Phantom accessible triangulated categories

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Given a category $\mathcal{K}$ and its small full subcategory $\mathcal{A}$, the *canonical functor*

$$E_{\mathcal{A}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

assigns to each object K the restriction

$$E_{\mathcal{A}}(K) = \hom(-, K)/\mathcal{A}^{\mathrm{op}}$$

of its hom-functor $\hom(-, K) : \mathcal{K} \to \mathbf{Set}$ to $\mathcal{A}^{\mathrm{op}}$.

If $E_{\mathcal{A}}$ is a full embedding then $\mathcal{A}$ is called *dense*. It means that each object K is a colimit of its *canonical diagram* $D : \mathcal{A} \downarrow K \to \mathcal{K}$.

$E_{\mathcal{A}}$ full iff each object K is a weak colimit of its canonical diagram.

$E_{\mathcal{A}}$ full and reflects isomorphisms iff each object K is a minimal weak colimit of its canonical diagram.

An object K is a *weak colimit* of D if it is equipped with a cocone $(\delta_d : Dd \to K)_{d \in \mathcal{D}}$ such that for each cocone $(\gamma_d : Dd \to L)_{d \in \mathcal{D}}$ there is a morphism $k : K \to L$ such that $k\delta_d = \gamma_d$ for each d in $\mathcal{D}$.

A weak colimit K is called a *minimal weak colimit* if each morphism $k : K \to K$ with $k\delta_d = \delta_d$ for each d in $\mathcal{D}$ is an isomorphism.

A minimal weak colimit is a retract of each weak colimit. Thus minimal weak colimits are unique up to an isomorphism. A colimit is a minimal weak colimit.

If $\mathcal{A}$ has weak λ-small colimits (λ is a regular cardinal) then canonical diagrams of objects of $\mathcal{K}$ are λ-filtered. Thus

$$E_{\mathcal{A}} : \mathcal{K} \to Ind_{\lambda}\, \mathcal{A}$$

where $Ind_{\lambda}\, \mathcal{A}$ is the free completion of $\mathcal{A}$ under λ-filtered colimits, i.e., the full subcategory of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ consisting of λ-filtered colimits of hom-functors.

Categories $Ind_{\lambda}\, \mathcal{A}$ are precisely λ-*accessible* categories. More precisely, $\mathcal{K}$ is λ-accessible iff

$$E_{\mathcal{A}} : \mathcal{K} \to Ind_{\lambda}\, \mathcal{A}$$

is an equivalence for some small full subcategory $\mathcal{A}$.

We say that $\mathcal{K}$ is *phantom λ-accessible*, where $\mathcal{A}$ is its small full subcategory and λ a regular cardinal, provided that

$$E_{\mathcal{A}} : \mathcal{K} \to Ind_\lambda \, \mathcal{A}$$

is a *representation equivalence*, i.e., it is full, reflects isomorphisms and is essentially surjective on objects.

A representation equivalence is an equivalence if and only if it is faithful. But a representation equivalence E admits distinct morphisms $f, g : K_1 \to K_2$ with $Ef = Eg$. In the additive case, it is the same as the existence of $f \neq 0$ with $Ef = 0$. From the point of view of $\mathcal{A}$, f is a phantom.

Let $\mathcal{T}$ be a λ-well generated triangulated category and $\mathcal{T}_\lambda$ its full subcategory of λ-compact objects. We denote $E_{\mathcal{T}_\lambda}$ by E_λ. This functor preserves coproducts and reflects isomorphisms. $\mathcal{T}$ will be called λ-*Brown* if E_λ is full.

Since $\mathcal{T}_\lambda$ has weak λ-small colimits,

$$E_\lambda : \mathcal{T} \to Ind_\lambda \, \mathcal{T}_\lambda$$

The classical case is $\lambda = \omega$. Here $Ind_\omega \, \mathcal{T}_\omega$ is the category of homological functors $\mathcal{T}_\omega^{\mathrm{op}} \to \mathbf{Ab}$.

A. Beligiannis showed that if E_ω is full then it is a representation equivalence. Thus $\mathcal{T}$ is phantom ω-accessible.

Theorem 1. Let $\mathcal{M}$ be a combinatorial (stable) model category. There are arbitrarily large regular cardinals λ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "P"] \arrow[dr, "Ind_\lambda\, P_\lambda"'] & & Ho\,\mathcal{M} \arrow[dl, "E_{\mathcal{A}}"] \\
& Ind_\lambda\,\mathcal{A} &
\end{tikzcd}$$

Moreover, $E_{\mathcal{A}}$ preserves coproducts (and reflects isomorphisms).

P_λ is the restriction of P on the full subcategory $\mathcal{M}_\lambda$ of λ-presentable objects in $\mathcal{M}$ and $\mathcal{A}$ the full image of P_λ in $Ho\,\mathcal{M}$. We denote $H_\lambda = Ind_\lambda P_\lambda$. We use that $\mathcal{M} = Ind_\lambda\,\mathcal{M}_\lambda$.

We say that $\mathcal{M}$ is λ-*good* if the triangle above commutes.

$\mathcal{M}$ is λ-good provided that it is locally λ-presentable and the cofibrant and fibrant replacement functors are strongly λ-accessible; it means that they preserve λ-filtered colimits and λ-presentable objects.

If $\mathcal{M}$ is λ-good and $\lambda \trianglelefteq \mu$ then $\mathcal{M}$ is μ-good.

Sp is λ-good for each $\omega_1 \trianglelefteq \lambda$. It is not ω-good because finitely presentable spectra have ω_1-presentable fibrant replacements.

We have $\omega_1 \triangleleft \omega_n$, $n = 2, 3, \dots$ but not $\omega_1 \triangleleft \omega_{\omega+1}$.

Theorem 2. Let $\mathcal{M}$ be a stable combinatorial model category. Then $Ho\,\mathcal{M}$ is well generated.

$Ho\,\mathcal{M}$ is well λ-generated if $\mathcal{M}$ is λ-good. Each object from $\mathcal{A}$ is λ-compact.

The reason is that

$$\begin{aligned} \coprod_{i\in I} K_i &\cong \operatorname{colim}_J \coprod_{j\in J} K_j \\ &\cong \operatorname{colim}_J \coprod_j \operatorname{colim}_{n\in N_j} K_{jn} \\ &\cong \operatorname{colim}_J \operatorname{colim}_{(n_j)} \coprod_j K_{jn_j} \end{aligned}$$

where J runs over subsets of I of cardinality $< \lambda$.

Problem. Let $\mathcal{M}$ be a stable combinatorial λ-good model category. Is then $E_{\mathcal{A}}$ full?

I claimed that the answer is positive but my argument contains a gap.

Let $\mathcal{M}$ be a stable combinatorial λ-good model category. Consider M in $\mathcal{M}$ and express it as a λ-filtered colimit $(\delta_d : D \to M)$ of its canonical diagram $D : \mathcal{D} \to \mathcal{M}_\lambda$. It means that

$$\begin{tikzcd}
Dd \arrow[d, "u_e"'] \arrow[dr, "v_d"] & & \\
\coprod\limits_{e:d\to d'} Dd \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \coprod\limits_{d} Dd \arrow[r, "g"] & M \\
Dd \arrow[u, "u_e"] \arrow[r, "De"'] & Dd' \arrow[u, "v_{d'}"'] &
\end{tikzcd}$$

where g is a pushout

$$\begin{tikzcd}
\coprod\limits_{d} Dd \arrow[r, "g"] & M \\
\coprod\limits_{e} Dd \amalg \coprod\limits_{d} Dd \arrow[u, "(p,\mathrm{id})"] \arrow[r, "(q,\mathrm{id})"'] & \coprod\limits_{d} Dd \arrow[u, "g"']
\end{tikzcd}$$

If we replace the pushout above by a homotopy pushout, we get $(\overline{\delta}_d : D \to \overline{M})$. It is not a cocone in $\mathcal{M}$ but $(P\overline{\delta}_d : PD \to P\overline{M})$ is a weak colimit in $Ho\,\mathcal{M}$. It will be called *standard*. There is a comparison morphism $t : \overline{M} \to M$ such that $t\overline{\delta}_d = \delta_d$ for each d. Since H_λ preserves λ-filtered colimits, there is a morphism $u : H_\lambda M \to H_\lambda \overline{M}$ such that $uH\delta_d = H\overline{\delta}_d$ for each d. We have

$$H_\lambda(t)u = \mathrm{id}$$

because

$$H_\lambda(t)uH_\lambda\delta_d = H_\lambda(t\overline{\delta}_d) = H_\lambda\delta_d$$

We will show that $E_{\mathcal{A}}$ is full iff, for each M in $\mathcal{M}$, $u = E_{\mathcal{A}}\overline{u}$ for some $\overline{u} : PM \to P\overline{M}$.

We get the same situation for each λ-filtered diagram $D : \mathcal{D} \to \mathcal{M}$. So, homotopy λ-filtered colimits are weak colimits provided that $E_{\mathcal{A}}$ is full.

Consider a morphism $h : H_\lambda M_1 \to H_\lambda M_2$. There is a cocone $\gamma_d : PD_1 d \to P\overline{M}_2$ from PD_1 such that

$$E_\mathcal{A}\gamma_d = u_2 h H_\lambda \delta_{1d} : H_\lambda D_1 d \to H_\lambda \overline{M}_2$$

for each d in $\mathcal{D}_1$. Thus there is a morphism

$$\overline{h} : \overline{M}_1 \to \overline{M}_2$$

such that

$$\overline{h} P \overline{\delta}_{1d} = \gamma_d$$

for each d in $\mathcal{D}_1$. Hence

$$E_\mathcal{A}(\overline{h}) u_1 H_\lambda \delta_{1d} = E_\mathcal{A}(\overline{h}) H_\lambda \overline{\delta}_{1d} = E_\mathcal{A}\gamma_d = u_2 h H_\lambda \delta_{1d}$$

for each d in $\mathcal{D}_1$. Thus

$$E_\mathcal{A}(\overline{h}) u_1 = u_2 h$$

Put $h' = t_2 \overline{h} \overline{u}_1$. We have

$$E_\mathcal{A}(h') = E_\mathcal{A}(t_2 \overline{h}) u_1 = E_\mathcal{A}(t_2) u_2 h = h$$

Let $\mathcal{A}$ be a small full subcategory of a category $\mathcal{K}$. By an $\mathcal{A}$-*pure monomorphism* is meant a morphism $f : K \to L$ such that given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ K & \xrightarrow[f]{} & L \end{array}$$

with A and B in $\mathcal{A}$, then u factorizes through g, i.e., $u = tg$ for some $t : B \to K$.

By an $\mathcal{A}$-*pure epimorphism* is meant a morphism $f : K \to L$ such that every morphism $v : B \to L$ factorizes through f, i.e., $v = gt$ for some $t : B \to K$.

f is an $\mathcal{A}$-pure epimorphism iff $E_{\mathcal{A}}(f)$ is an epimorphism.

An object will be called $\mathcal{A}$-*pure projective* if it is projective w.r.t. $\mathcal{A}$-pure epimorphisms.

Lemma. $\mathcal{A}$-pure projective objects are precisely retracts of coproducts of objects from $\mathcal{A}$.

It follows from $\coprod\limits_{d \in \mathcal{D}} Dd \to K$ being an $\mathcal{A}$-pure epimorphism.

In a triangulated category, a morphism will be called a λ-*pure monomorphism* if it is a pure monomorphism w.r.t. λ-compact objects. Analogously, we have λ-*pure epimorphisms* and λ-*phantom morphisms*.

Proposition 1. Let $\mathcal{K}$ be a well λ-generated triangulated category. Then f is a λ-pure monomorphism iff $E_\lambda(f)$ is a monomorphism.

Proposition 2. Let $\mathcal{K}$ be a well λ-generated triangulated category. For an exact triangle

$$K \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} \Sigma K$$

the following are equivalent

(1) f is a λ-phantom,

(2) g is a λ-pure monomorphism,

(3) h is a λ-pure epimorphism.

ω-pure monomorphisms in compactly generated triangulated categories were introduced by H. Krause as morphisms f such that $E_\omega(f)$ is a monomorphism.

We say that a triangulated category has the λ-*pure global dimension less or equal to 1* if each object K admits an exact triangle

$$Q \xrightarrow{f} P \xrightarrow{g} K \xrightarrow{h} \Sigma Q$$

where P and Q are λ-pure projective and g is a λ-pure epimorphism.

Proposition 3. Let $\mathcal{K}$ be a λ-Brown triangulated category. Then $\mathcal{K}$ has λ-*pgldim* ≤ 1.