A Primer on Q-Curvature

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Disclaimer: These are rough notes only, aimed at setting the scene and promoting discussion at the American Institute of Mathematics Research Conference Center Workshop 'Conformal Structure in Geometry, Analysis, and Physics,' 12^{th}–16^{th} August 2003. For simplicity, we have omitted all references. Curvature conventions are in an appendix. Conversations with Tom Branson and Rod Gover have been extremely useful.

Let M be an oriented even-dimensional Riemannian n-manifold. Branson's Q-curvature is a canonically defined n-form on M. It is not conformally invariant but enjoys certain natural properties with respect to conformal transformations.

When $n = 2$, the Q-curvature is a multiple of the scalar curvature. With conventions as in the appendix $Q = -\frac{1}{2}R$. Under conformal rescaling of the metric, $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$ we have

$$\widehat{Q} = Q + \Delta \log \Omega,$$

where $\Delta = \nabla^a \nabla_a$ is the Laplacian.

When $n = 4$, the Q-curvature is given by

$$Q = \tfrac{1}{6}R^2 - \tfrac{1}{2}R^{ab}R_{ab} - \tfrac{1}{6}\Delta R. \tag{1}$$

Under conformal rescaling,

$$\widehat{Q} = Q + P \log \Omega,$$

where P is the Paneitz operator

$$Pf = \nabla_a \left[\nabla^a \nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab}\right] \nabla_b f. \tag{2}$$

For general even n, the Q-curvature transforms as follows:–

$$\widehat{Q} = Q + P \log \Omega, \tag{3}$$

where P is a linear differential operator from functions to n-forms whose symbol is $\Delta^{n/2}$. It follows from this transformation law that P is conformally invariant. To see this, suppose that

$$\widehat{g}_{ab} = \Omega^2 g_{ab} \quad \text{and} \quad \widehat{\widehat{g}}_{ab} = e^{2f}\widehat{g}_{ab} = (e^f\Omega)^2 g_{ab}.$$

Then

$$\widehat{\widehat{Q}} = \widehat{Q} + \widehat{P}\log e^f = Q + P\log\Omega + \widehat{P}f$$

but also

$$\widehat{\widehat{Q}} = Q + P\log(e^f\Omega) = Q + Pf + P\log\Omega.$$

Therefore, $\widehat{P}f = Pf$. With suitable normalisation, P is the celebrated Graham-Jenne-Mason-Sparling operator. Thus, Q may be regarded as more primitive than P and, therefore, is at least as mysterious.

Even when M is conformally flat, the existence of Q is quite subtle. We can reason as follows. When M is actually flat then Q must vanish. Therefore, in the conformally flat case, locally if we write $g_{ab} = \Omega^2\eta_{ab}$ where η_{ab} is flat, then (3) implies that

$$Q = \Delta^{n/2}\log\Omega, \tag{4}$$

where Δ is the ordinary Laplacian in Euclidean space with η_{ab} as metric. An immediate problem is to verify that this purported construction of Q is well-defined. The problem is that there is some freedom in writing g_{ab} as proportional to a flat metric. If also $g_{ab} = \widehat{\Omega}^2\widehat{\eta}_{ab}$, then we must show that

$$\Delta^{n/2}\log\Omega = \widehat{\Delta}^{n/2}\log\widehat{\Omega}.$$

This easily reduces to two facts:–

fact 1: $\Delta^{n/2}$ is conformally invariant on flat space.

fact 2: if g_{ab} is itself flat, then $\Delta^{n/2}\log\Omega = 0$.

The second of these is clearly necessary in order that (4) be well-defined. For $n = 2$ it is immediate from (17). For $n \geq 4$ it may be verified by direct calculation as follows. If g_{ab} and η_{ab} are both flat then

$$\nabla_a\Upsilon_b = \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon^c\Upsilon_c g_{ab}, \tag{5}$$

where $\Upsilon_a = \nabla_a\log\Omega$. Therefore,

$$\nabla_c(\Upsilon^a\Upsilon_a)^k = 2k(\Upsilon^a\Upsilon_a)^{k-1}\Upsilon^a\nabla_c\Upsilon_a = k(\Upsilon^a\Upsilon_a)^k\Upsilon_c$$

and

$$\nabla_b\nabla_c(\Upsilon^a\Upsilon_a)^k = k^2(\Upsilon^a\Upsilon_a)^k\Upsilon_b\Upsilon_c + k(\Upsilon^a\Upsilon_a)^k(\Upsilon_b\Upsilon_c - \tfrac{1}{2}\Upsilon^a\Upsilon_a g_{bc})$$

whence

$$\Delta(\Upsilon^a\Upsilon_a)^k = k(k+1-\tfrac{n}{2})(\Upsilon^a\Upsilon_a)^{k+1}. \tag{6}$$

Taking the trace of (5) gives

$$\Delta\log\Omega = \nabla^a\Upsilon_a = (1-\tfrac{n}{2})\Upsilon^a\Upsilon_a$$

and now (6) gives, by induction,

$$\Delta^{k+1}\log\Omega = k!(1-\tfrac{n}{2})(2-\tfrac{n}{2})\cdots(k+1-\tfrac{n}{2})(\Upsilon^a\Upsilon_a)^{k+1}.$$

In particular, $\Delta^{n/2}\log\Omega = 0$, as advertised.

That $\Delta^{n/2}$ is conformally invariant on flat space is well-known. It may also be verified directly by a rather similar calculation. For example, here is the calculation when $n = 4$. For general conformally related metrics $\widehat{g}_{ab} = \Omega^2 g_{ab}$ in dimension 4,

$$\begin{aligned}\widehat{\Delta}^2 f &= \Delta^2 f + 2\Upsilon^a\Delta\nabla_a f - 2\Upsilon^a\nabla_a\Delta f\\ &\quad + 4(\nabla^a\Upsilon^b)\nabla_a\nabla_b f - 2(\nabla^a\Upsilon_a)\Delta f - 4\Upsilon^a\Upsilon^b\nabla_a\nabla_b f\\ &\quad + 2(\Delta\Upsilon^a)\nabla_a f - 4(\nabla^a\Upsilon^b)\Upsilon_a\nabla_b f - 4(\nabla^a\Upsilon_a)\Upsilon^b\nabla_b f.\end{aligned}$$

If g_{ab} is flat then the third order terms cancel leaving

$$\begin{aligned}\widehat{\Delta}^2 f &= \Delta^2 f + 4(\nabla^a\Upsilon^b)\nabla_a\nabla_b f - 2(\nabla^a\Upsilon_a)\Delta f - 4\Upsilon^a\Upsilon^b\nabla_a\nabla_b f\\ &\quad + 2(\Delta\Upsilon^a)\nabla_a f - 4(\nabla^a\Upsilon^b)\Upsilon_a\nabla_b f - 4(\nabla^a\Upsilon_a)\Upsilon^b\nabla_b f.\end{aligned}$$

If $\widehat{g}_{ab}$ is also flat, then (5) implies

$$\nabla^a\Upsilon^b = \Upsilon^a\Upsilon^b - \tfrac{1}{2}\Upsilon^c\Upsilon_c g^{ab} \quad \text{and} \quad \nabla^a\Upsilon_a = -\Upsilon^a\Upsilon_a$$

whence the second order terms cancel and the first order ones simplify:–

$$\widehat{\Delta}^2 f = \Delta^2 f + 2(\Delta\Upsilon^b)\nabla_b f + 2\Upsilon^a\Upsilon_a\Upsilon^b\nabla_b f.$$

But using (5) again,

$$\begin{aligned}\Delta\Upsilon^b &= \nabla_a(\Upsilon^a\Upsilon^b - \tfrac{1}{2}\Upsilon^c\Upsilon_c g^{ab})\\ &= (\nabla^a\Upsilon_a)\Upsilon^b + (\nabla^a\Upsilon^b)\Upsilon_a - (\nabla^b\Upsilon^a)\Upsilon_a = -\Upsilon^a\Upsilon_a\Upsilon^b\end{aligned}$$

and the first order terms also cancel leaving $\widehat{\Delta}^2 f = \Delta^2 f$, as advertised.

Conundrum: Deduce **fact 2** from **fact 1** or vice versa. Both are consequences of (5). Alternatively, construct a Lie algebraic proof of **fact 2**. There is a Lie algebraic proof of **fact 1**. It corresponds to the existence of a homomorphism between certain generalised Verma modules for $\mathfrak{so}(n+1,1)$.

What about a formula for Q, even in the conformally flat case? We have a recipe for Q, namely (4), but it is not a formula. We may proceed as follows.

If $\widehat{g}_{ab} = \Omega g_{ab}$ and g_{ab} is flat, then (16) implies that

$$\nabla_a \Upsilon_b = -\widehat{\mathrm{P}}_{ab} + \Upsilon_a \Upsilon_b - \tfrac{1}{2} g_{ab} \Upsilon^c \Upsilon_c. \tag{7}$$

Taking the trace yields

$$\Delta \log \Omega = \nabla^a \Upsilon_a = -\widehat{\mathrm{P}} - \tfrac{1}{2}(n-2)\Upsilon^a \Upsilon_a. \tag{8}$$

This identity is also valid when $n = 2$: it is (17). Dropping the hat gives $Q = -\mathrm{P} = \frac{1}{2}R$. This is the simplest of the desired formulae.

To proceed further we need two identities. If ϕ has conformal weight w, then as described in the appendix,

$$\widehat{\nabla}_a \phi = \nabla_a \phi + w \Upsilon_a \phi,$$

which we rewrite as

$$\nabla_a \phi = \widehat{\nabla}_a \phi - w \Upsilon_a \phi. \tag{9}$$

Similarly, if ϕ_a has weight w, then

$$\nabla^a \phi_a = \widehat{\nabla}^a \phi_a - (n + w - 2)\Upsilon^a \phi_a \tag{10}$$

and, if ϕ_{ab} is symmetric and has weight w, then

$$\nabla^a \phi_{ab} = \widehat{\nabla}^a \phi_{ab} - (n + w - 2)\Upsilon^a \phi_{ab} + \Upsilon_b \phi^a{}_a. \tag{11}$$

The quantities in (8) have weight -2. Therefore, applying (9) gives

$$\nabla_a \Delta \log \Omega = -\widehat{\nabla}_a \widehat{\mathrm{P}} - 2\Upsilon_a \widehat{\mathrm{P}} - (n-2)\Upsilon^b \nabla_a \Upsilon_b$$

wherein we may use (7) to replace $\nabla_a \Upsilon_b$ to obtain

$$\nabla_a \Delta \log \Omega = -\widehat{\nabla}_a \widehat{\mathrm{P}} - 2\Upsilon_a \widehat{\mathrm{P}} + (n-2)\Upsilon^b \widehat{\mathrm{P}}_{ab} - \tfrac{1}{2}(n-2)\Upsilon_a \Upsilon^b \Upsilon_b.$$

We may now apply ∇^a, using (9), (10), and (11) to replace ∇^a by $\widehat{\nabla}^a$ on the right hand side and (7) to replace derivatives of Υ_a. We obtain an expression involving only complete contractions of $\widehat{\mathrm{P}}_{ab}$, its hatted derivatives, and Υ_a:–

$$\begin{aligned}\Delta^2 \log \Omega \;=\;& -\widehat{\Delta}\widehat{\mathrm{P}} - (n-2)\widehat{\mathrm{P}}^{ab}\widehat{\mathrm{P}}_{ab} + 2\widehat{\mathrm{P}}^2 \\ &+ (n-6)\Upsilon^a\widehat{\nabla}_a\widehat{\mathrm{P}} + (n-2)\Upsilon^a\widehat{\nabla}^b\widehat{\mathrm{P}}_{ab} + 2(n-4)\Upsilon^a\Upsilon_a\widehat{\mathrm{P}} \\ &- (n-2)(n-4)\Upsilon^a\Upsilon^b\widehat{\mathrm{P}}_{ab} + \tfrac{1}{4}(n-2)(n-4)\Upsilon^a\Upsilon_a\Upsilon^b\Upsilon_b.\end{aligned}$$

Using the Bianchi identity $\widehat{\nabla}^b\widehat{\mathrm{P}}_{ab} = \widehat{\nabla}_a\widehat{\mathrm{P}}$, we may rewrite this as

$$\begin{aligned}\Delta^2 \log \Omega \;=\;& -\widehat{\Delta}\widehat{\mathrm{P}} - (n-2)\widehat{\mathrm{P}}^{ab}\widehat{\mathrm{P}}_{ab} + 2\widehat{\mathrm{P}}^2 \\ &+ 2(n-4)\Upsilon^a\widehat{\nabla}_a\widehat{\mathrm{P}} + 2(n-4)\Upsilon^a\Upsilon_a\widehat{\mathrm{P}} \\ &- (n-2)(n-4)\Upsilon^a\Upsilon^b\widehat{\mathrm{P}}_{ab} + \tfrac{1}{4}(n-2)(n-4)\Upsilon^a\Upsilon_a\Upsilon^b\Upsilon_b\end{aligned} \tag{12}$$

and, in particular, conclude that when $n = 4$,

$$Q = 2\mathrm{P}^2 - 2\mathrm{P}^{ab}\mathrm{P}_{ab} - \Delta\mathrm{P}. \tag{13}$$

Though it is only guaranteed that this formula is valid in the conformally flat case, in fact it agrees with the general expression (1) in dimension 4.

Of course, we may continue in the vein, further differentiating (12) to obtain a formula for $\Delta^k \log \Omega$ expressed in terms of complete contractions of $\widehat{\mathrm{P}}_{ab}$, its hatted derivatives, and Υ_a. With increasing k, this gets rapidly out of hand. Moreover, it is only guaranteed to give Q in the conformally flat case. Indeed, when $n = 6$ this naïve derivation of Q fails for a general metric. Nevertheless, there are already some questions in the conformally flat case.

Conundrum: Find a formula for Q in the conformally flat case. Show that the procedure outlined above produces a formula for Q.

In fact, there is a tractor formula for the conformally flat Q. This is not the place to explain the tractor calculus but, for those who know it already:–

$$\Box \begin{bmatrix} n-2 \\ 0 \\ -\mathrm{P} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix}$$

where

$$\Box = D_A \cdots D_B(\Delta - \frac{n-2}{4(n-1)}R)\underbrace{D^B \cdots D^A}_{(n-4)/2}.$$

Unfortunately, this formula hides a lot of detail and does not seem to be of much immediate use. It is not valid in the curved case.

Recall that, like Q, the Pfaffian is an n-form canonically associated to a Riemannian metric on an oriented manifold in even dimensions. It is defined as a complete contraction of $n/2$ copies of the Riemann tensor with two copies of the volume form. For example, in dimension four it is

$$E = \epsilon^{abpq}\epsilon^{cdrs}R_{abcd}R_{pqrs},$$

where ϵ_{abcd} is the volume form normalised, for example, so that

$$\epsilon^{abcd}\epsilon_{abcd} = 4! = 24.$$

Therefore, in four dimensions,

$$\begin{aligned}
E &= 4R_{ab}{}^{ab}R_{cd}{}^{cd} - 16R_{ab}{}^{ac}R_{cd}{}^{bd} + 4R_{ab}{}^{cd}R_{cd}{}^{ab} \\
&= 4R^2 - 16R_b{}^cR_c{}^b + 4C_{abcd}C^{abcd} + 32\mathrm{P}^{ab}\mathrm{P}_{ab} + 16\mathrm{P}^2 \\
&= 144\mathrm{P}^2 - 16(4\mathrm{P}^{ab}\mathrm{P}_{ab} + 8\mathrm{P}^2) + 4C_{abcd}C^{abcd} + 32\mathrm{P}^{ab}\mathrm{P}_{ab} + 16\mathrm{P}^2 \\
&= 32\mathrm{P}^2 - 32\mathrm{P}^{ab}\mathrm{P}_{ab} + 4C_{abcd}C^{abcd}.
\end{aligned}$$

The integral of the Pfaffian on a compact manifold is a multiple of the Euler characteristic. In dimension 4, for example,

$$\int_M E = 128\pi^2\,\chi(M).$$

Notice the simple relationship between Q and E in dimension 4:–

$$Q = \tfrac{1}{16}E - \tfrac{1}{4}C^{abcd}C_{abcd} - \Delta\mathrm{P}.$$

Of course, it follows from (3) that $\int_M Q$ is a conformal invariant. Also, in the conformally flat case, it follows from a theorem of Branson, Gilkey, and Pohjanpelto that Q must be a multiple of the Pfaffian plus a divergence. However, the link between Q and the Pfaffian is extremely mysterious.

Conundrum: Find a direct link between Q and the Pfaffian in the conformally flat case. Prove directly that $\int_M Q$ is a topological invariant in this case.

Conundrum: Is it true that, on a general Riemannian manifold, Q may be written as a multiple of the Pfaffian plus a local conformal invariant plus a divergence?

Recall the conventions for Weyl structures as in the appendix. In particular, a metric in the conformal class determines a 1-form α_a. In fact, a Weyl structure may be regarded as a pair (g_{ab}, α_a) subject to equivalence under the simultaneous replacements

$$g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab} \quad \text{and} \quad \alpha_a \mapsto \widehat{\alpha}_a = \alpha_a + \Upsilon_a \quad \text{where } \Upsilon_a = \nabla_a \Omega.$$

A Riemannian structure induces a Weyl structure by taking the equivalence class with $\alpha_a = 0$ but not all Weyl structures arise in this way. A Weyl structure gives rise to a conformal structure by discarding α_a. We may ask how Q-curvature is related to Weyl structures. From the transformation property (3), it follows that Q may be defined for a Weyl structure as follows. Since Q is a Riemannian invariant, the differential operator P is necessarily of the form $f \mapsto S^a \nabla_a f$ for some Riemannian invariant linear differential operator from 1-forms to n-forms. Now, if $[g_{ab}, \alpha_a]$ is a Weyl structure, choose a representative metric g_{ab} and consider the n-form

$$Q - S^a \alpha_a,$$

where Q is the Riemannian Q-curvature associated to g_{ab} and α_a is the 1-form associated to g_{ab}. If $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then

$$\begin{aligned} \widehat{Q} - \widehat{S}^a \widehat{\alpha}_a &= Q + S^a \Upsilon_a - \widehat{S}^a \alpha_a - \widehat{S}^a \Upsilon_a \\ &= Q + P \log \Omega - \widehat{S}^a \alpha_a - \widehat{P} \log \Omega \\ &= Q - \widehat{S}^a \alpha_a. \end{aligned} \tag{14}$$

In dimension 4 we can proceed further as follows. From (2) we see that

$$S^a \alpha_a = \nabla_b \left[\nabla^b \nabla^a + 2R^{ab} - \tfrac{2}{3} R g^{ab} \right] \alpha_a = \nabla_b \left[\nabla^b \nabla^a + 4\mathrm{P}^{ab} - 2\mathrm{P} g^{ab} \right] \alpha_a$$

and so we may calculate

$$\widehat{S}^a \alpha_a = S^a \alpha_a + 4 \nabla^a (\Upsilon^b \nabla_{[a} \alpha_{b]}).$$

In combination with (14) we obtain

$$\widehat{Q} - \widehat{S}^a \widehat{\alpha}_a = Q - S^a \alpha_a - 4 \nabla^a (\Upsilon^b \nabla_{[a} \alpha_{b]}).$$

However,

$$\widehat{\nabla}^a(\widehat{\alpha}^b\widehat{\nabla}_{[a}\widehat{\alpha}_{b]}) = \nabla^a(\widehat{\alpha}^b\nabla_{[a}\alpha_{b]}) = \nabla^a(\alpha^b\nabla_{[a}\alpha_{b]}) + \nabla^a(\Upsilon^b\nabla_{[a}\alpha_{b]})$$

and, therefore,

$$\mathbf{Q} = Q - S^a\alpha_a + 4\nabla^a(\alpha^b\nabla_{[a}\alpha_{b]}) \tag{15}$$

is an invariant of the Weyl structure that agrees with Q when the Weyl structure arises from a Riemannian structure.

Conundrum: Can we find such a $\mathbf{Q}$ in general even dimensions? Presumably, this would restrict the choice of Riemannian Q.

Though $\mathbf{Q}$ given by (15) is an invariant of the Weyl structure, it is not manifestly so. Better is to rewrite it as follows. Using conventions from the appendix, we may write the Schouten tensor (18) of the Weyl structure in terms of the Schouten tensor of a representative metric g_{ab}:–

$$\mathbf{P}_{ab} = \mathrm{P}_{ab} + \nabla_a\alpha_b + \alpha_a\alpha_b - \tfrac{1}{2}\alpha^c\alpha_c g_{ab}.$$

In particular,

$$\begin{aligned}
\mathbf{P} &= \mathrm{P} + \nabla^a\alpha_a - \alpha^a\alpha_a \\
\mathbf{P}^{ab}\mathbf{P}_{ab} &= \mathrm{P}^{ab}\mathrm{P}_{ab} + (\nabla^a\alpha^b)(\nabla_a\alpha_b) + (\alpha^a\alpha_a)^2 + 2\mathrm{P}^{ab}\nabla_a\alpha_b \\
&\quad\; + 2\mathrm{P}^{ab}\alpha_a\alpha_b - \mathrm{P}\alpha^a\alpha_a + 2(\nabla^a\alpha^b)\alpha_a\alpha_b - (\nabla^a\alpha_a)\alpha^b\alpha_b \\
D^aD_a\mathbf{P} &= D^a(\nabla_a\mathbf{P} + 2\alpha_a\mathbf{P}) = \nabla^a(\nabla_a\mathbf{P} + 2\alpha_a\mathbf{P}) \\
&= \Delta\mathbf{P} + 2(\nabla^a\alpha_a)\mathbf{P} + 2\alpha_a\nabla^a\mathbf{P} \\
&= \Delta\mathrm{P} + \Delta\nabla^b\alpha_b - 2(\Delta\alpha^b)\alpha_b - 2(\nabla^a\alpha^b)(\nabla_a\alpha_b) \\
&\quad\; + 2(\nabla^a\alpha_a)\mathrm{P} + 2(\nabla^a\alpha_a)\nabla^b\alpha_b - 2(\nabla^a\alpha_a)\alpha^b\alpha_b \\
&\quad\quad + 2\alpha_a\nabla^a\mathrm{P} + 2\alpha_a\nabla^a\nabla^b\alpha_b - 4\alpha_a(\nabla^a\alpha^b)\alpha_b \\
\mathbf{P}^2 &= \mathrm{P}^2 + (\nabla^a\alpha_a)^2 + (\alpha^a\alpha_a)^2 \\
&\quad\; + 2\mathrm{P}\nabla^a\alpha_a - 2\mathrm{P}\alpha^a\alpha_a - 2(\nabla^a\alpha_a)(\alpha^b\alpha_b).
\end{aligned}$$

Therefore, recalling the formula (13) for Q in dimension 4,

$$\begin{aligned}
Q = 2\mathbf{P}^2 &- 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} \\
&+ 4\mathrm{P}^{ab}\nabla_a\alpha_b + 4\mathrm{P}^{ab}\alpha_a\alpha_b + \Delta\nabla^b\alpha_b - 2(\Delta\alpha^b)\alpha_b \\
&\quad + 2\alpha_a\nabla^a\mathrm{P} + 2\alpha_a\nabla^a\nabla^b\alpha_b - 2\mathrm{P}\nabla^a\alpha_a + 2\mathrm{P}\alpha^a\alpha_a
\end{aligned}$$

whence, from (15),

$$\begin{aligned}
\mathbf{Q} &= 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} \\
&\qquad + 4\mathrm{P}^{ab}\nabla_a\alpha_b + 4\mathrm{P}^{ab}\alpha_a\alpha_b + \Delta\nabla^b\alpha_b - 2(\Delta\alpha^b)\alpha_b \\
&\qquad\quad + 2\alpha_a\nabla^a\mathrm{P} + 2\alpha_a\nabla^a\nabla^b\alpha_b - 2\mathrm{P}\nabla^a\alpha_a + 2\mathrm{P}\alpha^a\alpha_a \\
&\qquad\qquad - \nabla_b\left[\nabla^b\nabla^a + 4\mathrm{P}^{ab} - 2\mathrm{P}g^{ab}\right]\alpha_a + 4\nabla^a(\alpha^b\nabla_{[a}\alpha_{b]}) \\
&= 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} \\
&\qquad + 4\mathrm{P}^{ab}\alpha_a\alpha_b + 2\alpha_a(\nabla^a\nabla^b - \nabla^b\nabla^a)\alpha_b + 2\mathrm{P}\alpha^a\alpha_a \\
&\qquad\quad + 2(\nabla^a\alpha^b)\nabla_a\alpha_b - 2(\nabla^a\alpha^b)\nabla_b\alpha_a \\
&= 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} + 2(\nabla^a\alpha^b)\nabla_a\alpha_b - 2(\nabla^a\alpha^b)\nabla_b\alpha_a.
\end{aligned}$$

However,

$$2(\nabla^a\alpha^b)\nabla_a\alpha_b - 2(\nabla^a\alpha^b)\nabla_b\alpha_a = 4(\nabla^{[a}\alpha^{b]})\nabla_{[a}\alpha_{b]} = 4\mathbf{P}^{ab}\mathbf{P}_{[ab]}$$

and so

$$\mathbf{Q} = 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ba} - D^aD_a\mathbf{P}$$

a manifest invariant of the Weyl structure, as required.

Conundrum: Did we really need to go through this detailed calculation? What are the implications, if any, for the operator S : 1-forms $\to$ 4-forms?

Conundrum: Can we characterise the Riemannian Q by sufficiently many properties? Do Weyl structures help in this regard?

Tom Branson has suggested that, for two metrics g and $\widehat{g} = \Omega^2 g$ in the same conformal class on a compact manifold M, one should consider the quantity

$$\mathcal{H}[\widehat{g}, g] = \int_M (\log\Omega)(\widehat{Q} + Q).$$

That it is a cocycle,

$$\mathcal{H}[\widehat{\widehat{g}}, \widehat{g}] + \mathcal{H}[\widehat{g}, g] = \mathcal{H}[\widehat{\widehat{g}}, g],$$

is easily seen to be equivalent to the GJMS operators P being self-adjoint.

Conundrum: Are there any deeper properties of Branson's cocycle $\mathcal{H}[\widehat{g}, g]$?

One possible rôle for Q is in a curvature prescription problem:–

Conundrum: On a given manifold M, can one find a metric with specified Q?

One can also ask this question within a given conformal class or within the realm of conformally flat metrics though, of course, if M is compact, then $\int_M Q$ must be as specified by the conformal class and the topology of M. There is also the question of uniqueness:–

Conundrum: When does Q determine the metric up to constant rescaling within a given conformal class?

Since we know how Q changes under conformal rescaling (3), this question is equivalent to

Conundrum: When does the equation $Pf = 0$ have only constant solutions?

On a compact manifold in two dimensions this is always true: harmonic functions are constant. In four dimensions, though there are conditions under which $Pf = 0$ has only constant solutions, there are also counterexamples, even on conformally flat manifolds. The following counterexample is due to Michael Singer and the first author. Consider the metric in local coördinates

$$\frac{dx^2 + dy^2}{(x^2 + y^2 + 1)^2} + \frac{ds^2 + dt^2}{(s^2 + t^2 - 1)^2}.$$

It is easily verified that it is conformally flat, scalar flat, and has

$$R_{ab} = 4\frac{dx^2 + dy^2}{(x^2 + y^2 + 1)^2} - 4\frac{ds^2 + dt^2}{(s^2 + t^2 - 1)^2}.$$

From (2) we see that if f is a function of (x, y) alone, then $Pf = L(L+8)f$, where L is the Laplacian for the two-dimensional metric

$$\frac{dx^2 + dy^2}{(x^2 + y^2 + 1)^2}.$$

More specifically, in these local coördinates

$$L = (x^2 + y^2 + 1)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

It is easily verified that $L+8$ annihilates the following functions:–

$$\frac{x}{x^2+y^2+1}, \qquad \frac{y}{x^2+y^2+1}, \qquad \frac{x^2+y^2-1}{x^2+y^2+1}.$$

In fact, (x,y) are stereographic coördinates on the sphere and these three functions extend to the sphere to span the spherical harmonics of minimal non-zero energy. On then other hand, the metric

$$\frac{ds^2+dt^2}{(s^2+t^2-1)^2}$$

is the hyperbolic metric on the disc. We conclude that the Paneitz operator has at least a 4-dimensional kernel on $S^2 \times H^2$. The same conclusion applies to $S^2 \times \Sigma$ where Σ is any Riemann surface of genus ≥ 2 equipped with constant curvature metric as a quotient of H^2. (In fact, the dimension in this case is exactly 4.)

APPENDIX: Curvature Conventions

Firstly, our conventions for conformal weight. A density f of conformal weight w may be identified as a function for any metric in the conformal class. At the risk of confusion, we shall also write this function as f. If however, our choice of metric g_{ab} is replaced by a conformally equivalent $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then the function f is replaced by $\widehat{f} = \Omega^w f$. Quantities that are not conformally invariant can still have a conformal weight with respect to constant rescalings. For example, the scalar curvature has weight -2 in this respect. Explicit conformal rescalings are generally suppressed.

The Riemann curvature is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = R_{abc}{}^d \omega_d.$$

The Ricci and scalar curvatures are

$$R_{ac} = R_{abc}{}^b \quad \text{and} \quad R = R^a{}_a,$$

respectively. The Schouten tensor is

$$\mathrm{P}_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)} g_{ab}\right)$$

and transforms under conformal rescaling by

$$\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2} g_{ab} \Upsilon^c \Upsilon_c. \tag{16}$$

In particular, if $\widehat{\eta}_{ab} = \Omega^2 \eta_{ab}$ are two flat metrics, then

$$\nabla_a \Upsilon_b = \Upsilon_a \Upsilon_b - \tfrac{1}{2} g_{ab} \Upsilon^c \Upsilon_c,$$

a tensor version of the Riccati equation. When $n = 2$, the Schouten tensor itself is not defined but its trace is well-defined:–

$$\mathrm{P} = \tfrac{1}{2} R \qquad \widehat{\mathrm{P}} = \mathrm{P} - \nabla^a \Upsilon_a = \mathrm{P} - \Delta \log \Omega \tag{17}$$

and so, if $\widehat{\eta}_{ab} = \Omega^2 \eta_{ab}$ are two flat metrics, then $\Delta \log \Omega = 0$.

A Weyl structure is a conformal structure together with a choice of torsion-free connection D_α preserving the conformal structure. In other words, if we choose a metric g_{ab} in the conformal class, then

$$D_a g_{bc} = 2\alpha_a g_{bc},$$

determining a smooth 1-form α_a. Conversely, α_a determines D_a:

$$D_a \phi_b = \nabla_a \phi_b + \alpha_a \phi_b + \alpha_b \phi_a - \alpha^c \phi_c g_{ab},$$

where ∇_a is the Levi-Civita connection for the metric g_{ab}. Let W_{ab} denote the Ricci curvature of the connection D_a:–

$$(D_a D_b - D_b D_a) V^c = W_{ab}{}^c{}_d V^d \qquad W_{ab} = W_{ca}{}^c{}_b.$$

We may compute these curvatures in terms of α_a and ∇_a, for a chosen metric in the conformal class:–

$$\begin{aligned} D_a D_b V^c \;=\; & \nabla_a(\nabla_b V^c - \alpha_b V^c + \alpha^c V_b - \alpha_d V^d \delta_b{}^c) \\ & + \alpha_a(\nabla_b V^c - \alpha_b V^c + \alpha^c V_b - \alpha_d V^d \delta_b{}^c) \\ & + \alpha_b(\nabla_a V^c - \alpha_a V^c + \alpha^c V_a - \alpha_d V^d \delta_a{}^c) \\ & + \alpha^e(\nabla_e V^c - \alpha_e V^c + \alpha^c V_e - \alpha_d V^d \delta_e{}^c) g_{ab} \\ & - \alpha_a(\nabla_b V^c - \alpha_b V^c + \alpha^c V_b - \alpha_d V^d \delta_b{}^c) \\ & + \alpha^c(\nabla_b V_a - \alpha_b V_a + \alpha_a V_b - \alpha_d V^d g_{ba}) \\ & - \alpha_e(\nabla_b V^e - \alpha_b V^e + \alpha^e V_b - \alpha_d V^d \delta_b{}^e) \delta_a{}^c \end{aligned}$$

so

$$\begin{aligned}(D_aD_b - D_bD_a)V^c &= (\nabla_a\nabla_b - \nabla_b\nabla_a)V^c \\ &\quad - (\nabla_a\alpha_b)V^c + (\nabla_a\alpha^c)V_b - (\nabla_a\alpha_d)V^d\delta_b{}^c \\ &\quad + (\nabla_b\alpha_a)V^c - (\nabla_b\alpha^c)V_a + (\nabla_b\alpha_d)V^d\delta_a{}^c \\ &\quad + \alpha^c\alpha_aV_b + \alpha_b\delta_a{}^c\alpha_eV^e - \alpha_e\alpha^e\delta_a{}^cV_b \\ &\quad - \alpha^c\alpha_bV_a - \alpha_a\delta_b{}^c\alpha_eV^e + \alpha_e\alpha^e\delta_b{}^cV_a\end{aligned}$$

whence

$$\begin{aligned}W_{ab}{}^c{}_d &= R_{ab}{}^c{}_d - 2\delta^c{}_d\nabla_{[a}\alpha_{b]} - 2g_{d[a}\nabla_{b]}\alpha^c + 2\delta_{[a}{}^c\nabla_{b]}\alpha_d \\ &\quad + 2\alpha^c\alpha_{[a}g_{b]d} + 2\delta_{[a}{}^c\alpha_{b]}\alpha_d - 2\alpha_e\alpha^e\delta_{[a}{}^cg_{b]d}\end{aligned}$$

and

$$W_{ab} = R_{ab} + (n-1)\nabla_a\alpha_b - \nabla_b\alpha_a + g_{ab}\nabla^c\alpha_c + (n-2)\alpha_a\alpha_b - (n-2)\alpha^c\alpha_cg_{ab}$$

whose trace is

$$W = R + 2(n-1)\nabla^c\alpha_c - (n-1)(n-2)\alpha^c\alpha_c.$$

Therefore,

$$\frac{1}{n-2}\left(W_{ab} - \frac{W}{2(n-1)}g_{ab}\right) = \mathrm{P}_{ab} + \nabla_a\alpha_b + \alpha_a\alpha_b - \tfrac{1}{2}\alpha^c\alpha_cg_{ab} + \tfrac{2}{n-2}\nabla_{[a}\alpha_{b]}.$$

If two Weyl structures have the same underlying conformal structure, then we may, without loss of generality, represent them as (g_{ab}, α_a) and $(g_{ab}, \alpha_a - \Upsilon_a)$ for the same metric g_{ab} and an arbitrary 1-form Υ_a. If we write hatted quantities to denote those computed with respect to $(g_{ab}, \alpha_a - \Upsilon_a)$, then for

$$\mathbf{P}_{ab} = \frac{1}{n-2}\left(W_{ab} - \frac{2}{n}W_{[ab]} - \frac{W}{2(n-1)}g_{ab}\right) \tag{18}$$

we have the convenient transformation law

$$\widehat{\mathbf{P}}_{ab} = \mathbf{P}_{ab} - D_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}g_{ab}\Upsilon^c\Upsilon_c. \tag{19}$$

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