Toward a Characterization of Algebraic Exactness

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November 4, 2002

Abstract

It is well known that the morphisms between varieties of algebras (as objects) induced by morphisms of algebraic theories are precisely the algebraically exact functors, and they can be completely characterized as the finitary, continuous and exact functors. We prove that this characterization extends to morphisms between algebraically exact categories (forming an "equational hull" of the category of all varieties). And among categories with finite coproducts, the algebraically exact ones are proved to be precisely the precontinuous completely exact categories.

I. Introduction

It is well-known that varieties of (finitary, many-sorted) algebras, considered as categories, enjoy a number of "exactness" properties relating limits and colimits. In particular:

(i) filtered colimits commute with finite limits,

(ii) filtered colimits distribute over products (see I.4 below),

(iii) regular epimorphisms are closed under product

and

*Supported by the Grant Agency of the Czech Republic under the grant No. 201/01/0148.

†Supported by the Ministry of Education of the Czech Republic under the project MSM 143100009.

(iv) exactness: regular epimorphisms are stable under pullback, and every equivalence relation is effective (i.e., is a kernel of some morphism).

It is our conjecture that these are all the exactness properties, i.e., every other relationship between limits and colimits in all varieties follows from (i)-(iv) (and from general properties of limits and colimits in categories). We have proved this in [ARV] for all categories with a small regular generator. In the present paper this is proved for all categories with finite coproducts. A reader acquainted with [ARV] can skip the rest of the Introduction until I.6 where we explain our new result concerning functors preserving exactness properties.

I.1 The 2-Category of Varieties. A duality between (finitary, many-sorted) varieties of algebras and algebraic theories, described by F. W. Lawvere and the present authors in [ALR_1], leads to the following 2-category *VAR* of

- objects: all varieties,
- morphisms: all functors which are

  (i) continuous (i.e., preserve limits),

  (ii) finitary (i.e., preserve filtered colimits)

  and

  (iii) exact (i.e., preserve regular epimorphisms)
- 2-cells: all natural transformations.

Recall that, under (i), (iii) is equivalent to preservation of coequalizers of equivalence relations. Later we have shown in [ALR_2] that

"algebra is not algebraic"

in the sense that the (non-full) inclusion

$$VAR \hookrightarrow \text{CAT} \qquad \text{(the 2-category of all categories)}$$

is not pseudomonadic. And we have described an equational hull of VAR. Before we recall this hull, we need the following concept:

I.2 Sifted Colimits. Recall that the classical concept of a filtered category can be characterized as follows: a small category $\mathcal{D}$ is filtered iff $\mathcal{D}$-colimits commute with finite limits in **Set**. In [AR] we have introduced the concept of a *sifted* category: a small category $\mathcal{D}$ is sifted if $\mathcal{D}$-colimits commute with finite products in **Set**. (Earlier, Lair [L] has called a category "tamisant" if colimits over it commute with nonempty finite products in **Set**, and he has characterized these categories.) Colimits of diagrams with small sifted domains are called *sifted colimits*. Besides filtered colimits, also reflexive coequalizers are an important example of sifted colimits. Recall that a parallel pair $f_1, f_2 : A \to B$ is called *reflexive* if there is $d : B \to A$ with $f_1 d = f_2 d = \mathrm{id}_B$, and coequalizers of such pairs are called *reflexive coequalizers*.

Analogously to the famous free completion $Ind\,\mathcal{A}$ of a category $\mathcal{A}$ under filtered colimits of Grothendieck, we have introduced in [AR] a free completion $Sind\,\mathcal{A}$ of $\mathcal{A}$ under sifted colimits. For more on *Sind* see II.1 below.

I.3 Algebraically Exact Categories. A category $\mathcal{A}$ with sifted colimits is always endowed with an (essentially unique) functor

$$C : Sind\,\mathcal{A} \to \mathcal{A}$$

of computation of sifted colimits in $\mathcal{A}$. In [ALR$_2$] we call $\mathcal{A}$ *algebraically exact* provided that it has limits and sifted colimits, and C is continuous. For example, every variety is algebraically exact, and so is $Sind\,\mathcal{A}$ for every complete category $\mathcal{A}$.

A functor between algebraically exact categories is called *algebraically exact* if it preserves limits and sifted colimits. We obtain a 2-category ALG of all algebraically exact categories, algebraically exact functors, and natural transformations.

We have proved in [ALR$_2$] that ALG is an equational hull of *VAR*. However, the definition of algebraic exactness is somewhat unsatisfactory: what does it mean in "classical" categorical terms? Before dealing with this (still partially open) question, let us recall an analogous result, formulating a complete answer in case varieties are generalized to the locally finitely presentable categories of Gabriel and Ulmer [GU]:

I.4 Equational Hull of LFP**.** The "right morphisms" between locally finitely presentable categories, as follows from the Gabriel-Ulmer duality [GU], are the continuous and finitary functors. We denote by LFP the 2-category of all locally finitely presentable categories, all continuous and finitary functors, and all natural transformations. This 2-category also fails to be equational, see [ALR$_3$] where its equational hull has been described as follows:

For every category $\mathcal{A}$ with filtered colimits we denote by $C : Ind\,\mathcal{A} \to \mathcal{A}$ the functor of computation of filtered colimits in $\mathcal{A}$. We call $\mathcal{A}$ *precontinuous* provided that it has limits and filtered colimits, and C is continuous. Thus, all varieties (in fact, all algebraically exact categories) are precontinuous; also all quasivarieties are precontinuous. Fortunately, precontinuity has been translated into "classical" terms in [ARV]. There we have proved that a category $\mathcal{A}$ with limits and filtered colimits is precontinuous iff its filtered colimits

(a) commute with finite limits

and

(b) distribute over products. That is given a small collection $D_i : \mathcal{D}_i \to \mathcal{A}$ $(i \in I)$ of filtered diagrams, then the following filtered diagram

$$D : \prod_{i\in I} D_i \to \mathcal{A}\,, \quad (d_i) \mapsto \prod D_i\, d_i$$

has a colimit canonically isomorphic to $\prod\limits_{i\in I} colim\, D_i$

Now the 2-category PREC of all precontinuous categories, all continuous and finitary functors, and all natural transformations is an equational hull of LFP, as proved in [ALR$_3$]. This implies that all the exactness properties of locally finitely presentable categories are consequences (a) and (b) above.

I.5 Complete Exactness. Recall from [B$_2$] that a category $\mathcal{A}$ is called *exact* provided that

(i) it has finite limits,

(ii) it has regular factorization, i.e., every morphism factors as a regular epimorphism followed by a monomorphism,

(iii) regular epimorphisms stable under pullback

and

(iv) equivalence relations are effective.

Here we are going to use a stronger condition:

Definition. A category is called *completely exact* provided that it is complete and exact and a product $\prod e_i : \prod A_i \to \prod B_i$ of regular epimorphism $e_i : A_i \to B_i$ $(i \in I)$ is always a regular epimorphism.

Example. Every variety is completely exact (and precontinuous). More generally, every algebraically exact category has these two properties.

Open problem. Is every precontinuous, completely exact category algebraically exact?

Theorem. ([ARV]) *If a category $\mathcal{A}$ has a regular generator, then the following conditions are equivalent:*

(i) $\mathcal{A}$ *is algebraically exact,*

(ii) $\mathcal{A}$ *is a complete localization of a variety, i.e., a reflective subcategory of a variety such that the reflector functor is continuous,*

and

(iii) $\mathcal{A}$ *is precontinuous and completely exact.*

Remark. Categories satisfying (ii) above are obviously cocomplete. One of the results of our paper is the proof of the equivalence of (i) and (iii) for a much wides class of categories: all categories with finite coproducts.

I.6 Algebraically Exact Functors. As we have mentioned in I.1, "morphisms between varieties" are precisely the continuous, finitary and exact functors. Or, equivalently, precisely the algebraically exact functors. The first result of our paper is an analogous characterization of morphisms of the category ALG: we prove that a functor between algebraically exact categories is algebraically exact iff it is continuous, finitary, and exact.

II. Algebraically Exact Functors

II.1 Completions *Ind* and *Sind*. What is the relationship between Grothendieck's completion *Ind* under filtered colimits and the above completion *Sind* under sifted colimits? A precise answer can be given for all categories $\mathcal{C}$ which have either finite coproducts or limits:

$$\mathit{Sind}\,\mathcal{C} = \mathit{Ind}\,(\mathrm{Rec}\;\mathcal{C})$$

where $\mathrm{Rec}\,\mathcal{C}$ is a free completion under reflexive coequalizers. More precisely, we denote below by

$$\eta^S : \mathcal{C} \to \mathit{Sind}\,\mathcal{C}$$

and

$$\eta^R : \mathcal{C} \to \mathrm{Rec}\;\mathcal{C}$$

free completions of $\mathcal{C}$ under sifted colimits and reflexive coequalizers, respectively. And by

$$\eta^I : \mathrm{Rec}\ \mathcal{C} \to \mathit{Ind}\,(\mathrm{Rec}\ \mathcal{C})$$

a free completion of Rec $\mathcal{C}$ under filtered colimits.

II.2 Theorem. [(AR), 2.8, [ARV] 5.1] *If $\mathcal{C}$ has finite coproducts or is complete, then Ind*(Rec $\mathcal{C}$) *is a free completion of $\mathcal{C}$ under sifted colimits w.r.t. the universal arrow*

$$\eta^I \cdot \eta^R : \mathcal{C} \to \mathit{Ind}\,(\mathrm{Rec}\ \mathcal{C})\,.$$

II.3 Remark. This result does not hold for general categories $\mathcal{C}$, a counterexample is shown in [AR], 2.3 (4).

A precise relationship between sifted colimits and the combination of filtered colimits and reflexive coequalizers is not known. In particular, we have the following

II.4 Open problem. Does every category with filtered colimits and reflexive coequalizers have sifted colimits?

The answer is affirmative whenever the category has finite coproducts — then the existence of filtered colimits implies that coproducts exist. And then the category is cocomplete: the well-known procedure of computing colimits via coproducts and coequalizers, see [M], uses in fact reflexive coequalizers.

II.5 Equivalence Relations. Recall from [B_2] that in an exact category $\mathcal{C}$ a relation on an object C is a subobject of $C \times C$ represented by a monomorphism $r : R \rightarrowtail C \times C$, or, equivalently, by a monomorphic pair of morphisms $r_1, r_2 : R \to C$. The pair $r_2, r_1 : R \to C$ represents the inverse relation, r^{-1}. A composite of r with a relation $s : S \rightarrowtail C \times C$ is given by forming the pullback P of r_2 and s_1:

$$\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, dashed, "r_1"'] \arrow[dr, "r_2"'] & & S \arrow[dl, "s_1"] \arrow[dr, dashed, "s_2"] & \\
C & & C & & C
\end{tikzcd}$$

And factoring $\langle r_1 p_1, r_2 p_2 \rangle : P \to C \times C$ as a regular epimorphism followed by a subobject of $C \times C$ – that subobject, then, represents the composite $r \circ s$.

A relation r is called

(i) reflexive if it contains the diagonal of $C \times C$,

(ii) symmetric if it contains r^{-1}

and

(iii) transitive if it contains $r \circ r$.

Relations satisfying (i)-(iii) are called *equivalence relations*.

Assuming that $\mathcal{C}$ has colimits of ω-chains which commute with finite limits, every reflexive relation r has an *equivalence hull*, the least equivalence relation $\overline{r}$ containing r, which we now describe. Define an ω-chain

$$\begin{tikzcd} R_0 \arrow[r] \arrow[dr, "r_0"'] & R_1 \arrow[r] \arrow[d, "r_1"] & R_1 \arrow[r] \arrow[dl, "r_2"] & \cdots \\ & C \times C & & \end{tikzcd}$$

of relations by induction as follows:

$$r_0 = r \circ r^{-1}$$
$$r_{n+1} = r_n \circ r_n$$

Factor $\mathit{colim}\, r_n : \mathit{colim}\, R_n \to C \times C$ as a regular epimorphism followed by a subobject $\overline{r} : \overline{R} \to C \times C$. It has been proved by M. Barr [B_1] that

(i) $\overline{r}$ is an equivalence relation containing r,

and

(ii) the parallel pairs $r_1, r_2 : R \to C$ and $\overline{r}_1, \overline{r}_2 : \overline{R} \to C$ are coequalized by the same morphisms of $\mathcal{C}$.

In particular, if $\mathcal{C}$ has coequalizers of equivalence relations, then it has reflexive coequalizers.

II.6 Theorem. *A functor between algebraically exact categories is algebraically exact iff it is continuous, finitary, and exact.*

Proof. Every algebraically exact functor obviously has all the three properties. Let $F : \mathcal{A} \to \mathcal{B}$ be a continuous, finitary, exact functor between algebraically exact categories. We prove below that F preserves reflexive coequalizers. From this, we derive the algebraic exactness of F as follows: since $\mathcal{A}$ is algebraically exact, the functor

$$C : \mathit{Sind}\,\mathcal{A} \to \mathcal{A}$$

characterized by preserving sifted colimits and fulfilling $C\eta_{\mathcal{A}} \cong \mathit{Id}_{\mathcal{A}}$, is continuous. Let

$$F' : \mathit{Sind}\,\mathcal{A} \to \mathcal{B}$$

be the essentially unique functor preserving sifted colimits with $F \cong F'\eta^S$. Then

$$F' \cong FC : \mathit{Sind}\,\mathcal{A} \to \mathcal{B}\,.$$

In fact, recall from II.1 that $\mathit{Sind}\,\mathcal{A} = \mathit{Ind}\,\mathrm{Rec}\,\mathcal{A}$. Both sides of $F' \cong FC$ are functors extending F:

$$F'\eta^S \cong F \cong (FC)\eta^S\,,$$

and both sides preserve reflexive coequalizers and filtered colimits; consequently, the two sides are naturally isomorphic.

For every sifted diagram D in $\mathcal{A}$ we thus have natural isomorphisms

$$\begin{aligned} \mathit{colim}(FD) &\cong \mathit{colim}(F'\eta^S D) \\ &\cong F'\,\mathit{colim}(\eta^S D) \\ &\cong FC\,\mathit{colim}(\eta^S D) \\ &\cong F(\mathit{colim}\,C\eta^S D) \\ &\cong F\,\mathit{colim}\,D \end{aligned}$$

We prove that F preserves reflexive coequalizers. In fact, let $f_1, f_2 : A \to B$ be a reflexive pair. Factor $\langle f_1, f_2\rangle : A \to B \times B$ into a regular epimorphism $e : A \to R$ followed by a monomorphism $r : R \to B \times B$ (since $\mathcal{A}$ is exact, it has regular factorizations). Then r is a reflexive relation on the object B and its components have the same coequalizer as f_1 and f_2. Since F is exact, it preserves regular factorizations. Thus, Fr is a binary relation on FB whose components have the same coequalizer as Ff_1 and Ff_2. Consequently, it is sufficient to prove that F preserves coequalizers of reflexive relations. First observe that F, preserving finite limits and regular factorizations, preserves the calculus of relations: if $r : R \to B\times B$ represents a relation on B then $Fr : FR \to FB \times FB$ represents a relation on FB, and F preserves relation inverses and composition of relations. We know that F preserves coequalizers of equivalence relations. We conclude that it preserves coequalizers of reflexive relations, r, by forming the hull $\overline{r}$ as in II.1

and observing that the functor F preserves the operations $r \mapsto \overline{r}$ on reflexive relations:

$$\begin{aligned} Fr_0 &= (Fr) \circ (Fr)^{-1}\,, \\ Fr_{n+1} &= (Fr_n) \circ (Fr_n)\,, \end{aligned}$$

and

$$F\overline{r} = \underset{n\in\omega}{colim}(Fr_n) = \overline{Fr}\,.$$

Consequently, Fr and $F\overline{r}$ have the same coequalizer. Thus, since F preserves coequalizers of equivalence relations, it preserves the coequalizer of r. ☐

III. Algebraically Exact Categories

III.1 Theorem. *A category with finite coproducts is algebraically exact iff it is completely exact and precontinuous.*

Proof. Let $\mathcal{C}$ be a completely exact, precontinuous category with finite coproducts. Then the embedding

$$\eta^S : \mathcal{C} \xrightarrow{\eta^R} \operatorname{Rec}\mathcal{C} \xrightarrow{\eta^I} Ind(\operatorname{Rec}\mathcal{C}) = Sind\mathcal{C}$$

(cf. II.2) preserves finite coproducts because η^R has this property and η^I preserves all finite colimits. Hence $Sind\mathcal{C}$ has all coproducts of objects $\eta^S(X)$ where X is in $\mathcal{C}$ (because coproducts are filtered colimits of finite coproducts). Since objects $\eta^S(X)$, X in $\mathcal{C}$, are regularly projective in $Sind\mathcal{C}$, all their coproducts are regularly projective too. Moreover, every object in $Sind\mathcal{C}$ is a small colimit of objects from $\eta^S(\mathcal{C})$ and thus a regular quotient of coproducts of objects from $\eta^S(\mathcal{C})$. Therefore $Sind\mathcal{C}$ has enough regularly projective objects and, moreover, every regularly projective object in $Sind\mathcal{C}$ is a retract of a coproduct of objects from $\eta^S(\mathcal{C})$. Let $\overline{\mathcal{C}}$ be the full subcategory of $Sind\mathcal{C}$ consisting of all regularly projective objects. Since $Sind\mathcal{C}$ is exact, we have that

$$Sind\mathcal{C} = \overline{\mathcal{C}}_{ex}$$

is the exact completion of $\overline{\mathcal{C}}$ (following [CV] Theorem 16).

We are to prove that the functor

$$C : Sind\mathcal{C} \to \mathcal{C}$$

preserves limits. Following [ARV] 5.3, it suffices to show that it preserves finite limits, which, following [CV] Theorem 29, is equivalent to the fact that the domain restriction

$$\overline{C} : \overline{\mathcal{C}} \to \mathcal{C}$$

of C is left covering, i.e., for every finite diagram D in $\overline{\mathcal{C}}$ and every weak limit L of D in $\overline{\mathcal{C}}$ the factorizing morphism $L \to \lim \overline{C}D$ is a regular epimorphism.

Let $D : \mathcal{D} \to \overline{\mathcal{C}}$ be a finite diagram and

$$l_d : L \to Dd \qquad (d \in \mathcal{D})$$

be a limit of D in $Sind\mathcal{C}$. Since $\overline{\mathcal{C}} = Ind\mathcal{C}$, L belongs to $Ind\mathcal{C}$ and, due to precontinuity of $\mathcal{C}$,

$$C(L) = \lim CD\,.$$

Since every weak limit of D in $\overline{\mathcal{C}}$ is given by a regularly projective cover

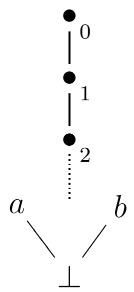
$$e : X \to L$$

(i.e., $l_d \cdot e$ is the corresponding weak limit cone) and C preserves regular epimorphisms, the canonical morphism

$$C(e) : C(X) \to \lim CD$$

is a regular epimorphism. Hence $\overline{C}$ is left covering. $\square$

III.2 Remark. As mentioned in the Introduction, we do not know whether Theorem II.5 holds without the assumption of finite coproducts. In any case let us observe that an algebraically exact category does not have finite coproducts, in general. For example, consider the ordered class $\mathcal{C}$ obtained from $Ord^{\,\mathrm{op}}$ (the transfinite chain of all ordinals ordered dually to the usual well-order) by adding three new elements $\perp$, the smallest element of $\mathcal{C}$, and uncomparable elements a, b which are lower bounds of all ordinals:



Then a coproduct of a, b does not exist, but $\mathcal{C}$ is easily seen to the algebraically exact.

References

[ALR_1] J. Adámek, F. W. Lawvere and J. Rosický, *On the duality between varieties and algebraic theories*, to appear in Alg. Univ.

[ALR_2] J. Adámek, F. W. Lawvere and J. Rosický, *Continuous categories revisited*, to appear.

[ALR_3] J. Adámek, F. W. Lawvere and J. Rosický, *How algebraic is algebra?*, Theory Appl. Cat. **8** (2001), 253–283.

[AR] J. Adámek and J. Rosický, *On sifted colimits and generalized varieties*, Theory Appl. Cat. **8** (2001), 33–53.

[ARV] J. Adámek, J. Rosický and E. M. Vitale, *On algebraically exact categories and essential localizations of varieties*, J. Algebra **244** (2001), 450–477.

[AGV] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lect. Notes in Math. **269**, Springer-Verlag, Berlin 1972.

[B_1] M. Barr, *Right exact functors*, J. Pure Appl. Algebra **4** (1974), 1–8.

[B_2] M. Barr, *Exact categories*, In: M. Barr, P. A, Grillet and D. Van Osdol (editors), Exact categories and categories of sheaves, Lect. Notes in Math. **236**, 1–120, Springer-Verlag, Berlin 1971.

[CV] A. Carboni and E. M. Vitale, *Regular and exact completions*, J. Pure Appl. Alg. **125** (1998), 79–117.

[GU] P. Gabriel and F. Ulmer: *Lokal präsentiebare Kategorien*, Springer Lecture Notes Math. **221**, Springer-Verlag Berlin, 1971.

[L] C. Lair, *Sur le genre d'esquissibilité des catégories modelables (accessibles) possédant les produits de deux*, Diagrammes **35** (1996), 25–52.

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