Notes on Tractor Calculi

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Abstract. These notes present elementary introduction to tractors based on classical examples, together with glimpses towards modern invariant differential calculus related to vast class of Cartan geometries, the so called parabolic geometries.

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This is a survey article based on the lectures given by the first author at the Summer School Wisła 19, 19-29 August, 2019, captured by the second author. The exposition aims at quick understanding of basic principles, omitting many proofs or at least their details. The reader might find a lot of further information in the cited sources throughout the text. In particular, our approach has been heavily inspired by [12], while the general background on Cartan geometries including the tractors can be found in [11].

The six sections of the article roughly correspond to the six lectures (about 100 minutes each). We first introduce some elements of the tractor calculus in quite general situation. Then we focus rather on the overall structure of the invariant linear differential operators and we do not present much of the tractor calculus itself. In this sense, these lecture notes are complementary to [12], where the reader should look for the genuine calculus.

The audience was assumed to have basic knowledge of differential geometry as well as some representation theory (Lie groups and algebras, their representations, principal and associated bundles, connections, tensors, etc.). All this background can be found e.g. in [17] and [11].

1 Tracy Thomas' conformal tractors

Let us start with a quick review of the two very well known geometries, the Riemannian and the conformal Riemannian ones.

1.1 Riemannian sphere

There are many ways how to view the standard sphere

$$S^n = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

as a homogeneous space. Perhaps the most common one is to consider the orthogonal group $G = \mathrm{O}(n+1)$ which keeps S^n invariant and its subgroup H of the maps fixing a given point $o \in S^n$, isomorphic to $\mathrm{O}(n)$. Clearly $S^n = \mathrm{O}(n+1)/\mathrm{O}(n)$. The Lie algebras $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{h} = \operatorname{Lie} H$ enjoy the nice matrix $(1, n)$ block structure

$$\mathfrak{g} = \begin{pmatrix} 0 & -v^{\mathrm{T}} \\ v & X \end{pmatrix}, \quad \mathfrak{h} = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}.$$

The tangent spaces are $T_x S^n = \{y \in \mathbb{R}^{n+1} | \ \langle x, y \rangle = 0\}$ and the action of G on $\mathbb{R}^{n+1}$, $x \mapsto Ax$, preserves both S^n and TS^n, i.e. $y \mapsto Ay$ maps $T_x S^n \mapsto T_{Ax} S^n$. Moreover, they preserve the scalar products on the tangent spaces and thus S^n enjoys $\mathrm{O}(n+1)$ as isometries of the natural structure of a Riemannian manifold (S^n, g).

Observation 1. There are no other isometries of S^n apart from $\mathrm{O}(n+1)$.

The standard way to see the above observation holds true is the following. Consider a unit vector $e_1 \in \mathbb{R}^{n+1}$ and an isometry $\phi \in \operatorname{Isom}(S^n, g)$. Then $\phi(e_1) \in S^n$ and $\phi(e_1) = A(e_1)$ for some $A \in \mathrm{O(n+1)}$. Moreover, elements of the form $A^{-1} \circ \phi$ are in the isotropy group of e_1. As well known the Riemannian isometries are (on connected components) uniquely determined by their differential in one point. Thus the latter map coincides with its differential at e_1 and we are finished.

Another possibility is based on the Maurer-Cartan form. It is more complicated, but much more conceptual.

Consider the principal H-bundle $G \xrightarrow{p} G/H$ over $S^n \cong G/H$ equipped with the Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$. Since $\mathfrak{g}$ splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ (as $\mathfrak{h}$-module), the Maurer-Cartan form also splits as $\omega = \omega_{\mathfrak{h}} \oplus \omega_{\mathfrak{n}}$, where the first part is the principal connection $\omega_{\mathfrak{h}}$ on G and the second part is the soldering form $\omega_{\mathfrak{n}}$. The soldering form provides for all $A \in G$ isomorphisms

$$\omega_{\mathfrak{n}} \colon T_A G / V_A G \xrightarrow{\cong} T_{p(A)} S^n \cong \mathbb{R}^n ,$$

where $V_A G := \ker \omega_{\mathfrak{n}}$ is the vertical subspace. This means $\omega_{\mathfrak{n}}$ makes $G \to S^n$ into a reduction of the linear frame bundle $P^1 S^n$ to H.

Now any isometry ϕ lifts to the level of frame bundles and can be restricted to G and thus we have a lift $\tilde{\phi} \colon G \to G$ such that $\tilde{\phi}^* \omega_{\mathfrak{n}} = \omega_{\mathfrak{n}}$. Because $\omega_{\mathfrak{h}}$ is principal connection preserving the metric we also have $\tilde{\phi}^* \omega_{\mathfrak{h}} = \omega_{\mathfrak{h}}$. We see that $\tilde{\phi}^* \omega = \omega$, i.e. $\tilde{\phi}$ preserves the Maurer-Cartan form.

Notice that in this setting, $\omega_{\mathfrak{h}}$ must be the only torsion free metric connection on S^n. Thus, we arrived at the canonical Cartan connection on S^n in the sense of the so called *Cartan geometry* as defined below.

For any principal fiber bundle $\mathcal{G}$ with structure group P, we shall write r^g for the principal right action of elements in P and ζ_X means the fundamental vector field, $\zeta_X(u) = \frac{d}{dt}_{|0} r^{\exp tX}(u)$.

Definition 1. *For a pair $H \subset G$ of a Lie group and its Lie subgroup, a Cartan geometry is a principal H-bundle $p : \mathcal{P} \to M$ endowed with a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ satisfying for all $h \in H, X \in \mathfrak{h}, u \in \mathcal{P}$ the following three properties*

$$\mathrm{Ad}(h^{-1}) \circ \omega = (r^h)^* \omega, \tag{1}$$
$$\omega(\zeta_X(u)) = X, \tag{2}$$
$$\omega(u) \colon T_u\mathcal{P} \xrightarrow{\cong} \mathfrak{g}. \tag{3}$$

Now, our observation follows from a general result:

Theorem 1 (Fundamental theorem of calculus). *Let ω_G be the Maurer-Cartan form of a Lie group G with the Lie algebra $\mathfrak{g}$, M a smooth manifold endowed with a 1-form $\omega \in \Omega^1(M, \mathfrak{g})$. Then for each $x \in M$ there is a neighborhood $U \ni x$ and $f \colon U \to G$ such that $f^*\omega_G = \omega$, if and only if*

$$\mathrm{d}\,\omega + \frac{1}{2}[\omega, \omega] = 0. \tag{4}$$

If M is connected and $f_1, f_2 \colon M \to G$ with $f_1^*\omega_G = f_2^*\omega_G$ on M, then there exists a unique $c \in G$ such that $f_2 = cf_1$ on M.

Under the additional requirement that $Tf : T_xM \to T_{f(x)}G$ is a linear isomorphism for each point x, the theorem shows that the local Lie group structure is uniquely determined by the Maurer-Cartan form satisfying (4).

The theorem is proved by building the graph of the mapping f, see [11, section 1.2.4]. Notice, in dimension one the condition is empty and so with the additive group $G = \mathbb{R}$ we obtain just the existence of primitive functions up to a constant. If G was the multiplicative group $\mathbb{R}_+$, the theorem would show how the logarithmic derivatives prescribe the functions, up to a constant multiple.

In our case each isomorphism $\phi : S^n \to S^n$ lifts to the unique map $f : G \to G$ satisfying $f^*\omega = \omega$. The Maurer-Cartan forms on all Lie groups satisfy the condition in (4) and thus, even locally, f can differ from the identity map only by an element of G.

1.2 Conformal Riemannian sphere

A conformal Riemannian manifold $(M, [g])$ is a manifold M with a conformal class of metrics. Two metrics g, $\tilde{g}$ are representatives of the same conformal class if they differ by some positive function, $\tilde{g} = \Omega^2 g$, $\Omega \in C^\infty(M)$. Conformal isometry is a diffeomorphism, whose differentials at all points belong to the conformal orthogonal group $\mathrm{CO}(n)$ for the given structures on the tangent spaces.

The conformal sphere is $(S^n, [g])$ where $[g]$ includes the standard round metric. Let us discuss the following question: *What is the group of all conformal isomorphisms on S^n making it into a homogeneous space G/P?*

Option 1. We can go the ‘brutal force’ way. Take $\mathbb{R}^n$ with the conformal class containing the Euclidean metric and write down the PDEs for an arbitrary locally defined conformal isomorphism ϕ, i.e., we request the differentials of ϕ are in $\mathrm{CO}(n)$ at all points. There is the famous Liouville theorem saying that each such ϕ is generated by the Euclidean motions and the sphere inversions. An elementary (but tricky) proof can be found in [20, section 5.4]. In particular, if we compactify $\mathbb{R}^n$ by the one point at infinity, we can extend all such local diffeomorphisms to globally defined conformal maps on S^n.

Let us try to do it in a smart way. Consider $\mathbb{R}^{n+2}$ with the pseudo-Euclidean metric $Q(x,x) = 2x_0x_{n+1} + x_1^2 + \cdots + x_n^2$ of signature $(n+1,1)$ and define C to be the null-cone of this metric. Now, we may identify the sphere S^n with the projectivization $\mathbb{P}C$ of this cone and write down the action of all the latter maps in projective coordinates on $\mathbb{P}\mathbb{R}^{n+2}$. We can represent the null-vectors of the affine $\mathbb{R}^n \subset S^n$ as $(1 : x : -\frac{1}{2}\|x\|^2)$, while the remaining infinite point in S^n is $(0:0:1)$. Now we may easily identify the above conformal maps as actions of particular matrices in $\mathrm{O}(n+1,1)$ on the projectivized cone $\mathbb{P}C$.

$$\begin{pmatrix} 1 \\ x \\ -\frac{1}{2}\|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ x \\ -\frac{1}{2}\|x\|^2 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} 1 \\ x \\ -\frac{1}{2}\|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v & E & 0 \\ -\frac{1}{2}\|v\|^2 & -v^T & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ -\frac{1}{2}\|x\|^2 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} 1 \\ x \\ -\frac{1}{2}\|x\|^2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & -2 \\ 0 & E & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ -\frac{1}{2}\|x\|^2 \end{pmatrix} \tag{7}$$

Notice in the last line that the sphere inversion $\sigma \in \mathrm{O}(n+1,1)$ is in a different component than the unit, while the nontrivial maps fixing the origin and having the identity as differential there are obtained by composing $\sigma \circ \tau_v \circ \sigma$, with the translation $\tau_v \in \mathrm{O}(n+1,1)$ from (6), cf. [20, section 5.10].

Option 2. Similarly to the Riemannian case, we first choose the right homogeneous space $S^n = G/P$ with $G = \mathrm{O}(n+1,1)$, P the isotropy group of one fixed point in S^n, and show that G is just the group of all conformal isomorphisms. Again, we can achieve that by building a reasonably normalized Cartan geometry for each conformal Riemannian manifolds. Then the Maurer-Cartan form ω_G of G will be preserved by all conformal morphisms and thus the Theorem 1 applies.

We shall come back to such normalizations of Cartan geometries later in the fifth lecture.

At the level of Lie algebras, $\mathfrak{g} = \mathrm{Lie}\, G$ decomposes as $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{p}$, where $\mathfrak{g}_{-1}$ are the infinitesimal translations with matrices

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ v & 0 & 0 \\ 0 & -v^T & 0 \end{pmatrix} \right\}$$

while

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \Bigg\{ \underbrace{\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}}_{\mathfrak{co}(n)} \oplus \underbrace{\begin{pmatrix} 0 & w & 0 \\ 0 & 0 & -w^T \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbb{R}^n} \Bigg\}.$$

Clearly, $\mathfrak{g}_1$ is a $\mathfrak{p}$-submodule (actually an ideal), while $\mathfrak{g}_0$ is identified with the $\mathfrak{p}$-module $\mathfrak{p}/\mathfrak{g}_1$ with the trivial action of $\mathfrak{g}_1$. This is the well known decomposition of $\mathfrak{p}$ into the reductive quotient and the nilpotent submodule.

At the level of Lie groups, this corresponds to the splitting of the isotropy group P into the semi-direct product of $\mathrm{CO}(n)$ containing the conformal isomorphisms fixing the origin and determined by their first derivatives, and the nilpotent normal subgroup $P_+ \subset P$ of those conformal isomorphisms fixing the origin with trivial first differential and determined by the second order derivatives. We shall also write $G_0 = P/P_+$ and this reductive group decomposes further into the semisimple part $\mathrm{O}(n)$ and the center $\mathbb{R}$.

1.3 Towards tractors

The conformal Riemannian structure on S^n can be read off the standard metric Q on $\mathbb{R}^{n+2}$ as follows. Any choice of a non-zero section of the null-cone C (e.g. we may choose one of the non-zero components C_+ of C and consider sections there), seen as line bundle over its projectivization S^n, provides the identification of the tangent bundle T_pS^n with the quotient of

$$T_pC_+ = \{z \in \mathbb{R}^{n+2} \mid Q(z,p) = 0\} = \langle p \rangle^\perp$$

by the line $\langle p \rangle$ (notice p is null, so this line is in the tangent space). Clearly $\langle p \rangle^\perp / \langle p \rangle$ is linearly isomorphic to T_pS^n and since p is null, Q induces a positive definite metric on T_pS^n. If we multiply p by a constant $a \neq 0$, then the induced metric will change by the positive multiple a^2. By the very construction, this conformal structure is invariant with respect to the natural action of $\mathrm{O}(n+1,1)$ on the cone C. Thus, we may also view C_+ as the square root of the line bundle of the conformal metrics in this class.

Of course, there is no preferred affine connection on S^n in this picture. But if we consider the flat affine connection ∇ on $\mathbb{R}^{n+2}$, then we can consider the parallel (constant) vector fields in the trivial vector bundle $C \times \mathbb{R}^{n+2}$ along the null-lines in C and view them as fields in the trivial vector bundle $\mathcal{T}S^n = S^n \times \mathbb{R}^{n+2}$.

The slight problem with this point of view is that we should expect that the fibers of $\mathcal{T}S^n$ split into the 'vertical part' along the null-lines in C, the 'tangent part' to S^n and the complementary 1-dimensional part in $\mathbb{R}^{n+2}$. While the vertical part is well defined, such a splitting clearly depends on the choice of the identification of S^n with a section of $C \to S^n$. Moreover, we should hope to inherit an invariant connection from the flat connection ∇ on $\mathbb{R}^{n+2}$. Before answering these questions, we are going to indicate a much simpler abstract description of such objects and we come back to these functorial objects and constructions in the fourth lecture.

Let $H \subset G$ be a Lie subgroup and $G \to G/H$ the corresponding Klein geometry. Notice that $G \to G/H$ is a principal H-bundle. Consider any linear representation $\mathbb{V}$ of G and the associated bundle $\mathcal{V} = G \times_H \mathbb{V}$, i.e., the classes of the equivalence relations on $G \times \mathbb{V}$ given by $(u, v) \sim (u \cdot h, h^{-1} \cdot v)$.[1]

In particular, we may identify the class $[\![u, v]\!]$ with the couple $(u \cdot H, u \cdot v)$. Indeed, taking another representative, we arrive at

$$((u \cdot h) \cdot H, u \cdot h \cdot (h^{-1} \cdot v)) = (u \cdot H, u \cdot v)$$

and thus $\mathcal{V}$ is the trivial bundle on M

$$\mathcal{V} = G/H \times \mathbb{V}.$$

Moreover, there is the Maurer-Cartan form ω on G. Extending $G \to G/H$ to the principle G-bundle $\tilde{G} = G \times_H G \to G/H$, the form ω uniquely extends to a principal connection form $\tilde{\omega}$ on $\tilde{G}$. Finally, we can further identify $\mathcal{V}$ as the associated space $\mathcal{V} = \tilde{G} \times_G \mathbb{V}$. Thus we see that there is the induced connection ∇ on all such bundles $\mathcal{V}$.

1.4 Tracy Thomas' tractors $\mathcal{T}$

Now we come back to the conformal sphere and we apply the above abstract construction. Thus, $G = \mathrm{O}(n+1, 1)$, and $H = P \subset G$ is the isotropy group of the fixed origin $(1 : 0 : 0)$, i.e., the Poincare subgroup in G with the Lie algebra $\mathfrak{p}$ as discussed above. Further, we may take $\mathbb{T} = \mathbb{R}^{n+2}$ with the standard action of G.

The final ingredients we need are the weights of line bundles or more general tensor bundles on conformal manifolds. Consider $\mathbb{R}[w]$ as the representation of P such that P_+ and $\mathrm{O}(n)$ act trivially, while the central element $\lambda = \exp(aE) \in G_0$ acts as $\lambda \cdot x = \mathrm{e}^{-aw}\, x$. Here E is the so called grading element in $\mathfrak{g}$, i.e.,

$$\lambda = \exp \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{a} & 0 & 0 \\ 0 & \mathrm{Id}_n & 0 \\ 0 & 0 & \mathrm{e}^{-a} \end{pmatrix}.$$

Now, the line bundles of weights w are defined as

$$\mathcal{E}[w] = G \times_P \mathbb{R}[w].$$

At the level of the infinitesimal action, the central element $aE \in \mathfrak{g}_0$ will act as $a \cdot x = -wax$. Notice the minus sign convention – this is because we want the line bundle of the conformal metrics to get the weight two.

[1] Here $u \cdot h$ is multiplication in G and $h^{-1} \cdot v$ is the left-action of H or G on $\mathbb{V}$ given by the chosen representation.

Taking tensor products with line bundles, we arrive at the general weighted bundles $\mathcal{V}[w] = \mathcal{V} \otimes \mathcal{E}[w]$ for any representation $\mathbb{V}$ of P or even G.

Now, look at the action (5) on the components of $\mathbb{T}$. We see immediately that the representation space $\mathbb{T} = \mathbb{R}^{n+2}$ splits as G_0-module into

$$\mathbb{T} = \mathbb{R}[1] \oplus \mathbb{R}^n[-1] \oplus \mathbb{R}[-1],$$

where the right ends are P-submodules. In particular, $\mathbb{R}[-1]$ is a P-submodule, while $\mathbb{R}[1]$ is the projecting component. Thus, the trivial bundle $\mathcal{T}$ splits (once a section of C_+ and thus one of the metrics in the class are fixed):

$$\mathcal{T} = \mathcal{E}[1] \oplus \underbrace{TS^n[-1] \oplus \overbrace{\mathcal{E}[-1]}^{VC_+}}_{TC_+}.$$

As mentioned above, the bundle $\mathcal{T}$ comes equipped with the canonical metric induced by Q and $TC_+ = (VC_+)^\perp$. Thus, there is the positive definite metric $\mathbf{g} : TS^n[-1] \times TS^n[-1] \to \mathcal{E}$, i.e. $\mathbf{g}$ is a section of $S^2(T^*S^n)[2]$. This is the conformal class of metrics on S^n viewed as the section of a weighted metric bundle and it allows us to raise and lower tensor indices of arbitrary tensors exactly as in the Riemannian case, but at the expense of adding or subtracting the weight 2. For example, we may write $\mathcal{T} = \mathcal{E}[1] \oplus T^*S^n[1] \oplus \mathcal{E}[-1]$.

Finally, any section σ of the projecting part $\mathcal{E}[1]$ provides the Riemannian metric $g = \sigma^{-2}\mathbf{g}$.

2 Conformal to Einstein and the tractor connection

Tracy Thomas came across his conformal tractors in [22], when constructing basic invariants of conformal geometry via a linear connection on a suitable vector bundle (instead of building an absolute parallelism in the Cartan's approach). He succeeded in finding the simplest of such vector bundles, together with an invariant linear connection. He also worked out the necessary transformation properties based on the so called Schouten tensor.

All these objects were reinvented in [1] and here the authors also discussed the following question: *Given a conformal class* $[g]$ *on a manifold, is there a representative of the class which is an Einstein metric?* We shall follow this development and thus we shall find the Thomas' tractors when prolonging a conformally invariant geometric PDE (also following [12]).

In the sequel, we shall use the abstract index formalism. Moreover we shall mostly not distinguish between the bundles $\mathcal{V}M$ and the spaces of their sections $\Gamma(\mathcal{V}M)$. Thus, we shall talk about vector fields in $\mathcal{E}^a$ or one-forms in $\mathcal{E}_a$. Similarly, η_{ab} is either a two-form in $\mathcal{E}_{ab}$ or a $\mathcal{E}_a$-valued one-form. As usual, repeated indices at different positions (lower versus upper) mean the relevant trace.

2.1 The Einstein scales

Recall that the curvature $R_{ab}{}^c{}_d$ of the unique torsion-free metric connection ∇ decomposes into the trace-free Weyl tensor $W_{ab}{}^c{}_d$ and the Ricci tensor R_{ab}. We shall see later why a trace-adjusted version of R_{ab}, the so called Schouten tensor,

$$P_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}Rg_{ab}\right)$$

where $R = g^{ab}R_{ab}$ is the scalar curvature, is very useful. (Notice, here we use the opposite sign convention for the Schouten tensor P than in [11], i.e. it is the same as in [12].)

Of course, we should believe there is an overdetermined distinguished PDE system on the scales, i.e. the choices of the metrics in the class, whose solutions correspond to the Einstein scales. We can write down all such PDEs with the help of any of the metrics in the class and, as a matter of fact, the equation must be independent of our choice, i.e. *conformally invariant.* A straightforward check reveals (we shall come to such techniques later) that the following equation on (the square roots of) the scales σ in $\mathcal{E}[1]$ is invariant

$$\boxed{\nabla_{(a}\nabla_{b)_0}\sigma + P_{(ab)_0}\sigma = 0} \tag{8}$$

Now, if σ is a nowhere zero solution, we may write the equation using the metric connection corresponding to σ and thus, σ is parallel and we arrive at $P_{(ab)_0} = 0$. This is exactly the condition to be Einstein, i.e., the trace-free part of Ricci vanishes and $\sigma^{-2}\mathbf{g}_{ab}$ is Einstein.

We are going to apply the classical method of prolongation of overdetermined systems of PDEs to show that solutions of (8) are equivalent to *parallel tractors* in $\mathcal{T}$.

First, we add trace part $\rho\mathbf{g}$ to the equation (8), i.e, ρ is a new (-1)-weighted quantity $\rho \in \mathcal{E}[-1]$ and the new equation becomes

$$\nabla_{(a}\nabla_{b)}\sigma + P_{(ab)}\sigma + \mathbf{g}_{ab}\rho = 0. \tag{9}$$

Moreover, we know that the Ricci curvature is symmetric for all scales, i.e. the Levi-Civita connections of the metrics in the class, and thus the Schouten tensor is symmetric too. Finally, the antisymmetric part of the second order derivative is given by the action of the curvature $R_{ab}{}^c{}_d$ as a 2-form valued in the Lie algebra $\mathfrak{so}(n,\mathbb{R})$ and these values have no central component to act on the densities $\mathcal{E}[w]$. Thus our equation becomes

$$\nabla_a\nabla_b\sigma + P_{ab}\sigma + \mathbf{g}_{ab}\rho = 0. \tag{10}$$

In the next step, we give the derivative $\nabla_a\sigma$ the new name $\mu_a = \nabla_a\sigma \in \mathcal{E}_a[1]$.[2] Thus the latter equation (10) can be rewritten as the system of two first order

[2] We know that μ_a must be of weight 1 because covariant differentiation does not alter weights and σ is already of weight 1.

equations

$$\boxed{\begin{aligned} \nabla_a \sigma - \mu_a &= 0 \\ \nabla_a \mu_b + P_{ab}\sigma + \mathbf{g}_{ab}\rho &= 0 \end{aligned}} \tag{11}$$

This system is not yet closed since there is still the uncoupled variable ρ. Thus we have to prolong the system and we need some computational preparation first.

Recall the invariant conformal metric $\mathbf{g}$ is covariantly constant in all scales and differentiate (10):

$$\nabla_a \nabla_b \nabla_c \sigma + \mathbf{g}_{bc} \nabla_a \rho + (\nabla_a P_{bc})\sigma + P_{bc} \nabla_a \sigma = 0 \ . \tag{12}$$

Contract (12) by hitting it with $\mathbf{g}^{ab}$ and $\mathbf{g}^{bc}$, respectively:

$$\Delta(\nabla_c \sigma) + \nabla_c \rho + \nabla^a P_{ac}\sigma + P^a{}_c \nabla_a \sigma = 0, \tag{13}$$

$$\nabla_a(\Delta\sigma) + n\nabla_a \rho + \nabla_a P\sigma + P\nabla_a \sigma = 0, \tag{14}$$

where P is the trace of P_{ab}. Next, contracting the Bianchi identity and some straightforward computations lead to

$$\nabla^a P_{ac} = \nabla_c P \tag{15}$$

$$[\nabla_c, \Delta] = R_{cb}{}^b{}_d \nabla^d \ . \tag{16}$$

Subtracting (13) from (14) and using (15) and (16) we arrive at

$$(n-1)\nabla_c \rho + P\nabla_c \sigma - P^a{}_c \nabla_a \sigma + R_{cb}{}^b{}_d \nabla^d \sigma = 0. \tag{17}$$

Further notice

$$R_{cb}{}^b{}_d \nabla^d \sigma = -R_{ca} \nabla^a \sigma = (2-n) P_c{}^a \nabla_a \sigma - \nabla_c P \sigma$$

which together with (17) yields, up to the constant factor $n-1$

$$\boxed{\nabla_c \rho - P_c{}^a \mu_a = 0} \tag{18}$$

and our system of equations closes up. Summarizing, the Einstein scales correspond to nowhere zero solutions of our system of three first order equations coupling σ, μ_a, and ρ and all this should be understood in terms of conformally invariant objects and operations.

Indeed, this is the content of the following theorem. For now, we formulate it only for the solutions to our equations on the sphere S^n, although our discussion on the equations has concerned general conformal Riemannian manifolds.

Theorem 2. *Let $\mathcal{T} = \mathcal{E}[1] \oplus T^*S^n[1] \oplus \mathcal{E}[-1]$ be the bundle of the Thomas' tractors on the conformal sphere. Define the following operator on $\mathcal{T}$*

$$\nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathbf{g}_{ab}\rho + P_{ab}\sigma \\ \nabla_a \rho - P_{ab}\mu^b \end{pmatrix}, \tag{19}$$

where ∇_a *on the right-hand side refers to the Levi-Civita connection of the metric* $\sigma^{-2}\mathbf{g}$.

The operator $\nabla^{\mathcal{T}}$ *is a linear connection on* $\mathcal{T}$ *which is conformally invariant. Moreover, solutions to (8) are in bijective correspondence with parallel tractors, i.e. with sections* $t \in \Gamma(\mathcal{T})$ *such that* $\nabla^{\mathcal{T}} t = 0$.

Notice that on the sphere, $\mathcal{T}$ is the trivial bundle $S^n \times \mathbb{T}$ and we shall see soon that $\nabla^{\mathcal{T}}$ is the flat connection there which we mentioned earlier. So we also postpone the proof of this theorem. (Actually, we see immediately that this is a connection, but we should check its curvature and, in particular, how it depends on the choice of the fixed metric.)

Obviously, all the parallel tractors are determined uniquely by their values in $\mathbb{T}$ in the origin. In particular, we managed to compute all Einstein metrics on the conformal sphere.

2.2 Conformal invariance

What do we really mean when saying that objects or operations are *conformally invariant*?

The intuitively obvious answer should be that they are independent of our choice of the metric in the conformal class. So we should start to look how the covariant derivative changes if we change the scale. Consider the change of our metric by taking $\hat{g} = \Omega^2 g$ with a positive smooth function Ω, and write $\Upsilon_a = \Omega^{-1}\nabla_a \Omega$.

Lemma 1. *Let* $\hat{\nabla}$ *be the Levi-Civita connection for the rescaled metric* $\hat{g}$. *Then for all* $v \in \mathcal{E}^a$, $\alpha \in \mathcal{E}_a$, $\rho \in \mathcal{E}[w]$

$$\hat{\nabla}_a v^b = \nabla_a v^b + \Upsilon_a v^b - \Upsilon^b v_a + \Upsilon^c v_c \delta_a^b \tag{20}$$

$$\hat{\nabla}_a \alpha_b = \nabla_a \alpha_b - \Upsilon_a \alpha_b - \Upsilon_b \alpha_a + \Upsilon^c \alpha_c g_{ab} \tag{21}$$

$$\hat{\nabla}_a \rho = \nabla_a \rho + w \Upsilon_a . \tag{22}$$

Proof. Recall the Christoffel symbols of Levi-Civita connection are expressed in any coordinates via the derivative of the metric coefficients (we write ∇_i for the partial derivatives here)

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell}(\nabla_k g_{\ell j} + \nabla_j g_{\ell k} - \nabla_\ell g_{jk}) \ . \tag{23}$$

Conformal rescaling of the metric $g \mapsto \hat{g} = \Omega^2 g$ affects all other objects derived from metric, e.g. the new inverse metric is $\hat{g}^{-1} = \Omega^{-2} g$. Thus, the Christoffels (23) change

$$\begin{aligned} \hat{\Gamma}^i{}_{jk} &= \Gamma^i{}_{jk} + \frac{1}{\Omega}\left(\delta^i_j \nabla_k \Omega + \delta^i_k \nabla_j \Omega - g_{jk} \nabla^i \Omega\right) \\ &= \Gamma^i{}_{jk} + \delta^i_j \Upsilon_k + \delta^i_k \Upsilon_j - g_{jk} \Upsilon^i . \end{aligned} \tag{24}$$

Now recall, the covariant derivative is in coordinates given as the directional derivative modified by the action of the Christoffels viewed as $\mathfrak{o}(n)$-valued one-form. Thus the latter formula provides exactly the three formulae in the statement.

The formulae from the lemma allow to compute easily the changes of conformal derivatives on all weighted tensor bundles.

For example, considering possible first order operators on weighted forms $\mathcal{E}_a[w]$, we get

$$\hat{\nabla}_a \alpha_b = \nabla_a \alpha_b + (w-1)\Upsilon_a \alpha_b - \Upsilon_b \alpha_a + \Upsilon^c \alpha_c g_{ab},$$

and we immediately see that the antisymmetric part $\nabla_{[a}\alpha_{b]}$ is invariant for the weight $w = 0$ (this is the exterior differential on one-forms), the trace-free part of the symmetrization $\nabla_{(a}\alpha_{b)_0}$ is invariant for $w = 2$ (we may view this as the operator on the vector fields in $\mathcal{E}^a = \mathcal{E}_a[2]$ and the kernel describes the conformal Killing vector fields), and finally the trace $\nabla^a \alpha_a$ is invariant for $w = 2 - n$ (this is the divergence of vector fields with weight $-n$).

Let us look at the geometric objects next. On the conformal sphere S^n, the category of natural objects was defined in 1.3 – those are the homogeneous bundles $G \times_P \mathbb{V}$ corresponding to any representation of P. If the representation comes from a representation of $G_0 = \mathrm{CO}(n)$, extended by the trivial representation of $P_+ = \exp \mathfrak{g}_1$, the corresponding bundles extend to all conformal Riemannian manifolds. Indeed, since general conformal Riemannian manifolds are given as reduction of the linear frame bundles to the structure group G_0, such bundles are well defined on all of them.

If we deal with more general P-representations, then we arrive at sums of the latter bundles as soon as we fix a metric g in the conformal class, but the components are not given invariantly. We shall explain the general procedure in the next lecture and, in particular, we shall see how this behavior extends and defines such bundles on all conformal Riemannian manifolds. For now, just believe that in the case of the Thomas' tractor bundles we face the following transformation rule

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \sigma \Upsilon_a \\ \rho - \Upsilon^c \mu_c - \frac{1}{2}\Upsilon^c \Upsilon_c \sigma \end{pmatrix} \tag{25}$$

Of course, a straightforward (and really tedious) computation can reveal that considering the formulae for the transformations of covariant derivatives from the above Lemma and (25), the linear tractor connection $\nabla_a^{\mathcal{T}}$ is a well defined conformally invariant operator on the tractors. Fortunately, we do not have to check this the pedestrian way and can wait for general reasons.

3 Parabolic geometries

We met the general Cartan geometries with the model G/H in the Definition 1. If the Lie group G is semisimple and the subgroup H is a parabolic subgroup in G, we talk about the *parabolic geometries*. This class of Cartan geometries includes many very important examples and provides a unified theory for all of them. In this lecture we shall introduce some basic features and clarify many phenomena in the conformal case on the way. Detailed exposition of the background, including the necessary representation theory is available in [11].

In general, the definition of the parabolic subgroups is a little subtle. For us, the simplest approach is via their Lie algebras. The parabolic ones are those which contain a Borel subalgebra and the choices of parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ correspond to graded decompositions of the semisimple Lie algebras

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k.$$

This means that Lie brackets respect the grading, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is the decomposition into the reductive quotient $\mathfrak{g}_0$ and nilpotent subalgebra $\mathfrak{p}_+$. Moreover, there always is the unique *grading element* E in the center of $\mathfrak{g}_0$ with the property $[E, X] = jX$ for all $X \in \mathfrak{g}_j$.

The closed Lie subgroups $P \subset G$ are called parabolic if their algebras $\mathfrak{p} = \operatorname{Lie} P$ are parabolic.

If G is a complex semisimple Lie subgroup, then there is a nice geometric description: $P \subset G$ is parabolic if and only if G/P is a compact manifold (and then it is a compact Kähler projective variety), see e.g., [24, Section 1.2]. In the real setting, the so called generalized flag varieties G/P with parabolic P are always compact.

3.1 |1|-graded parabolic geometries

For the sake of simplicity, we shall restrict ourselves to the so called |1|-graded cases here, i.e., $k = 1$. Thus we shall deal with Lie groups with the algebras

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}_{\mathfrak{p}} \tag{26}$$

where $\mathfrak{p}$ refers to the parabolic subalgebra.

Now, we consider Cartan geometries modelled on $G \to G/P$, i.e. principal P-bundles $\mathcal{G} \to M$ with Cartan connections $\omega \in \Omega^1(\mathcal{G}, \mathfrak{p})$. Such a connection ω splits due to (26) as

$$\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1.$$

We shall further consider all reductions of the principal bundles $\mathcal{G}$ to the structure group $G_0 = P/\exp \mathfrak{g}_1$, i.e. we are interested in all equivariant mappings

$$\sigma \colon \mathcal{G}_0 = \mathcal{G}/\exp \mathfrak{g}_1 \to \mathcal{G}$$

with respect to the right principal actions. The diagram below summarizes our situation (notice we are also fixing the subgroup G_0 in the semidirect product $P = G_0 \ltimes \exp \mathfrak{g}_1$, following the splitting of the Lie algebra)

$$G_0 \subset P \rightharpoondown \mathcal{G} \underset{\sigma}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{G}_0 \longrightarrow M.$$

The Cartan connection ω allows us to identify the cotangent bundle T^*M with $\mathcal{G} \times_P \mathfrak{g}_1$, where the action of P_+ is trivial. Similarly $TM \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, where $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_{-1}$, again with trivial action of P_+. The duality is provided by the Killing form on $\mathfrak{g}$.

Recall that all sections ϕ of associated bundles $\mathcal{G} \times_P \mathbb{V}$ are identified with equivariant functions $f : \mathcal{G} \to \mathbb{V}$, i.e. $\phi(x) = [\![u(x), f(u(x))]\!]$ and so

$$f(u \cdot p) = p^{-1} \cdot f(u).$$

Once we restrict the structure group to the reductive part of P, the pullback of the Cartan connection along σ splits into the so called soldering form valued in $\mathfrak{g}_{-1}$, principal connection form valued in $\mathfrak{g}_0$, and the one-form valued one-form P (which we shall see is the general analog of the Schouten tensor P_{ab} from conformal geometry)

$$\sigma^*\omega = \theta \oplus \sigma^*\omega_0 \oplus \mathsf{P}\,.$$

We can also take the other way round – since P_+ is contractible (as the exponential image of a nilpotent algebra), we may start with $\mathcal{G} = \mathcal{G}_0 \times P_+$, fix one of the (reasonably normalized) pullbacks $\sigma^*\omega_0$ and use some suitable P to define the Cartan connection on the entire $\mathcal{G}$. We shall see later, the Schouten tensor (with the opposite sign, see the comment in the beginning of the second lecture) is the right choice to get the normalized Cartan connection in the case of conformal Riemannian geometries, taking one of the Levi-Civita connections for $\sigma^*\omega_0$. But we shall stay at the level of general Cartan connections now, so P is just the relevant pullback.

Two such reductions differ by a one form Υ, viewed as equivariant function $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$:

$$\hat{\sigma} = \sigma \cdot \exp \Upsilon.$$

3.2 Natural bundles and Weyl connections

For each representation $\mathbb{V}$ of P there is the functorial construction of the bundles $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ and the morphisms of the Cartan geometries act on them in the obvious way.

Actually we do not need the Cartan connection for this definition, but notice that the morphisms of the principal bundles respecting the Cartan connections are rather rigid in the following sense. If we fix their projection to the base manifolds, the freedom in covering them is described by the kernel K of the homogeneous models, i.e. the subgroup of P acting trivially on G/P, see [11, section 1.5.3]. This means two such morphisms may differ only by right principal action of elements from K. Usually K is trivial or discrete.

Consider now a representation $\mathbb{V}$ of P and its decomposition as a G_0-module. The action of the grading element $E \in \mathfrak{g}_0$ provides the splitting

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k,$$

where the action of $\mathfrak{g}_1$ moves elements from $\mathbb{V}_i$ to $\mathbb{V}_{i+1}$. Clearly, any section v of $\mathcal{V}$ decomposes into the components $v_i : \mathcal{G}_0 \to \mathbb{V}_i$ as soon as we fix our reduction.

Further, fixing our reduction σ we have the affine connection $\omega_\sigma = \sigma^*(\omega_{\leqslant 0})$ (which is a Cartan connection, i.e., an absolute parallelism, on the linear frame bundle obtained as the sum of the soldering form and connection form). As well known, the corresponding covariant derivative is obtained via the constant vector fields:

$$\nabla^\sigma_\xi v(u) = \omega_\sigma^{-1}(X) \cdot v(u)$$

where $X \in \mathfrak{g}_{-1}$ corresponds to the vector ξ in a frame $u \in \mathcal{G}_0$, i.e., we simply differentiate a function in the direction of a vector (the horizontal lift of ξ to $\mathcal{G}_0$). Notice that this connection ∇^σ always respects the decomposition of $\mathcal{V}$ given by the same reduction. We call all these connections the *Weyl connections* (and we obtain the genuine Weyl connections in the conformal case with the Schouten tensor $-\mathsf{P}$, i.e. all the torsion free connections preserving the conformal Riemannian structure).

Our next theorem says, how the splitting of $\mathbb{V}$, the covariant derivative, and also the one-form P change if we change the reduction σ.

In order to formulate the results, let us introduce some further conventions. Recall, tangent vectors $\xi \in T_xM$ can be identified with right-equivariant functions X on the frames over x valued in $\mathfrak{g}_{-1} = \mathfrak{g}/\mathfrak{p}$. This identification can be written down with the help of the Cartan connection, $X = \omega_{-1}(u)(\tilde{\xi})$ for any lift $\tilde{\xi}$ of ξ to $T_u\mathcal{G}$. By abuse of notation we shall write the same symbol ξ for the vector in TM and the corresponding element X in $\mathfrak{g}_{-1}$. Similarly we shall deal with the one forms Υ represented by elements in $\mathfrak{g}_1$, and also the endomorphisms of TM represented by elements in $\mathfrak{g}_0$.

For instance, $\operatorname{ad}\Upsilon(\xi)\cdot v$ means we take the Lie algebra valued functions Υ and ξ, take the Lie bracket of their values and act by the result on the value of the function v via the representation of $\mathfrak{g}_0$ in question. Of course, we may use only such operations which ensure the necessary equivariance (which is guaranteed when taking the adjoint action within the Lie algebra).

Theorem 3. *Consider $\hat{\sigma} = \sigma \cdot \exp \Upsilon$ and use the hat to indicate all the transformed quantities. For every section $v = v_0 \oplus \cdots \oplus v_k$ in the representation space of the representation $\lambda : \mathfrak{p} \to \mathfrak{gl}(\mathbb{V})$, and vector ξ in the tangent bundle,*

$$\hat{v}_\ell = (\lambda(\exp(-\Upsilon))(v))_\ell = \sum_{i+j=\ell} \frac{(-1)^i}{i!} \lambda(\Upsilon)^i(v_j). \tag{27}$$

If λ is a completely reducible P-representation, then

$$\hat{\nabla}_\xi v = \nabla_\xi v - \operatorname{ad}\Upsilon(\xi) \cdot v. \tag{28}$$

Finally, the one-form P transforms

$$\hat{\mathsf{P}} = \mathsf{P}(\xi) + \nabla_\xi \Upsilon + \frac{1}{2}(\operatorname{ad}\Upsilon)^2(\xi). \tag{29}$$

Proof. The formula (27) is just a direct consequence of our definitions and reflects the fact that by changing the reduction σ, the equivariant function $v : \mathcal{G} \to \mathbb{V}$

is restricted to another subset, shifted by the right action of $\exp(\Upsilon)$. Thus, the values have to get corrected by the action of $(\exp \Upsilon)^{-1} = \exp(-\Upsilon)$. The formula then follows by collecting the terms with the right homogeneities.

The transformation of the derivative is also not too difficult. Consider a section $v : \mathcal{G}_0 \to \mathbb{V}$ and recall $\nabla_\xi v$ is given with the help of any lift $\tilde{\xi}$ to $\mathcal{G}_0$:

$$\nabla_\xi v = \tilde{\xi} \cdot v(u) - \omega_0(T_u\sigma \cdot \tilde{\xi}) \cdot v(u). \tag{30}$$

Writing $r^p = r(\ ,p)$ and $r_u = r(u,\ )$ for the right action,

$$T_u\hat{\sigma} \cdot \tilde{\xi} = T_{\sigma(u)} r^{\exp \Upsilon(u)} \cdot T_u\sigma \cdot \tilde{\xi} + T_{\exp \Upsilon(u)} r_{\sigma(u)} \cdot T_u \exp \Upsilon \cdot \tilde{\xi}. \tag{31}$$

The second term in (31) is vertical in $\mathcal{G} \to \mathcal{G}_0$ and thus

$$\omega_0(T_u\hat{\sigma} \cdot \tilde{\xi}) = \omega_0(T_{\sigma(u)} r^{\exp \Upsilon(u)} \cdot T_u\sigma \cdot \tilde{\xi}).$$

By equivariancy of the Cartan connection ω, this equals to the $\mathfrak{g}_0$ component of $\mathrm{Ad}((\exp \Upsilon(u))^{-1})(\omega(T_u\sigma \cdot \tilde{\xi}))$. Now, notice $\omega_{-1}(T_u\sigma \cdot \tilde{\xi})$ is exactly the coordinate function representing the vector ξ. Thus, the only $\mathfrak{g}_0$ component of the latter expression is $\mathrm{ad}(-\Upsilon(u))(\xi)$ and this has to act on v in our transformation formula.

The transformation of the P tensor is also deduced from (31), but it is more technical and we refer to the detailed proof in [11, section 5.1.8].

Similar formulae are available for general parabolic geometries and their Weyl connections. Just the non-trivial gradings of TM and T^*M make them much more complicated. The complete exposition can be read from [11, sections 5.1.5 through 5.1.9].

Notice also that we are allowing all reductions σ. But some of them are nicer than others – we may reduce the structure group to the semisimple part G_0^{ss} of G_0. These further reductions correspond to sections of the line bundle $\mathcal{L} = \mathcal{G}_0/G_0^{ss}$, which can be viewed as the associated bundle $\mathcal{G}_0 \times_{G_0} \exp\{wE\}$ carrying the natural structure of a principal bundle with structure group $\mathbb{R}_+$. This is the line bundle of scales and its sections correspond to Weyl connections inducing flat connections on $\mathcal{L}$. In the conformal case, these are just the choices of metrics in the conformal class. The induced connection on $\mathcal{L}$ has got the antisymmetric part of P as its curvature and thus, we can recognize such more special reductions by the fact that for these the Rho-tensor is symmetric.

3.3 Higher order derivatives

Notice, in Theorem 3 we provided the formula for the change of the Weyl connections for completely reducible P-modules only. This is because the formulae get very nasty for modules with nontrivial $\mathfrak{g}_1$ actions. But even dealing with tensorial bundles, iterating the derivatives always leads to such modules.

In order to avoid at least part of these hassles, we should seek for better linear connections related to our reductions σ and the fixed Cartan connection ω. An obvious choice seems to be the following one. Fixing a reduction σ consider the principal connections $\mathcal{G}$ with the connection form $\gamma^\sigma \in \Omega^1(\mathcal{G}, \mathfrak{p})$,

$$\gamma^\sigma(\sigma(u)\cdot g)(\xi) = \omega_{\mathfrak{p}}(\sigma(u))(Tr^{g^{-1}}\cdot \xi)$$

for all $u \in \mathcal{G}_0$, $\xi \in T_{\sigma(u)\cdot g}$, $g \in P_+$. In words, we restrict the $\mathfrak{p}$-component of ω to the image of σ and extend it the unique way to a principal connection form.

Clearly, this connection form defines the associated linear connections on all natural bundles, we call them the *Rho-corrected Weyl connections* ∇^{P}. They were perhaps first introduced in [21] and exploited properly in [6]. In the case of conformal Riemannian structures, these concepts are closely related to the so called Wünsch's conformal calculus, cf. [23].

Theorem 4. *Consider natural bundle $\mathcal{V} = \mathcal{G}\times_P \mathbb{V}$ and a reduction with the Weyl connection ∇ and its Rho-corrected derivative ∇^{P}. Then*

$$\nabla^{\mathsf{P}}_\xi v = \nabla_\xi v + \mathsf{P}(\xi)\cdot v \tag{32}$$

$$\hat\nabla^{\mathsf{P}}_\xi v = \nabla^{\mathsf{P}}_\xi v + \sum_{i\geqslant 1}\frac{(-1)^i}{i!}(\operatorname{ad}\Upsilon)^i(\xi)\cdot v. \tag{33}$$

Proof. In order to see the difference between ∇ and ∇^{P}, we can inspect the expression (30) with the choice of the horizontal vector field $\tilde\xi$ lifting ξ. Thus $\nabla_\xi v = \tilde\xi\cdot v$. On the other hand, choosing the lift $T\sigma\cdot\tilde\xi$ on $\mathcal{G}$, we obtain $\omega_0(T\sigma\cdot\tilde\xi) = 0$ and $\omega_1(T\sigma\cdot\tilde\xi)$ represents $\mathsf{P}(\xi)$. Equivariancy of v then implies our formula (32) along the entire image of σ.

Let us now consider the horizontal lift $\tilde\xi$ of ξ on $\mathcal{G}$ with respect to γ^σ. Then $\nabla^{\mathsf{P}}_\xi v$ is represented by $\tilde\xi\cdot v$, while $\tilde\xi\cdot v + \gamma^{\hat\sigma}(\tilde\xi)\cdot v$ represents $\hat\nabla_\xi v$. By the very definition, $\omega(\sigma(u))(\tilde\xi) \in \mathfrak{g}_{-1}$. Thus,

$$\gamma^{\hat\sigma}(\sigma(u))(\tilde\xi) = \omega_{\mathfrak{p}}(Tr^{\exp\Upsilon(u)}\cdot\tilde\xi(\sigma(u))),$$

which is just the component of $\operatorname{Ad}((\exp\Upsilon(u))^{-1})(\omega(\sigma(u))(\tilde\xi))$. Now, notice that $\omega(\sigma(u))(\tilde\xi)$ represents ξ by values in $\mathfrak{g}_{-1}$ and the requested formula follows.

We should notice that the Weyl connections and the Rho corrected ones coincide on bundles coming from representations with trivial action of P_+. Of course, the transformation formulae coincide in this case, too.

3.4 A few examples

We shall go through a few homogeneous models and comment on the general 'curved' situations. In all cases the actual geometric structures are given by the reductions of the linear frame bundles and the construction of the right Cartan geometry is a separate issue. We shall come back to this in the fifth lecture and work with the general choices of the Cartan connections ω here.

Conformal Riemannian geometry. The relevant Cartan geometry can be modelled by the choice $G = \mathrm{O}(n+1,1,\mathbb{R})$ (there is some freedom in the choice of the group with the given graded Lie algebra $\mathfrak{g}$) and the parabolic subgroup P as we saw in detail in the first lecture.

It is a simple exercise now to recover the formulae from Lemma 1 by computing the brackets in the Lie algebra. In our conventions using the coordinate functions instead of fields, we can rewrite them as (notice α is valued in $\mathfrak{g}_1$, while η, ξ have got values in $\mathfrak{g}_{-1}$, and s sits in $\mathbb{R}[w]$)

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta - [\Upsilon, \xi] \cdot \eta \tag{34}$$

$$\hat{\nabla}_\xi \alpha = \nabla_\xi \alpha - [\Upsilon, \xi] \cdot \alpha \tag{35}$$

$$\hat{\nabla}_\xi s = \nabla_\xi s - (-\Upsilon(\xi)) w s \tag{36}$$

where we have picked up just the central component of the bracket in the last line, viewed as the multiple of the grading element.

Another, but still much more tedious exercise would be to check the conformal invariance of the tractor connection on $\mathcal{T}$. We shall develop much better tools for that in the next lecture.

We shall also enjoy much better tools to discuss second or higher order operators. For example, considering second order operators on densities $s \in \mathcal{E}[w]$, we may iterate the Rho-corrected derivative to obtain

$$\mathbf{g}^{ab} \nabla^{\mathsf{P}}_a \nabla^{\mathsf{P}}_b s = \nabla^a \nabla_a s - w \mathbf{g}^{ab} \mathsf{P}_{ab} s$$

and check that this gets an invariant operator for $w = 1 - \frac{n}{2}$, which is the famous conformally invariant Laplacian, the so called Yamabe operator

$$Y : \mathcal{E}[1 - \frac{n}{2}] \to \mathcal{E}[-1 - \frac{n}{2}].$$

Projective geometry. The choice of the homogeneous model is obtained from the algebra of trace free real matrices $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ with the grading

$$\left(\begin{array}{c|c} \mathfrak{z} & \mathbb{R}^{n*} \\ \hline \mathbb{R}^n & \mathfrak{gl}(n, \mathbb{R}) \end{array} \right) \begin{array}{l} 1 \\ n \end{array}$$

Here $\mathfrak{z} = \mathbb{R}$ is the center, the grading element E corresponds to $\frac{n}{n+1}$ and $-\frac{1}{n+1} \operatorname{id}_{\mathbb{R}^n}$ on the diagonal. We may take $G = \mathrm{SL}(n+1, \mathbb{R})$ and P the subgroup of block upper triangular matrices. The homogeneous model is then the real projective space $G/P = \mathbb{RP}^n$. On the homogeneous model, the Weyl connections transform as

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta$$

and so they clearly share the geodesics.

For general projective structures on manifolds M, the space of Weyl connections has to be chosen as a class of all affine connections sharing geodesics with a given one and they transform then the same way. We shall see, that projective geometries are rare exceptions of parabolic geometries not given by a first order structure on the manifold.

The analog of the Thomas' tractors is the natural bundle corresponding to the standard representation of $\mathrm{SL}(n+1,\mathbb{R})$ on $\mathbb{T}=\mathbb{R}^{n+1}$. The injecting part of $\mathcal{T}$ is the line bundle $\mathcal{T}^1$ with the action of the grading element by $\frac{n}{n+1}$. The usual convention says this is the line bundle $\mathcal{E}[-1]$. Then the projecting component is the weighted tangent bundle $TM[-1]$.

Almost Grassmannian geometry. This is essentially a continuation of the previous example. We take $G=\mathrm{SL}(p,q)$ and the splitting of the matrices into blocks of sizes p and q, say $2\leqslant p\leqslant q$. Unlike the projective case, here the geometry is determined by reducing the structure group of the tangent bundle to $\mathbb{R}\times\mathrm{SL}(p,\mathbb{R})\times\mathrm{SL}(q,\mathbb{R})$. This corresponds to identifying the tangent bundle with the tensor product of the auxiliary bundles $\mathcal{V}^*$ and $\mathcal{W}$ of dimensions p and q, together with the identification of their top degree forms $\Lambda^p\mathcal{V}\simeq\Lambda^q\mathcal{W}^*$.

Thus, we may use the abstract indices and write $\mathcal{V}=\mathcal{E}^A$, $\mathcal{W}=\mathcal{E}^{A'}$. Then the tangent bundle is $\mathcal{E}_A^{A'}$ and the formula for the brackets in the Lie algebra says $[[\Upsilon,\xi],\eta]_A^{A'}=-\xi_B^{A'}\Upsilon_{B'}^B\eta_A^{B'}-\xi_A^{B'}\Upsilon_{B'}^B\eta_B^{A'}$. The Weyl connections are tensor products of connections on $\mathcal{V}^*$ and $\mathcal{W}$ (but not all of them). The right formula for the change of the Weyl connections is

$$\hat{\nabla}_{A'}^A\eta_B^{B'}=\nabla_{A'}^A\eta_B^{B'}+\delta_{A'}^{B'}\Upsilon_{C'}^A\eta_B^{C'}+\delta_B^A\Upsilon_{A'}^C\eta_C^{B'}.$$

The analog to the Thomas' tractors comes from the standard representation of G on $\mathbb{T}=\mathbb{R}^{p+q}=\mathcal{V}\oplus\mathcal{W}$. Thus, fixing a Weyl connection, we get the tractors as couples $(v^A,w^{A'})$ with the transformation rules

$$\hat{v}^A=v^A-\Upsilon_{B'}^A w^{B'},\quad \hat{w}^{A'}=w^{A'}.$$

Notice the special case $p=q=2$ which provides (the split real form of) the Penrose's spinor presentation of tangent bundle and the two-component four-dimensional twistors $\mathcal{T}$. Indeed, $\mathfrak{so}(6,\mathbb{C})=\mathfrak{sl}(4,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$ splits into sum of two $\mathfrak{sl}(2,\mathbb{C})$ components. Thus, up to the choice of the right real form, the almost Grassmannian geometries with $p=q=2$ correspond to the four-dimensional conformal Riemannian geometries.

The twistor parallel transport (connection) is then given by the formula

$$(\nabla^{\mathcal{T}})_{A'}^A\begin{pmatrix}v^B\\ w^{B'}\end{pmatrix}=\begin{pmatrix}\nabla_{A'}^A v^B+\mathsf{P}_{A'C'}^{AB}w^{C'}\\ \nabla_{A'}^A w^{B'}+\delta_{A'}^{B'}v^A\end{pmatrix}$$

and we shall see that this is the right formula for the standard tractor connection for the almost Grassmannian geometries in all dimensions.

The reader can find many further explicit examples in the last two chapters of [11], including those with nontrivial gradings on TM.

4 Elements of tractor calculus

In order to show how simple and general the basic functorial constructions and objects are, we shall focus for a while on general Cartan geometries with Klein models $G\to G/H$ without any further assumptions. But we shall come back to the parabolic and, in particular, conformal geometries in the end of this lecture.

4.1 Natural bundles and tractors

Let us come back to the functorial constructions on homogeneous spaces $G \to G/H$ mentioned in the first lecture. As always, $\mathfrak{h} \subset \mathfrak{g}$ are the Lie algebras of H and G.

For any Klein geometry G/H, there is the category of the homogeneous vector bundles, where the objects are the associated bundles $\mathcal{V} = G \times_H \mathbb{V}$. All morphisms on G/H are the actions of elements of G and these are mapped to the obvious actions on $\mathcal{V}$. Further morphisms in this category are the linear mappings intertwining the actions of the elements of G.

Clearly, there is the functor from the category of H-modules mapping the modules $\mathbb{V}$ to the associated bundles $\mathcal{V} = G \times_H \mathbb{V}$, while any module homomorphism $\phi : \mathbb{V} \to \mathbb{W}$ provides the morphisms $[\![u, v]\!] \mapsto [\![u, \phi(v)]\!]$ between these bundles.

The latter functorial construction extends obviously to the entire category $\mathcal{C}_{G/H}$ of all Cartan geometries modelled on G/H. The morphisms have to respect the Cartan connections ω on the principal fiber bundles.

In this setting, a natural bundle is a functor $\mathcal{V}\colon \mathcal{C}_{G/H} \to \mathcal{VB}$ valued in the category of vector bundles. The functor sends every Cartan geometry $(\mathcal{G} \to M, \omega)$ to the vector bundle $\mathcal{V}M \to M$ over the same base (so it is a special case of the so called gauge-natural bundles, see [17]). Moreover, $\mathcal{V}$ has the property that whenever there is a morphism between objects of $\mathcal{C}_{G/H}$, $\Phi\colon (\mathcal{G} \to M, \omega) \to (\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ covering $f\colon M \to \tilde{M}$, then there is the corresponding vector bundle morphism $\mathcal{V}\Phi\colon \mathcal{V}M \to \mathcal{V}\tilde{M}$ covering f. This is just an explicit description of the functoriality property with respect to the category of Cartan geometries. The main point is that each representation of H produces such a functor for all general Cartan geometries of the given type G/H.

At the same time, the Maurer-Cartan equation $\mathrm{d}\,\omega + \frac{1}{2}[\omega, \omega] = 0$, valid on the homogeneous model, is no more true in general and we obtain the definition of the *curvature* κ of the Cartan geometries $(\mathcal{G} \to M, \omega)$ instead:

$$\kappa = \mathrm{d}\,\omega + \frac{1}{2}[\omega, \omega]. \tag{37}$$

The fundamental Theorem 1 immediately reveals that a general Cartan geometry is locally isomorphic to its homogeneous model, if and only if its curvature vanishes identically.

We should also notice that there is the projective component of the curvature in $\mathfrak{g}/\mathfrak{h}$ which we call the *torsion*. Thus, the Cartan geometry is *torsion-free* if the values of its curvature κ are in $\mathfrak{h}$. We shall see later that the normalizations of Cartan geometries consist in prescribing more complicated curvature restrictions, which always depend on the algebraic features of the Klein models.

As already mentioned, we are interested in specific functors on Cartan geometries $(\mathcal{G}, \omega)$ of the form $\mathcal{G} \times_H -$, referring to the associated bundle construction given for each fixed representation of H. See [11, section 1.5.5] for a detailed discussion on the topic of natural bundles on Cartan geometries. Specializing to representations of H which come as restrictions of representations of the whole group G leads to the following definition of *tractor bundles* below.

Recall the sections v of natural bundles $\mathcal{V}$ are identified with equivariant functions $v : \mathcal{G} \to \mathbb{V}$, i.e. $v(u \cdot g) = g^{-1} \cdot v(u)$. In particular, consider $\mathbb{V} = \mathfrak{g}/\mathfrak{h} = \mathbb{R}^n$ with the truncated adjoint action of H (i.e. the induced action on the quotient). The Cartan connection ω allows us to identify every tangent vector $\xi \in T_xM$ with the equivariant function $v : \mathcal{G} \to \mathbb{V}$, $u \mapsto \omega(\tilde{\xi}(u))$ for an arbitrary lift $\tilde{\xi}$ of ξ. This is the identification of the tangent bundle $TM \simeq \mathcal{V}M$. (And it completely justifies our earlier quite sloppy usage of elements in $\mathfrak{g}_{-1}$ instead of tangent vectors etc.)

So in this way, the Cartan connection provides soldering of the tangent bundle, i.e. each element $u \in \mathcal{G}$ in the fiber over $x \in M$ can be viewed as a frame of T_xM. In general, different elements u may represent the same frame, depending on whether the truncated adjoint action of H on $\mathfrak{g}/\mathfrak{h}$ has got a non-trivial kernel.

Definition 2. *The tractor bundles are natural vector bundles associated to the Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $G \to G/H$, via restrictions of a representations of G to the subgroup H.*

The unique principal connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}) \to \mathfrak{g}$ on the extended principal G-bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_H G$ extending the Cartan connection ω on $\mathcal{G}$ induces the so called tractor connections $\nabla^{\mathcal{V}}$ *on all tractor bundles $\mathcal{V}M$.*

Notice that $\tilde{\mathcal{G}}$ is indeed a G-principal fiber bundle with the action of G defined by the right multiplication on the standard fiber G. Moreover, $u \mapsto [\![u, e]\!]$ provides the canonical inclusion of the principal fiber bundles $\mathcal{G} \subset \tilde{\mathcal{G}}$. The requested invariance of $\tilde{\omega}$, together with the reproduction of the fundamental vector fields, define the values of $\tilde{\omega}$ completely from its restriction $\tilde{\omega} = \omega$ on $T\mathcal{G}$.

In fact, we can equivalently define the tractor connections on the tractor bundles directly (by specifying their special properties), instead of referring to the Cartan connections on $\mathcal{G}$. This was also the approach by Thomas in [22]. The equivalence of such approaches for |1|-graded parabolic geometries was noticed and exploited in [14]. In full generality, the construction, normalization and properties of tractor connections were derived in [8] (see also [11, Sections 1.5 and 3.1.22]).

4.2 Adjoint tractors

A prominent example of tractor bundles arises when considering the Ad representation of the Lie group G on its Lie algebra $\mathfrak{g}$ and restricting it to H. Applying the corresponding associated bundle construction $\mathcal{G} \times_H -$ on the following short exact sequence of Lie algebras (with the obvious Ad actions)

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

we obtain

$$0 \to \mathcal{G} \times_H \mathfrak{h} \to \mathcal{AM} \xrightarrow{\pi} TM \to 0 , \tag{38}$$

where we have identified $TM \cong \mathcal{G} \times_H \mathfrak{g}/\mathfrak{h}$. The middle term $\mathcal{AM} := \mathcal{G} \times_H \mathfrak{g}$ is called the *adjoint tractor bundle.*

Let us come back to the curvature (37) of the Cartan geometry now. Clearly we may evaluate κ on the so called *constant vector fields* $\omega^{-1}(X)$ for all $X \in \mathfrak{g}$. Consider $X \in \mathfrak{h}$ and any $Y \in \mathfrak{g}$. Then $\omega^{-1}(X)$ is the fundamental vector field ζ_X and $\mathrm{d}\,\omega(\zeta_x, -) = i_{\zeta_X}\,\mathrm{d}\,\omega = \mathcal{L}_{\zeta_X}\omega = -\operatorname{ad}(X)\circ\omega$, by the equivariancy of ω. Thus,

$$\kappa(\omega^{-1}(X), \omega^{-1}(Y)) = \kappa(\omega^{-1}(X), \omega^{-1}(Y)) = -\operatorname{ad}(X)(Y) + [X, Y] = 0.$$

We have concluded that, actually, the curvature is a horizontal 2-form which can be represented by the equivariant *curvature function*

$$\begin{aligned} \kappa : \mathcal{G} &\to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}, \\ \kappa(X, Y)(u) &= -\omega([\omega^{-1}(X), \omega^{-1}(Y)](u)) + [X, Y]. \end{aligned} \tag{39}$$

In particular, we understand that the curvature descends to a genuine 2-form on the base manifold M valued in the adjoint tractors, i.e. $\kappa \in \Omega^2(M, \mathcal{A}M)$.

There is much more to say about the adjoint tractors, we shall summarize several observations in the following two theorems (both were derived in [8], see also [11]).

Theorem 5. *1. There is the (algebraic) Lie bracket* $\{\ ,\ \}\colon \mathcal{AM} \times \mathcal{AM} \to \mathcal{AM}$ *inherited from the Lie bracket on* $\mathfrak{g}$.

2. *The adjoint tractors are in bijective correspondence with the right-equivariant vector fields in* $\mathcal{X}(\mathcal{G})^H$, *and the Lie bracket of vector fields on* $\mathcal{G}$ *equips* $\mathcal{A}M$ *with the differential Lie bracket* $[\ ,\ ]$, *which is compatible with the Lie bracket on the tangent bundle* TM, *i.e.* $\pi[\zeta, \eta] = [\pi\zeta, \pi\eta]$.[3]
3. *If* $\mathcal{V}$ *is a tractor bundle, then there is the natural map* $\bullet\colon \mathcal{AM} \times \mathcal{VM} \to \mathcal{VM}$, *corresponding to the action of* $\mathfrak{g}$ *given by the* G*-representation* $\mathbb{V}$. *Moreover,* $\{s_1, s_2\} \bullet t = s_1 \bullet s_2 \bullet t - s_2 \bullet s_1 \bullet t$.
4. *The bracket* $\{\ ,\ \}$ *and the actions* $\bullet$ *are parallel with respect to the tractor connections* $\nabla^{\mathcal{A}}$, $\nabla^{\mathcal{V}}$, *i.e. for* $s \in \mathcal{AM}$ *and* $v \in \mathcal{VM}$ *we know*

$$\begin{aligned} \nabla^{\mathcal{A}}_\xi \{s_1, s_2\} &= \{\nabla^{\mathcal{A}}_\xi s_1, s_2\} + \{s_1, \nabla^{\mathcal{A}}_\xi s_2\}, \\ \nabla^{\mathcal{V}}_\xi (s \bullet v) &= (\nabla^{\mathcal{A}}_\xi s) \bullet v + s \bullet (\nabla^{\mathcal{V}}_\xi v). \end{aligned}$$

5. *For every tractor bundle* $\mathcal{V}$, *the value of the curvature* $R^{\mathcal{V}}$ *of the tractor connection* $\nabla^{\mathcal{V}}$ *is (for all vector fields* ξ, η *on* M *and sections* v *of* $\mathcal{V}M$*)*

$$R^{\mathcal{V}}(\xi, \eta)(v) = \kappa(\xi, \eta) \bullet v,$$

where $\kappa \in \Omega^2(M, \mathcal{A}M)$ *is the curvature of the Cartan connection.*

[3] Recall that $\pi\colon \mathcal{AM} \to TM$ is the projection from sequence (38).

Proof. The first claim is obvious just by definition. The Lie bracket on the Lie algebra is Ad-equivariant.

The adjoint tractors are smooth equivariant functions $\mathcal{G} \to \mathfrak{g}$. At the same time ω makes $T\mathcal{G}$ trivial. Now all $\xi \in \mathcal{G}$ correspond to $\omega \circ \xi : \mathcal{G} \to \mathfrak{g}$ and the right invariant fields ξ correspond just to the adjoint tractors. Since the Lie brackets of related fields is again related (here with respect to the principal actions of the elements in H), the Lie bracket restricts to $\mathcal{X}(G)^H$. Moreover, the right-invariant fields are projectable onto vector fields on M, and the same argument applies to brackets of the projections.

The third claim also follows directly from the definitions. Indeed, writing λ for the representation $\lambda : H \to \mathrm{GL}(\mathbb{V})$, and λ' for its differential at the unit, we recall $\exp(t\,\mathrm{Ad}(g)(X)) = g\exp(tX)g^{-1}$ and thus, differentiating we arrive at

$$\lambda'(\mathrm{Ad}(g)(X))(\lambda(g)(v)) = \lambda(g)(\lambda'(X)(v)).$$

Consequently, the bilinear map $\mathfrak{g} \times \mathbb{V} \to \mathbb{V}$ defined by λ' is G equivariant, it induces the map $\bullet : \mathcal{A}M \times \mathcal{V}M \to \mathcal{V}M$ and the bracket formula is just the defining property of a Lie algebra representation, in this picture.

The next claim is a straightforward consequence of the fact that both $\{\ ,\ \}$ and $\bullet$ are operations induced by G-equivariant maps. Thus we may view them as living on the associated bundles to the extended G-principal fiber bundle $\tilde{\mathcal{G}}$. The formulae are just simple properties of the induced linear connections associated to a principal connection.

The same argument holds true for the last claim as well.

Notice also the definition of the operation $\bullet$ extends to all natural bundles $\mathcal{V}$, if we restrict the tractors only to the natural subbundle $\ker\pi \subset \mathcal{A}M$ of all vertical right invariant vector fields on $\mathcal{G}$, including the bracket compatibility property.

4.3 Fundamental derivative

Consider the natural bundle $\mathcal{V} := \mathcal{G} \times_H \mathbb{V}$ associated to an H-representation λ on $\mathbb{V}$. Then, viewing the adjoint tractors as right invariant vector fields on $\mathcal{G}$, we can define the differential operator $\mathrm{D}\colon \mathcal{A}\mathcal{M} \times \mathcal{V}M \to \mathcal{V}M$ by the formula

$$\mathrm{D}_s\, v = s \cdot v,$$

where $s \in \mathcal{A}\mathcal{M}$ is any tractor in $\mathcal{X}(\mathcal{G})^H$ differentiating the function $v : \mathcal{G} \to \mathbb{V}$. A simple check,

$$s(u \cdot h) \cdot v = (Tr^h \cdot s(u)) \cdot v = s(u) \cdot (v \circ r^h) = s(u) \cdot (\lambda_{h^{-1}} \cdot v) = \lambda_{h^{-1}}(s(u) \cdot v),$$

reveals that the result is again a smooth $\mathbb{V}$-valued H-equivariant mapping on $\mathcal{G}$. We call this operator D the *fundamental derivative.*

Notice that extending the tangent bundle to the adjoint tractors, we always have a canonical way of 'differentiating' on all natural bundles for all Cartan geometries. As we may expect, there will be a lot of redundancy in such differentiation, since the vertical tractors in the kernel of the projection $\mathcal{A}M \to TM$ must act in an algebraic way due to the equivariance of the functions v.

Let us summarize some simple but very useful consequences of our definitions:

Theorem 6. *1. The fundamental derivative on the smooth functions (i.e. we consider the trivial representation $\mathbb{V} = \mathbb{R}$) is just the derivative in the direction of the projection:*

$$D_s f = \pi(s) \cdot f\,.$$

2. If the adjoint tractor s is vertical, i.e. $\pi(s) = 0$, then for every section v of a natural bundle $\mathcal{V}M$,

$$\mathrm{D}_s\, v = -s \bullet v.$$

3. The fundamental derivative D is compatible with all natural operations on natural bundles (i.e. those coming from H-invariant maps between the corresponding representation spaces). For example, having sections v, v^*, and w of natural bundles $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{W}$, and a function f

$$\begin{aligned} \mathrm{D}_s(fv) &= (\pi(s)\cdot f)v + f\,\mathrm{D}_s\, v \\ \mathrm{D}_s(v\otimes w) &= \mathrm{D}_s\, v\otimes w + v\otimes \mathrm{D}_s\, w \\ \pi(s)\cdot v^*(v) &= (\mathrm{D}_s\, v^*)(v) + v^*(\mathrm{D}_s\, v) \end{aligned}$$

4. If $\mathbb{V}$ is a G-representation, i.e. $\mathcal{V}$ is a tractor bundle, then

$$\nabla^{\mathcal{V}}_{\pi(s)} v = \mathrm{D}_s\, v + s \bullet v.$$

Proof. The equivariant functions $\mathcal{G} \to \mathbb{R}$ are just the compositions of functions f on the base manifold M with the projection $p : \mathcal{G} \to M$. Thus the first property is obvious, $s \cdot (f \circ p) = (Tp \cdot s) \cdot f = \pi(s) \cdot f$.

If s is vertical, then $s(u) = \zeta_Z(u)$, where ζ_Z is a fundamental vector field given by $Z \in \mathfrak{h}$. Thus,

$$s(u) \cdot v = \frac{d}{dt}_{|0} r^{\exp tZ}(u) \cdot v = -\lambda'(Z)(v(u)) = (-s \bullet v)(u).$$

The third property is again obvious – as long as the natural operations come from (multi)linear H-invariant maps, these will be compatible with the differentiations of functions valued in those spaces, in the directions of the right-invariant vector fields.

In order to see the last formula, consider a vector $\xi \in T_u\mathcal{G} \subset T_u\tilde{\mathcal{G}}$, covering a vector $\tau \in T_xM$. Then the horizontal lift of τ at the frame $u \in \mathcal{G} \subset \tilde{\mathcal{G}}$ is $\xi - \zeta_{\tilde\omega(\xi)} = \xi - \zeta_{\omega(\xi)}$. But the tractor connection is defined as the derivative of the equivariant function v in any frame of $\tilde{\mathcal{G}}$ in the direction of the horizontal lift and we obtain exactly the requested formula interpreting ξ as the value of the right-invariant vector field s (i.e. the adjoint tractor viewed as the equivariant function at u is expressed just via $\omega(\xi)$).

If we leave the slot for the adjoint tractor in the fundamental derivative free, we obtain the operator $\mathrm{D} : \mathcal{V}M \to \mathcal{A}^*M \otimes \mathcal{V}M$, and this can be obviously iterated,

$$\mathrm{D}^k : \mathcal{V}M \to \otimes^k \mathcal{A}^*M \otimes \mathcal{V}M$$

Of course, there is a lot of redundancy in these higher order operators compared to standard jet spaces of the sections. In the case of the first order, we can identify the first jet prolongations $J^1\mathcal{V}$ of natural bundles as the natural bundles associated to the representations $J^1\mathbb{V}$ which are much smaller H-submodules in the modules $\mathbb{V} \oplus \mathrm{Hom}(\mathfrak{g}, \mathbb{V})$ corresponding to the values of the fundamental derivative. This is a useful observation because it implies that all invariant first order differential operators on the homogeneous models extend naturally to the entire category of the Cartan geometries with this model.

Before returning to the parabolic special cases, let us remark two more facts. The proofs are using similar arguments as above and the reader can find them in [11, sections 1.5.8, 1.5.9].

Expanding the formula for the exterior differential in the defining equation of the curvature κ, we can express the differential bracket on $\mathcal{A}M$:

$$\begin{aligned} [s_1, s_2] &= \mathrm{D}_{s_1} s_2 - \mathrm{D}_{s_2} s_1 - \kappa(\pi(s_1), \pi(s_2)) + \{s_1, s_2\} \\ &= \nabla^{\mathcal{A}}_{\pi(s_1)} s_2 - \nabla^{\mathcal{A}}_{\pi(s_2)} s_1 - \kappa(\pi(s_1), \pi(s_2)) - \{s_1, s_2\}. \end{aligned} \tag{40}$$

There is the generalization of the well known Bianchi identities for curvature in the general Cartan geometry setting:

$$\sum_{\text{cyclic}} \big(\nabla^{\mathcal{A}}_{\xi_1}(\kappa(\xi_2, \xi_3)) - \kappa([\xi_1, \xi_2], \xi_3) \big) = 0 \tag{41}$$

for all vector fields ξ_1, ξ_2, ξ_3, or its equivalent form for triples of adjoint tractors:

$$\sum_{\text{cyclic}} \big(\{s_1, \kappa(s_2, s_3)\} - \kappa(\{s_1, s_2\}, s_3) + \kappa(\kappa(s_1, s_2), s_3) + (\mathrm{D}_{s_1} \kappa)(s_2, s_3) \big) = 0. \tag{42}$$

Similarly, the Ricci identity has got the general form for every section v of a natural bundle $\mathcal{V}$:

$$(\mathrm{D}^2 v)(s_1, s_2) - (\mathrm{D}^2 v)(s_2, s_1) = -\mathrm{D}_{\kappa(s_1, s_2)} v + \mathrm{D}_{\{s_1, s_2\}} v. \tag{43}$$

Notice, how easy we can read the classical identities for the affine connections from the latter two. Since the Cartan geometry is modelled on $\mathbb{R}^n = \mathrm{Aff}(n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{R})$ and the Lie algebra decomposes into direct sum of $\mathfrak{gl}(n, \mathbb{R})$-modules $\mathfrak{g}_{-1} = \mathbb{R}^n$ and $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$, all the formulae decompose by homogeneities, $\mathcal{A}M = TM \oplus P^1M$ (here P^1M is the linear frame bundle of TM), the bracket $\{\ ,\ \}$ becomes trivial on TM, while the mixed bracket is just the evaluation. Thus, the Bianchi identity can be evaluated on tangent vectors and it decomposes into the two classical Bianchi identities for the torsion free connections, while it gets the more complex quadratic form in general. Similarly for Ricci, evaluated on s_1 and s_2 in TM. If the torsion is zero, κ has got only vertical values and thus the first term on the right hand side is the algebraic action of the curvature (with plus sign), while the other one vanishes.

4.4 Back to parabolic geometries

Recall the parabolic cases always come with the splitting

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p},$$

where $\mathfrak{g}_-$ is a subalgebra (but only a $\mathfrak{g}_0$ submodule). As before, we shall restrict ourselves to the $|1|$-graded case, although the below formulae easily extend to the general case.

Consider the category of parabolic geometries with the model G/P and a P-representation $\mathbb{V}$ which decomposes with respect to the action of the grading element $E \in \mathfrak{g}_0$ into $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k$. The adjoint tractor bundle has got the composition series

$$\mathcal{A}M = TM \oplus \operatorname{End} TM \oplus T^*M$$

where the middle term is a subbundle in $T^*M \otimes TM$ corresponding to the group $G_0 = P/P_+$. Again, T^*M is the injecting part while TM is the projecting part, and the algebraic bracket $\{\ ,\ \}$ maps $T^*M \times TM \to \operatorname{End} TM$.

Once we fix a Weyl connection ∇, the Rho-tensor becomes a one-form valued in $T^*M \subset \mathcal{A}M$, we get the Rho-corrected derivative ∇^{P}, all P-modules get split into G_0-irreducible components which can be grouped according to the actions of the grading element in $\mathfrak{g}_0$ etc.

Theorem 7. *The fundamental derivative* D *on* $\mathcal{V}$ *is given in terms of any choice of Weyl connection by*

$$(\mathrm{D}_s\, v)_i = (\nabla^{\mathsf{P}}_{\pi(s)} v)_i - s_0 \bullet v_i - s_1 \bullet v_{i-1} = \nabla_{\pi(s)} v_i - s_0 \bullet v_i + (\mathsf{P}(\pi(s)) - s_1) \bullet v_{i-1}$$

where $s = (\pi(s), s_0, s_1)$ *and we indicate the splitting* $\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_k$ *with respect to the action of the grading element by the extra lower indices.*

If $\mathcal{V}$ *is a tractor bundle, then the tractor connection is given by*

$$(\nabla^{\mathcal{V}}_\xi v)_i = (\nabla^{\mathsf{P}}_\xi v)_i + \xi \bullet v_{i+1} = \nabla_\xi v_i + \mathsf{P}(\xi) \bullet v_{i-1} + \xi \bullet v_{i+1}.$$

Proof. Both formulae are direct consequences of the general formulae and the definitions. The reader may also consult [11, section 5.1.10].

4.5 Towards effective calculus for conformal geometry

Now, with the general concepts and formulae at hand, it is obvious that the Thomas' tractors come equipped with the nice tractor connection on all conformal Riemannian manifolds in the sense of Cartan geometries and the connection will be always given by the formulae in Theorem 2, which are manifestly invariant. Moreover, we know that the curvature of the Thomas' tractor connection on the sphere (with the Maurer-Cartan form ω) is zero.

But we still cannot be happy enough, for at least two reasons. First, we want to define the geometries by a structure on the tangent bundle and we shall come to that question in the next lecture. Second, we need some more effective manifestly natural operators than the fundamental derivative.

We shall only briefly comment on the latter problem and advise the readers to look at [12] for much more information.

Already Tracy Thomas constructed the differential operator D which is invariant for $\sigma \in \mathcal{E}[1]$, with values in $\mathcal{T}$ (we follow the usual convention of [12] and write the projecting part in the top, while the injecting part is in the bottom of the column vector). We may follow our prolongation of the 'conformal to Einstein' equation from the second lecture. Starting with σ in $\mathcal{E}[1]$, we first put $\mu_a = \nabla_a \sigma$ in $\mathcal{E}_a[1]$ and then, contracting the equation $\nabla_a \nabla_b \sigma + P_{ab}\sigma + g_{ab}\rho = 0$ we see $-n\rho = \nabla^a \nabla_a \sigma + P^a{}_a \sigma$. Thus, adjusting the $1/n$ factor, we arrive at the operator $D : \mathcal{E}[1] \to \mathcal{T}$

$$\sigma \xrightarrow{D} \begin{pmatrix} n\sigma \\ n\nabla_a \sigma \\ -(\nabla^a \nabla_a + P^a{}_a)\sigma \end{pmatrix}. \tag{44}$$

This *Thomas' D-operator* extends to all densities $\mathcal{E}[w]$. For $f \in \mathcal{E}[w]$ we define Df in $\mathcal{T}[w-1]$ as

$$Df = \begin{pmatrix} (n+2w-2)wf \\ (n+2w-2)\nabla_a f \\ -(\nabla^a \nabla_a + wP^a{}_a)f \end{pmatrix}. \tag{45}$$

In particular, we should notice the following facts. For $w = 0$, the first nonzero slot in the column is $(n-2)\nabla_a f$. Thus, this operator must be invariant and we have recovered the usual differential of functions.

A much more interesting choice is $w = 1 - \frac{n}{2}$ since this kills the first two components and the third one gets manifestly invariant. This way we get the second order operator $\nabla^a \nabla_a + \frac{2-n}{2} P^a{}_a$ and we recognize the celebrated Yamabe operator mentioned already in the third lecture. (Just checking the pedestrian way the invariance of this operator shows that the general theory was worth the effort!)

This example indicates where the genuine tractor calculus goes with the aim to construct manifestly invariant operators in an effective way.

5 The (co)homology and normalization

We shall continue with parabolic $P \subset G$ and the Klein model $G \to G/P$, mainly restricting to $|1|$-graded $\mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+ = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Recall that any choice of the reduction $\sigma : \mathcal{G}_0 = \mathcal{G}/\exp \mathfrak{g}_1 \to \mathcal{G}$ of the structure group of a Cartan geometry $(\mathcal{G} \to M, \omega)$ provides the pullback $\sigma^*(\omega)$ which splits into the soldering form $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$ (independent of the choice of σ), the Weyl connection ∇_a, and the Rho-tensor P_{ab}, which is a T^*M valued one-form on M. Moreover, the adjoint tractor bundle splits as

$$\mathcal{A}M = TM \oplus \mathcal{A}_0 M \oplus T^*M.$$

Our aim is now to find some suitable normalization allowing to construct a natural Cartan connection from the data on $\mathcal{G}_0$. Once we succeed, the tractor calculus related to this Cartan connection will become a natural part of the geometry defined on $\mathcal{G}_0$. We shall see that the crucial tool at our disposal is related to the cohomological properties of the Lie algebras in question. There are two equivalent ways: either to normalize the curvature of the Cartan connection, or to normalize the curvature of a suitable tractor connection. We shall show the first one, the other one was first achieved in [8], and both are explained in full generality in [11, chapter 3].

5.1 Deformations of Cartan connections

The obvious idea is to quest for normalizations which will make the curvatures of the Cartan connections as small as possible. In particular, this will ensure that the right Cartan connections on homogeneous models will be the Maurer-Cartan forms.

Consider two Cartan connections on the same principal bundle $\mathcal{G} \to M$, ω and $\tilde{\omega}$. Then their difference $\Phi = \tilde{\omega} - \omega$ clearly vanishes on all vertical vectors and is right-invariant. Thus, we deal with a one-form $\Phi \in \Omega^1(M, \mathcal{A}M)$.

In the $|1|$-graded case, let us understand the 'geometry' on M as the choice of the G_0-principal bundle $\mathcal{G}_0$ together with the soldering form θ, i.e. we adopt the most classical concept of a G-structure as a reduction of the first order linear frame bundle P^1M to the structure group G_0. (We already mentioned in the examples in lecture 3, that the projective geometries are different.) It is obvious from our definitions that the two Cartan connections will define the same structure in the latter sense if and only if their difference has got values in $\mathfrak{p}$. Thus, in our $|1|$-graded cases, Φ should be in $\Omega^1(M, \mathcal{A}_0M \oplus T^*M)$.

In the general situation with longer gradings, we have to be much more careful with the definition of the G_0-structure which has to be generalized to the filtered manifolds. In brief, the tangent space inherits the filtration by $\mathfrak{p}$-submodules of $\mathfrak{g}_-$ and a full analog of the classical G-structure has to be considered on the associated graded vector bundle $\operatorname{Gr} TM$. We shall not go to any details here, the reader can find a detailed exposition in [11, chapter 3].

As we know, the curvature can be also viewed as the curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, and $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}_+$ via the Killing form on $\mathfrak{g}$. Thus, we should like to know how κ changes if we deform the Cartan connection by Φ in $\Omega^1(M, \mathcal{A}_0M \oplus T^*M)$.

Let us write κ_ℓ for the component of the curvature function of homogeneity ℓ, i.e. $\kappa_\ell \in \Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{\ell-2}$ for the $|1|$-graded parabolic geometries.

Lemma 2. *Assume $\Phi \in \Omega^1(M, \mathcal{A}_0M \oplus T^*M)$ is of homogeneity $\ell = 1$ or $\ell = 2$. Then the components of the curvature of homogeneities lower than ℓ remain unchanged, while the corresponding change of the $\mathfrak{g}_{-1}$ or $\mathfrak{g}_0$ component of the curvature, viewed as function valued in $\Lambda^2\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_i$ with $i = -1$ or 0, respectively, is given by the formula*

$$(\tilde{\kappa} - \kappa)_i(X, Y) = [X, \phi(Y)] - [Y, \phi(X)]$$

where ϕ *is the equivariant function* $\mathcal{G} \to \mathfrak{g}_{-1}^* \otimes (\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ *representing* Φ.

Proof. Considering vector fields ξ, $\eta \in T\mathcal{G}$,

$$\tilde{\omega}(\xi) = \omega(\xi) + \phi(\omega(\xi)).$$

Thus, hitting the equation with the exterior derivative, we obtain

$$\mathrm{d}\,\tilde{\omega}(\xi, \eta) = \mathrm{d}\,\omega(\xi, \eta) + \mathrm{d}\,\phi(\xi)(\omega(\eta)) - \mathrm{d}\,\phi(\eta)(\omega(\xi)) + \phi(\mathrm{d}\,\omega(\xi, \eta)),$$

while

$$[\tilde{\omega}(\xi), \tilde{\omega}(\eta)] = [\omega(\xi), \omega(\eta)] + [\phi(\omega(\xi)), \omega(\eta)] + [\omega(\xi), \phi(\omega(\eta))] + [\phi(\omega(\xi)), \phi(\omega(\eta))].$$

Comparing the curvatures (as $\mathfrak{g}$-valued two forms on $\mathcal{G}$),

$$\begin{aligned}(\tilde{\kappa} - \kappa)(\xi, \eta) = \mathrm{d}\,\phi(\xi)(\omega(\eta)) &- \mathrm{d}\,\phi(\omega(\eta))(\xi) + \phi(\mathrm{d}\,\omega(\xi, \eta)) \\ &- [\phi(\omega(\xi)), \omega(\eta)] + [\omega(\xi), \phi(\omega(\eta))] + [\phi(\omega(\xi)), \phi(\omega(\eta))].\end{aligned}$$

Now, inspecting the homogeneities for ϕ valued in $\mathfrak{g}_i$ ($i = 0$ corresponds to homogeneity 1, while $i = 1$ yields homogeneity 2), the first three terms will land in $\mathfrak{g}_i$, while the very last term is either zero (if $i = 1$) or sits in $\mathfrak{g}_i$ again (if $i = 0$). Thus only the two remaining brackets have got the values in $\mathfrak{g}_{i-1}$ and we obtain just the requested result if we write the vector fields as functions on $\mathcal{G}$ with the help of ω.

5.2 Homology and cohomology

The formula for the lowest homogeneity deformation of the curvature is a special instance of a general algebraic construction, which works for arbitrary Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module $\mathbb{V}$. We define the k-chains $C_k(\mathfrak{g}, \mathbb{V})$ as

$$C_k(\mathfrak{g}, \mathbb{V}) := \Lambda^k \mathfrak{g} \otimes \mathbb{V}.$$

For each $k > 0$ we define the linear operator $\delta_k \colon C_k \to C_{k-1}$

$$\begin{aligned}\delta_k(X_1 \wedge \cdots \wedge X_k \otimes v) = &\sum_i (-1)^i \underbrace{X_1 \wedge \cdots \wedge X_k}_{\text{omit } i\text{-th}} \otimes X_i \cdot v \\ &+ \sum_{i<j} (-1)^{i+j} [X_i, X_j] \wedge \underbrace{X_1 \wedge \cdots \wedge X_k}_{\text{omit } i\text{-th, } j\text{-th}} \otimes v\,.\end{aligned}$$

Then $\delta^2 = 0$ and thus δ acts on the chain complex $C(\mathfrak{g}, \mathbb{V})$ as a boundary operator. A direct check reveals that δ is always a $\mathfrak{g}$-module homomorphism. Therefore we can define the homology groups

$$H_k(\mathfrak{g}, \mathbb{V}) = \frac{\ker \delta_k}{\operatorname{im} \delta_{k+1}}$$

and they are again $\mathfrak{g}$-modules.

Note that $C_0(\mathfrak{g}, \mathbb{V}) = \mathbb{V}$ and $C_1(\mathfrak{g}, \mathbb{V}) \xrightarrow{\delta_1} C_0(\mathfrak{g}, \mathbb{V})$ is given by $\delta_1(X \otimes v) = X \cdot v$ which implies $H_0(\mathfrak{g}, \mathbb{V}) = \mathbb{V}/\langle X \cdot v \rangle = \mathbb{V}/\mathfrak{g} \cdot \mathbb{V}$.

In particular, considering the adjoint representation, $H_0(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Similarly to the homology, we can consider the dual construction for cochains $C^k(\mathfrak{g}, \mathbb{V}) = \Lambda^k \mathfrak{g}^* \otimes \mathbb{V}$ and coboundaries $\partial_k \colon C^k(\mathfrak{g}, \mathbb{V}) \to C^{k+1}(\mathfrak{g}, \mathbb{V})$ given by

$$\begin{aligned}\partial_k \varphi(X_0, \dots, X_k \otimes v) &= \sum_i (-1)^i X_i \varphi(\underbrace{X_0, \dots, X_k}_{\text{omit } i\text{-th}}) \otimes X_i \cdot v \\ &\quad + \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], \underbrace{X_1, \dots, X_k}_{\text{omit } i\text{-th and } j\text{-th}}) \otimes v \ .\end{aligned}$$

Then ∂ provides a coboundary operator on the complex of cochains, i.e. $\partial^2 = 0$. The operators ∂ are again $\mathfrak{g}$-module homomorphisms and we define the cohomology groups

$$H^k(\mathfrak{g}, \mathbb{V}) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k-1}}.$$

Again, the zero cohomology is easy to compute. Clearly $\partial_0(v)(X_0) = X_0 \cdot v$, while

$$\partial_1 \psi(X, Y) = X \cdot \psi(Y) - Y \cdot \psi(X) - \psi([X, Y]).$$

Thus, $H^0(\mathfrak{g}, \mathbb{V}) = \mathbb{V}^{\mathfrak{g}} \subset \mathbb{V}$ is the kernel of the $\mathfrak{g}$-action. If we choose $\mathbb{V} = \mathfrak{g}$ with the adjoint action then $H^1(\mathfrak{g}, \mathfrak{g}) = \{\text{all derivatives}\}/\{\text{inner derivatives}\}$.

Now, the crucial observation is that Lemma 2 expresses the lowest homogeneity of the deformation of the curvature of our Cartan geometries, caused by $\phi \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_i$, via the coboundary differential $\partial\phi$ (the third term is not there in our case since we deal with $|1|$-graded geometries).

For general parabolic geometries we also consider the curvature as an equivariant function $\kappa : \mathcal{G} \to C^2(\mathfrak{g}_- \otimes \mathfrak{g})$ and $\mathfrak{g}$ is a $\mathfrak{g}_-$-module with the adjoint action. Even in full generality, the Lemma 2 holds true, i.e. the lowest homogeneity of the curvature deformation caused by ϕ is given by $\partial\phi$, see [11, section 3.1.10].

5.3 Normalization of parabolic geometries

We should be interested in the cohomologies $H^k(\mathfrak{g}_-, \mathfrak{g})$, in particular in the second degree since the curvature has got the values in the second degree cochains. Recall Lemma 2 which discussed how all possible deformations of the Cartan curvature (with positive homogeneities) impact the curvature. In particular, we learned there that the available deformation of the curvature fill the image of ∂ in the second degree cochains (in the lowest non-trivial homogeneity).

Now the crucial moment comes. Consider parabolic geometries with the homogeneous model G/P, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and a $\mathfrak{g}$-module $\mathbb{V}$. Recall $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{g}^*_- \cong \mathfrak{p}_+$. Thus, the dual of the space of cochains $C^k(\mathfrak{g}_-, \mathbb{V})$ is $C^k(\mathfrak{p}_+, \mathbb{V}^*)$ and there is the dual mapping $\partial^* : C^{k+1}(\mathfrak{p}_+, \mathbb{V}^*) \to C^k(\mathfrak{p}_+, \mathbb{V}^*)$. It was Kostant who noticed in his celebrated paper [18], that there always is a scalar product

$\langle\ ,\ \rangle$ on the space of cochains $C^k(\mathfrak{p}_+, \mathbb{V}^*)$ such that, identifying $C^k(\mathfrak{p}_+, \mathbb{V}^*)$ with $C^k(\mathfrak{g}_-, \mathbb{V})$, the latter dual map ∂^* becomes the adjoint operator to ∂. Moreover its formula then coincides with the boundary operator δ. We shall follow the (confusing) convention by many authors and call this adjoint ∂^* the *codifferential*. In particular, ∂^* is a P-module homomorphism.

Now, we equivalently consider

$$H^k(\mathfrak{g}_-, \mathbb{V}) = H^k(\mathfrak{p}_+, \mathbb{V}^*) = \frac{\ker \partial^*}{\operatorname{im} \partial^*}$$

and, applying the standard algebraic Hodge theory, we get the decompositions (of G_0-modules)

$$C^k(\mathfrak{g}_-, \mathbb{V}) = \operatorname{im} \partial^* \oplus \ker \partial = \ker \partial^* \oplus \operatorname{im} \partial = \operatorname{im} \partial^* \oplus \ker \Box \oplus \operatorname{im} \partial, \qquad (46)$$

where $\Box \equiv \partial\partial^* + \partial^*\partial$ (thus the intersection of the kernels of ∂ and ∂^*). This means that the cohomology $H^k(\mathfrak{p}_+, \mathbb{V}^*) = H^k(\mathfrak{g}_-, \mathbb{V})$ equals to the kernel of the algebraic Hodge Laplacian operator $\Box$.

Further, we see that $\ker \partial^*$ is always the complementary subspace to $\operatorname{im} \partial$ and in view of Lemma 2 we adopt the following normalization.

Notice ∂^* is a P-module homomorphism and so it induces natural transformations between the corresponding natural bundles. In particular, it makes sense to apply ∂^* to the curvatures of our Cartan connections, i.e. there is the natural algebraic operator

$$\partial^* : \Lambda^2 T^*M \otimes \mathcal{A}M \to T^*M \otimes \mathcal{A}M$$

which preserves the homogeneities.

Definition 3. *Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry with the homogeneous model $G \to G/P$, $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$. We say that ω is a* regular parabolic geometry*, if its curvature κ has got only positive homogeneities. The geometry is called* normal*, if its curvature is co-closed, i.e. $\partial^*\kappa = 0$.*

Let us stress the following observation. The curvature of any normal parabolic geometry lies in the kernel of ∂^* and thus it projects to the natural bundle defined by the cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$. This is the so called *harmonic curvature* $\kappa^H \in \mathcal{G} \times_P H^2(\mathfrak{g}_-, \mathfrak{g})$.

Let us restrict again our attention to $|1|$-graded geometries. First notice, the regularity condition is empty in this case. Indeed, the decomposition of κ into its homogeneity components coincides with the decomposition by its values, i.e. values in $\mathfrak{g}_i$ are of homogeneity $i + 2$, $i = -1, 0, 1$.

Further, there is a nice consequence of the Bianchi identity (42). Consider the component κ_i of the lowest homogeneity ℓ. Then the four terms in (42) are of homogeneity at least, $\ell - 1$, $\ell - 1$, ℓ, ℓ, respectively. But each homogeneity component in (42) has to vanish independently. Finally, the first two terms represent exactly the differential $\partial\kappa$.

We conclude that the lowest homogeneity non-zero component of the curvature should be closed and thus, for normal geometries it must coincide with its harmonic projection. Moreover, if all these harmonic components are zero, then we conclude (by induction using the previous result) that the entire curvature κ must vanish, too. These results hold true even for general parabolic geometries, the reader may consult [11, section 3.1.12].

Now we are ready to manage the normalization of the $|1|$-graded parabolic geometries with $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Given any G_0-principal bundle $\mathcal{G}_0 \to M$ with the soldering form $\theta \in \Omega^1(M, \mathfrak{g}_{-1})$, i.e. a classical G_0-structure, we consider the fiber bundle $\mathcal{G} = \mathcal{G}_0 \times \exp \mathfrak{g}_1$ and equip it with the obvious principal action of $P = G_0 \ltimes \exp \mathfrak{g}_1$.

If we choose any principal connection γ on $\mathcal{G}_0$, then $\theta \oplus \gamma$ is a Cartan connection on $\mathcal{G}_0 \subset \mathcal{G}$ and choosing any $\mathsf{P} \in \Omega^1(M, T^*M)$, there is exactly one Cartan connection ω on $\mathcal{G}$ coinciding with $\theta \oplus \gamma \oplus \mathsf{P}$ on $T\mathcal{G}_0 \subset T\mathcal{G}$.

The connection is automatically regular and the lowest component of its curvature can have homogeneity 1. It is a simple exercise to see that this component will coincide with the torsion T of the connection γ (e.g. viewed as the torsion part of the curvature of the Cartan connection $\theta \oplus \gamma$). Moreover, changing the inclusion of $\mathcal{G}_0 \to \mathcal{G}$, i.e. choosing a Weyl connection for ω, this torsion part does not change at all.

We know that for the normal Cartan connections, this torsion has to coincide with its harmonic part. Moreover, Lemma 2 says that we can modify the Cartan connection ω by a homogeneity one deformation Φ so that this condition will be satisfied.

In fact, this only recovers the very classical results about the distinguished connections with special torsions on G-structures.

For example, in conformal Riemannian geometry, there is no cohomology in homogeneity one and thus we may always find torsion free connections. This is, of course, no surprise since we may take any Levi-Civita connection of one of the metrics in the class. But for the almost Grassmannian geometries with $p \geqslant q \geqslant 3$, all the cohomology appears in homogeneity one only (with two irreducible components) and thus connections with torsions are unavoidable in general, unless we deal with the homogeneous models.

Next, we may assume that we have chosen the above connection γ in such a way, that its torsion is harmonic. In order to see the link between the curvature of γ and the curvature κ of ω, consider the Cartan connection $\tilde{\omega}$ on $\mathcal{G}$ which would be given by the choice $\mathsf{P} = 0$. The Cartan connections $\theta \oplus \gamma$ and $\tilde{\omega}$ are related by the inclusion $\mathcal{G}_0 \to \mathcal{G}$ and thus the curvature $\tilde{\kappa}$, restricted to $\mathcal{G}_0$ coincides with the curvature $T + R$ of $\theta \oplus \gamma$. Thus, Lemma 2 says (with the deformation $\mathsf{P} = \omega - \tilde{\omega}$) that the homogeneity two component of the curvature of ω is

$$\kappa_0 = R + \partial \mathsf{P}.$$

Hitting this equality by ∂^* gives

$$\partial^* \kappa_0 = \partial^* R + \partial^* \partial \mathsf{P}.$$

But by homogeneity argument, $\partial^*\mathsf{P}$ would have values in $\mathfrak{g}_2$ and thus vanishes automatically. Thus, the second term in the latter equation equals $\Box\mathsf{P}$ and the normalization condition will be satisfied if we choose P such that

$$\Box\mathsf{P} = -\partial^* R. \tag{47}$$

The final crucial observation is that the Laplacian acts by non-zero constant multiples on all irreducible components, except the harmonic ones. But we want to invert $\Box$ on $\operatorname{im}\partial^*$, which cannot include any harmonic components. The final formula for P is

$$\mathsf{P} = -\Box^{-1}\partial^* R. \tag{48}$$

Summarizing, in order the construct the normal Cartan connection ω on a manifold equipped with the relevant G_0-structure, we first choose any connection γ with harmonic torsion. Then we consider its curvature R, apply the codifferential and compute the right coefficients for each of its irreducible components. There are effective tools in the representation theory allowing to compute them easily via the so called Casimir operators. We have no space to go into details here.

Finally, there is the question about the uniqueness of our construction. The answer is again hidden in cohomologies. If there are no positive homogeneity components in $H^1(\mathfrak{g}_{-1}, \mathfrak{g})$, all our choices of the deformations in both steps were unique. This is the case for nearly all $|1|$-graded geometries. The only exceptions are the projective geometries (and their complex versions), where we have to choose one of the connections in the first step to define the structure. Then the Cartan connection is already given uniquely via the next step in our construction.

In the categorical language, there is the subcategory of the regular and normal Cartan geometries, and this subcategory is equivalent to the category of the infinitesimal G_0-structures on manifolds, up to some rare exceptions due to the existence of positive homogeneities in first cohomologies in some examples (where a similar equivalence exists, too).

In conformal Riemannian geometry, i.e. $\mathfrak{g} = \mathfrak{so}(n+1,1)$, there is no positive homogeneity first cohomology, while the entire second cohomology is concentrated in homogeneity two (except of dimension $n = 3$, where it is homogeneity three). The operator ∂^* is just the trace, so the image on the curvature of a Levi-Civita connection is the Ricci tensor. The formula for P reflects the right choices of the constants in the action of $\Box$, while the invariant Weyl part of the curvature (shared by all Weyl connections) is $R + \partial\mathsf{P}$, the harmonic component in all dimensions $n > 3$. Of course, the geometry is locally isomorphic to the conformal sphere if and only if this Weyl curvature vanishes.

We do not have space in this lecture to inspect further examples and detailed computations. The readers may look up many of them in [11], a few hundreds of pages of examples and details for general parabolic geometries are there in chapters 3 through 5.

6 The BGG machinery

As well known, the linearized theories in Physics usually appear as locally exact complexes of differential operators. A lot of attention was devoted to this phenomenon in Mathematics, too. Already in the early days people around Gelfand or Kostant knew that on the Klein models, the existence of such complexes is an algebraic phenomenon related to homomorphisms of Verma modules (which were understood as topological duals of the infinite jet prolongations of the natural bundles), cf. [4,19].

The main message of this series of lectures is to show how remarkably the algebraic features and phenomena from the Klein models extend to the categories of Cartan geometries. The so called BGG machinery does exactly this – extends the complexes of the differential operators from the homogeneous models to sequences on all Cartan geometries of the given type.

In this last lecture we comment on this exciting development and we shall also come back to the solutions of the 'conformal to Einstein' equation (8) in terms of constant tractors. On the way we shall touch the general construction of the latter sequences of operators and identify the equation (8) as one of the so called 1st BGG operators.

6.1 The twisted de-Rham complexes

Denote by $H^k_{\mathbb{V}}M$ the natural bundle associated to the P-module $H^k(\mathfrak{g}_-, \mathbb{V})$ of cohomologies with coefficients in a G-module $\mathbb{V}$. Notice, that by the Kostant's complete description of the cohomologies, [18], the latter cohomology module is a G_0 module with trivial action of P_+ and thus, it is completely reducible. In particular, $H^0_{\mathbb{V}}M$ is the bundle coming from the projecting part of $\mathbb{V}$ which can be viewed as the orbit of the lowest weight vector in $\mathbb{V}$ under the $\mathfrak{g}_0$-action. Our goal is to come to the following diagram of operators

$$\begin{array}{ccccc} \Omega^0(M,\mathcal{V}M) & \xrightarrow{\mathrm{d}_{\mathbb{V}}} & \Omega^1(M,\mathcal{V}M) & \xrightarrow{\mathrm{d}_{\mathbb{V}}} & \dots \\ {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle \mathrm{L}} & & {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle \mathrm{L}} & & \\ H^0_{\mathbb{V}}M & \overset{\mathrm{D}}{\dashrightarrow} & H^1_{\mathbb{V}}M & \overset{\mathrm{D}}{\dashrightarrow} & \dots \end{array} \tag{49}$$

where all the arrows have to be yet explained. As usual, we write $\mathcal{V}M$ for the tractor bundle over the manifold M corresponding to $\mathbb{V}$, and notice that ∂^* is the adjoint of ∂ which is a P-module homomorphism and thus, it gives rise to the natural algebraic operator $\partial^* : \Omega^k(M,\mathcal{V}M) \to \Omega^{k-1}(M,\mathcal{V}M)$. Clearly, the projections π are well defined only on the kernel of ∂^*. We shall have to be careful about this.

The ideas presented below go back to [2] and [3], and they were further developed in [10].

Let us discuss the upper line in (49) now. First, restrict to the parabolic Klein model $G \to G/P$. Together with the G-module $\mathbb{V}$, consider a P-module $\mathbb{W}$. Then there is the following identification of the sections of the tensor product bundle

$\mathcal{V} \otimes \mathcal{W}$. For any section s of $\mathcal{W}$, i.e. an equivariant mapping $s : \mathcal{G} \to \mathbb{W}$, and $v \in \mathbb{V}$ consider the map

$$s \otimes v \mapsto (\underbrace{g \mapsto s(g) \otimes g^{-1} \cdot v}_{\text{equivariant } G \to \mathbb{W} \otimes \mathbb{V}}),$$

which provides a natural isomorphism of the G-modules of sections

$$\Gamma(\mathcal{W}) \otimes \mathbb{V} \cong \Gamma(\mathcal{W} \otimes \mathcal{V}). \tag{50}$$

Thus, if $F \colon \mathcal{W}_1 \to \mathcal{W}_2$ is an arbitrary differential operator between the homogeneous vector bundles, then $F \otimes \mathrm{id}_{\mathbb{V}} = F_{\mathbb{V}}$ provides the *twisted operator* $F_{\mathbb{V}} \colon \mathcal{W}_1 \otimes \mathcal{V} \to \mathcal{W}_2 \otimes \mathcal{V}$.

Considering the exterior differential $\mathrm{d} : \Lambda^k T^*M \to \Lambda^{k+1} T^*M$, this explains the whole first line in (49), at least on the homogeneous model. On zero-degree forms, the exterior differential is just the covariant derivative of the sections.

Let us look more carefully on this example. At the level of first order jets, we can express the twisted operator by means of the algebraic P-homomorphism

$$J^1(\Lambda^k \mathfrak{p}_+ \otimes \mathbb{V}) \to \Lambda^{k+1} \mathfrak{p}_+ \otimes \mathbb{V}, \quad (f_0, Z \otimes f_1) \mapsto \partial f_0 + (k+1) Z \wedge f_1. \tag{51}$$

In general, if we write $J^r(\mathbb{W})$ and $\bar{J}^r(\mathbb{W})$ for the standard fibers of the holonomic and semi-holonomic jet prolongations $J^r(\mathcal{W})$, $\bar{J}^r(\mathcal{W})$,[4] then the isomorphism (50) must hold true at the jet level, e.g. $\bar{J}^r(\mathbb{W}) \otimes \mathbb{V} \cong \bar{J}(\mathbb{W} \otimes \mathbb{V})$.

Now the crucial observation comes: Although the jet prolongations $J^r\mathcal{W}$ are no more natural bundles associated to $\mathcal{G}$ in general, there is still no problem with the first jets. Thus, $J^1(\mathcal{W}) = \mathcal{G} \times_P J^1(\mathbb{W})$ and iterating this procedure, we conclude that the semi-holonomic jet prolongations are natural bundles again, i.e., $\bar{J}^r(\mathcal{W}M) = \mathcal{G} \times_P \bar{J}^r(\mathbb{W})$ for the relevant P-module $\bar{J}^r(\mathbb{W})$ (the standard fiber over the origin in G/P as the module with the action of the isotropy group P). Moreover, we can construct a universal differential operator $\mathcal{W}M \to \bar{J}^r(\mathcal{W}M)$ based on the iterated fundamental derivative, which allows one to extend many invariant operators from the homogeneous model to all Cartan geometries of this type.

Therefore, the so called *strongly invariant operators*, i.e. those coming from algebraic P-module homomorphisms $\bar{J}^r(\mathbb{W}_1) \to \mathbb{W}_2$, enjoy a canonical extension to all Cartan geometries by means of the formulae obtained on the homogeneous model.

A careful exposition of the algebraic structure of the semiholonomic jets and their links to the strongly invariant operators can be found in [13].

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[4] We iterate the first jet prolongation. Considering the first jets of sections of a bundle $\mathcal{W}$, the jets in a fiber of $J^1(J^1\mathcal{W})$ look in coordinates as 4-tuples $(y^p, y^p_i, Y^p_j, Y^p_{ij})$ were Y^p_{ij} do not need to be symmetric. These are the non-holonomic 2-jets. The semi-holonomic ones remove part of the redundancy by requesting that the two natural projections to 1-jets coincide, i.e. $y^p_i = Y^p_i$. This construction extends to all orders and the semi-holonomic jets look in coordinates nearly as the holonomic ones, just loosing the symmetry of the derivatives. See e.g. [17] for detailed exposition.

This in particular applies for all first order operators and we are done with the first line in (49), which is called the *twisted de-Rham* sequence. Obviously, there are many other ways for twisting the de-Rham. For example, we could take the covariant exterior differential d^ω of the tractor valued k-forms with respect to the tractor connection on $\mathcal{V}$. A straightforward computation reveals

$$\mathrm{d}^\omega \varphi = \mathrm{d}_{\mathbb{V}} \varphi + \iota_{\kappa_-} \varphi \ , \tag{52}$$

where κ_- is the torsion part of the curvature $\kappa = \mathrm{d}\,\omega + \frac{1}{2}[\omega, \omega]$.

6.2 BGG machinery

Next, let us focus on the vertical arrows in (49). We already know about the projections π, so we have to deal with L's.

$$\begin{array}{ccccc} \Omega^0(M,\mathbb{V}) & \xrightarrow{\mathrm{d}_{\mathbb{V}}} & \Omega^1(M,\mathbb{V}) & \xrightarrow{\mathrm{d}_{\mathbb{V}}} & \dots \\ {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle \mathrm{L}} & & {\scriptstyle\pi}\big\downarrow\big\uparrow{\scriptstyle \mathrm{L}} & & \\ H^0_{\mathbb{V}}M & & H^1_{\mathbb{V}}M & & \end{array}$$

The quite straightforward idea is to seek for differential operators L, such that $\mathrm{d}_{\mathbb{V}} \circ \mathrm{L}$ are requested to be algebraically co-closed. Then the composition with the projection π makes sense and we could arrive at operators D

$$H^k_{\mathbb{V}}M \xrightarrow{\mathrm{D}=\pi\circ \mathrm{d}_{\mathbb{V}}\circ \mathrm{L}} H^{k+1}_{\mathbb{V}}M.$$

The most important (and demanding) step in the original construction of the sequence of those operators in (49) was the following lemma in [10]. Notice, [5] suggests a different and more efficient construction of these operators.

Lemma 3. *On each irreducible component of $H^k_{\mathbb{V}}M$, there is the unique strongly invariant operator* L *with values in* $\ker \partial^*$ *and splitting the projection π,*

$$H^k_{\mathbb{V}}M \underset{\pi}{\overset{\mathrm{L}}{\rightleftarrows}} \Omega^k(M, \mathcal{V}) \ ,$$

such that $\mathrm{d}_{\mathrm{V}} \circ L \in \ker \partial^*$.

The proof in [10] is very technical and there are many later improvements in the literature, starting with [5].

The resulting sequence of operators

$$H^0_{\mathbb{V}}M \xrightarrow{\mathrm{D}_0} H^1_{\mathbb{V}}M \xrightarrow{\mathrm{D}_1} H^2_{\mathbb{V}}M \xrightarrow{\mathrm{D}_2} \dots$$

is called the *BGG sequence* associated with the tractor bundle $\mathcal{V}$.

Theorem 8. *For each G-module $\mathbb{V}$, the BGG sequence is well defined on each Cartan geometry modelled on G/P and it restricts to the celebrated BGG resolution on the homogeneous model.*

If the twisted de-Rham sequence on a Cartan geometry is a complex, then also the BGG sequence is a complex, and they both compute the same cohomology of the underlying manifold.

A good example is the case when the Cartan geometry is torsion-free and the curvature values act trivially on $\mathbb{V}$. Then the comparison (52) of the twisted exterior differential and the covariant exterior differential implies that the twisted de-Rham sequence will be exact.

Often only a part of the whole BGG sequence is exact and many celebrated complexes known in differential geometry can be recovered this way.

6.3 The first BGG operators

Finally, we are coming back to the first operators in BGG sequences. They are always overdetermined operators $\mathrm{D} : H^0_{\mathbb{V}}M \to H^1_{\mathbb{V}}M$. Moreover, by the very construction, its space of solutions is in bijection with the space of the parallel tractors on the homogeneous model. Unfortunately, this is not true in general and the so called *normal solutions* are those sections in the kernel of D which correspond to parallel tractors. See [9] for interesting results on the normal solutions. Because of lack of space in this last lecture, we shall just report briefly on the available results.

As carefully explained in [16], the normalization condition on the canonical tractor connections can be written as $\partial^*(R^{\mathcal{V}}) = 0$, considered on the space of 2-forms valued in endomorphisms $\mathcal{V} \otimes \mathcal{V}^*$. At the same time, the normalization necessary for keeping the 1-1 correspondence between the solutions and the parallel tractors is rather $\partial^*_{\mathbb{V}} R^{\mathcal{V}} = 0$, where the codifferential is modified, see [16].

So, although the values of our operator L on the harmonic curvature are always algebraically co-closed, this is not enough.

The paper [16] answers positively the question: Can we modify the Cartan connection so that $\partial^*_{\mathbb{V}} \circ \mathrm{d}_{\mathbb{V}} \circ \mathrm{L}(\kappa) = 0$ and thus the 1-1 correspondence will hold true for all Cartan geometries?

The first useful observation is the fact that the BGG machinery construction survives without any changes if we restrict the deformations to the class of connections:

$$C = \{\tilde{\nabla} = \nabla + \Phi \mid \Phi \in \ker \partial^*_{\mathbb{V}} \otimes \mathrm{id}_{\mathbb{V}}, \Phi \text{ has homogeneity} \geqslant 1\}$$

The main theorem of [16] says:

Theorem 9. *There is precisely one $\tilde{\nabla} \in C$ providing the $1-1$ correspondence between $\ker \mathrm{D}_0$ and $\tilde{\nabla}$-parallel tractors.*

At the very end, let us look again at the case of the 'conformal to Einstein' equation (8), which is the first BGG operator for the choice of $\mathcal{V}$ equal to the conformal standard tractors $\mathcal{T}$.

Clearly, $H^0_{\mathbb{T}}M$ is the projecting part $\mathcal{E}[1]$ of the tractors. Further, a straightforward check reveals that the operator (44) satisfies the conditions on $\mathrm{L} : H^0_{\mathbb{T}}M \to \Omega^1(TM, \mathcal{T}M)$. Indeed, the entire space of zero-forms is in the kernel of ∂^*, the exterior derivative d^ω is just the covariant derivative (19) of the tractor, its projecting slot vanishes, ∂^* maps the injecting slot to zero by the homogeneity, and ∂^* is given by the trace in the middle slot, which vanishes, too. Since the geometry is torsion free, the exterior covariant derivative coincides with the twisted derivative, see (52). Finally, the projection of $\mathrm{d}_{\mathbb{T}} \circ \mathrm{L}$ to the harmonic component provides just the right operator (8) on $\mathcal{E}[1]$.

In this very special case, there is no need to modify the tractor connection in the above sense and thus there always is the 1-1 correspondence between the solutions and the parallel tractors, which is again realized directly by the operator L.

As already mentioned, many of important overdetermined operators appear as the first BGG operators. A vast supply of interesting examples of the first order ones appear in relation with the generalization of the classical problem of metrizability of a projective geometry into the realm of filtered manifolds and parabolic geometry. The projective case goes back to 19th century, the generalization was recently worked out in [7].

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References

1. T.N. Bailey, M.G. Eastwood, A. Rod Gover, *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math., 24 (1994), 1191-1217.
2. R.J. Baston, Almost Hermitian symmetric manifolds, I: Local twistor theory, II: Differential Invariants, Duke Math. J. 63 (1991), 81-111, 113-138.
3. R.J. Baston, M.G. Eastwood, The Penrose Transform: Its Interaction with Representation Theory, Courier Dover Publications, (2nd edition, 2016), 256pp.
4. I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Differential operators on the base affine space and a study of g-modules, in "Lie Groups and their Representations" (ed. I.M. Gelfand) Adam Hilger 1975, 21-64.
5. D.M.J. Calderbank,, T. Diemer, *Differential invariants and curved Bernstein-Gelfand-Gelfand sequences.* J. Reine Angew. Math. 537 (2001), 67-103.

6. D.M.J. Calderbank, T. Diemer, V. Souček, *Ricci-corrected derivatives and invariant differential operators*, Diff. Geom. Appl. 23 (2005) 149-175.
7. D.M.J. Calderbank, J. Slovák, V. Souček, *Subriemannian metrics and the metrizability of parabolic geometries*, Journal of Geometric Analysis (2019), 32pp. (https://doi.org/10.1007/s12220-019-00320-1).
8. A. Čap, A.R. Gover, *Tractor calculi for parabolic geometries*, Trans. Amer. Math. Soc. 354 (2002), no. 4, 1511-1548.
9. A. Čap, A. R. Gover, Matthias Hammerl, *Normal BGG solutions and polynomials*, Internat. J. Math. 23 (2012), no. 11, 1250117, 29 pp.
10. A. Čap, J. Slovák, V. Souček, *Bernstein-Gelfand-Gelfand sequences*, Annals of Mathematics, Princeton University: The Johns Hopkins University Press, 2001, vol. 154, No 1, p. 97-113. (extended, more understandable version as ESI preprint No 722, see www.esi.ac.at).
11. A. Čap, J. Slovák, *Parabolic Geometries I, Background and General Theory*, Providence, RI, USA: American Mathematical Society, 2009. 628 pp. Mathematical Surveys and Monographs, 154.
12. S. Curry, R. Gover, *An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity*, arXiv:1412.7559v2, 74pp.
13. M.G. Eastwood, J. Slovák *Semiholonomic Verma modules*, Journal of Algebra, 1997, vol. 197, No 2, p. 424-448.
14. A.R. Gover, J. Slovák, *Invariant local twistor calculus for quaternionic structures and related geometries*, Journal of Geometry and Physics, Amsterdam: Elsevier Science, 1999, vol. 32, No 1, p. 14-56.
15. M. Hammerl, P. Somberg, V. Souček, J. Šilhan, *Invariant prolongation of overdetermined PDEs in projective, conformal, and Grassmannian geometry*, Ann. Global Anal. Geom. 42 (2012), no. 1, 121-145.
16. M. Hammerl, P. Somberg, V. Souček, J. Šilhan, *On a new normalization for tractor covariant derivatives*, J. Eur. Math. Soc. (JEMS) 14 (2012), no. 6, 1859-1883.
17. I. Kolář, P.W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Berlin-Heidelberg-New York: Springer-Verlag, 1993. 434 pp.
18. Bertrand Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. 74 (1961) 329-387.
19. J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, J. of Algebra 49 (1977), 496-511.
20. J. Slovák, Natural operators on conformal manifolds, Research Lecture Notes, University of Vienna, 1992, 138pp. (http://www.math.muni.cz/~slovak/Papers/habil.pdf)
21. J. Slovák, *Parabolic Geometries*, Research Lecture Notes, Adelaide, 1997, 70pp., (http://www.math.muni.cz/~slovak/Papers/examples.pdf)
22. T.Y. Thomas, *On conformal geometry.* Proc. N.A.S. 12 (1926), 352-359.
23. V. Wünsch, *On Conformally Invariant Differential Operators*, Mathematische Nachrichten, 129 (1986), 269-281.
24. R. Zierau, Representations in Dolbeault Cohomology, in Representation Theory of Lie Groups edited by Jeffrey Adams, David Vogan, IAS/Park City Mathematics Series, American Math. Soc., vol. 8 (2015), 89-146.