Generality of the quaternionic contact structures

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joint work with **Ivan Minchev**, Brno / Sofia

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1. Our motivation
2. Geometric structures as EDS
3. The qc structures
4. The Cartan test for qc structures

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Claude LeBrun, Olivier Biquard

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He interpreted the disk $B^{4n+4} \simeq Sp(n+1,1)/Sp(1)Sp(n)$ and his class was parameterized by one holomorphic function on the twistor space $\mathcal{Z} \subset \mathbb{CP}^(2n+3)$.

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A bit later, Biquard desribed the induced structures, the so called quaternionic contact structures on the boundary more carefully, cf. *Métriques d'Einstein asymptotiquement symétriques*, Astérisque 265 (2000).
He also showed that each real analytic qc structure on a manifold M is the conformal boundary at innity of a (germ) unique quaternionic Kähler metric deffined in a small neighborhood of M.
In particular, it was immediately known that there are large families of examples there.

Robert Bryant – Srní 2015

In the three lectures *G-structures with prescribed symmetry* at the Geometry and Physics in Srni, January 2015, Bryant delivered a detailed exposition how to use the Cartan-Kähler theory when describing the generality of geometric structures. See http://conference.math.muni.cz/srni/index.php?id=2015 for the full texts of the lectures.

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Ivan Minchev thought one could proceed this way in the case of the qc-structures. This meant, he wanted to recover completely the Chern-Moser approach to CR-structures in order to get the necessary data for the Cartan's test.

I will describe the result of this quest.

1 Our motivation

2 Geometric structures as EDS

3 The qc structures

4 The Cartan test for qc structures

We adopt the following ranges of indices: $1 \leq a, b, c, d, e \leq n$, $1 \leq s \leq \ell$, where ℓ and n are some fixed positive integers.

Problem: *Given a set of real analytic functions $C^a_{bc} : \mathbb{R}^\ell \to \mathbb{R}$ with $C^a_{bc} = -C^a_{cb}$, find linearly independent one-forms ω^a, defined on a domain $\Omega \subset \mathbb{R}^n$, and a mapping $u = (u^s) : \Omega \to \mathbb{R}^\ell$ so that the equations*

$$d\omega^a = -\frac{1}{2} C^a_{bc}(u)\, \omega^b \wedge \omega^c \tag{1}$$

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The problem is diffeomorphism invariant in the sense that if (ω^a, u) is any solution of (1) defined on $\Omega \subset \mathbb{R}^n$ and $\Phi : \Omega^{'} \to \Omega$ is a diffeomorphism, then $(\Phi^*(\omega^a), \Phi^*(u))$ is a solution of (1) on $\Omega^{'}$. We regard any such two solutions as equivalent and we are interested in the following question:

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How many non-equivalent solutions does a given problem of this type admit?

EDS reformulation

We set

$$N = GL(n,\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^\ell$$

with the projections

$$p = (p^a_b) : N \to GL(n,\mathbb{R}),\ x = (x^a) : N \to \mathbb{R}^n,\ u = (u^s) : N \to \mathbb{R}^\ell$$

.

Further set $\omega^a \stackrel{def}{=} p^a_b\, dx^b$, and consider the differential ideal $\mathcal{I}$ on N generated by the set of two-forms

$$\Upsilon^a \stackrel{def}{=} d\omega^a + \frac{1}{2} C^a_{bc}(u)\, \omega^b \wedge \omega^c.$$

Then, the solutions of (1) are precisely the n-dimensional integral manifolds of $\mathcal{I}$ on which the restriction of the n-form $\omega^1 \wedge \cdots \wedge \omega^n$ is nowhere vanishing.

In our equations $d\omega^a = -\frac{1}{2} C^a_{bc}(u)\, \omega^b \wedge \omega^c$, the functions C^a_{bc} play the role curvature. Their derivatives are driven by the Bianchi identities and, thus, they are quadratic. In order to employ the Cartan-Kähler theory we need to replace the quadratic terms by some linear objects. Thus we posit the following two assumptions:

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Assumption I: *Let us assume that there exist a real analytic mapping $F = (F^s_a) : \mathbb{R}^\ell \to \mathbb{R}^{\ell n}$ for which*

$$d\big(C^a_{bc}\omega^b \wedge \omega^c\big) = \frac{\partial C^a_{bc}(u)}{\partial u^s}\big(du^s + F^s_d(u)\, \omega^d\big) \wedge \omega^b \wedge \omega^c. \qquad (2)$$

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Assumption II: *On the integral manifolds of* $\mathcal{I}$*, the RHS of* (2) *has to vanish, which is a system of algebraic equations for the unknown 1-forms* du^s *(for a fixed* u*). We assume that it is non-degenerate, i.e, that equations yield* $du^s \in span\{\omega^a\}$*. As a consequence, at any* u*, the set of all solutions* du^s *is parameterized by a certain vector space. We will assume that the dimension of this vector space is a constant* D *(independent of* u*).*

$\mathcal{I}$ is a differential ideal, thus it is algebraically generated by the forms Υ^a and $d\Upsilon^a$. By (2), we have

$$2d\Upsilon^a = \frac{\partial C^a_{bc}(u)}{\partial u^s}\left(du^s + F^s_d(u)\omega^d\right)\wedge\omega^b\wedge\omega^c + 2C^a_{bc}\,\Upsilon^b\wedge\omega^c \quad (3)$$

and therefore, $\mathcal{I}$ is algebraically generated by Υ^a and the 3-forms

$$\Xi^a \stackrel{def}{=} \frac{\partial C^a_{bc}(u)}{\partial u^s}\left(du^s + F^s_d(u)\omega^d\right)\wedge\omega^b\wedge\omega^c.$$

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If we take Ω^a to be some other basis of one-forms for the vector space span$\{\omega^a\}$, we can express the forms Ξ^a as

$$\Xi^a = \Pi^a_{bc}\wedge\Omega^b\wedge\Omega^c, \quad (4)$$

where Π^a_{bc} are linear combinations of the linearly independent one-forms

$$\{du^s + F^s_d(u)\omega^d \;:\; s = 1,\dots,n\}.$$

The Cartan's test

Next, define $v_1(u), v_2(u), \ldots, v_n(u)$, non-negative integers for any fixed u, as follows:

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$$- \ \operatorname{rank}\Big\{\Pi^a_{bc} \ : \ a = 1, \dots, n, \ 1 \le b < c \le d-1\Big\};$$

for $1 < d \le n-1$, and

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If, for every $u \in \mathbb{R}^\ell$, one can find a basis Ω^a of $\mathrm{span}\{\omega^a\}$ for which the Cartan's Test

$$v_1(u) + 2v_2(u) + \cdots + nv_n(u) = D, \qquad (5)$$

is satisfied (remind D is the constant dimension from the above Assuption II), then the system (1) is said to be *in involution*.

The latter method of computation for the Cartan's character sequence of an ideal is based on the big Bryant et al EDS book, Proposition 1.15.

Theorem (Cartan, essentially, see the Bryant et al EDS book)

If the system is in involution, then for any u_0, there exists a solution (ω^a, u) of (1) defined on a neighborhood Ω of $0 \in \mathbb{R}^n$ for which $u(0) = u_0$ and

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Moreover, in certain sense (see again the EDS book for a more precise formulation):

The generality of the solutions

Different solutions (ω^a, u) of (1), modulo diffeomorphisms, depend on $v_k(u)$ functions of k variables, where $v_k(u)$ is the last non-vanishing integer in the Cartan's sequence $v_1(u), \dots, v_n(u)$.

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The flat model

The spheres appear as Klein models $G \to G/P$ in many ways, e.g. the conformal Riemannian sphere $S^n \subset \mathbb{R}P^{n+1}$, the CR-sphere $S^{2n+1} \subset \mathbb{C}P^{n+1}$, and the quaternionic contact sphere $S^{4n+3} \subset \mathbb{H}P^{n+1}$, respectively, or other nice homogeneous spaces in the cases of other than positive definite signatures.

All these geometries appear as boundaries of domains, carrying a lot of information – let us mention the conformal horizons in mathematical physics, the boundaries of domains in complex analysis and function theory, and the boundaries of quaternionic-Kähler domains.

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The regular Cartan geometries modelled on the hypersphere in $\mathbb{H}P^{n+1}$ are called the *quaternionic contact geometries*.

The corresponding Lie algebras enjoy very similar algebraic structures with $|2|$-gradings where $\mathfrak{g}_0$ further splits as $\mathfrak{h} \oplus \mathfrak{g}_0'$, as indicated symbolically in the matrix (the $*$ entries mean those computed from the symmetries of the matrix)

$$\left(\begin{array}{c|c|c} \mathfrak{h} & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0' & * \\ \hline \mathfrak{g}_{-2} & * & * \end{array}\right)$$

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The corresponding Lie algebras $\mathfrak{g}$ are $\mathfrak{so}(p+1, q+1)$, $\mathfrak{su}(p+1, q+1)$, and $\mathfrak{sp}(p+1, q+1)$. Thus viewing them as matrix algebras over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, they always have columns and rows of width $1, n, 1$, respectively, and $\mathfrak{h} = \mathbb{K}$, $\mathfrak{g}_{-1} = \mathbb{K}^n$, $\mathfrak{g}_1 = \mathbb{K}^{n*}$, $\mathfrak{g}_2$ is the imaginary part of $\mathbb{K}$ (thus vanishing in the case $\mathbb{K} = \mathbb{R}$) and $\mathfrak{g}_0'$ is the algebra of the same type as $\mathfrak{g}$ of signature (p, q).

Quaternionic contact geometries

Following the work of Biquard, this is a type of geometric structure describing the Carnot-Carathéodory geometry of the boundary at infinity of quaternionic Kähler manifolds. From the point of view of Cartan geometry, this is an instance of parabolic geometries.

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Notice, the first cohomology $H^1(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ appears only in negative homogeneities, thus the entire geometry is completely defined by the distribution $T^{-1}M \subset TM$ of codimension three. This means that if there were a pre-quaternionic vector space structure on $T^{-1}M$ for which an algebraic bracket $[\ ,\ ]_{\mathrm{alg}} : \Lambda^2 T^{-1}M \to TM/T^{-1}M$ would be an imaginary part of a hermitian form, then this structure is essentially unique.

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Notice, the latter assumption is quite rigid. For example, on hypersurfaces in $\mathbb{H}^{n+1}$ we obtain the Levi form compatible with the inherited quaternionic structures if and only if the hypersurface is locally isomorphic to the 3-Sasakian sphere S^{4n+3}.

The definition of the geometry

Let M be a $(4n+3)$-dimensional manifold and H be a smooth distribution on M of codimension three. The pair (M,H) is said to be a quaternionic contact structure if around each point of M there exist 1-forms η_1, η_2, η_3 with the common kernel H, a positive definite inner product g on H, and endomorphisms I_1, I_2, I_3 of H, satisfying

$$
\begin{aligned}
&(I_1)^2 = (I_2)^2 = (I_3)^2 = -\mathrm{id}_H, \qquad && I_1 I_2 = -I_2 I_1 = I_3, \\
&d\eta_s(X,Y) = 2g(I_s X, Y) \qquad && \text{for all } \; X, Y \in H.
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$$d\eta_s(X, Y) = 2g(I_s X, Y) \qquad \text{for all } \ X, Y \in H.$$

As well known, if $\dim(M) > 7$, one can always find, locally, a triple ξ_1, ξ_2, ξ_3 of Reeb vector fields on M satisfying for all $X \in H$,

$$\eta_s(\xi_t) = \delta^s_t, \qquad \eta_s(\xi_t, X) = -d\eta_t(\xi_s, X) \qquad (7)$$

(δ^s_t being the Kronecker delta).
In dimension 7, the existence of Reeb vector fields is our additional integrability condition on the qc structure.

The Cartan connection (I.M. & J.S., AGAG, 2017)

The existence of the unique Cartan connection with co-closed curvature is well known. But in order to apply the Cartan-Kähler theory, we need explicit formulae and knowledge of the structure of the complete curvature and its derivatives.

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The harmonic curvature has got only one component, which is of the type $\kappa_H : \Lambda^2\mathfrak{g}_{-1} \mapsto \mathfrak{g}_0$. The complete curvature is given as a value of a differential operator on κ_H.

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By general arguments (due to A. Cap, including BGG machinery), or by direct experiment, it turns out that the curvature cannot have non-trivial values in all the slots indicated by zero:

$$\left(\begin{array}{c|c|c} 0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline 0 & \mathfrak{g}_0' & * \\ \hline 0 & * & * \end{array}\right)$$

The procedure

Consequently, we may proceed in the following steps:

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(All this is available in the AGAG paper.)

Brief account on the computations

The small Greek indices $\alpha, \beta, \gamma, \dots$ have got the range $1, \dots, 2n$, whereas the indices s, t, k, l, m are running from 1 to 3.

$\mathbb{R}^{4n}$ is considered with its standard inner product $\langle , \rangle$ and a quaternionic structure induced by the identification $\mathbb{R}^{4n} \cong \mathbb{H}^n$ with the quaternion coordinate space $\mathbb{H}^n$, i.e. we fix triple J_1, J_2, J_3 of complex structures which are Hermitian with respect to $\langle , \rangle$ and satisfy $J_1 J_2 = -J_2 J_1 = J_3$.
The complex vector space $\mathbb{C}^{4n}$ then splits as a direct sum of $+i$ and $-i$ eigenspaces, $\mathbb{C}^{4n} = \mathcal{W} \oplus \overline{\mathcal{W}}$, with respect to the complex structure J_1. The complex 2-form π,

$$\pi(u, v) \stackrel{def}{=} \langle J_2 u, v \rangle + i \langle J_3 u, v \rangle, \qquad u, v \in \mathbb{C}^{4n},$$

has type $(2, 0)$ with respect to J_1, i.e., it satisfies $\pi(J_1 u, v) = \pi(u, J_1 v) = i\pi(u, v)$.

Further, fix an $\langle,\rangle$-orthonormal basis

$$\{\mathfrak{e}_\alpha \in \mathcal{W}, \mathfrak{e}_{\bar{\alpha}} \in \overline{\mathcal{W}}\}, \qquad \mathfrak{e}_{\bar{\alpha}} = \overline{\mathfrak{e}_\alpha},$$

with dual basis $\{\mathfrak{e}^\alpha, \mathfrak{e}^{\bar{\alpha}}\}$ so that
$\pi = \mathfrak{e}^1 \wedge \mathfrak{e}^{n+1} + \mathfrak{e}^2 \wedge \mathfrak{e}^{n+2} + \cdots + \mathfrak{e}^n \wedge \mathfrak{e}^{2n}$. Then

$$\langle,\rangle = g_{\alpha\bar{\beta}}\, \mathfrak{e}^\alpha \otimes \mathfrak{e}^{\bar{\beta}} + g_{\bar{\alpha}\beta}\, \mathfrak{e}^{\bar{\alpha}} \otimes \mathfrak{e}^\beta, \qquad \pi = \pi_{\alpha\beta}\, \mathfrak{e}^\alpha \wedge \mathfrak{e}^\beta$$

with

$$g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}, \quad \pi_{\alpha\beta} = -\pi_{\beta\alpha} = \begin{cases} 1, & \text{if } \alpha + n = \beta \\ -1, & \text{if } \alpha = \beta + n \\ 0, & \text{otherwise.} \end{cases}$$

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$$\langle , \rangle = g_{\alpha\bar{\beta}}\, \mathfrak{e}^\alpha \otimes \mathfrak{e}^{\bar{\beta}} + g_{\bar{\alpha}\beta}\, \mathfrak{e}^{\bar{\alpha}} \otimes \mathfrak{e}^\beta, \qquad \pi = \pi_{\alpha\beta}\, \mathfrak{e}^\alpha \wedge \mathfrak{e}^\beta$$

with

$$g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}, \quad \pi_{\alpha\beta} = -\pi_{\beta\alpha} = \begin{cases} 1, & \text{if } \alpha + n = \beta \\ -1, & \text{if } \alpha = \beta + n \\ 0, & \text{otherwise.} \end{cases}$$

We also introduce a complex antilinear endomorphism $\mathfrak{j}$ of tensor algebra of $\mathbb{R}^{4n}$, which takes a tensor with components $T_{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_l \ldots}$ to

$$(\mathfrak{j}T)_{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_l \ldots} = \sum_{\bar{\sigma}_1 \ldots \bar{\sigma}_k \tau_1 \ldots \tau_l \ldots} \pi^{\bar{\sigma}_1}_{\alpha_1} \ldots \pi^{\bar{\sigma}_k}_{\alpha_k} \pi^{\tau_1}_{\bar{\beta}_1} \ldots \pi^{\tau_l}_{\bar{\beta}_l} \ldots T_{\bar{\sigma}_1 \ldots \bar{\sigma}_k \tau_1 \ldots \tau_l \ldots}.$$

The P_0 step

If (M, H) is a qc manifold, defined in terms of η_s, I_s, g, and $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3$, then there exists a natural principle bundle $\pi_0 : P_0 \to M$ with structure group $CSO(3) = \mathbb{R}^+ \times SO(3)$ whose local sections are precisely the triples of 1-forms (η_1, η_2, η_3) satisfying (6). Further, on P_0 we obtain a global triple of canonical one-forms which we denote again by (η_1, η_2, η_3). The exterior derivatives of the canonical one-forms are

$$\begin{cases} d\eta_1 = -\varphi_0 \wedge \eta_1 - \varphi_2 \wedge \eta_3 + \varphi_3 \wedge \eta_2 + 2ig_{\alpha\bar{\beta}}\,\theta^\alpha \wedge \theta^{\bar{\beta}} \\ d\eta_2 = -\varphi_0 \wedge \eta_2 - \varphi_3 \wedge \eta_1 + \varphi_1 \wedge \eta_3 + \pi_{\alpha\beta}\,\theta^\alpha \wedge \theta^\beta + \pi_{\bar{\alpha}\bar{\beta}}\,\theta^{\bar{\alpha}} \wedge \theta^{\bar{\beta}} \\ d\eta_3 = -\varphi_0 \wedge \eta_3 - \varphi_1 \wedge \eta_2 + \varphi_2 \wedge \eta_1 - i\pi_{\alpha\beta}\,\theta^\alpha \wedge \theta^\beta + i\pi_{\bar{\alpha}\bar{\beta}}\,\theta^{\bar{\alpha}} \wedge \theta^{\bar{\beta}}, \end{cases} \tag{8}$$

where $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ are some (local, non-unique) real one-forms on P_0, θ^α are some (local, non-unique) complex and semibasic one-froms on P_0.

Towards the P_1 prolongation

One can show that, if $\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\theta}^\alpha$ are any other one-forms (with the same properties as $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$) that satisfy (8), then

$$\begin{cases} \tilde{\theta}^\alpha = U^\alpha_\beta \theta^\beta + i r^\alpha \eta_1 + \pi^\alpha_{\bar{\sigma}} r^{\bar{\sigma}}(\eta_2 + i\eta_3) \\ \tilde{\varphi}_0 = \varphi_0 + 2U_{\beta\bar{\sigma}} r^{\bar{\sigma}} \theta^\beta + 2U_{\bar{\beta}\sigma} r^\sigma \theta^{\bar{\beta}} + \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3 \\ \tilde{\varphi}_1 = \varphi_1 - 2iU_{\beta\bar{\sigma}} r^{\bar{\sigma}} \theta^\beta + 2iU_{\bar{\beta}\sigma} r^\sigma \theta^{\bar{\beta}} + 2r_\sigma r^\sigma \eta_1 - \lambda_3\eta_2 + \lambda_2\eta_3, \\ \tilde{\varphi}_2 = \varphi_2 - 2\pi_{\sigma\tau} U^\sigma_\beta r^\tau \theta^\beta - 2\pi_{\bar{\sigma}\bar{\tau}} U^{\bar{\sigma}}_{\bar{\beta}} r^{\bar{\tau}} \theta^{\bar{\beta}} + \lambda_3\eta_1 + 2r_\sigma r^\sigma \eta_2 - \lambda_1\eta_3, \\ \tilde{\varphi}_3 = \varphi_3 + 2i\pi_{\sigma\tau} U^\sigma_\beta r^\tau \theta^\beta - 2i\pi_{\bar{\sigma}\bar{\tau}} U^{\bar{\sigma}}_{\bar{\beta}} r^{\bar{\tau}} \theta^{\bar{\beta}} - \lambda_2\eta_1 + \lambda_1\eta_2 + 2r_\sigma r^\sigma \eta_3, \end{cases} \tag{9}$$

where $U^\alpha_\beta, r^\alpha, \lambda_s$ are some appropriate functions; $\lambda_1, \lambda_2, \lambda_3$ are real, and $\{U^\alpha_\beta\} \in Sp(n) \subset End(\mathbb{R}^{4n})$.
The functions U^α_β, r^α and λ_s give a parametrization of a Lie group diffeomorphic to $Sp(n) \times \mathbb{R}^{4n+3}$.

The P_1 bundle and the canonical coframe

There is a canonical principle bundle $\pi_1 : P_1 \to P_0$ whose local sections are the one-forms $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$ on P_0 satisfying (8). We use $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$ to denote also the induced canonical (global) one-forms on the principal bundle P_1. Then, on P_1, there exists a unique set of complex one-forms $\Gamma_{\alpha\beta}, \phi^\alpha$ and real one-forms ψ_1, ψ_2, ψ_3 so that

$$\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}, \qquad (\mathrm{j}\Gamma)_{\alpha\beta} = \Gamma_{\alpha\beta}. \tag{10}$$

and the following equations hold true:

$$\begin{cases} d\theta^\alpha = -i\phi^\alpha \wedge \eta_1 - \pi^\alpha_{\bar\sigma}\phi^{\bar\sigma} \wedge (\eta_2 + i\eta_3) - \pi^{\alpha\sigma}\Gamma_{\sigma\beta} \wedge \theta^\beta - \frac{1}{2}(\varphi_0 + i\varphi_1) \wedge \theta^\alpha \\ \qquad\qquad -\frac{1}{2}\pi^\alpha_{\bar\beta}(\varphi_2 + i\varphi_3) \wedge \theta^{\bar\beta} \\ d\varphi_0 = -\psi_1 \wedge \eta_1 - \psi_2 \wedge \eta_2 - \psi_3 \wedge \eta_3 - 2\phi_\beta \wedge \theta^\beta - 2\phi_{\bar\beta} \wedge \theta^{\bar\beta} \\ d\varphi_1 = -\varphi_2 \wedge \varphi_3 - \psi_2 \wedge \eta_3 + \psi_3 \wedge \eta_2 + 2i\phi_\beta \wedge \theta^\beta - 2i\phi_{\bar\beta} \wedge \theta^{\bar\beta} \\ d\varphi_2 = -\varphi_3 \wedge \varphi_1 - \psi_3 \wedge \eta_1 + \psi_1 \wedge \eta_3 - 2\pi_{\sigma_\beta}\phi^\sigma \wedge \theta^\beta - 2\pi_{\bar\sigma\bar\beta}\phi^{\bar\sigma} \wedge \theta^{\bar\beta} \\ d\varphi_3 = -\varphi_1 \wedge \varphi_2 - \psi_1 \wedge \eta_2 + \psi_2 \wedge \eta_1 + 2i\pi_{\sigma_\beta}\phi^\sigma \wedge \theta^\beta - 2i\pi_{\bar\sigma\bar\beta}\phi^{\bar\sigma} \wedge \theta^{\bar\beta}, \end{cases} \tag{11}$$

The complete structure equations on P_1

The latter one-forms $\{\eta_s\}$, $\{\theta^\alpha\}$, $\{\varphi_0\}$, $\{\varphi_s\}$, $\{\Gamma_{\alpha\beta}\}$, $\{\phi^\alpha\}$, $\{\psi_s\}$ represent the components of the canonical Cartan connection corresponding to a fixed splitting of the relevant Lie algebra

$$sp(n+1,1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathbb{R} \oplus sp(1) \oplus sp(n)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

We have seen big part of the structure equations already, the remaining ones will display the curvatures components

$$\mathcal{S}_{\alpha\beta\gamma\delta},\ \mathcal{V}_{\alpha\beta\gamma},\ \mathcal{L}_{\alpha\beta},\ \mathcal{M}_{\alpha\beta},\ \mathcal{C}_\alpha,\ \mathcal{H}_\alpha,\ \mathcal{P},\ \mathcal{Q},\ \mathcal{R}$$

satisfying:

(I) Each of the arrays $\{\mathcal{S}_{\alpha\beta\gamma\delta}\}, \{\mathcal{V}_{\alpha\beta\gamma}\}, \{\mathcal{L}_{\alpha\beta}\}, \{\mathcal{M}_{\alpha\beta}\}$ is totally symmetric in its indices.

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(II) We have

$$\begin{cases} (\mathrm{j}\mathcal{S})_{\alpha\beta\gamma\delta} = \mathcal{S}_{\alpha\beta\gamma\delta} \\ (\mathrm{j}\mathcal{L})_{\alpha\beta} = \mathcal{L}_{\alpha\beta} \\ \overline{\mathcal{R}} = \mathcal{R}. \end{cases} \tag{12}$$

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(III) The exterior derivatives of the connection one-forms $\Gamma_{\alpha\beta}$, ϕ_α and ψ_s are given by

$$\begin{aligned} d\Gamma_{\alpha\beta} \;=\; & -\pi^{\sigma\tau}\Gamma_{\alpha\sigma}\wedge\Gamma_{\tau\beta} + 2\pi^{\bar\sigma}_{\alpha}(\phi_\beta\wedge\theta_{\bar\sigma} - \phi_{\bar\sigma}\wedge\theta_\beta) + 2\pi^{\bar\sigma}_{\beta}(\phi_\alpha\wedge\theta_{\bar\sigma} - \phi_{\bar\sigma}\wedge\theta_\alpha) \\ & + \pi^{\sigma}_{\bar\delta}\,\mathcal{S}_{\alpha\beta\gamma\sigma}\,\theta^\gamma\wedge\theta^{\bar\delta} + \left(\mathcal{V}_{\alpha\beta\gamma}\,\theta^\gamma + \pi^{\bar\sigma}_{\alpha}\,\pi^{\bar\tau}_{\beta}\,\mathcal{V}_{\bar\sigma\bar\tau\bar\gamma}\,\theta^{\bar\gamma}\right)\wedge\eta_1 \\ & - i\pi^{\sigma}_{\bar\gamma}\,\mathcal{V}_{\alpha\beta\sigma}\,\theta^{\bar\gamma}\wedge(\eta_2+i\eta_3) + i(\mathrm{j}\mathcal{V})_{\alpha\beta\gamma}\,\theta^\gamma\wedge(\eta_2-i\eta_3) \\ & - i\mathcal{L}_{\alpha\beta}\,(\eta_2+i\eta_3)\wedge(\eta_2-i\eta_3) + \mathcal{M}_{\alpha\beta}\,\eta_1\wedge(\eta_2+i\eta_3) + (\mathrm{j}M)_{\alpha\beta}\,\eta_1\wedge(\eta_2-i\eta_3), \end{aligned} \tag{13}$$

$$
\begin{aligned}
d\phi_\alpha \;=\;& \frac{1}{2}(\varphi_0 + i\varphi_1)\wedge\phi_\alpha + \frac{1}{2}\pi_{\alpha\gamma}(\varphi_2 - i\varphi_3)\wedge\phi^\gamma - \pi_\alpha^{\bar\sigma}\,\Gamma_{\bar\sigma\bar\gamma}\wedge\phi^{\bar\gamma} - \frac{i}{2}\,\psi_1\wedge\theta_\alpha \\
&- \frac{1}{2}\,\pi_{\alpha\gamma}(\psi_2 - i\psi_3)\wedge\theta^\gamma - i\pi_{\bar\delta}^{\sigma}\,\mathcal{V}_{\alpha\gamma\sigma}\,\theta^\gamma\wedge\theta^{\bar\delta} + \mathcal{M}_{\alpha\gamma}\,\theta^\gamma\wedge\eta_1 + \pi_\alpha^{\bar\sigma}\,\mathcal{L}_{\bar\sigma\bar\gamma}\,\theta^{\bar\gamma}\wedge\eta_1 \\
&+ i\mathcal{L}_{\alpha\gamma}\,\theta^\gamma\wedge(\eta_2 - i\eta_3) - i\pi_{\bar\gamma}^{\sigma}\,\mathcal{M}_{\alpha\sigma}\,\theta^{\bar\gamma}\wedge(\eta_2 + i\eta_3) - \mathcal{C}_\alpha(\eta_2 + i\eta_3)\wedge(\eta_2 - i\eta_3) \\
&+ \mathcal{H}_\alpha\,\eta_1\wedge(\eta_2 + i\eta_3) + i\pi_{\alpha\sigma}\,\mathcal{C}^\sigma\,\eta_1\wedge(\eta_2 - i\eta_3),
\end{aligned}
$$

$$
\begin{aligned}
d\psi_1 \;=\;& \varphi_0\wedge\psi_1 - \varphi_2\wedge\psi_3 + \varphi_3\wedge\psi_2 - 4i\phi_\gamma\wedge\phi^\gamma + 4\pi_{\bar\delta}^{\sigma}\,\mathcal{L}_{\gamma\sigma}\,\theta^\gamma\wedge\theta^{\bar\delta} + 4\mathcal{C}_\gamma\,\theta^\gamma\wedge\eta_1 \\
&+ 4\mathcal{C}_{\bar\gamma}\,\theta^{\bar\gamma}\wedge\eta_1 - 4i\pi_{\bar\gamma\bar\sigma}\,\mathcal{C}^{\bar\sigma}\,\theta^{\bar\gamma}\wedge(\eta_2 + i\eta_3) + 4i\pi_{\gamma\sigma}\,\mathcal{C}^\sigma\,\theta^\gamma\wedge(\eta_2 - i\eta_3) \\
&+ \mathcal{P}\,\eta_1\wedge(\eta_2 + i\eta_3) + \overline{\mathcal{P}}\,\eta_1\wedge(\eta_2 - i\eta_3) + i\mathcal{R}\,(\eta_2 + i\eta_3)\wedge(\eta_2 - i\eta_3),
\end{aligned}
$$

$$
\begin{aligned}
d\psi_2 + i\,d\psi_3 \;=\;& (\varphi_0 - i\varphi_1)\wedge(\psi_2 + i\psi_3) + i(\varphi_2 + i\varphi_3)\wedge\psi_1 \\
&+ 4\pi_{\gamma\delta}\phi^\gamma\wedge\phi^\delta + 4i\pi_\gamma^{\bar\sigma}\,\mathcal{M}_{\bar\sigma\bar\delta}\,\theta^\gamma\wedge\theta^{\bar\delta} \\
&+ 4i\pi_\gamma^{\bar\sigma}\,\mathcal{C}_{\bar\sigma}\,\theta^\gamma\wedge\eta_1 - 4\mathcal{H}_{\bar\gamma}\,\theta^{\bar\gamma}\wedge\eta_1 \\
&- 4i\mathcal{C}_{\bar\gamma}\,\theta^{\bar\gamma}\wedge(\eta_2 + i\eta_3) - 4i\pi_\gamma^{\bar\sigma}\,\mathcal{H}_{\bar\sigma}\,\theta^\gamma\wedge(\eta_2 - i\eta_3) \\
&\quad i\mathcal{R}\,\eta_1\wedge(\eta_2 + i\eta_3) + \overline{\mathcal{Q}}\,\eta_1\wedge(\eta_2 - i\eta_3) - \overline{\mathcal{P}}\,(\eta_2 + i\eta_3)\wedge(\eta_2 - i\eta_3).
\end{aligned}
$$

Structure of the curvature

All the potentially nonzero curvature components of κ, as just deduced, are listed in the table:

homogeneity	the cochains	object in structure eq.
2	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{sp}(n)$	$\mathcal{S}_{\alpha\beta\gamma\delta}$
3	$\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \to \mathfrak{sp}(n)$	$\mathcal{V}_{\alpha\beta\gamma}$
3	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{1}$	$\mathcal{V}_{\alpha\beta\gamma}$
4	$\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-2} \to \mathfrak{sp}(n)$	$\mathcal{L}_{\alpha\beta}, \mathcal{M}_{\alpha\beta}$
4	$\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_{1}$	$\mathcal{L}_{\alpha\beta}, \mathcal{M}_{\alpha\beta}$
4	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{2}$	$\mathcal{L}_{\alpha\beta}, \mathcal{M}_{\alpha\beta}$
5	$\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-2} \to \mathfrak{g}_{1}$	$\mathcal{C}_{\alpha}, \mathcal{H}_{\alpha}$
5	$\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_{2}$	$\mathcal{C}_{\alpha}, \mathcal{H}_{\alpha}$
6	$\mathfrak{g}_{-2} \wedge \mathfrak{g}_{-2} \to \mathfrak{g}_{2}$	$\mathcal{P}, \mathcal{Q}, \mathcal{R}$

Notice that it is the $\partial^* \kappa = 0$ normalization which enforces several potentially different components to coincide.

1. Our motivation
2. Geometric structures as EDS
3. The qc structures
4. The Cartan test for qc structures

Essentially finished followup paper with I.M.

Now we come to the Cartan test quest.

In order to show that the Assumptions I and II on the geometric structures via EDS holds true, we may observe that the Bianchi identities imply that the one-forms expressing the differentials of the curvature functions belong to the linear span of η_1, η_2, η_3, θ^{α}, $\theta^{\bar{\alpha}}$ (quite lengthy and technical, all already in the AGAG paper).

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In order to show that the Assumptions I and II on the geometric structures via EDS holds true, we may observe that the Bianchi identities imply that the one-forms expressing the differentials of the curvature functions belong to the linear span of η_1, η_2, η_3, θ^α, $\theta^{\bar{\alpha}}$ (quite lengthy and technical, all already in the AGAG paper).

Furthermore, if we are considering the Bianchi identities as a system of algebraic equations for the unknown one-forms expressing the differentials of the curvature functions, then the solutions may be parametrized by elements of a certain vector space.

More explicitly, on P_1, there exist unique, globally defined, complex valued functions

$$\begin{aligned} &\mathcal{A}_{\alpha\beta\gamma\delta\epsilon},\ \mathcal{B}_{\alpha\beta\gamma\delta},\ \mathcal{C}_{\alpha\beta\gamma\delta},\ \mathcal{D}_{\alpha\beta\gamma},\ \mathcal{E}_{\alpha\beta\gamma},\ \mathcal{F}_{\alpha\beta\gamma},\ \mathcal{G}_{\alpha\beta},\ \mathcal{X}_{\alpha\beta},\\ &\mathcal{Y}_{\alpha\beta},\ \mathcal{Z}_{\alpha\beta},\ (\mathcal{N}_1)_\alpha,\ (\mathcal{N}_2)_\alpha,\ (\mathcal{N}_3)_\alpha,\ (\mathcal{N}_4)_\alpha,\ (\mathcal{N}_5)_\alpha,\ \mathcal{U}_s,\ \mathcal{W}_s \end{aligned} \tag{14}$$

so that:
(I) Each of the arrays $\{\mathcal{A}_{\alpha\beta\gamma\delta\epsilon}\}$, $\{\mathcal{B}_{\alpha\beta\gamma\delta}\}$, $\{\mathcal{C}_{\alpha\beta\gamma\delta}\}$, $\{\mathcal{D}_{\alpha\beta\gamma}\}$, $\{\mathcal{E}_{\alpha\beta\gamma}\}$, $\{\mathcal{F}_{\alpha\beta\gamma}\}$, $\{\mathcal{G}_{\alpha\beta\gamma}\}$, $\{\mathcal{X}_{\alpha\beta}\}$, $\{\mathcal{Y}_{\alpha\beta}\}$, $\{\mathcal{Z}_{\alpha\beta}\}$ is totally symmetric in its indices.
(II) The differentials of all the curvature components are expressed in a linear way by means of the latter tensors and the curvature and the coframe.

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(II) The differentials of all the curvature components are expressed in a linear way by means of the latter tensors and the curvature and the coframe.

This concludes, that the qc structures might be discussed via the Cartan test.

For the (real) dimension D of the vector space determined by (14), we calculate

$$D = 2\binom{2n+4}{5} + 4\binom{2n+3}{4} + 6\binom{2n+2}{3} + 8\binom{2n+1}{2} + 20n + 12$$
$$= \frac{2}{15}(2n+5)(2n+3)(n+3)(n+2)(n+1).$$

In order to show that our exterior differential system $\mathcal{I}$ is in involution — which would allow us to apply the Cartan's Third Theorem to it — we need to compute the character sequence $v_1, v_2, v_3, \ldots, v_{d_1}$ of the system and show that the Cartan's test

$$D = v_1 + 2v_2 + 3v_3 + \cdots + d_1 v_{d_1}$$

is satisfied, where $d_1 = (2n+5)(n+2)$ is the dimension of P_1.

A lengthy and technical computation shows $(1 \leq \lambda \leq n)$

$$\begin{cases} v_\lambda = \frac{1}{2}(\lambda - 1)(\lambda - 2n - 4)(\lambda - 2n - 5) \\ v_{n+\lambda} \ = \ \frac{1}{2}(n + \lambda - 1)(n - \lambda + 4)(n - \lambda + 5) \\ v_{2n+1} \ = \ 12n \\ v_{2n+2} \ = \ 6n + 3 \\ v_{2n+3} \ = \ 2n + 2 \\ v_{2n+4} \ = \ v_{2n+5} \ = \ \dots \ = \ v_{d_1} \ = \ 0. \end{cases}$$

Moreover, **the Cartan test works!!!**

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Moreover, **the Cartan test works!!!**

$n = 1$:
$D = \frac{2}{15} \cdot 7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 112$
$0 \cdot 1 + 1 \cdot 20 + 2 \cdot 10 + 3 \cdot 12 + 4 \cdot 9 + 5 \cdot 4 = 112.$

A lengthy and technical computation shows $(1 \leq \lambda \leq n)$

$$\begin{cases} v_\lambda = \frac{1}{2}(\lambda - 1)(\lambda - 2n - 4)(\lambda - 2n - 5) \\ v_{n+\lambda} \;=\; \frac{1}{2}(n + \lambda - 1)(n - \lambda + 4)(n - \lambda + 5) \\ v_{2n+1} \;=\; 12n \\ v_{2n+2} \;=\; 6n + 3 \\ v_{2n+3} \;=\; 2n + 2 \\ v_{2n+4} \;=\; v_{2n+5} \;=\; \ldots \;=\; v_{d_1} \;=\; 0. \end{cases}$$

Moreover, **the Cartan test works!!!**

$n = 1$:
$D = \frac{2}{15} \cdot 7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 112$
$0 \cdot 1 + 1 \cdot 20 + 2 \cdot 10 + 3 \cdot 12 + 4 \cdot 9 + 5 \cdot 4 = 112.$
$n = 2$:
$D = \frac{2}{15} \cdot 9 \cdot 7 \cdot 5 \cdot 4 \cdot 3 = 504$
$1 \cdot 0 + 2 \cdot 21 + 3 \cdot 30 + 4 \cdot 30 + 5 \cdot 24 + 6 \cdot 15 + 7 \cdot 6 = 504.$

The generality of the quaternionic contact structures

Theorem

The quaternionic contact structures in dimension $4n+3$ depend on $2n+2$ functions of $2n+3$ variables.

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Moreover, given some arbitrary complex numbers $\mathcal{S}^\circ_{\alpha\beta\gamma\delta}$, $\mathcal{V}^\circ_{\alpha\beta\gamma}$, $\mathcal{L}^\circ_{\alpha\beta}$, $\mathcal{M}^\circ_{\alpha\beta}$, $\mathcal{C}^\circ_{\alpha}$, $\mathcal{H}^\circ_{\alpha}$, $\mathcal{P}^\circ$, $\mathcal{Q}^\circ$, $\mathcal{R}^\circ$, $\mathcal{A}^\circ_{\alpha\beta\gamma\delta\epsilon}$, $\mathcal{B}^\circ_{\alpha\beta\gamma\delta}$, $\mathcal{C}^\circ_{\alpha\beta\gamma\delta}$, $\mathcal{D}^\circ_{\alpha\beta\gamma}$, $\mathcal{E}^\circ_{\alpha\beta\gamma}$, $\mathcal{F}^\circ_{\alpha\beta\gamma}$, $\mathcal{G}^\circ_{\alpha\beta}$, $\mathcal{X}^\circ_{\alpha\beta}$, $\mathcal{Y}^\circ_{\alpha\beta}$, $\mathcal{Z}^\circ_{\alpha\beta}$, $(\mathcal{N}^\circ_1)_\alpha$, $(\mathcal{N}^\circ_2)_\alpha$, $(\mathcal{N}^\circ_3)_\alpha$, $(\mathcal{N}^\circ_4)_\alpha$, $(\mathcal{N}^\circ_5)_\alpha$, $\mathcal{U}^\circ_s$, $\mathcal{W}^\circ_s$ depending totally symmetrically on the indices $1 \leq \alpha, \beta, \gamma, \delta \leq 2n$ and satisfiyng $(\mathrm{j}\mathcal{S}^\circ)_{\alpha\beta\gamma\delta} = \mathcal{S}^\circ_{\alpha\beta\gamma\delta}$, $(\mathrm{j}\mathcal{L}^\circ)_{\alpha\beta} = \mathcal{L}^\circ_{\alpha\beta}$, $\overline{\mathcal{R}^\circ} = \mathcal{R}^\circ$, there exists a real analytic qc structure defined in a neighborhood Ω of $0 \in R^{4n+3}$ such that for some point $u \in P_1$ with $\pi_o(\pi_1(u)) = 0$ the curvature functions and their derivatives are given by these complex numbers.

The generality of the quaternionic contact structures

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Moreover, given some arbitrary complex numbers $\mathcal{S}^\circ_{\alpha\beta\gamma\delta}$, $\mathcal{V}^\circ_{\alpha\beta\gamma}$, $\mathcal{L}^\circ_{\alpha\beta}$, $\mathcal{M}^\circ_{\alpha\beta}$, $\mathcal{C}^\circ_{\alpha}$, $\mathcal{H}^\circ_{\alpha}$, $\mathcal{P}^\circ$, $\mathcal{Q}^\circ$, $\mathcal{R}^\circ$, $\mathcal{A}^\circ_{\alpha\beta\gamma\delta\epsilon}$, $\mathcal{B}^\circ_{\alpha\beta\gamma\delta}$, $\mathcal{C}^\circ_{\alpha\beta\gamma\delta}$, $\mathcal{D}^\circ_{\alpha\beta\gamma}$, $\mathcal{E}^\circ_{\alpha\beta\gamma}$, $\mathcal{F}^\circ_{\alpha\beta\gamma}$, $\mathcal{G}^\circ_{\alpha\beta}$, $\mathcal{X}^\circ_{\alpha\beta}$, $\mathcal{Y}^\circ_{\alpha\beta}$, $\mathcal{Z}^\circ_{\alpha\beta}$, $(\mathcal{N}^\circ_1)_\alpha$, $(\mathcal{N}^\circ_2)_\alpha$, $(\mathcal{N}^\circ_3)_\alpha$, $(\mathcal{N}^\circ_4)_\alpha$, $(\mathcal{N}^\circ_5)_\alpha$, $\mathcal{U}^\circ_s$, $\mathcal{W}^\circ_s$ depending totally symmetrically on the indices $1 \leq \alpha, \beta, \gamma, \delta \leq 2n$ and satisfiyng $(\mathrm{j}\mathcal{S}^\circ)_{\alpha\beta\gamma\delta} = \mathcal{S}^\circ_{\alpha\beta\gamma\delta}$, $(\mathrm{j}\mathcal{L}^\circ)_{\alpha\beta} = \mathcal{L}^\circ_{\alpha\beta}$, $\overline{\mathcal{R}^\circ} = \mathcal{R}^\circ$, there exists a real analytic qc structure defined in a neighborhood Ω of $0 \in R^{4n+3}$ such that for some point $u \in P_1$ with $\pi_o(\pi_1(u)) = 0$ the curvature functions and their derivatives are given by these complex numbers.

It is interesting to compare this to an old result by Claude LeBrun (Duke Math. J., August 1991) where he finds a nontrivial deformation of the quaterninic contact structure on the sphere