Enriched weakness *

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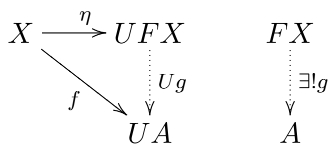
Abstract

The basic notions of category theory, such as limit, adjunction, and orthogonality, all involve assertions of the existence and uniqueness of certain arrows. Weak notions arise when one drops the uniqueness requirement and asks only for existence. The enriched versions of the usual notions involve certain morphisms between hom-objects being invertible; here we introduce enriched versions of the weak notions by asking that the morphisms between hom-objects belong to a chosen class of "surjections". We study in particular injectivity (weak orthogonality) in the enriched context, and illustrate how it can be used to describe homotopy coherent structures.

The basic notions of category theory, such as limit, colimit, free object, adjunction, and factorization system, involve assertions of the existence of a unique morphism with certain properties. For example an object FX is free on X with respect to a functor $U : \mathscr{A} \to \mathscr{X}$ when there is a morphism $\eta : X \to UFX$, as in the diagram



with the property that for any morphism $f : X \to UA$ there is a unique morphism $g : FX \to A$ such that $Ug.\eta = f$. It turns out that these can be expressed by saying that certain induced functions between hom-sets are invertible — in this case by saying that the function $\mathscr{A}(FX, A) \to \mathscr{X}(X, UA)$ obtained by applying U and then composing with $\eta : X \to UFX$ is invertible for all $A \in \mathscr{A}$. It is this formulation in terms of hom-sets which makes enriched category theory possible — you can replace these bijections for hom-sets with isomorphisms of hom-objects lying in the monoidal category $\mathscr{V}$ over which categories are being enriched.

Weak notions arise when one asks just for the existence of a morphism with given properties, not the uniqueness. Thus in the example considered above, we might ask that for every $f : X \to UA$ there exists a $g : FX \to A$ with $Ug.\eta f$. Once again this can be reformulated; this time by saying that the induced function $\mathscr{A}(FX, A) \to \mathscr{X}(X, UA)$ is *surjective* for all $A \in \mathscr{A}$.

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In order to obtain an enriched version of these weak notions, it is necessary to choose a class of morphisms in $\mathscr{V}$ playing the role of the surjective functions. There are various possibilities: one might consider the epimorphisms, or the regular epimorphisms, or the split epimorphisms; in fact we shall develop the basic notions using an abstract class $\mathscr{E}$ of morphisms in $\mathscr{V}$, although to make much progress we shall have to start making some assumptions about this class.

This work began as a more-or-less technical investigation, within the context of our broader investigation of the homotopy theory of enriched categories. The role of weakness in homotopy theory is well-known: for instance weak factorization systems play a key role in the theory of model categories, and indeed weak limits were first considered in the homotopy context. In general, however, weak limits are not the same as homotopy limits, and in homotopical situations, it is generally the latter that are more important. Nonetheless, as we began to develop the theory and some specific examples, the project grew to be more than just technical. A key turning point was the example $\mathscr{V} = \mathbf{Cat}$ where $\mathscr{E}$ consists of the *retract equivalences*.

Since retract equivalences of categories are not just surjections but rather some "homotopy" version of isomorphisms, this example looks much less like "weak category" theory, as ordinarily understood, and much more like some sort of homotopy theory. So in fact in this instance we are brought closer to a different, more recent, use of the word "weak" in category theory, meaning something like "up to coherent homotopy". We illustrate this in Section 10 by sketching how certain sorts of homotopy coherent structures can be described in terms of $\mathscr{E}$-injectivity classes.

The various weak notions we shall define will all say that some naturally defined map or collection of maps lies in the class $\mathscr{E}$. Thus the smaller the class $\mathscr{E}$, the stronger the notion. In particular, the smallest possible class $\mathscr{E}$ is just the isomorphisms, and then our notion of weakness is not really weak at all, and we recover the classical (non-weak!) theory of enriched categories. The larger $\mathscr{E}$ becomes, the weaker our weak notions really are.

The goal then, is to develop the basic ingredients of weak category theory, such as can be found in [2, Section 4A] for example, in this setting of $\mathscr{V}$-enriched category theory with a specified class $\mathscr{E}$ of morphisms in $\mathscr{V}$ giving the weakness. In particular, we shall look at weak colimits, weak adjunctions, injectivity, and the basic relationships between these notions. We hope to use this later in developing the homotopy theory of enriched categories, and in particular a homotopy version of locally presentable enriched categories.

The examples we have in mind are:

(a) If we take $\mathscr{E}$ to be the isomorphisms, we obtain the "non-weak notion of weakness": weak colimits are ordinary colimits, weak adjunctions are ordinary adjunctions, injectivity is orthogonality, and so on.

(b) The ordinary notion of weakness, for $\mathscr{V} = \mathbf{Set}$, is where $\mathscr{E}$ is the surjective functions. We generalize this to the case of a locally finitely presentable closed category $\mathscr{V}$ by taking $\mathscr{E}$ to be the pure epimorphisms, whose definition is recalled below.

(c) If $\mathscr{V} = \mathbf{Cat}$, we may take $\mathscr{E}$ to be the retract equivalences.

(d) If $\mathscr{V}$ is a monoidal model category, we may take $\mathscr{E}$ to be the trivial fibrations.

(e) $\mathscr{V}$ is the category of fibrant objects in a monoidal model category, and $\mathscr{E}$ is the weak equivalences.

We prove our basic results for the first three cases. In the non-weak setting of (a) all of this is known. In case (b) it is known for the case $\mathscr{V} = \mathbf{Set}$, but new in general. Case (c) is new. Although we have (essentially) uniform statements of results across these three cases, we have not been able to unify completely the proofs: while certain parts of the arguments are completely formal, others are treated on a case-by-case basis. Case (d) is in fact a common generalization of the other three cases. We are not able to prove all our results in the level of generality given in (d), but it would be interesting to find conditions on a monoidal model category for which the results can be proved. (In fact we are not really using the full model structure: it is enough to have a cofibrantly generated weak factorization system, suitably compatible with the monoidal structure.)

Case (e) is something like the setting of our broader (and on-going) investigations into the homotopy theory of enriched categories. It is much more delicate, since in general the fibrant objects in a monoidal model category will not be complete or cocomplete, or even closed under the monoidal structure. We shall have nothing to say about this case here, and indeed the results presented here have to be reformulated to deal with this case; nonetheless this case has influenced much of what we present below.

We start, in Section 1, by recalling a few key facts about enriched category theory. We then make the basic definitions in the general setting: injectivity in Section 2, weak left adjoints in Section 3, and weak colimits in Section 4; in the latter we also define a weakly locally presentable category to be an accessible category with weak colimits. These are the main objects of study. In the classical case, the following conditions on a category $\mathscr{K}$ are equivalent:

(i) $\mathscr{K}$ is weakly locally presentable;

(ii) $\mathscr{K}$ is accessible and has products;

(iii) $\mathscr{K}$ is the full subcategory of a presheaf category consisting of those objects injective to a given small set of morphisms;

(iv) $\mathscr{K}$ is a weakly reflective subcategory of a presheaf category, closed under retracts and under λ-filtered colimits for some regular cardinal λ;

(v) $\mathscr{K}$ is the category of models of a limit-epi sketch.

We give analogous characterizations in the enriched context in each of the three main cases (a), (b), and (c) listed above. This is done in the three Sections 7, 8, and 9. Before this, in Section 5, we describe the general form of our results, and in Section 6 the common parts of the proofs. We shall need a technical result involving enriched accessible categories; we state and prove it in Section 11. In Section 10, we describe various examples of weakly locally presentable 2-categories, using $\mathscr{V} = \mathbf{Cat}$ and $\mathscr{E}$ the retract equivalences. These arise using the approach to coherent structures initiated by Segal in [12]. We describe in detail an example using bicategories.

1 Our base category $\mathscr{V}$

We work over a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$, with tensor $\otimes$ and unit I. We follow the general conventions of [7], and write $\mathscr{V}_0$ for the underlying ordinary category of $\mathscr{V}$. Later on we shall suppose that $\mathscr{V}$ is locally finitely presentable as a closed category in the sense of Kelly [6]: this means that $\mathscr{V}_0$ is locally finitely presentable, the unit object I is finitely presentable, and the tensor product of two finitely presentable objects is finitely presentable.

We shall use heavily the notion of power (cotensor). For an object A in a $\mathscr{V}$-category $\mathscr{K}$ and an object X of $\mathscr{V}$, the power $X \pitchfork A$ is defined by the universal property

$$\mathscr{K}(-, X \pitchfork A) \cong \mathscr{V}(X, \mathscr{K}(-, A)).$$

Dually the copower (or tensor) of an object $A \in \mathscr{K}$ by $X \in \mathscr{V}$ is an object $X \cdot A$ defined by the universal property

$$\mathscr{K}(X \cdot A, -) \cong \mathscr{V}(X, \mathscr{K}(A, -)).$$

All notions should be understood as $\mathscr{V}$-enriched unless specified otherwise. For instance, if we speak of a subcategory or a functor, this should always be understood to mean a sub-$\mathscr{V}$-category or $\mathscr{V}$-functor, as the case may be. Similarly limits or colimits will always be understood in the enriched sense, defined in terms of some isomorphism of $\mathscr{V}$-valued homs.

If we do wish to speak of ordinary unenriched notions, we shall say so. We shall sometimes consider ordinary diagrams in enriched categories: given a $\mathscr{V}$-category $\mathscr{K}$ and an ordinary category $\mathscr{J}$, a diagram in $\mathscr{K}$ of shape $\mathscr{J}$ is an ordinary functor from $\mathscr{J}$ to the underlying ordinary category of $\mathscr{K}$ (although this is equivalent to giving a $\mathscr{V}$-functor from the free $\mathscr{V}$-category on $\mathscr{J}$ to $\mathscr{K}$). The (conical) colimit of such a diagram S in a $\mathscr{V}$-category $\mathscr{K}$ is defined by a natural isomorphism

$$\mathscr{K}(\mathrm{colim} S, A) \cong [\mathscr{J}, \mathscr{K}](S, \Delta A)$$

in $\mathscr{V}$. As explained in [7], the universal property of a conical colimit in the underlying ordinary category $\mathscr{K}_0$ is weaker than the that of a colimit in $\mathscr{K}$; but if the colimit in $\mathscr{K}$ is known to exist, then the universal property of the colimit in $\mathscr{K}_0$ is enough to detect it. As a consequence, if $\mathscr{K}$ and $\mathscr{L}$ have conical colimits of a certain type, then a $\mathscr{V}$-functor $F : \mathscr{K} \to \mathscr{L}$ will preserve these colimits provided that the underlying ordinary functor $F_0 : \mathscr{K}_0 \to \mathscr{L}_0$ does so.

Furthermore, there is an important special case in which the universal property of the colimit in $\mathscr{K}_0$ suffices: let $\mathscr{G}$ be a strong generator for $\mathscr{V}_0$, and suppose that $\mathscr{K}$ has powers by all $G \in \mathscr{G}$. Then there is no difference between the colimit of S in $\mathscr{K}$, and the colimit in $\mathscr{K}_0$. In particular, this is the case if $\mathscr{V}_0$ is locally finitely presentable and $\mathscr{K}$ has powers by finitely presentable objects of $\mathscr{V}_0$.

We follow [7] by using *filtered colimit* to mean the (conical) colimit of a diagram $\mathscr{J} \to \mathscr{K}$ with $\mathscr{J}$ an ordinary filtered category; the case of λ-filtered colimits, for a regular cardinal λ, is similar. We say that a $\mathscr{V}$-category $\mathscr{A}$ is λ-*accessible*, if it is the free completion under λ-filtered colimits of a small $\mathscr{V}$-category $\mathscr{C}$.

Since conical colimits can be defined in terms of the underlying categories in the presence of enough powers, we have:

Proposition 1.1 *If a $\mathscr{V}$-category has all $\mathscr{G}$-powers for some strong generator $\mathscr{G}$ of $\mathscr{V}_0$, then $\mathscr{A}$ is λ-accessible if and only if the underlying ordinary category $\mathscr{A}_0$ is so.*

Remark 1.2 The words *filtered* and *accessible* were used in a different sense, in relation to $\mathscr{V}$-categories, in [4]; there filtered colimits included all weighted colimits commuting with finite limits, and accessibility was defined accordingly.

We shall also fix a class $\mathscr{E}$ of morphisms in $\mathscr{V}$, which to start with is assumed only to be closed under composition and to contain the isomorphisms. As explained in the introduction, "weak category theory" corresponds to the case $\mathscr{V} = \mathbf{Set}$ and $\mathscr{E}$ the surjections, while the non-weak case corresponds to taking $\mathscr{E}$ to consist of the isomorphisms.

2 Injectivity in $\mathscr{V}$-categories

Let $f : A \to B$ be a morphism in a $\mathscr{V}$-category $\mathscr{K}$. We say that an object $C \in \mathscr{K}$ is *f-injective over $\mathscr{E}$*, or just f-injective if $\mathscr{E}$ is understood, when the induced morphism

$$\mathscr{K}(B,C) \xrightarrow{\mathscr{K}(f,C)} \mathscr{K}(A,C)$$

is in $\mathscr{E}$. More generally, if $\mathscr{F}$ is any class of morphisms in $\mathscr{K}$, we say that C is $\mathscr{F}$-injective over $\mathscr{E}$ if it is f-injective for all $f \in \mathscr{F}$. We write $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$ for the full subcategory of $\mathscr{K}$ consisting of those objects which are f-injective over $\mathscr{E}$ for all $f \in \mathscr{F}$, and call such a full subcategory an *injectivity class*, or *$\mathscr{E}$-injectivity class* for emphasis. We call it a small-injectivity class if the class $\mathscr{F}$ of morphisms is small. We write $\mathrm{Inj}_0(\mathscr{F})$for the full subcategory of $\mathscr{K}$ consisting of those objects which are f-injective in the ordinary category $\mathscr{K}_0$ in the ordinary sense, for all $f \in \mathscr{F}$.

Obviously this notion of injectivity depends heavily on both $\mathscr{V}$ and $\mathscr{E}$, as the following examples show:

Example 2.1 Ordinary injectivity is the case $\mathscr{V} = \mathbf{Set}$ and $\mathscr{E}$ the surjections.

Example 2.2 We can also obtain *orthogonality* as an example, by taking $\mathscr{E}$ to be the isomorphisms. This works in either the enriched or the unenriched contexts.

Example 2.3 Let $\mathscr{V} = \mathbf{Cat}$ and $\mathscr{E}$ be the equivalences. Let 1 and 2 be discrete categories with one and two objects, respectively, and let $\mathscr{I}$ be the free-living isomorphism, consisting of two objects and an invertible arrow between them. Let $f : 1 \to 2$ be an injection, and $g : \mathscr{I} \to 1$ the unique functor into the terminal category. Since g is an equivalence, all objects of $\mathbf{Cat}$ are g-injective over $\mathscr{E}$; on the other hand very few categories are g-injective in $\mathbf{Cat}_0$ — in particular, $\mathscr{I}$ is not. Since f is split mono, all categories are f-injective in $\mathbf{Cat}_0$, while very few are f-injective over $\mathscr{E}$— in particular, 2 is not.

We do have the following positive result, which requires the unit object I of $\mathscr{V}$ to be *$\mathscr{E}$-projective*, in the sense that $\mathscr{V}_0(I,-) : \mathscr{V}_0 \to \mathbf{Set}$ sends morphisms in $\mathscr{E}$ to surjections; more explicitly, this says that for any $e : X \to Y$ in $\mathscr{E}$ and any $y : I \to Y$, there exists an $x : I \to X$ such that $ex = y$.

Proposition 2.4 *If I is $\mathscr{E}$-projective, then injectivity over $\mathscr{E}$ implies ordinary injectivity in the underlying ordinary category.*

PROOF: To say that C is f-injective over $\mathscr{E}$ is to say that the morphism $\mathscr{K}(f,C) : \mathscr{K}(B,C) \to \mathscr{K}(A,C)$ in $\mathscr{V}$ lies in $\mathscr{E}$. Under the conditions of the proposition, however, this implies that $\mathscr{V}_0(\mathscr{K}(f,C))$ is surjective, and this is just $\mathscr{K}_0(f,C) : \mathscr{K}_0(B,C) \to \mathscr{K}_0(A,C)$, so surjectivity of $\mathscr{K}_0(f,C)$ is just ordinary injective of C with respect to f. $\square$

This condition on the unit object will hold in all the main examples we study. We shall see that in many cases $\mathscr{F}$-injectivity over $\mathscr{E}$ is equivalent to ordinary injectivity with respect to some other class $\mathscr{F}'$ of maps. In this case, if also I is $\mathscr{E}$-projective, then $\mathscr{E}$-injectivity classes reduce to injectivity classes in the usual sense.

Definition 2.5 *We say that a class of limits is $\mathscr{E}$-*stable *if $\mathscr{E}$ is closed in $\mathscr{V}^{\mathbf{2}}$ under these limits.*

Proposition 2.6 *The full subcategory of $\mathscr{K}$ consisting of the $\mathscr{F}$-injective objets is closed under $\mathscr{E}$-stable limits.*

PROOF: Let $S : \mathscr{D} \to \mathscr{K}$ be any diagram in $\mathscr{K}$ for which SD is f-injective for all $D \in \mathscr{D}$. We consider a limit (possibly weighted) of S. In the diagram

$$\begin{array}{ccc} \mathscr{K}(B, \lim S) & \longrightarrow & \lim \mathscr{K}(B, SD) \\ {\scriptstyle \mathscr{K}(f,\lim S)} \downarrow & & \downarrow {\scriptstyle \lim \mathscr{K}(f,SD)} \\ \mathscr{K}(A, \lim S) & \longrightarrow & \lim \mathscr{K}(A, SD) \end{array}$$

the horizontal arrows are invertible, since representables preserve limits. Thus the left hand vertical is in $\mathscr{E}$ if the right hand vertical is. □

Proposition 2.7 *The full subcategory of $\mathscr{K}$ consisting of the $\mathscr{F}$-injective objects is also closed under any class Φ of colimits for which*

(*i*) *$\mathscr{E}$ is closed under Φ-colimits*

(*ii*) *$\mathscr{K}(A, -)$ preserves Φ-colimits for any object A which is the domain or codomain of a morphism in $\mathscr{F}$.*

PROOF: Let $S : \mathscr{D} \to \mathscr{K}$ be any diagram in $\mathscr{K}$ for which SD is f-injective for all $D \in \mathscr{D}$. We consider a limit (possibly weighted) of S. In the diagram

$$\begin{array}{ccc} \mathrm{colim}\mathscr{K}(B, S) & \longrightarrow & \mathscr{K}(B, \mathrm{colim}S) \\ {\scriptstyle \mathrm{colim}\mathscr{K}(f,S)} \downarrow & & \downarrow {\scriptstyle \mathscr{K}(f,\mathrm{colim}S)} \\ \mathrm{colim}\mathscr{K}(A, S) & \longrightarrow & \mathscr{K}(A, \mathrm{colim}S) \end{array}$$

the horizontal arrows are invertible, since $\mathscr{K}(B, -)$ and $\mathscr{K}(A, -)$ are assumed to preserve the colimits in question, and the right hand vertical is in $\mathscr{E}$, since the $\mathscr{E}$'s assumed to be closed under the colimits in question. Thus the left hand vertical is in $\mathscr{E}$. □

3 Weak left adjoints

Let $U : \mathscr{A} \to \mathscr{K}$ be a $\mathscr{V}$-functor, and K an object of $\mathscr{K}$. We say that a morphism $\eta : K \to UFK$ exhibits FK as a weak left adjoint to U at K, when for any $A \in \mathscr{A}$ the induced map

$$\mathscr{A}(FK, A) \xrightarrow{U} \mathscr{K}(UFK, UA) \xrightarrow{\mathscr{K}(\eta, UA)} \mathscr{K}(K, UA)$$

is in $\mathscr{E}$.

A special case is where we actually have a functor $F : \mathscr{K} \to \mathscr{A}$, and a natural transformation $\eta : 1 \to UF$, for which

$$\mathscr{A}(FK, A) \xrightarrow{U} \mathscr{K}(UFK, UA) \xrightarrow{\mathscr{K}(\eta, UA)} \mathscr{K}(K, UA)$$

is in $\mathscr{E}$. We then say that U has a *natural weak left adjoint.*

If there is a weak left adjoint to U at every object of $\mathscr{K}$, we say simply that U has a weak left adjoint. We shall be particularly interested in the case where U is fully faithful and has a weak left adjoint, in which case we say that $\mathscr{A}$ is *weakly reflective* in $\mathscr{K}$ (or $\mathscr{E}$-weakly reflective if we wish to emphasize $\mathscr{E}$).

Example 3.1 Suppose that $\mathscr{E}$ contains the retractions. If $W : \mathscr{L} \to \mathscr{K}$ has a left adjoint H, and we factorize W as a bijective on objects functor $P : \mathscr{L} \to \mathscr{A}$ followed by a fully faithful $U : \mathscr{A} \to \mathscr{K}$, then let $F = PH$ and $\eta : 1 \to UF = WH$ be the unit of the adjunction. We claim that this exhibits F as a natural weak left adjoint (a natural weak reflection) to U. To see this, we shall show that

$$\mathscr{A}(FK, A) \xrightarrow{U} \mathscr{K}(UFK, UA) \xrightarrow{\mathscr{K}(\eta, UA)} \mathscr{K}(K, UA)$$

has a section. To do this, observe that each $A \in \mathscr{A}$ has the form PL for a unique $L \in \mathscr{L}$, and that the counit $\epsilon L : HWL \to L$ gives a map $P\epsilon L : HUPL = HWL \to PL$ so that we obtain (non-natural) maps $\pi A : HUA \to A$ with $\pi PL = P\epsilon L$. Now the required section is given by

$$\mathscr{K}(K, UA) \xrightarrow{F} \mathscr{A}(FK, FUA) \xrightarrow{\mathscr{A}(FK, \pi A)} \mathscr{A}(FK, A)\ .$$

The following proposition says roughly that weak left adjoints compose.

Proposition 3.2 *Let $U : \mathscr{A} \to \mathscr{B}$ and $V : \mathscr{B} \to \mathscr{C}$ be $\mathscr{V}$-functors. Suppose that $\eta : C \to VGC$ exhibits GC as a weak left adjoint to V at C, and $\beta : GC \to UFGC$ exhibits FGC as a weak left adjoint to U at GC, then the composite*

$$C \xrightarrow{\eta} VGC \xrightarrow{V\beta} VUFGC$$

exhibits FGC as a weak left adjoint to VU at C.

Proof: For any $A \in \mathscr{A}$ we have

$$\begin{array}{ccccc}
\mathscr{A}(FGC, A) & \xrightarrow{U} & \mathscr{B}(UFGC, UA) & \xrightarrow{V} & \mathscr{C}(VUFGC, VUA) \\
 & \searrow & \downarrow{\scriptstyle \mathscr{B}(\beta, UA)} & & \downarrow{\scriptstyle \mathscr{C}(V\beta, VUA)} \\
 & & \mathscr{B}(GC, UA) & \xrightarrow{V} & \mathscr{C}(VGC, VUA) \\
 & & & \searrow & \downarrow{\scriptstyle \mathscr{C}(\eta, 1)} \\
 & & & & \mathscr{C}(C, VUA)
\end{array}$$

and the two diagonals are in $\mathscr{E}$ so the composite diagonal is in $\mathscr{E}$. ☐

There is a corresponding result for natural weak left adjoints.

4 Weak colimits

Let $S : \mathscr{C} \to \mathscr{K}$ be a $\mathscr{V}$-functor. There is an induced $\mathscr{V}$-functor

$$\mathscr{K} \xrightarrow{\mathscr{K}(S,1)} [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$$

sending an object $A \in \mathscr{K}$ to $\mathscr{K}(S-, A)$. We sometimes write $\widetilde{S}$ for $\mathscr{K}(S, 1)$.

Now let $H : \mathscr{C}^{\text{op}} \to \mathscr{V}$ be a $\mathscr{V}$-functor. We define a weak H-weighted colimit of S to be a weak left adjoint to $\mathscr{K}(S, 1)$ at H. Explicitly, this means we have an object $C \in \mathscr{K}$ and a natural transformation $\gamma : H \to \mathscr{K}(S, C)$, such that for all $A \in \mathscr{K}$ the induced map

$$\mathscr{K}(C, A) \xrightarrow{\mathscr{K}(S,1)} [\mathscr{C}^{\text{op}}, \mathscr{V}](\mathscr{K}(S, C), \mathscr{K}(S, A)) \xrightarrow{[\mathscr{C}^{\text{op}},\mathscr{V}](\gamma,1)} [\mathscr{C}^{\text{op}}, \mathscr{V}](H, \mathscr{K}(S, A))$$

in $\mathscr{V}$ is in $\mathscr{E}$. We may sometimes write $H *_{\mathscr{E}} S$ or $H *_w S$ for the weak colimit.

Example 4.1 If $\mathscr{E}$ is the class of isomorphisms, then this reduces to the usual notion of weighted colimit.

Example 4.2 If $\mathscr{V} = \mathbf{Set}$ and $\mathscr{E}$ is the class of surjections, a weak colimit $H *_w S$ consists of an object C equipped with a natural transformation $\gamma : H \to \mathscr{K}(S, C)$ such that for any A and any natural transformation $\alpha : H \to \mathscr{K}(S, A)$ there exists a morphism $f : C \to A$, not necessarily unique, such that $\mathscr{K}(S, f)\gamma = \alpha$. In particular, this reduces to ordinary weak (conical) colimits when the weight $H : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ is constant at the terminal object of $\mathbf{Set}$.

Recall that $S : \mathscr{C} \to \mathscr{K}$ is said to be dense when the induced map $\mathscr{K}(S, 1) : \mathscr{K} \to [\mathscr{C}^{\text{op}}, \mathscr{V}]$, sending $A \in \mathscr{K}$ to $\mathscr{K}(S-, A) : \mathscr{C}^{\text{op}} \to \mathscr{V}$, is fully faithful.

Proposition 4.3 *If $S : \mathscr{C} \to \mathscr{K}$ is dense, then $\mathscr{K}(S, 1)$ has a weak left adjoint if and only if $\mathscr{K}$ has weak colimits.*

Proof: By definition, $\mathscr{K}(S, 1)$ has a weak left adjoint if $\mathscr{K}$ has all weak colimits $H *_w S$. This gives one direction. For the converse, suppose that $\mathscr{K}(S, 1)$ has a weak left adjoint, and that $R : \mathscr{A} \to \mathscr{K}$ is any diagram. Form the composite $\mathscr{K}(S, R) : \mathscr{A} \to [\mathscr{C}^{\text{op}}, \mathscr{V}]$ and consider the composite

$$\mathscr{K} \xrightarrow{\mathscr{K}(S,1)} [\mathscr{C}^{\text{op}}, \mathscr{V}] \xrightarrow{\widetilde{\mathscr{K}(S,R)}} [\mathscr{A}^{\text{op}}, \mathscr{V}]$$

where the second functor sends $G : \mathscr{C}^{\text{op}} \to \mathscr{V}$ to the functor $\mathscr{A}^{\text{op}} \to \mathscr{V}$ sending $A \in \mathscr{A}$ to $[\mathscr{C}^{\text{op}}, \mathscr{V}](\mathscr{K}(S-, RA), G-)$. Now $\mathscr{K}(S, 1)$ has a weak left adjoint by assumption, while $\widetilde{\mathscr{K}(S, R)}$ has an actual left adjoint, since $[\mathscr{C}^{\text{op}}, \mathscr{V}]$ is cocomplete. So the composite has a weak left adjoint. But $\mathscr{K}(S, 1)$ sends an object $A \in \mathscr{K}$ to $\mathscr{K}(S, A)$, and $\widetilde{\mathscr{K}(S, R)}$ now sends this to $[\mathscr{C}^{\text{op}}, \mathscr{V}](\mathscr{K}(S, R), \mathscr{K}(S, A))$; but since $\mathscr{K}(S, 1)$ is fully faithful, this is just $\mathscr{K}(R, A)$, and so the composite is really just $\mathscr{K}(R, 1)$. We have therefore shown that $\mathscr{K}(R, 1)$ has a weak left adjoint, and so that $\mathscr{K}$ has weak colimit $H *_w R$ for all weights H. ☐

As in the case of $\mathbf{Set}$, we define a $\mathscr{V}$-category $\mathscr{A}$ to be *weakly locally presentable* if it is accessible and has weak colimits.

In the case of $\mathbf{Set}$, the weakly locally presentable categories are precisely the categories of models of limit-epi sketches; we shall also prove enriched versions of this characterization. For convenience, the limit part of our sketches will be taken to be limit theories (that is, small $\mathscr{V}$-categories with α-small limits for some regular cardinal α).

A *limit-$\mathscr{E}$ sketch* is a small $\mathscr{V}$-category $\mathscr{T}$ with α-small limits, and a specified collection $\mathscr{F}$ of morphisms in $\mathscr{T}$. The $\mathscr{V}$-category of models of the sketch is the full subcategory of $[\mathscr{T}, \mathscr{V}]$ consisting of those $\mathscr{V}$-functors which preserve α-small limits and send morphisms in $\mathscr{F}$ to morphisms in $\mathscr{E}$.

5 The structure of the theorems

In this section we explain briefly the form of our results. At this stage we suppose only that the monoidal category $\mathscr{V}$ is locally presentable, which we henceforth assume. We describe the results in the form of three Theorem-Schemas, but should point out straight away that they do not hold without further assumptions on $\mathscr{V}$ and $\mathscr{E}$. In this section we show that the second and third of these Theorem-Schema follow from the first. In the sections that follow, we shall describe the various examples in which we can prove the first.

Theorem-Schema A *Let $\mathscr{K}$ be a locally presentable $\mathscr{V}$-category, and $\mathscr{A}$ a full subcategory. The following conditions are equivalent:*

- *(i) $\mathscr{A}$ is the category of objects injective to a small class of maps in $\mathscr{K}$;*
- *(ii) $\mathscr{A}$ is accessible, accessibly embedded, closed under $\mathscr{E}$-stable limits;*
- *(iii) $\mathscr{A}$ is accessibly embedded and $\mathscr{E}$-weakly reflective.*

Following [2], we say that a full subcategory of an accessible category is *accessibly embedded* if it is closed under α-filtered colimits for some regular cardinal α. We understand this to include the fact that the subcategory is *replete*, meaning that it if it contains an object A then it contains any object isomorphic to A.

Theorem-Schema B *For any $\mathscr{V}$-category $\mathscr{A}$, the following conditions are equivalent:*

- *(i) $\mathscr{A}$ is an $\mathscr{E}$-weakly reflective, accessibly embedded, full subcategory of some presheaf category $[\mathscr{C}, \mathscr{V}]$;*
- *(ii) $\mathscr{A}$ is (equivalent to) a small-injectivity class in some locally presentable $\mathscr{V}$-category $\mathscr{K}$;*
- *(iii) $\mathscr{A}$ is accessible and has $\mathscr{E}$-stable limits*
- *(iv) $\mathscr{A}$ is accessible and has $\mathscr{E}$-weak colimits.*

We then say that $\mathscr{A}$ is weakly locally presentable*.*

Proof: We shall prove the equivalence, given that Theorem-Schema A holds. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow immediately from Theorem-Schema A. To see that $(iii) \Rightarrow (i)$, suppose that $\mathscr{A}$ is accessible and has $\mathscr{E}$-stable limits. Choose a regular cardinal λ for which $\mathscr{A}$ is λ-accessible, and let $\mathscr{A}_\lambda$ be the full subcategory of $\mathscr{A}$ consisting of the λ-presentable objects. Then $\mathscr{A}$ is a full subcategory of $[\mathscr{A}_\lambda^{\mathrm{op}}, \mathscr{V}]$, closed under λ-filtered colimits and $\mathscr{E}$-stable limits. By Theorem-Schema A it is weakly reflective, and so (i) holds.

If the first three conditions hold, then since $\mathscr{A}$ is $\mathscr{E}$-weakly reflective in the cocomplete $[\mathscr{C}, \mathscr{V}]$, it is $\mathscr{E}$-weakly cocomplete. Thus (4) holds.

Conversely, if $\mathscr{A}$ is accessible with $\mathscr{E}$-weak colimits, choose a regular cardinal λ such that $\mathscr{A}$ is λ-accessible, and consider the embedding $\mathscr{A} \to [\mathscr{A}_\lambda^{\mathrm{op}}, \mathscr{V}]$. Since $\mathscr{A}$ is $\mathscr{E}$-weakly cocomplete, it is $\mathscr{E}$-weakly reflective by Proposition 4.3. This gives (i). $\square$

Finally there is a description in terms of sketches. We shall not consider the most general notion of sketch, but restrict ourselves to the case where the limit-part of the sketch is in fact a theory.

In other words, we consider small $\mathscr{V}$-categories $\mathscr{T}$ with with certain specified limit diagrams, and with a chosen class $\mathscr{T}_{\mathscr{E}}$ of morphisms. A model is then a $\mathscr{V}$-functor from $\mathscr{T}$ to $\mathscr{V}$ which sends the specified limits to limits in $\mathscr{V}$, and the morphisms in $\mathscr{T}_{\mathscr{E}}$ to morphisms in $\mathscr{E}$.

Theorem-Schema C *A $\mathscr{V}$-category $\mathscr{A}$ is weakly locally presentable if and only if it is equivalent to the category of models of a (limit,$\mathscr{E}$)-sketch.*

PROOF: Once again, prove this, given that Theorem-Schema A holds. Suppose first that $\mathscr{T}$ is a small $\mathscr{V}$-category with certain specified limit diagrams, and that $\mathscr{F}$ is a set of morphisms in $\mathscr{T}$. Let $\mathscr{K}$ be the full subcategory of $[\mathscr{T},\mathscr{V}]$ sending the chosen limits in $\mathscr{T}$ to limits in $\mathscr{V}$. Then $\mathscr{K}$ is locally presentable. Now a morphism $M : \mathscr{T} \to \mathscr{V}$ in $\mathscr{K}$ sends the morphisms in $\mathscr{F}$ to $\mathscr{E}$ if and only if it is injective in $\mathscr{K}$ with respect to the morphisms $\mathscr{T}(f,-) : \mathscr{T}(B,-) \to \mathscr{T}(A,-)$ for all $f : A \to B$ in $\mathscr{F}$. Thus the models of a (limit,$\mathscr{E}$)-sketch are a small-injectivity class.

For the converse, suppose that $\mathscr{A} = \mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$ for some small set $\mathscr{F}$ of morphisms in a locally presentable $\mathscr{V}$-category $\mathscr{K}$. Choose a regular cardinal λ sufficiently large that $\mathscr{K}$ is locally λ-presentable, and all domains and codomains of morphisms in $\mathscr{F}$ are λ-presentable in $\mathscr{K}$. Let $\mathscr{C}$ be the opposite of the category of λ-presentable objects in $\mathscr{K}$. Then $\mathscr{C}$ has λ-small limits, and $\mathscr{K}$ is equivalent to the category of λ-continuous functors from $\mathscr{C}$ to $\mathscr{V}$. Furthermore, $\mathscr{F}$ can be seen as a set of morphisms in $\mathscr{C}$, and the objects of $\mathscr{A}$ are those objects of $\mathscr{K}$ which are $\mathscr{E}$-injective with respect to the morphisms in $\mathscr{F}$. Thus $(\mathscr{C},\mathscr{F})$ is a (limit,$\mathscr{E}$)-theory whose $\mathscr{V}$-category of models is $\mathscr{A}$. □

6 General aspects of the proofs

In this section we introduce further assumptions which allow us to prove the equivalences in Theorem-Schema A. The first assumption is easy: we suppose that I is $\mathscr{E}$-projective; recall that this means that $\mathscr{V}_0(I,-) : \mathscr{V}_0 \to \mathbf{Set}$ sends the maps in $\mathscr{E}$ to surjections.

Proposition 6.1 *If I is $\mathscr{E}$-projective then the implication $(iii) \Rightarrow (i)$ in Theorem-Schema A holds.*

PROOF: For each object $K \in \mathscr{K}$, we may choose a weak reflection $r_K : K \to K^*$ into $\mathscr{A}$. The universal property of the weak reflection says that each object $A \in \mathscr{A}$ is $\mathscr{E}$-injective with respect to these maps r_K. Conversely, if $K \in \mathscr{K}$ is $\mathscr{E}$-injective with respect to the single map $r_K : K \to K^*$, then since I is projective with respect to $\mathscr{E}$, it follows that K is injective in $\mathscr{K}_0$ with respect to r_K, and so that K is a retract of K^*. But $\mathscr{A}$ is accessibly embedded and replete, so closed under retracts, thus $K \in \mathscr{A}$.

Thus $\mathscr{A}$ consists of all objects which are $\mathscr{E}$-injective with respect to all the r_K. The only problem is that this is a large class of maps; to prove the proposition, we must show that it can be replaced by a small one.

Choose a regular cardinal λ such that $\mathscr{A}$ is λ-accessible, and the inclusion $\mathscr{A} \to \mathscr{K}$ preserves λ-filtered colimits and λ-presentable objects. Let $\mathscr{F}$ consist of all the $r_K : K \to K^*$ for which K is λ-presentable in $\mathscr{K}$. Certainly $\mathscr{A}$ is contained in $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$; we must show that the reverse inclusion holds.

Suppose then that $X \in \mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$. Let $\mathscr{J}$ be the full subcategory of $\mathscr{K}_0/X$ consisting of all morphisms into X with λ-presentable domain. Then $\mathscr{J}$ is λ-filtered, and X is the colimit of the canonical map $\mathscr{J} \to \mathscr{K}$.

Let $\mathscr{J}'$ be the full subcategory of $\mathscr{J}$ consisting of those $K \to X$ for which K is not just λ-presentable, but also in $\mathscr{A}$. We shall show that $\mathscr{J}'$ is final in $\mathscr{J}$. Then $\mathscr{J}'$ will still be λ-filtered, and X will be a λ-filtered colimit of objects in $\mathscr{A}$, and so itself will be in $\mathscr{A}$.

Since $\mathscr{J}$ is λ-filtered, $\mathscr{J}'$ will be final provided that for each $J \in \mathscr{J}$, there is a morphism $J \to J'$ with $J' \in \mathscr{J}'$. So let $f : K \to X$ in $\mathscr{J}$ be given. Since $\mathscr{A}$ is λ-accessible, K^* is a λ-filtered colimit of λ-presentable objects of $\mathscr{A}$. Since K is λ-presentable, $r : K \to K^*$ factorizes as $s : K \to B$ followed by $g : B \to K^*$ for some λ-presentable object $B \in \mathscr{A}$. Since $r : K \to K^*$ is a weak reflection, $f = f'r$ for some $f' : K^* \to X$, and $f = f'r = f'gs$. We have an object $f'g : B \to X$ of $\mathscr{J}'$, and a morphism s from $f : K \to X$ to $f'g : B \to X$ in $\mathscr{J}$. This proves that $\mathscr{J}'$ is final in $\mathscr{J}$, and so completes the proof. □

Next we introduce a condition that allows enriched injectivity to be reduced to ordinary injectivity. We say that the class $\mathscr{E}$ is *cofibrantly generated* if there is a small set $\mathscr{J}$ of of morphisms in $\mathscr{V}$ such that $\mathscr{E}$ consists of those morphisms with the right lifting property with respect to $\mathscr{J}$; in other words, those e with the property that for any commutative square as in the solid part of the diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle j}\downarrow & \overset{w}{\nearrow} & \downarrow{\scriptstyle e} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array}$$

with j in $\mathscr{J}$, there exists a "fill-in" w making the two triangles commute. (It then follows that $\mathscr{E}$ forms part of a cofibrantly generated weak factorization system.)

Proposition 6.2 *If $\mathscr{E}$ is cofibrantly generated and $\mathscr{F}$ is a small set of morphisms in $\mathscr{K}$ then there is a small set $\mathscr{F}'$ of morphisms in $\mathscr{K}$ for which $\mathscr{F}$-injectivity over $\mathscr{E}$ is equivalent to ordinary $\mathscr{F}'$-injectivity; that is,* $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F}) = \mathrm{Inj}_0(\mathscr{F}')$.

PROOF: An object $A \in \mathscr{K}$ is $\mathscr{F}$-injective over $\mathscr{E}$ if and only if $\mathscr{K}(f, A)$ is in $\mathscr{E}$ for all $f : B \to C$ in $\mathscr{F}$. But this says that $\mathscr{K}(f, A)$ has the right lifting property with respect to all j in $\mathscr{J}$; in other words, given the solid part of the diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & \mathscr{K}(C, A) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \mathscr{K}(f,A)} \\ Y & \xrightarrow[v]{} & \mathscr{K}(B, A) \end{array}$$

there exists a diagonal fill-in. But to give u and v is equivalently to give $u' : X \cdot C \to A$ and $v' : Y \cdot B \to A$ making the square

$$\begin{array}{ccc} X \cdot B & \xrightarrow{X \cdot f} & X \cdot C \\ {\scriptstyle j \cdot B}\downarrow & & \downarrow{\scriptstyle u'} \\ Y \cdot B & \xrightarrow[v']{} & A \end{array}$$

commute; or equivalently a morphism

$$X \cdot C +_{X \cdot B} Y \cdot B \xrightarrow{w} A$$

out of the pushout of $X \cdot f$ and $j \cdot B$. A diagonal fill-in is then equivalent to the existence of an extension of w along the canonical map

$$X \cdot C +_{X \cdot B} Y \cdot B \xrightarrow{j \Box f} Y \cdot C.$$

Thus A is $\mathscr{F}$-injective over $\mathscr{E}$ if and only if it is $\mathscr{F}'$-injective in the ordinary sense, where $\mathscr{F}'$ consists of all $j \Box f$ with $j \in \mathscr{J}$ and $f \in \mathscr{F}$. $\Box$

We now state a second implication from Theorem-Schema A; the proof depends on the notion of pure subobject, but we defer this aspect to Section 11.

Corollary 6.3 *If $\mathscr{E}$ is cofibrantly generated, then the implication $(i) \Rightarrow (ii)$ in Theorem-Schema A holds.*

PROOF: Let $\mathscr{K}$ be locally presentable, and $\mathscr{F}$ a small set of morphisms in $\mathscr{K}$. We have already seen that $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$ is closed under $\mathscr{E}$-stable limits and under λ-filtered colimits, where λ is a regular cardinal for which the domain and codomain of any morphism in $\mathscr{F}$ is λ-presentable. It remains only to prove that $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$ is accessible. Now the underlying ordinary category $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})_0$ of $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$ is $\mathrm{Inj}_0(\mathscr{F}')$ by the previous proposition, and by the classical theory this is an accessible and accessibly embedded subcategory of the locally finitely presentable ordinary category $\mathscr{K}_0$. Thus by [2, Corollary 2.36], $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})_0$ is closed in $\mathscr{K}_0$ under μ-pure subobjects for some regular cardinal μ, and now $\mathrm{Inj}_{\mathscr{E}}(\mathscr{F})$ is accessible by Theorem 11.2. $\Box$

Thus if $\mathscr{E}$ is cofibrantly generated and I is $\mathscr{E}$-projective then we have the implications $(iii) \Rightarrow (i) \Rightarrow (ii)$, and it remains only to prove $(ii) \Rightarrow (iii)$. Rather than describing general sufficient conditions for this to be true, we treat it on a case-by-case basis in the examples that follow.

7 Case 1: $\mathscr{E}$ is the isomorphisms

This case is entirely classical, we merely state the results, to show what they give in this context. In this case injectivity becomes orthogonality, weak reflectivity is ordinary reflectivity, and weak colimits are just colimits.

Observe that I is indeed $\mathscr{E}$-projective, since $\mathscr{V}_0(I, -)$ sends the isomorphisms not just to surjections but to bijections (as indeed does any functor). Also the class $\mathscr{E}$ is cofibrantly generated: if α is some regular cardinal for which $\mathscr{V}_0$ is locally α-presentable, then the α-presentable objects form a strong generator for $\mathscr{V}_0$, and a morphism $e : A \to B$ in $\mathscr{V}_0$ is in $\mathscr{E}$ (that is, invertible) if and only if it has the right lifting property with respect to the unique map $0 \to G$ and the codiagonal $G + G \to G$ for all α-presentable objects G.

Theorem 7.1 *Let $\mathscr{K}$ be a locally presentable $\mathscr{V}$-category, and $\mathscr{A}$ a full subcategory. The following conditions are equivalent:*

(i) $\mathscr{A}$ is the category of objects orthogonal to a small class of maps;

(ii) $\mathscr{A}$ is accessible, accessibly embedded, and closed under limits;

(iii) $\mathscr{A}$ is replete, accessibly embedded, and reflective.

By the general results of the previous section, we need only prove that (ii) implies (iii). Suppose then that $\mathscr{A}$ is accessible, accessibly embedded, and closed under limits. Let $K \in \mathscr{K}$ be given; we shall construct a reflection of K into $\mathscr{A}$. Let $J : \mathscr{A} \to \mathscr{K}$ be the inclusion, and $\mathscr{K}(K, J) : \mathscr{A} \to \mathscr{V}$ be the $\mathscr{V}$-functor sending $A \in \mathscr{A}$ to the hom $\mathscr{K}(K, JA)$. If λ is any regular cardinal for which $\mathscr{A}$ is closed in $\mathscr{K}$ under λ-filtered colimits and K is λ-presentable, then $\mathscr{K}(K, J)$ will preserve λ-filtered colimits. We may choose λ so that $\mathscr{A}$ is also λ-accessible, and now $\mathscr{K}(K, J)$ is the left Kan extension of its restriction to the λ-presentable objects. Thus $\mathscr{K}(K, J)$-weighted limits exist in $\mathscr{A}$ and are preserved by J. The $\mathscr{K}(K, J)$-weighted limit of the identity $\mathscr{A} \to \mathscr{A}$ is now the desired reflection of K into $\mathscr{A}$.

The other two theorems now follow as in the previous section:

Theorem 7.2 *For a $\mathscr{V}$-category $\mathscr{A}$, the following are equivalent:*

- *(i) $\mathscr{A}$ is a reflective, accessibly embedded subcategory of $[\mathscr{C}, \mathscr{V}]$ for some small $\mathscr{V}$-category $\mathscr{C}$;*
- *(ii) $\mathscr{A}$ is equivalent to a small orthogonality class in some locally presentable $\mathscr{V}$-category;*
- *(iii) $\mathscr{A}$ is accessible and complete;*
- *(iv) $\mathscr{A}$ is accessible and cocomplete.*

Theorem 7.3 *A $\mathscr{V}$-category is locally presentable if and only if it is equivalent to the $\mathscr{V}$-category of models of a limit sketch.*

The general form of this last theorem would be that a $\mathscr{V}$-category is $\mathscr{E}$-weakly locally presentable if and only if it is the $\mathscr{V}$-category of models of a (limit,$\mathscr{E}$)-sketch. But when $\mathscr{E}$ is the isomorphisms $\mathscr{E}$-weakly means not weakly at all; and $\mathscr{E}$-specifications are just iso-specifications, which do not require any sort of colimits.

8 Case 2: $\mathscr{E}$ is the pure epimorphisms

In this section, we suppose that $\mathscr{V}$ is locally finitely presentable as a closed category [6]; recall that this means that the underlying ordinary category $\mathscr{V}_0$ is locally finitely presentable, and the full subcategory of finitely presentable objects contains the unit and is closed under the tensor product.

Recall that an epimorphism $p : X \to Y$ in a locally finitely presentable (ordinary) category $\mathscr{K}$ is said to be *pure* [1], if $\mathscr{K}(G, p) : \mathscr{K}(G, X) \to \mathscr{K}(G, Y)$ is surjective for all finitely presentable objects G.

We now take as our class $\mathscr{E}$ the pure epimorphisms in $\mathscr{V}_0$: those morphisms for which $\mathscr{G}(G, p)$ is surjective for all finitely presentable G. Equivalently, $\mathscr{E}$ consists of those morphisms with the right lifting property with respect to the unique map $0 \to G$ for any finitely presentable objects, so $\mathscr{E}$ is cofibrantly generated. Note also that I is finitely presentable, and so is $\mathscr{E}$-projective. Thus all our theorems will hold provided that the implication $(ii) \Rightarrow (iii)$ in Theorem-Schema A does, as we shall see below.

As usual, we regard $\mathscr{E}$ as a full subcategory of $\mathscr{V}^{\mathbf{2}}$.

Proposition 8.1 *The pure epimorphisms are closed in $\mathscr{V}^{\mathbf{2}}$ under products, retracts, and finite powers.*

Proof: Let $\Pi_i p_i : \Pi_i X_i \to \Pi_i Y_i$ be a product of pure epimorphisms, let G be finitely presentable, and let $f : G \to \Pi_i Y_i$ be given. Then f is determined by components $f_i : G \to Y_i$ for each $i \in I$. Since p_i is pure epi and G is finitely presentable, there is a lifting $f_i = p_i g_i$ of f_i through p_i for each i, and so a lifting of f through $\Pi_i p_i$. This proves that the pure epimorphisms are closed under products. The case of retracts is similar.

To see that the pure epimorphisms are closed under finite powers, let $p : X \to Y$ be a pure epimorphism, and $H \in \mathscr{V}$ a finitely presentable object. We must show that $H \pitchfork p : H \pitchfork X \to H \pitchfork Y$ is a pure epimorphism. Suppose then that $f : G \to H \pitchfork Y$ is given; this amounts to giving $f' : G \otimes H \to Y$. Since G and H are finitely presentable, and $\mathscr{V}$ is locally finitely presentable as a closed category, $G \otimes H$ is also finitely presentable, and so f' lifts through p, say as $g' : G \otimes H \to X$. Now g' determines a unique $g : G \to H \pitchfork X$, and $(H \pitchfork p)g = f$, which proves that $H \pitchfork p$ is a pure epimorphism. □

Remark 8.2 There are analogues to all results in this section for higher cardinals λ. This would involve a $\mathscr{V}$ which is locally λ-presentable as a closed category, and taking $\mathscr{E}$ to be the λ-pure epis (that is, the morphisms $p : X \to Y$ with $\mathscr{K}(G, p)$ surjective for all λ-presentable G). We shall not bother to spell these out.

Proposition 8.3 ([1]) *Pure epimorphisms are closed under filtered colimits.*

Proof: Consider a filtered colimit $\mathrm{colim}_i p_i : \mathrm{colim}_i X_i \to \mathrm{colim}_i Y_i$ in $\mathscr{V}^{\mathbf{2}}$ of pure epimorphisms. Any $f : G \to \mathrm{colim}_i Y_i$ with G finitely presentable lands in some Y_i, and then lifts through X_i, to give a lifting of f itself through $\mathrm{colim}_i p_i$. □

It now follows, for any class $\mathscr{F}$, that the $\mathscr{F}$-injective objects are closed under products, retracts, and finite powers. Furthermore, they are closed under λ-filtered colimits for any regular cardinal λ large enough that all domains and codomains of maps in $\mathscr{F}$ are λ-presentable. Since $\mathscr{V}$ is locally presentable, such λ will exist if $\mathscr{F}$ is small.

It is also convenient to state

Proposition 8.4 *For $f : A \to B$ in $\mathscr{K}$ and $C \in \mathscr{K}$, the following are equivalent:*

(i) C is f-injective in $\mathscr{K}$

(ii) $G \pitchfork C$ is f-injective in $\mathscr{K}_0$, for all finitely presentable G

(iii) C is $G \cdot f$-injective in $\mathscr{K}_0$, for all finitely presentable G.

Now turn to weak left adjoints.

Proposition 8.5 *Let $\mathscr{A}$ and $\mathscr{K}$ be $\mathscr{V}$-categories with finite powers, and let $U : \mathscr{A} \to \mathscr{K}$ be a $\mathscr{V}$-functor which preserves finite powers. Then $\eta : K \to UFK$ exhibits FK as a weak left adjoint to U at K if and only if it exhibits FK as a weak left adjoint to $U_0 : \mathscr{A}_0 \to \mathscr{K}_0$ at K.*

Proof: Observe that

$$\mathscr{A}(FK, A) \xrightarrow{U} \mathscr{K}(UFK, UA) \xrightarrow{\mathscr{K}(\eta, UA)} \mathscr{K}(K, UA)$$

is in $\mathscr{E}$ if and only if

$$\mathscr{V}_0(G,\mathscr{A}(FK,A)) \xrightarrow{\mathscr{V}_0(G,U)} \mathscr{V}_0(G,\mathscr{K}(UFK,UA)) \xrightarrow{\mathscr{V}_0(G,\mathscr{K}(\eta,UA))} \mathscr{V}_0(G,\mathscr{K}(K,UA))$$

is surjective, which in turn is the case if and only if

$$\mathscr{A}_0(FK,G\pitchfork A) \xrightarrow{U_0} \mathscr{K}_0(UFK,U(G\pitchfork A)) \xrightarrow{\cong} \mathscr{K}_0(UFK,G\pitchfork UA) \xrightarrow{\mathscr{K}_0(\eta,G\pitchfork UA)} \mathscr{K}_0(K,G\pitchfork UA)$$

is surjective. □

It is also useful to note

Proposition 8.6 *Let $U:\mathscr{A}\to\mathscr{B}$ be a fully faithful functor whose image is closed under retracts, and which has a weak left adjoint $\eta B: B\to UFB$ at every object $B\in\mathscr{B}$. Then C is in the image of U if and only if it is injective with respect to all ηB.*

Proof: If C is in the image of U, then injectivity with respect to the ηB is what it means for the ηB to give a weak left adjoint.

Suppose conversely that C is injective with respect to all the ηB. In particular, it is injective with respect to ηC, and so

$$\mathscr{K}(UFC,C) \xrightarrow{\mathscr{K}(\eta C,C)} \mathscr{K}(C,C)$$

is in $\mathscr{E}$. Now I is finitely presentable in $\mathscr{V}_0$, so the identity $j:I\to\mathscr{K}(C,C)$ lifts to give a map $k:I\to\mathscr{K}(UFC,C)$ which is a retraction of ηC. This shows that C is a retract of UFC and so is in the image of U. □

Corollary 8.7 *Any weakly reflective subcategory closed under retracts is closed under products and finite powers.*

Theorem-Schema A, in the current setting of $\mathscr{E}$ the pure epimorphisms, contains [2, Theorem 4.8] as the special case where $\mathscr{V}=\mathbf{Set}$, and indeed our proof of the remaining implication $(ii)\Rightarrow(iii)$ amounts to reducing the general case to the special one, using the existence of finite powers:

Theorem 8.8 *Let $\mathscr{K}$ be a locally presentable $\mathscr{V}$-category, and $\mathscr{A}$ a full subcategory. The following conditions are equivalent:*

- (i) *$\mathscr{A}$ is equivalent to the category of injective objects for a small class of maps;*
- (ii) *$\mathscr{A}$ is accessible, accessibly embedded, and closed under products and finite powers;*
- (iii) *$\mathscr{A}$ is accessibly embedded and weakly reflective.*

Proof: It remains only to prove that $(ii)\Rightarrow(iii)$. Suppose then that $\mathscr{A}$ is accessible, accessibly embedded, and closed under products and finite powers. We must show that it is weakly reflective.

Since $\mathscr{A}_0$ is accessible, and accessibly embedded and closed under products in $\mathscr{K}_0$ it follows by [2, Theorem 4.8] that $\mathscr{A}_0$ is weakly reflective in $\mathscr{K}_0$. Thus for each $K\in\mathscr{K}$ there is an object $K^*\in\mathscr{A}$ and a morphism $r:K\to K^*$ such that for any $A\in\mathscr{A}$, the morphism $\mathscr{K}_0(r,A)$ is surjective. But $\mathscr{A}$ is closed under finite powers, so for any finitely presentable $G\in\mathscr{V}$, the power$G\pitchfork A$ in $\mathscr{K}$ lies in

$\mathscr{A}$. Thus also $\mathscr{K}_0(r, G \pitchfork A)$ is surjective. But $\mathscr{K}_0(r, G \pitchfork A)$ is just $\mathscr{V}_0(G, \mathscr{K}(r, A))$, and surjectivity of this says that $\mathscr{K}(r, A)$ is a pure epi. This proves that $r : K \to K^*$ is not just a weak reflection of K into $\mathscr{A}_0$ but also an $\mathscr{E}$-weak reflection of K into $\mathscr{A}$. $\square$

Theorem-Schema B in this setting generalizes [2, Theorem 4.11], and follows immediately from the previous theorem and Section 5:

Theorem 8.9 *For a $\mathscr{V}$-category $\mathscr{A}$, the following are equivalent:*

- *(i) $\mathscr{A}$ is a weakly reflective, accessibly embedded subcategory of $[\mathscr{C}, \mathscr{V}]$ for some small $\mathscr{V}$-category $\mathscr{C}$;*
- *(ii) $\mathscr{A}$ is equivalent to a small injectivity class in some locally presentable $\mathscr{V}$-category;*
- *(iii) $\mathscr{A}$ is accessible and has products and finite powers;*
- *(iv) $\mathscr{A}$ is accessible and weakly cocomplete.*

$\square$

A $\mathscr{V}$-category satisfying these conditions is called $\mathscr{E}$-weakly locally presentable, or just weakly locally presentable when $\mathscr{E}$ is understood.

Theorem-Schema C generalizes [2, Theorem 4.13]:

Theorem 8.10 *A $\mathscr{V}$-category is $\mathscr{E}$-weakly locally presentable if and only if it is equivalent to the $\mathscr{V}$-category of models of a (limit,$\mathscr{E}$)-sketch.* $\square$

We end this section by spelling out a little what $\mathscr{E}$-weak colimits are in this context. Let $S : \mathscr{D} \to \mathscr{A}$ be a $\mathscr{V}$-functor, and $F : \mathscr{D}^{\mathrm{op}} \to \mathscr{V}$ a presheaf. An $\mathscr{E}$-weak colimit of S weighted by F consists of an object $F *_w S$ and a $\mathscr{V}$-natural transformation $\eta : F \to \mathscr{A}(S, F *_w S)$ for which the induced map

$$\mathscr{A}(F *_w S, A) \longrightarrow [\mathscr{D}^{\mathrm{op}}, \mathscr{V}](F, \mathscr{A}(S, A))$$

in $\mathscr{V}$ lies in $\mathscr{E}$ for all $A \in \mathscr{A}$. This in turn says that for any finitely presentable object $G \in \mathscr{V}$ and any y as in the solid part of the diagram

$$\begin{array}{ccc} \mathscr{A}(F *_w S, A) & \longrightarrow & [\mathscr{D}^{\mathrm{op}}, \mathscr{V}](F, \mathscr{A}(S, A)) \\ & \overset{x}{\nwarrow} & \uparrow y \\ & & G \end{array}$$

there exists a lifting x. If $\mathscr{A}$ has finite powers, then this says that any $\mathscr{V}$-natural $F \to \mathscr{A}(S, A)$ arises from some map $F *_w S \to A$ in $\mathscr{A}$.

In the case of weak conical colimits, where $\mathscr{D}$ is an ordinary category and $S : \mathscr{D} \to \mathscr{A}$ an ordinary functor, if $\mathscr{A}$ has finite powers, a weak colimit of S in the enriched sense is just a weak colimit in the ordinary sense.

9 Case 3: $\mathscr{E}$ is the retract equivalences

Here we treat the case $\mathscr{V} = \mathbf{Cat}$, with $\mathscr{E}$ the retract equivalences: these are the functors $f : A \to B$ for which there exists a functor $g : B \to A$ with $fg = 1$ and $gf \cong 1$. The fact that the unit object 1 is $\mathscr{E}$-projective amounts to the fact that retract equivalences are surjective on objects; the fact the retract equivalences are cofibrantly generated is well-known: see [8] for example. Thus once again we shall only have to check the implication $(ii) \Rightarrow (iii)$.

Note that every retract equivalences is in particular a retract, and so is certainly a pure epimorphism. Thus the notion of weakness considered in this section is "less weak" than the notion of weakness for 2-categories arising from the pure epimorphisms.

The retract equivalences are closed under filtered colimits, and they are closed under products, powers, and retracts; more generally, they are closed under *flexible limits*, in the sense of [3]. These flexible limits were shown in [3] to be be all those limits which can be constructed using products, splitting of idempotents, and two 2-categorical limits called inserters and equifiers. We have already observed that the retract equivalences are closed under products and splittings of idempotents, and it is not too hard to check that they are also closed under inserters and equifiers.

There is also another perspective on this, based on the theory developed in [9]. Recall that **Cat** has a model structure for which the trivial fibrations are the retract equivalences and the weak equivalences are the equivalences. For any small 2-category $\mathscr{A}$, the functor 2-category $[\mathscr{A}^{\text{op}}, \mathbf{Cat}]$ has a "projective" model structure for which the trivial fibrations and weak equivalences are defined "pointwise": thus a 2-natural $\alpha : F \to G$ is a trivial fibration if and only if each component $\alpha A : FA \to GA$ is a trivial fibration in **Cat** (that is, a retract equivalence). Now a 2-functor $E : \mathscr{A}^{\text{op}} \to \mathbf{Cat}$ is cofibrant in this model structure if and only if it is flexible as a weight, and the fact that the hom 2-functor $[\mathscr{A}^{\text{op}}, \mathbf{Cat}](E, -)$ sends trivial fibrations in $[\mathscr{A}^{\text{op}}, \mathbf{Cat}]$ to trivial fibrations in **Cat** is part of the fact that the model structure on $[\mathscr{A}^{\text{op}}, \mathbf{Cat}]$ is not just a model category structure but a model 2-category structure: see [9]. Finally to say that $[\mathscr{A}^{\text{op}}, \mathbf{Cat}](E, -)$ sends pointwise trivial fibrations to trivial fibrations, for all cofibrant E, is precisely to say that the retract equivalences are closed under flexible limits.

A related notion is that of pseudolimit. If $F : \mathscr{D}^{\text{op}} \to \mathbf{Cat}$ and $S : \mathscr{D} \to \mathscr{K}$ 2-functors, the *pseudolimit* of S weighted by F is an object $\{F, S\}_{ps}$ of $\mathscr{K}$ equipped with a 2-natural isomorphism

$$\mathscr{K}(A, \{F, S\}_{ps} \cong \text{Ps}(\mathscr{D}^{\text{op}}, \mathbf{Cat})(F\mathscr{K}(A, S))$$

where we have replaced the usual presheaf 2-category $[\mathscr{D}^{\text{op}}, \mathbf{Cat}]$ appearing in the definition of limit with the 2-category $\text{Ps}(\mathscr{D}^{\text{op}}, \mathbf{Cat})$ whose objects are still 2-functors from $\mathscr{D}^{\text{op}} \to \mathbf{Cat}$ but whose morphisms are pseudonatural transformations, and whose 2-cells are modifications. In general the pseudolimit $\{F, S\}_{ps}$ is different (non-equivalent) to $\{F, S\}$, but it turns out that the pseudolimit $\{F, S\}_{ps}$ can be calculated as an actual weighted limit $\{F', S\}$ for a different weight F' (see [3]) and this weight F' is flexible. Thus pseudolimits are special case of flexible limits.

Finally, there is the weighted bilimit $\{F, \mathscr{S}\}_b$, which is defined by a pseudonatural equivalence

$$\mathscr{K}(A, \{F, S\}_b \simeq \text{Ps}(\mathscr{D}^{\text{op}}, \mathbf{Cat})(F\mathscr{K}(A, S)).$$

Since every 2-natural isomorphism is a pseudonatural equivalence, a pseudolimit, if it exists, is also a bilimit. Putting all this together, we see that if a 2-category has flexible limits, then it has pseudolimits, and so bilimits: see [3] once again.

Turning to our weak notions, we first consider injectivity. Let $f : A \to B$ be a morphism in a $\mathscr{V}$-category $\mathscr{K}$. To say that C is f-injective over the equivalences is to say that $\mathscr{K}(f, C) : \mathscr{K}(B, C) \to \mathscr{K}(A, C)$ is a retract equivalence of categories. More explicitly, this means that for each morphism $a : A \to C$ there exists a $b : B \to C$ with $bf = a$, and for any two $b, b' : B \to C$ and any $\alpha bf \to b'f$ there exists a unique $\beta : b \to b'$ with $\beta f = \alpha$.

Next we turn to a "weak" version of the fact that any accessible category with limits has an initial object (of course it is in fact cocomplete). We shall only need it in the case where the accessible 2-category has flexible limits, but it is no harder to prove under the weaker assumption of bilimits.

Lemma 9.1 *Let $\mathscr{A}$ be an accessible 2-category with bilimits. Then $\mathscr{A}$ has a bi-initial object.*

Proof: First we construct the object. Let λ be a regular cardinal for which $\mathscr{A}$ is λ-accessible, and $\mathscr{A}_\lambda$ the full subcategory of λ-presentable objects. Then the bilimit L of the inclusion $J : \mathscr{A}_\lambda \to \mathscr{A}$ will be our bi-initial object.

We must show that $\mathscr{A}(L, A)$ is equivalent to the terminal category for all objects A. To see that $\mathscr{A}(L, A)$ is non-empty, observe that any $A \in \mathscr{A}$ is a λ-filtered colimit of objects in $\mathscr{A}_\lambda$, so in particular, we can find an object $C \in \mathscr{A}_\lambda$ with a morphism $f : C \to A$. Then L has a projection $\pi_C : L \to C$ and so we have a map $f\pi_C : L \to A$.

Claim: For any $c : C \to L$ with C λ-presentable, $c\pi_C \cong 1$. To see this, observe that if D is any other λ-presentable object, then pseudonaturality of the projections gives an isomorphism $\pi^{\pi_D c} : \pi_D c \pi_C \cong \pi_D$. These $\pi^{\pi_D c}$ are natural in D, so there is a unique invertible 2-cell $\gamma : c\pi_C \cong 1$ with $\pi_D \gamma = \pi^{\pi_D c}$ for all D. **This proves the claim.**

We now show that any two objects of $\mathscr{A}(L, A)$ are isomorphic. Let $g_1, g_2 : L \to A$ be any two maps. Since L is also a λ-filtered colimit of λ-presentables, we can find a λ-presentable object C with a morphism $c : C \to L$. Now $g_1 c, g_2 c : C \to A$ are a pair of morphisms with λ-presentable domain, so we can find a λ-presentable D with a morphism $d : D \to A$ and factorizations $g_1 c = dh_1$ and $g_2 c = dh_2$. Let $\pi_C : L \to C$ be the projection of the limit L, and observe that by naturality of the projections $h_1 \pi_C \cong \pi_D \cong h_2 \pi_C$; thus $g_1 c \pi_C = dh_1 \pi_C \cong dh_2 \pi_C = g_2 c \pi_C$, and so finally $g_1 \cong g_1 c \pi_C \cong g_2 c \pi_C \cong g_2$.

Finally we show that for any two maps $g_1, g_2 : L \to A$ there is a unique 2-cell between them. But we already know that all maps $L \to A$ are isomorphic, so we may as well suppose that both g_1 and g_2 are given by $c\pi_C$ where $c : C \to A$ is some map into A with λ-presentable domain. Suppose then that $\varphi_1, \varphi_2 : c\pi_C \to c\pi_C$ are any two 2-cells: we must show that $\varphi_1 = \varphi_2$. Let $d : D \to L$ be any map into L with λ-presentable domain. Now since D is λ-presentable, the map $c\pi_C d : D \to A$ and 2-cells $\varphi_1 d, \varphi_2 d : c\pi_C d \to c\pi_C d$ factorize through some $e : E \to A$ with E a λ-presentable object, say as $c\pi_C d = ef$ and $\varphi_1 d = e\psi_1$ and $\varphi_2 d = e\psi_2$, with $f : D \to E$ and $\psi_1, \psi_2 : f \to f$.

Now E, D, f, ψ_1, and ψ_2 are all in $\mathscr{A}_\lambda$, and so by pseudonaturality of the cone π, we have $\psi_1 \pi_D = \psi_2 \pi_D$, and so $\varphi_1 d \pi_D = e\psi_1 \pi_D = e\psi_2 \pi_D = \varphi_2 d \pi_D$; finally $d\pi_D$ by the Claim, and so $\varphi_1 = \varphi_2$ as required. □

Theorem 9.2 *Let $\mathscr{K}$ be a locally finitely presentable 2-category, and $\mathscr{A}$ a full sub-2-category of $\mathscr{K}$. Then the following conditions are equivalent:*

(i) $\mathscr{A}$ is a small-injectivity class;

(ii) $\mathscr{A}$ is accessible, accessibly embedded, and closed under flexible limits;

(iii) $\mathscr{A}$ *is accessibly embedded and weakly reflective.*

PROOF: Note that we have stated *(ii)* in terms of flexible limits rather than $\mathscr{E}$-stable limits; as observed above, all flexible limits are $\mathscr{E}$-stable. It remains only to prove the implication $(ii) \Rightarrow (iii)$.

Suppose then that $\mathscr{A}$ is accessible, accessibly embedded, and closed under flexible limits. Let $K \in \mathscr{K}$ be given. Consider the slice 2-category $K/\mathscr{A}$ whose objects are morphisms $K \to A$ in $\mathscr{K}$ with $A \in \mathscr{A}$. A morphism in $K/\mathscr{A}$ form $f : K \to A$ to $g : K \to B$ is a morphism $x : A \to B$ with $xf = g$. A 2-cell from x to y in $K/\mathscr{A}$ is a 2-cell $x \to y$ in $\mathscr{K}$ whose restriction along f is the identity. Now $K/\mathscr{A}$ will have any colimits that $\mathscr{A}$ does, in particular, it will have λ-filtered colimits for any sufficiently large λ. Similarly, $K/\mathscr{A}$ has powerssince $\mathscr{A}$ does (powersare flexible). Thus $K/\mathscr{A}$ will be accessible provided that its underlying ordinary category $(K/\mathscr{A})_0$ is so; but this $(K/\mathscr{A})_0$ is just the slice category $K/\mathscr{A}_0$ of $\mathscr{A}_0$, which is accessible since $\mathscr{A}$ is.

Furthermore, $K/\mathscr{A}$ has flexible limits, since $\mathscr{A}$ and $\mathscr{K}$ do and the inclusion preserves them. It follows by Lemma 9.1 that $K/\mathscr{A}$ has a bi-initial object $r : K \to K^*$. The universal property of the bi-initial property is that for any object $a : K \to A$ of $K/\mathscr{A}$, the hom-category $(K/\mathscr{A})((K^*, r), (A, a))$ is equivalent to the terminal category 1. Now this hom-category can be constructed as a pullback as in

$$\begin{array}{ccc} (K/\mathscr{A})((K^*, r), (A, a)) & \longrightarrow & \mathscr{K}(K^*, A) \\ \downarrow & & \downarrow{\scriptstyle \mathscr{K}(r,A)} \\ 1 & \xrightarrow[a]{} & \mathscr{K}(K, A) \end{array}$$

Thus the universal property says that the left vertical is an equivalence (and so a retract equivalence) and we are to prove that the right vertical is a retract equivalence.

Now retract equivalences are stable under pullback, so if we knew that the right vertical were a retract equivalence it would follow immediately that the left vertical was one, but here we need to go in the other direction. To do this, we use the fact that $\mathscr{A}$ is closed in $\mathscr{K}$ under powers. For any category G, the functor $[G, \mathscr{K}(r, A)] : [G, \mathscr{K}(K^*, A)] \to [G, \mathscr{K}(K, A)]$ is isomorphic to $\mathscr{K}(r, G \pitchfork A) : \mathscr{K}(K^*, G \pitchfork A) \to \mathscr{K}(K, G \pitchfork A)$, and any pullback of this along a functor $1 \to \mathscr{K}(K^*, G \pitchfork A)$ is a retract equivalence. Thus $\mathscr{K}(r, A)$ satisfies the conditions of the following lemma, and so is a retract equivalence. $\square$

Lemma 9.3 *Let $p : E \to B$ a functor with the property that for every category C and every functor $g : C \to B$, if we form the pullback*

$$\begin{array}{ccc} P & \longrightarrow & [C, E] \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle [C,p]} \\ 1 & \xrightarrow[g]{} & [C, B] \end{array}$$

the resulting functor q is a retract equivalence. Then p is a retract equivalence.

PROOF: Taking $C = 1$ gives the fact that p is surjective on objects, and that if $e, e' \in E$ with $pe = pe'$ then there is a unique isomorphism $\epsilon : e \cong e'$ sent by p to the identity.

Suppose now that $e_1, e_2 \in E$ with $\beta : pe_1 \to pe_2$. Then β determines a map $g : 1 \to [\mathbf{2}, B]$, and now we can find $\alpha : e'_1 \to e'_2$ with $pe'_1 = pe_1$, $pe'_2 = pe_2$, and $p\alpha = \beta$. There are now unique isomorphism $\epsilon_1 : e_1 \cong e'_1$ and $\epsilon_2 : e_2 \cong e'_2$ sent by p to identities, and now the composite

$$e_1 \xrightarrow{\epsilon_1} e'_1 \xrightarrow{\alpha} e'_2 \xrightarrow{\epsilon_2^{-1}} e_2$$

is sent by p to β. This proves that p is full.

It remains to show that p is faithful. Suppose then that $\gamma : e_1 \to e_2$ is *any* morphism in E with $p\gamma = \beta$. We must show that γ equals the composite displayed above, or equivalently that $\epsilon_2\gamma = \alpha\epsilon_1$.

Now $\epsilon_2\gamma$ and $\alpha\epsilon_1$ can be seen as objects of $[\mathbf{2}, E]$ which are sent by $[\mathbf{2}, p]$ to the same object of $[\mathbf{2}, B]$. Thus they must be isomorphic, via unique isomorphisms in $[\mathbf{2}, E]$ sent by $[\mathbf{2}, p]$ to identities. But such an isomorphism in $[\mathbf{2}, E]$ would have components $\theta : e_1 \cong e_1$ and $\varphi : e'_2 \cong e'_2$ satisfying $\varphi\epsilon_2\gamma = \alpha\epsilon_1\theta$ and being sent by p to identities. But then θ and the identity on e_1 are both isomorphisms $e_1 \to e_1$ lying over the identity, so are equal. Similarly φ is equal to the identity, and it follows that $\epsilon_2\gamma = \alpha\epsilon_1$ as required. □

Theorem 9.4 *For any 2-category $\mathscr{A}$, the following conditions are equivalent:*

- *(i) $\mathscr{A}$ is a weakly reflective, accessibly embedded, full subcategory of some presheaf 2-category $[\mathscr{C}, \mathbf{Cat}]$;*
- *(ii) $\mathscr{A}$ is (equivalent to) a small-injectivity class in some locally finitely presentable 2-category $\mathscr{K}$;*
- *(iii) $\mathscr{A}$ is an accessible 2-category with flexible limits;*
- *(iv) $\mathscr{A}$ is an accessible 2-category with weak colimits.*

Proof: We have only departed slightly from Theorem-Schema B, by using flexible limits rather that $\mathscr{E}$-stable ones. But this is consistent with the formulation of Theorem-Schema A in Theorem 9.2 above, and so the result follows.

□

A 2-category $\mathscr{A}$ satisfying the conditions of the theorem is called *weakly locally presentable.*

We define a limit-$\mathscr{E}$ sketch to be a small category $\mathscr{C}$ with finite limits, equipped with a class $\mathscr{F}$ of morphisms. A model of the sketch is a finite-limit-preserving 2-functor from $\mathscr{C}$ to **Cat** which sends the morphisms in $\mathscr{F}$ to retract equivalences. (It is also possible, in the usual way, to consider more general presentations for such sketches, where we do not assume the existence of all finite limits.) The models of the sketch are taken to be a full subcategory of the functor 2-category $[\mathscr{C}, \mathbf{Cat}]$.

Remark 9.5 Being a retract equivalence is a purely equational structure: to say that f is a retract equivalence is to say that there is a g with $fg = 1$ and $gf \cong 1$, and then any 2-functor will send f to a retract equivalence. Thus it might seem that the class $\mathscr{F}$ makes no difference when it comes to sketching structures. But there is a subtlety here: if g and the isomorphism $gf \cong 1$ were included in the sketch then morphisms would have to be strictly natural with respect to them; by merely requiring f to be an equivalence we only require our morphisms to be strictly natural with respect to f. (It will then follow that they are *pseudonatural* with respect to g, but not necessarily strictly natural.)

The following theorem follows immediately from the others, as in Section 5.

Theorem 9.6 *A $\mathscr{V}$-category $\mathscr{A}$ is the $\mathscr{V}$-category of models of a limit-$\mathscr{E}$ sketch if and only if it is a small-injectivity class in a locally presentable $\mathscr{V}$-category $\mathscr{K}$.* □

10 Examples of weakly locally presentable 2-categories

In this section we focus on the case where $\mathscr{V} = \mathbf{Cat}$ and $\mathscr{E}$ is the class of retract equivalences, and exhibit some of the sorts of examples which can arise as weakly locally presentable 2-categories.

10.1 2-categories of fibrant objects

Let $\mathscr{K}$ be a locally presentable 2-category, equipped with a model 2-category structure [9]; that is, a **Cat**-model structure in the sense of [5] for the "categorical" or "natural" model structure on **Cat**.

Explicitly, this means that there is a model structure on the underlying ordinary category $\mathscr{K}_0$ satisfying the following condition. Let $i : A \to B$ be a cofibration and $p : C \to D$ a fibration, and form the pullback of $\mathscr{K}(i, D); \mathscr{K}(B, D) \to \mathscr{K}(A, D)$ and $\mathscr{K}(A, p) : \mathscr{K}(A, C) \to \mathscr{K}(A, D)$. Then the induced functor

$$\mathscr{K}(B, C) \longrightarrow \mathscr{K}(A, C) \times_{\mathscr{K}(A,D)} \mathscr{K}(B, D)$$

is a fibration in **Cat**, trivial if either i or p is a weak equivalence.

Now consider the full subcategory $\mathscr{A}$ of $\mathscr{K}$ consisting of the fibrant objects. These are the objects C for which $C \to 1$ is a fibration; equivalently, they are characterized by the property that for each trivial cofibration $i : A \to B$, the function $\mathscr{K}_0(i, C) : \mathscr{K}_0(B, C) \to \mathscr{K}_0(A, C)$ is surjective, or in other words the functor $\mathscr{K}(i, C) : \mathscr{K}(B, C) \to \mathscr{K}(A, C)$ is surjective on objects. But by the model 2-category condition above, this functor is not just surjective on objects but a retract equivalence.

Thus $\mathscr{A}$ is an $\mathscr{E}$-injectivity class in $\mathscr{K}$. If moreover the model structure on $\mathscr{K}$ is cofibrantly generated, then $\mathscr{A}$ is a small-$\mathscr{E}$-injectivity class, and so is weakly locally presentable.

10.2 Coflexible presheaves

For a small 2-category $\mathscr{C}$, we write $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ for the 2-category of 2-functors, 2-natural transformations, and modifications; and we write $\mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ for the 2-category of 2-functors, pseudonatural transformations, and modifications. The inclusion $J : [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}] \to \mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ has a left adjoint, sending a presheaf $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ to a presheaf F' with the property that pseudonatural transformations from F to G are in bijection with 2-natural transformations from F' to G (as well as a 2-dimensional aspect to this universal property, involving the modifications). There is a canonical 2-natural transformation $q : F' \to F$, which is the component at F of the counit of the adjunction. F is flexible when this q has a 2-natural section s. It is then a consequence that q is a retract equivalence; see [3] for more details, or [9] for the fact that these flexible presheaves are the cofibrant objects for the *projective* model structure on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$: this is the model structure for which a 2-natural transformation $f : F \to G$ is a weak equivalence or a fibration if and only if $fC : FC \to GC$ is one for each object C of $\mathscr{C}$.

There is also a dual version of these results, using the *injective* model structure [11, Proposition A.3.3.2] on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ for which $f : F \to G$ is a weak equivalence or a *co*fibration if and only if $fC : FC \to GC$ is one for each object C of $\mathscr{C}$. The inclusion $J : [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}] \to \mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ has a right adjoint, whose image at a presheaf F we shall call $\check{F}$, and the component at F of the unit is a 2-natural transformation $j_F : F \to \check{F}$. The components $j_F A : FA \to \check{F}A$ of j are all trivial cofibrations in **Cat**, and so j is an equivalence in $\mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$. A presheaf F for which this j_F has a 2-natural retraction will be called *coflexible.*

Proposition 10.1 *Let $\mathscr{C}$ be a small 2-category. For a presheaf $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ the following conditions are equivalent:*

(i) *F is coflexible*

(ii) $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](j_F, F) : [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](\check{F}, F) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](F, F)$ *is a retract equivalence*

(iii) $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](j_G, F) : [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](\check{G}, F) \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](G, F)$ *is a retract equivalence for all G*

(iv) *F is fibrant in the injective model structure on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$.*

Proof: Here *(iii)* and *(iv)* agree by definition of fibrant object, and the implication *(iii)* $\Rightarrow$ *(ii)* is trivial. To see that *(ii)* $\Rightarrow$ *(i)*, observe that if $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](j_F, F)$ is a retract equivalence then in particular it is surjective on objects, and so there is some 2-natural $r : \check{F} \to F$ with $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](j_F, F)(r) = 1$; that is, $r j_F = 1$. Thus F is coflexible.

So it remains only to prove that *(i)* $\Rightarrow$ *(iii)*. As observed above, $j_G : G \to \check{G}$ is an equivalence in $\mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$, thus for any G we have the commutative square

$$\begin{array}{ccc} [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](\check{G}, F) & \longrightarrow & \mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(\check{G}, F) \\ {\scriptstyle [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](j_G, F)} \downarrow & & \downarrow {\scriptstyle \mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(j_G, F)} \\ [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](G, F) & \longrightarrow & \mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})(G, F) \end{array}$$

in which the right vertical is an equivalence, and the horizontals are the fully faithful inclusions. Thus the left vertical is also fully faithful, and so will be a retract equivalence if and only if it is surjective on objects.

Let $r : \check{F} \to F$ be a 2-natural retraction of j_F. Any 2-natural $u : G \to F$ can be extended to $\check{u} : \check{G} \to \check{F}$, and now $r\check{u}$ satisfies $r\check{u} j_G = r j_F u = u$, so $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}](j_G, F)$ is surjective on objects. $\square$

Since the injective model structure on $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ is cofibrantly generated (see [11, Appendix A.3.3] again) we are in the situation of the previous section, and the 2-category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{fib}}$ of coflexible presheaves is weakly locally presentable.

One reason to be interested in coflexible presheaves is the following:

Proposition 10.2 *For any small 2-category $\mathscr{C}$, the composite inclusion*

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{fib}} \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}] \longrightarrow \mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$$

is a biequivalence of 2-categories.

Proof: First of all, both inclusions are locally fully faithful (fully faithful on the hom-categories). If F and G are presheaves, with G coflexible, any pseudonatural transformation $F \to G$ is isomorphic to a 2-natural one; in particular this is the case if F and G are both coflexible. This proves that the composite is essentially surjective on the hom-categories.

Finally, in any model category, every object is weakly equivalent to a fibrant one; thus in $[\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ every presheaf is weakly equivalent to coflexible one; but weakly equivalent presheaves are pseudonaturally equivalent; that is, equivalent in $\mathrm{Ps}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$. This proves that the composite inclusion is biessentially surjective on objects, and so a biequivalence. □

10.3 Bicategories

This example can be further developed by starting not just with all presheaves $\mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$, but only those which preserve some class of limits. This allows various algebraic structures to be described. Then one could, following [12], consider those functors which preserve products only up to homotopy. For example, if $\mathscr{C}$ has finite coproducts, one could consider functors $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ for which the canonical comparisons $F(C + D) \to FC \times FD$ and $F0 \to 1$ are retract equivalences. These are the presheaves which are $\mathscr{E}$-injective with respect the maps $\mathscr{C}(-, C) + \mathscr{C}(-, D) \to \mathscr{C}(-, C + D)$ and $0 \to C(-, 0)$. Once again one would also want to restrict to something like the coflexible presheaves.

In this section, however, we have chosen to work through a similar but different example, involving the structure of bicategory. We start with the 2-category $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$, of simplicial objects in $\mathbf{Cat}$. It was shown in [10] that there is a full sub-2-category of $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$ which can be identified with 2-category **NHom** of bicategories, normal homomorphisms of bicategories, and icons.

The objects of this full sub-2-category were defined by the following four requirements, in which we write X for a typical simplicial object in $\mathbf{Cat}$, and X_n for the category of n-simplices. For each n, we may form the n-fold fibre product X_1^n of n copies of X_1 over X_0; this represents the "composable n-tuples", and comes equipped with a canonical map $X_n \to X_1^n$ often called the Segal map. The four conditions for X to be in the subcategory **NHom** are:

(i) The simplicial object X is 3-coskeletal: this means that is the right Kan extension of its restriction to the full subcategory of Δ^{op} containing the objects [0], [1], [2], [3];

(ii) The category X_0 of 0-simplices is discrete;

(iii) The maps $c_2 : X_2 \to (\mathrm{Cosk}_1 X)_2$ and $c_3 : X_3 \to (\mathrm{Cosk}_1 X)_3$ are discrete isofibrations (see below);

(iv) The Segal map $X_n \to X_1^n$ is a retract equivalences for all n.

Here condition (i) says that each X_n with $n > 3$ is canonically a limit of X_3, X_2, X_1, and X_0; this is a limit condition so imposing this restriction does not take us outside of the world of locally presentable categories. Once again, condition (ii) is a limit condition, since it can be seen as saying that the canonical map $X_0 \to X_0^{\mathbf{2}}$ is invertible. A functor $f : A \to B$ is called a discrete isofibration if for each object $a \in A$ and each isomorphism $\beta : b \cong fa$ in B, there is a unique isomorphism $\alpha : a' \cong a$ lying over β. Once again this is a limit condition: it says that the diagram

$$\begin{array}{ccc} A^{\mathrm{Iso}} & \xrightarrow{\mathrm{cod}} & A \\ {\scriptstyle f^{\mathrm{Iso}}}\downarrow & & \downarrow{\scriptstyle f} \\ B^{\mathrm{Iso}} & \xrightarrow[\mathrm{cod}]{} & B \end{array}$$

is a pullback, where A^{Iso} is the category of isomorphisms in A, and cod the codomain map. Thus the full sub-2-category of $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$ consisting of the objects satisfying conditions (i), (ii), and (iii) is still locally presentable. Finally, condition (iv) is an injectivity condition. For example, to say that the Segal map $X_2 \to X_1 \times_{X_0} X_1$ is a retract equivalence is to say that X is $\mathscr{E}$-injective with respect to the map induced by

$$\begin{array}{ccc} \Delta_0 \cdot I & \xrightarrow{\delta_0 \cdot I} & \Delta_1 \cdot I \\ {\scriptstyle \delta_1 \cdot I}\downarrow & & \downarrow{\scriptstyle \delta_0 \cdot I} \\ \Delta_1 & \xrightarrow[\delta_2 \cdot I]{} & \Delta_2 \end{array}$$

from the pushout of the top and left maps into the bottom corner.

Thus the 2-category **NHom** is weakly locally presentable.

11 Enriched accessibility and pure morphisms

We recall from [2, Section 2D] that for a locally λ-presentable ordinary category $\mathscr{K}$, a morphism $f : A \to B$ is said to be λ-pure if for each commutative square

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[f]{} & B \end{array}$$

in which C and D are λ-presentable, there exists a morphism $w : B \to A$ with $wg = u$. These can be characterized as the closure in $\mathscr{K}^{\mathbf{2}}$ under λ-filtered colimits of the split monomorphisms. Such a pure morphism is necessarily a monomorphism, and so we have the notion of pure subobject, which plays an important role in the theory of accessible categories.

Here we shall use this notion in the context of enriched categories; the notion itself remains unchanged in our enriched context. First we state [2, Theorem 2.34], combined with the remark that immediately follows it as:

Theorem 11.1 (Adámek-Rosický) *For any λ-accessible category $\mathscr{K}$, the (non-full) subcategory $\mathrm{Pure}_\lambda \mathscr{K}$ of $\mathscr{K}$ consisting of all objects and all λ-pure morphisms is accessible, and closed in $\mathscr{K}$ under λ-filtered colimits.*

We apply this to the case of a locally λ-presentable $\mathscr{V}$-category $\mathscr{K}$: then $\mathrm{Pure}_\lambda \mathscr{K}_0$ is accessible, and closed in $\mathscr{K}_0$ under λ-filtered colimits. Since not just $\mathscr{K}_0$ but $\mathscr{K}$ has λ-filtered colimits, the inclusion $\mathrm{Pure}_\lambda \mathscr{K}_0$ in $\mathscr{K}_0$ sends λ-filtered colimits in $\mathrm{Pure}_\lambda \mathscr{K}_0$ to λ-filtered colimits in $\mathscr{K}$.

Theorem 11.2 *Let $\mathscr{K}$ be a locally λ-presentable $\mathscr{V}$-category and $\mathscr{A}$ a full subcategory closed under λ-filtered colimits. If $\mathscr{A}_0$ is closed in $\mathscr{K}_0$ under λ-pure subobjects, then $\mathscr{A}$ is accessible.*

Proof: The proof follows that of [2, Corollary 2.36], merely taking care to use enriched notions where necessary. By the previous theorem, we know that $\mathrm{Pure}_\lambda \mathscr{K}_0$ is accessible, and that λ-filtered colimits in $\mathrm{Pure}_\lambda \mathscr{K}_0$ are λ-filtered colimits in $\mathscr{K}$. Let μ_0 be some regular cardinal greater than or equal to λ for which $\mathrm{Pure}_\lambda \mathscr{K}_0$ is μ_0-accessible. Now choose a regular cardinal $\mu \triangleright \mu_0$ such that each μ_0-presentable object in $\mathrm{Pure}_\lambda \mathscr{K}_0$ is μ-presentable in $\mathscr{K}$: this is possible since $\mathscr{K}$ is locally presentable, and the set of all μ_0-presentable objects in $\mathrm{Pure}_\lambda \mathscr{K}_0$ is small. Then each μ-presentable object of $\mathrm{Pure}_\lambda \mathscr{K}_0$ is a μ-small μ_0-filtered colimit of μ_0-presentable objects, and so is a μ-small colimit in $\mathscr{K}$ of μ-presentable objects, and so is μ-presentable.

For any object $A \in \mathscr{A}$, we can write A as a μ-filtered colimit in $\mathrm{Pure}_\lambda \mathscr{K}_0$ of μ-presentable objects. This is equally a μ-filtered colimit in $\mathscr{K}$ of μ-presentable objects (in $\mathscr{K}$!) Furthermore, each vertex of the diagram is λ-pure subobject of A, so is in $\mathscr{A}$, and so finally we have written A as a μ-filtered colimit in $\mathscr{A}$ of μ-presentable objects. Since $\mathscr{A}$ has not just μ-filtered colimits but λ-filtered colimits, it follows that $\mathscr{A}$ is μ-accessible. □

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