

GENERALIZED PLANAR CURVES AND QUATERNIONIC GEOMETRY

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ABSTRACT. Motivated by the analogies between the projective and the almost quaternionic geometries, we first study the generalized planar curves and mappings. We follow, recover, and extend the classical approach, see e.g. [9, 10]. Then we exploit the impact of the general results in the almost quaternionic geometry. In particular we show, that the natural class of \mathbb{H} -planar curves coincides with the class of all geodesics of the so called Weyl connections and preserving this class turns out to be the necessary and sufficient condition on diffeomorphisms to become morphisms of almost quaternionic geometries.

Various concepts generalizing geodesics of affine connections have been studied for almost quaternionic and similar geometries. Let us point out the generalized geodesics defined via generalizations of normal coordinates, cf. [2] and [3], or more recent [4, 11]. Another class of curves was studied in [10] for the hypercomplex structures with additional linear connections. The latter authors called a curve c quaternionic planar if the parallel transport of each of its tangent vectors $\dot{c}(t_0)$ along c was quaternionic colinear with the tangent field \dot{c} to the curve. Yet another natural class of curves is given by the set of all unparameterized geodesics of the so called Weyl connections, i.e. the connections compatible with the almost quaternionic structure with normalized minimal torsion. The latter connections have remarkably similar properties for all parabolic geometries, cf. [3], and so their name has been borrowed from the conformal case. In the setting of almost quaternionic structures, they were studied first in [7] and so they are also called Oproiu connections, see [1].

The first author showed in [6] that actually the concept of quaternionic planar curves was well defined for the almost quaternionic geometries and their Weyl connections. Moreover, it did not depend on the choice of a particular Weyl connection and it turned out that the quaternionic planar curves were just all unparameterized geodesics of all Weyl connections.

The aim of this paper is to find further analogies of Mikeš' classical results in the realm of the almost quaternionic geometry. On the way we

simplify, recover, and extend results on generalized planar mappings, explain results from [6], and finally we show that morphisms of almost quaternionic geometries are just those diffeomorphisms which leave invariant the class of all unparameterized geodesics of Weyl connections.

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1. MOTIVATION AND BACKGROUND ON QUATERNIONIC GEOMETRY

There are many equivalent definitions of almost quaternionic geometry to be found in the literature. Let us start with the following one:

Definition 1.1. *Let M be a smooth manifold of dimension $4n$. An almost hypercomplex structure on M is a triple (I, J, K) of smooth affinors in $\Gamma(T^*M \otimes TM)$ satisfying*

$$I^2 = J^2 = -E, \quad K = I \circ J = -J \circ I$$

where $E = \text{id}_{TM}$.

An almost quaternionic structure is a rank four subbundle $\mathcal{Q} \subset T^*M \otimes TM$ locally generated by the identity E and a hypercomplex structure. An (almost) quaternionic manifold is a manifold equipped with an (almost) quaternionic structure.

An almost complex geometry on a $2m$ -dimensional manifold M is given by the choice of the affnor J satisfying $J^2 = -E$. Let us observe, that such a J is uniquely determined within the rank two subbundle $\langle E, J \rangle \subset TM$, up to its sign. Indeed, if $\hat{J} = aE + bJ$, then the condition $\hat{J}^2 = -E$ implies $a = 0$ and $b = \pm 1$.

Thus we may view the almost quaternionic geometry as a straightforward generalization of this case. Here, a similar simple computation reveals that the rank three subbundle $\langle I, J, K \rangle$ is invariant of the choice of the generators and this is the definition we may find in [1]. More explicitly, different choices will always satisfy $\hat{I} = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$, and similarly for J and K . Let us also remark that the 4-dimensional almost quaternionic geometry coincides with 4-dimensional conformal Riemannian geometries.

1.2. The frame bundles. Equivalently, we can define an almost quaternionic structure \mathcal{Q} on M as a reduction of the linear frame bundle P^1M

to an appropriate structure group, i.e. as a G -structure with the structure group of all automorphisms preserving the subbundle \mathcal{Q} . We may view such frames as linear mappings $T_x M \rightarrow \mathbb{H}^n$ which carry over the multiplications by $i, j, k \in \mathbb{H}$ onto some of the possible choices for I, J, K . Thus, a further reduction to a fixed hypercomplex structure leads to the structure group $GL(n, \mathbb{H})$ of all quaternionic linear mappings on \mathbb{H}^n . Additionally, we have to allow morphisms which do not leave the affinors I, J, K invariant but change them within the subbundle \mathcal{Q} . As well known, the resulting group is

$$(1) \quad G_0 = GL(n, \mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$$

where $Sp(1)$ are the unit quaternions in $GL(1, \mathbb{H})$, see e.g. [8].

We shall write $\mathcal{G}_0 \subset P^1 M$ for this principal G_0 -bundle defining our structure.

The simplest example of such a structure is well understood as the homogeneous space

$$\mathbb{P}_n \mathbb{H} = G/P$$

where $G_0 \subset P$ are the subgroups in $G = SL(n + 1, \mathbb{H})$

$$(2) \quad G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}; A \in GL(n, \mathbb{H}), \operatorname{Re}(a \det A) = 1 \right\},$$

$$(3) \quad P = \left\{ \begin{pmatrix} a & Z \\ 0 & A \end{pmatrix}; \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in G_0, Z \in (\mathbb{H}^n)^* \right\}.$$

Since P is a parabolic subgroup in the semisimple Lie group G , the almost quaternionic geometry is an instance of the so called parabolic geometries. All these geometries enjoy a rich and quite uniform theory similar to the classical development of the conformal Riemannian and projective geometries, but we shall not need much of this here. We refer the reader to [3] and the references therein.

1.3. Weyl connections. The classical prolongation procedure for G -structures starts with finding a minimal available torsion for a connection belonging to the structure on the given manifold M . Unlike the projective and conformal Riemannian structures where torsion free connections always exist, the torsion has to be allowed for the almost quaternionic structures in general in dimensions bigger than four. The standard normalization comes from the general theory of parabolic geometries and we shall not need this in the sequel. The details may be found for example in [5], [2], for another and more classical point of view see [8]. The only essential point for us will be that all connections compatible with the given geometry sharing the unique normalized torsion are parameterized by smooth one-forms on M . In analogy to the

conformal Riemannian geometry we call them *Weyl connections* for the given almost quaternionic geometry on M (although the name Oproiu connections has been used too, see e.g. [7] and [1]).

The almost quaternionic structures on a manifold M admitting a linear connection compatible with the structure and without torsion are called *quaternionic structures*. In this case, the Weyl connections on M are just all connections without torsion and compatible with the G -structure. In general, the Weyl connections are all compatible connections sharing the unique normalized torsion. The normalization is easily described either at the complexified tangent bundle $TM \otimes \mathbb{C} = S \otimes S'$ or in the cohomological language. In the first case the torsion has to be completely tracefree, the other description reads that the torsion has to be a harmonic chain of homogeneity one with respect to the Lie algebra cohomology $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$.

From the point of view of prolongations of G -structures, the class of all Weyl connections defines a reduction \mathcal{G} of the semiholonomic second order frame bundle over the manifold M to a principal subbundle with the structure group P , while the individual Weyl connections represent further reductions of \mathcal{G} to principal subbundles $\mathcal{G}_0 \subset \mathcal{G}$ with the structure group $G_0 \subset P$. Thus, the Weyl connections ∇ are in bijective correspondence with G_0 -equivariant sections $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$ of the natural projection.

As mentioned above, the difference of two Weyl connections is naturally identified with an one-form and we shall explain this in 1.4 below. In full analogy to the conformal geometry, there are neat formulae for the change of the covariant derivatives of two such connections $\hat{\nabla}$ and ∇ in terms of their difference $\Upsilon = \hat{\nabla} - \nabla$. As we shall see below, there is a natural identification of such a difference with a one-form denoted by the same symbol $\Upsilon \in \Omega^1(M)$.

1.4. Adjoint tractors. In order to understand the latter formulae, we introduce the so called adjoint tractors. They are sections of the vector bundle (called usually *adjoint tractor bundle*) over M

$$\mathcal{A} = \mathcal{G}_0 \times_{G_0} \mathfrak{g},$$

see (1) through (3) for the definition of the groups. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$ carries the G_0 -invariant grading

$$(4) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}; A \in \mathfrak{gl}(n, \mathbb{H}), a \in \mathbb{H} \operatorname{Re}(a + \operatorname{Tr} A) = 0 \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}; Z \in (\mathbb{H}^n)^* \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}; X \in \mathbb{H}^n \right\}.$$

Moreover, $TM = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$, $T^*M = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$, and we obtain on the level of vector bundles

$$\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 = TM \oplus \mathcal{A}_0 \oplus T^*M$$

where $\mathcal{A}_0 = \mathcal{G}_0 \times \mathfrak{g}_0$ is the adjoint bundle of the Lie algebra \mathfrak{g}_0 .

The key feature of \mathcal{A} is that all further G_0 -invariant objects on \mathfrak{g} are carried over to the adjoint tractors, too. In particular, the Lie bracket on G induces an algebraic bracket $\{ , \}$ on \mathcal{A} such that each fibre has the structure of the graded Lie algebra (4).

Now we may write down easily the transformation formula. Let $\hat{\nabla}$ and ∇ be two Weyl connections, $\hat{\nabla} - \nabla = \Upsilon \in \Gamma(\mathcal{A}_1)$. Then for all tangent vector fields $X, Y \in \Gamma(\mathcal{A}_{-1})$, i.e. sections of the \mathfrak{g}_{-1} -component of the adjoint tractor bundle,

$$(5) \quad \hat{\nabla}_X Y = \nabla_X Y + \{ \{ X, \Upsilon \}, Y \} = \nabla_X Y + \Upsilon(X)Y,$$

where $\Upsilon(X) \in \mathfrak{g}_0 = \mathfrak{sp}(1) + \mathfrak{gl}(n, \mathbb{H})$, see [3] or [2, 5] for the proof. Notice that the internal bracket results in an endomorphism on TM , while the external bracket is just the evaluation of this endomorphism on Y (all this is read off the brackets in the Lie algebra easily).

2. GENERALIZED PLANAR CURVES AND MAPPINGS

Various geometric structures on manifolds are defined as smooth distributions in the vector bundle $T^*M \otimes TM$ of all endomorphisms of the tangent bundle. We have seen the two examples of almost complex and almost quaternionic structures above. Let us extract some formal properties from these examples.

Definition 2.1. *Let A be a smooth ℓ -dimensional vector subbundle in $T^*M \otimes TM$, such that the identity affinor $E = \operatorname{id}_{TM}$ restricted to $T_x M$ belongs to $A_x M \subset T_x^* M \otimes T_x M$ at each point $x \in M$. We say that M is equipped by an A -structure.*

For any tangent vector $X \in T_x M$ we shall write $A(X)$ for the vector subspace

$$A(X) = \{ F(X); F \in A_x M \} \subset T_x M$$

and we call $A(X)$ the A -hull of the vector X . Similarly, the A -hull of a vector field will be the subbundle in TM obtained pointwise. Notice

that the dimension of such a subbundle in TM may vary pointwise. For every smooth parameterized curve $c : \mathbb{R} \rightarrow M$ we write \dot{c} and $A(\dot{c})$ for the tangent vector field and its A -hull along the curve c .

For any vector space V , we say that a vector subspace $A \subset V^* \otimes V$ of endomorphisms is of *generic rank* ℓ if the dimension of A is ℓ , and the subset of vectors $(X, Y) \in V \times V$, such that the A -hulls $A(X)$ and $A(Y)$ generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ , is open and dense. An A -structure is said to be of generic rank ℓ if $A_x M$ has this property for each point $x \in M$.

Let us point out some examples:

- The $\langle E \rangle$ -structure is of generic rank one on all manifolds of dimensions at least 2.
- Any almost complex structure or almost product structure $\langle E, J \rangle$ is of generic rank two on all manifolds of dimensions at least 4.
- Any almost quaternionic structure is of generic rank four on all manifolds of dimensions at least 8.

Similar conclusions apply to paracomplex and parahermitian structures.

Definition 2.2. *Let M be a smooth manifold with a given A -structure and a linear connection ∇ . A smooth curve $c : \mathbb{R} \rightarrow M$ is told to be A -planar if*

$$\nabla_{\dot{c}} \dot{c} \in A(\dot{c}).$$

Clearly, A planarity means that the parallel transport of any tangent vector to c has to stay within the A -hull $A(\dot{c})$ of the tangent vector field \dot{c} along the curve. Moreover, this concept does not depend on the parametrization of the curve c .

Definition 2.3. *Let M be a manifold with a linear connection ∇ and an A -structure, while N be another manifold with a linear connection $\hat{\nabla}$ and a B -structure. A diffeomorphism $f : M \rightarrow N$ is called (A, B) -planar if each A -planar curve c on M is mapped onto the B -planar curve $F \circ c$ on N .*

Example 2.4. *The 1-dimensional $A = \langle E \rangle$ structure must be given just as the linear hull of the identity affiner E , by the definition. Obviously, the $\langle E \rangle$ -planar curves on a manifold M with a linear connection ∇ are exactly the unparameterized geodesics. Moreover, two connections ∇ and $\bar{\nabla}$ without torsion are projectively equivalent (i.e. they share the same unparameterized geodesics) if and only if their difference satisfies $\bar{\nabla}_X Y - \nabla_X Y = \alpha(X)Y + \alpha(Y)X$ for some one-form α on M . The latter condition can be rewritten as*

$$(6) \quad \bar{\nabla} - \nabla \in \Gamma(T^*M \odot \langle E \rangle) \subset \Gamma(S^2 T^*M \otimes TM)$$

where the symbol \odot stays for the symmetrized tensor product.

The latter condition on projective structures may be also rephrased in the terms of morphisms: A diffeomorphism $f : M \rightarrow M$ is called geodesical (or an automorphism of the projective structure) if $f \circ c$ is an unparameterized geodesic for each geodesic c and this happens if and only if the symmetrization of the difference $f^*\nabla - \nabla$ is a section of $T^*M \odot \langle E \rangle$. We are going to generalize the above example in the rest of this section.

In the case $A = \langle E \rangle$, the $(\langle E \rangle, B)$ -planar mappings are called simply B -planar. They map each geodesic curve on (M, ∇) onto a B -planar curve on $(N, \hat{\nabla}, B)$.

Each ℓ dimensional A structure $A \subset T^*M \otimes TM$ determines the distribution $A^{(1)}$ in $S^2T^*M \otimes TM$, given at any point $x \in M$ by

$$A_x^{(1)}M = \{\alpha_1 \odot F_1 + \cdots + \alpha_\ell \odot F_\ell; \alpha_i \in T_x^*M, F_i \in A_xM\}.$$

Theorem 2.5. *Let M be a manifold with a linear connection ∇ , let N be a manifold of the same dimension with a linear connection $\hat{\nabla}$ and an A -structure of generic rank ℓ , and suppose $\dim M \geq 2\ell$. Then a diffeomorphism $f : M \rightarrow N$ is A -planar if and only if*

$$(7) \quad \text{Sym}(f^*\hat{\nabla} - \nabla) \in f^*(A^{(1)})$$

where Sym denotes the symmetrization of the difference of the two connections.

The proof is based on a purely algebraical lemma below. Let us first observe that the entire claim of the theorem is of local character. Thus, identifying the objects on N with their pullbacks on M , we may assume that $M = N$ and $f = \text{id}_M$.

Next, let us observe that the A -planarity of $f : M \rightarrow N$ does not at all depend on the possible torsions of the connection. Indeed, we always test expressions of the type $\nabla_c \dot{c}$ for a curve c and thus deforming ∇ into $\nabla' = \nabla + T$ by adding some torsion will not effect the results. Thus, without any loss of generality, we may assume that the connections ∇ and $\hat{\nabla}$ share the same torsion, and then we may omit the symmetrization from equation (7).

Finally, we may fix some (local) basis $E = F_0, F_i, i = 1, \dots, \ell - 1$, of A , i.e. $A = \langle F_0, \dots, F_{\ell-1} \rangle$. Then the condition in equation (7) says

$$(8) \quad \hat{\nabla} = \nabla + \sum_{i=0}^{\ell-1} \alpha_i \odot F_i$$

for some suitable one-forms α_i on M . Of course, the existence of such forms does not depend on our choice of the basis of A .

The quite simplified statement we now have to prove is:

Proposition 2.6. *Let M be a manifold of dimension at least 2ℓ , ∇ and $\hat{\nabla}$ two connections on M with the same torsion, and consider an A -structure of generic rank ℓ on M . Then each geodesic curve with respect to ∇ is A -planar with respect to $\hat{\nabla}$ if and only if there are one-forms α_i satisfying equation (8).*

Assume first we have such forms α_i , and let c be a geodesic for ∇ . Then equation (8) implies $\hat{\nabla}_c \dot{c} \in A(\dot{c})$ so that c is A -planar, by definition.

The other implication is the more difficult one. Assume each (unparameterized) geodesic c is A -planar. This implies that the symmetric difference tensor $P = \hat{\nabla} - \nabla \in \Gamma(S^2 T^* M \otimes TM)$ satisfies

$$P(\dot{c}, \dot{c}) = \hat{\nabla}_c \dot{c} \in \langle \dot{c}, F_1(\dot{c}), \dots, F_{\ell-1}(\dot{c}) \rangle.$$

In fact, the main argument of the entire proof boils down to a purely algebraic claim:

Lemma 2.7. *Let $A \subset V^* \otimes V$ be a vector subspace of generic rank ℓ , and assume that $P(X, X) \in A(X)$ for some fixed symmetric tensor $P \in V^* \otimes V^* \otimes V$ and each vector $X \in V$. Then the induced mapping $P : V \rightarrow V^* \otimes V$ has values in A .*

Proof. Let us fix a basis $F_0 = \text{id}_V, F_1, \dots, F_{\ell-1}$ of A . Since A is of generic rank ℓ , there is the open and dense subset $\mathcal{V} \subset V$ of all vectors $X \in V$ for which $\{F_0(X), F_1(X), \dots, F_{\ell-1}(X)\}$ are linearly independent. Now, for each $X \in \mathcal{V}$ there are the unique coefficients $\alpha_i(X) \in \mathbb{R}$ such that

$$(9) \quad P(X, X) = \sum_{i=0}^{\ell-1} \alpha_i(X) F_i(X).$$

The essential technical step in the proof of our Lemma is to show that all functions α_i are in fact restrictions of one-forms on V . Let us notice, that P is a symmetric bilinear mapping and thus it is determined by the restriction of $P(X, X)$ to arbitrarily small open non-empty subset of the arguments X in V .

Claim 1. *If a symmetric tensor $P \in V^* \otimes V^* \otimes V$ is determined over the above defined subset \mathcal{V} by (9), then the functions $\alpha_i : \mathcal{V} \rightarrow \mathbb{R}$ are smooth and their restrictions to the individual rays (half-lines) generated by vectors in \mathcal{V} are linear.*

Let us fix a basis $e_i \in V$, the dual basis e^i , and consider the induced dual bases e_I and e^I on the multivectors and exterior forms. Let us

consider the smooth mapping

$$\chi : \Lambda^\ell V \setminus \{0\} \rightarrow \Lambda^\ell V^*, \quad \chi(\sum a_I e^I) = \sum \frac{a_I}{\sum a_I^2} e_I.$$

Now, for all non-zero tensors $\Xi = \sum a_I e^I$, the evaluation $\langle \Xi, \chi(\Xi) \rangle$ is the constant function 1, while $\chi(k \cdot \Xi) = k^{-1} \chi(\Xi)$.

Next, we define for each $X \in \mathcal{V}$

$$\tau(X) = \chi(X \wedge F_1(X) \wedge \cdots \wedge F_{\ell-1}(X))$$

and we may compute the unique coefficients α_i from (9):

$$\begin{aligned} \alpha_0(X) &= \langle P(X, X) \wedge F_1(X) \wedge F_2(X) \wedge \cdots \wedge F_{\ell-1}(X), \tau(X) \rangle \\ \alpha_1(X) &= \langle X \wedge P(X, X) \wedge F_2(X) \wedge \cdots \wedge F_{\ell-1}(X), \tau(X) \rangle \\ &\vdots \\ \alpha_{\ell-1}(X) &= \langle X \wedge F_1(X) \wedge F_2(X) \wedge \cdots \wedge P(X, X), \tau(X) \rangle. \end{aligned}$$

In particular, this proves the first part our Claim 1.

Let us now consider a fixed vector $X \in \mathcal{V}$. The defining formula (9) for α_i implies $\alpha_i(kX) = k\alpha_i(X)$, for each real number $k \neq 0$. Passing to zero with positive k shows that α does have the limit 0 in the origin and so we may extend the definition of α_i 's (and validity of formula (9)) to the entire cone $\mathcal{V} \cup \{0\}$ by setting $\alpha_i(0) = 0$ for all i .

Finally, along the ray $\{tX; t > 0\} \subset \mathcal{V}$, the derivative $\frac{d}{dt}\alpha(tX)$ has the constant value $\alpha(X)$. This proves the rest of Claim 1.

Now, in order to complete the proof of Lemma 2.7, we have to prove the following assertion.

Claim 2. *If a symmetric tensor P is determined over the above defined subset $\mathcal{V} \cup \{0\}$ by (9), then the coefficients α_i are linear one-forms on V and the tensor P is given by*

$$P(X, Y) = \frac{1}{2} \sum_{i=0}^{\ell-1} (\alpha_i(Y) F_i(X) + \alpha_i(X) F_i(Y)).$$

The entire tensor P is obtained through polarization from its evaluations $P(X, X)$, $X \in TM$,

$$(10) \quad P(X, Y) = \frac{1}{2} (P(X + Y, X + Y) - P(X, X) - P(Y, Y)),$$

and again, the entire tensor is determined by its values on arbitrarily small non-empty open subset of X and Y .

The summands on the right hand side have values in the following subspaces:

$$\begin{aligned} P(X + Y, X + Y) &\in \langle X + Y, F_1(X + Y), \dots, F_{\ell-1}(X + Y) \rangle \subset \\ &\quad \langle X, F_1(X), \dots, F_{\ell-1}(X), Y, F_1(Y), \dots, F_{\ell-1}(Y) \rangle, \\ P(X, X) &\in \langle X, F_1(X), \dots, F_{\ell-1}(X) \rangle, \\ P(Y, Y) &\in \langle Y, F_1(Y), \dots, F_{\ell-1}(Y) \rangle. \end{aligned}$$

Since we have assumed that A has generic rank ℓ , the subset $\mathcal{W} \subset V \times V$ of vectors (X, Y) such that all the values

$$\{X, F_1(X), \dots, F_{\ell-1}(X), Y, F_1(Y), \dots, F_{\ell-1}(Y)\}$$

are linearly independent is open and dense. Clearly $\mathcal{W} \subset \mathcal{V} \times \mathcal{V}$. Moreover, if $(X, Y) \in \mathcal{W}$ then $F_0(X + Y), \dots, F_{\ell-1}(X + Y)$ are independent, i.e. $X + Y \in \mathcal{V}$. Inserting (9) into (10), we obtain

$$P(X, Y) = \sum_{i=0}^{\ell-1} (d_i(X, Y)F_i(X) + e_i(X, Y)F_i(Y)).$$

For all $(X, Y) \in \mathcal{W}$, the coefficients $d_i(X, Y) = \frac{1}{2}(\alpha_i(X + Y) - \alpha_i(X))$ at $F_i(X)$, and $e_i(X, Y) = \frac{1}{2}(\alpha_i(X + Y) - \alpha_i(Y))$ at $F_i(Y)$ in the latter expression are uniquely determined. The symmetry of P implies $d_i(X, Y) = e_i(Y, X)$. If $(X, Y) \in \mathcal{W}$ then also $(sX, tY) \in \mathcal{W}$ for all non-zero reals s, t and the linearity of P in the individual arguments yields for all real parameters s, t

$$std_i(X, Y) = sd_i(sX, tY).$$

Thus the functions α_i satisfy

$$\alpha_i(sX + tY) - \alpha_i(sX) = t(\alpha_i(X + Y) - \alpha_i(X)).$$

Since $\alpha_i(tX) = t\alpha_i(X)$, in the limit $s \rightarrow 0$ this means

$$\alpha_i(Y) = \alpha_i(X + Y) - \alpha_i(X).$$

Thus α_i are additive over the open and dense set $(X, Y) \in \mathcal{W}$. Choosing a basis of V such that each couple of basis elements is in \mathcal{W} , this shows that α_i are restrictions of linear forms, as required. \square

Now the completion of the proof of Theorem 2.5 is straightforward. Following the equivalent local claim in the Proposition 2.6 and the pointwise algebraic description of P achieved in Lemma 2.7, we just have to apply the latter Lemma to individual fibers over the points $x \in M$ and verify, that the linear forms α_i may be chosen in a smooth way. But this is obvious from the explicit expression for the coefficients α_i in the proof of Claim 1 above.

Theorem 2.8. *Let M be a manifold with a linear connection ∇ and an A -structure, N be a manifold of the same dimension with a linear connection $\hat{\nabla}$ and a B -structure with generic rank ℓ . Then a diffeomorphism $f : M \rightarrow N$ is (A, B) -planar if and only if f is B -planar and $A(X) \subset (f^*(B))(X)$ for all $X \in TM$.*

Proof. As in the proof of Theorem 2.5, we may restrict ourselves to some open submanifolds, fix generators F_i for B , assume that $f = \text{id}_M$ and both connections ∇ and $\hat{\nabla}$ share the same torsion, and prove the equivalent local assertion to our theorem:

Claim. *Each A -planar curve c with respect to ∇ is B -planar with respect to $\hat{\nabla}$, if and only if the symmetric difference tensor $P = \hat{\nabla} - \nabla$ is of the form (9) with smooth one-forms α_i , $i = 0, \dots, \ell - 1$ and $A(X) \subset B(X)$ for each $X \in TM$.*

Obviously, the condition in this statement is sufficient. So let us deal with its necessity.

Since every (A, B) -planar mapping is also B -planar, Theorem 2.5 (or the equivalent Proposition in its proof) says that

$$P(X, X) = \sum_{j=0}^{\ell} \alpha_j(X) F_j(X)$$

for uniquely given smooth one-forms α_i .

Now, consider a fixed $F \in A$ and suppose $F(X) \notin B(X)$. Since we assume that all $\langle E, F \rangle$ -planar curves c in M are B -planar, we may proceed exactly as in the beginning of the proof of Theorem 2.5 to deduce that

$$(11) \quad P(X, X) = \sum_{j=0}^{\ell} \alpha_j(X) F_j(X) + \beta(X) F(X)$$

on a neighborhood of X , with some unique functions α_i and β .

The comparison of the latter two unique expressions for $P(X, X)$ shows that $\beta(X)$ vanishes. But since $F(X) \notin B(X)$, there definitely are curves which are $\langle E, F \rangle$ -planar and tangent to X , but not $\langle E \rangle$ -planar. Thus, the assumption in the theorem would lead to $\beta(X) \neq 0$. Consequently, our choice $F(X) \notin B(X)$ cannot be achieved and we have proved $A(X) \subset B(X)$ for all $X \in TM$. \square

3. RESULTS ON QUATERNIONIC GEOMETRIES

The main result of this section is:

Theorem 3.1. *Let $f : M \rightarrow M'$ be a diffeomorphism between two almost quaternionic manifolds of dimension at least eight. Then f is*

a morphism of the geometries if and only if it preserves the class of unparameterized geodesics of all Weyl connections on M and M' .

This theorem will follow easily from the results of Section 2 and its proof requires only a few quite simple formal steps. On the way we shall also give a complete description of all geodesics of the Weyl connections in terms of the Q -planar curves where Q is the almost quaternionic structure.

Lemma 3.2. *A curve $c : \mathbb{R} \rightarrow M$ is Q -planar with respect to at least one Weyl connection ∇ on M if and only if c is Q -planar with respect to all Weyl connections on M .*

Proof. For a Weyl connection ∇ and a curve $c : \mathbb{R} \rightarrow M$, the defining equation for Q -planarity reads $\nabla_{\dot{c}}\dot{c} \in Q(\dot{c})$. If we choose some hypercomplex structure within Q , we may rephrase this condition as: $\nabla_{\dot{c}}\dot{c} = \dot{c} \cdot q$ where $\dot{c}(t)$ is a curve in the tangent bundle TM while $q(t)$ is a suitable curve in quaternions \mathbb{H} . Now the formula (5) for the deformation of the Weyl connections implies

$$\hat{\nabla}_{\dot{c}}\dot{c} = \nabla_{\dot{c}}\dot{c} + \{\{\dot{c}, \Upsilon\}, \dot{c}\} = \nabla_{\dot{c}}\dot{c} + 2\dot{c} \cdot \Upsilon(\dot{c}).$$

Indeed, this is the consequence of the computation of the Lie bracket in \mathfrak{g} of the corresponding elements $\dot{c} \in \mathfrak{g}_{-1}$, $\Upsilon \in \mathfrak{g}_1$:

$$\begin{aligned} [[\dot{c}, \Upsilon], \dot{c}] &\simeq \left[\left[\begin{pmatrix} 0 & 0 \\ \dot{c} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Upsilon \\ 0 & 0 \end{pmatrix} \right], \begin{pmatrix} 0 & 0 \\ \dot{c} & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & 0 \\ 2\dot{c} \cdot \Upsilon(\dot{c}) & 0 \end{pmatrix} \simeq 2\dot{c} \cdot \Upsilon(\dot{c}), \end{aligned}$$

where $\Upsilon(\dot{c})$ is the standard evaluation of the linear form $\Upsilon \in \mathfrak{g}_1 = (\mathbb{H}^n)^*$ on the vector $\dot{c} \in \mathfrak{g}_{-1} = \mathbb{H}^n$. Thus we see that if there is such a quaternion q for one Weyl connection, then it exists also for all of them. \square

Definition 3.3. *A curve $c : \mathbb{R} \rightarrow M$ is called \mathbb{H} -planar if it is Q -planar with respect to each Weyl connection ∇ on M .*

Theorem 3.4. *Let M be a manifold with an almost quaternionic structure. Then a curve $c : \mathbb{R} \rightarrow M$ is \mathbb{H} -planar if and only if c a geodesic of some Weyl connection, up to parametrization.*

Proof. Let us remark that c is a geodesic for ∇ if and only if $\nabla_{\dot{c}}\dot{c} = 0$. Thus, the statement follows immediately from the computation in the proof of Lemma 3.2. Indeed, if c is \mathbb{H} -planar, then choose any Weyl connection ∇ and pick up Υ so that $\hat{\nabla}_{\dot{c}}\dot{c}$ vanishes. \square

3.5. Proof of Theorem 3.1. Every morphism of almost quaternionic geometries preserves the class of Weyl connections and thus also the class of their geodesics.

We have to prove the oposit implication. This means, we have got two manifolds with almost quaternionic structures (M, Q) , (N, Q') and a diffeomorphism $f : M \rightarrow N$ which is (Q, Q') -planar. Then Theorem 2.8 implies that Tf maps quaternionic lines in $T_x M$ to quaternionic lines in $T_{f(x)} N$, i.e. $Q(X) = (f^* Q')(X)$ for each $X \in TM$ (since they both have the same dimension). In order to conclude the Theorem, we have to verify $Q = f^* Q'$ instead, since this is exactly the requirement that f preserves the defining subbundles Q and Q' . Fortunately, the very same arguments as in the proof of Theorem 2.5 will provide the verification.

Indeed, let us look at the subsets of all second jets of \mathbb{H} -planar curves. As we have seen a few lines above in the proof of Theorem 3.4, the accelerations of all \mathbb{H} -planar curves fill at each point just the complete Q -hulls of the velocities. Thus, for each generator F in Q and for any given point $x \in M$ we may locally choose a smoothly parameterized system c_X of \mathbb{H} -planar curves with parameter $X \in T_x M$ such that $\dot{c}_X = X$ and $\nabla_{\dot{c}_X} \dot{c}_X = \beta(X)F(X)$ where F is the chosen generator of Q and β is a non-zero 1-form. Then

$$(\nabla_{\dot{c}_X} - \hat{\nabla}_{\dot{c}_X})\dot{c}_X = \beta(X)F(X) + \sum_k \alpha^k(X)F_k(X)$$

where F_k are the generators of Q' and α^k are smooth 1-forms. Thus we are again in the situation as in the proof of Theorem 2.5, cf. the equation (11). Therefore, except of the zero set of β , $F(X) = \sum_k \gamma^k(X)F_k(X)$ for some smooth one-forms γ^k . Since F is linear in X , γ^k have to be constants and we are done. \square

3.6. Final remarks. All curves in the four-dimensional quaternionic geometries are \mathbb{H} -planar by the definition. Thus this concept starts to be interesting in higher dimensions only, and all of them are covered by Theorem 3.1.

The class of the unparameterized geodesics of Weyl connections is well defined for all parabolic geometries. Our result for the quaternionic geometries suggests the question, whether a similar statement holds for other geometries as well.

REFERENCES

- [1] Alekseevsky, D.V.; Marchiafava, S.; Pontecorvo, M., Compatible complex structures on almost quaternionic manifolds, *Trans. Amer. Math. Soc* 351 (1999), no. 3, 997–1014.
- [2] Bailey, T.N.; Eastwood, M.G., Complex paraconformal manifolds – their differential geometry and twistor theory, *Forum Math.*, 3 (1991), 61–103.
- [3] Čap, A.; Slovák, J., Weyl structures for parabolic geometries, *Math. Scand.* 93 (2003), 53–90.
- [4] Čap, A.; Slovák, J.; Žádník, V.; On distinguished curves in parabolic geometries, *Transform. Groups*, Vol. 9, No. 2, (2004), 143–166.
- [5] Gover, A.R.; Slovák, J., Invariant local twistor calculus for quaternionic structures and related geometries, *J. Geom. Physics*, 32 (1999), no. 1, 14–56.
- [6] Hrdina, J., \mathbb{H} -planar curves, *Proceedings Int. Conf. Differential Geometry and Applications*, Prague, August 2004, Charles University Press, 2005.
- [7] Oproiu, V., Integrability of almost quaternionic structures, *An. st. Univ. "Al. I. Cuza" Iasi* 30 (1984), 75–84.
- [8] Salamon, S.M.; Differential geometry of quaternionic manifolds, *Ann. Scient. ENS* 19 (1987), 31–55.
- [9] Mikes J., Sinyukov N.S. On quasilplanar mappings of spaces of affine connection, *Sov. Math.* 27, No.1 (1983), 63–70.
- [10] Mikeš, J.; Němčíková, J.; Pokorná, O., On the theory of the 4-quasilplanar mappings of almost quaternionic spaces, *Rediconti del circolo matematico di Palermo, Serie II, Suppl.* 54 (1998), 75–81.
- [11] Žádník, V., Generalized geodesics, Ph.D. thesis, Masaryk university in Brno, 2003.

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