Ann. Global Anal. Geom.
Vol. 6, No. 3 (1988), 273—283

Peetre Theorem for Nonlinear Operators

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Some generalizations of the well-known Peetre theorem on the locally finite order of support non-increasing **R**-linear operators, [9, 11], has become a useful tool for various geometrical considerations, see e.g. [1, 5]. A nonlinear version of Peetre theorem came into question during author's investigations of the order of natural operators between natural bundles, see [12].

However, the ideas used there have turned out to be efficient even for the study of the order of natural bundles and this originated the present paper. Since the obtained general results might also be of an independent interest, they are formulated in a pure analytical form. Some possible applications are discussed at the end of the paper. In particular, we sketch an alternative proof of the well-known result on the finiteness of the order of natural bundles, [4, 10], see Remark 4. We also give countre-examples showing that our results are the best possible in such a general setting.

The author is greatful to Prof. I. Kolář for his encouragement and to Prof. P. W. Michor for several valuable discussions.

1. Preliminaries

Let us first formulate a general concept of local operators transforming continuous maps between topological spaces. We write $C(X, Y)$ for the set of all continuous maps $f: X \to Y$.

Definition 1. Let X, Y, Z, W be topological spaces, $\pi \in C(Z, X)$. A mapping A defined on a subset $\mathscr{D} \subset C(X, Y)$ with values in $C(Z, W)$ is called a *π-local operator* if for any map $f \in \mathscr{D}$ and any point $x \in X$ the restriction $Af \mid \pi^{-1}(x)$ depends on the germ of f in the point x only.

In the sequel, we shall deal with manifolds X, Y of class C^s, $0 < s \leqq \infty$, and metric spaces Z, W. A π-local operator $A: \mathscr{D} \subset C^s(X, Y) \to C(Z, W)$ is said to be of the order r, $0 \leqq r \leqq s \leqq \infty$, if for any $f \in \mathscr{D}$, $z \in Z$ the value $Af(z)$ depends on the r-jet $j^r f(\pi(z))$ only. The r-th order operators seem to be a very special case of local operators. Nevertheless, we deduce some Peetre-like results under certain additional assumptions.

First of all, some requirements on the codomain of the operators are necessary. But we at most assume that the values are locally Hölder-continuous. Let us recall that a map $f: Z \to W$ is said to be locally Hölder-continuous if for any compact subset $K \subset Z$ there are positive real constants C_K, λ_K, such that for all $x, y \in K$ the following inequality holds

$$\varrho_W(f(x), f(y)) \leqq C_K(\varrho_Z(x, y))^{\lambda_K},$$

where ϱ_Z or ϱ_W are the metrics on Z or W, respectively. We write $HC(Z, W)$ for the subset of all locally Hölder-continuous maps in $C(Z, W)$. Clearly, dealing with smooth manifolds (with fixed Riemannian metrics) every one-differentiable map is locally Hölder-continuous.

The map π is assumed to be locally non-constant, i.e. for any open set $U \subset Z$, $\pi(U)$ contains at least two points.

The most delicate requirement is the s-extendability of the domain $\mathscr{D} \subset C^s(X, Y)$, $0 < s \leqq \infty$, defined below. We could avoid this rather technical assumption dealing with the whole $C^s(X, Y)$ or local s-diffeomorphisms only, but it might be useful for applications to point out what is really needed (cf. Remarks 3, 4).

Definition 2. A subset $\mathscr{D} \subset C^s(X, Y)$, $0 \leqq s \leqq \infty$, is called *s-extendable* if for any compact set $K \subset X$ and any limit point $x \in K$ the following statement holds. Given maps $f_a \in \mathscr{D}$ for all $a \in K$ in such a way that there exists a map $g \in C^s(X, Y)$ satisfying

$$j^s f_a(a) = j^s g(a) \tag{1}$$

for all $a \in K$, then there also is a map $h \in \mathscr{D}$ satisfying (1) with g replaced by h for all a from some neighbourhood of x.

Roughly speaking, $\mathscr{D}$ is s-extendable if and only if Whitney extension theorem locally holds for maps from $\mathscr{D}$. Using this fact, we can easily find some s-extendable domains, e.g. all smooth sections of a fibred manifold, local diffeomorphisms between two manifolds, all fibred morphisms between two fibred manifolds, orientation preserving morphisms between two oriented manifolds.

2. The Main Results

Let us recall we consider metric spaces Z or W with metrics ϱ_Z or ϱ_W, respectively. Since only local results can be expected, we deal with $X = \mathbf{R}^n$, $Y = \mathbf{R}^m$ without any loss of generality. The symbol $\|\cdot\|$ refers to the usual Euclidean norm and we use current notation for jets and multiindices.

Theorem 1. *Let $\mathscr{D} \subset C^s(\mathbf{R}^n, \mathbf{R}^m)$ be s-extendable, $0 < s \leqq \infty$, $\pi \in C(Z, \mathbf{R}^n)$ be locally non-constant. Let $A: \mathscr{D} \to C(Z, W)$ be a π-local operator. Then for any point $x \in \mathbf{R}^n$ and for any map $f \in \mathscr{D}$ the restriction $Af \mid \pi^{-1}(x)$ depends on the s-jet $j^s f(x)$ only.*

Proof. Let us first assume $s = \infty$. Consider $f, g \in \mathscr{D}$, $j^\infty f(x) = j^\infty g(x)$, and a point $y \in \pi^{-1}(x)$. We choose a sequence y_k tending to y in Z, $\pi(y_k) = x_k$, and neighbourhoods

U_k of x_k in such a way that

$$\|a - x\| > 2\|b - x\| \quad \text{for all } a \in \bar{U}_k,\ b \in \bar{U}_{k+1}, \tag{2}$$

$$\|D^\alpha f(a) - D^\alpha g(a)\| / \|a - x\|^m \leqq 1/k \quad \text{for all } a \in \bar{U}_k,\ |\alpha| + m \leqq k. \tag{3}$$

This is possible by induction using Taylor formula and the fact that π is locally non-constant. Condition (2) implies

$$\|a - x\| < 2\|a - b\| \quad \text{for all } a \in \bar{U}_k,\ b \in \bar{U}_j,\ k \neq j. \tag{4}$$

Whitney extension theorem, [9, 14] ensures the existence of a map $h \in C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ satisfying for all $k \in \mathbf{N}$

$$h \mid \bar{U}_{2k} = f \mid \bar{U}_{2k}, \qquad h \mid \bar{U}_{2k+1} = g \mid \bar{U}_{2k+1}, \qquad j^\infty h(x) = j^\infty f(x). \tag{5}$$

Indeed, a sufficient condition is that

$$\sum_{|\beta| \leqq m} \frac{1}{\beta!} D^{\alpha+\beta} h(a)(b - a)^\beta = D^\alpha h(b) + o(\|b - a\|^m) \tag{6}$$

holds uniformly for $a, b \in \{x\} \cup \bigcup_k \bar{U}_k$, $\|a - b\| \to 0$, and for every fixed m, α, where the values of $D^\alpha h$ are prescribed by (5) (including boundaries). But using Taylor formula and (3), (4), the verification of (6) is easy.

According to our requirement on $\mathscr{D}$, we may assume $h \in \mathscr{D}$ and use (5) for large k's only. But now, the π-locality and the continuity of Ah imply $Af(y) = Ag(y)$.

In the case $0 < s < \infty$ we only have to modify the previous proof. Actually, we consider $f, g \in \mathscr{D}$, $j^s f(x) = j^s g(x)$, $y \in \pi^{-1}(x)$, and we construct a suitable sequence $y_k \to y$, open neighbourhoods U_k of $\pi(y_k) = x_k$ and a map $h \in \mathscr{D}$ satisfying (5) for large k's. But in contrast to the case $s = \infty$, a sufficient condition for the existence of h is

$$\sum_{|\beta| \leqq s - |\alpha|} \frac{1}{\beta!} D^{\alpha+\beta} h(a)(b - a)^\beta = D^\alpha h(b) + o(\|b - a\|^{s-|\alpha|})$$

uniformly for $a, b \in \{x\} \cup \bigcup_k \bar{U}_k$, $\|a - b\| \to 0$ and for any fixed multiindex α, $0 \leqq |\alpha| \leqq s$. Therefore, we replace (3) by the condition

$$\|D^\alpha f(a) - D^\alpha g(a)\| / \|a - x\|^{s-|\alpha|} \leqq 1/k$$

for all $a \in \bar{U}_k$, $0 \leqq |\alpha| \leqq s$. The proof is completed as before. QED.

According to Theorem 1, every operator $A : \mathscr{D} \subset C^s(\mathbf{R}^n, \mathbf{R}^m) \to C(Z, W)$, $0 < s < \infty$, has a finite order less than or equal to s. The main result of this paper is the next theorem giving more informations about the case $s = \infty$.

Theorem 2. *Let $\mathscr{D} \subset C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ be ∞-extendable, $\pi \in C(Z, \mathbf{R}^n)$ be locally non-constant and $A : \mathscr{D} \to HC(Z, W)$ be a π-local operator. For any fixed map $f \in \mathscr{D}$ and for any compact subset $K \subset Z$ there exist a natural number r and a smooth function $\varepsilon : \pi(K) \to \mathbf{R}$ that is* *strictly positive with a possible exception of a finite set of points in $\pi(K)$, such that the following statement holds. For any point $z \in K$ and for any maps $g_1, g_2 \in \mathcal{D}$ satisfying*

$$\|D^\alpha(g_i - f)(\pi(z))\| \leqq \varepsilon(\pi(z)), \qquad i = 1, 2, \qquad 0 \leqq |\alpha| \leqq r,$$

the condition $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$ implies $Ag_1(z) = Ag_2(z)$.

Proof. Theorem 2 is implied by Lemma 1 below and by the standard compactness arguments.

Remark 1. The assertion of Theorem 2 could be interpreted as a local finiteness of the order (locality with respect to Z and to the C^∞-topology), provided a strictly positive function ε may ever be chosen. However, the "exceptional zero points" can appear even at classical operators with smooth values, cf. Example 2, and in order to avoid them, we need some additional assumptions, see the next section. Example 1 demonstrates that the Hölder continuity cannot be weakened to continuity.

Lemma 1. *Let $\mathcal{D}$ and π be as before, $A: \mathcal{D} \to HC(Z, W)$ be a π-local operator, $z_0 \in Z$ be a point, $\pi(z_0) = x_0$, and $f \in \mathcal{D}$. We define a function ε: $\mathbf{R}^n \to \mathbf{R}$*

$$\varepsilon(x) = \begin{cases} \exp(-1/\|x - x_0\|) & \text{for } x \neq x_0, \\ 0 & \text{for } x = x_0. \end{cases}$$

There exist a neighbourhood V of the point $z_0 \in Z$ and a natural number r such that for any $z \in V$ and any maps $g_1, g_2 \in \mathcal{D}$ satisfying

$$\|D^\alpha(g_i - f)(\pi(z))\| \leqq \varepsilon(\pi(z)), \qquad i = 1, 2, \qquad 0 \leqq |\alpha| \leqq r,$$

the condition $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$ implies $Ag_1(z) = Ag_2(z)$.

Proof. We assume Lemma 1 does not hold and we deduce a contradiction. Under this assumption, we can construct a sequence $z_k \to z_0$, $\pi(z_k) = x_k$ and maps $f_k, g_k \in \mathcal{D}$ satisfying for all $k \in \mathbf{N}$

$$\|D^\alpha(f_k - f)(x_k)\| \leqq \varepsilon(x_k), \quad \|D^\alpha(g_k - f)(x_k)\| \leqq \varepsilon(x_k), \quad 0 \leqq |\alpha| \leqq k, \tag{7}$$

$$j^k f_k(x_k) = j^k g_k(x_k), \qquad Af_k(z_k) \neq Ag_k(z_k). \tag{8}$$

By passing to subsequences, we may assume

$$\|x_k - x_0\| \geqq 2\|x_{k+1} - x_0\| \tag{9}$$

and either $x_k \neq x_0$ or $x_k = x_0$ for all $k \in \mathbf{N}$. We shall deduce a contradiction in both the possibilities.

In the first case, we choose further points $y_k \in Z$, $y_k \to z_0$, $\pi(y_k) = \bar{x}_k$, $\bar{x}_k \neq x_j$ satisfying for all $k \in \mathbf{N}$

$$\|x_k - \bar{x}_k\| < \frac{1}{k}\|x_k - x_0\|, \tag{10}$$

$$\|D^\alpha(f_k - f)(\bar{x}_k)\| \leqq 2\varepsilon(x_k), \qquad 0 \leqq |\alpha| \leqq k, \tag{11}$$

$$\varrho_W(Ag_k(z_k), Af_k(y_k)) \geqq k(\varrho_Z(z_k, y_k))^{1/k}. \tag{12}$$

Moreover we shall need for all α, m

$$\sum_{|\beta|\leqq m} \frac{1}{\beta!} D^{\alpha+\beta} f_k(\bar{x}_k)(x_k - \bar{x}_k)^\beta = D^\alpha g_k(x_k) + o(\|x_k - \bar{x}_k\|^m),$$
$$\sum_{|\beta|\leqq m} \frac{1}{\beta!} D^{\alpha+\beta} g_k(x_k)(\bar{x}_k - x_k)^\beta = D^\alpha f_k(\bar{x}_k) + o(\|\bar{x}_k - x_k\|^m). \tag{13}$$

All these requirements can be satisfied. Indeed, all conditions, except (12), are satisfied for all x_k from suitable neighbourhoods of the points x_k (we use Taylor formula for (13)). By virtue of (8), there are also neighbourhoods of the points z_k in Z ensuring (12). Hence we are able to choose appropriate points $y_k \in Z$ using the fact that π is continuous and locally non-constant.

The aim of conditions (7), (9), (10), (11), (13) is to guarantee the existence of a map $h \in C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ satisfying

$$j^\infty h(x_k) = j^\infty g_k(x_k), \qquad j^\infty h(\bar{x}_k) = j^\infty f_k(\bar{x}_k), \qquad j^\infty h(x_0) = j^\infty f(x_0). \tag{14}$$

By virtue of our requirements on $\mathscr{D}$, we may assume $h \in \mathscr{D}$, provided we use equalities (14) for large k's only. But then applying A to h, inequality (12) and Theorem 1 imply for large k's

$$\varrho_W(Ah(z_k), Ah(y_k)) \geqq k(\varrho_Z(z_k, y_k))^{1/k}$$

and we have obtained a contradiction with $Ah \in HC(Z, W)$ and $y_k \to z_0$, $z_k \to z_0$.

According to Whitney extension theorem, a sufficient condition for the existence of such a map h is that

$$\sum_{|\beta|\leqq m} \frac{1}{\beta!} D^{\alpha+\beta} h(a)(b - a)^\beta = D^\alpha h(b) + o(\|b - a\|^m) \tag{15}$$

holds uniformly for a, $b \in B = \{x_k, \bar{x}_k, x_0; k \in \mathbf{N}\}$, $\|a - b\| \to 0$; for all fixed $m \in \mathbf{N}$ and multiindices α, and the values of $D^\alpha h$ on B are prescribed by (14). The verification of (15) can be easily done separately for a finite number of special types of sequences $(a_k, b_k) \in B \times B$ using Taylor formula and conditions (7), (9), (10), (11), (13).

Consider the other possibility now. Since $x_k = x_0$ for all k, the definition of the function ε, (7) and (8) imply

$$j^k f_k(x_0) = j^k g_k(x_0) = j^k f(x_0), \tag{16}$$
$$Af_k(z_k) \neq Ag_k(z_k). \tag{17}$$

Hence we may assume $Af_k(z_k) \neq Af(z_k)$ and we continue the proof analogously to the first case. We choose further points y_k tending to z_0 in Z, $\pi(y_k) = \bar{x}_k$ in such a way that for all $k \in \mathbf{N}$

$$\varrho_W(Af_k(y_k), Af(z_k)) \geqq k(\varrho_Z(y_k, z_k))^{1/k},$$
$$\|\bar{x}_k - x_0\| > 2\|\bar{x}_{k+1} - x_0\|,$$
$$\|D^\alpha(f_k - f)(\bar{x}_k)\| / \|\bar{x}_k - x\|^m \leqq 1/k \quad \text{for all } |\alpha| + m \leqq k.$$

This is possible as before using (16), (17) and Taylor formula. Once more, using Whitney extension theorem and our assumptions, we verify the existence of a map $h \in \mathcal{D}$ satisfying

$$j^\infty h(\bar{x}_k) = j^\infty f_k(\bar{x}_k), \qquad j^\infty h(x_0) = j^\infty f(x_0)$$

for large k's. Hence

$$\varrho_W(Ah(y_k), Ah(z_k)) \geqq k(\varrho_Z(y_k, z_k))^{1/k}$$

which is a contradiction with $Ah \in HC(Z, W)$ and $y_k \to z_0$, $z_k \to z_0$, QED.

Remark 2. The basic idea of the proof of Theorem 2 can also be applied to any operator A: $\mathcal{D} \subset C^\infty(\mathbf{R}^n, \mathbf{R}^m) \to C(Z, W)$. However, in this case an essentially weaker version of Theorem 2 holds, see [12] and Example 1. The difference is that the resulting order and function ε from Theorem 2 only assure that the distance $\varrho_W(Ag_1(z), Ag_2(z))$ is less than a positive constant chosen in advance. We remark that our methods are partially similar to those used in [2], but we deduce essentially stronger results in a much more general situation. Let us also remark, that the operator constructed in [3, p. 632] has only continuous values in general, so that this is not contradictory to Theorem 2.

3. Regularity Condition

Definition 3. An operator $A: \mathcal{D} \subset C^s(X, Y) \to C(Z, W)$, $0 < s \leqq \infty$, is said to be *HC-regular* or *C-regular* if any s-differentiably parametrized system of maps from $\mathcal{D}$ is transformed into a Hölder continuously or continuously parametrized system of maps in $C(Z, W)$, respectively.

It is well known that even operators invariant with respect to a large set of transformations and having finite order need not to be C-regular, see [3, p. 638]. On the other hand, dealing with operators some regularity condition is often assumed because of its geometrical character. But under this additional requirement, there is a stronger version of Theorem 2. For the sake of simplicity, the following theorem is formulated for operators defined on $C^\infty(X, Y)$, but it is clear from the proof how to get more general results.

Theorem 3. *Let X, Y be smooth manifolds, Z, W be metric spaces. Let $\pi \in C(Z, X)$ be locally non-constant and $A: C^\infty(X, Y) \to HC(Z, W)$ be a π-local and HC-regular operator. For any fixed map $f \in C^\infty(X, Y)$ and for any compact subset $K \subset Z$, there exist a natural number r and a neighbourhood $V \subset C^r(X, Y)$ of the map f in the compact C^r-topology such that for any point $z \in K$ and any smooth maps $g_1, g_2 \in V$, the condition $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$ implies $Ag_1(z) = Ag_2(z)$.*

Proof. Let us first deal with $X = \mathbf{R}^n$, $Y = \mathbf{R}^m$ and assume Theorem 3 does not hold for some A, f, K. We set

$$V_k = \{g \in C^k(\mathbf{R}^n, \mathbf{R}^m); \|D^\alpha(g - f)(x)\| < e^{-k}, 0 \leqq |\alpha| \leqq k, x \in \pi(K)\}$$

and we choose a sequence y_k in K and smooth maps $g_k, f_k \in V_k$ satisfying for all $k \in \mathbf{N}$

$$j^k g_k(\pi(y_k)) = j^k f_k(\pi(y_k)) \quad \text{and} \quad Ag_k(y_k) \neq Af_k(y_k).$$

By passing to subsequences, we may assume that the sequence y_k tends to $y \in K$ and $\|x_k - x\| \geqq 2\|x_{k+1} - x\|$, where $x_k = \pi(y_k)$, $x = \pi(y)$. In order to deduce a contradiction, we construct a smoothly parametrized system of maps $s(t, x)\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^m$ satisfying

$$j^\infty s(2^{-k}, -)(x_k) = j^\infty f_k(x_k), \qquad j^\infty s(0, -)(x) = j^\infty f(x). \tag{18}$$

As before, this is possible using Whitney extension theorem after defining $D^\alpha s(t, a) = 0$ whenever $\alpha = (\alpha_1, \dots, \alpha_{n+1})$, $\alpha_1 \neq 0$, and $(t, a) = (2^{-k}, x_k)$ or $(t, a) = (0, x)$.

Given $h \in C^\infty(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^m)$ and $(t, z) \in \mathbf{R} \times Z$, we set

$$\tilde{A}h(t, z) = A(h(t, -))(z).$$

Since A is HC-regular, we have defined an operator $\tilde{A}\colon C^\infty(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^m) \to HC(\mathbf{R} \times Z, W)$ in this way. Moreover $\tilde{A}$ is $(\mathrm{id}_{\mathbf{R}} \times \pi)$-local, so that we can apply Lemma 1 to the above constructed map s and to the point $(0, y) \in \mathbf{R} \times Z$. Consider maps $\tilde{g}_k\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^m$, $\tilde{g}_k(t, x) = g_k(x)$, $r \in \mathbf{N}$. We have $j^k s(2^{-k}, x_k) = j^k \tilde{g}_k(2^{-k}, x_k)$, so that Lemma 1, Theorem 1 and (18) imply

$$Af_k(y_k) = \tilde{A}s(2^{-k}, y_k) = \tilde{A}\tilde{g}_k(2^{-k}, y_k) = Ag_k(y_k)$$

for large k's. Hence we have proved Theorem 3 in the case $X = \mathbf{R}^n$, $Y = \mathbf{R}^m$.

The proof can be completed by the standard compactness arguments. Indeed, given f, K we take some open covers (U_i, φ_i) and (V_i, ψ_i) of X and Y by local coordinates in such a way that $f(U_i) \subset V_i$ and the images of open unit discs form open covers, too. Let $D_i \subset U_i$ be the images of closed unit discs and consider the compact sets $K_i = \pi^{-1}(D_i) \cap K$. We choose a finite cover $K_1, \dots, K_s$ of K, $K_1 \subset U_1, \dots, K_s \subset U_s$. Using the preceding part of the proof, we find some natural numbers $r_1, \dots, r_s$ and neighbourhoods of f in the compact C^{r_i}-topologies. Finally we take the maximum r of these orders and an intersection of suitable neighbourhoods of f in the compact C^r-topology.

4. Counter-Examples

Example 1. We construct an operator $A\colon C^\infty(\mathbf{R}, \mathbf{R}) \to C(\mathbf{R}, \mathbf{R})$ which essentially depends on infinite jets. Given $f \in C^\infty(\mathbf{R}, \mathbf{R})$ we set

$$Af(x) = \sum_{k=0}^{\infty} 2^{-k}(\arctan \circ (\mathrm{d}^k f/\mathrm{d}x^k)(x))$$

for all $x \in \mathbf{R}$.

Example 2. We present a translational invariant C-regular and $\mathrm{id}_{\mathbf{R}}$-local operator $A\colon C^\infty(\mathbf{R}, \mathbf{R}) \to C^\infty(\mathbf{R}, \mathbf{R})$ for which Theorem 3 does not hold. Let $g\colon \mathbf{R}^2 \to \mathbf{R}$ be a function with the following three properties:

(i) g is smooth in all points $x \in \mathbf{R}^2 \setminus \{(0, 1)\}$,

(ii) $\limsup_{x \to (0,1)} g(x) = \infty$,

(iii) g is identically zero on closed unit discs with centres in $(-1, 1)$ and $(1, 1)$.

Let $a: \mathbf{R}^2 \to \mathbf{R}$ be a smooth function satisfying $a(t, x) \neq 0$ if and only if $|x| > t > 0$.

Given $f \in C^\infty(\mathbf{R}, \mathbf{R})$, $x \in \mathbf{R}$, we set

$$Af(x) = f(x) \cdot \arctan\left(\sum_{k=0}^{\infty} \left(a(k, -) \circ g \circ \left(f \times \frac{df}{dx}\right)(x) \cdot \frac{d^k f}{dx^k}(x)\right)\right).$$

First of all, we have to prove $Af \in C^\infty(\mathbf{R},\mathbf{R})$. But the sum is locally finite if $g \circ (f \times (df/dx))$ is locally bounded. Hence Af is well defined and smooth if $g \circ (f \times (df/dx))$ is smooth. The only difficulty may happen if we deal with some $f \in C^\infty(\mathbf{R}, \mathbf{R})$ and $x \in \mathbf{R}$ satisfying $f(x) = 0$, $(df/dx)(x) = 1$. However, in this case we have

$$\lim_{y \to x} \frac{(df/dx)(y) - 1}{f(y)} = (d^2 f/dx^2)(x),$$

so that property (iii) of g implies $g \circ (f \times (df/dx)) = 0$ on a neighbourhood of x.

Now we are going to verify the C-regularity. Consider a manifold P and a continuously parametrized system of smooth maps $s: P \times \mathbf{R} \to \mathbf{R}$. We only have to be careful at limit points $(p, x) \in P \times \mathbf{R}$ with $s(p, x) = 0$, $(\partial s/\partial x)(p, x) = 1$. But there is an estimate

$$0 \leqq |A(s(q, -))(y)| \leqq \frac{\pi}{2} |s(q, y)|,$$

so that for any sequence (p_k, y_k) tending to (p, x) the values $A(s(p_k, -))(y_k)$ tend to $A(s(p, -))(x) = 0$.

It remains to show that Theorem 3 does not hold for the operator A. Consider $f \in C^\infty(\mathbf{R}, \mathbf{R})$ and $x \in \mathbf{R}$, $f(x) = 0$, $(df/dx)(x) = 1$. Given an arbitrary real number $\varepsilon > 0$ and order $k \in \mathbf{N}$, there are such functions h_1, $h_2 \in C^\infty(\mathbf{R}, \mathbf{R})$ that $j^k h_1(x) = j^k h_2(x)$, $\|D^\alpha(h_1 - f)(x)\| < \varepsilon$, $0 \leqq |\alpha| \leqq k$, and $Ah_1(x) \neq Ah_2(x)$. This is caused by property (ii) of the function g.

5. Remarks on Applications

Remark 3. We are going to discuss the relations to the classical Peetre theorem, [9, 11] (in base extending case see [5]). This follows from Theorem 2 in a simple way, as it is sufficient to prove Peetre theorem for trivial vector bundles. Hence in our setting we have to deal with $X = Z = \mathbf{R}^n$, $Y = \mathbf{R}^m$, $W = \mathbf{R}^k$, $\pi = \mathrm{id}_{\mathbf{R}^n}$ and a π-local $\mathbf{R}$-linear map $A: C^\infty(\mathbf{R}^n, \mathbf{R}^m) \to C^\infty(\mathbf{R}^n, \mathbf{R}^k)$ (π-local is equivalent to support non-increasing in this case). But Peetre theorem is then obtained by applying Theorem 2 to the identically zero map in $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ (we can even use $\varepsilon \equiv 0$, cf. [2]). Using our general theorem and homotheties, we also obtain a base extending multilinear version of Peetre theorem, cf. [1]. Indeed, the domain of a multilinear support non-increasing operator A is formed by p-tuples of maps $f_1 \in C^\infty(X_1, Y_1), \dots, f_p \in C^\infty(X_p, Y_p)$, where X_i, Y_i are vector spaces, and all these p-tuples clearly form an ∞-extendable subset $\mathscr{D} \subset C^\infty(X_1 \times \dots \times X_p, Y_1 \times \dots \times Y_p)$. Hence we can apply Theorem 2 to a π-local operator $A: \mathscr{D} \to C^\infty(Z, W)$ and to the p-tuple of identically zero maps. Chosen a compact set $K \subset Z$ we obtain some order r and function ε. If we take arbitrary p-tuples $(f_1, \dots, f_p)$, $(g_1, \dots, g_p)$ from $\mathscr{D}$ and a point $x \in \pi(K)$, $\varepsilon(x) > 0$, then all derivatives of f_i, g_i, $i = 1, \dots, p$, in the point x up to the order r can be sufficiently "pressed" to zero using a suitable homothety. Therefore, if $j^r f_i(x) = j^r g_i(x)$, $i = 1, \dots, p$, then for any $z \in K$, $\pi(z) = x$ and a suitable $c > 0$, $c \in \mathbf{R}$,

$$\begin{aligned} A(cf_1, \dots, cf_p)(z) &= c^p A(f_1, \dots, f_p)(z) = c^p A(g_1, \dots, g_p)(z) \\ &= A(cg_1, \dots, cg_p)(z) . \end{aligned}$$

But ε can be chosen in such a way that the set $\{x \in X; \varepsilon(x) = 0\}$ is discrete, so that the proof can be completed using the continuity of the values of the operator A in a similar way to the classical proof of Peetre theorem.

Remark 4. We shall indicate some possible applications to geometrical functors. In order to illustrate how to prove finite order theorems, we sketch an alternative proof of the well-known result on natural bundles, due to Palais, Terng, [10] and Epstein, Thurston, [4]. For the sake of simplicity, we shall investigate the smooth case only. That is to say, we consider a covariant functor F defined on the category $\mathscr{MF}_n$ of smooth n-dimensional manifolds and smooth local diffeomorphisms with the values in the category $\mathscr{MF}$ of smooth manifolds and smooth maps. Moreover, we assume that a natural transformation $\pi: F \to \mathrm{Id}_{\mathscr{MF}_n}$ is given and that the following locality axiom holds. If $i: U \to M$ is an inclusion of an open submanifold, then Fi gives a smooth diffeomorphism between FU and $\pi_M^{-1}(U)$. By virtue of this axiom, the functor is completely determined by its values on the group $\mathrm{Diff}(\mathbf{R}^n)$ of all diffeomorphisms $f: \mathbf{R}^n \to \mathbf{R}^n$. In order to profit from Theorem 2, we only have to realize that we deal with an η-local operator F: $\mathscr{D} \subset C^\infty(\mathbf{R}^n, \mathbf{R}^n) \to C^\infty(F\mathbf{R}^n, F\mathbf{R}^n)$, where the domain $\mathscr{D}$ is formed by all local diffeomorphisms and $\eta = \pi_{\mathbf{R}^n}$. Let us choose an arbitrary relatively compact open submanifold $U \subset F\mathbf{R}^n$. According to Theorem 2, there is a natural number r_U such that for any $f \in \mathscr{D}$, $z \in U$ the condition $j^r f(\eta(z)) = j^r \mathrm{id}_{\mathbf{R}^n}(\eta(z))$ implies $Ff(z) = z$. Because of the functoriality, the operator F induces an action of $\mathrm{Diff}(\mathbf{R}^n)$ on $F\mathbf{R}^n$. The orbit $V_U = \bigcup_{f \in \mathrm{Diff}(\mathbf{R}^n)} Ff(U)$ is an open submanifold of $F\mathbf{R}^n$ which is invariant under the action of $\mathrm{Diff}(\mathbf{R}^n)$. Consider $f, g \in \mathrm{Diff}(\mathbf{R}^n)$, $z \in V_U$, $\eta(z) = x$, $j^r f(x) = j^r g(x)$. Then there are some $h \in \mathrm{Diff}(\mathbf{R}^n)$ and $\bar{z} \in U$ such that $z = Fh(\bar{z})$. Hence $j^r(f \circ h)(\bar{x}) = j^r(g \circ h)(\bar{x})$ and so $f(h^{-1} \circ f^{-1} \circ g \circ h)(\bar{z}) = \bar{z}$. But this implies $Fg(z) = Ff(z)$. In this way we have proved that the induced action of $\mathrm{Diff}(\mathbf{R}^n)$ on the manifold V_U depends on r-jets only. Using the considerations from [8, sections 1 and 2], we deduce that this induced action is continuous and the proof can be completed using results due to A. Zajtz, [15]. Indeed, Zajtz has proved that any continuous action of the Lie group $G(n, k)$ (the group of invertible k-jets of maps $\mathbf{R}^n \to \mathbf{R}^n$

keeping zero fixed) on an m-dimensional manifold factorises to an action of $G(n, r)$, where

$$r \leqq \max\left\{\frac{m}{n-1}, \frac{m}{n}+1\right\} \quad \text{for} \quad n \geqq 2\,,$$

$$r \leqq 2m + 1 \qquad\qquad \text{for } n = 1\,.$$

Our way of reasoning seems to have some advantages. First of all we avoid any manipulation with infinite dimensional Lie groups and the proof of the "local finiteness" of the order does not need any continuity. Moreover, we can investigate functors defined on various categories, because only the ∞-extendability of the domain of the induced operator was essentially used.

Remark 5. Our general result can be applied to the study of geometrical operators. In particular, various types of natural operators between certain natural bundles can be proved to have finite order, see [12, 13]. This enables a complete classification of some types of natural operators without any assumption on the order. Results of this kind can be found in [6, 7].

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(Received January 15, 1988)