ON THE DUALITY BETWEEN VARIETIES AND ALGEBRAIC THEORIES

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Abstract. Every variety $\mathcal{V}$ of finitary algebras is proved to have an essentially unique algebraic theory $Th(\mathcal{V})$ which is Cauchy complete, i.e. all idempotents split in $Th(\mathcal{V})$. This defines a duality between varieties (and algebraically exact functors) and Cauchy complete theories (and theory morphisms). Algebraically exact functors are precisely the right adjoints preserving filtered colimits and regular epimorphisms; or, more succintly: the functors preserving limits and sifted colimits.

Introduction

A variety of algebras has many equational presentations, and this has led the second author in his dissertation [Law_1] (see also [Law_2]) to propose a categorical approach in which a presentation-independent description of a variety $\mathcal{V}$ via an *algebraic theory* $\mathcal{T}$ is considered: $\mathcal{T}$ is a small category with finite products such that algebras of $\mathcal{V}$ correspond to *models* of $\mathcal{T}$, i.e., functors in $\mathbf{Set}^{\mathcal{T}}$ preserving finite products, and homomorphisms of algebras correspond to natural transformations of models. More precisely, the category $\mathcal{V}$ is equivalent to the full subcategory

$$\mathbf{Mod}\,\mathcal{T}$$

of $\mathbf{Set}^{\mathcal{T}}$ of all models of $\mathcal{T}$. In the classical case of one-sorted varieties, $\mathcal{T}$ is generated by a single object K, i.e., objects of $\mathcal{T}$ are powers K^n $(n \in \omega)$, and evaluation at K gives a forgetful functor

$$\mathbf{Mod}\,\mathcal{T} \to \mathbf{Set}.$$

An important feature of the mentioned equivalence $\mathcal{V} \cong \mathbf{Mod}\,\mathcal{T}$ is that it is *concrete*, i.e., commutes with naturally defined forgetful functors (up to a natural isomorphism). For the readers' convenience we recall this approach shortly in Part I of the present paper.

If a variety $\mathcal{V}$ is viewed as an abstract category, i.e., without reference to underlying sets and mappings, algebraic theories turn out not to be entirely satisfactory: every non-trivial variety has many (non-equivalent) algebraic theories. For example the variety $\mathbf{Set}$ of algebras with no operations is equivalent to $\mathbf{Mod}\,\mathcal{T}$ where $\mathcal{T}$ is the full subcategory of $\mathbf{Set}^{\mathrm{op}}$ whose objects are all natural numbers. But at is also equivalent to $\mathbf{Mod}\,\mathcal{T}'$ where $\mathcal{T}'$ is the full subcategory of $\mathbf{Set}^{\mathrm{op}}$ whose objects are just the even numbers. An aim of the present note is to propose a remedy for this:

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the algebraic theories $\mathcal{T}$ are to be required to have split idempotents (or, shortly, be Cauchy complete), i.e., given $f : X \to X$ with $f^2 = f$ there exist $e : X \to Y$ and $m : Y \to X$ with $f = me$ and $\mathrm{id}_Y = em$. We are going to prove that every variety has a *canonical algebraic theory* which is Cauchy complete, and each pair of Cauchy complete algebraic theories is equivalent. We introduce here the category of all varieties, whose morphisms are called algebraically exact functors (to be distinguished from the usual concept of an algebraic functor, see e.g. [B], Vol. 2., which are precisely the concrete functors between varieties). Algebraically exact functors are precisely the functors between varieties which are induced by morphisms of their theories. A full characterization of these functors is presented in Part III: they are the right adjoints preserving filtered colimits and regular epimorphisms. Then in the last part we prove a duality between the 2-category VAR of all varieties and the 2-category TH of all Cauchy complete algebraic theories. The duality closely follows the well-known Gabriel-Ulmer duality between locally finitely presentable categories and left exact theories (small categories with finite limits), see [GU]. It is interesting that, although Gabriel and Ulmer have indeed observed the above existence and uniqueness of Cauchy complete algebraic theories, see 11.10 in [GU], they have not pursued this any further, thus, the duality presented here seems to be new.

A detailed description of the algebraic theory $\mathcal{T}$ considered for a variety $\mathcal{V}$ (as a concrete category) in [Law$_1$] is: $\mathcal{T}$ is the full subcategory of $\mathcal{V}^{\mathrm{op}}$ whose objects are all $\mathcal{V}$-free algebras on finitely many generators. A detailed description of the canonical algebraic theory is: $\mathcal{T}$ is the full subcategory of $\mathcal{V}^{\mathrm{op}}$ whose object are the finitely presentable regular projectives, or, equivalently, the retracts of all $\mathcal{V}$-free algebras on finitely many generators.

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I. S-sorted Algebraic Theories

Let $\mathcal{V}$ be a variety of finitary, one-sorted algebras. We consider $\mathcal{V}$ as a category whose morphisms are all homomorphisms and we denote by

$$U_{\mathcal{V}} : \mathcal{V} \to \mathbf{Set}$$

the natural forgetful functor.

In the dissertation [Law$_1$] an *algebraic theory* of an (arbitrary) functor $U : \mathcal{V} \to \mathbf{Set}$ is defined as the category $\mathcal{T}(U)$ whose objects are finite powers K^n $(n \in \omega)$ of a given object K, and morphisms from K^n to K^k are natural transformations from U^n to U^k. (Thus a morphism $\sigma : K^n \to K$ in $\mathcal{T}(U_{\mathcal{V}})$ assigns to every algebra $A \in \mathcal{V}$ an n-ary operation $\sigma_A : A^n \to A$ preserved by all homomorphisms – in other words, σ is a term in $\mathcal{V}$ of n variables. And a morphism from K^n to K^k is a k-tuple of terms in n variables.) The composition of morphisms in $\mathcal{T}(U)$ is just the usual horizontal composition of natural transformations (and corresponds to the clone composition in case of a variety.) It is obvious that the object K^n is a product of n

copies of K in $\mathcal{T}(U)$ with projections corresponding to the n projections $U^n \to U$. Thus, $\mathcal{T}(U)$ has finite products.

Every object $A \in \mathcal{V}$ defines a model $M(A)$ of the theory $\mathcal{T}(U)$, i.e., a functor $\mathcal{T}(U) \to \mathbf{Set}$ preserving finite products, as follows: $M(A)n = U(A)^n$ and $M(A)\sigma = \sigma_A : U^n A \to U^k A$ for all $\sigma : U^n \to U^k$. Further, every morphism $f : A \to B$ in $\mathcal{V}$ defines a model-morphism $M(f) : M(A) \to M(B)$ with $M(f)_n = U^n f$. This establishes a "comparison" functor $M : \mathcal{V} \to \mathbf{Mod}\,\mathcal{T}(U)$. Moreover, $\mathbf{Mod}\,\mathcal{T}(U)$ is equipped with a forgetful functor

$$\overline{U} : \mathbf{Mod}\,\mathcal{T}(U) \to \mathbf{Set}$$

of evaluation at the object K, i.e. $\overline{U}M = MK$ for every model M, $\overline{U}f = f_K$ for every natural transformation $f : M \to M'$ in $\mathbf{Mod}\,(\mathcal{T}(U))$. And the following triangle

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "M"] \arrow[dr, "U"'] & & \mathbf{Mod}\,\mathcal{T}(U) \arrow[dl, "\overline{U}"] \\ & \mathbf{Set} & \end{tikzcd}$$

commutes.

The crucial result of [Law$_1$] is the following

Theorem 1.1. (1) *If $\mathcal{V}$ is a variety of finitary, one-sorted algebras then M is an equivalence.*

(2) *If M is an equivalence then $\mathcal{V}$ is concretely equivalent to a variety $\mathcal{V}'$ of finitary, one-sorted algebras (i.e., there exists an equivalence $E : \mathcal{V} \to \mathcal{V}'$ with $U \cong U_{\mathcal{V}'}E$).*

Remark 1.2. (1) For a variety $\mathcal{V}$ denote by $F_{\mathcal{V}} : \mathbf{Set} \to \mathcal{V}$ the $\mathcal{V}$-free algebra functor, i.e., a left adjoint of $U_{\mathcal{V}}$. Then the algebraic theory $\mathcal{T}(U_{\mathcal{V}})$ is equivalent to the full subcategory of $\mathcal{V}^{\mathrm{op}}$ whose objects are the free algebras $F_{\mathcal{V}}n$ $(n \in \omega)$. In fact, a homomorphism from $F_{\mathcal{V}}1$ to $F_{\mathcal{V}}n$ is just an element of $U_{\mathcal{V}}F_{\mathcal{V}}n$, i.e, a $\mathcal{V}$-term of n variables – and this is precisely a morphism from K^n to K in $\mathcal{T}(U_{\mathcal{V}})$. And, more generally, homomorphisms from $F_{\mathcal{V}}k$ to $F_{\mathcal{V}}n$ correspond to k-tuples of terms.

(2) As morphisms betweeen (finitary, one-sorted) varieties are usually considered the *algebraic functors*, i.e., precisely the concrete functors. Observe that since the forgetful functor of a variety both preserves and creates

(a) limits

(b) filtered colimits

and

(c) regular epimorphisms,

it follows that every algebraic functor preserves and creates (a) – (c) as well.

Let us further note that given varieties $\mathcal{V}$ and $\mathcal{W}$ then every functor

$$H : \mathcal{T}(U_{\mathcal{V}}) \to \mathcal{T}(U_{\mathcal{W}})$$

which (a) preserves finite products and (b) is the identity map on objects gives rise to an algebraic functor

$$\overline{H} : \mathcal{W} \to \mathcal{V}$$

corresponding to the functor $\mathbf{Mod}\,\mathcal{T}(U_{\mathcal{W}}) \to \mathbf{Mod}\,\mathcal{T}(U_{\mathcal{V}})$ of precomposition with H (i.e., $M \mapsto M \cdot H$ for models M of $\mathcal{T}(U_{\mathcal{W}})$). And every algebraic functor is naturally isomorphic to one of this form. In fact, algebraic functors preserve free algebras, now apply (1).

Examples of algebraic functors: the forgetful functor $U_{\mathcal{V}}$, the functor $Rng \to Ab$ which forgets the multiplication of rings.

For *many-sorted*, finitary algebras the situation is completely analogous. Given a functor $U : \mathcal{V} \to \mathbf{Set}^S$ (where S is a set "of sorts"), denote by $U^s : \mathcal{V} \to \mathbf{Set}$ the s-component of U for $s \in S$. The S-sorted algebraic theory of U is the category $\mathcal{T}(U)$ whose objects are finite products of given objects K_s $(s \in S)$, and whose morphisms from $K_{s_1} \times K_{s_2} \times \cdots \times K_{s_n}$ to $K_{t_1} \times K_{t_2} \times \cdots \times K_{t_k}$ are natural transformations from $U_{\mathcal{V}}^{s_1} \times U_{\mathcal{V}}^{s_2} \times \cdots \times U_{\mathcal{V}}^{s_n}$ to $U_{\mathcal{V}}^{t_1} \times U_{\mathcal{V}}^{t_2} \times \cdots \times U_{\mathcal{V}}^{t_k}$. Composition is the horizontal composition of natural transformations.

Now let $\mathcal{V}$ be a variety of S-sorted algebras and let $U_{\mathcal{V}} : \mathcal{V} \to \mathbf{Set}^S$ be the forgetful functor. Then $\mathcal{T}(U_{\mathcal{V}})$ has as morphisms tuples of terms of $\mathcal{V}$ (see [AR] for details). This category $\mathcal{T}(U_{\mathcal{V}})$ has finite products. We form the category $\mathbf{Mod}\,\mathcal{T}(U_{\mathcal{V}})$ of all models (i.e. functors into $\mathbf{Set}$ preserving finite products) equipped with the forgetful functor

$$\overline{U} : \mathbf{Mod}\,\mathcal{T}(U_{\mathcal{V}}) \to \mathbf{Set}^S$$
$$\overline{U}(M) = (Ms)_{s\in S}.$$

Every algebra $A \in \mathcal{V}$ gives rise to a model $M(A)$ with

$$M(A)(K_{s_1} \times \cdots \times K_{s_n}) = A_{s_1} \times \cdots \times A_{s_n} = (U^{s_1} \times \cdots \times U^{s_n})A,$$

and this defines a comparison functor $M : \mathcal{V} \to \mathbf{Mod}\,\mathcal{T}(U_{\mathcal{V}})$ with $U = \overline{U} \cdot M$. This comparison functor is an equivalence of categories (and, conversely, when building up a theory from a functor $U : \mathcal{V} \to \mathbf{Set}^S$ such that the comparison functor is an equivalence, then $\mathcal{V}$ is concretely equivalent to an S-sorted variety of algebras).

Morphisms between S-sorted varieties are usually called *algebraic functors*. They can be defined as precisely those functors $F : \mathcal{V} \to \mathcal{W}$ which are induced by a theory-morphism which is identity on objects. That is, we are given a functor

$$H : \mathcal{T}(U_{\mathcal{W}}) \to \mathcal{T}(U_{\mathcal{V}})$$

preserving finite products and satisfying $H(M) = M$ for all objects, and F is naturally isomorphic to the induced functor of composition with H:

$$- \cdot H : \mathcal{V} \cong \mathbf{Mod}\,(\mathcal{T}(U_{\mathcal{V}})) \to \mathcal{W} \cong \mathbf{Mod}\,(\mathcal{T}(U_{\mathcal{W}})).$$

It is easy to verify that a functor $F : \mathcal{V} \to \mathcal{W}$ is algebraic iff it is concrete, i.e., $U_{\mathcal{W}} \cdot F \cong U_{\mathcal{V}}$. These functors preserve and create limits, filtered colimits, and regular epimorphisms (because the forgetful functors of varieties create all these).

II. Canonical Algebraic Theory

From now on, by a *variety* we will mean a variety of many-sorted, finitary algebras. We have seen in Part I that each variety $\mathcal{V}$ has an algebraic theory $\mathcal{T}(U_{\mathcal{V}})$ given by of all finitely generated $\mathcal{V}$-free algebras (as a full subcategory of $\mathcal{V}^{\mathrm{op}}$). This is not satisfactory as far as we want to consider $\mathcal{V}$ as an abstract category since the concept of free algebra depends of the underlying functor from $\mathcal{V}$ to $\mathbf{Set}^S$. We obtain a theory independent of an underlying set functor if we add also all retracts of $\mathcal{V}$-free algebras, i.e., take the Cauchy completion of $\mathcal{T}(U_{\mathcal{V}})$.

Recall that an object V of a category $\mathcal{V}$ is called *finitely presentable* provided that the hom-functor $\mathcal{V}(V,-)$ preserves filtered colimits. This is, for any variety $\mathcal{V}$, equivalent to the usual algebraic concept of presenting V by finitely many generators and relations, see [AR]. Next, recall that an object V is called a *regular projective* provided that the hom-functor $\mathcal{V}(V,-)$ preserves regular epimorphisms.

Lemma 2.1. *For an object V of a variety $\mathcal{V}$ the following conditions are equivalent:*

(i) *V is a finitely presentable regular projective;*
(ii) *V is a retract of a $\mathcal{V}$-free algebra on finitely many generators;*

Proof. *ii* $\to$ *i*. This is trivial since finitely generated $\mathcal{V}$-free algebras are obviously both finitely presentable and regularly projective, and both of these properties transfer to retracts.

i $\to$ *ii*. Since V is finitely presentable, it is isomorphic to $\Phi(X)/\sim$ where X is a finite set of variables, $\Phi(X)$ denotes a corresponding $\mathcal{V}$-free algebra and $\sim$ is a finitely presented congruence on $\Phi(X)$. Let $e : \Phi(X) \to V$ be the quotient homomorphism. Then $\mathcal{V}(V,-)$ maps e to an epimorphism in $\mathbf{Set}$, thus, for $\mathrm{id}_V \in \mathcal{V}(V,V)$ there exists $m \in \mathcal{V}(V,\Phi(X))$ with $\mathrm{id}_V = em$ – consequently, V is a retract of $\Phi(X)$. □

Remark 2.2. Every category $\mathcal{K}$ has a *Cauchy completion*, i.e., a full embedding $E : \mathcal{K} \to \mathcal{K}^*$ into a Cauchy complete category $\mathcal{K}^*$ such that for any functor $F : \mathcal{K} \to \mathcal{L}$ into a Cauchy-complete category $\mathcal{L}$ there is a unique functor $F^* : \mathcal{K} \to \mathcal{L}$ with $F^* \cdot E = F$. See [B], Vol. 1.

Definition 2.3. By a *canonical algebraic theory* of a variety $\mathcal{V}$ we understand a Cauchy-complete theory of $\mathcal{V}$.

Corollary 2.4. *For every variety $\mathcal{V}$ the category*

$$Th(\mathcal{V})$$

dual to the full subcategory of $\mathcal{V}$ on finitely presentable regular projectives is a canonical algebraic theory of $\mathcal{V}$.

Proof. $Th(\mathcal{V})$ is a Cauchy complete algebraic theory. In fact the category $\mathcal{V}^{\mathrm{op}}$ is complete, and $Th(\mathcal{V})$ is closed under finite products and retracts in $\mathcal{V}^{\mathrm{op}}$, and has a small set of representatives w.r.t. isomorphism. $Th(\mathcal{V})$ is a Cauchy completion of the algebraic theory $\mathcal{T}(U_{\mathcal{V}})$ as follows from the fact that the latter is the full subcategory of $\mathcal{V}^{\mathrm{op}}$ on finitely generated free algebras, thus, by Lemma 2.1, every object of $Th(\mathcal{V})$ is a retract of some object of $\mathcal{T}(U_{\mathcal{V}})$. □

Examples 2.5. (i) The canonical algebraic theory for the trivial variety $\mathbf{Set}$ is the category dual to $\mathbf{Set}_{fin}$, the category of finite sets and mappings.

(ii) The canonical theory of Abelian groups is the dual of the category of all groups Z^n $(n \in \omega)$. In fact, every retract of Z^n is isomorphic to some Z^m. (Thus, $\mathcal{T}(U_{\mathbf{Ab}}) = Th(\mathbf{Ab})$.)

(iii) Let R be a commutative ring. Projective R-modules do not in general coincide with free ones (e.g., for $R = \mathbb{Z}[\sqrt{-5}]$, see [La]). Hence the canonical algebraic theory of the variety R-**Mod** of R-modules is generally larger than its algebraic theory.

Proposition 2.6. *For every variety $\mathcal{V}$, all algebraic theories have the same Cauchy completion.*

Proof. Let $\mathcal{T}$ be an algebraic theory of $\mathcal{V}$. Since $\mathcal{V} \cong \mathbf{Mod}\,\mathcal{T}$ and $Th(\mathcal{V})$ is a Cauchy completion of $\mathcal{T}$, we conclude

$$\mathbf{Mod}\,(Th(\mathcal{V})) \cong \mathbf{Mod}\,\mathcal{T} \cong \mathcal{V}\,.$$

In fact, it is easy to check that, for every model $M : \mathcal{T} \to \mathbf{Set}$, the extension $M^* : Th(\mathcal{V}) \to \mathbf{Set}$ to the Cauchy-completion preserves finite products. And the passage $M \mapsto M^*$ is an equivalence of $\mathbf{Mod}\,\mathcal{T}$ and $\mathbf{Mod}\,(Th(\mathcal{V}))$. $\square$

Corollary 2.7. *(see [GU] 11.10) Every variety has a canonical algebraic theory, unique up to equivalence.*

Remark 2.8. There is a natural restriction of the above to one-sorted varieties: the corresponding algebraic theories are precisely the *singly generated* algebraic theories, i.e., categories $\mathcal{T}$ with finite products which have an object T such that all objects of $\mathcal{T}$ are retracts of finite products of T.

Every one-sorted variety has a Cauchy complete, singly generated algebraic theory, unique up-to equivalence of categories.

III. Algebraically Exact Functors

We have seen in Part I that every variety $\mathcal{V}$ is equivalent to the category $\mathbf{Mod}\,\mathcal{T}$ of models of an algebraic theory $\mathcal{T}$, i.e., a small category with finite products. The converse is also true: for every algebraic theory $\mathcal{T}$, the category $\mathbf{Mod}\,\mathcal{T}$ is equivalent to a variety, see [AR]. Observe that for every *theory morphism*, i.e., every functor $H : \mathcal{T} \to \mathcal{T}'$ between algebraic theories preserving finite products, we have a functor

$$\overline{H} : \mathbf{Mod}\,\mathcal{T}' \to \mathbf{Mod}\,\mathcal{T}$$

of precomposition with H (i.e., $\overline{H}(M) = M \cdot H$ for models M of $\mathcal{T}'$). We call $\overline{H}$ the functor *induced* by the theory morphism H. If varieties are viewed, as in Part I, as concrete categories, the canonical choice of morphisms between them are the algebraic functors. We now want to study varieties as abstract categories, i.e., without reference to underlying sets. Our choice of morphisms has been influenced by the following

Theorem 3.1. *A functor between varieties is induced by a theory morphism iff it preserves limits, filtered colimits and reflexive coequalizers.*

Remark. (i) Explicitly, a functor $G : \mathcal{V}_1 \to \mathcal{V}_2$ between varieties $\mathcal{V}_i \cong \mathbf{Mod}\,\mathcal{T}_i$ is induced by a theory morphism $H : \mathcal{T}_2 \to \mathcal{T}_1$ if there are equivalence functors $E_1 : \mathcal{V}_1 \to \mathbf{Mod}\,\mathcal{T}_1$ and $E_2 : \mathbf{Mod}\,\mathcal{T}_2 \to \mathcal{V}_2$ such that G is naturally isomorphic to $E_2\overline{H}E_1$.

$$\begin{tikzcd}
\mathcal{V}_1 \arrow[rr, "G"] \arrow[d, "E_1"'] & & \mathcal{V}_2 \\
& \cong \Uparrow & \\
\mathbf{Mod}\,\mathcal{T}_1 \arrow[rr, "\overline{H}"'] & & \mathbf{Mod}\,\mathcal{T}_2 \arrow[uu, "E_2"']
\end{tikzcd}$$

(ii) Recall that reflexive coequalizers are coequalizers of pairs $f_1, f_2 : A \to B$ for which there exists $d : B \to A$ such that $f_1 d = f_2 d = \mathrm{id}$.

Proof. I. Sufficiency. Let $G : \mathcal{V} \to \mathcal{W}$ preserve limits and filtered colimits (therefore be a right adjoint, see 1.66 in [AR]) as well as reflexive coequalizers. Let $L : \mathcal{W} \to \mathcal{V}$ be a left adjoint of G. Since G preserves filtered colimits, L preserves the finitely presentable objects. Since G preserves regular epimorphisms, L preserves regular projectives. Consequently, $L^{\mathrm{op}}(Th(\mathcal{W})) \subseteq Th(\mathcal{V})$, and we denote by

$$L_0 : Th(\mathcal{W}) \to Th(\mathcal{V})$$

the domain-codomain restriction of L^{op}. Then L_0 is morphism of algebraic theories (since L preserves finite coproducts, being a left adjoint), hence, we obtain the induced functor

$$G_0(-) = - \cdot L_0 : \mathbf{Mod}\,(Th(\mathcal{V})) \to \mathbf{Mod}\,(Th(\mathcal{W}))\,.$$

We will prove that $G \cong G_0$ by verifying that G_0, like G, is a right adjoint of L. More precisely, a right adjoint of the corresponding functor $L' : \mathbf{Mod}\,(Th(\mathcal{W})) \to \mathbf{Mod}\,(Th(\mathcal{V}))$ given by

$$L'\big(\mathcal{W}(-,X)/Th(\mathcal{W})\big) = \mathcal{V}(-,LX)/Th(\mathcal{V})\,.$$

Thus, given models $\widehat{X} = \mathcal{W}(-,X)/Th(\mathcal{W})$ and $\widehat{Y} = \mathcal{V}(-,Y)/Th(\mathcal{V})$ for algebras $X \in \mathcal{W}$ and $Y \in \mathcal{V}$, we want to establish a bijection

$$\frac{L'\widehat{X} \longrightarrow \widehat{Y}}{\widehat{X} \longrightarrow G_0\widehat{Y}}$$

natural in $\widehat{X}$ and $\widehat{Y}$.

If we restrict X to range through the finitely presentable regular projectives of $\mathcal{W}$, then this natural bijection is obvious:

$$\begin{array}{c}
L'\widehat{X} = \mathcal{V}(-,L_0X) \longrightarrow \widehat{Y} \\
\hline
\widehat{Y}(L_0X) \\
\hline
\begin{array}{ccc}
\mathcal{W}(-,X) & \longrightarrow & \widehat{Y} \cdot L_0 \\
\parallel & & \parallel \\
\widehat{X} & & G_0\widehat{Y}
\end{array}
\end{array}$$

For general X we just use the fact that X is a canonical colimit of finitely presentable regular projectives X_i with a canonical colimit cocone $x_i : X_i \to X$ $(i \in I)$ representing $Th(\mathcal{W}) \downarrow X$: it follows that $\widehat{X}$ is a canonical colimit of $\widehat{X}_i$ $(i \in I)$ and, since L' is a left adjoint, then $(L\widehat{X}_i \stackrel{L'x_i}{\to} LX)_{i \in I}$ is a colimit in $\mathbf{Mod}\,(Th(\mathcal{V}))$. Thus, to give a morphism $f : L'\widehat{X} \to Y$ means to give a collection $f_i : L'\widehat{X}_i \to Y$ $(i \in I)$ of morphisms compatible in the sense that for any $g : X_i \to X_j$ with $x_i = x_j g$ we have $f_i = f_j \cdot L'g$. By the above established natural bijection this is the same as giving a collection $f_i^* : \widehat{X}_i \to G_0\widehat{Y}$ $(i \in I)$ compatible in the sense that $x_i = x_j g$ implies $f_i^* = f_j^* \cdot g$, and the latter is equivalent to giving a morphism $f^* : \widehat{X} \to G_0\widehat{Y}$. This proves that $L' \vdash G_0$.

II. Necessity. Suppose that

$$G \cong \overline{H}$$

for some morphism $H : \mathcal{W}_0 \to \mathcal{V}_0$ of algebraic theories with $\mathcal{W} = \mathbf{Mod}\,\mathcal{W}_0$ and $\mathcal{V} = \mathbf{Mod}\,\mathcal{V}_0$. Observe that the functor $(-) \cdot H : \mathbf{Set}^{\mathcal{W}_0} \to \mathbf{Set}^{\mathcal{V}_0}$ preserves limits and colimits (since they are computed objectwise in presheaf categories). Consequently, $(-) \cdot H$ preserves all limits and colimits under which the model categories are always closed in the presheaf categories. These are all limits, all filtered colimits and all reflexive coequalizers: recall that models are precisely the presheaves preserving finite products, and finite products commute with all three (in **Set** and thus) in all presheaf categories. □

Definition 3.2. A functor between varieties is called *algebraically exact* provided that it has the equivalent properties of Theorem 3.1.

Remark. Observe that algebraically exact functors are exact, in the sense of Barr [Ba], i.e., preserve finite limits and regular epimorphisms.

Examples 3.3. (1) Every algebraic functor is, of course, algebraically exact.

(2) The constant functor $C_1 : \mathcal{V}_1 \to \mathcal{V}_2$ whose value is a terminal (one-element per sort) algebra is algebraically exact but not algebraic (if $\mathcal{V}$ is not trivial).

(3) Let $\mathcal{V}$ be a one-sorted variety. For any free algebra $F_{\mathcal{V}}n$ on n generators, the hom-functor $\hom(F_{\mathcal{V}}n, -) : \mathcal{V} \to \mathbf{Set}$ is algebraically exact. It is algebraic only for $n = 1$.

Remark 3.4. There is a more compact way of expressing preservation of filtered colimits and reflexive coequalizers. Observe that both of those kinds of colimits commute with finite products in **Set**. In fact, filtered colimits are known to be *characterized* by the property that they commute with finite limits in **Set**. This has been generalized by C. Lair, who introduced in [L] the following concept.

Definition 3.5. A small category $\mathcal{D}$ is called *sifted* provided that $\mathcal{D}$-colimits commute with finite products in **Set**. Colimits of diagrams whose schemes are sifted are called *sifted colimits.*

Examples 3.6. (1) Every filtered category is sifted. For posets, the two concepts coincide.

(2) Reflexive coequalizers are sifted. That is, the free category on the following graph

$$\begin{tikzcd} x \arrow[r, shift left=3, "f_1"] \arrow[r, shift right=3, "f_2"'] & y \arrow[l, "d"'] \end{tikzcd}$$

modulo $f_1 d = f_2 d = \mathrm{id}$ is sifted. This follows easily from the fact that given a set-valued functor on the above category and forming a coequalizer c of Df_1, Df_2, then for each pair (u, u'), (v, v') in $Dy \times Dy$ with $c(u) = c(u')$ and $c(v) = c(v')$ we can find zig-zags connecting u, u' and v, v' of the same length (by using Dd).

(3) Every category with (weak) finite coproducts is sifted. This follows from the following

Proposition 3.7. *A small category $\mathcal{D}$ is sifted iff it is non-empty and for every pair d_1, d_2 of objects the category of all co-spans*

$$d_1 \xrightarrow{\delta_1} x \xleftarrow{\delta_2} d_2$$

is connected.

Remark 3.8. (1) By the category of all co-spans we mean the obvious one: a morphism from $d_1 \xrightarrow{\delta_1} x \xleftarrow{\delta_2} d_2$ into $d_1 \xrightarrow{\delta_1'} x' \xleftarrow{\delta_2'} d_2$ is a morphism $h : x \longrightarrow x'$ of $\mathcal{D}$ such that the diagram

$$\begin{tikzcd}
 & x \arrow[dd, "h"] & \\
d_1 \arrow[ur, "\delta_1"] \arrow[dr, "\delta_1'"'] & & d_2 \arrow[ul, "\delta_2"'] \arrow[dl, "\delta_2'"] \\
 & x' &
\end{tikzcd}$$

commutes. Thus, sifted categories are characterized (among small non-empty categories) by the fact that for every pair d_1, d_2 of objects the category of all these co-spans has just 1 connected component, i.e.,

(i) a co-span $d_1 \longrightarrow x \longleftarrow d_2$ exists

and

(ii) any two of such co-spans are connected by a zig-zag (of co-spans).

(2) The sufficiency of the above condition for $\mathcal{D}$ to be sifted has been proved by C. Lair [L] who used, instead of our ”sifted” the French version ”tamisante”.

The necessity is trivial: if $\mathcal{D}$ is sifted, consider the diagram $D(d_i, -) : \mathcal{D} \to \mathbf{Set}$. It has colimit $\cong 1$ in $\mathbf{Set}$. Consequently, the diagram $D = D(d_1, -) \times D(d_2, -)$ assigning to every x the set of all co-spans $d_1 \longrightarrow x \longleftarrow d_2$ has colimit $\cong 1$. This means precisely that (i) and (ii) above hold.

Proposition 3.9. *Algebraically exact functors are precisely those functors between varieties that preserve limits and sifted colimits.*

Proof. Sufficiency follows from 3.6. (1), (2); necessity is proved as in 3.1. $\square$

Remark 3.10. Algebraically exact functors are fully characterized as precisely those exact right adjoints preserving sifted colimits. (The reader may recall here that the category of locally finitely presentable categories, introduced in [GU], has as morphisms right adjoints preserving filtered colimits.) In fact, in the proof of Theorem 3.1 we only use preservation of regular epimorphisms.

Observe that algebraically exact functors also preserve the passage from a reflexive coequalizer

$$R \underset{f_2}{\overset{f_1}{\rightrightarrows}} X \xrightarrow{\ c\ } Y$$

to the equivalence relation (kernel pair of c) it induces because the latter is the colimit of the chain of composites $R_0 = R \circ R^{-1}$, $R_{n+1} = R_n \circ R_n$. And exact functors preserve relational calculus, see [Ba].

IV. Duality of VAR and TH

Denote by VAR the 2-category of

(0) all varieties (as objects, or 0-cells)
(1) all algebraically exact functors (as morphisms, or 1-cells)

and

(2) all natural transformations (as 2-cells).

Observe that VAR fails badly to be locally small: there certainly exists a large collection of functors $G : \mathbf{Set} \to \mathbf{Set}$ naturally isomorphic to $Id_{\mathbf{Set}}$ (in fact, a collection indexed by $\exp(Ord)$, the collection of all subclasses of the class Ord of all ordinals – see [AP]). Each such functor is a 1-cell in VAR.

Denote further by TH the 2-category of

(0) all Cauchy complete algebraic theories (0-cells)
(1) all functors preserving finite products (1-cells)

and

(2) all natural transformations (2-cells).

We are going to prove that VAR is "essentially equivalent" to the dual of TH. These categories are, of course, not equivalent on the nose because TH (whose objects are *small* categories) is locally small. The precise statement to be proved is that the following 2-functor

$$\mathbf{Mod} : \mathrm{TH}^{\mathrm{op}} \to \mathrm{VAR}$$

defined on objects by $\mathcal{T} \mapsto \mathbf{Mod}\,\mathcal{T}$, on morphisms by

$$(H : \mathcal{T} \to \mathcal{T}') \mapsto (- \cdot H : \mathbf{Mod}\,\mathcal{T}' \to \mathbf{Mod}\,\mathcal{T})$$

and 2-cells by

$$(f : H \to H') \mapsto ((Mf)_{M \in \mathbf{Mod}\,\mathcal{T}'} : - \cdot H \to - \cdot H')$$

is a biequivalence. This concept, introduced by R. Street [S], means that (i) every object of VAR is equivalent to one in the image of $\mathbf{Mod}$ and (ii) for every pair $\mathcal{T}_1$, $\mathcal{T}_2$ of objects of $\mathrm{TH}^{\mathrm{op}}$ the corresponding functor $\mathrm{TH}^{\mathrm{op}}(\mathcal{T}_1, \mathcal{T}_2) \to \mathrm{VAR}(\mathbf{Mod}\,\mathcal{T}_1, \mathbf{Mod}\,\mathcal{T}_2)$ is a biequivalence of categories.

Thus, $\mathbf{Mod}$ is a "equivalence of categories up-to natural isomorphisms".

Theorem 4.1. *The 2-category VAR is dually biequivalent to the 2-category TH.*

Proof. We shall verify that the 2-functor **Mod** satisfies the above conditions (i) and (ii) for biequivalences.

(i) Every variety $\mathcal{V}$ is equivalent to $\mathbf{Mod}\,(Th(\mathcal{V}))$: see Corollary 2.5.

(ii) For every pair $\mathcal{T}_1$, $\mathcal{T}_2$ of Cauchy complete algebraic theories the functor

$$\mathbf{Mod}_{\,\mathcal{T}_1,\mathcal{T}_2} : \mathrm{TH}(\mathcal{T}_1, \mathcal{T}_2) \to \mathrm{VAR}(\mathcal{T}_2, \mathcal{T}_1)$$

given by

$$H : \mathcal{T}_1 \to \mathcal{T}_2 \mapsto - \cdot H : \mathbf{Mod}\,\mathcal{T}_2 \to \mathbf{Mod}\,\mathcal{T}_1$$

is an equivalence of categories:

a. $\mathbf{Mod}_{\mathcal{T}_1,\mathcal{T}_2}$ is full and faithful. This is a standard argument completely analogous to that used in the Gabriel-Ulmer duality; see e.g. [AP] for a careful exposition of the latter.

b. Every object of $\mathrm{VAR}(\mathcal{T}_2, \mathcal{T}_1)$ is (naturally) isomorphic to one in the image of $\mathbf{Mod}_{\mathcal{T}_1,\mathcal{T}_2}$: this follows from the definition of an algebraically exact functor. □

Remark. Since **Mod** is a biequivalence, it has a biadjoint biequivalence VAR $\to$ $\mathrm{TH}^{\mathrm{op}}$: this is a bifunctor of the formation of canonical algebraic theories.

Remark 4.2. Returning to the above case of 1-sorted varieties and algebraic functors, observe that this 2-category is obviously biequivalent to the full subcategory of the comma-category

$$\mathrm{VAR} \downarrow \mathbf{Set}$$

formed by all faithful functors. On the level of theories, denote by $\mathbf{Set}_{\mathrm{fin}}^{\mathrm{op}}$ the dual of the category of finite sets (a theory of the trivial variety **Set**). Then $\mathrm{VAR} \downarrow \mathbf{Set}$ is, by our duality, dually biequivalent to the comma-category $\mathbf{Set}_{\mathrm{fin}}^{\mathrm{op}} \downarrow \mathrm{TH}$. And the above 2-category of 1-sorted varieties is equivalent to the full subcategory of $\mathbf{Set}_{\mathrm{fin}}^{\mathrm{op}} \downarrow \mathrm{TH}$ formed by all functors which are identity maps on object sets.

Analogously for S-sorted varieties and $\mathbf{Set}_{\mathrm{fin}}^{\mathrm{op}} \downarrow \mathbf{Set}^S$.

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