Remarks on curvature in sub-Riemannian geometry

Jan Slovák

Masaryk University, Brno, Czech Republic
joint work with D. Alekseevsky, A. Medvedev, nearly finished ...

June 5, 2017
Tromsø

Subriemannian geometry

Definition

Subriemannian geometry (M, D, S) on a manifold M is given by a distribution D, and (positive definite) metric S on D.

Subriemannian geometry

Definition

Subriemannian geometry (M, D, S) on a manifold M is given by a distribution D, and (positive definite) metric S on D.

Sheaf $\mathcal{D}^{-1} = \mathcal{D}$ of vector fields valued in D generates the filtration by sheafs

$$\mathcal{D}^j = \{[X, Y], X \in \mathcal{D}^{j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = -2, -3, \dots .$$

We say that D is a bracket generating distribution if for some k, $\mathcal{D}^k$ is the sheaf of all vector fields on M.

Bracket generating distribution D defines the filtration of subspaces

$$T_xM = D_x^k \supset \cdots \supset D_x^{-1}$$

at each point $x \in M$.
The associated graded tangent space

$$\mathrm{gr}\, T_xM = T_xM/D_x^{k+1} \oplus \cdots \oplus D_x^{-1}$$

comes equipped with the structure of a nilpotent Lie algebra.

Bracket generating distribution D defines the filtration of subspaces

$$T_x M = D_x^k \supset \cdots \supset D_x^{-1}$$

at each point $x \in M$.
The associated graded tangent space

$$\mathrm{gr}\, T_x M = T_x M / D_x^{k+1} \oplus \cdots \oplus D_x^{-1}$$

comes equipped with the structure of a nilpotent Lie algebra.

Definition

(M, D, S) is a *sub-Riemannian geometry with constant symbol* if D is bracket generating, and the nilpotent algebra $\mathrm{gr}\, T_x M$, together with the metric, is isomorphic to a fixed graded Lie algebra

$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$$

with a fixed metric σ on $\mathfrak{g}_{-1}$.

Prolongation of subriemannian geometries

Let $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_{-1})$ be the Lie algebra of the Lie group G_0 of all automorphisms of the graded nilpotent algebra $\mathfrak{g}_-$ preserving the metric σ on $\mathfrak{g}_{-1}$.
The action of the derivations from $\mathfrak{g}_0$ on $\mathfrak{g}_-$ extends the Lie algebra structure on $\mathfrak{g}_-$ to the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

Prolongation of subriemannian geometries

Let $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_{-1})$ be the Lie algebra of the Lie group G_0 of all automorphisms of the graded nilpotent algebra $\mathfrak{g}_-$ preserving the metric σ on $\mathfrak{g}_{-1}$.
The action of the derivations from $\mathfrak{g}_0$ on $\mathfrak{g}_-$ extends the Lie algebra structure on $\mathfrak{g}_-$ to the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

Observation 1

The Tanaka prolongation of $\mathfrak{g}$ is finite.[a]

[a] Corollary 2 of Theorem 11.1 in *Tanaka, N.*, On differential systems, graded Lie algebras and pseudo-groups, *J. Math. Koyto Univ., 10, 1 (1970), 1-82.*

Observation 2

Already the first prolongation is trivial.[a] Thus $\mathfrak{g}$ is the full prolongation of $\mathfrak{g}_-$.

[a] Yatsui, T., *On pseudo-product graded Lie algebras*, Hokkaido Math. J., 17 (1988), 333-343.

Observation 2

Already the first prolongation is trivial.[a] Thus $\mathfrak{g}$ is the full prolongation of $\mathfrak{g}_-$.

[a] Yatsui, T., *On pseudo-product graded Lie algebras*, Hokkaido Math. J., 17 (1988), 333-343.

Theorem

For each subriemannian manifold (M, D, S) with constant symbol, there is the unique Cartan connection $(\mathcal{G} \to M, \omega)$ of type $(\mathfrak{g}, G_0)$ with the curvature function $\kappa : \mathcal{G} \to \mathfrak{g} \otimes \Lambda^2 \mathfrak{g}_-^*$ satisfying $\partial^* \kappa = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection κ_H, i.e. the component with $\partial \kappa_H = 0$ as well.[a]

[a] Morimoto, T., *Cartan connection associated with a subriemannian structure*, Differential Geometry and its Applications 26 (2008), 75-78.

The distribution D on M itself is often a finite type geometry. defines a nice finite type filtered geometry which enjoys a canonical Cartan connection, too.
Many of them belong to the class of the parabolic geometries, for which the full Tanaka prolongation of $\mathfrak{g}_-$ is a semisimple Lie algebra

$$\bar{\mathfrak{g}} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_k$$

and $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is the opposite nilpotent radical to the parabolic subalgebra $\mathfrak{p} = \bar{\mathfrak{g}}_0 \oplus \cdots \oplus \bar{\mathfrak{g}}_k \subset \bar{\mathfrak{g}}$, with $\mathfrak{g}_0 \subset \bar{\mathfrak{g}}_0$.

Fix one such graded semisimple $\bar{\mathfrak{g}}$ and consider the frame bundle $\mathcal{G}_0 \to M$ of gr TM giving a parabolic geometry. Often the structure group G_0 of $\mathcal{G}_0$ is the full group of graded automorphisms of $\mathfrak{g}_-$.[1] Adding a metric S on D, we have got two Cartan connections there:

[1]See Čap, A., Slovák, J., Parabolic Geometries I, Background and General Theory, AMS, Math. Surveys and Monographs 154, x+628pp. for details.

Fix one such graded semisimple $\bar{\mathfrak{g}}$ and consider the frame bundle $\mathcal{G}_0 \to M$ of gr TM giving a parabolic geometry. Often the structure group G_0 of $\mathcal{G}_0$ is the full group of graded automorphisms of $\mathfrak{g}_-$.[1] Adding a metric S on D, we have got two Cartan connections there:

Theorem

Consider a bracket generating distribution D on M with the constant symbol equal to the negative part of a graded semisimple Lie algebra $\bar{\mathfrak{g}}$ and the corresponding frame bundle $\mathcal{G}_0 \to M$ of gr TM. Then there is the unique Cartan connection $(\bar{\mathcal{G}} \to M, \omega)$ of type $(\bar{\mathfrak{g}}, P)$ with the curvature function $\bar{\kappa} : \bar{\mathcal{G}} \to \bar{\mathfrak{g}} \otimes \Lambda^2 \mathfrak{g}_-^*$ satisfying $\partial^* \bar{\kappa} = 0$. Via the Bianchi identities, the entire curvature is obtained from its harmonic projection $\bar{\kappa}_H$, i.e. the component with $\partial \bar{\kappa}_H = 0$ as well.

[1] See Čap, A., Slovák, J., Parabolic Geometries I, Background and General Theory, AMS, Math. Surveys and Monographs 154, x+628pp. for details.

Consider a parabolic geometry (M, D) equipped by the metric S on D, assume (M, D, S) has got constant symbol.
Thus we have got:

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$
$$\bar{\mathfrak{g}} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_k$$

This is an instance of a $\mathfrak{g}_-$-submodule W of $\mathfrak{g}_-$-module V.

The short exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

induces the short exact sequence of differential complexes

$$0 \longrightarrow C^\bullet(\mathfrak{g}_-, W) \xrightarrow{i} C^\bullet(\mathfrak{g}_-, V) \xrightarrow{\pi} C^\bullet(\mathfrak{g}_-, V/W) \longrightarrow 0$$

and thus the long exact sequence in cohomologies

$$\longrightarrow H^n(\mathfrak{g}_-, W) \xrightarrow{i} H^n(\mathfrak{g}_-, V) \xrightarrow{\pi} H^n(\mathfrak{g}_-, V/W) \xrightarrow{\delta} H^{n+1}(\mathfrak{g}_-, W) \xrightarrow{i} H^{n+1}(\mathfrak{g}_-, V) \xrightarrow{\pi} H^{n+1}(\mathfrak{g}_-, V/W) \longrightarrow$$

The gradings on $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ induce the gradings on the corresponding spaces of chains, the differential ∂ respects this grading, thus we get grading on the cohomology spaces, too.
Clearly, we may consider the sequences for the individual homogeneities separately.

The gradings on $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ induce the gradings on the corresponding spaces of chains, the differential ∂ respects this grading, thus we get grading on the cohomology spaces, too.
Clearly, we may consider the sequences for the individual homogeneities separately.
We are interested in geometries described via the filtration induced by the distribution D and we declare its symbol to be equal to the Lie algebra $\mathfrak{g}_-$ at all points. Thus, all the curvatures have to vanish in all nonpositive homogeneities.

The gradings on $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ induce the gradings on the corresponding spaces of chains, the differential ∂ respects this grading, thus we get grading on the cohomology spaces, too.
Clearly, we may consider the sequences for the individual homogeneities separately.
We are interested in geometries described via the filtration induced by the distribution D and we declare its symbol to be equal to the Lie algebra $\mathfrak{g}_-$ at all points. Thus, all the curvatures have to vanish in all nonpositive homogeneities.

Theorem

Assume that $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}) = 0$. The cohomology $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is a direct sum of 2 parts:

1. $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$,
2. $\ker \pi_2 \colon H^2_+(\mathfrak{g}_-, \bar{\mathfrak{g}}) \to H^2_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$.

Remark

$\bar{\mathfrak{g}}_0$ equals to all derivations on the graded algebra $\mathfrak{g}_-$ if and only if all the non-negative homogeneities $H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$ vanish.
If $H^1_0(\mathfrak{g}_-, \bar{\mathfrak{g}}) \neq 0$, then we need further reduction of the algebra of all derivations to $\bar{\mathfrak{g}}_0$ in order to get a canonical Cartan connection.
In particular, the technical assumption in the theorem is not much restrictive.

Remark

$\bar{\mathfrak{g}}_0$ equals to all derivations on the graded algebra $\mathfrak{g}_-$ if and only if all the non-negative homogeneities $H^1_{\geq 0}(\mathfrak{g}_-, \bar{\mathfrak{g}})$ vanish.
If $H^1_0(\mathfrak{g}_-, \bar{\mathfrak{g}}) \neq 0$, then we need further reduction of the algebra of all derivations to $\bar{\mathfrak{g}}_0$ in order to get a canonical Cartan connection. In particular, the technical assumption in the theorem is not much restrictive.

Remark

The projection π_2 is zero whenever the cochains representing the cohomology are valued in $\mathfrak{g}$. Actually, the structure of $H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$ is quite well known and positive homogeneities in the curvature are rather exceptional. Only a very few of those in the list allow for curvature components valued in $\bar{\mathfrak{g}}_{\geq 0}$. Except for the length $k = 1$ and contact cases, there are just five exceptions.

Proof.

The first rows of the long exact sequence are

$$\cdots \longrightarrow H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}) = 0 \xrightarrow{\pi_1} H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) \xrightarrow{\delta} H^2_+(\mathfrak{g}_-, \mathfrak{g}) \xrightarrow{i_2} H^2_+(\mathfrak{g}_-, \bar{\mathfrak{g}}) \xrightarrow{\pi_2} H^2_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) \longrightarrow$$

Notice the connecting homomorphism δ is essentially given by ∂. The first part of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$, which is mapped by δ injectively into $H^2_+(\mathfrak{g}_-, \mathfrak{g})$. The second part is $\operatorname{im} i_2 \colon H^2_+(\mathfrak{g}_-, \mathfrak{g}) \to H^2_+(\mathfrak{g}_-, \bar{\mathfrak{g}})$. Exactness of the sequence implies $\operatorname{im} i_2 = \ker \pi_2$. □

There are further helpful technical claims for computation of $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$. We write $\bar{\mathfrak{g}}^i$ for the "left $\mathfrak{g}_-$-invariant ends" of $\bar{\mathfrak{g}}$.

Lemma

For $i \geq 0$ we have $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^i/\mathfrak{g})/\delta(\bar{\mathfrak{g}}^{i+1})$.

There are further helpful technical claims for computation of $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$. We write $\bar{\mathfrak{g}}^i$ for the "left $\mathfrak{g}_-$-invariant ends" of $\bar{\mathfrak{g}}$.

Lemma

For $i \geq 0$ we have $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^i/\mathfrak{g})/\delta(\bar{\mathfrak{g}}^{i+1})$.

Lemma

$H^1_1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = \mathfrak{g}^*_{-1} \otimes (\bar{\mathfrak{g}}_0/(\mathfrak{g}_0 \oplus \mathbb{R}Z))$ *where Z is the grading element of the parabolic geometry $(\bar{\mathfrak{g}}, \bar{\mathfrak{g}}_{\geq 0})$.*

There are further helpful technical claims for computation of $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$. We write $\bar{\mathfrak{g}}^i$ for the "left $\mathfrak{g}_-$-invariant ends" of $\bar{\mathfrak{g}}$.

Lemma

For $i \geq 0$ we have $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^i/\mathfrak{g})/\delta(\bar{\mathfrak{g}}^{i+1})$.

Lemma

$H^1_1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = \mathfrak{g}^*_{-1} \otimes (\bar{\mathfrak{g}}_0/(\mathfrak{g}_0 \oplus \mathbb{R}Z))$ *where Z is the grading element of the parabolic geometry $(\bar{\mathfrak{g}}, \bar{\mathfrak{g}}_{\geq 0})$.*

Lemma

For $j < i$, $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^j/\mathfrak{g}) = 0$.

There are further helpful technical claims for computation of $H^1_+(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$. We write $\bar{\mathfrak{g}}^i$ for the "left $\mathfrak{g}_-$-invariant ends" of $\bar{\mathfrak{g}}$.

Lemma

For $i \geq 0$ we have $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^i/\mathfrak{g})/\delta(\bar{\mathfrak{g}}^{i+1})$.

Lemma

$H^1_1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = \mathfrak{g}^*_{-1} \otimes (\bar{\mathfrak{g}}_0/(\mathfrak{g}_0 \oplus \mathbb{R}Z))$ *where Z is the grading element of the parabolic geometry $(\bar{\mathfrak{g}}, \bar{\mathfrak{g}}_{\geq 0})$.*

Lemma

For $j < i$, $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}^j/\mathfrak{g}) = 0$.

Lemma

If k is the length of the grading for $\mathfrak{g}$ then for $i \geq k+1$ $H^1_{i+1}(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g}) = 0$.

1. Cartan geometries
   - Subriemannian prolongation
   - Underlying parabolic geometries

2. Cohomologies

3. Free step 2 distributions

4. Constant curvature spaces

Let M be a manifold of dimension $n(n+1)$. We say that distribution D of dimension n is a free (step 2) distribution on M if $D + [D, D] = TM$.
This is a nice parabolic geometry, of type $(\bar{\mathfrak{g}}, \bar{P})$ with the Lie algebras of the form

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} A & X & Y \\ -Z^t & 0 & -X^t \\ T & Z & -A^t \end{pmatrix} \right\}, \quad \bar{\mathfrak{p}} = \left\{ \begin{pmatrix} A & 0 & 0 \\ -Z^t & 0 & 0 \\ T & Z & -A^t \end{pmatrix} \right\},$$

where $A, Y, T \in \mathrm{Mat}_n(\mathbb{R})$, $X, Z \in \mathbb{R}^n$, $Y + Y^t = T + T^t = 0$.

Let M be a manifold of dimension $n(n+1)$. We say that distribution D of dimension n is a free (step 2) distribution on M if $D+[D,D]=TM$.
This is a nice parabolic geometry, of type $(\bar{\mathfrak{g}},\bar{P})$ with the Lie algebras of the form

$$\bar{\mathfrak{g}}=\left\{\begin{pmatrix} A & X & Y \\ -Z^t & 0 & -X^t \\ T & Z & -A^t \end{pmatrix}\right\}, \quad \bar{\mathfrak{p}}=\left\{\begin{pmatrix} A & 0 & 0 \\ -Z^t & 0 & 0 \\ T & Z & -A^t \end{pmatrix}\right\},$$

where $A,Y,T\in \mathrm{Mat}_n(\mathbb{R})$, $X,Z\in\mathbb{R}^n$, $Y+Y^t=T+T^t=0$.
We introduce the obvious basis $e^{[ij]}$, e^j, e^i_j, e_j, $e_{[ij]}$ in $\bar{\mathfrak{g}}$.
The commutation relations are given by:

$$[e^{[ij]},e_{[jk]}]=-e^i_k-\delta^i_k e^j_j=\begin{cases}-e^i_k, & k\neq i\\ -e^i_i-e^j_j, & k=i\end{cases}$$

The metric S defines a reduction of $\bar{P}$-principle bundle $\bar{\mathcal{G}}$ to $G_0 = SO_n(\mathbb{R})$-principle bundle $\mathcal{G}$ of orthogonal frames. The sub-Riemannian structure in the background can be given in terms of orthonormal frame $X_1, \dots, X_n$ on D.

We define $X_{[ij]} = -[X_i, X_j]$. Due to the fact that D is a free distribution the graded symbol of $\{X_i, X_{[jk]}\}$ is given by e_i, $e_{[jk]}$ with the same relations as in $\bar{\mathfrak{g}}$.

The infinitesimal model is given by

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \langle e_{[ij]} \rangle \oplus \langle e_k \rangle \oplus \langle a^i_j \rangle.$$

Theorem

The $H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$ part of $H^2(\mathfrak{g}_-, \mathfrak{g})$ is the entire $H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$, i.e. the subspace of totally trace-free elements in

$$\mathrm{Hom}(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2}, \mathfrak{g}_{-2}).$$

The $H^1(\mathfrak{g}_-, \bar{\mathfrak{g}}/\mathfrak{g})$ part of $H^2(\mathfrak{g}_-, \mathfrak{g})$ consists of 2 subspaces:

- *in degree 1 it is generated by symmetric and traceless in (i, j) tensors*

$$\alpha^k_{(ij)} = \left(e_j \otimes e_i^* + e_i \otimes e_j^* + \sum_t (e_{[jt]} \otimes e^*_{[it]} + e_{[it]} \otimes e^*_{[jt]})\right) \wedge e_k^*$$

- *in degree 2 it is generated by symmetric in (p, q) tensors*

$$\alpha_{(pq)} = \sum_t e_t \otimes (e^*_{[tp]} \wedge e^*_q + e^*_{[tq]} \wedge e^*_p) + \sum_{t,r} e_{[tr]} \otimes e^*_{[tp]} \wedge e^*_{[qr]}.$$

Constant curvature subriemannian geometries are those with curvature in a submodule with the trivial $\mathfrak{g}_0$ action.
Thus we aim at finding all submodule.

Constant curvature subriemannian geometries are those with curvature in a submodule with the trivial $\mathfrak{g}_0$ action.
Thus we aim at finding all submodule.

Theorem

Assume $n \geq 4$. The only constant curvature models for free step 2 sub-Riemannian geometries are defined on $SO(n+1)$ and $SO(n,1)$ with orthonormal frame given by the elements of $\mathfrak{so}_{n+1}$ of the form

$$\begin{pmatrix} 0 & A_i^t \\ -A_i & 0_n \end{pmatrix},$$

and by the elements of $\mathfrak{so}_{n,1}$ of the form

$$\begin{pmatrix} 0 & A_i^t \\ A_i & 0_n \end{pmatrix},$$

where the only non-zero element in A_i is on the place i.

We have to check the individual invariants components of the harmonic curvature for the trivial submodules in the $\mathfrak{so}_n$ decomposition.

We have to check the individual invariants components of the harmonic curvature for the trivial submodules in the $\mathfrak{so}_n$ decomposition.
While there are no such trivial submodules in the totally tracefree part of $\mathrm{Hom}(\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2}, \mathfrak{g}_{-2})$, and in the homogeneity one traceless in (i,j) tensors

$$\alpha^k_{(ij)} = \left(e_j \otimes e_i^* + e_i \otimes e_j^* + \sum_t (e_{[jt]} \otimes e^*_{[it]} + e_{[it]} \otimes e^*_{[jt]}) \right) \wedge e_k^*,$$

there is just one such module in

$$\alpha_{(pq)} = \sum_t e_t \otimes (e^*_{[tp]} \wedge e_q^* + e^*_{[tq]} \wedge e_p^*) + \sum_{t,r} e_{[tr]} \otimes e^*_{[tp]} \wedge e^*_{[qr]}.$$

.
The models with positive and negative curvature are just those in the theorem.