Subriemannian metrizability of some Parabolic Geometries

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Definition

Parabolic Geometry $(\mathcal{G}, \omega)$ is a curved deformation of the homogeneous model $G \to G/P$ with G semisimple, $P \subset G$ parabolic.
Thus, $\mathcal{G} \to M$ is a principal fiber budle, $\omega : T\mathcal{G} \to \mathfrak{g}$ is an absolute parallelism and they mimic the properties of the Maurer Cartan form on G as much as possible.

(i) ω reproduces the fundamental vector fields, and
(ii) ω is Ad-equivariant with respect to the principal action of P.
NB:

$$P = P_0 \ltimes P_+$$

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+ = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k.$$

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- $\gamma = \sigma^*\omega_0 : T\mathcal{G}_0 \to \mathfrak{p}_0$
- $\mathsf{P} = \sigma^*\omega_+ = \mathsf{P}_1 + \cdots + \mathsf{P}_k : T\mathcal{G}_0 \to \mathfrak{p}_+ = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$.

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On the underlying manifold M we obtain:

- θ is a soldering form for M
- γ is a linear connection form on M, $\theta + \gamma$ is the affine connection, called *Weyl connection*
- P is a one–form on M valued in T^*M

Transformation properties of Weyl connections

Let us start with examples of $|1|$-graded Lie algebras $\mathfrak{g}$.
In this case, the change of the Weyl connection ∇ by a one-form Υ deforms the covariant derivatives $\nabla_\xi \eta$ by the action of the Lie bracket

$$\{\xi, \Upsilon\}$$

on η, i.e. the above formulae represent the Lie brackets

$$[[\,\cdot\,, \Upsilon], \eta]$$

in $\mathfrak{g}$ with the free slot for the value of ξ (and we consider vectors and forms as functions on $\mathcal{G}_0$ valued in $\mathfrak{n}$ and $\mathfrak{p}_1$, respectively).

Examples – projective geometry

Choice of a linear connection ∇_a defining the projective class $[\nabla]$ of connections sharing the same geodesics (as unparameterized curves).

$$\hat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b + \Upsilon_c X^c \delta_a^b \tag{1}$$

for all vector fields X^b and a given differential form Υ_a.
In particular we see, that the freedom in the choice is parameterized by one-forms.
The normalized Cartan connections ω are of type G/P with $G = PGL(m+1, \mathbb{R})$ and P the maximal parabolic subgroup containing all upper block triangular matrices with blocks of sizes 1 and m. Further, $\mathfrak{n} = \mathbb{R}^m$, $\mathfrak{p}_0 = \mathfrak{gl}(m, \mathbb{R})$, and $\mathfrak{p}^1 = \mathbb{R}^{m*}$.

Examples – complex projective geometry

Consider the complex homogeneous space G/P with $\mathfrak{g} = \mathfrak{s}l(m+1, \mathbb{C})$ and $\mathfrak{p} \subset \mathfrak{g}$ block-wise upper triangular as above, viewed as real algebras.

The geometric structure is determined by the choice of an almost complex structure $J \in \operatorname{End} TM$ on a $2m$-dimensional manifold M and a connection ∇ keeping J convariantly constant and having the Nijenhuis tensor of J as torsion.

If we want to translate the previous formula to the real setup of $\mathfrak{g} \subset \mathfrak{gl}(2m, \mathbb{R})$, we get

$$\hat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b - \Upsilon_c J^c_a J^b_d X^d + \Upsilon_c X^c \delta^b_a - \Upsilon_c J^c_d X^d J^b_a, \quad (2)$$

where J^b_a means the almost complex structure. The homogeneous model for this geometry is the complex projective space $\mathbb{C}P^m = G/P$, viewed as real manifold with a complex structure.

Example - Grassmannian 2-planes

Consider again $\mathfrak{g} = \mathfrak{sl}(m+2, \mathbb{R})$ as in the case of projective geometries, but with another choice of block-wise upper triangular matrices in P, those with sizes 2 and m.
This choice leads to special G-structures, where the reduction to P_0 means identification of the tangent space with the tensor product of two auxiliary vector bundles E^* and F of dimensions 2 and m (and identifying $\Lambda^2 E^* \simeq \Lambda^m F$). These geometries are called almost Grasmannian geometries of 2-planes.
The abstract index formalism introduces $X_{A'}$ for a section of E, while X_A lives in F. Thus we write $X_A^{A'}$ for a vector field, $\eta_{A'}^B$ is a one-form, etc. Again, the G-structure on a manifold M determines uniquely the normal Cartan connection ω on $\mathcal{G} \to M$.

The connections from the distinguished class respect the structures, thus they are of the form $\nabla^A_{A'}$ (i.e. a tensor product of two connections on E^* and F) and the formula for the freedom in their choice is

$$\hat{\nabla}^A_{A'} X^{B'}_B = \nabla^A_{A'} X^{B'}_B + \delta^{B'}_{A'} \Upsilon^A_{C'} X^{C'}_B + \delta^A_B \Upsilon^C_{A'} X^{B'}_C . \tag{3}$$

The homogenous model G/P is the space of 2-planes in $\mathbb{R}^{m+2}$. Choosing $m = 2n$, we can consider the other real form of the same algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$, and we get the well known almost quaternionic structures which may be understood as the quaternionic version of the projective structures above.

CR and quaternionic contact geometries

Consider hypersurfaces M in $\mathbb{C}^{m+1}$ or $\mathbb{H}^{m+1}$.
The conjugation on the tangent spaces TM cut a distribution $\mathcal{H} = \overline{TM} \cap TM$ which is, in generic situations, of codimension 1 in the complex case and codimension 3 in the quaternionic case.
The geometries are called *Cauchy-Riemann (CR)* or *quaternionic contact*, respectively.
The resulting symbol algebras are, in the generic case, isomorphic to the complex or quaternionic Heisenberg algebras $\mathfrak{n}$, the entire algebras $\mathfrak{g}$ are $\mathfrak{su}(p+1, q+1)$ or $\mathfrak{sp}(p+1, q+1)$, respectively.

In both cases, the parabolic subalgebras are the block-wise upper triangular matrices of block sizes 1, $m = p + q$, 1, of complex or quaternionic hermitian skew-symmetric matrices (with respect to the anti-diagonals and the suitable Hemitian forms of the given signatures $(p + 1, q + 1)$).
In particular, the length of the grading is 2 and the $\mathfrak{g}_{-2}$ component consists of purely imaginary items, thus 1-dimensional in the CR case, while 3-dimensional for the quaternionic contact geometries.
The homogeneous models are the CR sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$ and the quaternionic sphere $S^{4m+3} \subset \mathbb{H}^{m+1}$, respectively.

More $|2|$-graded examples

Actually, the block-wise structure of the matrices with blocks of sizes k, n, and k appears in many more interesting geometric structures which we shall discuss, all of them with filtrations of length two on the manifolds.

The so called free distributions and free CR distributions are good examples with blocks of sizes n, 1, n and Lie agebras $\mathfrak{so}(n+1,n)$ and $\mathfrak{su}(n+1,n)$, respectively.

The quaternionic analogue is available for the choice of $\mathfrak{so}^*(2n)$ with n odd.

Weyl connections

$$\begin{tikzcd}
& & \mathcal{L} \arrow[dr] & \\
\mathcal{G} \arrow[r, "\pi"'] & \mathcal{G}_0 \arrow[l, dotted, bend right, "\text{Weyl connections}"'] \arrow[ur] \arrow[dr] & & M \arrow[ul, dotted, bend right, "\text{exact Weyl connections}"'] \\
& & \mathcal{G}_0/\exp tE \arrow[ur] \arrow[ul, dotted, bend right, "\text{reductions to } P_0'"'] &
\end{tikzcd}$$

- $\theta = \sigma^* \omega_- = \theta_{-k} + \cdots + \theta_{-1} : T\mathcal{G}_0 \to \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$
- $\gamma = \sigma^* \omega_0 : T\mathcal{G}_0 \to \mathfrak{p}_0$
- $\mathsf{P} = \sigma^* \omega_+ = \mathsf{P}_1 + \cdots + \mathsf{P}_k : T\mathcal{G}_0 \to \mathfrak{p}_+ = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k.$

Exact Weyl connections – reductions to P_0', normal Weyl connections – exponential coordinates.

Projective metrizability

The question is: can we find a connection in a given projective class which would leave some metric parallel?

This was discussed in projective geometry very long back, including the following procedure due to Roger Liouville (1889):
Given an inverse metric $\eta \in S^2(TM)$ on a projective manifold $(M, [\nabla])$ of dimension n, we have seen the change of its derivative under a deformation $\hat{\nabla} = \nabla + \Upsilon$ in the class $[\nabla]$

$$\hat{\nabla}_Z \eta = \nabla_Z \eta - \{\Upsilon, Z\} \cdot \eta$$

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Consider the first order operator $\mathcal{D} : S^2 TM \to \ker b$, i.e. we project the covariant derivative to the kernel of the trace $b : S^2 TM \otimes T^* M \to TM$ and choose the weights so that this operator $\mathcal{D} = \pi \circ \nabla$ be invariant.
Thus, if $\mathcal{D}(\eta)$ is zero, the change in $\hat{\nabla}\eta$ may appear only in the complement to the kernel of the trace.
Finally, we can compute Υ so that we make $\hat{\nabla}\eta = 0$ and this is the requested ∇ with parallel η.

The sub-riemannian parabolic version

Consider a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) with the bracket generating (horizontal) distribution $\mathcal{H} \subset TM$.
There is the class $[\nabla_{|\mathcal{H}}]$ of partial Weyl connections on M and we look for metrics on $\mathcal{H}$ parallel in the $\mathcal{H}$-directions for one of the connections in the class.

The homogeneous model

On $M = G/P$, the exponential coordinates identify the big cell $\exp \mathfrak{g}_- \subset M$ with the nilpotent group. This identification also yields the reduction to the Levi factor of P and thus the (very flat) Weyl connection ∇. Any metric on $\mathfrak{h} \subset T_O M$ in the origin can be uniquely extended by left shifts and this provides a parallel metric on the big cell.

Notation: $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{h} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$
The choice of $\mathfrak{p}$ will be indicated by the crosses over the nodes Σ_0 in Dynkin diagram for $\mathfrak{g}$.

The algebraic linearization principle

Let $c \colon \mathfrak{h}^* \otimes S^2\mathfrak{h} \to \mathfrak{h}$ be the natural contraction. We consider an invariant subspace B of $S^2\mathfrak{h}$ which contains nondegenerate elements. Given such a B, let $b \colon \mathfrak{h}^* \otimes B \to \mathfrak{h}$ be the restriction of c. We then say:
B satisfies the *algebraic linearization condition* (ALC) iff b is surjective, and its kernel has at most $\#\Sigma_0$ irreducible components.

Thus we may write $\mathfrak{h}^* \otimes B = \ker b \oplus \zeta(\mathfrak{h})$ where $\zeta \colon \mathfrak{h} \to \mathfrak{h}^* \otimes B$ is a $\mathfrak{p}_0$-invariant map with $b \circ \zeta = \mathrm{id}_{\mathfrak{h}}$.
Let $\pi = \mathrm{id}_{\mathfrak{h}^* \otimes B} - \zeta \circ b$ be the projection onto $\ker b$.

NB:

The ALC is quite restrictive. It implies that B cannot have many irreducible components; on the other hand each such component must lie in a single weight space of $\mathfrak{z}(\mathfrak{p}_0)$, with weight $\alpha + \beta$ for weights α and β of $\mathfrak{h}$, hence in the image $\mathfrak{h}_{\alpha\beta}$ of $\mathfrak{h}_\alpha \otimes \mathfrak{h}_\beta \to S^2\mathfrak{h}$ for the corresponding weight spaces. Now the restriction of c to $\mathfrak{h}^* \otimes \mathfrak{h}_{\alpha\beta}$ has image $\mathfrak{h}_\alpha + \mathfrak{h}_\beta$, so B must have sufficiently many irreducible components to cover all weight spaces of $\mathfrak{h}$.

These competing requirements are most easily satisfied when $\#\Sigma_0 \leq 2$.

The linearization principle

We want to construct weighted inverse metric, i.e. a section η of $S^2\mathcal{H} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$. We suppose η is a section of $\mathcal{B} \otimes \mathcal{L}$, where $\mathcal{B} = \mathcal{G} \times_P B$ and $B \subset S^2\mathfrak{h}$ satisfies the ALC.
Assume, that there is an invariant first order linear operator $\mathcal{D}$ from $\Gamma(\mathcal{B} \otimes \mathcal{L})$ to $\Gamma(\ker b)$ with $\mathcal{D} = \pi \circ \nabla_{|\mathcal{H}}$ for any Weyl structure ∇.
Solutions of $\mathcal{D}\eta = 0$ are characterized by the fact that for some (hence any) Weyl structure ∇, there is a section X^∇ of $\mathcal{H} \otimes \mathcal{L}$ such that

$$\nabla_{|\mathcal{H}}\eta = \zeta(X^\nabla).$$

Next, suppose $\tilde{\nabla}|_{\mathcal{H}} = \nabla|_{\mathcal{H}} + \Upsilon$ with Υ in $\mathcal{H}^*$.
Then for any $Z \in \Gamma\mathcal{H}$

$$\tilde{\nabla}_Z \eta = \nabla_Z \eta - \{\Upsilon, Z\} \cdot \eta$$

and $\{\Upsilon, \cdot\} \bullet \eta$ is in the image of ζ by the invariance of $\mathcal{D}$. Hence by Schur's lemma

$$\{\Upsilon, \cdot\} \bullet \eta = (\zeta \circ b)(\{\Upsilon, \cdot\} \bullet \eta) = (\zeta \circ b)\Big(\textstyle\sum_{i \in \Sigma_0} \ell_i \Upsilon_i \otimes \eta\Big)$$

for some scalars ℓ_i, where $\Upsilon = \sum_{i \in \Sigma_0} \Upsilon_i$ with $\Upsilon_i \in \mathfrak{h}_i^*$. We define $\sharp_\eta(\Upsilon) = \sum_i \ell_i \Upsilon_i \lrcorner \eta$, and

$$\tilde{\nabla}|_{\mathcal{H}} \eta = \nabla|_{\mathcal{H}} \eta - \zeta(\sharp_\eta(\Upsilon)).$$

Now if η is a nondegenerate solution of $\mathcal{D}\eta = 0$, with $\nabla|_{\mathcal{H}}\eta = \zeta(X^{\nabla})$ for some Weyl structure ∇ and $X^{\nabla} \in \Gamma(\mathcal{H} \otimes \mathcal{L})$, we may take

$$\Upsilon = \sharp_\eta^{-1}(X^{\nabla})$$

to obtain

$$\tilde{\nabla}|_{\mathcal{H}}\eta = \zeta(X^{\nabla}) - \zeta(\sharp_\eta(\Upsilon)) = 0.$$

Hence η is (inverse to) a horizontal compatible metric.

The first order operators on parabolic geometries are very well understood in [V. Souček, JS, First order invariant differential operators for parabolic geometries. In Seminaires & Congres. France: French Math. Soc., 2000. s. 249-273].
They are given in the following way: taking V_λ an irreducible representation of the Levi factor, $\lambda = \lambda' + \lambda^0$, let $\alpha = \alpha' + \alpha^0$ be the highest weight of one of the components V_α in $\mathfrak{h}^* \otimes \mathbb{C}$, and let $\mu = \mu' + \mu^0$ be the highest weight of a component V_μ in the tensor product $V_\lambda \otimes V_\alpha$.
The key observation is that there will be a first order invariant operator between sections of the bundles associated to V_λ and V_μ if and only if the scalar expression

$$c_{\lambda,\mu,\alpha} = \frac{1}{2}\big((\mu, \mu + 2\rho_0) - (\lambda, \lambda + 2\rho_0) - (\alpha, \alpha + 2\rho_0)\big)$$

vanishes, where $\rho_0 \in \mathfrak{t}'^*$ is half the sum of the positive roots of $\mathfrak{p}_0$.

Splitting the latter expression into contributions from the central and semisimple parts, we arrive at

$$c_{\lambda,\mu,\alpha} = c_{\lambda',\mu',\alpha'} + (\lambda^0, \alpha^0).$$

If we fix λ', α and μ', this decomposition provides one (real) linear equation on the central weight λ^0.
Finally we quite easily see that having more components, we might adjust the weight uniquely so that the operators will exist simultaneously.
Moreover, if the projection goes to the Cartan products of the weights, the resultign operator will always be the so called 'first BGG operator'. This will be the case with all our choices of components B satisfying the ALC condition.

The metric tractor bundle and first BGG operator

If $\mathcal{V}$ is a bundle coming from a G representation, then there is a sequence of invariant BGG operators between bundles coming from P-modules with $\mathfrak{p}$-dominant highest weights on the same affine orbit in the Weyl group of $\mathfrak{g}$ as the highest weight of $\mathbb{V}$. The first of such operators is defined on the bundle defined by the highest weight of $\mathbb{V}$ viewed as a weight for a P-module and it is always an overdetermined operator.

On the homogenous model G/P, the latter operator has got the maximal space of global solutions parametrized by the representation space $\mathbb{V}$ (the kernel of the operator is in bijective correspondence with the space of parallel sections of the tractor bundle in question). The solutions are always expressed in polynomial terms in the flat normal coordinates at the homogenous model, which holds true also for those solutions on general curved geometries which are determined by parallel sections of the corresponding tractor bundle, the so called normal solutions.

Example - the Grassmannian 2-planes

Let us show some formulae for the Grassmannian 2-plane geometry. Here S^2TM decomposes into two components $S^2E^* \otimes S^2F$ and the right choice is $B = \Lambda^2E^* \otimes \Lambda^2F$. Λ^2E^* thus we work with inverse metrics $\sigma^{[AB]}$.

$$\zeta \circ b(\psi_A^{A'[BC]}) = \zeta(\psi_B^{A'[BC]}) = \frac{1}{m-1}(\delta_D^B\psi_P^{A'PC} - \delta_D^C\psi_P^{A'PB}).$$

In particular, we obtain

$$X^\nabla = \frac{1-m}{2}\nabla_D^{C'}\sigma^{AD}.$$

The action of the expression $\{\Upsilon, \cdot\}$ on the field σ^{EF} is

$$(\Upsilon_C^{A'}\delta_A^D) \bullet \sigma^{EF} = \sigma^{CF}\delta_A^E\Upsilon_C^{A'} - \sigma^{CE}\delta_A^F\Upsilon_C^{A'}$$

The tractor bundle corresponding to our choice of B is the second exterior power of the standard representation of $SL(m+2,\mathbb{R})$ and we do not add any extra weight to our σ^{AB} and thus

$$\{\Upsilon, \cdot\} \bullet \sigma = \zeta \circ b\big((m-1)\Upsilon_A^{A'}\sigma^{EF}\big).$$

The second exterior power splits as

$$\Lambda^2\mathcal{V} = \mathcal{E}^{\alpha\beta} = \begin{pmatrix} \rho^{[AB]} \\ \tau^{AA'} \\ \nu \end{pmatrix}$$

and the action of $\mathfrak{h}$ provides the general polynomial solution (in coordinates $x^A_{A'}$) corresponding to the parallel tractor with the chosen value in the origin:

$$\sigma^{AB}(x) = \rho^{AB} + \tau^{A'[A}x^{B]}_{A'} + \frac{1}{2}\nu x^{[A}_{[A'}x^{B]}_{B']}.$$

For curved geometries, some of the solutions to our linearized metrizability problem may be still related to parallel tractors in the corresponding metric tractor bundle.
Then the same formulae apply in the so called normal coordinates.

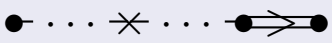
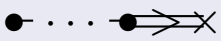
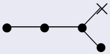
Theorem

The following list contains all real parabolic geometries with the Lie algebra $\mathfrak{g}$ of type A_n and irreducible $\mathfrak{h}$, such that the ALC condition holds true for non-trivial B.

1. $\times\!\!-\!\!\bullet \cdots \bullet$, $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$, $n > 1$.
2. $\bullet\!\!-\!\!\times\!\!-\!\!\bullet \cdots \bullet$, $\mathfrak{g} \simeq \mathfrak{sl}(n+1, \mathbb{R})$, $n \geq 4$.
3. $\bullet\!\!-\!\!\times\!\!-\!\!\bullet \cdots \bullet$, $\mathfrak{g} \simeq \mathfrak{sl}(p+1, \mathbb{H})$, $n = 2p+1 \geq 5$.
4. $\times\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\times$, $\mathfrak{g} \simeq \mathfrak{su}(p, q)$, $1 \leq p \leq q$, $p + q = n + 1$.
5. $\bullet \cdots \bullet\!\!-\!\!\times\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\times\!\!-\!\!\bullet \cdots \bullet$, $\mathfrak{g} \simeq \mathfrak{su}(p, q)$, $k \leq p \leq q$, $p + q = n + 1$, *the first cross at the k-th root, crosses at symmetric places.*
6. $\bullet \cdots \bullet\!\!-\!\!\times\!\!-\!\!\times\!\!-\!\!\bullet \cdots \bullet$, $\mathfrak{g} \simeq \mathfrak{su}(p, p+1), 2p = n$; *the first cross at the p-th root.*

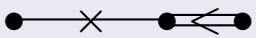
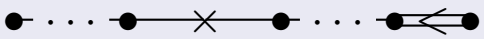
Theorem

The cases B_n or D_n with irreducible $\mathfrak{h}$ allowing for the ALC:

1.  *or* *, with the cross at the kth position, not in the ends. The Lie algebras are* $\mathfrak{g} \simeq \mathfrak{so}(p,q)$ *with* $k \leq p \leq q$ *(and* $p+q=2n+1$ *in the* B_n *case, while* $p+q=2n$ *for* D_n*).*
2. *, with the cross at the 2k-th position,* $2 \leq 2k \leq n-2$, $n \geq 4$*. The Lie algebras are* $\mathfrak{g} \simeq \mathfrak{so}^*(2n)$*.*
3. *,* $n \geq 2$*, the Lie algebra is the split form* $\mathfrak{so}(n, n+1)$*.*
4. *, the real Lie algebra is* $\mathfrak{so}(5,5)$
5. *, the real Lie algebra is* $\mathfrak{so}(p-1, p+1)$*.*
6. *, the real Lie algebra is* $\mathfrak{so}^*(2p)$*,* p *odd.*

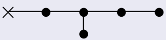
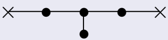
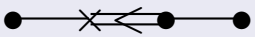
Theorem

The following list contains all real parabolic geometries with the Lie algebra $\mathfrak{g}$ *of type* C_n, $n \geq 3$, *and irreducible* $\mathfrak{h}$, *such that the ALC condition holds true for non-trivial* B.

1.  , *the Lie algebras are* $\mathfrak{sp}(8,\mathbb{R})$, $\mathfrak{sp}(2,2)$, *or* $\mathfrak{sp}(1,3)$.
2.  , *the cross at* k*-th position,* $k \geq 3$, *but not the last node. The Lie algebras are* $\mathfrak{sp}(2n,\mathbb{R})$, *and if* k *is even, then also* $\mathfrak{sp}(p,q)$, $k \leq p \leq q$.

Theorem

The following list contains all real parabolic geometries with the Lie algebra $\mathfrak{g}$ of the exotic types and irreducible $\mathfrak{h}$, such that the ALC condition holds true for non-trivial B.

1.  *, the real Lie algebra is of type EI or EIV (|1|-graded).*
2.  *, the real Lie algebra is of type EII (|2|-graded).*
3.  *, the split form FI (|4|-graded).*
4.  *, the split form (the (2,3,5) distributions).*