection on* M. In this case, there is at least the well defined concept of the torsion of the partial connection which is defined by the standard formula $T(\xi,\eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta]$ for all ξ, η in $\mathcal{D}$. This means that the torsion is a mapping $\mathcal{D} \wedge CalD \to TM$ and its composition with the quotient projection $TM \to TM/\mathcal{D}$ is just the algebraic projection $\mathcal{D} \wedge \mathcal{D} \to TM/\mathcal{D}$ of the Lie bracket of vector fields, i.e. the obstruction against the integrability of $\mathcal{D}$. We may also view the partial torsion T of the affine partial connection as the restriction of the full curvature of any extension of the partial connection ∇ to an affine connection ω which does not depend on the choice. Of course, if the distribution $\mathcal{D}$ is involutive, then the partial curvature is well defined but this is because the partial connections are then in fact families of standard connections on the individual leaves of the associated foliation.

1.3.8. Remarks. The next natural item in our chain of various versions of connections is the concept of the *Cartan connection*, see 1.5 below. Since we prefere to study homogeneous spaces in some details first, let us at least summarize our views at this point. The affine connections clearly may be understood as the general curved version of the flat affine space $\mathbb{R}^n$ with its standard flat linear connection. In this respect, the general linear frame bundle $\mathcal{P}^1M \to M$ plays the curved role of $A(n,\mathbb{R}) \to A(n,\mathbb{R})/GL(n,\mathbb{R}) = \mathbb{R}^n$ while the affine connection is the counterpart of the Maurer–Cartan form. Similarly, the connections on G–structures are special cases of affine conections whose flat models are $B \to B/H$, where B is the affine extension of the subgroup $H \subset GL(n, \mathbb{R})$.

The general point of our interest in these constructions is the existence of an absolute parallelism ω on the total space of the frame bundle $\mathcal{P}$, remarkably similar to that one given by the Maurer–Cartan form on the corresponding homogeneous model. As we shall see later, there is a general theory for such situations which allows to generalize much of the standard calculus for affine connections. This is the key feature which makes the difference to the general theory of principal connections. In our example of affine conections $\omega \in \Omega^1(\mathcal{P}^1M, \mathfrak{g})$, this amounts to the extra data encoded in the soldering form θ.

1.4. Geometry of homogeneous spaces

As stated in the introduction to this chapter, Cartan geometries are geometric structures associated to a homogeneous space, which is then called the *homogeneous model* of the Cartan geometry. One of the nice features of this point of view is that many general geometric properties can be read off directly from the homogeneous model. Otherwise put, many questions about Cartan geometries, like existence of canonical linear connections on certain vector bundles, or the local structure of the set of canonical curves, can be answered by looking at the homogeneous model only. Moreover, there is always the subclass of locally flat geometries, which on one hand for many structures is still quite rich and interesting, while on the other hand, for geometries in this subclass practically everything can be read off the homogeneous model.

Therefore, we study in this section geometric properties of homogeneous spaces, i.e. geometric questions on manifolds which are invariant under a transitive action of a Lie group.

1.4.1. Homogeneous bundles. Let G be a Lie group, $H \subset G$ a closed subgroup, $M := G/H$ the corresponding homogeneous space and $p : G \to G/H$ the canonical projection, which is an H–principal bundle, see 1.2.5. By $\ell : G \times M \to M$ we denote the canonical left action $\ell(g, g'H) := gg'H$. The first step towards G–invariant geometric objects on M is to get an appropriate class of fiber bundles over the homogeneous space.

Definition. (1) A *homogeneous bundle* over $M = G/H$ is a locally trivial fiber bundle $\pi : E \to M$ together with a left G–action $\tilde{\ell} : G \times E \to E$, which extends the action on M, i.e. which satisfies $\pi(\tilde{\ell}(g, e)) = \ell(g, \pi(e))$.
(2) A *homogeneous vector bundle* over M is a homogeneous bundle $\pi : E \to M$, which is a vector bundle and such that for each element $g \in G$ the bundle map $\tilde{\ell}_g : E \to E$ is linear in each fiber.
(3) A *morphism* of homogeneous bundles (respectively homogeneous vector bundles) is a G–equivariant bundle map (respectively G–equvivariant homomorphism of vector bundles) which covers the identity on M.

If there is no risk of confusion, we will always denote all actions simply by dots, so the definition of a homogeneous bundle simply reads as $\pi(g \cdot e) = g \cdot \pi(e)$. From the definitions it is also clear that homogeneous fiber bundles (respectively homogeneous vector bundles) form a category $\mathrm{Fib}_G(G/H)$ (respectively $\mathrm{Vect}_G(G/H)$).

Example. There are two obvious sources of homogeneous bundles. We shall see that the first one is a special case of the second one and that there are no other homogeneous bundles, up to isomorphisms.

(1) Let n be the dimension of the homogeneous space $M = G/H$ and suppose that F is a natural bundle on the category $\mathcal{M}_n$ of n–dimensional manifolds and local diffeomorphsims, see 1.2.7 or [**KMS**, section 14]. Now consider the bundle $p_M : F(M) \to M$. For any $g \in G$, the left action $\ell_g : M \to M$ of g on M is a diffeomorphism (with inverse $\ell_{g^{-1}}$), so we have the induced bundle map $F(\ell_g) : F(M) \to F(M)$ which covers $\ell_g : M \to M$. Since F is a functor, $F(\ell_e) = \mathrm{id}_{F(M)}$ and $F(\ell_{gg'}) = F(\ell_g \circ \ell_{g'}) = F(\ell_g) \circ F(\ell_{g'})$, and thus the map $\tilde{\ell} : G \times F(M) \to F(M)$ defined by $\tilde{\ell}(g, e) = F(\ell_g(u)$ defines a left action of G on $F(M)$, which is smooth by regularity of F. Thus, any natural bundle over a homogeneous space is a homogeneous bundle.

In particular, we can apply this line of argument to natural vector bundles, for which any of the bundle maps $F(f)$ is a vector bundle homomorphism, to get homogeneous vector bundles. In particular, all tensor bundles over G/H are homogeneous.

(2) The second basic source of homogeneous bundles is the canonical H–principal bundle $p : G \to G/H$. By definition of the action of G on G/H, this is a homogeneous bundle under the action $\tilde{\ell} : G \times G \to G$ which is just given by the multiplication map. Now assume that S is any smooth manifold with a smooth left action $H \times S \to S$. Then we can form the associated (or induced) bundle $E := G \times_H S \to M$. By definition (see 1.2.6) this is the space of orbits in $G \times S$ of the right action $(g, s) \cdot h = (gh, h^{-1} \cdot s)$ of H. We have denoted the orbit of (g, s) by $\{g, s\}$. Since left and right translations on a Lie group always commute, the map $G \times G \times S \to G \times_H S$ defined by $(g, g', s) \mapsto \{gg', s\}$ descends to a map $G \times (G \times_H S) \to G \times_H S$ which is smooth since $G \times G \times S \to G \times (G \times_H S)$ is a surjective submersion (even an H–principal bundle). From the construction it is clear that this defines a smooth left action of G on E, and since the projection $E \to M$ is simply given by $\{g, s\} \mapsto gH$, this action extends the canonical action of G on M. Thus, any associated bundle to the principal bundle $G \to G/H$ is canonically a homogeneous bundle. unify notation { , } and [,]!

If we start with a representation of H on a vector space V, then the induced bundle $G \times_H V$ is a vector bundle, and the linear structure is determined by $\{g, v\} + t\{g, w\} = \{g, v + tw\}$. Thus, the above constructions gives rise to an action of G on $G \times_H V$ by vector bundle homomorphisms, and thus $G \times_H V \to G/H$ is a homogeneous vector bundle.

1.4.2. Classification of homogeneous bundles. We shall come quickly to a complete description of homogeneous bundles and homogeneous vector bundles. Suppose that $p_E : E \to M$ is a homogeneous bundle over $M = G/H$. Consider the base point $o = eH \in G/H$ and the fiber $S := E_o$ of E over o. For $g \in G$ and $s \in S$ we get $p_E(g \cdot s) = g \cdot o = gH$ by definition of a homogeneous bundle. In particular, $p_E(h \cdot s) = o$ for all $h \in H$, so the left G–action on E restricts to a left action of H on S. Now consider the associated bundle $G \times_H S \to M$, and consider the map $G \times S \to E$ defined by $(g, s) \mapsto g \cdot s$, where we use the right action of G on E, and view s as an element of $E_o \subset E$. Since the action of H on S is just the restriction of the action of G on E, this mapping factorizes to a map $\Phi : G \times_H S \to E$ which is smooth since the projection $G \times S \to G \times_H S$

is a surjective submersion. Moreover, by definition $\Phi(\{g,s\}) = g \cdot s$, which on one hand shows that Φ is a bundle map covering the identity and on the other hand implies that Φ is G–equivariant, so $\Phi : G \times_H S \to E$ is a morphism of homogeneous bundles. If on the other hand $e \in E$ is any element, then choose an element $g \in G$ such that $g \cdot p_E(e) = o$. Then $p_E(g \cdot e) = o$, so $g \cdot e \in S = E_o$, and we may consider $\{g^{-1}, g \cdot e\} \in G \times_H S$. If $g' \in G$ is another choice, then $(g'g^{-1}) \cdot o = o$, so $g'g^{-1} \in H$, and thus $\{(g')^{-1}, g' \cdot e\} = \{(g')^{-1}g'g^{-1}, g(g')^{-1}g' \cdot e\} = \{g^{-1}, g \cdot e\}$, so we get a well defined map $\Psi : E \to G \times_H S$ in this way. Moreover, using a local smooth section σ of $G \to G/H$ we can locally use the choice $\sigma(p_E(e)) \in G$, which shows that Ψ is smooth, and one immediately verifies that Ψ is inverse to Φ, so these are inverse isomorphisms of homogeneous bundles.

Moreover, if $f : E \to E'$ is a morphism of homogeneous bundles, then $f(E_o) \subset E'_o$ and the restriction $f|_{E_o} : E_o \to E'_o$ is H–equivariant. Equivariancy of f now implies that for $g \in G$ and $s \in S = E_o$, we get $f(g \cdot s) = g \cdot f(s)$ which means that $f(\Phi(\{g,s\})) = \Phi'(\{g, f|_{E_o}(s)\})$, so we see that identifying E and E' with with $G \times_H E_o$ respectively $G \times_H E'_o$, the map f is induced by $\mathrm{id} \times f|_{E_o}$. Conversely, any H–equivariant map $S \to S'$ clearly induces a morphism $G \times_H S \to G \times_H S'$ of homogeneous bundles.

Finally note that for a homogeneous vector bundle, the above construction clearly produces an isomorphism $G \times_H E_o \to E$ of homogeneous vector bundles, and homomorphisms of homogeneous vector bundles correspond to linear H–equivariant maps between their standard fibers. Thus we have proved

Proposition. *The mapping $E \mapsto E_o$ and $f \mapsto f_{E_o}$, which assigns to any homogeneous bundle its standard fiber and to any morphism of homogeneous bundles its restriction to the standard fibers induces equivalencies between the category $\mathrm{Fib}_G(G/H)$ and the category of manifolds endowed with left H–actions and H–equivariant smooth maps, as well as between the category $\mathrm{Vect}_G(G/H)$ and the category of finite dimensional representations of H.*

Remark. The proposition also implies immediately that this equivalences of categories are compatible with various constructions. For example, the fibered product of homogeneous fiber bundles corresponds to the product of left H–spaces, the Whitney sum of homogeneous vector bundles corresponds to the direct sum of representations and the tensor product of homogeneous vector bundles corresponds to the tensor product of representations. This also works the other way round. If for example we start with two homogeneous vector bundles corresponding to indecomposable representations of H, then the decomposition of the tensor product of these two representations into a direct sum of indecomposable representations induces a decomposition of the tensor product of the two homogeneous vector bundles into a Whitney sum.

Example. Let us determine the H–representations corresponding to tensor bundles in a way which allows direct generalizations (cf. also 1.3.5(8)). We start by identifying the tangent bundle, so we have to determine $T_o(G/H)$ as a vector space and as an H–module. The projection map $p : G \to G/H$ is a submersion and thus for each $g \in G$ the tangent map $T_g p : T_g G \to T_{p(g)} G/H$ is surjective. In particular, we get a surjection $T_e p : \mathfrak{g} = T_e G \to T_o G/H$. Moreover, the representation of H on $T_o G/H$ is just obtained as the restriction of the action of G on the homogeneous vector bundle $T(G/H)$, so the action of $h \in H$ on $T_o(G/H)$ is just $T_o \ell_h$. By definition, the action on G/H is induced by left translations in

G, so $\ell_h \circ p = p \circ \lambda_h$ and thus $T_o\ell_h \circ T_e p = T_h p \circ T_e\lambda_h$. On the other hand, p commutes with right multiplications by elements from h, and differentiating this, we see that $T_h p = T_e p \circ T_h \rho^{h^{-1}}$. Since $\rho^{h^{-1}} \circ \lambda_h$ is just the conjugation by h, whose derivative is the adjoint action, we get $T_o\ell_h \circ T_e p = T_e p \circ \mathrm{Ad}(h)$. Hence $T_o(G/H)$ is a quotient of $\mathfrak{g}$, with the H–module structure defined by the restriction of the adjoint representation Ad to the subgroup $H \subset G$. On the other hand, the kernel of $T_e p$ is simply the subalgebra $\mathfrak{h}$, which also is an H–submodule by naturality of the adjoint action.

Thus the homogeneous vector bundle $T(G/H)$ corresponds to the H–representation $\mathfrak{g}/\mathfrak{h}$ given by the adjoint representation of G restricted to the subgroup H. By the proof of proposition 1.4.2 the isomorphism $G \times_H (\mathfrak{g}/\mathfrak{h}) \to T(G/H)$ maps $\{g, X+\mathfrak{h}\}$ to $T_g p \cdot T_e\lambda_g \cdot X$. The opposite isomorphism can be conveniently described using the Maurer Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ of the group G. Again by the proof of proposition 1.4.2, for a tangent vector $\xi_x \in T_x(G/H)$, we have to choose an element $g \in G$ such that $x = g^{-1}H$ and then view $T\ell_g \cdot \xi_x \in T_o(G/H)$ as an element of $\mathfrak{g}/\mathfrak{h}$. Having chosen g, we know that $T_{g^{-1}}p : T_{g^{-1}}G \to T_x(G/H)$ is surjective, so we can choose a tangent vector $\tilde{\xi} \in T_g G$ such that $\xi_x = Tp \cdot \tilde{\xi}$. But then $T\ell_g \cdot \xi_x = Tp \cdot T\lambda_g \cdot \tilde{\xi}$, and identifying $T_e G$ with $\mathfrak{g}$ and $T_o(G/H)$ with $\mathfrak{g}/\mathfrak{h}$, the projection Tp becomes just the canonical projection to the quotient. Since $T\lambda_g \cdot \tilde{\xi}$ by definition equals $\omega(\tilde{\xi})$, we see that the isomorphism $T(G/H) \to G \times_H (\mathfrak{g}/\mathfrak{h})$ maps ξ_x to $\{g, \omega(\tilde{\xi}) + \mathfrak{h}\}$.

Another way to rephrase this is to consider $\pi \circ \omega \in \Omega^1(G, \mathfrak{g}/\mathfrak{h})$, where $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical surjection. This map is clearly H–equivariant (since the Maurer–Cartan form is even G–equivariant) and its kernel in any point of G is exactly the vertical tangent space in that point. A straightforward extension of the arguments in 1.2.5 and 1.3.6 shows that there is a unique homomorphism $G \to \mathcal{P}^1(G/H)$ from the principal bundle $G \to G/H$ to the linear frame bundle $\mathcal{P}^1(G/H)$ of G/H such that $\pi \circ \omega$ is the pullback of the canonical soldering form on the frame bundle to G. Thus, via the Maurer Cartan form, we may view the principal bundle $G \to G/H$ as a generalized first order H_1–structure on G/H, where $H_1 \subset GL(\mathfrak{g}/\mathfrak{h})$ is the image of H under the representation from above.

By the naturality of the correspondence between homogeneous vector bundles and H–representations, this implies that the cotangent bundle $T^*(G/H)$ corresponds to the dual representation $(\mathfrak{g}/\mathfrak{h})^*$. Note that $(\mathfrak{g}/\mathfrak{h})^*$ is just the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$, so this is a subrepresentation of the restriction of the coadjoint representation $\mathfrak{g}^*$ to H. Moreover, again by naturality, the tensor bundle $\otimes^k T(G/H) \otimes \otimes^\ell T^*(G/H)$ corresponds to the representation $\otimes^k(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^\ell(\mathfrak{g}/\mathfrak{h})^*$.

1.4.3. Sections of homogeneous bundles. Let $p_E : E \to G/H$ be a homogeneous bundle over a homogeneous space G/H, and let $\Gamma(E)$ be the set of all sections of E. By definition, $\Gamma(E) = \{\sigma : G/H \to E : p_E \circ \sigma = \mathrm{id}\}$. Then we immediately get a left action of G on $\Gamma(E)$ defined by $g \cdot \sigma := \tilde{\ell}_g \circ \sigma \circ \ell_{g^{-1}}$, where $\tilde{\ell}$ and ℓ are the actions on E and G/H, respectively. Equivalently, this can be written as $(g \cdot \sigma)(x) = g \cdot (\sigma(g^{-1} \cdot x))$, and one immediately verifies that this really defines an action of G. In particular, if E is a homogeneous vector bundle, then $\Gamma(E)$ is a vector space, and the action of G on $\Gamma(E)$ is clearly linear, and thus we get a representation of G on the space $\Gamma(E)$. In view of proposition 1.4.2, the homogeneous vector bundle E is completely determined by the representation of H on its standard fiber $V = E_o$. The representation of G on $\Gamma(E)$ is then called the *induced* *representation* of G corresponding to the representation V of H, and denoted by $\mathrm{Ind}_H^G(V)$.

The correspondence between sections of an associated bundle and functions of the total space of the principal bundle, see 1.2.6, gives rise to another interpretation of the action of G on the set of sections of a homogeneous bundle, which is (in the case of homogeneous vector bundles) frequently used in representation theory. Namely, via the isomorphism $E \cong G \times_H E_o$ from proposition 1.4.2, we can identify $\Gamma(E)$ with $\Gamma(G \times_H E_o)$, which in turn by proposition 1.2.6 is in bijective corespondence with the set $C^\infty(G, E_o)^H = \{f : G \to E_o : f(gh) = h^{-1} \cdot f(g)\}$. In fact, the bijective correspondence of the latter set with $\Gamma(E)$ can also be easily established directly. To $\sigma : G/H \to E$, we associate the function $f : G \to E_o$ defined by $f(g) = g^{-1} \cdot \sigma(gH)$, which is visibly equivariant. Conversely, having given $f : G \to E_o$ such that $f(gh) = h^{-1} \cdot f(g)$, for a point $x \in G/H$ we choose an element $g \in G$ such that $x = gH$ and put $\sigma(x) = g \cdot f(g)$, which is independent of the choice of g by equivariancy of f and visibly defines a section of E. Since the two constructions are obviously inverse to each other, they establish the required bijection. Moreover, in the case of a natural vector bundle, the obtained bijection is clearly a linear isomorphism, since both for sections and functions the operations are pointwise, and all actions under consideration are by linear maps.

To get the action of G on $\Gamma(E)$ in the picture of equivariant functions, we only have to notice that by definition $(g \cdot \sigma)(g'H) = g \cdot \sigma(g^{-1}g'H)$. Consequently, the map $g \cdot f : G \to E_o$ corresponding to $g \cdot \sigma \in \Gamma(E_o)$ is given by

$$(g \cdot f)(g') = (g')^{-1} \cdot (g \cdot \sigma(g^{-1}g'H)) = (g^{-1}g')^{-1} \cdot \sigma(g^{-1}g'H) = f(g^{-1}g').$$

Our next result is a first version of Frobenius reciprocity, which states that in certain situations one can reduce questions about the (often infinite dimensional) space $\Gamma(E)$ to questions about the finite dimensional manifold E_o. In particular, determination of G–invariant sections of a homogeneous bundle always reduces to a finite dimensional problem.

THEOREM (geometric version of Frobenius reciprocity). *Let $E \to G/H$ be a homogeneous bundle with standard fiber E_o (viewed as an H–space), and let X be a smooth manifold endowed with a smooth left G–action. Then the evaluation at o induces a natural bijection between the set of G–equivariant maps $X \to \Gamma(E)$ and the set of H–equivariant maps $X \to E_o$. In particular, there is a natural bijection between the set $\Gamma(E)^G$ of G–invariant sections of E and the set $(E_o)^H$ of H–invariant elements in the standard fiber.*

If E is the natural vector bundle induced by an H–representation W, and V is a representation of G, then the bijections above are linear and respect the subspaces of linear equivariant maps. Denoting by $\mathrm{Res}_H^G(V)$ the space V viewed as an H–representation, this implies that we get a linear isomorphism $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)$, i.e. Res_H^G and Ind_H^G are adjoint functors.

PROOF. Suppose we have given a map $X \to \Gamma(E)$, which we write as $x \mapsto \sigma_x$. This map is G–equivariant if and only if $\sigma_{g\cdot x}(g'H) = g \cdot \sigma_x(g^{-1}g'H)$. In particular, if we consider the map $X \to E_o$ defined by $x \mapsto \sigma_x(o)$, then for $h \in H$, we get $\sigma_{h\cdot x}(o) = h \cdot \sigma_x(h^{-1} \cdot o) = h \cdot \sigma_x(o)$, so this is H–equivariant. Conversely, if $f : X \to E_o$ is H–equivariant, then for $x \in X$ we define $\sigma_x : G/H \to E$ by $\sigma_x(gH) = g \cdot f(g^{-1} \cdot x)$. This is well defined since $f(h^{-1}g^{-1} \cdot x) = h^{-1} \cdot f(g^{-1} \cdot x)$, and thus $gh \cdot f(h^{-1}g^{-1} \cdot x) = g \cdot f(g^{-1} \cdot x)$. Moreoverover, $\sigma_x(o) = f(x)$ and

$$\sigma_{g\cdot x}(g'H) = g' \cdot f((g')^{-1}g \cdot x) = g \cdot g^{-1}g' \cdot f((g')^{-1}g \cdot x) = g \cdot \sigma_x(g^{-1}g'H),$$

so $x \mapsto \sigma_x$ is G–equivariant. Since for any G–equivariant map $x \mapsto \sigma_x$ we must have $\sigma_x(gH) = g \cdot g^{-1} \cdot \sigma_x(g^{-1}gH) = g \cdot \sigma_{g\cdot x}(o)$, the two constructions are inverse bijections between the set of G–equivariant maps $X \to \Gamma(E)$ and the set of H–equivariant maps $X \to E_o$.

Taking $X = \{pt\}$ a single point space with the trivial G–action, a G–equivariant map $X \to \Gamma(E)$ is the same thing as a section $\sigma_{pt} \in \Gamma(E)$ such that $g \cdot \sigma_{pt} = \sigma_{pt}$, i.e. a G–invariant section, and similarly an H–equivariant map $X \to E_o$ is an H–invariant element in E_o. Thus, we get a bijection $\Gamma(E)^G \to (E_o)^H$.

Finally, if E is a natural vector bundle induced by an H–representation W and X is a G–representation V, then $L(V, \Gamma(E))$ and $L(V, E_o)$ are vector spaces under the pointwise operations, and the evaluation in o induces a linear map $L(V, \Gamma(E)) \to L(V, E_o)$. On the other hand, if we start from a linear map $f : V \to E_o$, the construction above produces $\sigma_v(gH) = g \cdot f(g^{-1} \cdot v)$, which is linear in v provided that f is linear in V and the action of G is by linear maps. □

Example. The geometric version of Frobenius reciprocity allows us immediately to reduce questions about the existence of invariant Rimannian metrics and invariant tensor fields to problems of finite dimensional representation theory: If $M = G/H$ is a homogeneous space, then from 1.4.2 we know that the tensor bundle $\otimes^k TM \otimes \otimes^\ell T^*M$ is the associated bundle to $p : G \to M$ corresponding to the representation $\otimes^k(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^\ell(\mathfrak{g}/\mathfrak{h})^*$. By Frobenius reciprocity, G–invariant sections of this bundle are in bijective correspondence with H–invariant elements in the representation. Since invariant elements in a representation are the same thing as linear maps from the trivial representation to the given representation, this can be rephrased in such a way that the dimension of invariant tensor fields of type $\binom{k}{\ell}$ on G/H equals the multiplicity of the trivial representation in $\otimes^k(\mathfrak{g}/\mathfrak{h}) \otimes \otimes^\ell(\mathfrak{g}/\mathfrak{h})^*$. Moreover, the proof of the theorem above gives an explicit construction of the invariant tensor field from the invariant element in the representation. Clearly, this whole construction respects symmetries of any kind, so invariant tensor fields having certain symmetries are in bijective correspondence to invariant elements in the representations having the same symmetries. Finally, if we consider (pointwise) questions of non–degeneracy they clearly reduce the the analogous non–degeneracy properties in the representation. In particular, the set of G–invariant Riemann metrics on G/H is in bijective correspondence with the set of H–invariant positive definit inner products on the vector space $\mathfrak{g}/\mathfrak{h}$. Hence we get

Corollary. *A homogeneous space G/H admits a G–invariant Riemannian metric if and only if the image $H_1 \subset GL(\mathfrak{g}/\mathfrak{h})$ of H under the map induced by the restriction to H of the adjoint representation of G is contained in a compact subgroup of $GL(\mathfrak{g}/\mathfrak{h})$.*

Proof. From above we know that existence of an invariant Riemann metric is equivalent to existence of an H_1–invariant positive definit inner product on $\mathfrak{g}/\mathfrak{h}$. If such an inner product exists, then H_1 is contained in the orthogonal group of this inner product, which is isomorphic to some $O(n)$ und thus compact. Conversely, if $K \subset GL(\mathfrak{g}/\mathfrak{h})$ is a compact subgroup containing H_1, then averaging any inner product on $\mathfrak{g}/\mathfrak{h}$ over K gives a K–invariant and thus H_1–invariant inner product. □

Consider the examples from 1.1 of viewing spheres as homogeneous spaces. In the case of the Riemannian sphere 1.1.1, we have $H = O(n)$ and $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ as an H–representation, so the set of $O(n+1)$–invariant Riemann metrics on S^n consits exactly of all positive multiples of the standard metric. The description of invariant Riemann metrics on $\mathbb{R}^n$, viewed as a homogeneous space of the Euclidean group $\mathrm{Euc}(n)$ is exactly the same. For the projective sphere from 1.1.3, we get $H_1 = GL(\mathfrak{g}/\mathfrak{h})$, so there certainly is no $SL(n+1,\mathbb{R})$–invariant metric on S^n. Similarly, one shows that in the other examples from 1.1.4–1.1.6 there are no invariant Riemann metrics.

Finally, we remark that invariance often allows to analyze the situation further in a pretty elementary way. For example, applying natural operators to invariant objects, we get invariant objects. Thus if φ is an invariant differential form on G/H, then the exterior derivative $d\varphi$ is invariant as well. Similarly, the curvature of an invariant Riemann metric has to be an invariant tensor field, and so on.

1.4.4. Invariant connections. The next topic we discuss is the question whether there are connections compatible with the action of G. In general, it is possible reduce the problem to the question of finding invariant sections of homogeneous bundles, since connections on a bundle can always be viewed as sections of a bundle. Our aim is however, to give a simpler discription in a few important special cases. The simplest situation one may hope for is the existence of a G–invariant principal connection on the canonical H–principal bundle $p : G \to G/H$. Since any homogeneous bundle is associated to that principal bundle, we get an induced connection on any homogeneous bundle in that case. Moreover, these connections are compatible with all usual constrcutions like fibered produts and Whitney sums or tensor products of vector bundles.

Let us start by assuming that we have given a G–invariant principal connection on $p : G \to G/H$. A principal connection on this bundle is determined by its connection form $\psi \in \Omega^1(G,\mathfrak{h})$, see 1.3.3, which has to satisfy $(\rho^h)^*\psi = \mathrm{Ad}(h)^{-1} \circ \psi$ and $\psi(\zeta_A) = A$ for all $A \in \mathfrak{h}$, where $\zeta_A \in \mathfrak{X}(G)$ is the fundamental vector field corresponding to A. These fundamental vector fields are simply the left–invariant vector fields, so $\zeta_A = L_A$. In particular, this implies that $\psi(e)$ is a projection $T_eG = \mathfrak{g} \to \mathfrak{h}$. The condition that the connection is G–invariant means that $\lambda_g^*\psi = \psi$ for all $g \in G$. Equivariancy under H implies that for $h \in H$ and $X \in \mathfrak{g}$ we have $\psi(h)(T_e\rho^h \cdot X) = (\rho^h)^*\psi(e)(X) = \mathrm{Ad}(h)^{-1}(\psi(e)(X))$. On the other hand, differentiating the equation $gh = hh^{-1}gh$ in $g = e$ gives $T_e\rho^h = T_e\lambda^h \circ \mathrm{Ad}(h)^{-1}$, so the element $\psi(h)(T_e\rho^h \cdot X)$ can also be written as $\psi(h)(T_e\lambda_h \cdot \mathrm{Ad}(h)^{-1}(X)) = \lambda_h^*\psi(e)(\mathrm{Ad}(h)^{-1}(X))$, which by left invariance equals $\psi(e)(\mathrm{Ad}(h)^{-1}(X))$. Thus we see that $\psi(e) : \mathfrak{g} \to \mathfrak{h}$ is an equivariant projection, where the action of H on both sides is induced from the adjoint action of G. Such a projection is completely determined by its kernel $\mathfrak{n}$, which is an H–invariant complement to $\mathfrak{h}$ in $\mathfrak{g}$.

Assume conversely that we have given a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ of the H–module $\mathfrak{g}$. Then the Maurer–Cartan form $\omega \in \Omega^1(G,\mathfrak{g})$ splits as $\omega = \omega_{\mathfrak{h}} \oplus \omega_{\mathfrak{n}}$ according to the splitting of $\mathfrak{g}$. The Maurer–Cartan form is left invariant, and right–equivariant (for the adjoint action of G). Thus, the component $\omega_{\mathfrak{h}}$ is still left invariant under G and since the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ is H–invariant, it is right equivariant under H. Since moreover, the Maurer–Cartan form reproduces the generators of left invariant vector fields, $\omega_{\mathfrak{h}}$ defines a G–invariant principal connection on the bundle $p : G \to G/H$. As we have noticed above, an invariant connection is determined by its value in $e \in G$, so the two constructions are clearly inverse to each other.

Hence we see that G–invariant principal connections on $p : G \to G/H$ are the same thing as H–equivariant projections $\mathfrak{g} \to \mathfrak{h}$. If such projections exist, then the difference of two of them factorizes to an H–equivariant linear map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{h}$, and conversely adding the composition of such a map with the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ to an H–equivariant projection $\mathfrak{g} \to \mathfrak{h}$ one again gets an H–equivariant projection $\mathfrak{g} \to \mathfrak{h}$. Once we have fixed an H–equivariant projection $\pi : \mathfrak{g} \to \mathfrak{h}$ we have the complementary subspace $\mathfrak{n} = \ker(\pi)$ which is isomorphic as an H–representation to $\mathfrak{g}/\mathfrak{h}$. In particular, we may then view the tangent bundle $T(G/H)$ as the associated bundle $G \times_H \mathfrak{n}$, and similarly for the tensor bundles over G/H.

The connection form of any G–invariant connection on $p : G \to G/H$ is given as $\pi_{\mathfrak{h}} \circ \omega$, where $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ is an H–invariant projection. But this immediately allows us to compute the curvature of such a connection. The key to this is the Maurer–Cartan equation, $0 = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. As a two form on G/H with values in $\mathfrak{h}$, the curvature of a principal connection is given by the covariant exterior derivative of the connection form. Moreover, the curvature must be G–invariant, so it suffices to determine its value at $o \in G/H$. To do this, it suffices to compute the covariant exterior derivative of the connection form in $e \in G$. But the horizontal lift of a tangent vector $X + \mathfrak{h} \in T_o(G/H) \cong \mathfrak{g}/\mathfrak{h}$ clearly is the unique element $X' \in \mathfrak{n} \subset \mathfrak{g} = T_eG$ such that $X' + \mathfrak{h} = X + \mathfrak{h}$. Clearly, $d(\pi_{\mathfrak{h}} \circ \omega) = \pi_{\mathfrak{h}} \circ d\omega$ and hence we see that viewing TG/H as $G \times_H \mathfrak{n}$, the curvature of $\pi_{\mathfrak{h}} \circ \omega$ is induced by the map $\mathfrak{n} \times \mathfrak{n} \to \mathfrak{h}$, $(X, Y) \mapsto -\pi_{\mathfrak{h}}([X, Y])$. In particular, the principal connection $\pi_{\mathfrak{h}} \circ \omega$ has zero curvature if and only if $\mathfrak{n} \subset \mathfrak{g}$ is a subalgebra.

On the other hand, since the tangent bundle $T(G/H)$ is associated to $p : G \to G/H$, we can also compute the torsion of the linear connection on $T(G/H)$ induced by $\pi_{\mathfrak{h}} \circ \omega$. From 1.3.6 we know that this torsion is induced by the covariant exterior derivative of the soldering form, and comparing with example 1.4.2 we see that viewing $T(G/H)$ as $G \times_H \mathfrak{n}$, the soldering form is just $\pi_{\mathfrak{n}} \circ \omega$, where $\pi_{\mathfrak{n}} = \mathrm{id} - \pi_{\mathfrak{h}}$. On the other hand, this torsion is clearly a G–invariant section of $\Lambda^2 T^*(G/H) \otimes G/H$, and hence corresponds to a skew symmetric bilinear map $\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$. As in the case of the curvature, the Maurer–Cartan equation immediately implies that this map is given by $(X, Y) \mapsto -\pi_{\mathfrak{n}}([X, Y])$. In particular, the induced connection on $T(G/H)$ is torsion free if and only if $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$. Thus we have proved:

THEOREM. *Let $M = G/H$ be a homogeneous space, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h} \subset \mathfrak{g}$ the subalgebra corresponding to the subgroup H. Then there exists a G–invariant principal connection on the H–principal bundle $p : G \to G/H$, if and only if there is an H–equivariant projection $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$, where the H–action is induced from the adjoint action of G. If this is the case, then the set of all such principal connections is an affine space modelled on the vector space $\mathrm{Hom}_H(\mathfrak{n}, \mathfrak{h})$, where $\mathfrak{n} = \ker(\pi_{\mathfrak{h}})$ is the H–invariant complement to $\mathfrak{h}$ determined by $\pi_{\mathfrak{h}}$.*

The curvature of the invariant connection is induced by the map $\mathfrak{n} \times \mathfrak{n} \to \mathfrak{h}$ given by $(X, Y) \mapsto -\pi_{\mathfrak{h}}([X, Y])$, while the torsion of the induced linear connection on $T(G/H)$ is induced by the map $\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ given by $(X, Y) \mapsto -\pi_{\mathfrak{n}}([X, Y])$.

EXAMPLE. (1) Suppose that the subgroup $H \subset G$ is compact. Then averaging any positive definit inner product on $\mathfrak{g}$ over H, we obtain an H–invariant positive definit inner product on $\mathfrak{g}$. Now, the orthogonal projection onto $\mathfrak{h}$ with respect to such an inner product is H–invariant, thus giving a G–invariant principal connection on $p : G \to G/H$. The complementary subset $\mathfrak{n}$ is then just the orthogonal complement to $\mathfrak{h}$ with respect to this inner product.

In particular, consider the case where $H = O(n)$ and $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{n} \cong \mathbb{R}^n$ as an H–module, which occurs both in the case of the Riemannian sphere from 1.1.1 and the Euclidean space, viewed as a homogeneous space of the group of Euclidean motions from 1.1.2. In this case we not only have a G–invariant principal connection on $p : G \to G/H$ but it is also uniquely determined. This is due to the fact that both $\mathbb{R}^n$ and $\mathfrak{o}(n)$ are irreducible $O(n)$–modules and thus the zero map is the only H–homomorphism $\mathbb{R}^n \to O(n)$. In the case of the Euclidean space 1.1.2, the complement $\mathfrak{n}$ is an abelian subalgebra, so $[\mathfrak{n}, \mathfrak{n}] = \{0\}$, so the principal connection is flat, and the induced connection on the tangent bundle is torsion free. On the other hand, in the case of the Riemannian sphere, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ derived in 1.1.1 is the unique H–invariant decomposition, and one immediately verifies that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$, but the bracket of two elements of $\mathfrak{n}$ is nonzero in general. Hence, the induced connection on the tangent bundle TS^n is torsion free but the principal connection is not flat. More precisely, one easily computes that the curvature is induced by the map $\mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{o}(n)$ defined by $(v, w) \mapsto vw^t - wv^t$. Otherwise put, identifying $\mathfrak{o}(n)$ with $\Lambda^2\mathbb{R}^n$, the curvature is simply induced by $(v, w) \mapsto v \wedge w$.
(2) Consider the example of the projective sphere from 1.1.3. In this case $G = SL(n + 1, \mathbb{R})$ and from the presentation of the subgroup H in 1.1.3 it follows immediately that the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} -\operatorname{tr}(A) & Z \\ 0 & A \end{pmatrix} : A \in \mathfrak{gl}(n, \mathbb{R}), Z \in \mathbb{R}^{n*} \right\}.$$

If there were an H–invariant complement $\mathfrak{n}$ to $\mathfrak{h}$ in $\mathfrak{g}$, then in particular, $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$. On the other hand, in order to be a vector space complement, for any $X \in \mathbb{R}^n$, $\mathfrak{n}$ would have to contain a unique element of the form $\begin{pmatrix} * & * \\ X & * \end{pmatrix}$. In particular, the zero matrix would be the only element of $\mathfrak{n}$, whose entry in the left lower block is zero. But one immediately computes that for $Z, W \in \mathbb{R}^{n*}$, $X \in \mathbb{R}^n$ and $B \in \mathfrak{gl}(n, \mathbb{R})$, we get $\left[\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\operatorname{tr}(A) & W \\ X & A \end{pmatrix} \right] = \begin{pmatrix} ZX & Z(B - \operatorname{tr}(B)\operatorname{id}) \\ 0 & XZ \end{pmatrix}$. This clearly is incompatible with the existence of $\mathfrak{n}$ as assumed. In particular, for the case of the projective sphere, there is no $SL(n + 1, \mathbb{R})$–invariant connection on the H–principal bundle $SL(n + 1, \mathbb{R}) \to S^n$. Similarly, one can show that for the examples of the projective contact sphere, the conformal sphere and the CR–sphere from 1.1.4–1.1.6 there never exists a G invariant principal connection.

1.4.5. Invariant affine connections. The second special case of invariant connections that we want to discuss is the question of invariant linear connections on the tangent bundle, which are also called affine connections. These are a little mor subtle to analyze than the invariant principal connections on $G \to G/H$. Let us first assume that we have given any smooth manifold M and a linear connection ∇ on the tangent bundle of M, i.e. a bilinear map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ written as $(\xi, \eta) \mapsto \nabla_\xi \eta$, such that for any $f \in C^\infty(M, \mathbb{R})$ we have $\nabla_{f\xi}\eta = f\nabla_\xi\eta$ and $\nabla_\xi(f\eta) = (\xi \cdot f)\eta + f\nabla_\xi\eta$, c.f. 1.3.1 and 1.3.5. Fixing a vector field $\xi \in \mathfrak{X}(M)$, we define the *Nomizu operator* $L_\xi : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $L_\xi(\eta) = \nabla_\xi\eta - [\xi, \eta]$. From the fact that both $\nabla_\xi\_$ and $[\xi, \_]$ satisfy a Leibnitz rule, it follows immediately, that

$L_\xi(f\eta)=fL_\xi(\eta)$, so for any $\xi\in\mathfrak{X}(M)$, the operator L_ξ is actually a bundle map $L_\xi:TM\to TM$. Moreover, it follows immediately that $L_{f\xi}\eta=fL_\xi\eta+(\eta\cdot f)\xi$.

Conversely, if we have an operator L associating to any vector field $\xi\in\mathfrak{X}(M)$ a vector bundle homomorphism $L_\xi:TM\to TM$ such that $L_{f\xi}\eta=fL_\xi\eta+(\eta\cdot f)\xi$, the formula $\nabla_\xi\eta=L_\xi(\eta)+[\xi,\eta]$ defines a linear connection on M. Thus the Nomizu operators just form an equivalent way of dealing with linear connections on the tangent bundle.

Consider now the case where $M=G/H$ is a homogeneous space and ∇ is an arbitrary linear connection on $T(G/H)$. Then it suffices to know the Nomizu operators for fundamental vector fields. Recall from 1.2.5 that via the left action of G on G/H we associate to any $X\in\mathfrak{g}$ the fundamental vector field $\zeta_X\in\mathfrak{X}(G/H)$, and since the action is transitive, the values $\zeta_X(gH)$ for all $X\in\mathfrak{g}$ always span the full tangent space $T_{gH}(G/H)$. Now suppose that we know the Nomizu operator $L_{\zeta_X}:T(G/H)\to T(G/H)$ for all $X\in\mathfrak{g}$. Then for any $\eta\in\mathfrak{X}(G/H)$ and any $X\in\mathfrak{g}$ we can compute $\nabla_{\zeta_X}\eta$. If ξ is another vector field on G/H, then for any point $gH\in G/H$ we find an element $X\in\mathfrak{g}$ such that $\xi(gH)=\zeta_X(gH)$ and then $\nabla_\xi\eta(gH)=\nabla_{\zeta_X}\eta(gH)$, so we can compute $\nabla_\xi\eta$.

If we now assume in addition that the connection ∇ is G–invariant, then we can further reduce the problem: Since the Lie bracket is invariant under all diffeomorphisms, G–invariance of ∇ is equivalent to the property $L_{\ell_g^*\xi}(\ell_g^*\eta)=\ell_g^*(L_\xi(\eta))$ for any $g\in G$ and all $\xi,\eta\in\mathfrak{X}(G/H)$. Moreover, for $g\in G$ and $x\in X$, we know from 1.2.5 that $\ell_g^*\zeta_X=\zeta_{\mathrm{Ad}(g^{-1})\cdot X}$. In particular, this implies that

$$(*)\qquad L_{\zeta_X}(gH)=T\ell_g\circ L_{\zeta_{\mathrm{Ad}(g^{-1})\cdot X}}(o)\circ T\ell_{g^{-1}},$$

so it suffices to know the Nomizu operators in o corresponding to all fundamental vector fields. Indentifying $T_o(G/H)$ with $\mathfrak{g}/\mathfrak{h}$, we end up with a map $\Phi:\mathfrak{g}\to L(\mathfrak{g}/\mathfrak{h},\mathfrak{g}/\mathfrak{h})$, $X\mapsto\Phi_X$, which is characterized by $\Phi_X(Y+\mathfrak{h})=L_{\zeta_X}(T_ep\cdot Y)$. For $h\in H$, recall that $T_e\ell_h$ corresponds to the action $\underline{\mathrm{Ad}}$ of H on $\mathfrak{g}/\mathfrak{h}$ induced by the adjoint action of G. Thus the equivariancy condition $(*)$ above immediately leads to $\Phi_{\mathrm{Ad}(h)\cdot X}=\underline{\mathrm{Ad}}(h)\circ\Phi_X\circ\underline{\mathrm{Ad}}(h^{-1})$, which exactly means that $\Phi:\mathfrak{g}\to L(\mathfrak{g}/\mathfrak{h},\mathfrak{g}/\mathfrak{h})$ is an H–homomorphism.

Finally, assume that $X\in\mathfrak{h}$. Then since $\mathfrak{h}$ is exactly the kernel of T_ep, we have $\zeta_X(o)=0$, and thus for any $\eta\in\mathfrak{X}(G/H)$, $L_{\zeta_X}(\eta)(o)=-[\zeta_X,\eta](o)$ (which depends only on $\eta(o)$, since $\zeta_X(o)=0$). But for $Y\in\mathfrak{g}$ arbitrary, we know from 1.2.5 that $-[\zeta_X,\zeta_Y]=\zeta_{[X,Y]}$, which immediately implies that for $X\in\mathfrak{h}$ we get $\Phi_X=\underline{\mathrm{ad}}(X)$, where $\underline{\mathrm{ad}}$ is the Lie algebra action corresponding to $\underline{\mathrm{Ad}}$, i.e. $\underline{\mathrm{ad}}(X)(Y+\mathfrak{h})=[X,Y]+\mathfrak{h}$. Putting all this together we see that a G–invariant connection ∇ on $T(G/H)$ gives us an H–homomorphism $\Phi:\mathfrak{g}\to L(\mathfrak{g}/\mathfrak{h},\mathfrak{g}/\mathfrak{h})$ such that $\Phi_X=\underline{\mathrm{ad}}(X)$ for all $X\in\mathfrak{h}$. In the case of an invariant connection on the tangent bundle induced by an invariant principal connection on $G\to G/H$ corresponding to an H–equivariant projection $\pi_\mathfrak{h}:\mathfrak{g}\to\mathfrak{h}$ as in 1.4.4 above, the map Φ turns out to be $\Phi_X=\underline{\mathrm{ad}}(\pi_\mathfrak{h}(X))$.

Since we will only use the characterization of invariant affine connections to prove non–existence results, we only briefly remark that conversely one may extend any map Φ as above to a Nomizu operator for fundamental vector fields and then recover from this an invariant affine connection on G/H. All this only needs straightforward verifications, which partly are a little tedious. Moreover, since the difference of two invariant affine connections is an invariant tensor field, one gets a description of the affine structure on the set of all invariant affine connections. Finally, it is straightforward to compute the H–homomorphisms corresponding to torsion respectively curvature of an invariant affine connection (which both clearly are invariant tensor fields) to get the following result, a complete proof of which can be found in **??**.

THEOREM. *On a homogeneous space G/H, there exists a G–invariant affine connection if and only if there is an H–homomorphism $\Phi : \mathfrak{g} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ such that $\Phi_X = \underline{\mathrm{ad}}(X)$ for all $X \in \mathfrak{h}$. If this is the case, then the set of all G–invariant affine connections on G/H forms an affine space modelled on the vector space $\mathrm{Hom}_H(\otimes^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}/\mathfrak{h})$.*

If ∇ is the G–invariant affine connection corresponding to a homomorphism Φ as above, then the H–homomorphims $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ corresponding to the torsion of ∇ is induced by $(X, Y) \mapsto \Phi_X(Y + \mathfrak{h}) - \Phi_Y(X + \mathfrak{h}) - ([X, Y] + \mathfrak{h})$, while the H–homomorphism $\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ corresponding to the curvature of ∇ is induced by $(X, Y) \mapsto \Phi_X \circ \Phi_Y - \Phi_Y \circ \Phi_X - \Phi_{[X,Y]}$.

EXAMPLE. Let us again consider the projective sphere from 1.1.3. We claim that on the tangent bundle of S^n there is no $SL(n+1, \mathbb{R})$ invariant linear connection. From 1.1.3 we know that for $\varphi \in \mathbb{R}^{n*}$, the element $h := \begin{pmatrix} 1 & \varphi \\ 0 & \mathbb{I} \end{pmatrix}$ lies in H, where $\mathbb{I}$ denotes the $n \times n$ identity matrix. For $X \in \mathbb{R}^n$ one easily computes that $\mathrm{Ad}(h) \cdot \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} \varphi(X) & -\varphi(X)\varphi \\ X & -X \otimes \varphi \end{pmatrix}$. But in view of the presentation of the Lie algebras in example (2) of 1.4.4, this also implies that $\underline{\mathrm{Ad}}(h) = \mathrm{id}_{\mathfrak{g}/\mathfrak{h}}$. In particular, if $\Phi : \mathfrak{g} \to L(\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ is H–equivariant, then we obtain

$$\Phi_{\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}} = \Phi_{\begin{pmatrix} \varphi(X) & -\varphi(X)\varphi \\ X & -X \otimes \varphi \end{pmatrix}}.$$

But if Φ comes from an invariant connection, then the difference of these two elements must be $\underline{\mathrm{ad}}\begin{pmatrix} \varphi(X) & -\varphi(X)\varphi \\ 0 & -X \otimes \varphi \end{pmatrix}$. If $Y \in \mathbb{R}^n$ is another element, then we get

$$\left[\begin{pmatrix} \varphi(X) & -\varphi(X)\varphi \\ 0 & -X \otimes \varphi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}\right] = \begin{pmatrix} -\varphi(X)\varphi(Y) & 0 \\ -\varphi(Y)X - \varphi(X)Y & \varphi(X)Y \otimes \varphi \end{pmatrix},$$

which shows that this difference is nonzero, which contradicts the existence of an invariant affine connection. We shall see in **??** that this argument extends systematically to the flat models of all parabolic geometries. Hence in particular, also the projective contact, conformal, and CR–spheres from 1.1.4–1.1.6 do not admit invariant affine connections.

1.4.6. Invariant differential operators. Consider a homogeneous space G/H and two homogeneous vector bundles E, F over G/H corresponding to representations V and W of the subgroup H. From 1.4.3 we get G–actions on the spaces $\Gamma(E)$ and $\Gamma(F)$ of sections of the homogeneous bundles, and we are looking for linear *invariant differential operators*, i.e. linear differential operators $D : \Gamma(E) \to \Gamma(F)$ which are equivariant for these actions.

The first step towards this problem is to pass to jet prolongations. For $k \in \mathbb{N}$ let $J^k(E)$ denote the k–th jet prolongation of E. This means that the fiber of $J^k(E)$ at $x \in G/H$ is exactly the vector space of all k–jets at x of sections of E. Then these fit together to form a vector bundle over G/H, c.f. 1.2.7. Moreover, since the construction of jets is functorial, this is again a homogeneous vector bundle. More precisely, the action of $g \in G$ on $j^k s(x)$ is simply defined by $g \cdot (j^k s(x)) := j^k(g \cdot s)(x)$. A differential operator $D : \Gamma(E) \to \Gamma(F)$ is of order $\leq k$ if and only if for any two sections $s, t \in \Gamma(E)$ and any any point $x \in G/H$, the equation $j^k s(x) = j^k t(x)$ implies $D(s)(x) = D(t)(x)$. If D is such an operator, then we get an induced map $\tilde{D} : J^k(E) \to F$ over the identity on G/H, defined by $\tilde{D}(j^k s(x)) := D(s)(x)$, where s is any representattive of the jet. Conversely, the same formula associates to any such map $\tilde{D}$ a differential operator D. Clearly, D is linear if and only if $\tilde{D}$ is a homomorphism of vector bundles, and D is an invariant linear differential operator if and only if $\tilde{D}$ is a homomorphism of homogeneous vector bundles, i.e. G–equvariant. Hence we have reduced determination of invariant differential operators to determination of homomorphisms between homogeneous vector bundles. Since a homomorphism $J^k(E) \to F$ of homogeneous vector bundles is the same thing as a G–invariant section of the homogeneous vector bundle $L(J^k(E), F)$, this reduces to representation theory of the group H by 1.4.3.

While this at least in principle solves the problem of determining invariant differential operators, it is not too adequate in most situations. On one hand, in several interesting cases (in particular in those related to parabolic geometries) the H–representations corresponding to jet prolongations of simple homogeneous vector bundles quickly become very complicated and unmanageable. We will discuss below how to proceed in that case. On the other hand, in simple situations, one can get a simpler and more explicit solution by a more direct approach.

Let us assume that there exists a G–invariant principal connection on the H–principal bundle $G \to G/H$, and let us fix one of these if there are several. Then on any homogeneous vector bundle $E \to G/H$ we get an induced G–invariant linear connection. Applying this linear connection to a section $s \in \Gamma(E)$, we get a section $\nabla s \in \Gamma(T^*(G/H) \otimes E)$. Since $T^*(G/H)$ is also associated to the principal bundle $G \to G/H$, we get an induced linear connection on that bundle as well, and thus a linear connection on any of the bundle $\otimes^i T^*(G/H) \otimes E$, so we can for iterated covariant derivatives $\nabla^i s \in \Gamma(\otimes^i T^*(G/H) \otimes E)$ for $i = 0, 1, 2, \dots$. By construction, any such iterated covariant derivative defines a G–invariant differential operator. Next, for any $\ell \leq k$ there is a canonical projection $\pi^k_\ell : J^k(E) \to J^\ell(E)$ defined by $\pi^k_\ell(j^k s(x)) = j^\ell s(x)$, which is a homomorphism of homogeneous vector bundles. In particular, we can consider π^k_{k-1}, whose kernel is a homogeneous subbundle of $J^k(E)$. Is is well known (see e.g. [**?**]) that this kernel is isomorphic to $S^k T^*(G/H) \otimes E$, where S^k denotes the k–th symmetric power, and associating to $j^k s(x)$ the symmetrization of the k–fold covariant derivative with respect to any arbitrary linear connections on E and $T^*(G/H)$ defines a splitting of the inclusion $S^k T^*(G/H) \otimes E \hookrightarrow J^k(E)$. If $\tilde{D} : J^k(E) \to F$ is the bundle map corresponding to a k–th order differential operator, then the restriction of $\tilde{D}$ to $S^k T^*(G/H) \otimes E$ is called the *symbol* $\sigma(D)$ of the operator D. By construction, if D is an invariant differential operator, then its symbol $\sigma(D)$ is an invariant bundle map, and thus an G–invariant section of $S^k T(G/H) \otimes E^* \otimes F$. For G–invariant linear connections on E and $T^*(G/H)$, the obtained splitting $J^k(E) \to S^k T^*(G/H) \otimes E$ is a homomorphism of homogeneous vector bundles, and thus we can realize any G–equivariant map $S^k T^*(G/H) \otimes E \to F$ as the symbol of a k–th order invariant differential operator.

To the extent that one understands the H–representations V, W and $\mathfrak{g}/\mathfrak{h}$, this even allows a determination of all invariant differential operators between the bundles $E = G \times_H V$ and $F = G \times_H W$ of order at most k, in the case where a G–invariant principal connection on $G \to G/H$ exists. One starts by determining all possible symbols of order k by determining all H–homorphisms $S^k(\mathfrak{g}/\mathfrak{h})^* \otimes V \to W$. For any such homomorphism there is the corresponding homomorphism $S^kT^*(G/H) \otimes E \to F$ of homogeneous vector bundles, and by the above procedure we find an invariant differential operator $D : \Gamma(E) \to \Gamma(F)$ of order $\leq k$ having this as its symbol. To this operator we can now only add lower order operators, which we may determine by the same method and in finitely many steps we come to an end. The same thing works at least for invariant operators on tensor bundles assumeing the existence of an invariant affine connection on G/H. What has to be emphasized at this point is that the representations one has to deal with in that procedure are in principle as well behaved as the representations one starts with. In particular, if one starts for example with completely reducible representations, then one stays in the category of completely reducible representations.

In the case where one has no invariant affine connection available, the situation is much more complicated. In the case of parabolic geometries which we are mainly interested in, there is another problem involved, which is that the Lie algebra $\mathfrak{g}$ is fairly easy to understand because it is simple, while the subalgebra $\mathfrak{h}$ is fairly complicated in this case. Surprisingly, there is still a possiblity to convert the question of existence of invariant differential operators into an algebraic problem which is fairly manageable. The first step to this end is to understand invariant differential operators on the trivial line bundle over the group G.

1.4.7. The universal enveloping algebra. Recall that if A is any associative algebra, then the commutator $[a, b] = ab - ba$ makes A into a Lie algebra. If $\mathfrak{g}$ is any Lie algebra, then one can associate to it a unital associative algebra $\mathcal{U}(\mathfrak{g})$, the *universal enveloping algebra* of $\mathfrak{g}$ together with a homomorphism $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ of Lie algebras, which has the following universal property: If A is any associative algebra and $\varphi : \mathfrak{g} \to A$ is a homomorphism of Lie algebras, then there is a unique homomorphism $\tilde{\varphi} : \mathcal{U}(\mathfrak{g}) \to A$ of associative algebras such that $\tilde{\varphi} \circ i = \varphi$. In particular, a representation of $\mathfrak{g}$ is a homomorphism to the associative algebra of endomorphisms of a vector space, viewed as a Lie algebra under the commutator, which implies that a representation of the Lie algebra $\mathfrak{g}$ is the same thing as a left module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Explicitly, $\mathcal{U}(\mathfrak{g})$ can be constructed as the quotient of the tensor algebra $\mathcal{T}(\mathfrak{g})$ by the ideal I generated by all elements of the form $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$. Then the inclusion $\mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g})$ induces a homomorphism $\mathfrak{g} \to \mathcal{T}(\mathfrak{g})/I$ of Lie algebras and the universal property easily follows from the universal property of the tensor algebra.

From this explicit description one sees that the algebra $\mathcal{U}(\mathfrak{g})$ admits a canonical filtration $\mathcal{U}_0(\mathfrak{g}) \subset \cdots \subset \mathcal{U}_i(\mathfrak{g}) \subset \mathcal{U}_{i+1}(\mathfrak{g}) \subset \ldots$ by linear subspaces such that it becomes a filtered algebra, i.e. the product an element of $\mathcal{U}_i(\mathfrak{g})$ and an element of $\mathcal{U}_j(\mathfrak{g})$ lies in $\mathcal{U}_{i+j}(\mathfrak{g})$. To get this filtration, recall that the tensor algebra $\mathcal{T}(\mathfrak{g})$ is canonically graded, $\mathcal{T}(\mathfrak{g}) = \oplus_{n\in\mathbb{N}}\mathcal{T}^n(\mathfrak{g})$, where $\mathcal{T}^0(\mathfrak{g}) = \mathbb{K}$, the ground field and $\mathcal{T}^n(\mathfrak{g})$ the tensor product of n copies of $\mathfrak{g}$ for $n \geq 1$. Associated to this grading is an increasing filtration $\mathcal{T}_0(\mathfrak{g}) \subset \mathcal{T}_1(\mathfrak{g}) \subset \ldots$, where $\mathcal{T}_n(\mathfrak{g}) = \oplus_{i\leq n}\mathcal{T}^i(\mathfrak{g})$. Since the generators of the ideal I from above are linear combintations of two elements of degree two and one element of degree one, I is not homogeneous with respect to the grading of $\mathcal{T}(\mathfrak{g})$, so the quotient $\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I$ is not a graded algebra, but the filtration passes to the quotient. Otherwise put, an element $x \in \mathcal{U}(\mathfrak{g})$ lies in $\mathcal{U}_n(\mathfrak{g})$ if and only if it can be written as a linear combination of products of the form $X_1 \cdot X_2 \cdots X_\ell$ with $X_i \in \mathfrak{g}$ and $\ell \leq n$.

A basic result on universal enveloping algebras is the Poincaré–Birkhoff–Witt (PBW) theorem, see [**?**, ??], which for any Lie algebra $\mathfrak{g}$ gives a description of the associated graded algebra $\operatorname{gr}\mathcal{U}(\mathfrak{g})$. This associated graded algebra is defined by $\operatorname{gr}\mathcal{U}(\mathfrak{g}) := \oplus_{n\geq 0}(\mathcal{U}_n(\mathfrak{g})/\mathcal{U}_{n-1}(\mathfrak{g}))$, where by definition $\mathcal{U}_{-1}(\mathfrak{g}) = \{0\}$. In particular, as a vector space $\operatorname{gr}\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$. Note that by definition $\mathcal{U}_0(\mathfrak{g}) = \mathbb{K}$ and $\mathcal{U}_1(\mathfrak{g})/\mathcal{U}_0(\mathfrak{g}) = \mathfrak{g}$, so we have a canonical inclusion $\mathfrak{g} \hookrightarrow \operatorname{gr}\mathcal{U}(\mathfrak{g})$. The PBW theorem now states that this inclusion has the universal property of the symmetric algebra over $\mathfrak{g}$. In particular, this implies that if $\{X_i : i \in I\}$ is an ordered linear basis for $\mathfrak{g}$, then the set $\{X_{i_1} \cdot X_{i_2} \cdots X_{i_\ell} : \ell \in \mathbb{N}, i_1 \leq i_2 \leq \cdots \leq i_\ell\}$ is a linear basis for $\mathcal{U}(\mathfrak{g})$. This is another usual form of stating the PBW theorem.

To relate the universal enveloping algebra to invariant differential operators, consider a Lie group G as the homogeneous space $G/\{e\}$, and for $k \in \mathbb{N}$ consider the set $J^k_e(G,\mathbb{R})$ of k–jets in e of smooth functions $G \to \mathbb{R}$. As in 1.4.6 above, for $\ell \leq k$ we have the projections $\pi^k_\ell : J^k_e(G,\mathbb{R}) \to J^\ell_e(G,\mathbb{R})$, so in particular we may consider the directed system $\cdots \to J^k_e(G,\mathbb{R}) \to J^{k-1}_e(G,\mathbb{R}) \to \dots$, whose direct limit we denote as usual by $J^\infty_e(G,\mathbb{R})$. By construction, for any $k \in \mathbb{N}$ we have a natural projection $\pi_k : J^\infty_e(G,\mathbb{R}) \to J^k(G,\mathbb{R})$ and for $\pi_\ell = \pi^k_\ell \circ \pi_k$ for any $\ell \leq k$. In particular, a linear map defined on any of the spaces $J^k_e(G,\mathbb{R})$ canonically extends to $J^\infty_e(G,\mathbb{R})$. The advantage of passing to infinite jets is that part of the G action survives on infinite jets in one point. The action of G on $C^\infty(G,\mathbb{R})$ is given by $(g \cdot f)(g') = f(g^{-1}g')$. Hence the infinitesimal action of the Lie algebra $\mathfrak{g}$ can be computed as $(X \cdot f)(g') = \frac{d}{dt}|_{t=0} f(\exp(-tX)g') = (R_{-X} \cdot f)(g')$, see **??**. So the action of $\mathfrak{g}$ is given by differentiation along right invariant vector fields, which is well defined on infinite jets, so from our setting we get a canonical $\mathfrak{g}$–action on $J^\infty_e(G,\mathbb{R})$.

For $k \in \mathbb{N}$, $J^k_e(G,\mathbb{R})$ is just the fiber at $e \in G$ of the k–th jet prolongation of the trivial line bundle $\mathrm{pr}_1 : G \times \mathbb{R} \to G$, which is a homogeneous vector bundle over the homogeneous space $G = G/\{e\}$. In particular, the discussion from 1.4.6 implies that the space $L(J^k_e(G,\mathbb{R}),\mathbb{R})$ is isomorphic to the space of differential operators $D : C^\infty(G,\mathbb{R}) \to C^\infty(G,\mathbb{R})$ which are of order $\leq k$ and equivariant for the left G–action on $C^\infty(G,\mathbb{R})$. Hence, if we consider the space of all linear maps $J^\infty_e(G,\mathbb{R}) \to \mathbb{R}$ which factorize over $J^k_e(G,\mathbb{R})$ for some $k \in \mathbb{N}$, the resulting space is naturally isomorphic to the space $\mathrm{Diff}(C^\infty(G,\mathbb{R}))^G$ of left invariant differential operators on $C^\infty(G,\mathbb{R})$. Explicitily, the isomorphism maps a left invariant differential operator D to the linear mappint $j^\infty f(e) \mapsto D(f)(e)$. Note the the condition of factorizing over some $J^k_e(G,\mathbb{R})$ gives us a filtration $\mathrm{Diff}(C^\infty(G,\mathbb{R}))^G = \cup_{k\in\mathbb{N}} \mathrm{Diff}_k(C^\infty(G,\mathbb{R}))^G$ by the orders of operators. For any element $X \in \mathfrak{g}$, the left invariant vector field $L_X \in \mathfrak{X}(G)$ corresponding to X may be viewed as a first order differntial operator $C^\infty(G,\mathbb{R}) \to \mathbb{R}$. Moreover, left invariance of L_X can also be written as $L_X(f \circ \lambda_g) = (L_X \cdot f) \circ \lambda_g$ for all $g \in G$, which exactly means that $L_X \in \mathrm{Diff}_1(C^\infty(G,\mathbb{R}))^G$. Moreover, the Lie bracket of vector fields by definition corresponds to the commutator of differential operators, so since $[L_X, L_Y] = L_{[X,Y]}$, we see that $X \mapsto L_X$ is a homomorphism $\mathfrak{g} \to \mathrm{Diff}(C^\infty(G,\mathbb{R}))^G$ of Lie algebras. By the universal property of $\mathcal{U}(\mathfrak{g})$ we get an induced homomorphism

$\mathcal{U}(\mathfrak{g}) \to \mathrm{Diff}(C^\infty(G,\mathbb{R}))^G$ of associative algebras, which by construction is filtration preserving.

Passing back to the interpretation as linear maps on infinite jets, we see that we get a linear map $\mathcal{U}(\mathfrak{g}) \to L(J^\infty_e(G,\mathbb{R}),\mathbb{R})$ such that for an element $x \in \mathcal{U}_k(\mathfrak{g})$ the corresponding linear mapping factorizes over $J^k_e(G,\mathbb{R})$. We can also view this as a pairing $\langle\ ,\ \rangle : \mathcal{U}(\mathfrak{g}) \times J^\infty_e(G,\mathbb{R}) \to \mathbb{R}$, which for $X_i \in \mathfrak{g}$ and $f \in C^\infty(G,\mathbb{R})$ is explicitly given by

$$\langle X_1 \cdot X_2 \cdots X_k, j^\infty f(e)\rangle = (L_{X_1} \cdot L_{X_2} \cdots L_{X_k} \cdot f)(e).$$

We claim that this pairing is compatible with the $\mathfrak{g}$–actions on the two spaces. Recall that for $X \in \mathfrak{g}$ and $f \in C^\infty(G,\mathbb{R})$, we have $X \cdot (j^\infty f(e)) = j^\infty(R_{-X} \cdot f)(e)$. But now left invariant and right invariant vector fields commute, and thus $L_{X_1} \cdots L_{X_k} \cdot R_{-X} \cdot f = R_{-X} \cdot L_{X_1} \cdots L_{X_k} \cdot f$. Moreover, the value of the right hand side in e depends only on $R_{-X}(e) = -X = L_{-X}(e)$, so we see that for $x \in \mathcal{U}(\mathfrak{g})$, $X \in \mathfrak{g}$ and $f \in C^\infty(G,\mathbb{R})$, we get $\langle x, X \cdot (j^\infty f(e))\rangle = -\langle X \cdot x, j^\infty f(e)\rangle$, which shows that we get a pairing of $\mathfrak{g}$–modules.

Finally, we want to show that the pairing actually induces an isomorphism $\mathcal{U}(\mathfrak{g}) \cong \mathrm{Diff}(C^\infty(G,\mathbb{R}))^G$ of filtered algebras. For any $k \in \mathbb{N}$ the pairing gives us a mapping $\mathcal{U}_k(\mathfrak{g}) \to L(J^k_e(G,\mathbb{R}),\mathbb{R})$. The projection π^k_{k-1} includes $L(J^{k-1}_e(G,\mathbb{R}),\mathbb{R})$ into $L(J^k_e(G,\mathbb{R}),\mathbb{R})$, and restriction to the kernel of π^k_{k-1} induces an isomorphism between the quotient space and the dual space of this kernel. From 1.4.6 we see that this kernel is just the fiber over e of S^kT^*G, which is $S^k\mathfrak{g}^*$. Thus, we get a quotient map from $\mathcal{U}_k(\mathfrak{g})/\mathcal{U}_{k-1}(\mathfrak{g}) \to S^k\mathfrak{g}$, which in the language of differential operators is exactly given by mapping an operator of order $\leq k$ to its k–th order symbol. But from the construction it is clear that this map is induced by $X_1 \cdots X_k \mapsto X_1 \vee \cdots \vee X_k$ which is a linear isomorphism by the Poincaré–Birkhof–Witt theorem. Hence we see that the pairing from above induces an isomorphism of $\mathfrak{g}$–modules between $\mathcal{U}(\mathfrak{g})$ and the space of those linear maps $J^\infty_e(G,\mathbb{R}) \to \mathbb{R}$ which factorize over some $J^k_e(G,\mathbb{R})$, and we get an isomorphism of filtered associatve algebras between $\mathcal{U}(\mathfrak{g})$ and $\mathrm{Diff}(C^\infty(G,\mathbb{R}))^G$.

1.4.8. Induced modules. Now it is fairly easy to generalize this approach to arbitrary homogeneous spaces and arbitrary homogeneous vector bundles: First, let $\mathbb{V}$ be an arbitrary finite dimensional vector space, and let $\mathbb{V}^*$ be its dual. Then as in 1.4.7 above we get a pairing $\langle\ ,\ \rangle : (\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^*) \times J^\infty_e(G,\mathbb{V}) \to \mathbb{R}$. The same arguments as above show that this pairing induces an isomorphism of $\mathfrak{g}$–modules between $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^*$ and the space of those linear maps $J^\infty_e(G,\mathbb{V}) \to \mathbb{R}$ which factorize over some $J^k_e(G,\mathbb{V})$. Now let us in addition assume that $H \subset G$ is a closed subgroup and that $\mathbb{V}$ is a representation of H. Since sections of associated bundles correspond to H–equivariant mappings from G to the standard fiber, we next consider the space $J^\infty_e(G,\mathbb{V})^H$ of infinite jets at $e \in G$ of smooth functions $f : G \to \mathbb{V}$ such that $f(gh) = h^{-1} \cdot f(g)$. The pairing from above then gives a surjection from $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^*$ to the space of all linear maps $J^\infty_e(G,\mathbb{V})^H \to \mathbb{R}$ which facotrize over some J^k.

To understand the kernel of this mapping, note first that for $Y \in \mathfrak{h}$ the infintesimal version of H–equivariancy of $f : G \to \mathbb{V}$ is given by $L_Y \cdot f = \ell_{-Y} \circ f$, where ℓ denotes the action of H on $\mathbb{V}$. Similarly, $L_{Y_1} \cdots L_{Y_n} \cdot f = \ell_{-Y_n} \circ \ldots \circ \ell_{-Y_1} \circ f$ for $Y_1, \ldots, Y_1 \in \mathfrak{h}$. Since each ℓ_{Y_i} is linear, we further get $L_{X_1} \cdots L_{X_k} \cdot L_{Y_1} \cdots L_{Y_n} \cdot f = \ell_{-Y_n} \circ \ldots \circ \ell_{-Y_1} \circ L_{X_1} \cdots L_{X_k} \cdot f$. Moreover, for $\varphi \in \mathbb{V}^*$, by definition $\varphi(-Y \cdot (f(e))) =$

$(Y \cdot \varphi)(f(e))$. Thus we see that viewing $\mathcal{U}(\mathfrak{h})$ as a subalgebra of $\mathcal{U}(\mathfrak{g})$ and keeping in mind that the $\mathfrak{h}$–action on $\mathbb{V}^*$ extends to an action of $\mathcal{U}(\mathfrak{h})$, we see that for $x \in \mathcal{U}(\mathfrak{g})$, $y \in \mathcal{U}(\mathfrak{h})$ and $\varphi \in \mathbb{V}^*$ we get $\langle xy \otimes \varphi, j^\infty f(e)\rangle = \langle x \otimes (y \cdot \varphi), j^\infty f(e)\rangle$ for all $j^\infty f(e) \in J^\infty_e(G, \mathbb{V})^H$. Hence, all elements of the form $xy \otimes \varphi - x \otimes y \cdot \varphi$ are in the kernel of the above mapping. Thus, our pairing actually factorizes through the quotient of $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^*$ by the subspace of all elements of the above form. This quotient can be nicely described algebraically.

Since $\mathcal{U}(\mathfrak{h})$ is a subalgebra of $\mathcal{U}(\mathfrak{g})$, multiplication from the right makes $\mathcal{U}(\mathfrak{g})$ into a right $\mathcal{U}(\mathfrak{h})$–mdoule. On the other hand, $\mathbb{V}^*$ is canonically a left $\mathcal{U}(\mathfrak{h})$ module, and the quotient of $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^*$ by the space of all elements of the form $xy \otimes \varphi - x \otimes y \cdot \varphi$ with $x \in \mathcal{U}(\mathfrak{g})$, $y \in \mathcal{U}(\mathfrak{h})$ and $\varphi \in \mathbb{V}^*$ by definition is exactly $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*$. To see that the latter space is isomorphic as a $\mathfrak{g}$–mdoule to the space of those linear maps $J^\infty_e(G, \mathbb{V})^H \to \mathbb{R}$ which factorize over some $J^k_e(G, \mathbb{V})^H$, we only have to show that no element of this space acts trivially on all jets of H–euqivariant smooth functions. To do this, choose a linear basis $\{X_1, \dots, X_k, Y_1, \dots, Y_\ell\}$ for $\mathfrak{g}$ in such a way that $\{Y_1, \dots, Y_\ell\}$ is a basis for $\mathfrak{h}$. By the Poincaré–Birkhoff–Witt theorem the ordered monomials in these elements form a basis for $\mathcal{U}(\mathfrak{g})$ and passing to $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*$ all the Y' may be moved to the other side. This shows that for a basis $\{\varphi_1, \dots, \varphi_n\}$ for $\mathbb{V}^*$, all elements of the form $X_1^{i_1} \cdots X_k^{i_k} \otimes \varphi_j$ form a linear basis of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*$. On the other hand, denoting by $\mathfrak{k} \subset \mathfrak{g}$ the linear span of the X_i, by [**KMS**, 5.11] there is an open neighborhood $W \in \mathfrak{k}$ and such that the map $(X, h) \mapsto \exp(X)h$ is a diffeomorphism from $W \times H$ onto an open neighborhood of e in G. Using this, one easily constructs for each basis element as above a smooth H–equivariant map $f : G \to \mathbb{V}$ which pairs to one with the given basis element and to zero with all other basis elements.

There is one more point that we have to take into account here, namely that $\mathbb{V}$ is actually an H–module and not only a $\mathfrak{h}$–module. This may be incorporated into the picture using the notion of a $(\mathfrak{g}, H)$–mdoule. Recall that a $(\mathfrak{g}, H)$–module is a vector space V together with actions of $\mathfrak{g}$ and H which satisfy two compatibility conditions. First, the derivative of the H action should coincide with the restriction of the $\mathfrak{g}$–action to the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Secondly, for $X \in \mathfrak{g}$ and $h \in H$, we have $\mathrm{Ad}(h)X \in \mathfrak{g}$, and we require that $(\mathrm{Ad}(h)X) \cdot v = h \cdot (X \cdot (h^{-1} \cdot v))$.

We claim that both $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*$ and $J^\infty_e(G, \mathbb{V})^H$ can be canonically made into $(\mathfrak{g}, H)$–modules. For an element $g \in G$, the adjoint action $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, which by the universal property of $\mathcal{U}(\mathfrak{g})$ extends to an automorphism (again denoted by $\mathrm{Ad}(g)$) of the associative algebra $\mathcal{U}(\mathfrak{g})$. Explictly, this map $X_1 \cdots X_n$ to $\mathrm{Ad}(g)X_1 \cdots \mathrm{Ad}(g)X_n$. In particular, we may apply this to elements of H, thus arriving at a left action of H on $\mathcal{U}(\mathfrak{g})$ by algebra automorphisms. Moreover, this action clearly preserves the subalgebra $\mathcal{U}(\mathfrak{h})$. Now consider the tensor product action of H on $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^*$. For $x \in \mathcal{U}(\mathfrak{g})$, $y \in \mathcal{U}(\mathfrak{h})$ and $\varphi \in \mathbb{V}^*$ consider $\mathrm{Ad}(h)(xy) \otimes h \cdot v - \mathrm{Ad}(h)x \otimes h \cdot (y \cdot v)$. Since h acts by algebra automorphisms on $\mathcal{U}(\mathfrak{g})$, the first term may be rewritten as $(\mathrm{Ad}(h)x)(\mathrm{Ad}(h)y) \otimes h \cdot v$. On the other hand, for $Y \in \mathfrak{h}$ and $h \in H$ the equation $\exp(\mathrm{Ad}(h)Y) = h \exp(Y)h^{-1}$ implies that $(\mathrm{Ad}(h)Y) \cdot v = h \cdot (Y \cdot (h^{-1} \cdot v))$. From this, we condlude that $h \cdot (y \cdot v) = (\mathrm{Ad}(h)y) \cdot (h \cdot v)$. This implies that the left H–action descends to $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}$.

To see that the actions of $\mathfrak{g}$ and H satsify the compatibility conditions from above, note first, that since H acts on $\mathcal{U}(\mathfrak{g})$ by algebra automorphisms, the infinitesimal action of $\mathfrak{h}$ must be by derivations. Moreover, on $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$ the action of $Y \in \mathfrak{h}$ must be given by $X \mapsto [Y, X] = YX - XY$. From these two facts we immediatley conclude that the action is given by $x \mapsto Yx - xY$ for all $x \in \mathcal{U}(\mathfrak{g})$. Consequently, the action of Y on $x \otimes v$ is given by $Yx \otimes v - xY \otimes v + x \otimes Y \cdot v$, and the last two terms add up to zero in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}$, so the derivative of the H–action coincides with the restriction of the $\mathfrak{g}$–action. On the other hand, again since $\mathrm{Ad}(h)$ is an algebra homomorphism on $\mathcal{U}(\mathfrak{g})$, the action of $(\mathrm{Ad}(h)X)$ on $x \otimes v$ is given by $\mathrm{Ad}(h)(X\,\mathrm{Ad}(h^{-1})x) \otimes v$. On the other hand, acting on $x \otimes v$ with h^{-1}, we get $\mathrm{Ad}(h^{-1})x \otimes h^{-1} \cdot v$, and multiplying this from the left with X and then acting with h we get $\mathrm{Ad}(h)(X\,\mathrm{Ad}(h^{-1})x) \otimes v$, so the second condition is satisfied as well. The $(\mathfrak{g}, H)$–module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*$ is calle the $(\mathfrak{g}, H)$–module induced by the H–module $\mathbb{V}^*$.

On the other hand, recall that the $\mathfrak{g}$–action on $J^\infty_e(G, \mathbb{V})^H$ comes from the G–action on $C^\infty(G, \mathbb{V})^H$ defined by $g \cdot f = f \circ \lambda_{g^{-1}}$. For $h \in H$, equivariancy of f implies that $h \cdot f = \ell_h \circ f \,\mathrm{conj}_{h^{-1}}$, where ℓ denotes the action of H on $\mathbb{V}$. Since $\mathrm{conj}_{h^{-1}}$ maps e to itself, we see that the infinite jet of $h \cdot f$ in e depends only on the infinite jet of f in e, so we see that the H–action makes sense on $J^\infty_e(G, \mathbb{V})^H$. This also immediately implies that the derivative of the H–action coincides with the restriction of the $\mathfrak{g}$–action. The second compatibility condition immediately follows from the fact that $\exp(\mathrm{Ad}(h)X) = h \exp(X) h^{-1}$ for $X \in \mathfrak{g}$ and $h \in H$.

Finally, we claim that the pairing $\langle\ ,\ \rangle$ is also compatible with the H–mdoule structures. Recall that $h \cdot j^\infty f(e) = j^\infty(\ell_h \circ f \circ \mathrm{conj}_{h^{-1}})(e)$. For $X \in \mathfrak{g}$ consider the left invariant vector field $L_{\mathrm{Ad}(h)X}$. Since ℓ_h is linear, we may differentiate through it, while $\exp(t\,\mathrm{Ad}(h)X) = h \exp(tX) h^{-1}$ easily implies that $L_{\mathrm{Ad}(h)X} \cdot (f \circ \mathrm{conj}_{h^{-1}}) = (L_X \cdot f) \circ \mathrm{conj}_{h^{-1}}$. Thus we get

$$\begin{aligned}
&\langle \mathrm{Ad}(h)X_1 \cdots \mathrm{Ad}(h)X_n \otimes h \cdot \varphi, h \cdot j^\infty f(e)\rangle = \\
&\qquad (h \cdot \varphi)(L_{\mathrm{Ad}(h)X_1} \cdots L_{\mathrm{Ad}(h)X_n} \cdot (\ell_h \circ f \circ \mathrm{conj}_{h^{-1}})(e)) = \\
&\qquad (h \cdot \varphi)(h \cdot (L_{X_1} \cdots L_{X_n} \cdot f)(\mathrm{conj}_{h^{-1}}(e))) = \langle X_1 \cdots X_n \otimes \varphi, j^\infty f(e)\rangle.
\end{aligned}$$

The most important properties of the induced modules are colledted in the following theorem.

Theorem. *Let G be a Lie group, $H \subset G$ a closed subgroup and $\mathbb{V}$ and $\mathbb{W}$ finite dimensional representations of H. Let E and F denote the homogeneous vector bundles $G \times_H \mathbb{V}$ and $G \times_H \mathbb{W}$, respectively. Then we have:*
(1) The induced module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*$ is isomorphic to the space of those linear maps $J^\infty(E)_o \to \mathbb{R}$ which factor over some $J^k(E)_o$.
(2) (algebraic version of Frobenius reciprocity) There is a natural isomorphism between the space $\mathrm{Hom}_{(\mathfrak{g},H)}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{W})$ of homomorphism of $(\mathfrak{g}, H)$–mdoules and the space $\mathrm{Hom}_H(\mathbb{V}, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{W})$ of homomorphisms of H–mdoules.
(3) The space of invariant differential operators $D : \Gamma(E) \to \Gamma(F)$ is isomorphic to the space $\mathrm{Hom}_{(\mathfrak{g},H)}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{W}^*, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V}^*)$.

Proof. From 1.4.3 we know that $\Gamma(E) \cong C^\infty(G, \mathbb{V})^H$, where the isomorphism is given by $s(gH) = g \cdot f(g)$. In particular, this implies that we get an induced isomorphism between $J^k(E)_o$ and $J^k_e(G, \mathbb{V})^H$ and passing to the limit the analogous isomorphism for infinite jets. Thus, (1) is clear from the discussion above.
(2) Suppose that $\Phi : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{V} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{W}$ is a $(\mathfrak{g}, H)$–module homomorphism. Then we define $\varphi : \mathbb{V} \to \mathcal{U}(\mathfrak{g})_{\mathcal{U}(\mathfrak{h})}\mathbb{W}$ by $\varphi(v) := \Phi(1 \otimes v)$. Since $\mathrm{Ad}(h)(1) = 1$, this is an H–homomorphism. Conversley, assume that $\varphi : \mathbb{V} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{W}$ is an H–homomorphism. Then consider the bilinear map $\mathcal{U}(\mathfrak{g}) \times \mathbb{V} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{W}$

defined by $(x,v)\mapsto x\varphi(v)$. For $y\in\mathcal{U}(\mathfrak{h})$, we have $y\varphi(v)=\varphi(y\cdot v)$ since φ is an H–homomorphism and thus an $\mathfrak{h}$ and a $\mathcal{U}(\mathfrak{h})$–homomorphism. Consequently, the induced linear map on the tensor product factors to a linear map $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}\mathbb{V}\to\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}\mathbb{W}$, which by construction is a $\mathfrak{g}$–homomorphism. On the other hand, since $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}\mathbb{W}$ is a $(\mathfrak{g},H)$–module, the left action of $\mathrm{Ad}(h)X$ equals the compositions of the left actions of h, x and h^{-1}, so since φ is an H–homomorphism, we get $(\mathrm{Ad}(h)x)\varphi(h\cdot v)=h\cdot(x\cdot\varphi(v))$, which shows that we have defined a $(\mathfrak{g},H)$–homomorphism. Finally, one immediately checks that these two constructions are inverse to each other using that $\Phi(X\otimes v)=X\Phi(1\otimes v)$ for a $(\mathfrak{g},H)$–homomorphism Φ.

(3) As discussed in 1.4.6 the set of invariant differential operators $D:\Gamma(E)\to\Gamma(F)$ is in bijective correspondence with the set of those H–homomorphisms $J^\infty(E)_0\to\mathbb{W}$ which factor over some $J^k(E)_o$. The dual of such a homomorphism is an H–homorphism $\mathbb{W}^*\to L(J^k(E)_o,\mathbb{R})$, and the right hand module is a submodule of $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}\mathbb{V}^*$. Conversely, having given a homomorphism $\mathbb{W}^*\to\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}\mathbb{V}^*$, we find a k such that all linear maps in the image factor over $J^k(E)_o$, so this actually is a homomorphism $\mathbb{W}^*\to L(J^k(E)_o,\mathbb{R})$, whose dual map gives rise to an invariant differential operator. Thus we see that the set of invariant differential operators $D:\Gamma(E)\to\Gamma(F)$ is in bijective correspondence with the space $\mathrm{Hom}_H(\mathbb{W}^*,\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}\mathbb{V}^*)$, and the result follows from (2). □

1.4.9. Distinguished curves. The final notions that we want to discuss in this section are analogs of geodesics and normal coordinates. Until the end of this section we will assume that we have given a Lie group G, a closed subgroup $H\subset G$ and we have fixed a linear complement $\mathfrak{n}$ of $\mathfrak{h}$ in $\mathfrak{g}$. As before, the case when this complement is H–invariant will be particularly simple.

Once we have fixed the complement $\mathfrak{n}$, for any point $g\in G$ we have the notion of horizontal tangent vectors at g, namely $\xi\in T_gG$ is horizontal if and only if $\omega(\xi)=T\lambda_{g^{-1}}\cdot\xi\in\mathfrak{n}$, where ω is the Maurer–Cartan form on G. In particular, to any element $X\in\mathfrak{n}$, we may consider the left invariant vector field $L_X\in\mathfrak{X}(G)$, which satisfies $\omega(L_X)=X$, so this is a constant horizontal field. The distinguished curves on G/H are then exactly the projections of the flow lines of these constant horizontal vector fields. More precisely, a smooth curve $c:I\to G/H$ defined on an open intervall $I\subset\mathbb{R}$ is called distinguished if and only if there are a point $t_0\in I$ and elements $g\in G$ with $p(g)=c(t_0)$ and $X\in\mathfrak{n}$ such that $c(t)=p(\mathrm{Fl}^{L_X}_{t-t_0}(g))=p(g\exp((t-t_0)X))$ for all $t\in I$. Note that since left invariant vector fields are always complete, any distinguished curve can be canonically extended to a curve defined on $\mathbb{R}$. Note further, that by the flow property, if $c:I\to M$ is a distinguished curve then for any $t_0\in I$ there are $g\in G$ and $X\in\mathfrak{n}$ such that $c(t)=p(g\exp((t-t_0)X))$.

probably should be much earlier, maybe just before the invariant connections, i.e. 1.4.4

Proposition. *Let G be a Lie group, $H\subset G$ a closed subgroup and $\mathfrak{n}$ a complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then we have:*
(1) If $c:I\to M$ is a distinguished curve and $g\in G$ is any element, then $\ell_g\circ c:I\to M$ is a distinguished curve, too.
(2) For any point $x\in G/H$ and any tangent vector $\xi\in T_xG/H$ there is at least one distinguished curve $c:\mathbb{R}\to G/H$ such that $c(0)=x$ and $c'(0)=\xi$.
(3) If the complement $\mathfrak{n}$ is H–invariant, then the curve c in (2) is uniquely determined. This curve is called the geodesic with initial point x and initial direction ξ.

PROOF. (1) By definition, there are elements $t_0 \in I$, $g_0 \in G$ and $X \in \mathfrak{n}$ such that $c(t) = p(g_0 \exp((t-t_0)X))$. But then by definition of the action ℓ_g, we get $\ell_g(c(t)) = p(gg_0 \exp((t-t_0)X))$, so $\ell_g \circ c$ is distinguished, too.
(2) By (1) it suffices to consider the case $x = o$. Then for $\xi \in T_o G/H$ there is a unique element $X \in \mathfrak{n}$ such that $\xi = T_e p \cdot X$. Thus, $p(\exp(tX))$ is a distinguished curve as required.
(3) Again, we may confine ourselves to the case $x = 0$. As above, take $X \in \mathfrak{n}$ such that $T_e p \cdot X = \xi$. Since $p^{-1}(o) = H$, except from $p(\exp(tX))$ the other possibilities for distinguished curves are of the form $p(h\exp(tY))$ with $Y \in \mathfrak{n}$ and $T_h p \cdot L_Y(h) = \xi$ (and these two conditions determine Y uniquely). Thus we have

$$\xi = T_h p \circ T_e \lambda_h \cdot Y = T_h p \circ T_\rho^{h^{-1}} \circ T_e \lambda_h \cdot Y = T_e p \cdot \mathrm{Ad}(h) Y.$$

Thus, $X - \mathrm{Ad}(h)Y \in \mathfrak{h}$, but since $\mathfrak{n}$ is H–invariant, we have $\mathrm{Ad}(h)Y \in \mathfrak{n}$ and thus $Y = \mathrm{Ad}(h^{-1})X$. But then $h\exp(t\,\mathrm{Ad}(h^{-1})X) = hh^{-1}\exp(tX)h$, so $p(h\exp(tY)) = p(\exp(tX))$. □

1.5. Cartan connections

Now we come close to the heading of this introductory chapter. Cartan geometries may be studied from various points of view and we start with the simplest one, an absolute parallelism on a principal bundle. Of course, this does not illuminate the way how such objects are born in the realm of the Cartan's equivalence method. We shall come to this and similar points later. This section rather offers basic bricks for a calculus for Cartan connections.

1.5.1. Basic concepts. A *Cartan connection* of type $(\mathfrak{g}, H)$ on a manifold M is a principal fiber bundle $p : \mathcal{P} \to M$ with structure group H and a $\mathfrak{g}$–valued one–form $\omega \in \Omega^1(\mathcal{P}; \mathfrak{g})$ such that the Lie algebra $\mathfrak{h}$ of H is a subalgebra in $\mathfrak{g}$ and
(1) ω is H–equivariant, i.e. $(r^g)^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$,
(2) ω reproduces the fundamental vector fields, i.e. $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}$,
(3) ω is an absolute parallelism, i.e. $\omega_{|T_u\mathcal{P}} : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.

A flat model for the Cartan connection $(\mathcal{P}, \omega)$ is the homogeneous space $G \to G/H$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G and $H \subset G$. If the local model has been fixed, we say that the Cartan connection is of type G/H and it is called *effective*, *locally effective*, or *reductive* if its local model has the property in question.

Given a Cartan connection $(\mathcal{P}, \omega)$, there are the *constant vector fields* $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ defined for all $X \in \mathfrak{g}$ by $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{P}$. By the very definition,

$$\omega^{-1}(X)(u \cdot g) = Tr^g \cdot \omega^{-1}(\mathrm{Ad}_g(X))(u). \tag{4}$$

Of course, the Maurer–Cartan form is a Cartan connection on the flat model and the fields $\omega^{-1}(X)$ are the left invariant fields.

The *curvature form* K of a Cartan connection $(\mathcal{P}, \omega)$ is defined by the structure equation

$$K = d\omega + \tfrac{1}{2}[\omega, \omega]. \tag{5}$$

Clearly, K is a two–form and so we may use the absolute parallelism to express the curvature on the values $\omega^{-1}(X)(u)$. This leads to the *curvature function* κ :

$\mathcal{P} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ defined by $\kappa(u)(X,Y) = \omega\big(K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u))\big)$, and so the standard formula for d yields

$$\kappa(u)(X,Y) = [X,Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)](u)). \tag{6}$$

LEMMA. *The curvature form K is a two-form in $p^*\Omega^2(M;\mathfrak{g})$ where $p : \mathcal{P} \to M$ is the projection to the base manifold, while the curvature function may be viewed as $\kappa : \mathcal{P} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$. Moreover,*

$$(r^g)^*K = \mathrm{Ad}_{g^{-1}} \circ K \tag{7}$$

$$(r^g)^*\kappa = \lambda(g^{-1}) \circ \kappa \tag{8}$$

where λ is the tensor product of $\underline{\mathrm{Ad}}^*$ on $(\mathfrak{g}/\mathfrak{h})^*$ and Ad on $\mathfrak{g}$.

PROOF. We have just to imitate the computation from 1.3.3. The evaluation of the expression $d\omega + \frac{1}{2}[\omega,\omega]$ on two commuting vector fields $\xi = \omega^{-1}(X)$, $X \in \mathfrak{h}$, and $\eta \in \mathfrak{X}(\mathcal{P})$ arbitrary, yields

$$d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)] = \mathcal{L}_{\zeta_X}(\omega(\eta)) - \mathcal{L}_\eta(\omega(\zeta_X)) - \omega([\zeta_X,\eta]) + [X,\omega(\eta)]$$

and both the bracket $[\zeta_X,\eta]$ and the derivative of the constant mapping $\omega(\zeta_X)$ vanish. Now, $\mathcal{L}_{\zeta_X}(\omega(\eta)) = (\mathcal{L}_{\zeta_X}\omega)(\xi)$ since the fields commute, and the invariance of ω implies

$$d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)] = -\operatorname{ad}(X)(\omega(\eta)) + [X,\omega(\eta)] = 0.$$

Thus the curvature is strictly horizontal and so the values $K(\xi(u),\eta(u))$ depend exclusively on the projections $Tp.\xi(u)$, $Tp.\eta(u)$. Now also the second claim follows since the values $\kappa(u)(X,Y)$ depend only of the class in $\mathfrak{g}/\mathfrak{h}$ dtermined by X and Y.

The equivariance property of K follows directly from the structure equation and the intertwining of the pullback with the exterior differential d. In order to compute $(r^g)^*\kappa$ we have simply to evaluate $\kappa(u\cdot g)(X,Y)$. By definition, we get

$$\begin{aligned}
K(u\cdot g)(\omega^{-1}(X),\omega^{-1}(Y)) &= K(u\cdot g)(Tr^g\cdot\omega^{-1}(\mathrm{Ad}_g(X)), Tr^g\cdot\omega^{-1}(\mathrm{Ad}_g(Y))) \\
&= (r^g)^*K(u)(\omega^{-1}(\mathrm{Ad}_g(X)),\omega^{-1}(\mathrm{Ad}_g(X))) \\
&= \mathrm{Ad}_{g^{-1}} \circ \kappa(u)(\mathrm{Ad}_g(X),\mathrm{Ad}_g(Y))
\end{aligned}$$

and this is the required formula. □

EXAMPLE. All affine connections and connections on G–structures discussed in 1.3.5 and 1.3.6 are specific examples of reductive Cartan connections. The Riemannian manifolds are then more specific examples and the mutation of their flat models differ just by the notion of the curvature. The flat models with the Maurer–Cartan form ω have always curvature zero as the structure equations say.

1.5.2. Categories of Cartan connections. The objects of the category $\mathcal{C}_{(\mathfrak{g},H)}$ are Cartan connection of type $(\mathfrak{g},H)$ and morphisms consist of principal fiber bundle morphisms $\varphi : (\mathcal{P},\omega) \to (\mathcal{P}',\omega')$ such that $\varphi^*\omega' = \omega$.

LEMMA. *Let $\varphi : \mathcal{P} \to \mathcal{P}'$ be a morphism of principal fiber bundles. If ω' is a Cartan connection on $\mathcal{P}'$, then $\omega = \varphi^*\omega'$ is a Cartan connection. If ω' and ω are fixed Cartan connections, then $\varphi : (\mathcal{P},\omega) \to (\mathcal{P}',\omega')$ is a morphisms of Cartan connections if and only if φ preserves the constant vetor fields, i.e. $T\varphi \circ \omega^{-1}(X) = {\omega'}^{-1}(X) \circ \varphi$. In this case also the curvature forms K and K' are φ–related, and the curvature functions satisfy $\kappa = \kappa' \circ \varphi$.*

PROOF. The fundamental vectore fields are given by $\zeta_X(u) = \frac{\partial}{\partial t}_{|0} u \cdot \exp tX$ and so the pullback $\varphi^*\omega'$ by a homomorphism of principal fiber bundles reproduces the fundamental fields. Further, $(r^g)^*\varphi^*\omega' = \varphi^*(r^g)^*\omega'$ and so all three properties of Cartan connections are clearly fullfiled by ω.

The pullback $\varphi^*\omega'$ evaluates on a constant field as

$$\varphi^*\omega'(\omega^{-1}(X)(u)) = \omega'(\varphi(u))(T\varphi \circ \omega^{-1}(X)(u))$$

and the right hand side equals to X if and only if $T\varphi\circ\omega^{-1}(X)(u) = {\omega'}^{-1}(X)(\varphi(u))$. This proves the first claim. The relatedness of the curvature forms follows immediately form their definition via the structure equation. Finally, the relation between K and K' implies

$$\begin{aligned}\kappa(u)(X,Y) &= K(u)(\omega^{-1}(X), \omega^{-1}(Y)) = K'(\varphi(u))({\omega'}^{-1}(X), {\omega'}^{-1}(Y))\\ &= \kappa'(\varphi(u))(X,Y)\end{aligned}$$

for all $u \in \mathcal{P}$, $X, Y \in \mathfrak{g}$. □

Various interesting and useful subcategories in $\mathcal{C}_{(\mathfrak{g},H)}$ can be defined by restrictions on curvatures. In fact, this has been the basic idea behind the constructions of Cartan connections from simpler data on the manifolds M traced back to Cartan's equivalence method. The key feature is relation between the curvature functions $\kappa = \kappa' \circ \varphi$ for each morphism and the invariance of the cuvature functions. Thus, whenever we fix an H–submodule $\mathfrak{N}$ in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$, there is the full subcategory $\mathcal{C}^{\mathfrak{N}}_{(\mathfrak{g},H)}$ of objects whose curvature functions have values in $\mathfrak{N}$.

There are two distinguished choices: $\mathfrak{N} = 0$ and $\mathfrak{N} = \mathfrak{h}$. In the first case we call the geometries *locally flat* while in the other case we talk about *torsion free* Cartan geometries. Both names will be explained below.

PROPOSITION. *The curvature of a Cartan connection $(\mathcal{P}, \omega)$ on M vanishes identically if and only if there is a neighborhood U of each $x \in M$ such that the restriction $(p^{-1}(U), \omega)$ is isomorphic with a neighborhood of $o \in G/H$ in its flat model.*

PROOF. Assume $K = 0$ identically. Then theorem 1.2.4 implies that for each $u \in \mathcal{P}$, there is a neighborhood $u \in V \subset \mathcal{P}$ and a unique mapping $\varphi : V \subset \mathcal{P} \to G$ such that $\varphi(u) = e$ and $\varphi^*(\omega_G) = \omega$. In particular, φ respects the constant fields restricted to V. This implies that $\varphi(u.\exp t) - \varphi(u).\exp t$ remains constant on a neighborhood of $t = 0$, thus φ can be extended uniquely to a principal bundle morphism over a neighborhood U of $p(u)$. Thus, we see that the required isomorphism locally exists and is given uniquely, up to a choice of the frame u over x on the connected component of the fiber which includes u. □

1.5.3. Natural bundles. The general functorial construction of the associated bundles provides for each left action λ of H on a manifold S the *natural bundle* on the category $\mathcal{C}_{(\mathfrak{g},H)}$ whose values are fibred bundles denoted by the same script symbol $\mathcal{S} \to M$. The values on the flat models are just the homogeneous bundles dealt with in the beginning of section 1.4. In fact, we saw in 1.4.2 that each homogeneous budle is associated to $G \to G/H$ with respect to a left action λ and so it extends in this way to the whole category $\mathcal{C}_{(\mathfrak{g},H)}$. The additional data given by the Cartan connection result then in special properties and objects available on $\mathcal{S}$. For example, we have discussed the existence of an invariant principal connection on $G \to G/H$ in 1.4.5 with the result that each posibility to view the homogeneous model as a reductive Klein geometry leads to one such principal connection and there are no other ones. Thus, the $\mathfrak{h}$–component of any reductive Cartan connection $(\mathcal{P},\omega)$ is a natural principal connection which induces canonical connections on all natural bundles. The freedom in the choice of such a connection on a general $(\mathcal{P},\omega)$ is just in the choice of the H-invariant decomposition $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{h}$.

Since $T(G/H)$ was an example of a homogeneous bundle in 1.4.2, TM must be a natural bundle in general. Let us look at this case in more detail. Let M be a manifold with a Cartan connection $(\mathcal{P},\omega)$ of type $(\mathfrak{g},H)$ and write $\pi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ for the factor projection. For each $u\in\mathcal{P}$ and tangent vector $\xi\in T_{p(u)}M$ we may choose any $\tilde{\xi}\in T_u\mathcal{P}$ covering ξ and view $\pi\circ\omega(\tilde{\xi})$ as coordinates of ξ in the frame u. On the other hand, the couple $u\in\mathcal{P}$, $X+\mathfrak{h}\in\mathfrak{g}/\mathfrak{h}$ defines the vector $Tp.\omega^{-1}(X)(u)\in T_{p(u)}M$ and clearly $u\cdot g$ and $\underline{\mathrm{Ad}}_{g^{-1}}(X)$ define the same tangent vector since $Tp.\omega^{-1}(\underline{\mathrm{Ad}}_{g^{-1}}(X))(u\cdot g)=Tp.Tr^g.\omega^{-1}(X)(u)=Tp.\omega^{-1}(X)(u)$. This provides the identification $TM\simeq\mathcal{P}\times_{\underline{\mathrm{Ad}}}(\mathfrak{g}/\mathfrak{h})$.

Similarly, the associated bundle $\mathcal{P}\times_{\underline{\mathrm{Ad}}^*}(\mathfrak{g}/\mathfrak{h})^*$ provides the one–forms on M and the evaluation of a one–form $\alpha:\mathcal{P}\to(\mathfrak{g}/\mathfrak{h})^*$ on a vector field $\xi:\mathcal{P}\to(\mathfrak{g}/\mathfrak{h})$ is defined by the standard evaluation on the values.

There is a special class of natural bundles $\mathcal{S}_\lambda$ comming from left actions λ of the group G, restricted to H. In particular, if the action is a linear representation $\lambda:G\to GL(\mathbb{W})$, we call the natural vector bundles $\mathcal{W}\to M$ *tractors*. Let us consider the action of the group G on itself, i.e. the left multiplication on G. The corresponding natural budle $\tilde{\mathcal{P}}=\mathcal{P}\times_H G$ is the standard *extension* of $\mathcal{P}$ to the structure group G. Obviously, $\{u,g\}\cdot h=\{u,gh\}$ for any $u\in\mathcal{P}$, $g,h\in G$, defines the principal action of G on $\tilde{p}:\tilde{\mathcal{P}}\to M$ and $u\mapsto\{u,e\}$ is a homomorphism injecting $\mathcal{P}$ into $\tilde{\mathcal{P}}$. Now, each natural bundle $\mathcal{S}_\lambda=\mathcal{P}\times_H S$ is also associated to $\tilde{\mathcal{P}}$, $\mathcal{S}_\lambda=\tilde{\mathcal{P}}\times_G S$.

PROPOSITION. *The Cartan connection $\omega\in\Omega^1(\mathcal{P};\mathfrak{g})$ extends uniquely to the principal connection $\tilde{\omega}\in\Omega^1(\tilde{\mathcal{P}};\mathfrak{g})$ such that $\tilde{\omega}$ restricts to ω on $T\mathcal{P}$. Its horizontal subspaces lift each tangent vector $\{u,X+\mathfrak{h}\}\in TM$, $u\in\mathcal{P}$, $X\in\mathfrak{g}$, to the vector $\omega^{-1}(X)(u)-\zeta_X(u)\in T_u\tilde{\mathcal{P}}$, and its curvature form $\tilde{\rho}\in\tilde{p}^*\Omega^2(TM;\mathfrak{g})$ restricts to the curvature form K of the original Cartan connection ω on $\mathcal{P}$.*

Consequently on each natural bundle $\mathcal{S}_\lambda$ as above, $\tilde{\omega}$ induces connection and, in particular, there is a canonical linear connection on each tractor bundle. The curvatures of these induced tractor connections are given by the action of the values of the curvature K of the original Cartan connection ω.

PROOF. Each principal connection $\tilde{\omega}$ is defined uniquely along each fiber $\tilde{\mathcal{P}}_x$ by its restriction to one tangent space $T_u\tilde{\mathcal{P}}$. Let us choose $u\in\mathcal{P}$, and $\xi=\zeta_X+\eta\in T_u\tilde{\mathcal{P}}$ such that $\eta\in T_u\mathcal{P}$, and define $\tilde{\omega}(\xi)=X+\omega(\eta)$. The freedom in X and η does not effect the value $\tilde{\omega}(\xi)$ since another choice $\xi=\zeta_{X'}+\eta$ leads to $X-X'\in\mathfrak{h}$ and ω reproduces the fundamental vector fields. Next, if $\xi\in T_{u.g}\tilde{\mathcal{P}}$, we find $\xi'\in T_u\mathcal{P}$ with $Tr^g.\xi'=\xi$, and define $\tilde{\omega}(\xi)=\mathrm{Ad}_{g^{-1}}\circ\omega(\xi')$. Again, the freedom in the choice of ξ' does not effect the result because of the equivariance of ω.

The formula for the horizontal lift $\tilde{\xi}$ of $\xi=\{u,X\}\in TM$, $\tilde{\xi}=\omega^{-1}(X)(u)-\zeta_X(u)$, does not depend on the choice of a representative X of a class in $\mathfrak{g}/\mathfrak{h}$ and clearly projects down back to ξ. By the very definition of $\tilde{\omega}$ we obtain $\tilde{\omega}(\tilde{\xi})=0$, as required. The curvature is strictly horizontal and is defined by the same structure euqations, thus their values in the frames $u\in\mathcal{P}$ coincide.

All the remaining claims follow. □

1.5.4. Cartan's space $\mathcal{S}$. The left action of G on the homogeneous space $S = G/H$ provides one of the extremely interesting examples $\mathcal{S}$ of natural bundles with an induced connection. The first remarkable fact is that $x \mapsto O(x) = \{u, o\} \in \mathcal{P} \times_H G/H$ defines the canonical section O of $\mathcal{S} \to M$, and the restriction of the vertical tangent space $V\mathcal{S}$ to the invariant section O is clearly identified with the tangent space TM. Indeed,

$$\mathcal{P} \times_H (\mathfrak{g}/\mathfrak{h}) = TM \ni \{u, X + \mathfrak{h}\} \mapsto \{0_u, X + \mathfrak{h}\} \in T_{O(p(u))}\mathcal{S} \subset T\mathcal{P} \times_{TH} TS$$

is the required isomorphism. Thus we may view the Cartan's space $\mathcal{S}$ as a nonlinear version of the tangent space where the geometry in question is fully encoded by means of the parallel transport of the induced connection. This point of view goes really back to Cartan, cf. [**?**], and it was developed further in an abstract way in the second half of the 20th century (cf. [**?**]).

Let us illustrate the power of the parallelism on the classical concept of developments of curves. Each curve $c : I \subset \mathbb{R} \to M$, $I = [a, b]$, may be viewed as a parameterized curve in $\mathcal{S}$. Thus, for each fixed parameter $t_0 \in I$ and a neighborhood J of t_0 in I, the parallel transport $\tilde{c}_s$ of the point $O(c(s))$ along the inverse curve $c_s : t \mapsto c(t_0 + s - t)$ defines the curve $\operatorname{dev}(c, t_0) : J \to \mathcal{S}_{c(t_0)}$, $t \mapsto \tilde{c}_s(s)$, in the fiber over $c(t_0)$, called the *development* of c at t_0. If we fix a frame $u \in \mathcal{S}$ over $c(t_0)$, the development becomes a curve in the homogeneous space G/H passing through the origin o in the parameter t_0. Thus, each choice of an H–invariant class of curves through the origin in G/H results in the definition of a distinguished class of curves on all manifolds with Cartan connections of type G/H.

For example, for the affine connections on G–structures, there is the concept of straight lines in $G/H = \mathbb{R}^m$, the standard affine space. They are just the projections of left shifts of one–parametric subgroups $\exp tX$ in G with $X \in \mathbb{R}^m$ and so the class $\mathcal{C}$ of straight lines through the origin is clearly invariant under the left action of H. The class of curves on a manifold with an affine connection whose developments are curves in $\mathcal{C}$ is formed just by all geodesics of this connection. This is the classical result of affine differential geometry and we it follows easily as follows.

In general, the class of projections of all left shifts of one–parametric subgroups $\exp tX$ in G, $X \notin \mathfrak{h}$, by elements in H forms an H–invariant set of curves. The corresponding developments are the curves $\{u, [\exp tX]\} \subset \mathcal{S}$. Let us consider the projection $c(t)$ of the flow line $\mathrm{Fl}_t^{\omega^{-1}(X)}(u) \in \mathcal{P}$ to the manifold M. In view of proposition 1.5.3, its horizontal lift to $\tilde{\mathcal{P}}$ is $\mathrm{Fl}_t^{\omega^{-1}(X)}(u) \cdot \exp(-tX)$. Thus, the induced parallel transport of an element $\{u, [\exp tX]\} \in \mathcal{S}$ over $c(0)$ along $c(t)$ is clearly $\{\mathrm{Fl}_s^{\omega^{-1}(X)}(u), [\exp(t - s)X]\}$ and it reaches exactly $O(c(t))$ in time $s = t$. So we have proved that the curves $\{u, [\exp tX]\}$ are just the developments of the projections of flow lines of the non-vertical constant vector fields $\omega^{-1}(X)$ on $\mathcal{P}$.

Moreover, for any fixed complementary subspace $\mathfrak{n} \subset \mathfrak{g}$ to $\mathfrak{h}$, the class $\mathcal{C}_\mathfrak{n}$ of those curves with $X \in \mathfrak{n}$ is H–invariant. Thus, each such choice provides a distinguished class of curves on all manifolds M with Cartan connections of type (G, H). In the case of affine connections on manifolds, this complement $\mathfrak{n}$ is canonically fixed and the corresponding projections of flow lines are exactly the geodesics of the affine connection.

1.5.5. Adjoint tractors. Let us consider a Cartan geometry (P,ω) of type G/P on M. The adjoint action of $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defines the tractor bundle $\mathcal{A} \to M$, $\mathcal{A} = \mathcal{P} \times_H \mathfrak{g}$. We call it the *adjoint tractor bundle*. Obviously, the short exact sequence of G–modules $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ gives rise to the short exact sequence

$$0 \to \mathcal{P} \times_H \mathfrak{h} \to \mathcal{A} \to TM \to 0.$$

We may represent the adjoint tractors nicely as right invariant vector fields along the fibers of the total space of $\mathcal{P}$. Indeed, we associate the right invariant vector field $\xi \in \mathfrak{X}(\mathcal{P})$, determined by $\xi(u) = \omega^{-1}(X)(u)$, to the tractor $\{u, X\} \in \mathcal{A}$. The equivariance of the constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ implies $\omega^{-1}(\mathrm{Ad}_g^{-1} .X)(u.g) = Tr^g.\omega^{-1}(X)(u)$ and so the whole vector field ξ along the fiber does not depend on our choice of the representatives of $\{u, X\}$. Thus, the frame form $s : \mathcal{P} \to \mathfrak{g}$ of the tractor $\xi \in \mathfrak{X}(\mathcal{P})^H$ is simply the composition $s = \omega(\xi)$. The invariant subbundle in $\mathcal{A}$ in the short exact sequence is the space of right invariant vertical vector fields along the fibers in $\mathcal{P}$.

All H–invariant objects on $\mathfrak{g}$ give rise to objects on $\mathcal{A}$. In particular, the Lie bracket on $\mathfrak{g}$ defines the algebraic bracket $(\xi, \eta) \mapsto \{\xi, \eta\}_{\mathfrak{g}}$ on sections of $\mathcal{A}$,

$$\{\{u, X\}, \{u, Y\}\}_{\mathfrak{g}} = \{u, [X, Y]\}$$

for all $u \in \mathcal{P}$, $X, Y \in \mathfrak{g}$. Obviously, the bracket is well defined since the formula gives the same result for $\{u.g, \mathrm{Ad}_{g^{-1}} .X\}$, and $\{u.g, \mathrm{Ad}_{g^{-1}} .Y\}$. In terms of the right invariant vector fields on $\mathcal{P}$, this bracket is even more simple: if $\xi(u) = \omega^{-1}(X)(u)$, $\eta(u) = \omega^{-1}(Y)(u)$ then $\{\xi, \eta\}_g(u) = \omega^{-1}([X, Y])(u)$.

On the other hand, the standard Lie bracket of right invariant vector fields is again right invariant. Thus we obtain yet another bracket $(\xi, \eta) \mapsto [\xi, \eta]$ on the adjoint tractor fields, which is not algebraic.

Let us recall that the curvature function κ has values in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ and its equivariance was described in lemma 1.5.1. But this means that the curvature of a Cartan connection ω is an $\mathcal{A}$–valued two–form $K \in \Omega^2(M; \mathcal{A})$. More explicitly, taking two tractors $\xi = \{u, X\}$, $\eta = \{u, Y\}$ representing the tangent vectors $\pi(\xi)$, $\pi(\eta)$, the curvature function κ defines the tractor $K(\xi, \eta) = \{u, \kappa(u)(X, Y)\}$ which is well defined due to the equivariance of κ. Moreover, the torsion is given by the composition of κ with the projection $\mathfrak{g} \to (\mathfrak{g}/\mathfrak{h})$ and so it corresponds to the projection $\mathcal{A} \to TM$. In particular the torsion is a vector valued two–form $K_- \in \Omega^2(M; TM)$ and the curvatures of torsion–free Cartan geometries have values in the bundle $V\mathcal{P}/H$ of vertical right invariant vector fields along the fibers.

Now, the curvature $R^{\mathcal{A}}$ of the induced adjoint tractor connection is given by the action of the values of the curvature K of ω on tractors, cf. proposition 1.5.3 and 1.3.4(6). But in view of the definition of the algebraic bracket above, we can interpret the resulting formula as

$$R^{\mathcal{A}}(\xi, \eta)(\chi) = \{K(\pi(\xi), \pi(\eta)), \chi\}_{\mathfrak{g}}$$

where ξ, η, and χ are adjoint tractors in the same fiber and the result depends only on the projections $\pi(\xi)$, $\pi(\eta)$ to the tangent vectors.

Finally, let us notice that the adjoint action is always infinitesimally effective and so the induced linear connection $\nabla^{\tilde{\omega}}$ on $\mathcal{A}$ is fully equivalent to the principal connection $\tilde{\omega}$ on the extended bundle $\tilde{\mathcal{P}}$. This connection leaves invariant all objects in $\mathcal{A}$ which are defined by means of G–invariant objects on $\mathfrak{g}$. In particular, the algebraic bracket $\{,\}_{\mathfrak{g}}$ on $\mathcal{A}$ is covariantly constant, i.e.

$$\nabla^{\tilde{\omega}}_{\xi}\{\eta,\chi\} = \{\nabla^{\tilde{\omega}}_{\xi}\eta,\chi\} + \{\eta,\nabla^{\tilde{\omega}}\chi\}.$$

1.5.6. The fundamental D-operator. The constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ determined by the defining absolute parallelism $\omega \in \Omega^1(\mathcal{P},\mathfrak{g})$ of a Cartan geometry $(\mathcal{P},\omega)$ offer a simple differential operation on all sections of natural bundles $\mathcal{S} = \mathcal{P} \times_\lambda S$ called the *fundamental D-operator*. For each mapping $s \in C^\infty(\mathcal{P},S)$ we define its invariant derivative $Ds \in C^\infty(\mathcal{P},\mathfrak{g}^* \otimes TS)$ by

$$Ds(u)(X) = \omega^{-1}(X)\cdot s = Ts(\omega^{-1}(X)(u)).$$

We also write $D_X s(u)$ for $X \in \mathfrak{g}$. The equivariance of ω and sections s yield

$$\begin{aligned} D_X s(u.g) &= \omega^{-1}(X)(u.g)\cdot s = Ts\circ Tr^g(\omega^{-1}(\mathrm{Ad}_g .X)(u)) \\ &= T\lambda_{g^{-1}} \circ \nabla_{\mathrm{Ad}_g .X} s(u) \end{aligned}$$

and we have proved $Ds \in C^\infty(\mathcal{P},\mathfrak{g}^* \otimes TS)^H$ for all $s \in C^\infty(\mathcal{P},S)^H$. Thus the evaluation of the fundamental D-operator on a section s in a given direction $\xi \in \mathcal{A}$ is a section $D_\xi s$ of the vertical bundle $V\mathcal{S}$ covering the original section s. In particular, for natural vector bundles $\mathcal{E} = \mathcal{P}\times_H \mathbb{E}$ we identify $V\mathcal{E} = \mathcal{E}\oplus\mathcal{E}$ and since the second component of the action $T\lambda_g$ is the original linear mapping λ_g, the fundamental D-operator is interpreted as the mapping on the sections of natural vector bundles

$$D : C^\infty(\mathcal{P},\mathbb{E})^H \to C^\infty(\mathcal{P},\mathfrak{g}^*\otimes\mathbb{E})^H$$

or in other letters $D : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{A}^*\otimes\mathcal{E})$.

If $\mathcal{S}$ is a tractor bundle (or its non–linear version), then we can easily compare the fundamental D-operator with the induced connection $\nabla^{\tilde{\omega}}$ on $C^\infty(\mathcal{S})$. Indeed, since the horizontal vector $\tilde{\xi} \in T_u\tilde{\mathcal{P}}$, $u \in \mathcal{P}$, over a vector $\xi = \{u, X + \mathfrak{h}\} \in TM$ is $\omega^{-1}(X)(u) - \zeta_X(u)$, the induced covariant derivative $\tilde{\nabla}s$ of $s : \mathcal{P} \to S$ is given by

$$\nabla^{\tilde{\omega}}_{\xi} s = \{u, D_X s(u) - \zeta_X\cdot s(u)\} = D_X s(u) + X\cdot s(u)$$

where the latter formula is understood either in $T\mathcal{S}$ in the general case or in $\mathcal{S}$ in the case of a tractor bundle. Clearly, the formula does not depend on the choice of the representative X of the element in $\mathfrak{g}/\mathfrak{h}$, since $\omega^{-1}(X) = \zeta_X$ if $X \in \mathfrak{h}$.

Example. Let us look at the adjoint tractors in more detail. Clearly, the above comparison of the induced covariant derivative $\nabla^{\tilde{\omega}}$ and the fundamental D-operator yields

$$D_\xi\eta = \nabla^{\tilde{\omega}}_{\pi(\xi)}\eta - \{\xi,\eta\}_{\mathfrak{g}}$$

where $\pi : \mathcal{A} \to TM$ is the projection.

In the special case of the affine connections on G-structures, the adjoint tractors split as $\mathcal{A} = TM \oplus V\mathcal{P}/H$ and the restrictions of the fundamental D-operator and $\nabla^{\tilde{\omega}}$ to the tangent vector fields coincide and they provide the usual covariant derivative of vector fields. We have also seen that their commutator $D_\xi\eta - D_\eta\xi$ defines the torsion $T(\xi,\eta)$ of this covariant differentiation, up to the Lie bracket $[\xi,\eta]$. Of course, the latter bracket coincides with the bracket $[\ ,\ ]$ on adjoint tractors in this special case and the formula for the torsion extends to general adjoint tractors as well.

Let us start with the frame form $\omega([\xi,\eta]) : \mathcal{P} \to \mathfrak{g}$ of the Lie bracket of two right invariant vector fields ξ, $\eta \in \mathfrak{X}(\mathcal{P})$. The structure equation for the Cartan connection ω reveals

$$\omega([\xi,\eta]) = \mathcal{L}_\xi(\omega(\eta)) - \mathcal{L}_\eta(\omega(\xi)) + [\omega(\xi),\omega(\eta)] - \kappa(\omega(\xi),\omega(\eta)).$$

Now, $\mathcal{L}_\xi(\omega(\eta)) = D_\xi(\eta)$ by the definition and $\omega([\xi,\eta])$ yields the frame form of the bracket $[\xi,\eta]$ of the adjoint tractors. Thus

$$[\xi,\eta] = D_\xi\eta - D_\eta\xi + \{\xi,\eta\} - \kappa(\xi,\eta)$$

and we may also rewrite this formula in terms of the induced covariant derivative

$$[\xi,\eta] = \nabla^{\tilde\omega}_{\pi(\xi)}\eta - \nabla^{\tilde\omega}_{\pi(\eta)}\xi - \{\xi,\eta\} - \kappa(\xi,\eta).$$

These formulae for the curvature express the torsion part as well as the whole curvature by a simple first order formula, while the classical affine geometry finds the torsion as a first order object and curvature appears only at order two. This is not contradictory because the adjoint tractors themselves are not objects homogeneous in order. We shall realize this fact in connection with further differential identities below.

1.5.7. Bianchi and Ricci identities. A remarkable feature of the fundamental D–operators is that we may iterate them. Indeed, for each natural vector bundle $\mathcal{V}$ we have a sequence of differential operators $D^k : C^\infty(\mathcal{V}) \to C^\infty(\otimes^k\mathcal{A}^*\otimes\mathcal{V})$. We are going to look at some differential identities on the curvatures of Cartan geometries and the iterated D–operators.

PROPOSITION. *Let $\mathcal{V}$ be a natural bundle over the manifold M endowed with the Cartan geometry $(\mathcal{P},\omega)$ with curvature $K \in \Omega^2(M;\mathcal{A}) \subset C^\infty(\Lambda^2\mathcal{A}^*\otimes\mathcal{A})$.*
(1) (Bianchi) The curvature K satisfies for all ξ, η, $\chi \in C^\infty(\mathcal{A})$

$$\sum_{cycl}\Big(\{K(\xi,\eta),\chi\}_{\mathfrak{g}} + K(\{\xi,\eta\}_{\mathfrak{g}},\chi) - K(K(\xi,\eta),\chi) - D_\chi K(\xi,\eta)\Big) = 0$$

where the sum runs over all cyclic permutations of the arguments, $\{\ ,\ \}_{\mathfrak{g}}$ is the canonical algebraic bracket on $\mathcal{A}$, and the curvature K depends only on the projections of its arguments to TM.
(2) (Ricci) The alternation of the 2nd order fundamental D-operator is given by

$$(D^2)s(u)(\xi,\eta) - (D^2)s(u)(\eta,\xi) = D_{K(\xi,\eta)}s(u) - D_{\{\xi,\eta\}_{\mathfrak{g}}}s.$$

PROOF. In the frame form, the Bianchi identity expresses the differential symmetry of the curvature function κ by the following identity

$$\begin{aligned}
0 = \sum_{\text{cycl}}\Big([\kappa(u)(X,Y),Z] &+ \kappa(u)([X,Y],Z) - \\
&\kappa(u)\big(\kappa(u)(X,Y),Z\big) - \omega^{-1}(Z)\cdot\kappa(u)(X,Y)\Big)
\end{aligned} \tag{3}$$

at all frames $u \in \mathcal{P}$, and for all X, Y, $Z \in \mathfrak{g}$. Let us evaluate the structure equation on the vector fields $[\tilde X,\tilde Y]$ and $\tilde Z$, where $\tilde X = \omega^{-1}(X)$, $\tilde Y = \omega^{-1}(Y)$, and $\tilde Z = \omega^{-1}(Z)$. Remember in particular $\kappa(X,Y) = K(\tilde X,\tilde Y)$ and $[\tilde X,\tilde Y] = \omega^{-1}([X,Y]) - \omega^{-1}(K(\tilde X,\tilde Y))$. Thus

$$\begin{aligned}
K([\tilde X,\tilde Y],\tilde Z) &= -\mathcal{L}_{\tilde Z}\omega([\tilde X,\tilde Y]) - \omega([[\tilde X,\tilde Y],\tilde Z] + [\omega([\tilde X,\tilde Y]),Z]) \\
&= \mathcal{L}_{\tilde Z}(\kappa(X,Y)) - \omega([[\tilde X,\tilde Y],\tilde Z]) + [[X,Y],Z] - [\kappa(X,Y),Z]
\end{aligned}$$

and the left–hand side equals $\kappa([X,Y],Z)-\kappa(\kappa(X,Y),Z)$. Now, let us perform the cyclic permutation over X, Y, and Z. Leaving the two terms which disappear by virtue of the Jacobi identity for vector fields and the Lie algebra $\mathfrak{g}$ on left–hand side, while collecting all remaining terms on the other side of the equality, we obtain exactly the required identity.

In order to see the Ricci identity, we first prove that the iterated fundamental D-operator $D^i s(\xi_1,\dots,\xi_i)$ equals for all i to

$$D^i s(\xi_1,\dots,\xi_i)=\{u,\mathcal{L}_{\omega^{-1}(X_i)}\circ\mathcal{L}_{\omega^{-1}(X_{i-1})}\circ\dots\circ\mathcal{L}_{\omega^{-1}(X_1)}s(u)\} \tag{4}$$

where $u\in\mathcal{P}$, $\xi_i=\{u,X_i\}\in\mathcal{A}$ and we use the same letter s for the section and its frame form. Obviously, this is just the definition for $i=1$. Next, $D(Ds)(u)(X,Y)=\mathcal{L}_{\omega^{-1}(Y)}(\nabla s(u))(X)=\mathcal{L}_{\omega^{-1}(Y)}(\nabla_X s)(u)$ since the invariant differential is linear in X and so we have verified the formula for $i=2$. Now, replacing the section s by its $(i-1)$st invariant derivative, we deduce the general formula from the case $i=2$.

In view of formula (4), the alternation of the second invariant derivative is given by

$$\begin{aligned}\operatorname{Alt} D^2 s(\xi,\eta)&=(\mathcal{L}_{\tilde X}\circ\mathcal{L}_{\tilde Y}-\mathcal{L}_{\tilde Y}\circ\mathcal{L}_{\tilde X})s=\mathcal{L}_{[\tilde Y,\tilde X]}s\\&=-\mathcal{L}_{\omega^{-1}([X,Y])}s+\mathcal{L}_{\omega^{-1}(K(\tilde X,\tilde Y))}s\\&=-Ds(\{\xi,\eta\}_{\mathfrak{g}})+Ds(K(\xi,\eta)).\end{aligned}$$

□

We shall discuss the links to the classical Bianchi and Ricci identity in the end of the next subsection.

1.5.8. Splitted Cartan geometries. For many Lie algebras $\mathfrak{h}\subset\mathfrak{g}$, there is a preferred split $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{h}$ into subalgebras. We called the Klein geometries with such a fixed decomposition *splitted*, see 1.2.5, and we use the same adjective for the Cartan geometries with such models. The special case of reductive splitted geometries when the subalgebra $\mathfrak{n}$ happens to be an H–submodule was discussed in terms of the affine connections on G–structures already in 1.3.6. Since we shall meet the splitted geometries nearly exclusively in the rest of the book, let us specify some of the discussion above right now.

The first point to make is that the quotient $\mathfrak{g}/\mathfrak{h}$ gets identified with the subalgebra $\mathfrak{n}$ and the subalgebra inherits the restricted adjoint action $\underline{\mathrm{Ad}}$ of H in this way. Thus, with this H–module structure in mind, we may view TM as $\mathcal{P}\times_H\mathfrak{n}$.

The main new input from the splitting consists in the existence of the distinguished *horizontal distribution* $\omega^{-1}(\mathfrak{n})\subset T\mathcal{P}$ spanned by the *horizontal constant fields* $\omega^{-1}(X)$, $X\in\mathfrak{n}$. In particular, the Cartan connection ω is now a general connection with many specific properties, cf. 1.3.2. Of course, the curvature provides the obstruction to the involutivity of the horizontal distribution, but it splits itself into the horizontal and vertical parts. The horizontal projection $\pi_{\mathrm{hor}}:T\mathcal{P}\to\omega^{-1}(\mathfrak{n})$ composed with the standard Lie bracket of vector fields provides the horizontal Lie bracket of horizontal fields $\mathfrak{X}_{\mathrm{hor}}(\mathcal{P})\subset\mathfrak{X}(\mathcal{P})$, $[\ ,\ ]_{\mathrm{hor}}:\mathfrak{X}_{\mathrm{hor}}\wedge\mathfrak{X}_{\mathrm{hor}}\to\mathfrak{X}_{\mathrm{hor}}$. A simple check reveals that the torsion of ω vanishes if and only if the mapping $\omega^{-1}:\mathfrak{n}\to\mathfrak{X}_{\mathrm{hor}}$ is a Lie algebra homomorphism with respect to this bracket. Indeed, $\pi_{\mathrm{hor}}(\omega^{-1}(K(\omega^{-1}(X),\omega^{-1}(Y))))=\omega^{-1}([X,Y])-\pi_{\mathrm{hor}}([\omega^{-1}(X),\omega^{-1}(Y)])$.

Next, the curvature function κ is understood as $\kappa:\mathcal{P}\to\Lambda^2(\mathfrak{n}^*)\otimes(\mathfrak{n}\oplus\mathfrak{h})$. Let us notice however that while the torsion part is clearly the first component

$\kappa : \mathcal{P} \to \Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$, the other component does not build an invariant object until the torsion vanishes.

The fundamental D-operator restricts to an operation

$$\nabla^\omega : C^\infty(\mathcal{P}, S) \to C^\infty(\mathcal{P}, \mathfrak{n}^* \otimes TS)$$

called *invariant derivative*, which is more similar to the usual covariant derivative but fails to transform sections into sections in general. We often write $\nabla^\omega_X s$ for the evaluation $D_X s$ at the constant $X \in \mathfrak{n}$. Notice $\nabla^\omega_X s(u)$ means the derivative $D_{\{u,X\}} s(u)$ so that we differentiate in directions of different adjoint tractors for different frames u in the fiber of $\mathcal{P}$, in general. This is also the reason, why the invariance gets lost, except $\mathfrak{n}$ is H–invariant.

PROPOSITION. *Let $(\mathcal{P}, \omega)$ be a splitted Cartan geometry modeled over G/H, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, and let $\mathcal{E} = \mathcal{P} \times_H \mathbb{E}$ be a natural vector bundle, $s \in C^\infty(\mathcal{E})$.*
(1) (Bianchi identity) The curvature function $\kappa = \kappa_- + \kappa_{\mathfrak{h}} : \mathcal{P} \to \Lambda^2\mathfrak{n}^* \otimes (\mathfrak{n} \oplus \mathfrak{h})$ satisfies for all X, Y, $Z \in \mathfrak{n}$

$$0 = \sum_{cycl} \big([\kappa(X,Y), Z] + \kappa([X,Y], Z) - \kappa(\kappa_-(X,Y), Z) - \omega^{-1}(Z) \cdot \kappa(X,Y) \big).$$

(2) (Ricci identity) The alternation of the 2nd invariant derivative is given by

$$\nabla_Y \nabla_X s(u) - \nabla_X \nabla_Y s(u) = \nabla_{\kappa_-(u)(X,Y)} s(u) - \nabla_{[X,Y]} s(u) - \kappa_{\mathfrak{h}}(u)(X,Y) \cdot s(u).$$

PROOF. (1) The restriction of the frame form of Bianchi identity yields exactly the required equality, cf. the beginning of the proof of proposition 1.5.6.
(2) The restriction of the Ricci identity from 1.5.6 and the equivariance of the frame form s of the section provides exactly our formula. □

In the special case of affine connections on G–structures, we have recovered the standard properties of the torsion, curvature and covariant derivatives ∇. We shall write T^i_{jk}, R^i_{jkl}, $T^i_{jk;m}$, and $R^i_{jkl;m}$ for the frame forms of the torsion, curvature and their covariant derivatives, respectively.

COROLLARY (Bianchi identity). *Let (P, ω) be a reductive Cartan geometry with $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ and let the H–submodule $\mathfrak{n}$ be an abelian subalgebra. Then the invariant derivative is the frame form expression of the standard covariant derivative ∇ of all induced connections on the natural bundles $\mathcal{E}$, the Bianchi identity splits into two parts*

$$0 = R^i_{\{jk\ell\}} + T^p_{\{jk} T^i_{\ell\}p} - T^i_{\{jk;\ell\}} \tag{3}$$

$$0 = R^i_{p\{m} T^p_{k\ell\}} + R^i_{j\{kl;m\}} \tag{4}$$

where $\{...\}$ denotes the cyclic sum over the indicated indices.

PROOF. The fact that ∇^ω coincides with the covariant derivatives was discussed already in 1.3.5 and 1.3.6. The general Bianchi identity is an expression valued in $\mathfrak{n} \oplus \mathfrak{h}$ and so both of its components have to vanish simultaneously. Moreover, the brackets in $\mathfrak{n}$ vanish by our assumptions and formulae (4) and (5) follow. □

In order to understand the Ricci identity in the more classical terms, we have to notice the difference between the second iterated covariant differential $(\nabla^\omega)^2$ evaluated on two vector fields on M and the consecutive covariant differentiation $\nabla_\xi \nabla_\eta$ in the directions of two vector fields. While the former operation means to differentiate the full first order covariant derivative in the direction of the first vector field and then to evaluate the result on the other vector field, the latter operation evaluates first the covariant derivative in the direction of the second vector field and then repeats the same on the result with the first vector field. Thus, for each section s of an associated vector bundle $\mathcal{S}$ of P^1M,

$$(\nabla^\omega)^2 s(\xi,\eta) = \nabla^\omega_\xi(\nabla^\omega_\eta s) - \nabla^\omega_{\nabla^\omega_\xi \eta} s. \tag{5}$$

Corollary (Ricci identity). *The curvature $R(\xi,\eta) \in T^*M \otimes TM$ of the affine connection ω is defined by*

$$-R(\xi,\eta).\chi = (\nabla^\omega_\xi \nabla^\omega_\eta - \nabla^\omega_\eta \nabla^\omega_\xi)\chi - 2\nabla_{T(\xi,\eta)}\chi - \nabla_{[\xi,\eta]}\chi$$

where ξ, η, χ are arbitrary vector fields and T is the torsion of ω.

Proof. Straightforward computation using the general Ricci identity, the formula for the torsion (cf. 1.3.6) and formula (5) above yields

$$\begin{aligned}\mathrm{Alt}(\nabla^\omega)^2\chi(\xi,\eta) &= (\nabla^\omega_\xi \nabla^\omega_\eta - \nabla^\omega_\eta \nabla^\omega_\xi)\chi - \nabla^\omega_{(T(\xi,\eta)+[\xi,\eta])}\chi \\ &= \nabla^\omega_{T(\xi,\eta)}\chi - R(\xi,\eta)(\chi).\end{aligned}$$

This provides the required formula □

In the torsion–free case, the Bianchi and Ricci identities get particularly simple. The alternation of the 2nd differential is then just the algebraic action of the values of curvature, up to the sign. This Ricci identity specialized to covariant derivatives of vector fields reveals one of the most classical definitions of the curvature of a torsion–free linear connection on a manifold. The two Bianchi identities (4) and (5) appear then in the most classical form $R^i_{\{jk\ell\}} = 0$ and $R^i_{j\{k\ell;m\}} = 0$.

1.5.9. Jet prolongations. All first order differential operators are linked closely to the first jet prolongation of the bundles in question. The fundamental D–operators and the invariant derivatives are somewhat special in this respect and the next lemma identifies the first jet prolongation $J^1\mathcal{V}$ of each natural vector bundle with a natural subbundle in $\mathcal{V} \oplus (\mathcal{A}^* \otimes \mathcal{V})$.

Lemma. *Let $(\mathcal{P},\omega)$ be a Cartan geometry modeled over $(\mathfrak{g},H)$ and let $\mathcal{V}$ be a natural bundle corresponding to the H–module $\mathbb{V}$. The mapping*

$$C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V} \oplus (\mathcal{A}^* \otimes \mathcal{V})), \quad s \mapsto (s, Ds)$$

determines an injective map $J^1\mathcal{V} \to \mathcal{V} \oplus (\mathcal{A}^* \otimes \mathcal{V})$ whose image is the natural subbundle with standard fiber $J^1\mathbb{V} = \mathbb{V} \oplus ((\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{V}) \subset \mathbb{V} \oplus (\mathfrak{g}^* \otimes \mathbb{V})$ consisting of all couples (φ_0,φ_1) which satisfy $\varphi_1(X) + X \cdot \varphi_0 = 0$ for all $X \in \mathfrak{h}$. In particular, $J^1\mathbb{V}$ is the standard fiber of the homogenoues bundle $J^1(G \times_H \mathbb{V})$, as an H–module.

In particular, if the Cartan geometry is splitted then the restriction of $s \mapsto (s, Ds)$ to the invariant derivative $(s, \nabla^\omega s)$ defines the identification $J^1\mathcal{V} = \mathcal{P} \times J^1\mathbb{V}$, and the action of $Z \in \mathfrak{h}$ on $(s, (X \mapsto \nabla^\omega_X s) \in J^1\mathbb{V}$ is

$$Z \cdot (s, X \mapsto \nabla^\omega_X s) = (Z \cdot s, (X \mapsto Z \cdot (\nabla^\omega_X s) - \nabla^\omega_{[Z,X]_\mathfrak{n}} + [Z,X]_\mathfrak{h} \cdot s)).$$

Proof. Let us first recall the H–module structure on the standard fiber J^1_oV of the first jet prolongation of the homogeneous bundle $V = G \times_H \mathbb{E}$ which we described only quite implicitly in 1.4.6.

We shall view jets $j^1_o s$ as jets of H–equivariant functions $s : G \to V_o$ at the unit $e \in G$ and so we understand them as couples $(s(e), (X \mapsto Ts(X)))$ where the domain for X in a fixed complementary vector space $\mathfrak{n} \subset \mathfrak{g}$ of $\mathfrak{h}$. Since $g \cdot j^1_o s$ is given by $j^1_e s \circ \ell_{g^{-1}}$, we may compute with the help of the Maurer–Cartan form ω on G as follows

$$\begin{aligned} g \cdot j^1_o s &= \big(s(g^{-1}), (X \mapsto \omega^{-1}(X) \cdot (s \circ \ell_{g^{-1}}))\big) \\ &= \big(\lambda(g)(s(e)), (X \mapsto Ts(\omega^{-1}(X)(g^{-1})))\big) \\ &= \big(\lambda(g)(s(e)), (X \mapsto Ts \circ Tr^{g^{-1}}(\omega^{-1}(\mathrm{Ad}_{g^{-1}}.X)(e)))\big) \\ &= \big(\lambda(g)(s(e)), (X \mapsto T\lambda_g \circ Ts(\omega^{-1}(\mathrm{Ad}_{g^{-1}}.X)(e)))\big). \end{aligned}$$

Now, clearly the mapping $s \mapsto (s, Ds)$ depends only on first jets in each point $x \in M$. Moreover, its symbol is clearly the inclusion $T^*M \otimes \mathcal{V} \to \mathcal{A}^* \otimes \mathcal{V}$ defined by the dual of the canonical projection $\mathcal{A} \to TM$. By the properties of the invariant differential the image must be contained in the subspace from the lemma and since the two bundles have the same rank, the map is surjective onto this subbbundle too. Finally, the above computation for the action of H on the standard fiber of the hmogenenous space $J^1(G \times_H \mathbb{V})$ shows that also the H–module structures fit togehter.

Assume now that the Cartan geometry is splitted, i.e. $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. As an H–module, $\mathfrak{n}^* = (\mathfrak{g}/\mathfrak{h})^*$ is a submodule in $\mathfrak{g}^*$. Thus, once we have fixed the complement $\mathfrak{n}$ of $\mathfrak{h}$, the image of the embedding of $J^1\mathcal{V}$ into the natural bundle $\mathcal{P} \times_H (\mathbb{V} \oplus (\mathfrak{g}^* \otimes \mathbb{V}))$ clearly is the subbundle $J^1\mathbb{V} = \mathbb{V} \oplus (\mathfrak{n}^* \otimes \mathbb{V})$. In order to compute the $\mathfrak{h}$–module structure, we have to continue in the above computation of the action on $J^1\mathbb{V}$. Due to the splitting of $\mathfrak{g}$ and the equivariancy of s, the expression $X \mapsto T\lambda_g \circ Ts(\omega^{-1}(\mathrm{Ad}_{g^{-1}}).X)(e)$ yields exactly the required formula. □

Remark. Let us notice that the jet prolongation of the H–modules is of functorial character. Indeed, for each H–equivariant mapping $f : \mathbb{V} \to \mathbb{W}$ of H–modules, we define $J^1 f(v_0, X \otimes v_1) = (f(v_0), X \otimes f(v_1))$. At the same time, the mapping $J^1 f$ defines also the mapping $\mathcal{P} \times_H J^1\mathbb{V} \to \mathcal{P} \times_H J^1\mathbb{W}$ which coincides exactly with jet prolongation of the induced fiber bundle map $\mathcal{V} \to \mathcal{W}$. A simple check that this defines an H–module homomorphism is left to the reader.

Similarly as above, we may also embed the kth jet prolongation $J^k\mathcal{V}$ into $\oplus_{i=0}^k (\otimes^i \mathcal{A}^* \otimes \mathcal{V})$. However, the image of this embedding is by far not so easy to understand as in the case $k = 1$. The reason is hidden in the general Bianchi and Ricci identities. While the result is quite unpleasant to formulate in general, the things get reasonably simple in the framework of the splitted geometries and we obtain a nice algorithmic tool for the discussion of the invariant differential operators at the same time.

1.5.10. Higher order jets and invariant operators. As we have seen in section 1.4, the invariant operators on the homogeneous bundles are classified by means of equivariant mappings between the jet prolongations of homogeneous bundles and, in order to make this classification at least a bit tractable, we had to pass to representation theoretical tools. Now, we may pose several questions: (1) Given an invariant operator Q between two homogeneous bundles, is there an invariant operator between all natural bundles which restricts to Q in the homogeneous case? (2) If the answer to (1) is 'yes', how many operators with this property are there? (3) What are the reasonable algebraic tools to handle questions (1) and (2)?

We give an indication to partial answers to all these questions in the next proposition, but first we have to introduce the so called *semi–holonomic jet prolongation* of bundles and representations. For each fiber bundle $Y \to M$, there are the kth non–holonomic jet prolongations $J^1(J^1 \dots (J^1Y))$. This means, instead of taking higher order jets of sections, we iterate the construction of jets of sections k times. Of course, the (holonomic) kth jet prolongations J^kY are canonically embedded in these bundles. The Ricci identity for the invariant derivatives indicates that the candidate for higher order 'invariant jet' of a section of a natural bundle for a splitted Cartan geometry,

$$(s, \nabla^\omega s, \dots, (\nabla^\omega)^k s) : \mathcal{P} \to \oplus_{i=0}^k (\otimes^i \mathfrak{n}^* \otimes \mathbb{V})$$

has values in the standard fiber $J^k\mathbb{V}$ of the kth jet prolongation of the corresponding homogeneous bundle over the flat model only if the curvature of the Cartan connection ω vanishes. On the other hand, the full redundancy in the non–holonomic jets is not necessary.

Let us look more closely at the second iteration $J^1(J^1\mathcal{V})$. Obviously, there are two morphisms from the latter bundle to $J^1\mathcal{V}$. The first one is just the target projection of the outer J^1, while the other one is the action of the functor J^1 on the target projection of the inner J^1. Both of these projections are understood nicely on the level of the H–modules, $J^1(J^1\mathbb{V}) \to J^1\mathbb{V}$. As a vector space the domain is $\mathbb{V} \oplus (\mathfrak{n}^* \otimes \mathbb{V}) \oplus (\mathfrak{n}^* \otimes \mathbb{V}) \oplus (\mathfrak{n}^* \otimes \mathfrak{n}^* \otimes \mathbb{V})$, the former projection $\pi_{J^1\mathbb{V}}$ forgets the last two components, while the latter projection $J^1\pi_\mathbb{V}$ maps $(v_0, X \otimes v_1, Y \otimes v_2, Z_1 \otimes Z_2 \otimes v_3)$ to $(v_0, Y \otimes v_2)$. Thus, the equalizer of these two natural transformations $\pi_{J^1\mathbb{V}}$ and $J^1(\pi_\mathbb{V})$ is the H–module which looks as $\mathbb{V} \oplus (\mathfrak{n}^* \otimes \mathbb{V}) \oplus (\mathfrak{n}^* \otimes \mathfrak{n}^* \otimes \mathbb{V})$ as vector space. On the level of the fiber bundles, we have recovered the definition of the second order semi–holonomic jet prolongations $\bar{J}^2$ due to Ehresmann, and we call the corresponding functor $\bar{J}^2$ on the H–modules the second semi–holonomic prolongation of the modules.

In general, we may adopt an inductive assumption that the H–module $\bar{J}^k\mathbb{V}$ has been constructed together with the canonical projection $\pi_k : \bar{J}^k\mathbb{V} \to \bar{J}^{k-1}\mathbb{V}$ such that $\bar{J}^k\mathbb{V} = \oplus_{i=0}^k (\otimes^i \mathfrak{n}^* \otimes \mathbb{V})$ as vector space and the projection π_k just forgets the last summand. Moreover, we may assume that $\mathcal{P} \times_H \bar{J}^k\mathbb{V}$ is canonically isomorphic to the semi–holonomic jet prolongation $\bar{J}^k\mathcal{V}$, and π_k is the common restriction of the two projections $\bar{J}^k\mathbb{V} \subset J^1(\bar{J}^{k-1}\mathbb{V}) \to \bar{J}^{k-1}\mathbb{V} \subset J^1(\bar{J}^{k-2}\mathbb{V})$ to their domain $\bar{J}^k\mathbb{V}$. Of course, these assumptions are satisfied with $k = 2$ and the properties mimic those of the standard semi–holonomic jet prolongations of fiber bundles. Thus, we consider the obvious two projections $J^1(\bar{J}^k\mathbb{V}) \to J^1(\bar{J}^{k-1}\mathbb{V})$ and define the $(k+1)$st semi–holonomic prolongation as their equalizer. According to the properties of the projections π_k, the common restriction $\pi_{k+1} : \bar{J}^{k+1}\mathbb{V} \to \bar{J}^k\mathbb{V}$ of the two projections above enjoys again all the required properties, now with the order increased by one.

Proposition. *Let $(\mathcal{P}, \omega)$ be a splitted Cartan geometry of type G/H and consider an H–module $\mathbb{V}$. Then the kth semi–holonomic prolongation $\bar{J}^k\mathcal{V}$ is canonically isomorphic to the natural bundle $\mathcal{P} \times_H (\bar{J}^k\mathbb{V})$ and the iterated invariant differential ∇^ω induces the canonical embedding of the kth (holonomic) jet prolongation $J^k\mathcal{V}$ into the bundle of semi–holonomic jets $\bar{J}^k\mathcal{V}$,*

$$j^k_u s \mapsto (s(u), \nabla^\omega s(u), \dots, (\nabla^\omega)^k s(u))$$

where s is the frame form of a section of $\mathcal{V}$. In particular, each H–module homomorphism $\Phi : \bar{J}^k\mathbb{V} \to \mathbb{W}$ provides an invariant linear operator D_Φ defined by

$D_\Phi s(u) = \Phi(s(u), \dots, (\nabla^\omega)^k s(u))$ for all section s of $\mathcal{V}$. If $\mathcal{V}$ is the homogeneous bundle $G \times_H \mathbb{V}$ then D_Φ is given by the restriction of Φ to $J^k\mathbb{V} \subset \bar{J}^k\mathbb{V}$. Moreover, if Φ is any H–equivariant mapping we still obtain the (nonlinear) invariant operator D_Φ by the same formula.

Proof. The first claim has been proved already, but we still have to discuss why is the 'invariant jet' an equivariant mapping from $\mathcal{P}$ to $\bar{J}^k\mathbb{V}$. This has been proved in lemma 1.5.9 for $k = 1$. Thus the equivariance of the iterative application of the invariant derivative

$$j^k_u s \mapsto (\dots, ((s, \nabla^\omega s), \nabla^\omega(s, \nabla^\omega s)) \dots)$$

is clear as well. But the co–restriction of this highly redundant mapping to the H–submodule $\bar{J}^k\mathbb{V}$ mimics exactly the embedding of the semi–holonomic jets into the non-holonomic ones. In particular, the mapping under question has to be equivariant too.

Given such an H-module homomorphism Φ, and a frame form $s : \mathcal{P} \to \mathbb{V}$ of a section of $\mathcal{V}$, the definition $D_\Phi s(u) = \Phi(s(u), \dots, (\nabla^\omega)^k s(u))$ makes sense. By the invariance properties of the two compounds of this mapping, the whole operator has to be invariant. In the case of the homogeneous bundle, the formula factors through the holonomic jet prolongation of the H–module $\mathbb{V}$ since the curvature vanishes identically. Clearly, D_Φ is linear if Φ is a homomorphism. □

We should underline, that the result of the proposition is not completely satifactory in view of possible classifications of the invariant operators. Of course, also the general question 'how to find the homomorphisms Φ?' is difficult to answer for most choices of homogeneous models G/H, but even more serious problems are related to the fact that the semiholonomic jets are still too big. They accomodate the invariantly built jets for all possible background curvatures and they even do not involve the obvious but non–linear restrictions on the possible images of the embedding $J^k\mathcal{V} \to \bar{J}^k\mathcal{V}$ determined by the Ricci and Bianchi identities. Therefore, there are homomorphisms which are nontrivial in general but restrict to zero or to lower order operators in the homogeneous case, and there are non–zero homomorphisms which produce identically vanishing operators everywhere. But the relation between the operators and homomorphisms may happen to be even worse than that, since there are even nontrivial invariant operators for the homogeneous space for which such a Φ does *not* exist. We shall see explicit instances of all these unpleasant possibilities later on.

On the other hand, we shall also see that the tools offered by the proposition allow to construct vast classes of higly interesting operators.

1.5.11. Automorphisms of Cartan geometries. The aim of this subsection is to understand that the homogeneous models admit the largest group of possible symmetries among all Cartan geometries of a fixed type, the whole Lie group G. We shall start with a classical theorem due to [**?**].

Proposition. *Let G be a group of diffeomorphisms of a smooth manifold M and let $S \subset \mathfrak{X}(M)$ be the subset of all vector fields ξ whose flows Fl^ξ_t are globally defined and $\mathrm{Fl}^\xi_t \in G$ for all $t \in \mathbb{R}$. If S generates a finite dimensional Lie subalgebra in $\mathfrak{X}(M)$, then G is a Lie group of transformations of M and S is the Lie algebra of G.*

PROOF. We shall present a brief formulation of the proof following [?, theorem 3.1]. Let us write $\mathfrak{g} \subset \mathfrak{X}(M)$ for the Lie algebra of vector fields generated by S and consider the connected and simply connected group Lie group $\tilde{G}$ with the Lie algebra $\mathfrak{g}$. The exponential mapping $\exp : \mathfrak{g} \to \tilde{G}$ defines the local action of $\tilde{G}$ on M, i.e. there is a mapping $\lambda : \tilde{G} \times M \to M$ defined on a neighborhood U of $\{e\} \times M \subset \tilde{G} \times M$ such that $\lambda(\exp tX, x) = \mathrm{Fl}_t^X(x)$ for all $(X, x) \in U$.

The first claim we shall deal with is that S *generates* $\mathfrak{g}$ *as a vector space*. Thus let us write $V \subset \mathfrak{g}$ for the vector space generated by S. In fact we have to prove $[V, V] \subset V$. Let us notice that the Lie bracket of the vector fields in V is given by the action ad of $\mathfrak{g}$ on V. Since $\mathfrak{g}$ is generated by S, the group $\tilde{G}$ is generated by $\exp S$. If we suppose S is invariant with respect to the adjoint action of $\tilde{G}$ then also V must be invariant with respect to Ad, but this implies that V is invariant with respect to ad as required. Thus it remains to prove $\mathrm{ad}(\exp tX)(Y) \in S$ for all X, $Y \in S$. The element $Z = \mathrm{ad}(\exp X)Y \in \mathfrak{g}$ is defined by the condition $\exp tZ = \exp X \exp tY \exp(-X)$. Now, the letter condition means $\lambda(\exp tZ, x) = \mathrm{Fl}_1^X \circ \mathrm{Fl}_t^Y \circ \mathrm{Fl}_{-1}^X(x)$ which exists for all t and x. In particular $Z \in S$ and the claim has been proved.

Now, let us choose some basis $X_1, \dots, X_k$ of the vector space $\mathfrak{g}$ consisting of elements of S only, and consider the locally defined diffeomorphism $\mathfrak{g} \to \tilde{G}$,

$$\sum c^i X_i \mapsto \exp(c^1 X_1) \dots \exp(c^k X_k)$$

onto a negborhood of the unit $e \in \tilde{G}$. Thus, for each Y fixed and for all $t \in \mathbb{R}$ close enough to zero there are unique constants $c^1(t), \dots, c^k(t)$ satisfying $\exp tY = \exp(c^1(t)X_1) \dots \exp(c^k(t)X_k)$, and so we may define the action of $\exp tY$ on M by $x \mapsto \mathrm{Fl}_1^{c^1(t)X_1} \circ \dots \circ \mathrm{Fl}_1^{c^k(t)X_k}(x)$ which is well defined for all t and x, the action clearly has the properties of the flow of Y, and it produces an element in G for each t. Thus each element $Y \in \mathfrak{g}$ is in S.

Now, we know $S = \mathfrak{g}$ and so all fields in $\mathfrak{g}$ are defined by one–parametric subgroups of global doffeomorphisms on M. In particular, the group G_e of diffeomorphisms generated by $\mathfrak{g}$ is a Lie group of transformations and $G_0 \subset G$ at the same time. So it looks like G_0 should be the connected component of the unit in G. In order to see this, let us pick up any $\varphi \in G$ and a one–parametric subgroup ψ_t in G_0. Then $\varphi \psi_t \varphi^{-1}$ is again a one parametric subgroup contained in G, and so it has to be in G_0. Thus, each φ defines a contiunous automorphism $\iota_\varphi : G_0 \to G_0$ (clearly ι is a homomorphism which tranforms one–parametric subgroups into one–parametric subgroups and so it has to be continuous).

Finally, if we consider a basis $\{U_\alpha\}$ of the open neighborhoods of the unit $e \in G_0$, then for each $\varphi \in G$ the images $\{\varphi(U_\alpha)\}$ define the basis of the neighborhoods of φ in a well defined topology on G. In this topology, G_0 is open an normal in G and there is no other topology with this property. Now, G is a topological group whose connected component of the unit is the Lie group G_0. Thus, there is the unique smooth structure on G extending the smooth structure on G_0. □

THEOREM. *Let $(\mathcal{P}, \omega)$ be a Cartan geometry of type $(\mathfrak{g}, H)$ over a connected manifold M. Then the group Q of all automorphisms of $(\mathcal{P}, \omega)$ is a Lie group of dimension at most* $\dim \mathfrak{g}$.

PROOF. According to lemma 1.5.2, the automorphisms of $(\mathcal{P}, \omega)$ are those principal fiber bundle automorphisms which commute with the flows of the constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$. In particular, this implies that the set $\mathfrak{a}$ of all infinitesimal automorphisms contains exactly vector fields η with $\mathcal{L}_\eta\omega = 0$. The evaluation of $(\mathcal{L}_\eta\omega)(\omega^{-1}(X)) = 0 - \omega([\eta, \omega^{-1}(X)])$, $X \in \mathfrak{g}$, shows that $\mathfrak{a}$ constists of all fields commuting with every $\omega^{-1}(X)$, $X \in \mathfrak{g}$. Notice that all fields in $\mathfrak{a}$ are automatically right invariant vector fields and they form a subalgebra by the Jacobi identity. However, they need not have global flows in general.

Next, consider two fields ξ, $\eta \in \mathfrak{a}$ and $u \in \mathcal{P}$, such that $\xi(u) = \eta(u)$. Then

$$\begin{aligned}\xi(\mathrm{Fl}_s^{\omega^{-1}(X)}(u)) &= \frac{\partial}{\partial t}_{|0} \mathrm{Fl}_t^{\xi} \circ \mathrm{Fl}_s^{\omega^{-1}(X)}(u) = \frac{\partial}{\partial t}_{|0} \mathrm{Fl}_s^{\omega^{-1}(X)} \circ \mathrm{Fl}_t^{\xi}(u) \\ &= \frac{\partial}{\partial t}_{|0} \mathrm{Fl}_s^{\omega^{-1}(X)} \circ \mathrm{Fl}_t^{\eta}(u) = \frac{\partial}{\partial t}_{|0} \mathrm{Fl}_t^{\eta} \circ \mathrm{Fl}_s^{\omega^{-1}}(u) = \eta(\mathrm{Fl}_s^{\omega^{-1}(X)})(u).\end{aligned}$$

Thus, the vector fields ξ and η coincide on a whole connected component of $\mathcal{P}$. Since the fields are right invariant, this implies that they coincide over a whole conected component in M. In particular we have proved that the evaluation mapping $\mathfrak{a} \to T_u\mathcal{P}$ is injective and so the dimension of $\mathfrak{q}$ is always at most $\dim \mathfrak{g}$.

Next, let $S \subset \mathfrak{a}$ be the subset of all vector fields with global flows. By the definition, S and Q satisfy the assumptions of the proposition above and so Q is a finite dimensional Lie group and S equals to its Lie algebra $\mathfrak{q}$. Thus $\dim \mathfrak{q} \leq \dim \mathfrak{a} \leq \dim \mathfrak{g}$. □

1.5.12. Morphisms of Cartan geometries.

1.5.13. Infinitesimal automorphisms.

1.5.14. Generalized geodesics.

1.6. Conformal Riemannian structures

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