Towards categorical model theory

J. Rosický

(based on the joint papers with T. Beke and M. Lieberman)

Stockholm 2014

In order to define accessible categories one just needs the concept of a λ-directed colimit where λ is a regular cardinal. This is a colimit over a diagram $D : \mathcal{D} \to \mathcal{K}$ where $\mathcal{D}$ is a λ-directed poset, considered as a category. An object K of a category $\mathcal{K}$ is called λ-*presentable* if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves λ-directed colimits; here **Set** is the category of sets.

A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-directed colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

A category is *accessible* if it is λ-accessible for some regular cardinal λ.

An S-*sorted signature* is a set of (infinitary) S-sorted operation and relation symbols. Given cardinals κ and λ the language $L_{\kappa\lambda}(\Sigma)$ allows less than κ-ary conjunctions and disjunctions and less than λ-ary quantifications; here, Σ is assumed to be λ-ary. Given a theory T of $L_{\kappa\lambda}$, the category $\mathbf{Elem}(T)$ of T-models and $L_{\kappa\lambda}$-elementary embeddings is accessible.

We say that a category $\mathcal{K}$ is *well λ-accessible* if it is μ-accessible for each regular cardinal $\mu \geq \lambda$.
$\mathcal{K}$ is *well accessible* if it is well λ-accessible for some regular cardinal λ.

The category of λ-directed posets and embeddings is accessible but not well accessible.

Every finitely accessible category is well finitely accessible.

Any λ-accessible category with directed colimits is well λ-accessible.

Let λ be a regular cardinal. We say that an object K of a category $\mathcal{K}$ has *presentability rank* λ if it is λ-presentable but not μ-presentable for any regular cardinal $\mu < \lambda$. We write $\text{rank}(K) = \lambda$.

Lemma 1. Let $\mathcal{K}$ be a λ-accessible category with directed colimits and K an object of $\mathcal{K}$ which is not λ-presentable. Then $\text{rank}(K)$ is a successor cardinal.

Thus $\text{rank}(K) = |K|^+$ and $|K|$ is called the *size* of K.

An accessible category with directed colimits will be called λ-*LS-accessible* if it has objects of all sizes $\mu \geq \lambda$.

$\mathcal{K}$ is *LS-accessible* if it is well λ-accessible for some cardinal λ.

Problem 1. Is every accessible category with directed colimits LS-accessible?

We say that $\mathcal{K}$ is an accessible category with *concrete directed colimits* if $\mathcal{K}$ has directed colimits and is equipped with a faithful functor $U : \mathcal{K} \to \mathbf{Set}$ preserving directed colimits.

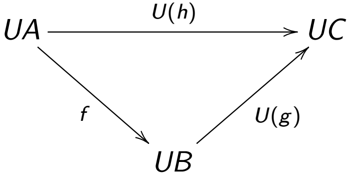
(1) Every finitely accessible category is an accessible category with concrete directed colimits.

(2) Let Σ be a (single sorted) finitary signature. Then the category $\mathbf{Elem}(T)$ is an accessible category with concrete directed colimits for each theory T of $L_{\kappa\omega}(\Sigma)$. These categories are called (∞, ω)-*elementary*.

(3) Every abstract elementary class is an accessible category with concrete directed colimits.

Theorem 1. Every large accessible category with concrete directed colimits is LS-accessible.

An accessible category $(\mathcal{K}, U)$ with concrete directed colimits is *coherent* if for each commutative triangle



there is $\overline{f} : A \to B$ in $\mathcal{K}$ such that $U(\overline{f}) = f$.

All examples (1-3) are coherent. These examples are in an ascending generality.

In a coherent accessible category with concrete directed colimits, the functor U preserves sizes starting from some cardinal μ.

Since sizes in **Set** equal to cardinalities, coherence implies that internal and external sizes coincide.

Let $(\mathcal{K}, U)$ be a λ-accessible category with concrete directed colimits. For a finite cardinal n we denote by U^n the functor

$$\mathbf{Set}(n, U(-)) : \mathcal{K} \to \mathbf{Set}\,.$$

Directed colimits preserving subfunctors R of U^n will be called *finitary relation symbols interpretable in* $\mathcal{K}$ and natural transformations $h : U^n \to U$ will be called *finitary function symbols interpretable in* $\mathcal{K}$.

Since they are both determined by their restrictions to the full subcategory $\mathcal{K}_\lambda$ of $\mathcal{K}$ consisting of λ-presentable objects, there is only a set of such symbols. They form the signature $\Sigma_{\mathcal{K}}$ which will be called the *canonical signature* of $\mathcal{K}$. We get the *canonical functor* $G : \mathcal{K} \to \mathbf{Str}(\Sigma_{\mathcal{K}})$ into $\Sigma_{\mathcal{K}}$-structures (with homomorphisms as morphisms).

Having a bijection $f : UA \to UB$, we get bijections $f^n : (UA)^n \to (UB)^n$. Assume that $R(f)$ is a bijection for each R and $h_B f^n = f h_A$ for each h above. This means that $f : GA \to GB$ is an isomorphism. We say that $\mathcal{K}$ is *iso-full* if $f = U(\overline{f})$ for each such f.

An abstract elementary class is a coherent and iso-full accessible category with concrete directed colimits whose morphisms are monomorphisms preserved by U.

This is a reformulation of Shelah's concept of an abstract elementary class, defined as a subcategory of $\mathbf{Str}(\Sigma)$, which is not only syntax-free but even signature-free.

The assumption about monomorphisms is not restrictive. We can pass from a (coherent) accessible category $\mathcal{K}$ with concrete directed colimits to its iso-full subcategory $\mathcal{K}_0$ on the same objects which is a (coherent) accessible category with directed colimits whose morphisms are monomorphisms.

Any finitely accessible category whose morphisms are monomorphisms is an AEC.

More generally, any (∞, ω)-*elementary* category whose morphisms are monomorphisms is an AEC.

I do not know any example of an abstract elementary class which is not (∞, ω)-elementary.

There are examples of abstract elementary classes $\mathcal{K} \subseteq \mathbf{Str}(\Sigma)$ which are not axiomatizable by any theory of $L_{\kappa\omega}(\Sigma)$. But it does not exclude an axiomatization in another signature.

Problem 2. Is the category of uncountable sets and monomorphisms (∞, ω)-elementary?

Let $\mathcal{K}$ be an accessible category with directed colimits and λ an infinite cardinal. $\mathcal{K}$ is λ-*categorical* if it has, up to isomorphism, precisely one object of size λ.

Shelah's Categoricity Conjecture claims that for every AEC $\mathcal{K}$ there is a cardinal κ such that $\mathcal{K}$ is either λ-categorical for all $\lambda \geq \kappa$ or $\mathcal{K}$ is not λ-categorical for any $\lambda \geq \kappa$.

This was conjectured by Loś for first-order theories in a countable language in 1954 and proved by Morley in 1965. In 1970, Shelah extended it for uncountable languages. SCC is the main test question for AECs.

Of course, SCC was formulated using external sizes, i.e., cardinalities of underlying sets. Since they coincide with internal sizes starting from some cardinal, SCC is the property of the category $\mathcal{K}$.

I do not know any example of an accessible category $\mathcal{K}$ with directed colimits violating SCC.

Having non-LS $\mathcal{K}$, then the disjoint union of $\mathcal{K}$ and **Set** would do it.

For a regular cardinal λ, an object K of a category $\mathcal{K}$ is called λ-*saturated* if it is injective with respect to morphisms between λ-presentable objects. This means that for any morphisms $f : A \to K$ and $g : A \to B$ where A and B are λ-presentable there is a morphism $h : B \to K$ such that $hg = f$.

If $\mathcal{K}$ is a λ-accessible category with directed colimits and the joint embedding property then any two λ-saturated λ^+-presentable objects are isomorphic.

If $\mathcal{K}$ is a λ-accessible category with directed colimits and the amalgamation property and $\lambda^{<\lambda} = \lambda$ then the full subcategory $\mathbf{Sat}_\lambda(\mathcal{K})$ of $\mathcal{K}$ consisting of λ-saturated objects is λ^+-accessible.

$\lambda^{<\lambda} = \lambda$ holds under GCH or if λ is inaccessible.

Consequently, assuming GCH or the existence of a proper class of inaccessible cardinals, there are arbitrarily large regular cardinals μ such that any μ^+-presentable object K has a morphism into a μ-saturated μ^+-presentable object $\overline{K}$.

Moreover, if $\mathcal{K}$ is large and its morphisms are monomorphisms then μ-saturated object cannot be μ-presentable. Thus $|\overline{K}| = \mu$.

μ-saturated objects of size μ are often called *monster* object.

Assume GCH. Let $\mathcal{K}$ be a λ-accessible category with directed colimits, the joint embedding property and the amalgamation property whose morphisms are monomorphisms. Let $\mathcal{K}$ be μ-categorical where $\mu > \lambda$ is a sufficiently large regular cardinal. Then a unique object of size μ is μ-saturated. Thus any object of size $\geq \mu$ is μ-saturated. Thus $\mathcal{K}$ is ν-categorical for $\mu < \nu$ if and only if every μ-saturated object is ν-saturated.

In classical model theory, types are maximal consistent sets of formulas in a single variable.

Shelah introduced (language-free) types for AECs in 1987. His definition makes sense in any accessible category $\mathcal{K}$ with concrete directed colimits.

Consider pairs (f, a) where $f : M \to N$ and $a \in UN$. Two pairs (f_0, a_0) and (f_1, a_1) are equivalent it there is an amalgamation

$$\begin{array}{ccc} N_0 & \xrightarrow{h_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle h_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

such that $U(h_0)(a_0) = U(h_1)(a_1)$.

The resulting equivalence classes are called *(Galois) types* over M. One needs the amalgamation property to get the equivalence relation.

Let $\mathcal{K}$ be an accessible category with concrete directed colimits, the amalgamation property and the joint embedding property which contains arbitrarily large monster objects. Then (f_0, a_0) and (f_1, a_1) are equivalent if and only if there is a square

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & L \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

and an isomorphism $s : L \to L$ such that $sg_0f_0 = g_1f_1$ and $U(sg_0)(a_0) = U(g_1)(a_1)$.

Thus types are orbits of automorphism groups. L is a sufficiently large monster object.

A type (f, a) where $f : M \to N$ is *realized* in K if there is a morphism $g : M \to K$ and $b \in U(K)$ such that (f, a) and (g, b) are equivalent.

Let λ be a regular cardinal. We say that K is λ-*Galois saturated* if for any $g : M \to K$ where M is λ-presentable and any type (f, a) where $f : M \to N$ there is $b \in U(K)$ such that (f, a) and (g, b) are equivalent.

Theorem 1. Let $\mathcal{K}$ be a coherent accessible category with concrete directed colimits, the amalgamation property and the joint embedding property which has arbitrarily large monster objects. Let λ be a sufficiently large regular cardinal. Then K is λ-Galois saturated if and only if it is λ-saturated.

Coherence appears to be indispensable in the "only if" part of the proof, i.e., in the element-by-element construction of morphisms.

Tameness was introduced by Grossberg and VanDieren in 2006 as a smallness property of Galois types for AECs.

Let $\mathcal{K}$ be an accessible category with concrete directed colimits and κ be a regular cardinal. We say that $\mathcal{K}$ is κ-*tame* if for two non-equivalent types (f_0, a_0) and (f_1, a_1) over M there is a morphism $u : X \to M$ with X κ-presentable such that the types $(f_0 u, a_0)$ and $(f_1 u, a_1)$ are not equivalent.
$\mathcal{K}$ is called *tame* if it is κ-tame for some κ.

Theorem.(Grossberg, VanDieren) Let $\mathcal{K}$ be a large, tame AEC with the amalgamation property and the joint embedding property. If $\mathcal{K}$ is λ^+-categorical for a sufficiently large cardinal λ then $\mathcal{K}$ is μ-categorical for all $\mu \geq \lambda^+$.

An uncountable cardinal κ is called *strongly compact* if every κ-complete filter can be extended to a κ-complete ultrafilter on the same set.

Equivalently, $L_{\kappa,\kappa}$ satisfies the compactness theorem.

Theorem.(Boney) Assuming the existence of arbitrarily large strongly compact cardinals, every AEC is tame.

Theorem 2. Assuming the existence of arbitrarily large strongly compact cardinals, every accessible category with concrete directed colimits is tame.

This generalization of Boney's theorem is the consequence of

Theorem.(Makkai, Paré) Assuming the existence of arbitrarily large strongly compact cardinals, every powerful image of an accessible functor is accessible.

The *powerful image* of a functor $G : \mathcal{K} \to \mathcal{L}$ is the smallest full subcategory of $\mathcal{L}$ containing $G(\mathcal{K})$ and closed under subobjects.

We expect that Grossberg-VanDieren theorem is valid for coherent accessible categories with concrete directed colimits, i.e., that one does not need iso-fullness.

Any accessible category $\mathcal{K}$ with concrete directed colimits admits an EM-functor, i.e., a faithful functor $E : \mathbf{Lin} \to \mathcal{K}$ preserving directed colimits.

One does not need coherence for this. In abstract elementary classes one gets this functor from the Shelah's Presentation Theorem which involves both the assumption of coherence and the reintroduction of language into the fundamentally syntax-free world of abstract elementary classes.