ADDENDUM TO "THE GEOMETRY OF HYPERBOLIC AND ELLIPTIC CR-MANIFOLDS OF CODIMENSION TWO"

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The aim of this article is to show how the individual harmonic components of the torsion of the canonical Cartan connection of embedded hyperbolic and elliptic CR-manifolds at a given point can be read off from the third order terms of the defining equation given in normal form. The general theory ensures that the vanishing of each of these one-dimensional components implies striking geometric consequences and we link each of them to an easily computable coefficient in the normal form. This allows to correct a mistake in [SS00] where it was claimed that four torsion components out of six vanish automatically for embedded CR-manifolds. The failure in that article appears already in Lemma 1.1 where the second order osculation was not dealt with carfully enough. At the same time, the rest of [SS00] is essentially worked out for abstract CR–structures and so the validity of the procedures and results has not been effected in general. In what follows, we use the terminology and notation of [SS00] without further comments.

In both, the elliptic and hyperbolic case, the 2-dimensional CR subspace H of the tangent space splits canonically into the direct sum of (complex) one-dimensional subspaces $H^L \oplus H^R$. This splitting allows to introduce a new CR structure $\tilde{J}$ on M by flipping the initial CR structure J to $\bar{J}$ at one of the summands, say H^R. (The other choice leads to the conjugate CR structure and so the new structure $\tilde{J}$ is uniquely given up to conjugation.) We will see that the 4 torsion components can be interpreted as integrability conditions for these almost direct product and almost CR structures.

Consider first the hyperbolic case. Let $M \subset \mathbb{C}^4$ be a hyperbolic manifold given by an equation that meets the normal form conditions (see [Lob88]) up to order 3:

$$\operatorname{Im} w^1 = |z_1|^2 + n_{11\bar{2}} z_1^2 \bar{z}_2 + n_{22\bar{2}} z_2^2 \bar{z}_2 + \bar{n}_{11\bar{2}} \bar{z}_1^2 z_2 + \bar{n}_{22\bar{2}} \bar{z}_1^2 z_2 + \cdots$$
$$\operatorname{Im} w^2 = |z_2|^2 + n_{11\bar{1}} z_1^2 \bar{z}_1 + n_{22\bar{1}} z_2^2 \bar{z}_2 + \bar{n}_{11\bar{1}} \bar{z}_1^2 z_1 + \bar{n}_{22\bar{1}} \bar{z}_1^2 z_1 + \cdots$$

where the dots indicate higher order terms.

Two out of the six torsion components, namely those corresponding to the cohomologies represented by cochains of the form

$$\mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-1} \to \mathfrak{g}^L_{-1} \quad \text{(antilinear in both arguments)}$$
$$\mathfrak{g}^R_{-1} \times \mathfrak{g}^L_{-1} \to \mathfrak{g}^R_{-1} \quad \text{(antilinear in both arguments)},$$

are responsible for the integrability of the CR structure and, therefore, vanish for embedded CR-manifolds.

It was shown in [SS00] that the remaining four torsion components can be interpreted as algebraic brackets on M. Let H^L and H^R be as above. These line bundles are determined by the condition

$$\pi[Z_1, \bar{Z}_2] = 0 \tag{1}$$

where Z_1 is a section of H^L, Z_2 is a section of H^R and π is the canonical projection $TM \to QM = TM/HM$.

The holomorphic part $H^{1,0}M \subset \mathbb{C} \otimes HM$ is generated by

$$X_1 = \tfrac{\partial}{\partial z_1} + 2\,\mathrm{i}(\bar{z}_1 + 2n_{11\bar{2}} z_1 \bar{z}_2)\tfrac{\partial}{\partial w_1} + 2\,\mathrm{i}(2n_{11\bar{1}}|z_1|^2 + \bar{n}_{11\bar{1}}\bar{z}_1^2 + \bar{n}_{22\bar{1}}\bar{z}_2^2)\tfrac{\partial}{\partial w_2}$$
$$X_2 = \tfrac{\partial}{\partial z_2} + 2\,\mathrm{i}(2n_{22\bar{2}}|z_2|^2 + \bar{n}_{22\bar{2}}\bar{z}_2^2 + \bar{n}_{11\bar{2}}\bar{z}_1^2)\tfrac{\partial}{\partial w_1} + 2\,\mathrm{i}(\bar{z}_2 + 2n_{22\bar{1}} z_2 \bar{z}_1)\tfrac{\partial}{\partial w_2}$$

(We wrote only the terms of order less than three.)

One can find generators Z_1 and Z_2 of H^L and H^R in the form

$$Z_1 = X_1 + A_{12} X_2$$
$$Z_2 = A_{21} X_1 + X_2$$

where A_{12}, A_{21} are chosen so that (1) be satisfied. We find

$$A_{12} = -2\bar{n}_{22\bar{1}}\ \bar{z}_2 + 4n_{11\bar{2}} n_{11\bar{1}} z_1^2 + 4n_{11\bar{2}}\bar{n}_{11\bar{1}}\ |z_1|^2$$
$$A_{21} = -2\bar{n}_{11\bar{2}}\ \bar{z}_1 + 4n_{22\bar{1}} n_{22\bar{2}} z_2^2 + 4n_{22\bar{1}}\bar{n}_{22\bar{2}}\ |z_2|^2.$$

Define

$$W_1 := \frac{1}{2\,\mathrm{i}}[Z_1, \bar{Z}_1], \qquad W_2 := \frac{1}{2\,\mathrm{i}}[Z_2, \bar{Z}_2].$$

These vector fields are well-defined up to multiple and summands from $\mathbb{C} \otimes H^+$ resp. $\mathbb{C} \otimes H^-$. Hence, they induce a splitting $TM = T^L \oplus T^R$ where T^L is generated by $\operatorname{Re} Z_1, \operatorname{Im} Z_1, W_1$ and T^R is generated by $\operatorname{Re} Z_2, \operatorname{Im} Z_2, W_2$.

The torsion components corresponding to the cohomologies

$$\mathfrak{g}^R_{-2} \times \mathfrak{g}^R_{-1} \to \mathfrak{g}^L_{-2}$$
$$\mathfrak{g}^L_{-2} \times \mathfrak{g}^L_{-1} \to \mathfrak{g}^R_{-2},$$

are responsible for the integrability of this almost direct product structure. T^L is integrable if and only if $[Z_1, W_1] \in T^L$. At 0 we obtain

$$[Z_1, W_1]|_0 = 2n_{11\bar{1}} W_2(0) + 2\,\mathrm{i}\,\bar{n}_{11\bar{1}} n_{11\bar{2}} Z_2(0),$$

and, analogously,

$$[Z_2, W_2]|_0 = 2n_{22\bar{2}} W_1(0) + 2\,\mathrm{i}\,\bar{n}_{2\bar{2}\bar{2}} n_{22\bar{1}} Z_1(0).$$

Hence, integrability of T^L or T^R at 0 is equivalent to vanishing of $n_{11\bar{1}}$ or $n_{22\bar{2}}$, respectively.

One can see here that the vanishing of the TM/HM component of the brackets above implies already vanishing of the complete brackets.

The remaining torsion corresponding to

$$\mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-1} \to \mathfrak{g}^R_{-1} \quad \text{(linear in 1st, antilinear in 2nd argument)}$$
$$\mathfrak{g}^R_{-1} \times \mathfrak{g}^L_{-1} \to \mathfrak{g}^L_{-1} \quad \text{(linear in 1st, antilinear in 2nd argument)}$$

can be interpreted by the algebraic brackets

$$S_R(Z_1, Z_2) = \pi_R[Z_1, \bar{Z}_2], \qquad S_L(Z_2, Z_1) = \pi_R[Z_2, \bar{Z}_1]$$

where π_R (π_L) is the projection to H^R (H^L) in HM. Vanishing of these brackets is equivalent to integrability of the flipped CR structure $\tilde{J}$. A simple calculation shows that

$$[Z_1, \bar{Z}_2]|_0 = 2\bar{n}_{22\bar{1}} Z_2(0) - 2n_{11\bar{2}} \bar{Z}_1(0).$$

The torsions S_R and S_L vanish at 0 if and only if $n_{22\bar{1}} = 0$, resp. $n_{11\bar{2}} = 0$. This corrects the claim in [SS00] that these torsions vanish automatically for embedded CR-manifolds. Thus, the assertion of Theorem 3.9. in [SS00] for embedded manifolds about splitting into a direct product of embedded hypersurfaces is only true if all four (remaining) torsion components vanish. With the provided geometric interpretation it is easy to see that this condition is necessary for this splitting, but we find remarkable that the vanishing of the four coefficients at each point is sufficient too.

Let now $M \subset \mathbb{C}^4$ be an elliptic CR-manifold given up to third order in normal form (see [ES96]) by

$$\frac{w_1 - \bar{w}_2}{2\,\mathrm{i}} = z_1 \bar{z}_2 + N_{21} + N_{12} + \cdots, \tag{2}$$

where $N_{21} = n_{11\bar{1}} z_1^2 \bar{z}_1 + n_{22\bar{1}} z_2^2 \bar{z}_1$ and $N_{12} = n_{2\bar{1}\bar{1}} z_2 \bar{z}_1^2 + n_{2\bar{2}\bar{2}} z_2 \bar{z}_2^2$.

As before, two of the six components correspond to the Nijenhuis tensor of the CR structure and therefore vanish automatically for an embedded CR-manifold.

We start with the calculation of the generators Z_1 and Z_2 of the distinguished line subbundles H^L and H^R of HM. The holomorphic part of the complexification $H^{1,0}M$ is spanned by

$$X_1 = \tfrac{\partial}{\partial z_1} + 2\,\mathrm{i}\left(\bar z_2 + \frac{\partial N_{12}}{\partial z_1} + \frac{\partial N_{21}}{\partial z_1}\right)\tfrac{\partial}{\partial w_1} + 2\,\mathrm{i}\left(\frac{\partial \bar N_{12}}{\partial z_1} + \frac{\partial \bar N_{21}}{\partial z_1}\right)\tfrac{\partial}{\partial w_2}$$
$$X_2 = \tfrac{\partial}{\partial z_2} + 2\,\mathrm{i}\left(\frac{\partial N_{12}}{\partial z_2} + \frac{\partial N_{21}}{\partial z_2}\right)\tfrac{\partial}{\partial w_1} + 2\,\mathrm{i}\left(\bar z_1 + \frac{\partial \bar N_{12}}{\partial z_2} + \frac{\partial \bar N_{21}}{\partial z_2}\right)\tfrac{\partial}{\partial w_2}$$

The vector fields that generate H^L and H^R are linear combinations of the fields $Z_1 = X_1 + A_{12}X_2$, and $Z_2 = A_{21}X_1 + X_2$ with $A_{12}(0) = A_{21}(0) = 0$.

From the conditions $\pi[Z_1, \bar Z_1] = 0$ and $\pi[Z_2, \bar Z_2] = 0$ we obtain

$$A_{12} = -2\bar n_{11\bar 1}\bar z_1 + 4n_{11\bar 1}\bar n_{2\bar 1\bar 1}z_1^2 + 4n_{11\bar 1}\bar n_{22\bar 1}z_1\bar z_2$$
$$A_{21} = -2n_{2\bar 2\bar 2}\bar z_2 + 4n_{22\bar 1}\bar n_{2\bar 2\bar 2}z_2^2 + 4n_{2\bar 1\bar 1}\bar n_{2\bar 2\bar 2}\bar z_1 z_2.$$

The torsion components corresponding to cohomologies with cochains

$$\mathfrak{g}^L_{-1} \times \mathfrak{g}^L_{-1} \to \mathfrak{g}^R_{-1} \quad \text{(sesquilinear)}$$
$$\mathfrak{g}^R_{-1} \times \mathfrak{g}^R_{-1} \to \mathfrak{g}^L_{-1} \quad \text{(sesquilinear)}$$

are responsible for the integrability of the H^L resp. H^R. We compute

$$[Z_1, \bar Z_1]|_0 = 2\bar n_{11\bar 1}Z_2(0) - 2n_{11\bar 1}\bar Z_2(0)$$
$$[Z_2, \bar Z_2]|_0 = 2n_{2\bar 2\bar 2}Z_1(0) - 2\bar n_{2\bar 2\bar 2}\bar Z_1(0).$$

Hence, these torsions do not vanish automatically for embedded manifolds, as claimed in Theorem 4.4. in [SS00] but they are represented by the terms $n_{11\bar 1}z_1^2\bar z_1$ and $n_{2\bar 2\bar 2}z_2\bar z_2^2$ in the normal form.

The remaining torsion components correspond to the cohomologies

$$\mathfrak{g}_{-2} \times \mathfrak{g}^L_{-1} \to \mathfrak{g}_{-2} \quad \text{(antilinear in both arguments)}$$
$$\mathfrak{g}_{-2} \times \mathfrak{g}^R_{-1} \to \mathfrak{g}_{-2} \quad \text{(antilinear in both arguments)}$$

and can be interpreted using the induced canonical almost complex structure $\hat J$ on M. On HM the operator $\hat J$ is defined by $JZ_1 = \mathrm{i}\,Z_1$ and $JZ_2 = -\,\mathrm{i}\,Z_2$. Consider $W := [Z_1, \bar Z_2] \in \mathbb{C} \otimes TM$. Define $\hat J W := \mathrm{i}\,W$ and $\hat J \bar W = -\,\mathrm{i}\,\bar W$. This is well-defined because another choice of Z_1 and Z_2 would modify W by addition of a multiple of Z_1 or $\bar Z_2$ on which $\hat J$ acts also by multiplication with i.

This almost complex structure has the characteristic property that

$$\pi[\hat J\xi, \eta] - J_Q\pi[\xi, \eta] \tag{3}$$

is antiholomorphic in η (with respect to $\hat J$) for an antiholomorphic argument ξ and $\pi(\hat J\xi) = J_Q\pi(\xi)$, where J_Q is the almost complex structure on $QM = TM/HM$ induced by the Levi bracket.

Expression (3) defines a tensor $S: QM \times HM \to QM$ which splits, according to the second argument into $S^{\pm}: QM \times H^{L,R} \to QM$. These tensors represent the remaining torsion components. A direct calculation shows that (up to second order terms)

$$W = -4(n_{11\bar{1}}\bar{n}_{22\bar{1}}z_1+\bar{n}_{11\bar{1}}\bar{n}_{2\bar{2}\bar{2}}z_2)\frac{\partial}{\partial z_2}-2\,\mathrm{i}\,\frac{\partial}{\partial w_1}-4\,\mathrm{i}(\bar{n}_{2\bar{1}\bar{1}}z_1+\bar{n}_{22\bar{1}}\bar{z}_2)\frac{\partial}{\partial w_2}+ \\ + 4(\bar{n}_{11\bar{1}}\bar{n}_{2\bar{2}\bar{2}}\bar{z}_1 + \bar{n}_{2\bar{1}\bar{1}}n_{2\bar{2}\bar{2}}\bar{z}_2)\frac{\partial}{\partial \bar{z}_1} - 4\,\mathrm{i}(\bar{n}_{2\bar{1}\bar{1}}z_1 + \bar{n}_{22\bar{1}}\bar{z}_2)\frac{\partial}{\partial \bar{w}_1} - 2\,\mathrm{i}\,\frac{\partial}{\partial \bar{w}_2}$$

Then

$$S(\bar{W}, \bar{Z}_1)|_0 = -4\,\mathrm{i}\,n_{2\bar{1}\bar{1}}W(0)$$
$$S(\bar{W}, Z_2)|_0 = -4\,\mathrm{i}\,n_{22\bar{1}}W(0).$$

(Here we identified QM with the subspace of TM spanned by $\mathrm{Re}\,W$ and $\mathrm{Im}\,W$.)

The geometric meaning of the vanishing of these tensors is integrability of $\hat{J}$:

$$[\bar{W}, \bar{Z}_1]|_0 = 2n_{2\bar{1}\bar{1}}W(0) + 4n_{11\bar{1}}\bar{n}_{22\bar{1}}\bar{Z}_2(0)$$
$$[\bar{W}, Z_2]|_0 = 2n_{22\bar{1}}W(0) - 4\bar{n}_{2\bar{1}\bar{1}}n_{2\bar{2}\bar{2}}Z_1(0)$$

vanish if $S|_0 = 0$.

References

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