ON DISTINGUISHED CURVES IN PARABOLIC GEOMETRIES

ANDREAS ČAP, JAN SLOVÁK, AND VOJTĚCH ŽÁDNÍK

Abstract. All parabolic geometries, i.e. Cartan geometries with homogeneous model a real generalized flag manifold, admit highly interesting classes of distinguished curves. The geodesics of a projective class of connections on a manifold, conformal circles on conformal Riemannian manifolds, and Chern–Moser chains on CR–manifolds of hypersurface type are typical examples. We show that such distinguished curves are always determined by a finite jet in one point, and study the properties of such jets. We also discuss the question when distinguished curves agree up to reparametrization and discuss the distinguished parametrizations in this case. We give a complete description of all distinguished curves for some examples of parabolic geometries.

Elie Cartan's idea of 'generalized spaces' as curved analogs of Felix Klein's geometries (i.e. homogeneous spaces) is a well understood geometrical concept, which, for a Lie subgroup $P \subset G$, generalizes the Maurer–Cartan form on the total space of the principal P–bundle $G \to G/P$ to Cartan connections on principal P–bundles, see e.g. the introductory book [17]. The concept of *parabolic geometries* refers to those cases where P is a parabolic subgroup in a (real or complex) semisimple Lie group G. In [9], Ch. Fefferman initiated a program to exploit the representation theory of parabolic subgroups in semisimple Lie groups in order to understand invariants of geometric structures like CR–geometries, projective geometries, or conformal Riemannian geometries. This approach has proved to be extremely powerful. First of all, all parabolic geometries can be described in terms of weaker analogies of classical G–structures on smooth manifolds and, similarly to the examples mentioned above, all such structures give rise to canonical normal Cartan connections, cf. [19, 14, 3]. In fact, these constructions express Cartan's method of equivalence using the language of the modern representation theory and natural cohomological reasoning. The existence of the Cartan connection provides an effective calculus to deal with invariant objects, see e.g. [5] and the references therein. To large extent, the understanding of the general (curved) geometries can be reduced to properties of the homogeneous model, and thus to purely algebraic questions.

The goal of this paper is to use this approach in order to understand invariantly defined systems of distinguished curves for parabolic geometries, which we call *(generalized) geodesics*. After recalling basic concepts of parabolic geometries, geodesics are introduced and discussed along the lines of the classical approach in affine geometry, which uses the development of curves. This approach may be found in similar context already in [17] and in [13]. In this way, many aspects of the study of the curves are reduced to the case of the homogeneous model. Thus the original 'smooth' question on curved manifolds can be transformed to an 'algebraic' problem, which is discussed in Section 2. In particular, we obtain estimates on the order of jets necessary to determine a geodesic, and this approach also leads to an algebraic description of all jets of geodesics in a point. The third section is devoted to the study of possible reparametrizations in the class of geodesics. Specializing the general results to $|1|$–graded Lie algebras, we obtain generalizations of some

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well known results on conformal, projective, and quaternionic geometries (see e.g. [1]). The final section provides further refinements for specific classes of curves, see in particular Theorems 4.2 and 4.3.

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1. General concepts

1.1. **Parabolic geometries.** Let us briefly recall the basic facts, more details can be found in [4] or [17], and the references therein. Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$, and $P \subset G$ a parabolic subgroup with Lie algebra $\mathfrak{p}$. A (real) *parabolic geometry* $(\mathcal{G}, \omega)$ of type (G, P) is a principal bundle $\mathcal{G}$ with structure group P over a manifold M, equipped with a smooth one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which satisfies

(1) $\omega(\zeta_Z)(u) = Z$ for all $u \in \mathcal{G}$ and fundamental fields ζ_Z, $Z \in \mathfrak{p} \subset \mathfrak{g}$, i.e. ω reproduces the generators of fundamental vector fields,

(2) $(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in P$, i.e. ω is P–equivariant with respect to the adjoint representation, and

(3) $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$, i.e. ω is an absolute parallelism on $\mathcal{G}$.

The curvature of a parabolic geometry $(\mathcal{G}, \omega)$ is the horizontal two–form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by the structure equations

$$K = d\omega + \tfrac{1}{2}[\omega, \omega], \text{ i.e. } K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

Clearly, the Maurer–Cartan form ω on the principal fiber bundle $G \to G/P$ is a parabolic geometry and the structure equations say that this geometry is *flat*, i.e. its curvature vanishes identically. $(G \to G/P, \omega)$ is called the *homogeneous model* for parabolic geometries of type (G, P).

Morphisms between Cartan geometries $(\mathcal{G}, \omega)$ and $(\mathcal{G}', \omega')$ are principal fiber bundle morphisms $\varphi : \mathcal{G} \to \mathcal{G}'$ such that $\varphi^*\omega' = \omega$. It is quite elementary to prove that a geometry is locally isomorphic to its homogeneous model if and only if its curvature vanishes identically, cf. [17].

Each smooth (left) action of the structure group P on a smooth manifold S leads to a functor $\mathcal{S}$ on the category of Cartan geometries of type (G, P). The value of $\mathcal{S}$ on $(\mathcal{G}, \omega)$ is the associated fiber bundle $\mathcal{G} \times_P S$ with respect to the action of P while a morphisms $\varphi : (\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ induces the fiber bundle morphism $\varphi \times_P \mathrm{id}_S : \mathcal{G} \times_P S \to \mathcal{G}' \times_P S'$. We call these bundles *natural bundles*. Moreover, this construction is functorial in the smooth action entry, because each equivariant mapping $\alpha : S \to S'$ induces the fiber bundle mapping $\mathrm{id}_{\mathcal{G}} \times_P \alpha : \mathcal{G} \times_P S \to \mathcal{G} \times_P S'$. Thus we have got a bifunctor on Cartan geometries and smooth left actions with values in fiber bundles.

In particular, linear representations of P lead to functors valued in vector bundles and their linear morphisms and the bifunctoriality of the construction extends all natural constructions like pairings, decompositions, and tensor products of representations to the natural bundles. Of course, all this is the obvious restriction of the usual functorial constructions over all principal fiber bundles to the category of Cartan geometries.

A central example, which also illustrates the role of the Cartan connection, is given by the representation of P on $\mathfrak{g}/\mathfrak{p}$ induced by the adjoint representation. This leads to the functor $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, and via the Cartan connection ω this associated bundle can be identified with the tangent bundle TM. Indeed, since ω defines an absolute parallelism, there are the corresponding 'constant' vector fields $\omega^{-1}(X) \in \mathcal{X}(\mathcal{G})$ for all $X \in \mathfrak{g}$, defined by $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{G}$. Denoting by $[\![u, X + \mathfrak{p}]\!]$ the class in $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ of $(u, X + \mathfrak{p}) \in \mathcal{G} \times \mathfrak{g}/\mathfrak{p}$ and by $\pi : \mathcal{G} \to M$ the bundle projection, one immediately verifies that $[\![u, X + \mathfrak{p}]\!] \mapsto T\pi(\omega^{-1}(X)(u))$ defines the claimed isomorphism.

For any parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, there is a grading $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$, and $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is the nilradical of $\mathfrak{g}$, see [20, 3]. In particular, this implies that $\mathfrak{g}_0$ is a reductive Levi component for $\mathfrak{p}$. Hence we obtain an identification $\mathfrak{n} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ with $\mathfrak{g}/\mathfrak{p}$, which is an isomorphism of P–modules if we endow $\mathfrak{n}$ with the 'truncated' adjoint action $\underline{\mathrm{Ad}}$. Via the Killing form, one further obtains an identification of $\mathfrak{n}^*$ with $\mathfrak{p}_+$, which induces the identification of the cotangent bundle T^*M with $\mathcal{G} \times_P \mathfrak{n}^*$. Thus all tensor bundles over M are identified with the natural bundles coming from tensor products of the representations $\mathfrak{n}$ and $\mathfrak{n}^*$. Moreover, the right hand ends $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ define a P–invariant filtration of $\mathfrak{g}$. Hence we obtain natural subbundles $T^iM \subset TM$ for all $i < 0$. The resulting filtration

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M \supset 0$$

is the most importing object underlying a parabolic geometry. This filtration is trivial for $|1|$–graded algebras and we call such parabolic geometries *irreducible*.

A very special case of the construction of natural bundles is the choice $S = G$ with the left action of P on G given by the group multiplication. This leads to the principal fiber bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$ with the principal action given by the usual right multiplication in G and the canonical inclusion $\mathcal{G} \subset \tilde{\mathcal{G}}$, $u \mapsto [\![u, e]\!]$, where $e \in G$ is the unit element. Now, the Cartan connection ω extends uniquely to a G–equivariant one–form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ reproducing the fundamental vector fields. One easily verifies that $\tilde{\omega}$ is a principal connection on $\tilde{\mathcal{G}}$. Whenever we have a left action of P on some manifold S which is the restriction of a left action of G, then we may view the natural bundle $\mathcal{G} \times_P S$ also as $\tilde{\mathcal{G}} \times_G S$. Hence on any natural bundle of this type, there is a canonical connection induced by $\tilde{\omega}$. Of course, if we consider restrictions of G–representations to P, then the resulting natural vector bundles, which are usually called *tractor bundles*, are equipped with canonical linear connections.

1.2. **Development of curves.** The notion of the development of curves is related to a particular instance of natural bundles associated to restrictions of G–actions to P, namely the case of the canonical left action on G/P. The resulting space $\mathcal{S} = \mathcal{G} \times_P G/P = \tilde{\mathcal{G}} \times_G G/P$ is called *Cartan's space* over the underlying manifold M of the Cartan geometry in question. Of course, $\mathcal{S} \to M$ is a fiber bundle with typical fiber G/P, and from 1.1 we know that the parabolic geometry induces a canonical connection on this fiber bundle.

Another remarkable fact about $\mathcal{S}$ is that for the point $o = eP \in G/P$, and a point $x \in M$, all points $u \in \mathcal{G}$ with $\pi(u) = x$ lead to the same class $O(x) = [\![u, o]\!] \in \mathcal{G} \times_P G/P$. Hence we obtain a canonical smooth section O of $\mathcal{S} \to M$ for every parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). Moreover, the vertical tangent bundle $V\mathcal{S}$ can be identified with the associated bundle $\mathcal{G} \times_P T(G/P)$. Since the base point $o \in G/P$ is a fix point for the action of P, we see that the restriction of $V\mathcal{S}$ to the image $O(M)$ of the canonical section is the associated bundle $\mathcal{G} \times_P T_o(G/P)$. Since $T_o(G/P)$ is canonically isomorphic with $\mathfrak{g}/\mathfrak{p}$ and $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ is naturally isomorphic to TM, we get a canonical isomorphism $V\mathcal{S}|_{O(x)} \cong TM$. Thus we may view the Cartan's space $\mathcal{S}$ as a nonlinear version of the tangent bundle in which the geometry in question is encoded by means of the local parallel transport of the induced connection. This point of view goes back to Cartan, and it was developed further in an abstract way in the second half of the 20th century (see e.g. [11]).

This canonical parallel transport provides a straightforward generalization of the classical concept of the development of curves. By composing with O, a curve $c : I \to M$ with $I = (a, b) \subseteq \mathbb{R}$ may be also viewed as a parametrized curve in $\mathcal{S}$. Fixing $t_0 \in I$ we find a neighborhood J of t_0 in I on which the parallel transport along $c : I \to M$ is well defined. Given $s \in J$, we may follow the curve $O \circ c$ from t_0 to s and then follow the parallel transport backward for time $t_0 - s$ to return to the fiber over t_0. More formally, we define a smooth curve $\operatorname{dev}(c, t_0)$ from an open neighborhood of 0 in $\mathbb{R}$ to $\mathcal{S}_{c(t_0)}$ by $\operatorname{dev}(c, t_0)(s) := \tilde{c}_s(s)$, where $\tilde{c}_s$ is the parallel curve in $\mathcal{S}$ lying over $t \mapsto c(t_0 + s - t)$ with the initial point $O(c(s))$. This curve is called the *development* of c at t_0. For a point $u \in \mathcal{G}$ over $c(t_0)$, there is a unique curve $\bar{c}(t)$ in G/P mapping $0 \in \mathbb{R}$ to $o \in G/P$ such that $\operatorname{dev}(c, t_0)(t) = [\![u, \bar{c}(t)]\!]$. Any other choice for the point in $\mathcal{G}$ has the form $u{\cdot}b$ for $b \in P$, and for that choice the curve changes to $\ell_{b^{-1}} \circ \bar{c}$.

Hence we conclude that each choice of a P–invariant class $\mathcal{C}$ of curves which map $0 \in \mathbb{R}$ to $o \in G/P$ leads to a distinguished class of curves on all manifolds endowed with a Cartan geometry of type (G, P). We say that a curve c on M is a *distinguished curve of type* $\mathcal{C}$ at a point $c(t_0) \in M$, if for some (and thus any) point $u \in \mathcal{G}$ the curve $\bar{c}$ constructed above lies in $\mathcal{C}$.

The natural choices for such sets $\mathcal{C}$ of curves, of course come from one–parameter subgroups in G: For a subset $A \subseteq \mathfrak{g}$, we can define a class $\mathcal{C}_A$ as $\{t \mapsto b\exp(tX)P : X \in A, b \in P\}$. So we take the one–parametric subgroups with generators in A, allow them to be shifted by left multiplications with elements of P, and project the resulting curves to G/P. Of course, for $X \in \mathfrak{p}$ this always leads to the constant curve o, so we may assume $A \cap \mathfrak{p} = \emptyset$. On the other hand, if we want to have curves in all directions in the class $\mathcal{C}_A$, then we have to assume that the restriction of the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ to A is surjective. The most obvious choice for A which satisfies this requirement is $A = \mathfrak{n}$. It should be noted that for $X \in \mathfrak{g} \setminus \mathfrak{p}$ the curve $t \mapsto b\exp(tX)P$ does not lie in $\mathcal{C}_\mathfrak{n}$ in general. Following the case of affine geometry and since we are mainly interested in having sets of distinguished curves which are as small as possible, we shall always assume $A \subseteq \mathfrak{n}$ in the sequel.

The parabolic subgroup $P \subset G$ always has a canonical closed subgroup G_0 which corresponds to the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{p}$. This group turns out to be reductive, and it can be characterized as the subgroup of those elements in G, whose adjoint action preserved the grading of $\mathfrak{g}$. In particular, the subspace $\mathfrak{n}$ is stable under the adjoint action of G_0. Now for $b \in G_0$ and $X \in \mathfrak{n}$, we of course have $b\exp(tX) = \exp(t\operatorname{Ad}_b X)b$, and thus $b\exp(tX)P = \exp(t\operatorname{Ad}_b X)P$. Thus it is natural to restrict attention to G_0–invariant subsets $A \subseteq \mathfrak{n}$, and the corresponding distinguished curves are called *(generalized) geodesics* of type $\mathcal{C}_A$. We often do not mention the type if $A = \mathfrak{n}$.

The generalized geodesics of type $\mathcal{C}_A$ are easily described explicitly by means of the constant vector fields $\omega^{-1}(X)$. Let us consider the projection $c(t)$ of the flow line $\mathrm{Fl}^{\omega^{-1}(X)}_t(u) \in \mathcal{G}$ to the manifold M. From the construction of the principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ one immediately concludes that the horizontal vectors for $\tilde{\omega}$ in points $u \in \mathcal{G}$ are $\omega^{-1}(X)(u) - \zeta_X(u)$ for all $X \in \mathfrak{n}$. Thus, the curve $t \mapsto \mathrm{Fl}^{\omega^{-1}(X)}_t(u){\cdot}\exp(-tX)$ must be the horizontal lift of c to $\tilde{\mathcal{G}}$. Now, the induced parallel transport of an element $[\![u, \exp tX]\!] \in \mathcal{S}$ along c is given at time s by $[\![\mathrm{Fl}^{\omega^{-1}(X)}_s(u), \exp(t - s)X]\!]$ and it reaches exactly the point $O(c(t))$ in the canonical embedding of M into $\mathcal{S}$ at time $s = t$. But this exactly means that for each $X \in \mathfrak{n}$ the curve $t \mapsto [\![u, \exp tX]\!]$ is the development of the projection of the flow line through u of the constant vector field $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$. Since the allowed developments for curves in $\mathcal{C}_A$ have the form $t \mapsto [\![u, \exp tX]\!]$ for $u \in \mathcal{G}$ and $X \in A$, we have proved the first part of:

1.3. **Proposition.** *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) and let $A \subseteq \mathfrak{n}$ be a G_0–invariant subset.*
(1) The geodesics of type $\mathcal{C}_A$ on M are exactly the projections of flow lines of the constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ with $X \in A$.
(2) Let $(p' : \mathcal{G}' \to M', \omega')$ be another parabolic geometry of type (G, P), $\varphi : \mathcal{G} \to \mathcal{G}'$ be a morphism of parabolic geometries covering $\varphi_0 : M \to M'$, and $c : I \to M$ a smooth curve. Then c is a geodesic of type $\mathcal{C}_A$ if and only if $\varphi_0 \circ c : I \to M'$ is a geodesic of type $\mathcal{C}_A$.

Proof. The curve c in M is a geodesics if an only if $c(t) = p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ for some $X \in A$ and $u \in \mathcal{G}$. Since $\varphi^*\omega' = \omega$, we get

$$p' \circ \mathrm{Fl}_t^{\omega'^{-1}(X)}(\varphi(u)) = p' \circ \varphi \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u) = \varphi_0 \circ p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$$

and the claim follows. □

Remark. (1) Our definition of geodesics and their general description is valid for arbitrary Cartan geometries. Though this is not a parabolic geometry, we may thus illustrate it in the case of affine connections on manifolds (i.e. G is the affine group $\mathbb{R}^m \times GL(m, \mathbb{R})$ and $P = GL(m, \mathbb{R})$). Here the complement $\mathfrak{n} = \mathbb{R}^m$ is P–invariant, and so any Cartan connection ω on $\mathcal{G}$ splits into the soldering form $\omega_{\mathfrak{n}} \in \Omega^1(\mathcal{G}, \mathbb{R}^m)$ and the principal connection form $\omega_{\mathfrak{p}} \in \Omega^1(\mathcal{G}, \mathfrak{p})$. Thus a Cartan geometry equips the underlying manifold M with the linear frame bundle $(\mathcal{G}, \omega_{\mathfrak{n}})$ and the principal connection $\omega_{\mathfrak{p}}$ on $\mathcal{G}$. The projections of flow lines of the constant vector fields are exactly the geodesics of the linear connection on TM induced by ω. Part (1) of the Proposition recovers the classical fact that the geodesics are those curves whose developments are straight lines in $\mathbb{R}^m = G/P$. On the other hand, if we choose $A = \mathfrak{g} \setminus \mathfrak{p}$, then more curves appear. For example, the following curves are projections of shifts of one–parametric subgroups in the affine group to the plane $\mathbb{R}^2$: $y = x \log x$ through $(1, 0)$, $y = \mathrm{e}^x$ through $(0, 1)$, $y = x^\alpha$ through $(1, 1)$, cf. [8].
(2) Exactly as in the homogeneous case, each choice of $u \in \mathcal{G}$ defines local coordinates around its projection $p(u) \in M$. Consider the mapping $X \mapsto p(\mathrm{Fl}_1^{\omega^{-1}(X)}(u))$, which is well defined on some neighborhood $U \subseteq \mathfrak{n}$ of 0. Choosing U sufficiently small, this becomes a diffeomorphism onto its image, thus gives rise to local coordinates on M. These are called normal coordinates for the Cartan geometry in question. Of course, in the setting of (1), we recover exactly the usual normal coordinates for affine connections on manifolds in this way. More information and a characterization of the normal coordinates can be found in [4].

We may rephrase our definition in terms of normal coordinates as follows: The geodesics of type $\mathcal{C}_A$ are those curves which are linearly parametrized straight lines through the origin with directions in $A \subseteq \mathfrak{n}$ in some normal coordinates. Again, this generalizes the standard facts on affine connections.

1.4. *Example.* Let us mention four well known examples of distinguished curves in parabolic geometries:
(1) $G = SL(m+1, \mathbb{R})$, P is the stabilizer of a line in $\mathbb{R}^{m+1}$. Normal parabolic geometries of type (G, P) are classical projective structures on m–dimensional manifolds. Generalized geodesics (of type $\mathcal{C}_{\mathfrak{n}}$) are exactly the geodesics of all connections in the projective class. They are determined by their 2–jet in one point as parametrized curves, but already determined by their direction in one point as unparametrized curves.

(2) $G = SL(m+1, \mathbb{H})$, P is the stabilizer of a quaternionic line. This choice leads to almost quaternionic geometries (the complex version of which is dealt with in [1]). Again generalized geodesics are determined by their 2–jet in one point, but they form more complicated systems of curves than in the projective case, cf. [1].
(3) $G = O(p+1, q+1)$, P is the stabilizer of a null line. This leads to conformal pseudo Riemannian geometries of signature (p, q). Here the (generalized) geodesics are the well known conformal circles, which owe their name to the fact that for the homogeneous model with signature $(n, 0)$ one obtains all circles on the sphere. For general signatures, the geodesics in null directions, which behave similarly to the projective case, form an interesting subclass.
(4) $G = SU(p+1, q+1)$, P the stabilizer of a (complex) null line. This Hermitian analog of (3) leads to non–degenerate CR–structures of hypersurface type with signature (p, q). Here the Lie algebra is 2–graded and the geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$ are the well known Chern–Moser chains.

2. Jets of distinguished curves

2.1. **The bundles of $\mathcal{C}_A$–velocities.** Let us recall the natural bundles T^r_k of rth order k–dimensional velocities on all smooth manifolds. By definition, $T^r_k M = J^r_0(\mathbb{R}^k, M)$, so this is the bundle of r–jets of parametrized k–dimensional (singular) submanifolds in M. In particular, r–jets of curves are elements in $T^r_1 M$. The action of all diffeomorphisms of M on $T^r_k M$ is defined by jet composition. Let us consider a category of Cartan geometries of fixed type (G, P) and a class of generalized geodesics $\mathcal{C}_A$, for a G_0–invariant subset A of $\mathfrak{n}$. Then the jets of distinguished curves of type $\mathcal{C}_A$ form a natural subbundle $T^r_{\mathcal{C}_A} \subset T^r_1$ on parabolic geometries of type (G, P). Clearly, $T^r_{\mathcal{C}_A}$ is a well defined functor, cf. Proposition 1.3(2) above, however their values are not smooth bundles in general, see the examples below. In the cases with G_0–invariant subsets $A \subset \mathfrak{n}$ we call the latter functors the *bundle of rth order velocities* of geodesics of type $\mathcal{C}_A$.

Our next goal is to prove that there always is a finite order r for which the entire geodesic is completely determined by a single value in $T^r_{\mathcal{C}_A}$.

2.2. **Jets of curves on G/P.** Using Cartan's space $\mathcal{S}$, the development of curves defines a bijection between smooth curves $c : I \to M$ defined on some neighborhood I of $0 \in \mathbb{R}$ such that $c(0) = x_0$, and smooth curves to G/P which map 0 to $o = eP$. Of course, this bijection is compatible with taking jets in x_0, i.e. two curves have the same ℓ–jet in x_0 if and only if the corresponding curves in G/P have the same ℓ–jet in o. By definition, this bijection also respects generalized geodesics of any type. Thus to prove that geodesics of some type $\mathcal{C}_A$ are determined by some jet in one point, it suffices to consider the homogeneous model G/P and the point o. We start by considering $A = \mathfrak{n}$ (which of course provides an estimate for any $A \subseteq \mathfrak{n}$). Thus, we have to study the curves $c^{b,X}(t) = b\exp(tX)P$, with $b \in P$ and $X \in \mathfrak{n}$, cf. 1.2.

Since $b\exp(tX) = \exp(t\,\mathrm{Ad}_b X)b$ we see that $c^{b,X}(t) = \exp(t\,\mathrm{Ad}_b \cdot X)P$. For any two curves $c(t)$ and $d(t)$ in G, there is a uniquely determined curve $u(t)$ in G such that $c(t) = d(t) \cdot u(t)$. The projections of $c(t)$ and $d(t)$ to G/P coincide if and only if $u(t) \in P$ for all t. Thus the curves c^{b_1,X_1} and c^{b_2,X_2} coincide if and only if the uniquely determined curve u such that

$$\exp(t\,\mathrm{Ad}_{b_1} X_1) = \exp(t\,\mathrm{Ad}_{b_2} X_2) \cdot u(t) \tag{1}$$

has values in P. Since exp is analytic, the curve u must be analytic, too, and hence it has values in P if and only if all derivatives $u^{(i)}(0) = (\frac{d}{dt})^i u(0)$ are tangent P. To formulate this precisely, we use left logarithmic derivative $\delta u : \mathbb{R} \to \mathfrak{g}$ of the curve $u : \mathbb{R} \to G$, see e.g. [10, p. 39]. In fact $\delta u : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathfrak{g}$, $\delta u(t) = T\lambda_{u(t)^{-1}} \circ T_t u$, but we shall identify the linear map $\delta u(t,\ ) : \mathbb{R} \to \mathfrak{g}$ with its value at the unit $1 \in T_t\mathbb{R}$. Since knowing δu is equivalent to knowing Tu, the following Lemma is a simple observation.

Lemma. *For each order $k \in \mathbb{N}$ we have $j^k_0 c^{b_1,X_1} = j^k_0 c^{b_2,X_2}$ if and only if the derivatives $(\delta u)^{(i)}(0)$ lie in $\mathfrak{p}$ for all $i \leq k-1$.*

2.3. **Some technicalities.** In order to compute the derivatives of δu from formula 2.2(1), we can use the Leibniz rule for the left logarithmic derivative,

$$\delta(f{\cdot}g)(x) = \delta g(x) + \operatorname{Ad}_{g(x)^{-1}} \delta f(x),$$

cf. [10, p. 39], so it remains to compute the left logarithmic derivative of the curve $t \mapsto \exp tX$. For later use, we shall compute this expression with an arbitrary curve $Y : \mathbb{R} \to \mathfrak{g}$ instead of the line tX. By definition, the logarithmic derivative $\delta(f \circ g)$ of the composition of two smooth maps $f : M \to G$, $g : N \to M$ is given by $\delta(f \circ g) = (\delta f) \circ Tg$. Thus, the key ingredient is the formula for $\delta(\exp) : T\mathfrak{g} \to \mathfrak{g}$. The proof of this formula for the right logarithmic derivative in [10, p. 39] can be easily adapted to our case, leading to

$$\delta(\exp)(Y) = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \operatorname{ad}(-Y)^p.$$

This proves:

Lemma. *Let $Y : \mathbb{R} \to \mathfrak{g}$ be a smooth curve with derivative $Y' : \mathbb{R} \to \mathfrak{g}$. Then*

$$\delta(\exp \circ Y)(t) = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \operatorname{ad}(-Y(t))^p {\cdot} Y'(t).$$

The first terms in the formula for $\delta(\exp Y(t))$ read as

$$Y'(t) - \tfrac{1}{2}[Y(t), Y'(t)] + \tfrac{1}{6}[Y(t), [Y(t), Y'(t)]] + \dots .$$

Notice that if Y has values in $\mathfrak{n}$, then also Y' has values in $\mathfrak{n}$, and compatibility of the grading of $\mathfrak{g}$ with the Lie bracket implies that at most k of these terms may be non–zero for $|k|$–graded $\mathfrak{g}$. Thus, for example,

$$\begin{aligned}
&\delta(\exp Y(t)) = Y'(t), \text{ if } k = 1,\\
&\delta(\exp Y(t)) = Y'(t) - \tfrac{1}{2}[Y(t), Y'(t)], \text{ if } k = 2,\\
&\delta(\exp Y(t)) = Y'(t) - \tfrac{1}{2}[Y(t), Y'(t)] + \tfrac{1}{6}[Y(t), [Y(t), Y'(t)]], \text{ if } k = 3.
\end{aligned}$$

On the other hand, if $Y(t) = \varphi(t)Y$ for some fixed $Y \in \mathfrak{g}$ and a smooth function φ, then $[Y(t), Y'(t)] = 0$ and hence we always get

$$\delta(\exp \varphi(t)Y) = \varphi'(t)Y. \tag{1}$$

Applying the left logarithmic derivative to equation 2.2(1) yields

$$\delta u(t) = \operatorname{Ad}_{b_1} X_1 - \operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{b_2} X_2. \tag{2}$$

In particular, $\delta u(0) = \operatorname{Ad}_{b_1} X_1 - \operatorname{Ad}_{b_2} X_2$, and this lies in $\mathfrak{p}$ if and only if $\operatorname{Ad}_{b_1} X_1$ and $\operatorname{Ad}_{b_2} X_2$ represent the same class in $\mathfrak{g}/\mathfrak{p}$, i.e. if the curves have the same tangent vector at 0.

Differentiating equation (2) at zero we obtain

$$(\delta u)'(0) = -\operatorname{ad}_{(-u'(0))} \operatorname{Ad}_{b_2} X_2 = [u'(0), \operatorname{Ad}_{b_2} X_2],$$

and $u'(0)$ is the image of $1 \in T_0\mathbb{R}$ by $\delta u(0)$. Substituting (2) yields $(\delta u)'(0) = [\delta u(0), \operatorname{Ad}_{b_1} X_1]$. Surprisingly, there is a general formula for $(\delta u)^{(i)}(t)$ for all $t \in \mathbb{R}$ and all orders i:

2.4. **Lemma.** *For all $i \geq 1$, $(\delta u)^{(i)}(t) = (\operatorname{ad}(-\operatorname{Ad}_{b_1} X_1))^i(\delta u(t))$.*

Proof. Let us start with the first order derivative, so we have to prove $(\delta u)'(t) = [\delta u(t), \operatorname{Ad}_{b_1} X_1]$. To do this, we have to compute the derivative of $t \mapsto \operatorname{Ad}_{u(t)^{-1}} : \mathbb{R} \to GL(\mathfrak{g})$. Clearly, $\frac{d}{dt}(t \mapsto \operatorname{Ad}_{u(t)^{-1}}) = (T \operatorname{Ad} \circ T\nu)(u'(t))$, where ν is the inversion in G and $T_t u = u'(t)$. First, we will express $T_g \nu$ and $T_g \operatorname{Ad}$ in general.

From $\rho_g \circ \nu \circ \lambda_g = \nu$ we have $T_{g^{-1}}\rho_g \circ T_g \nu \circ T_e \lambda_g = T_e \nu$, thus $T_g \nu = -T_e \rho_{g^{-1}} \circ T_g \lambda_{g^{-1}}$. Similarly, $\operatorname{Ad} \circ \lambda_g = \operatorname{Ad}_g \circ \operatorname{Ad}$ implies $T_g \operatorname{Ad} \circ T_e \lambda_g = \operatorname{Ad}_g \circ T_e \operatorname{Ad}$, so $T_g \operatorname{Ad} = \operatorname{Ad}_g \circ \operatorname{ad} \circ T_g \lambda_{g^{-1}}$. Altogether,

$$\tfrac{d}{dt} \operatorname{Ad}_{u(t)^{-1}} = (\operatorname{Ad}_{u(t)^{-1}} \circ \operatorname{ad} \circ T\lambda_{u(t)}) \circ (-T\rho_{u(t)^{-1}} \circ T\lambda_{u(t)^{-1}})(u'(t)).$$

Since $\operatorname{Ad}_g = T_e(\lambda_g \circ \rho_{g^{-1}})$ and $\delta u(t) = T\lambda_{u(t)^{-1}} \circ u'(t)$ the latter expression equals $(-\operatorname{Ad}_{u(t)^{-1}} \circ \operatorname{ad} \circ \operatorname{Ad}_{u(t)})(\delta u(t))$. Thus,

$$(\delta u)'(t) = \operatorname{Ad}_{u(t)^{-1}}[\operatorname{Ad}_{u(t)} \delta u(t), \operatorname{Ad}_{b_2} X_2] = [\delta u(t), \operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{b_2} X_2]$$

and substituting $\operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{b_2} X_2 = \operatorname{Ad}_{b_1} X_1 - \delta u(t)$ from 2.3(2) the claim follows.

Now, let $i > 1$ and assume that the formula is valid for all orders less then i. Then

$$(\delta u)^{(i)}(t) = \tfrac{d}{dt}\big|_t \left(\operatorname{ad}(-\operatorname{Ad}_{b_1} X)^{(i-1)} \delta u(t)\right)$$

and since $\operatorname{ad}(-\operatorname{Ad}_{b_1} X)^{(i-1)}$ is a linear map and we have computed $(\delta u(t))'$ already, we arrive at

$$(\delta u)^{(i)}(t) = \operatorname{ad}(-\operatorname{Ad}_{b_1} X)^{(i-1)}(\delta u(t))' = \operatorname{ad}(-\operatorname{Ad}_{b_1} X)^{(i)} \delta u(t),$$

which is the required formula. □

Let us notice that we have also derived the more general formula for the derivative of $\operatorname{Ad}_{u(t)^{-1}} Y(t)$ with $Y : \mathbb{R} \to \mathfrak{n}$. From the proof above we conclude

$$\tfrac{d}{dt}\big|_t \left(\operatorname{Ad}_{u(t)^{-1}} Y(t)\right) = \operatorname{Ad}_{u(t)^{-1}} Y'(t) - [\delta u(t), \operatorname{Ad}_{u(t)^{-1}} Y(t)]. \tag{1}$$

As a simple consequence of this Lemma, we can prove that any geodesic is determined by a finite jet in one point:

2.5. Proposition. *Let $\mathfrak{g}$ be a $|k|$–graded Lie algebra, and let $A \subseteq \mathfrak{n}$ be any G_0–invariant subset. If two geodesics of type $\mathcal{C}_A$ have the same $(k+2)$–jet in one point, then they coincide.*

Proof. As we have noticed in 2.2 it suffices to consider $A = \mathfrak{n}$, an we can complete the proof by showing that two curves c^{b_1,X_1} and c^{b_2,X_2} coincide if they have the same $(k+2)$–jet in 0. Denoting by $u : \mathbb{R} \to G$ the curve determined by equation 2.2(1), Lemma 2.4 tells us that $(\delta u)^{(i)}(0) = (\operatorname{ad}(-\operatorname{Ad}_{b_1} X_1))^i(\delta u(0))$. By Lemma 2.2, the assumption on the $(k+2)$–jet in 0 implies that $\operatorname{ad}(-\operatorname{Ad}_{b_1} X_1)^i(\delta u(0)) \in \mathfrak{p}$ for all $i \leq k+1$. Since $b_1 \in P$, we may hit this element with $\operatorname{Ad}_{b_1}^{-1}$, and the result remains in $\mathfrak{p}$. Putting $X = X_1 \in \mathfrak{n}$ and $Z = \operatorname{Ad}_{b_1^{-1}} \delta u(0) \in \mathfrak{p}$ we conclude that $\operatorname{ad}(-X)^i(Z) \in \mathfrak{p}$ for all $i = 1, \dots, k+1$. Since $Z \in \mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ and $-X \in \mathfrak{n} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$, compatibility of the bracket with the grading implies that $\operatorname{ad}(-X)^i(Z) \in \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{k-i}$. Putting $i = k+1$, we see that $\operatorname{ad}(-X)^{k+1}(Z)$ has to lie both in $\mathfrak{n}$ and in $\mathfrak{p}$, so it must be zero. This implies that $\delta u^{\ell}(0) = 0 \in \mathfrak{p}$ for all $\ell > k+1$, and thus $c^{b_1,X_1} = c^{b_2,X_2}$ and the claim follows. □

Let us remark at this point that the estimate $r = k+2$ on the jet needed to pin down a geodesic is not at all sharp and we will improve it heavily depending on a particular choice of the class of geodesics.

2.6. **Distinguished curves in a given direction.** The most natural way to approach the problem of distinguished curves usually is to fix a point $x \in M$ and a tangent vector $\xi \in T_xM$, and look for geodesics emanating from x in direction ξ. Given a G_0–invariant subset $A \in \mathfrak{n}$, the basic question then is how many geodesics of type $\mathcal{C}_A$ pass through x in direction ξ. Of course, it may happen that there are no such geodesics. As before, one may restrict the discussion to the point o in the homogeneous model G/P. Since the above question is perfectly geometric, the answer for a tangent vector $\xi \in T_o(G/P) \cong \mathfrak{g}/\mathfrak{p}$ will only depend on the P–orbit of ξ. Clearly, there is at least one geodesic of type $\mathcal{C}_A$ in direction X, if the image of A in $\mathfrak{g}/\mathfrak{p}$ meets the P–orbit of ξ. Otherwise put, if $X \in \mathfrak{n} \subset \mathfrak{g}$ is the unique element such that $\xi = X + \mathfrak{p}$, then there is at least one geodesic of type $\mathcal{C}_A$ in direction ξ if $\underline{\mathrm{Ad}}_b(X) \in A$ for some $b \in P$.

Second, suppose that $A, B \subset \mathfrak{n}$ are G_0–invariant subsets, and that for each $X \in A$ there is an element $b \in P$ such that $\mathrm{Ad}_b X \in B$, and vice versa. (Of course, this is a very restrictive condition, since we are using Ad_b, which does not leave $\mathfrak{n}$ invariant, but it happens in interesting cases.) Then this gives rise to a bijection between the sets $\mathcal{C}_A$ and $\mathcal{C}_B$ of curves in G/P, and consequently, geodesics of type $\mathcal{C}_A$ coincide with geodesics of type $\mathcal{C}_B$.

Fix a G_0–invariant subset $A \subseteq \mathfrak{n}$ and an element $X \in A$, and consider the tangent vector $\xi = X + \mathfrak{p} \in T_o(G/P)$. Clearly, $c^{e,X}(t) = \exp(tX)$ is a geodesic of type $\mathcal{C}_A$ in direction X, and any other geodesic of that type can be written as $c^{b,Y}$ with $b \in P$ and $Y \in A$. It is a general fact, see [3, 2.10] that there are unique elements $b_0 \in G_0$ and $Z \in \mathfrak{p}_+$ such that $b = b_0 \exp(Z) = \exp(\mathrm{Ad}_{b_0} Z) b_0$. From the definition of distinguished curves, we conclude that

$$c^{b_0 \exp Z, Y} = c^{\exp(\mathrm{Ad}_{b_0} Z), \mathrm{Ad}_{b_0} Y},$$

and $\mathrm{Ad}_{b_0} Y \in A$. Hence any geodesic of type $\mathcal{C}_A$ may be written as $c^{\exp(Z),Y}$ for $Z \in \mathfrak{p}_+$ and $Y \in A$. Hence we conclude that the set of geodesics of type $\mathcal{C}_A$ in direction $\xi = X + \mathfrak{p}$ can be equivalently described as

$$\{c^{\exp(Z),Y} : Z \in \mathfrak{p}_+, Y \in A, \underline{\mathrm{Ad}}_{\exp(Z)} \cdot Y = X\}.$$

Passing to a general curved geometry via developments as before, we obtain

Proposition. *Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P), $x \in M$ a point, $\xi \in T_xM$ a tangent vector, and let $A \subseteq \mathfrak{n}$ be a G_0–invariant subset. Then there is a geodesic of type $\mathcal{C}_A$ through x in direction ξ if and only if there are elements $u \in p^{-1}(x) \subset \mathcal{G}$ and $X \in A$ such that $\xi = T_u p \cdot \omega^{-1}(X)$. Moreover, for any such pair (u, X), one obtains a bijection between the set of geodesics of type $\mathcal{C}_A$ through x in direction ξ and the set $\{c^{\exp(Z),Y} : Z \in \mathfrak{p}_+, Y \in A, \underline{\mathrm{Ad}}_{\exp(Z)} \cdot Y = X\}$ of curves in G/P. This bijection is compatible with finite jets in 0 in the obvious sense.*

Finally note that the curves $c^{\exp(Z_1),Y_1}$ and $c^{\exp(Z_2),Y_2}$ have the same ℓ–jet in 0 respectively coincide if and only if the same is true for c^{e,Y_1} and $c^{\exp(Z_1)^{-1}\exp(Z_2),Y_2}$, and we can write $\exp(Z_1)^{-1}\exp(Z_2)$ as $\exp(Z)$ for some $Z \in \mathfrak{p}_+$. Hence we conclude that if for some ℓ and each $X \in A$ we can show that any curve $c^{\exp(Z),Y}$ with $Y \in A$ which has the same ℓ–jet in 0 as $c^{e,X}$ must actually equal $c^{e,X}$, then this implies that any geodesic of type $\mathcal{C}_A$ is uniquely determined by its ℓ–jet in a single point.

2.7. **The $|1|$–graded case.** For irreducible parabolic geometries we easily reach a complete description. So we assume $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $A = \mathfrak{n}$. The main simplification in the $|1|$–graded case comes from the fact that in this case $\mathfrak{p}_+$ acts trivially on $\mathfrak{g}/\mathfrak{p}$, so the P action on this quotient factorizes over G_0. In particular, for $Z \in \mathfrak{p}_+ = \mathfrak{g}_1$ and $Y \in \mathfrak{n} = \mathfrak{g}_{-1}$ we get $\underline{\mathrm{Ad}}_{\exp(Z)} Y = Y$, so in view of Proposition

2.6 it remains to compare the curves $c^{e,X}$ and $c^{\exp(Z),X}$ with $Z \in \mathfrak{g}_1$. For the corresponding curve u, we obviously get $\delta u(0) = -[Z, X] - \frac{1}{2}[Z, [Z, X]]$. For the two curves having the same two–jet in 0, we must have

$$(\delta u)'(0) = -[X_1, \delta u(0)] = [X_1, [Z, X_1]] + \tfrac{1}{2}[X_1, [Z, [Z, X_1]]] \in \mathfrak{p},$$

and thus $[X_1, [Z, X_1]] = 0$. But this implies $[X_1, [Z, [Z, X_1]]] = [Z, [X_1, [Z, X_1]]] = 0$, and so $(\delta u)^{(i)}(0) = 0$ for all $i \geq 2$. Thus, we have proved:

Proposition. *Each generalized geodesic in an irreducible parabolic geometry is uniquely determined by its 2–jet in one point.*

2.8. **The distinguished jets.** Using the procedures from above, one may compute explicitly the jets of all geodesics of type $\mathcal{C}_A$. For the sake of simplicity, we shall restrict ourselves again to the case of $|1|$–graded Lie algebras. Thus, the value in $T^2_1(G/P)$ over the origin will always determine a geodesic completely, and we shall compute explicitly the algebraic description of the standard fibers of $T^2_{\mathcal{C}_A}$. Understanding the higher jets of geodesics is an interesting problem, however the computations grow quickly out of hand.

Let us describe all distinguished curves in normal coordinates through the origin, i.e. we have to represent each geodesic in the form $t \mapsto \exp(Y(t))P$ for a smooth curve $Y : \mathbb{R} \to \mathfrak{g}_{-1}$ with $Y(0) = 0$. This means, that rather than with formula 2.2(1), we have to deal with

$$\exp(Y(t)) \cdot u(t) = \exp(t \operatorname{Ad}_{\exp Z} X)$$

for $Z \in \mathfrak{g}_1$ and $X \in A \subseteq \mathfrak{g}_{-1}$.

Using the results in 2.3 and formula 2.4(1), straightforward computations yield

$$\begin{aligned} \delta u(t) &= X + [Z, X] + \tfrac{1}{2}[Z, [Z, X]] - \operatorname{Ad}_{u(t)^{-1}}(Y'(t)) \\ (\delta u)'(t) &= \big[X + [Z, X] + \tfrac{1}{2}[Z, [Z, X]], \operatorname{Ad}_{u(t)^{-1}} Y'(t)\big] - \operatorname{Ad}_{u(t)^{-1}} Y''(t). \end{aligned}$$

The requirement $(\delta u)^{(i)}(0) \in \mathfrak{p}$, for $i = 0, 1$ immediately implies

$$\begin{aligned} Y'(0) &= X \\ Y''(0) &= [X, [X, Z]]. \end{aligned}$$

Now it is easy to describe the standard fiber of $T^2_{\mathcal{C}_A}$ as follows. The standard fiber of T^2_1 is the smooth manifold $J^2_0(\mathbb{R}, \mathfrak{g}_{-1})_0$, which is naturally identified with $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$. Hence the standard fiber of $T^2_{\mathcal{C}_A}$ is a subset in $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$, which we have computed to be

$$S = \left\{ \begin{pmatrix} X \\ [X, [X, Z]] \end{pmatrix} : X \in A, Z \in \mathfrak{g}_1 \right\}.$$

Recall that A is assumed to be G_0–invariant, but not necessarily a linear subspace. A good examples in which it is not a subspace is given by the null cone in $\mathbb{R}^{p+q}$ in the setting of Example 1.4(3). In that case, $[X, [Z, X]]$ happens to be a multiple of X for each Z, which corresponds to the fact that geodesics in null directions are conformally invariant up to parametrization.

For every parabolic geometry of type (G, P), there is the standard embedding $i : P \to G^2_m = \operatorname{inv} J^2_0(\mathbb{R}^m, \mathbb{R}^m)_0$, see e.g. [15, 17]. Further, the action of the structure group G^2_m on $J^2_0(\mathbb{R}, \mathbb{R}^m)_0$ transforms to the action on $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$, whose restriction to the subgroup $i(P)$ keeps the subset S invariant because the set $\mathcal{C}_A$ of all geodesics is P–invariant.

In fact, the action of G_0 obviously is the product of the adjoint actions on $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$, while the action of $P_+ = \exp \mathfrak{g}_1$ comes by the very definition of the curves from the left shift by the elements $\exp W$, $W \in \mathfrak{g}_1$. Since $\mathfrak{g}_1$ is an abelian subalgebra, the action by $\exp W$ is given by

$$\exp W\cdot \begin{pmatrix} Y' \\ Y'' \end{pmatrix} = \begin{pmatrix} Y' \\ Y'' + [Y',[Y',W]] \end{pmatrix}.$$

Hence we obtain an alternative description of the standard fiber as the P–orbit of the G_0–invariant subspace $A \times \{0\} \subseteq \mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$.

3. Reparametrizations

In this section we shall generalize our basic question to: *When are two distinguished curves equal up to a change of parametrization?* Thus we shall discuss the non–parametrized geodesics together with their preferred parametrizations.

3.1. **Technicalities.** In order to deal with this question, we have to modify our basic equation 2.2(1). The answer is positive if and only if there exist mappings $u : \mathbb{R} \to P$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\exp(\varphi(t) \operatorname{Ad}_{b_1} X_1) = \exp(t \operatorname{Ad}_{b_2} X_2)\cdot u(t), \tag{1}$$

where φ is a local reparametrization, i.e. we require $\varphi'(t) \neq 0$ and, for simplicity, $\varphi(0) = 0$. As discussed in the Remark 2.6, we may restrict ourselves to G_0–invariant subsets A, $b_1 = e$, $b_2 = \exp Z$ with $Z \in \mathfrak{p}_+$.

The left logarithmic derivative of (1) gives, cf. 2.3(1)

$$\delta u(t) = \varphi'(t) X_1 - \operatorname{Ad}_{u(t)^{-1}} \cdot \operatorname{Ad}_{\exp Z} X_2. \tag{2}$$

In particular, $\delta u(0) \in \mathfrak{p}$ if and only if tangent vectors of the two distinguished curves at 0 are equal up to a scalar multiple.

By formula 2.4(1), and the above equation (2), we get

$$(\delta u)'(t) = \varphi''(t) X_1 - \varphi'(t)[X_1, \delta u(t)]. \tag{3}$$

Now, similarly as in the parametrized case we prove a general iterative formula for $(\delta u)^{(i)}$:

3.2. **Lemma.** *For all $i \geq 1$ and at every $t \in \mathbb{R}$, with the notation as above*

$$(\delta u)^{(i)} = \varphi^{(i+1)} X_1 + \sum_{k=1}^{i} (-1)^k \big(\sum_{\mathbf{j},\mathbf{a}} c_{\mathbf{j},\mathbf{a}} (\varphi^{(j_1)})^{a_1} \dots (\varphi^{(j_s)})^{a_s} \big) (\operatorname{ad}_{X_1})^k (\delta u)$$

where the internal sum runs over all s–tuples of natural numbers $\mathbf{j} = (j_1, \dots, j_s)$, $j_1 < j_2 < \dots < j_s$, and s–tuples of arbitrary natural numbers $\mathbf{a} = (a_1, \dots, a_s)$ such that $a_1 j_1 + \dots + a_s j_s = i$ and $a_1 + \dots + a_s = k$, and the coefficients $c_{\mathbf{j},\mathbf{a}}$ are

$$c_{\mathbf{j},\mathbf{a}} = \frac{i!}{(j_1!)^{a_1} \dots (j_s!)^{a_s} a_1! \dots a_s!}.$$

Proof. In the case $i = 1$, the entire sum in the formula has just one possible term for $k = 1$, $j_1 = 1$ and $a_1 = 1$. As we have seen, this is the correct formula (3). The general case is proved by a tedious induction. □

Remark. As a hint for the induction mentioned in the proof above, let us describe in words, what the individual terms in the general formula mean. The value of k says how many times φ occurs in the term in question (and so many times X hits δu via the adjoint action and the sign is set appropriately), while the coefficients $c_{\mathbf{j},\mathbf{a}}$ express in how many different ways we may split i derivatives onto k copies of φ's in order to achieve the result $(\varphi^{(j_1)})^{a_1} \dots (\varphi^{(j_s)})^{a_s}$. Now, the differentiation of this formula and substitution from 3.1(3) means that we perform the last derivative on one of the φ's in the individual terms in the formula, or we attach a new φ to the existing terms which is differentiated only once. But this is exactly how all splittings of $i+1$ (distinguishable) hits of k (indistinguishable) targets are obtained from the answers to the same question for i derivatives and k or $k-1$ targets. Either the last hit has been to some existing one among k targets, i.e. we use the answer with i hits and k targets, or we have had to introduce a new target which was hit once, i.e. we used the answer with i hits and $k-1$ targets.

It is probably hard to deduce general results for all parabolic geometries and all classes of distinguished curves from this formula, but let us see how to use it in more specific situations.

3.3. **Irreducible parabolic geometries.** We are going to give a complete answer to our question for $|1|$–graded algebras $\mathfrak{g}$. In order to decide when two distinguished paths c^{b_1,X_1}, c^{b_2,X_2} parametrize the same curve we have to compute explicitly the consequences of $(\delta u)^{(i)}(0)\in\mathfrak{p}$ in relation to the necessary and sufficient conditions for the solution of the given problem. At the same time we shall get a complete and explicit description of the reparametrizations.

Lemma. *With the notation as above, $\delta u(0)\in\mathfrak{p}$ if and only if*

$$\varphi'(0)X_1 = X_2. \tag{1}$$

If $\delta u(0)\in\mathfrak{p}$, then $(\delta u)'(0)\in\mathfrak{p}$ if and only if

$$\frac{\varphi''(0)}{\varphi'(0)^2}X_1 = [X_1,[X_1,Z]], \tag{2}$$

and if $i\geq 2$ and $(\delta u)^{(j)}(0)\in\mathfrak{p}$ for all $j<i$, then $(\delta u)^{(i)}(0)\in\mathfrak{p}$ if and only if

$$\varphi^{(i+1)}(0) = \frac{(i+1)!}{2^i}\frac{\varphi''(0)^i}{\varphi'(0)^{i-1}},\ \textit{for all } i\geq 2. \tag{3}$$

Proof. Since our algebra $\mathfrak{g}$ is $|1|$–graded, all iterated adjoint actions by X_1 on $\delta u(0)$ vanish if the order is more then two. Thus only terms with $k\leq 2$ in Lemma 3.2 may survive and the general formula for $i\geq 1$ reads

$$\begin{aligned}(\delta u)^{(i)}(0) = {}&\varphi^{(i+1)}(0)X_1 - \varphi^{(i)}(0)[X_1,\delta u(0)] +\\ &\frac{1}{2}\sum_{\ell=1}^{i-1}\tfrac{i!}{\ell!(i-\ell)!}\varphi^{(\ell)}(0)\varphi^{(i-\ell)}(0))[X_1,[X_1,\delta u(0)]].\end{aligned}$$

Indeed, this can be either proved by inserting into the general formula from Lemma 3.2 or directly by induction.

Next, recall $\delta u(0) = \varphi'(0)X_1 - X_2 - [Z,X_2] - \frac{1}{2}[Z,[Z,X_2]]$. Thus (1) is obvious. We shall assume $\delta u(0)\in\mathfrak{p}$ and therefore

$$\delta u(0) = -\varphi'(0)([Z,X_1] + \tfrac{1}{2}[Z,[Z,X_1]]).$$

Now, (2) follows from the general formula with $i=1$. The most interesting step is the case $i=2$ (i.e. we deal with the third order jets of the curves, so that these must be determined by the lower order derivatives already). Indeed, the substitution of equalities in (1) and (2) into the general formula yields

$$\begin{aligned}(\delta u)^{(i)}(0) &= \varphi^{(i+1)}(0)X_1 + \varphi^{(i)}\varphi'(0)[X_1,[Z,X_1]] -\\ &\qquad \frac{1}{4}\sum_{\ell=1}^{i-1}\binom{i}{\ell}\varphi^{(\ell)}(0)\varphi^{(i-\ell)}(0)\varphi'(0)[X_1,[X_1,[Z,[Z,X_1]]]] + \text{term in } \mathfrak{g}_0\\ &= \left(\varphi^{(i+1)}(0) - \frac{\varphi^{(i)}(0)\varphi''(0)}{\varphi'(0)} - \frac{1}{4}\sum_{\ell=1}^{i-1}\binom{i}{\ell}\varphi^{(\ell)}(0)\varphi^{(i-\ell)}(0)\frac{\varphi''(0)^2}{\varphi'(0)^3}\right)X_1\\ &\qquad + \text{ term in } \mathfrak{g}_0.\end{aligned}$$

The structure of the latter equation implies that $\varphi^{(i+1)}(0)$ is determined uniquely in terms of the values $\varphi^{(k)}(0)$ with $k \le i$ and a direct computation checks that the formula in (3) is correct. □

Let us summarize what we have achieved so far. If the conditions of (1) and (2) are satisfied, than $\varphi(0)$, $\varphi'(0)$ and $\varphi''(0)$ are determined by the choice of the tangent vectors to the curve and by the element $Z \in \mathfrak{g}_1$ and we may define all other derivatives of φ by the formula (3). In particular, the special case $i = 2$ yields

$$\varphi'''(0) = \frac{3}{2}\frac{\varphi''(0)^2}{\varphi'(0)} \tag{4}$$

which reminds the well known Schwartzian differential equation. We shall see that the formulae for $\varphi^{(i)}(0)$ determine an analytic local solution for this equation.

If we denote $a = \varphi'(0)$ and $b = \varphi''(0)$, the Taylor development of the function φ at 0 must be

$$\varphi(t) = at + \frac{1}{2}bt^2 + \frac{1}{6}\frac{3}{2}\frac{b^2}{a}t^3 + \cdots + \frac{1}{(i+1)!}\frac{(i+1)!}{2^i}\frac{b^i}{a^{i-1}}t^{i+1} + \cdots .$$

Thus, we have obtained the geometric series $\varphi(t) = at\sum_{i=0}^{\infty}(\frac{bt}{2a})^i$ which converges locally around 0 and its value is

$$\varphi(t) = at(1 - \frac{b}{2a}t)^{-1}.$$

If we want to allow the reparametrizations with $\varphi(0) \ne 0$, we have just to replace equation 3.1(1) by $\exp(\varphi(t) - \varphi(0))\,\mathrm{Ad}_{b_1}X_1 = \exp(t\,\mathrm{Ad}_{b_2}X_2)\cdot u(t)$ and the result differs only by adding the value $\varphi(0)$ to the fraction above. In such a case the reparametrization takes a form

$$\varphi(t) = \frac{At+B}{Ct+D}, \text{ where } A = \varphi'(0) - \frac{\varphi''(0)}{2\varphi'(0)}\varphi(0),\ B = \varphi(0),\ C = -\frac{\varphi''(0)}{2\varphi'(0)},\ D = 1.$$

In particular, the solution with $\varphi''(0) = 0$ yields the affine reparametrization of the curve which of course have to be geodesics, too. The determinant of the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $\varphi'(0) \ne 0$, so we may normalize this to just 1 and we have proved:

3.4. **Proposition.** *Suppose that $\mathfrak{g}$ is $|1|$–graded. If the curves c^{b_1,X_1} and c^{b_2,X_2} coincide as unparametrized curves, then the corresponding local reparametrization φ has the form $\varphi(t) = \frac{At+B}{Ct+D}$, where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. Conversely, if $c = c^{b,X}$ is a parametrized geodesic then all curves $c \circ \varphi$ with reparametrizations $\varphi : \mathbb{R} \to \mathbb{R}$ of the latter form are again geodesics if and only if there is $Z \in \mathfrak{g}_1$ such that $[X,[X,Z]] = X$.*

Proof. It remains to prove the second statement. Obviously we may restrict ourselves to the case when $\varphi(0) = 0$. Then each φ satisfies all conditions from Lemma 3.3, provided there is a suitable Z for (2). □

Reparametrizations of the above type are called projective, see [2], where they are obtained as solutions of the Schwartzian differential equation $\varphi''' = \frac{3}{2}\frac{(\varphi'')^2}{\varphi'}$.

Corollary. *Suppose that $\mathfrak{g}$ is $|1|$–graded. Then the curves c^{e,X_1} and $c^{\exp Z,X_2}$ parametrize the same unparametrized geodesic if and only if there are $a \ne 0$ and b such that $X_2 = aX_1$ and $[X_2,[X_2,Z]] = bX_1$. This is equivalent to the existence of the projective local reparametrization φ which is uniquely determined by the initial condition $\varphi(0) = 0$, $\varphi'(0) = a$, and $\varphi''(0) = b$.*

3.5. *Example.* In the following examples we use the obvious fact that in the case of a $|1|$–grading, elements of P of the form $\exp(Z)$ for $Z \in \mathfrak{g}_1$ act trivially on $T_o(G/P) = \mathfrak{g}/\mathfrak{p}$. Then the P–action on this space factorizes over G_0.
(1) Conformal Riemannian structures correspond to $G = O(p+1, q+1)$ and the parabolic subgroup P as in 1.4(3). In an appropriate matrix representation, the grading of the Lie algebra $\mathfrak{g}$ has the form

$$\mathfrak{g}_{-1} = \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^t J & 0 \end{smallmatrix} \right) : X \in \mathbb{R}^{p+q} \right\}, \ \mathfrak{g}_0 = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{smallmatrix} \right) : A \in \mathfrak{o}(p,q), a \in \mathbb{R}, \right\},$$
$$\mathfrak{g}_1 = \left\{ \left(\begin{smallmatrix} 0 & Z & 0 \\ 0 & 0 & -JZ^t \\ 0 & 0 & 0 \end{smallmatrix} \right) : Z \in \mathbb{R}^{p+q*} \right\}.$$

Here J is the matrix defining the standard pseudo–metric of signature (p, q) on $\mathbb{R}^{p+q} = \mathfrak{g}_{-1}$.

A direct calculation shows that $[X, [X, Z]] = -2Z(X)X - ||X||^2 JZ^t$, where $||X||^2 = X^t JX$ and $Z(X) = ZX$ is a real number. Obviously, the space $\mathfrak{g}_{-1}$ splits into three different orbits of the action of G_0 according to the sign of $||X||^2$. The orbit of null–vectors is of particular interest, since $[X, [X, Z]] = -2Z(X)X$ in that case. This just means that all distinguished curves with the common tangent null–vector differ by a reparametrization which recovers the classical result that the null geodesics of the metrics in the conformal class together with the class of projective parametrizations are invariants of the conformal structure. Of course, these curves will have their tangent vectors null in all their points.

For all tangent vectors which are not null, the second derivative may be chosen arbitrarily. So that the standard fiber S in 2.8 has arbitrary entries in the bottom row if X is not null, but only multiples of X if X is null. On the other hand, there always is an element $Z \in \mathfrak{g}_1$ such that $[[Z, X], X] = X$, so all geodesics carry a natural projective structure.

(2) Almost Grassmannian structures. In this case, $G = SL(n + m, \mathbb{R})$ and the parabolic subgroup P is the stabilizer of $\mathbb{R}^n \subset \mathbb{R}^{n+m}$, so it consists of block upper triangular matrices with two blocks of sizes n and m. On the infinitesimal level,

$$\mathfrak{g}_{-1} = \{ \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix} \right) : X \in \mathbb{R}^{mn} \}, \ \mathfrak{g}_0 = \{ \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix} \right) : \operatorname{tr}(A) + \operatorname{tr}(B) = 0 \},$$
$$\mathfrak{g}_1 = \{ \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix} \right) : Z \in \mathbb{R}^{nm} \}.$$

First, it is easy to see that the subgroup G_0 consists of block diagonal matrices, and its action on $\mathfrak{g}_{-1}$ is given by $X \mapsto TXS^{-1}$, $(S, T) \in G_0$. Thus two elements of $\mathfrak{g}_{-1}$ lie in the same G_0–orbit if and only if they have the same rank. Further, the computation of the iterated bracket yields $[X, [X, Z]] = -2XZX$. In particular, the choice of the pseudoinverse matrix $Z = X^\dagger$ provides always a multiple of X and so all generalized geodesics enjoy the distinguished projective structure. If the rank of X is one, then we may choose X to be the matrix with the left upper element $x_{11} = 1$ and all other 0. Then $[X, [X, Z]]$ equals to $z_{11}X$ for all Z and so this behavior must be shared by all matrices of rank one. Thus, the directions corresponding to rank one matrices behave like null directions in pseudo–conformal geometries. The other extreme is that X has maximal rank. Then one gets a lot of freedom in the available second derivatives of the curves. The case that all elements of $\mathfrak{g}_{-1}$ are possible second derivatives occurs only if $m = n$ and X has rank n.

(3) Projective structures are the special case $n = 1$ of Example (2) above. In this case, the rank of $X \neq 0$ is always one. More explicitly, the product ZX is a real number, so the bracket $[X, [X, Z]]$ is always a multiple of X. From this it follows that all unparametrized distinguished curves are determined by the direction in a given point. This agrees with the classical definition of a projective structure as a class of affine connections sharing the same unparametrized geodesics. All such

connections are parametrized by smooth one–forms on the base manifold and they correspond to the Weyl connections defined in [4].

4. More refinements

In this section we improve the estimates on the jet in a point needed to pin down a geodesic for geodesics of certain types. The most general results is Theorem 4.3 but since the proofs of these results are a bit technical, we prefer to discuss two simpler special cases first.

4.1. **Curves tangent to $T^{-1}M$.** Let M be any manifold equipped with a parabolic geometry of some fixed type (G, P). A (generalized) geodesics with development of the form $c^{b,X}$ emanates in a direction in $T^{-1}M$ if and only if $X \in \mathfrak{g}_{-1}$. Thus we are dealing with distinguished curves of type $\mathcal{C}_{\mathfrak{g}_{-1}}$ and from Proposition 1.3 we see that they will be tangent to the distribution $T^{-1}M$ in all points.

To discuss geodesics of type $\mathcal{C}_{\mathfrak{g}_{-1}}$, by Proposition 2.6 we have to fix $X \in \mathfrak{g}_{-1}$ and study the curves $c^{\exp(Z),Y}$ for $Z \in \mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $Y \in \mathfrak{g}_{-1}$ such that $\underline{\mathrm{Ad}}(\exp(Z))(Y) = X$. Since $Y \in \mathfrak{g}_{-1}$ we get $\underline{\mathrm{Ad}}(\exp(Z))(Y) = Y$ for any Z, so we have to consider all curves of the form $c^{\exp(Z),X}$ with $Z \in \mathfrak{p}_+$. By [3, 2.10] we get a nicer presentation of $\exp(Z)$. Namely, there are unique elements $Z_i \in \mathfrak{g}_i$ for $i = 1, \dots, k$ such that $\exp(Z) = \exp(Z_1) \cdots \exp(Z_k)$. Since $\mathrm{Ad}(\exp(W)) = e^{\mathrm{ad}(W)}$ for each $W \in \mathfrak{g}$ we get

$$\mathrm{Ad}_{\exp Z} X = \sum_{i_1,\dots,i_k} \tfrac{1}{i_1!\cdots i_k!}(\mathrm{ad}\, Z_1)^{i_1} \cdots (\mathrm{ad}\, Z_k)^{i_k} X.$$

Moreover, since $X \in \mathfrak{g}_{-1}$ a summand in the right hand side lies in $\mathfrak{g}_\ell$ if and only if $i_1 + 2i_2 + \cdots + ki_k = \ell + 1$.

We need another observation for the proof: Suppose that $Y \in \mathfrak{g}$ is any element. The Jacobi identity reads as $\mathrm{ad}_X \circ \mathrm{ad}_Y = \mathrm{ad}_{[X,Y]} + \mathrm{ad}_Y \circ \mathrm{ad}_X$. Inductively, this implies that $\mathrm{ad}_X^n \circ \mathrm{ad}_Y$ can be written as a linear combination of terms of the form $\mathrm{ad}_{\mathrm{ad}_X^i(Y)} \circ \mathrm{ad}_X^j$ with $0 \leq i, j$ and $i + j = n$. In particular, if $\mathrm{ad}_X^{\ell+1}(Y) = 0$ for some $\ell \geq 0$, then for each $n > \ell$ there is a linear map φ such that $\mathrm{ad}_X^n \circ \mathrm{ad}_Y = \varphi \circ \mathrm{ad}_X^{n-\ell}$. Of course, it is not difficult to compute φ explicitly, but we will not need this explicit form.

Proposition. *A parametrized generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-1}}$ in a parabolic geometry corresponding to a $|k|$–grading of $\mathfrak{g}$ is uniquely determined by its $(k+1)$–jet in a single point.*

Proof. Of course, we have proved this for $k = 1$ in 2.7. In view of the above discussion and the last observation in 2.6 we have to show that for each fixed $X \in \mathfrak{g}_{-1}$ any curve of the form $c^{\exp(Z_1)\cdots\exp(Z_k),X}$ with $Z_i \in \mathfrak{g}_i$ which has the same $k+1$–jet in 0 as $c^{e,X}$ actually equals $c^{e,X}$.

Given $Z_1, \dots, Z_k$ define $W := \mathrm{Ad}(\exp(Z_1)\cdots\exp(Z_k))(X) - X \in \mathfrak{p}$. From the above discussion we see that

$$W = \sum_{i_1,\dots,i_k} \tfrac{1}{i_1!\cdots i_k!}(\mathrm{ad}\, Z_1)^{i_1} \cdots (\mathrm{ad}\, Z_k)^{i_k} X, \tag{1}$$

where the sum is over all $(i_1, \dots, i_k)$ such that $0 < i_1 + 2i_2 + \cdots + ki_k \leq k + 1$. Considering the curve $u(t)$ associated to $c^{e,X}$ and $c^{\exp(Z_1)\cdots\exp(Z_k),X}$ by equation 2.2(1), we see from 2.3 that $\delta u(0) = -W$ and Lemma 2.4 implies that $(\delta u)^{(i)}(0) = (-1)^{i+1}\,\mathrm{ad}_X^i(W)$. Consequently by Lemma 2.2 proving the result boils down to showing that $\mathrm{ad}_X^i(W) \in \mathfrak{p}$ for all $i \leq k$ implies $\mathrm{ad}_X^i(W) \in \mathfrak{p}$ for all $i \in \mathbb{N}$.

For each $\ell = 1, \dots, k$ define W'_ℓ to be the sum of those terms in the expression (1) for W for which all i_j with $j > \ell$ are zero, and put $W''_\ell = W - W'_\ell$. In particular, we have $W''_k = 0$, i.e. $W'_k = W$.

Claim: If $\operatorname{ad}^i_X(W) \in \mathfrak{p}$ for all $i \le \ell$, then for each $j \le \ell$ we have $\operatorname{ad}^{j+1}_X(Z_j) = 0$, and for each $n > \ell$ we get $\operatorname{ad}^n_X(W'_\ell) \in \mathfrak{p}$.

We prove this claim by induction on ℓ. If $\ell = 1$, we know that $\operatorname{ad}_X(W) \in \mathfrak{p}$. Looking at formula (1) for W and taking into account that $X \in \mathfrak{g}_{-1}$ we see that $\operatorname{ad}_X(W) \in \mathfrak{p}$ implies (and is actually equivalent to) $[X, [Z_1, X]] = 0$ and thus to $\operatorname{ad}^2_X(Z_1) = 0$. Hence it remains to show that $\operatorname{ad}^n_X(W'_1) \in \mathfrak{p}$ for all $n > 1$. By definition, $W'_1 = \sum_{i=1}^{k+1} \frac{1}{i!} \operatorname{ad}^i_{Z_1} X$. Thus $\operatorname{ad}^n_X(W'_1) \in \mathfrak{p}$ is equivalent to $\operatorname{ad}^n_X \circ \operatorname{ad}^i_{Z_1} X = 0$ for $i \le n$. From above we know that $\operatorname{ad}^2_X(Z_1) = 0$ implies that $\operatorname{ad}^n_X \circ \operatorname{ad}_{Z_1} = \varphi \circ \operatorname{ad}^{n-1}_X$, so inductively we conclude that $\operatorname{ad}^n_X \circ \operatorname{ad}^i_{Z_1} = \psi \circ \operatorname{ad}^{n-i+1}_X \circ \operatorname{ad}_Z$ for some linear map ψ and by assumption $n - i + 1 > 0$. Hence applying this element to X we get $\psi \circ \operatorname{ad}^{n-i+2}_X(Z)$ which vanishes since $n - i + 2 \ge 2$. This completes the proof of the case $\ell = 1$.

Assume inductively that $\ell > 1$ and we have proved the result for $\ell - 1$. Given that $\operatorname{ad}^i_X(W) \in \mathfrak{p}$ for all $i \le \ell$, we by induction conclude that $\operatorname{ad}^{j+1}_X(Z_j) = 0$ for $j = 1, \dots, \ell - 1$. Moreover, we know by induction that $\operatorname{ad}^\ell_X(W) \in \mathfrak{p}$ implies $\operatorname{ad}^\ell_X(W''_{\ell-1}) \in \mathfrak{p}$. By definition of $W''_{\ell-1}$ the only term in $\operatorname{ad}^\ell_X(W''_{\ell-1})$ which does not automatically lie in $\mathfrak{p}$ is $\operatorname{ad}^\ell_X([Z_\ell, X])$, so we conclude that $\operatorname{ad}^{\ell+1}_X(Z_\ell) = 0$. Hence it remains to show that $\operatorname{ad}^n_X(W'_\ell) \in \mathfrak{p}$ for all $n > \ell$. Since we know by induction that $\operatorname{ad}^n_X(W'_{\ell-1}) \in \mathfrak{p}$, it suffices to consider $\operatorname{ad}^n_X(W'_\ell - W'_{\ell-1})$. Now from the expression (1) for W we conclude that

$$W'_\ell - W'_{\ell-1} = \sum_{i_1, \dots, i_\ell} \tfrac{1}{i_1! \cdots i_\ell!} (\operatorname{ad} Z_1)^{i_1} \cdots (\operatorname{ad} Z_\ell)^{i_\ell} X,$$

with the sum going over $i_\ell > 0$ and $i_1 + 2i_2 + \dots + \ell i_\ell \le k+1$. Obviously, $\operatorname{ad}^n_X(W'_\ell - W'_{\ell-1}) \in \mathfrak{p}$ is equivalent to vanishing of $\operatorname{ad}^n_X \circ \operatorname{ad}^{i_1}_{Z_1} \circ \dots \circ \operatorname{ad}^{i_\ell}_{Z_\ell}$ for all multi–indices $(i_1, \dots, i_\ell)$ such that $i_1 + 2i_2 + \dots + \ell i_\ell \le n$. Since $\operatorname{ad}^{j+1}_X(Z_j) = 0$, we see from above that $\operatorname{ad}^m_X \circ \operatorname{ad}_{Z_j} = \varphi \circ \operatorname{ad}^{m-j}_X$ for $m > j$. Inductively we conclude that for $m > ji_j$ we get $\operatorname{ad}^m_X \circ \operatorname{ad}^{i_j}_{Z_j} = \psi \circ \operatorname{ad}^{m - ji_j}_X$ for some linear map ψ. Thus we conclude that

$$\operatorname{ad}^n_X \circ \operatorname{ad}^{i_1}_{Z_1} \circ \dots \circ \operatorname{ad}^{i_\ell}_{Z_\ell} = \tilde{\psi} \circ \operatorname{ad}^{n - i_1 - 2i_2 - \dots - \ell(i_\ell - 1)}_X \circ \operatorname{ad}_{Z_\ell},$$

and by assumption $n - i_1 - 2i_2 - \dots - \ell(i_\ell - 1) \ge \ell$. Thus applying the right hand side to X, we obtain $\operatorname{ad}^r_X(Z_\ell)$, and by construction $r \ge \ell + 1$, so this vanishes. Hence the proof of the claim is complete.

But taking the claim in the case $\ell = k$, we see that $\operatorname{ad}^i_X(W) = 0$ for all $i \le k$ implies that $\operatorname{ad}^n_X(W'_k) \in \mathfrak{p}$ for all $n > k$. Since we have observed above that $W'_k = W$, this completes the proof. □

4.2. **The case $A = \mathfrak{g}_{-k}$.** The other extreme class of geodesics on a manifolds M equipped with a parabolic geometry of type (G, P) with $|k|$–graded $\mathfrak{g}$ is provided by the generalized geodesics of type $\mathcal{C}_{\mathfrak{g}_{-k}}$. Of course, for a point $x \in M$ and a tangent vector $\xi \in T_xM$ one must have $\xi \in T_xM \setminus T^{-k+1}_x M$ in order to have a nontrivial geodesic of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ in direction ξ. On the other hand, this condition is not sufficient for such a geodesic, and the directions of these geodesics usually form a smaller cone in each tangent space.

An important special case is parabolic contact geometries, i.e. those geometries corresponding to $|2|$–gradings, such that $\mathfrak{g}_{-2}$ has dimension one and the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non–degenerate. These geometries always have an underlying contact structure. In these cases geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$ always exist for all directions in $TM \setminus T^{-1}M$. A very well known instance of this type of generalized geodesics is provided by the Chern–Moser chains on hypersurface type CR–structures. A slightly more general example of this type was studied for 6–dimensional CR–structures of codimension 2, in [16].

Let us recall that reparametrizations of the form $\varphi(t) = \frac{At+B}{Ct+D}$ with $A \neq 0$ and $AD - BC = 1$ are called projective.

Theorem. *Each generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ in a parabolic geometry of type (G, P) corresponding to a $|k|$–grading on $\mathfrak{g}$ is uniquely determined by its 2–jet in a single point. Moreover, if two of such curves coincide up to parametrization, then this reparametrization is projective. Conversely, given a generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ corresponding to $(u, X) \in \mathcal{G} \times \mathfrak{g}_{-k}$, every projective change of parametrization defines a geodesic of the same type if and only if there exists a $Z \in \mathfrak{g}_k$ such that $[X, [X, Z]] = X$.*

Proof. From 2.6 and 4.1 we know that for each $X \in \mathfrak{g}_{-k}$ we have to compare $c^{e,X}$ to all curves of the form $c^{b,Y}$ with $b = \exp(Z_1) \cdots \exp(Z_k)$ for $Z_i \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_{-k}$ and $\underline{\mathrm{Ad}}(b)(Y) = X$. The last condition immediately implies that $Y = X$. Expanding $W = \mathrm{Ad}(b)(X) - X$ as in equation 4.1(1), we conclude that if this expression has trivial component in $\mathfrak{g}_{-k+1}$, then $[Z_1, X] = 0$. Hence we may omit all terms in the expansion for which i_1 is the only nonzero index. Vanishing of the component in $\mathfrak{g}_{-k+2}$ then implies $[Z_2, X] = 0$, so we may omit terms in which only i_1 and i_2 are nonzero. Inductively, we get $[Z_\ell, X] = 0$ for all $\ell = 1, \dots, k-1$. Hence we conclude that $\delta u(0) = -[Z_k, X] - \frac{1}{2}[[Z_k, [Z_k, X]]$. Now $(\delta u)'(0) \in \mathfrak{p}$ implies $[X, [Z_k, X]] = 0$ and so $(\delta u)'(0) = 0$ exactly as in 2.7.

Concerning reparametrizations, we may adapt the proofs of Lemma 3.3 and Proposition 3.4 along the same lines. Using the notation from there, the condition $\delta u(0) \in \mathfrak{p}$ implies $X_2 = \varphi'(0)X_1$ and moreover $[Z_\ell, X_2] = 0$ for all $\ell \leq k-1$, inductively as above, and this is the only difference to the $|1|$–graded case. Further, $(\delta u)'(0) \in \mathfrak{p}$ if and only if $\varphi''(0)X_1 = \varphi'(0)^2[X_2, [X_2, Z_2]]$ and we finish the proof exactly as in the $|1|$–graded case. □

More generally, let us consider generalized geodesics of type $\mathcal{C}_{\mathfrak{g}_{-j}}$ with arbitrary j. Geodesics of this type are always curves with tangents in $T^{-j}M$ and they emanate from a given point in M in certain directions in $T^{-j}M \setminus T^{-j+1}M$.

4.3. **Theorem.** *Each generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-j}}$ in a parabolic geometry of type (G, P) with a $|k|$–graded $\mathfrak{g}$, $1 \leq j \leq k$ is uniquely determined by its r–jet in a single point provided that $rj \geq k+1$.*

Proof. This is a combination of the proofs of Theorem 4.2 and of Proposition 4.1 with minor generalizations, so we just outline the basic steps: For $X \in \mathfrak{g}_{-j}$ we have to compare $c^{e,X}$ to $c^{b,Y}$ for $b = \exp(Z_1) \cdots \exp(Z_k)$ with $Z_i \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_{-j}$ and $\underline{\mathrm{Ad}}(b)(Y) = X$. This immediately implies $Y = X$, and we put $W = \mathrm{Ad}(b)(X) - X$ and expand this as in 4.1(1). The proof boils down to showing that $\mathrm{ad}^i_X(W) \in \mathfrak{p}$ for $i \leq r$ implies the same result for all i. As in 4.2, $W \in \mathfrak{p}$ implies that $[Z_\ell, X] = 0$ for all $\ell < j$, so in the notation of the proof of Proposition 4.1 we obtain $W'_{j-1} = 0$.

The analog of the claim in the proof of Proposition 4.1 is that if $\mathrm{ad}^i_X(W) \in \mathfrak{p}$ for all $i \leq \ell$, then for each $s \leq \ell$ and $m < (s+1)j$, we get $\mathrm{ad}^{s+1}_X(Z_m) = 0$, and further $\mathrm{ad}^n_X(W'_{j\ell-1}) = 0$ for all $n > \ell$. This is proved by induction using the same arguments as in 4.1.

For $\ell = r$, we obtain $jr \geq k+1$, and as in 4.1, $W'_k = W$, and we conclude that $\mathrm{ad}^i_X(W) \in \mathfrak{p}$ for all $i \leq r$ implies the same property for all i as required. □

The following two examples expose the diversity of the possible behavior of various classes of distinguished curves in specific parabolic geometries. All claims may be checked by direct computations following the results above and their more detailed version may be also found in [21].

4.4. *Example.* Let us briefly illustrate the general results in the simplest cases of parabolic contact structures, so we are dealing with $|2|$–gradings such that $\mathfrak{g}_{-2}$ is one–dimensional and the bracket $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-2}$ is non–degenerate. As we have mentioned in 4.2, we get in each direction outside the contact subbundle geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$ which generalize the Chern–Moser chains for CR–structure. From 4.3 we know that they are determined by their two–jet in a point as parametrized curves, and it follows that they are uniquely determined by their direction in one point up to parametrization, by dimension reasons. Moreover, each such geodesic carries a natural projective structure of distinguished parametrizations.

Apart of these types of generalized geodesics, there are several other possibilities for non–equivalent types of geodesics as we may observe already at the simplest example of G being a real form of $SL(3,\mathbb{C})$ and P the Borel subgroup:
(1) $G=SL(3,\mathbb{R})$. The corresponding geometries are the Lagrangian contact structures on 3–dimensional manifolds, i.e. three dimensional contact structures endowed with a decomposition of the contact subbundle into a direct sum of two line subbundles, cf. [18]. Geometrically, there are four different classes of tangent vectors. First, we have vectors tangent to one of the two subbundles (two classes), then there are the remaining vectors in the contact subbundle, and finally those outside of the contact subbundle.

The subgroup P consist of all elements of G which are upper triangular, so on the Lie algebra level, we obtain $\mathfrak{n}$ as the subalgebra of strictly lower triangular matrices, with the two entries directly below the main diagonal corresponding to $\mathfrak{g}_{-1}$ and the entry in the lower left corner corresponding to $\mathfrak{g}_{-2}$. The action of the subgroup G_0 rescales each entry of a matrix in $\mathfrak{n}$ by a nonzero factor, so the G_0–orbits in $\mathfrak{n}$ are determined simply by the nonzero entries of a matrix.

First, there are two canonical invariant subspaces in $\mathfrak{g}_{-1}$ which correspond to the Lagrange subspaces of the contact distribution. They are $A_1=\left\{\left(\begin{smallmatrix}0&0&0\\ *&0&0\\ 0&0&0\end{smallmatrix}\right)\right\}$ and $A_2=\left\{\left(\begin{smallmatrix}0&0&0\\ 0&0&0\\ 0&*&0\end{smallmatrix}\right)\right\}$, respectively, where the star denotes a nonzero entry. Generalized geodesics of these types exist exactly in directions tangent to one of the two line subbundles, so the two classes are disjoint but have the same properties. In both cases they behave just like null–geodesics in conformal geometry, i.e. each such curve is determined by its 2–jet in one point and with a given tangent vector there is a 1–dimensional family of parametrized generalized geodesics determined by elements of the form $\left(\begin{smallmatrix}0&*&0\\ 0&0&0\\ 0&0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0&0\\ 0&0&*\\ 0&0&0\end{smallmatrix}\right)\in\mathfrak{p}_+$, respectively. Moreover, all curves from this family coincide up to a projective reparametrization.

For $A=\mathfrak{g}_{-1}$ we get directions in the contact distribution. From 4.1 we know that such curves are determined by their 3–jet in one point. There is a 3–dimensional family of parametrized generalized geodesics (corresponding to all elements in $\mathfrak{p}_+$) sharing a given tangent vector, which is not tangent to one of the two line subbundles. Admissible reparametrizations are the projective ones, so the dimension of the space of unparametrized generalized geodesics with the common direction in $T^{-1}M$ but outside of the Lagrange subspaces is two.

Now we discuss the curves emanating in directions which do not belong to the contact distribution. For $A=\mathfrak{g}_{-2}$ we obtain the analog of CR–chains as described in Theorem 4.3.

Besides these chains, there are another curves going in all directions except those in the contact distribution; this class of curves corresponds to the generic choice of $A=\left\{\left(\begin{smallmatrix}0&0&0\\ *&0&0\\ *&*&0\end{smallmatrix}\right)\right\}$. Curves of this type are determined by a 2–jet and to any tangent vector there is a 3–dimensional family of generalized geodesics. This set is parametrized by elements of $\mathfrak{p}_+$. In contrast to the previous cases, there are no two curves with the common tangent vector, which would be the same up to a reparametrization. So here only affine reparametrizations are allowed.

The two G_0–orbits in $\mathfrak{n}$, which have not yet been mentioned are $\left\{\begin{pmatrix} 0&0&0\\ *&0&0\\ *&0&0\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 0&0&0\\ 0&0&0\\ *&*&0\end{pmatrix}\right\}$. Any element of either of these can by mapped to $\mathfrak{g}_{-2}$ by some Ad_b with $b\in P$, and vice versa. Hence from 2.6 we know that these lead to the same curves as $A=\mathfrak{g}_{-2}$, and thus the discussion is complete.

(2) $G=SU(2,1)$. The corresponding geometries are non–degenerated strictly pseudoconvex 3–dimensional CR–structures. In contrast to the Lagrangian contact structures, there is no distinguished G_0–invariant subset in $\mathfrak{g}_{-1}$, so the discussion is similar as above, but easier, so we skip the details.

4.5. *Example.* Let us finish the paper with the discussion of generalized geodesics in the so called x–x–dot geometries (the name comes from the shape of the Dynkin diagram with crosses describing the corresponding parabolic subgroup in $\mathfrak{sl}(4,\mathbb{C})$). Such structures appear as correspondence spaces in classical twistor theory, and they are related to the geometric theory of ODE's.

Let us consider the group $G=SL(4,\mathbb{R})$ with the parabolic subgroup P which may be indicated as $P=\left\{\begin{pmatrix} *&*&*&*\\ 0&*&*&*\\ 0&0&*&*\\ 0&0&*&*\end{pmatrix}\right\}$. The following discussion may be also understood as a block–wise generalization of the discussion of the matrices in the example 4.4(1) which we shall call the 'x–x' case. The examples with more 'dots' in the Dynkin diagram and just two crosses over the first two nods on the left will behave quite similarly to the x–x–dot case.

The Lie algebra $\mathfrak{g}_-$ is described by block matrices of the form $\mathfrak{g}_-=\left\{\begin{pmatrix} 0&0&0\\ x_1&0&0\\ X_2&X_1&0\end{pmatrix}\right\}$, where the blocks x_1, X_1 generate the subalgebra $\mathfrak{g}_{-1}$ and X_2 belongs to $\mathfrak{g}_{-2}$. The truncated adjoint action of an element $\exp\begin{pmatrix} 0&z_1&Z_2\\ 0&0&Z_1\\ 0&0&0\end{pmatrix}\in P_+$ is given by the formula $\begin{pmatrix} 0&0&0\\ x_1&0&0\\ X_2&X_1&0\end{pmatrix}\mapsto\begin{pmatrix} 0&0&0\\ x_1+Z_1(X_2)&0&0\\ X_2&X_1-z_1X_2&0\end{pmatrix}$.

In accordance with the x–x case, there are two distinguished G_0–invariant subspaces in $\mathfrak{g}_{-1}$ corresponding to the blocks x_1 and X_1, respectively. The generalized geodesics emanating in the appropriate directions of the distribution $T^{-1}M$ have got the same properties as above. In particular, curves of this type are determined by a 2–jet but as unparametrized curves they are given by a direction. Parametrized geodesics of this type with the common tangent vector form a 1–dimensional family parametrized by the elements of the form $\left\{\begin{pmatrix} 0&z_1&0\\ 0&0&0\\ 0&0&0\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 0&0&0\\ 0&0&Z_1\\ 0&0&0\end{pmatrix}\right\}\Big/K$, respectively, where $K=\left\{\begin{pmatrix} 0&0&0\\ 0&0&Z_1\\ 0&0&0\end{pmatrix}:Z_1(X_1)=0\right\}$, briefly written as $K=\{Z_1(X_1)=0\}$. In the latter case, what really affects on the 2–jet is the value $Z_1(X_1)$ instead of Z_1, that is why the quotient appears.

Generalized geodesics with the generic directions in $T^{-1}M$ are determined by a 3–jet and to any tangent vector there is a 3–dimensional family of (projectively) parametrized geodesics described by elements of $\mathfrak{p}_+/K$, where $K=\{z_1=0, Z_1(X_1)=0, Z_2(X_1)=0\}$.

The only contrast with the x–x case appears in the directions not belonging to $T^{-1}M$. The analogy of chains, i.e. the curves from $\mathcal{C}_{\mathfrak{g}_{-2}}$, does not exhaust all directions out of the distribution $T^{-1}M$ but only a 4–dimensional 'cylinder' $\left\{\begin{pmatrix} 0&0&0\\ Z_1(X_2)&0&0\\ X_2&-z_1X_2&0\end{pmatrix}\right\}\subset\mathfrak{g}_-$ (at each point) according to the orbit of $\mathfrak{g}_{-2}$ with respect to the truncated adjoint action of P. Obviously, the complement is formed by all elements of $\mathfrak{g}_-$ such that vectors X_1 and X_2 are linearly independent; this set is G_0–invariant. Now, the discussion splits into two branches where the first one follows the x–x case, but the second one brings something new.

Let us start with the directions given by chains. First of all, it is easy to verify that the sets of curves given by the invariant subsets $A_1 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_2 & aX_2 & 0 \end{pmatrix}\right\}$ and $A_2 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ X_2 & 0 & 0 \end{pmatrix}\right\}$ are the same and both of these choices coincide with chains defined by $A = \mathfrak{g}_{-2}$. Of course, all chains depend on 2–jets in one point. For any tangent vector of this type there is a 1–dimensional family of parametrized chains, described by the elements of $\mathfrak{g}_2/\{Z_2(X_2) = 0\}$, all parameterizing the same curve.

Besides the chains, there is a 3–dimensional family of generalized geodesics emanating in the same directions as chains from a given point, defined by the subset $A = \left\{\begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ X_2 & aX_2 & 0 \end{pmatrix}\right\}$. This family is parametrized by the quotient $\mathfrak{p}_+/K$, where $K = \{z_1 = 0, Z_1(X_2) = 0, Z_2(X_2) = 0\}$. Curves of this type are also determined by a 2–jet and the admissible reparametrizations are affine.

Finally, we fix a tangent vector which does not belong to $T^{-1}M$ and is not tangent to a chain. By analogy to the previous case, there are two disjunct classes of generalized geodesics emanating in such directions, but having rather different properties than above. The first class corresponds to the invariant subset $A = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_2 & X_1 & 0 \end{pmatrix}\right\}$, where X_1 and X_2 are supposed to be linearly independent (we assume this in the rest of the example). Curves of this type are determined by a 2–jet, they allow projective reparametrizations, and to the given tangent vector there is a 3–dimensional family of parametrized geodesics described by elements of the form $\left\{\begin{pmatrix} 0 & 0 & Z_2 \\ 0 & 0 & Z_1 \\ 0 & 0 & 0 \end{pmatrix} : Z_1(X_2) = 0\right\}$. The last distinguished class of curves corresponds to the generic choice of $A = \left\{\begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ X_2 & X_1 & 0 \end{pmatrix}\right\}$. Again, curves of this type are determined by a 2–jet and allow the projective class of reparametrizations. The family of parametrized geodesics with the common tangent vector has got the maximal dimension 5 and it is described by all elements of $\mathfrak{p}_+$.

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A.C.: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A–1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Wien, Austria

J.S. and V.Ž.: Department of Algebra and Geometry, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic

E-mail address: andreas.cap@esi.ac.at, slovak@math.muni.cz, zadnik@math.muni.cz