SPECIAL REFLEXIVE GRAPHS IN MODULAR VARIETIES

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Abstract. We investigate a special kind of reflexive graphs in any congruence modular variety. When the variety is Maltsev these special reflexive graphs are exactly the internal groupoids, when the variety is distributive they are the internal reflexive relations. We use these internal structures to give some characterizations of Maltsev, distributive and arithmetical varieties.

Introduction

A variety of universal algebras is congruence *modular* if all algebras have modular lattice of congruences. An important aspect of modular varieties is that they admit a good theory of commutators of congruences [10], [9], [6], [15]. Any congruence is an internal reflexive graph, actually a groupoid, and the importance of internal categorical structures in commutator theory has been pointed out in various recents papers [2], [3], [5], [11], [12], [17], [18], [19]. The purpose of this paper is to prove some properties of these internal structures which make it possible to characterize important classes of modular varieties.

Given a modular variety $\mathcal{V}$ of universal algebras, one can consider the category $RG(\mathcal{V})$ of (internal) reflexive graphs in $\mathcal{V}$: the objects in this category are diagrams of the form

$$X_1 \; \overset{d}{\underset{c}{\rightrightarrows}} \; X_0 \qquad (e \colon X_0 \to X_1)$$

with $d \circ e = 1_{X_0} = c \circ e$. The homomorphism d represents the domain, c the codomain and e the reflexivity of the reflexive graph. In any modular variety a reflexive graph is equipped with at most one internal groupoid structure. A reflexive graph X as above is underlying a groupoid structure if and only if the kernel congruences $R[d]$ and $R[c]$ have trivial commutator and they permute [8]:

$$(1) \quad [R[d], R[c]] = \Delta_{X_1} \qquad \text{and} \qquad (2) \quad R[d] \circ R[c] = R[c] \circ R[d]$$

(Δ_{X_1} is the smallest congruence on X_1). Let us denote by $Grpd(\mathcal{V})$ the category of internal groupoids in $\mathcal{V}$ and by $RG^+(\mathcal{V})$ the category of reflexive graphs in $\mathcal{V}$ satisfying only property (1), that we call *special reflexive graphs*.

The characterisation of internal groupoids we just recalled says in particular that there are embeddings $U\colon Grpd(\mathcal{V}) \to RG^+(\mathcal{V})$ and $V\colon RG^+(\mathcal{V}) \to RG(\mathcal{V})$. It is not difficult to see that both these functors actually have left adjoints

$$Grpd(\mathcal{V}) \underset{U}{\overset{F}{\leftrightarrows}} RG^+(\mathcal{V}) \underset{V}{\overset{G}{\leftrightarrows}} RG(\mathcal{V}),$$

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so that U and V are inclusions of reflective subcategories.

In the present paper we prove that these adjunctions allow one to distinguish three classes of modular varieties. We first show that $RG^+(\mathcal{V})$ is always a subvariety of the variety of internal reflexive graphs $RG(\mathcal{V})$. As a consequence, we show that a modular variety is *Maltsev* precisely when $Grpd(\mathcal{V})$ is a subvariety of $RG(\mathcal{V})$, and it is *distributive* precisely when $RG^+(\mathcal{V})$ is equivalent to the category of reflexive relations. Thus, when a variety is *arithmetical*, namely when it satisfies both these conditions, the category $RG^+(\mathcal{V})$ is equivalent to the category $Cong(\mathcal{V})$ of congruences in $\mathcal{V}$. Finally, a characterization of arithmetical varieties can be given: a modular variety $\mathcal{V}$ is arithmetical if and only if $Cong(\mathcal{V})$ is a subvariety of $RG(\mathcal{V})$.

1. Internal reflexive graphs

In this section we briefly recall some basic internal categorical structures that will play a central role in the following. $\mathcal{C}$ will denote a category with finite limits.

An internal reflexive graph X is a diagram

$$X_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow[c]{} \end{array} X_0$$

in $\mathcal{C}$ satisfying the identities $d \circ e = 1_{X_0} = c \circ e$. An arrow in the category $RG(\mathcal{C})$ of (internal) reflexive graphs in $\mathcal{C}$ is a pair of arrows (f_0, f_1) in $\mathcal{C}$ making the diagram

$$\begin{array}{ccc} X_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow[c]{} \end{array} & X_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y_1 & \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\epsilon} \\ \xrightarrow[\gamma]{} \end{array} & Y_0 \end{array}$$

commute. An internal reflexive graph X has an internal groupoid structure when there is an arrow $m\colon X_1 \times_{X_0} X_1 \to X_1$, the composition, and an arrow $i\colon X_1 \to X_1$, the inversion, satisfying the usual axioms of a groupoid:

1. $m \circ (1_{X_1}, e \circ c) = 1_{X_1} = m \circ (e \circ d, 1_{X_1})$ (unit axiom)
2. $d \circ p_1 = d \circ m$ and $c \circ p_2 = c \circ m$ (m has the right domain and codomain)
3. $m \circ (1_{X_1} \times_{X_0} m) = m \circ (m \times_{X_0} 1_{X_1})$ (associativity)
4. $d \circ i = c$, $c \circ i = d$, $m \circ (1_{X_1}, i) = e \circ d$ and $m \circ (i, 1_{X_1}) = e \circ d$ (inverse).

An arrow in the category $Grpd(\mathcal{C})$ of internal groupoids in $\mathcal{C}$ is an arrow (f_0, f_1) in $RG(\mathcal{C})$ which is also required to respect the composition: this means that the diagram

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow[f_2]{} & Y_1 \times_{Y_0} Y_1 \\ \downarrow{\scriptstyle m_X} & & {\scriptstyle m_Y}\downarrow \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

must commute.

We shall be interested in a structure which is intermediate between the one of internal reflexive graph and the one of internal groupoid. This structure, that we call a special reflexive graph, is deeply related to the notion of pseudogroupoid which appears in commutator theory [12]. In order to explain this structure, we need to fix some notations.

Any reflexive graph X in $\mathcal{C}$ naturally determines the equivalence relations

$$R[d] \underset{p_2}{\overset{p_1}{\underset{\longleftarrow \varepsilon}{\rightrightarrows}}} X_1$$

and

$$R[c] \underset{p_2}{\overset{p_1}{\underset{\longleftarrow \varepsilon}{\rightrightarrows}}} X_1$$

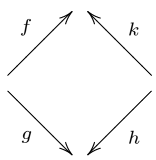
which are the kernel pairs of d and c, respectively.
We can then consider the following pullback:

$$\begin{array}{ccc} R[d]\square R[c] & \longrightarrow & R[c] \times R[c] \\ \downarrow & & \downarrow {\scriptstyle (p_1 \times p_2, p_1 \times p_2)} \\ R[d] \times R[d] & \xrightarrow[(p_1,p_1)\times(p_2,p_2)]{} & X_1 \times X_1 \times X_1 \times X_1 \end{array}$$

In the set-theoretical context $R[d]\square R[c]$ is the subobject of X_1^4 corresponding to the four-tuples (f,g,k,h) of elements of X_1 (= edges of the reflexive graph) with the appropriate domain and codomain:

$$fR[d]g, \quad kR[d]h, \quad fR[c]k \quad \text{and} \quad gR[c]h.$$

An element (f,g,k,h) in $R[d]\square R[c]$ will be then pictured as follows:



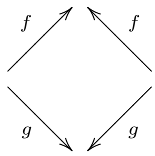
Via the obvious projections, the object $R[d]\square R[c]$ determines a double equivalence relation on $R[d]$ and $R[c]$, namely an equivalence relation in the category of equivalence relations:

$$\begin{array}{ccc} R[d]\square R[c] & \underset{\pi_{2,4}}{\overset{\pi_{1,3}}{\rightrightarrows}} & R[c] \\ {\scriptstyle \pi_{1,2}} \downarrow\downarrow {\scriptstyle \pi_{3,4}} & & {\scriptstyle p_1} \downarrow\downarrow {\scriptstyle p_2} \\ R[d] & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & X_1 \end{array}$$

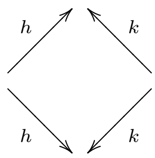
This double equivalence relation, which is the largest one on $R[d]$ and $R[c]$, is called the *parallelistic double equivalence relation on* $R[d]$ *and* $R[c]$ [5].

We shall denote by $s^d \colon R[d] \to R[d]\square R[c]$ and $s^c \colon R[c] \to R[d]\square R[c]$ the arrows giving the reflexivity of the double equivalence relation $R[d]\square R[c]$ on $R[d]$ and $R[c]$.

Accordingly, s^d applied to a pair of edges f and g with the same domain will give



while s^c applied to a pair of edges h and k with the same codomain will give

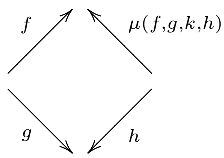


1.1. Definition. An internal reflexive graph in $\mathcal{C}$ is *special* if there is an arrow $\mu\colon R[d]\Box R[c] \to X_1$ in $\mathcal{C}$ satisfying the following properties:

(1) $d \circ \mu = d \circ p_2 \circ \pi_{3,4}$ and $c \circ \mu = c \circ p_1 \circ \pi_{1,2}$
(2) μ is independent of the third variable
(3) $\mu \circ s^d = p_1 \circ \pi_{1,2}$ and $\mu \circ s^c = p_1 \circ \pi_{1,2}$

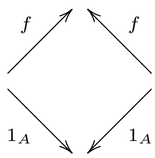
In terms of elements (via the Yoneda embedding) these axioms simply say that

(1) $\mu(f, g, k, h)$ is "parallel" to k, namely $\mu(f, g, k, h)$ has the same domain and the same codomain as k:

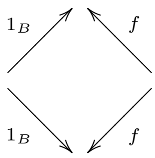


(2) $\mu(f, g, k, h) = \mu(f, g, k', h)$
(3) $\mu(f, g, f, g) = f$ and $\mu(f, f, k, k) = k$

1.2. Remark. From now on we shall call *arrows* the edges of a reflexive graph. Any arrow $f\colon A \to B$ in X_1 gives rise to two elements in $R[d]\Box R[c]$:



and



The arrow $\mu\colon R[d]\Box R[c] \to X_1$ allows one to "compose" f with 1_A and with 1_B: by axiom 3 one gets

$$\mu(f, 1_A, f, 1_A) = f \quad \text{and} \quad \mu(1_B, 1_B, f, f) = f,$$

namely the usual *identity axiom*. However, in general, one can only compose those composable pairs of arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

for which there exists an arrow k having the same domain as f and the same codomain as g:

$$A \xrightarrow{f} B \xrightarrow{g} C, \qquad k\colon A \dashrightarrow C$$

When this is the case, one defines $m(f, g) = \mu(g, 1_B, k, f)$.

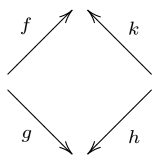
1.3. Remark. The notion of special reflexive graph is a simplified version of the notion of reflexive graph with a *pseudogroupoid structure* in the sense of Janelidze and Pedicchio on $R[d]$ and $R[c]$.

We shall denote by $RG^+(\mathcal{C})$ the category of special reflexive graphs in $\mathcal{C}$, where arrows are required to respect the structure: the diagram

$$\begin{array}{ccc} R[d]\Box R[c] & \xrightarrow[\overline{f}]{} & R[d]\Box R[c] \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_Y} \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

should commute.

Clearly, any internal groupoid is a special reflexive graph: if $m\colon X_1 \times_{X_0} X_1 \to X_1$ is the multiplication of the groupoid structure, i its inversion and



an element in $R[d]\Box R[c]$, one then sets $\mu(f, g, k, h) = m(f, m(i(g), h))$.

Via this natural interpretation one sees that the inclusion $U\colon Grpd(\mathcal{C}) \to RG^+(\mathcal{C})$ is always full, but the forgetful functor $V\colon RG^+(\mathcal{C}) \to RG(\mathcal{C})$ is not full, in general.

When the category $\mathcal{C}$ is locally finitely presentable, then the previous functors have left adjoints:

1.4. Proposition. *Let $\mathcal{C}$ be a locally finitely presentable category. Then we have the adjunctions*

$$Grpd(\mathcal{C}) \underset{U}{\overset{F}{\rightleftarrows}} RG^+(\mathcal{C}) \underset{V}{\overset{G}{\rightleftarrows}} RG(\mathcal{C}).$$

Proof. It is not difficult to check that any limit of special reflexive graphs in $\mathcal{C}$ is a special reflexive graph. Since filtered colimits commute with finite limits, this is also the case for a filtered colimit of special reflexive graphs. Now, when $\mathcal{C}$ is locally finitely presentable, so is the functor category $RG(\mathcal{C})$. The categories $Grpd(\mathcal{C})$ and $RG^+(\mathcal{C})$ can be seen as models for the respective finite limit sketches, and therefore they are locally finitely presentable (see Proposition 1.53 in [1]). Following Theorem 1.66 in [1] the functors $U\colon Grpd(\mathcal{C}) \to RG^+(\mathcal{C})$ and $V\colon RG^+(\mathcal{C}) \to RG(\mathcal{C})$ are right adjoints. □

2. Modular varieties

In this section we shall prove that, when $\mathcal{V}$ is a variety of universal algebras, then the category $RG^+(\mathcal{V})$ is itself equivalent to a variety, actually a subvariety of $RG(\mathcal{V})$.

Let us first recall that a variety $\mathcal{V}$ of universal algebras is congruence modular when, for any algebra X in $\mathcal{V}$, the lattice of congruences on X is modular: for any R, S, T in $Con(X)$ if $T \leq R$ then $R \wedge (S \vee T) = (R \wedge S) \vee T$.

2.1. Examples. Any *Maltsev* variety [20] is modular, hence in particular the varieties of groups, quasigroups, rings, Heyting algebras, Lie algebras and crossed modules. Also any *distributive* variety is modular, as for instance the varieties of lattices, or of median algebras [9].

In any modular variety, the condition characterizing special reflexive graphs can be expressed in terms of commutators [12] [2]. Indeed, a reflexive graph X is special if and only if the commutator is trivial: $[R[d], R[c]] = \Delta_{X_1}$. The internal groupoids can be also characterized by a simple condition:

2.2. Theorem. [8] *For a reflexive graph X the following conditions are equivalent:*

(1) *X has a unique groupoid structure;*
(2) $[R[d], R[c]] = \Delta_{X_1}$ *and* $R[d] \circ R[c] = R[c] \circ R[d]$.

We can now state our new results:

2.3. Proposition. *If $\mathcal{V}$ is a modular variety, then $RG^+(\mathcal{V})$ is a subvariety of the variety $RG(\mathcal{V})$.*

Proof. Whenever $\mathcal{V}$ is a variety of universal algebras, the category $RG(\mathcal{V})$ is itself equivalent to a variety. In order to prove it, let $\mathcal{W}$ be the variety whose theory has the same operations as $\mathcal{V}$ together with two additional unary operations s and t satisfying the following conditions: $s \circ t = t$, $t \circ s = s$ and both operations s and t are homomorphisms in $\mathcal{V}$. One can define a functor $K\colon RG(\mathcal{V}) \to \mathcal{W}$ sending a reflexive graph

$$X_1 \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} X_0$$

to the algebra X_1 in $\mathcal{W}$, where the two additional unary operations are $s = e \circ d$ and $t = e \circ c$. The functor K is defined on arrows by $K(f_0, f_1) = f_1$.

It is not difficult to check that K is essentially surjective on objects. For this, let us remark that given any algebra A in $\mathcal{W}$ equipped with unary operations s and

t, it is clear that s and t have the same regular image, that we denote by I. The algebra A is then isomorphic to $K(X)$ where X is the reflexive graph

$$\xymatrix{A \ar@<1.5ex>[r]^{\delta} \ar@<-1.5ex>[r]_{\gamma} & I \ar[l]|{\varepsilon}}$$

where $\varepsilon \circ \delta = s$ and $\varepsilon \circ \gamma = t$. Since K is clearly fully faithful, the functor K is an equivalence of categories.

On the other hand, in the modular context, a multiplication $\mu\colon R[d]\Box R[c] \to X_1$ is unique, when it exists (Theorem 2.12 in [2]). Moreover, the embedding $V\colon RG^{+}(\mathcal{V}) \to RG(\mathcal{V})$ is full, and its left adjoint $F\colon RG(\mathcal{V}) \to RG^{+}(\mathcal{V})$ is given by the quotient of X_1 by the congruence $[R[d], R[c]]$:

$$\xymatrix{X_1 \ar@<1.5ex>[r]^{d} \ar@<-1.5ex>[r]_{c} \ar[d]_{\eta} & X_0 \ar[l]|{e} \ar@{=}[d] \\ \frac{X_1}{[R[d],R[c]]} \ar@<1.5ex>[r]^{\delta} \ar@<-1.5ex>[r]_{\gamma} & X_0. \ar[l]|{\epsilon}}$$

Indeed, the universal property of the commutator asserts that $[R[d], R[c]]$ is the smallest congruence T with the property that in the canonical quotient $\phi\colon X_1 \to \frac{X_1}{T}$ one has that $[\phi(R[d]), \phi(R[c])] = \Delta_{\frac{X_1}{T}}$.

The unit $(1_{X_0}, \eta)$ of the adjunction is clearly a regular epimorphism in $RG(\mathcal{V})$, hence $RG^{+}(\mathcal{V})$ is closed in $RG(\mathcal{V})$ under subobjects.

Let us then prove that $RG^{+}(\mathcal{V})$ is also closed in $RG(\mathcal{V})$ under regular quotients. For this, let (f_0, f_1) be a regular epimophism from X to Y in $RG(\mathcal{V})$, with X a special reflexive graph:

$$\xymatrix{X_1 \ar@<1.5ex>[r]^{d} \ar@<-1.5ex>[r]_{c} \ar[d]_{f_1} & X_0 \ar[l]|{e} \ar[d]^{f_0} \\ Y_1 \ar@<1.5ex>[r]^{\delta} \ar@<-1.5ex>[r]_{\gamma} & Y_0 \ar[l]|{\epsilon}}$$

There is an arrow $\tilde{f}$ from $R[d]$ to $R[\delta]$ induced by the universal property of the kernel pair:

$$\xymatrix{R[d] \ar@<0.7ex>[r]^{p_1} \ar@<-0.7ex>[r]_{p_2} \ar[d]_{\tilde{f}} & X_1 \ar[r]^{d} \ar[d]_{f_1} \ar@{}[dr]|{(1)} & X_0 \ar[d]^{f_0} \\ R[\delta] \ar@<0.7ex>[r]^{p_1} \ar@<-0.7ex>[r]_{p_2} & Y_1 \ar[r]_{\delta} & Y_0}$$

Now, let $i \circ q$ be the regular epi-monomorphic factorization of $\tilde{f}$. The square (1) is easily checked to be a pushout because d, δ are split epimorphisms and f_0 and f_1 are regular epis. This implies that δ is the coequalizer of $p_1 \circ i$ and $p_2 \circ i$, so that $R[\delta] = \overline{f_1(R[d])}$, where the right-hand side of the equality represents congruence generated by $f_1(R[d])$. In any modular variety the commutator preserves the regular "direct images", namely if f_1 is a regular epi, then

$$\overline{f_1[R[d],R[c]]} = [\overline{f_1(R[d])}, \overline{f_1(R[c])}].$$

Accordingly,

$$\begin{aligned}
[R[\delta], R[\gamma]] &= [\overline{f_1(R[d])}, \overline{f_1(R[c])}] \\
&= \overline{f_1[R[d],R[c]]} \\
&= \overline{f_1(\Delta_{X_1})} \\
&= \Delta_{Y_1},
\end{aligned}$$

and Y is a special reflexive graph as well. □

2.4. Corollary. *For a modular variety $\mathcal{V}$ the following conditions are equivalent:*

(1) *$\mathcal{V}$ is Maltsev;*
(2) *$Grpd(\mathcal{V})$ is a subvariety of $RG(\mathcal{V})$.*

Proof. 1. ⇒ 2. Thanks to Theorem 2.2 we already know that $RG^+(\mathcal{V}) \simeq Grpd(\mathcal{V})$, since in a Maltsev variety the condition $R[d] \circ R[c] = R[c] \circ R[d]$ is always satisfied. One then concludes by the previous proposition.

2. ⇒ 1. If $Grpd(\mathcal{V})$ is closed in $RG(\mathcal{V})$ under subobjects, then in particular any reflexive relation

$$R \;\underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}}\; X$$

in $\mathcal{V}$ is then a groupoid, since it is a subobject of the largest congruence

$$X \times X \;\underset{p_2}{\overset{p_1}{\underset{\longleftarrow e}{\rightrightarrows}}}\; X$$

□

2.5. Example. We are now going to consider the important example when $\mathcal{V}$ is the Maltsev variety Grp of groups. Let us first recall that an object in the category PX-Mod of *precrossed modules* is an arrow

$$\alpha\colon A \to B$$

in the category of groups with an action of B on A, written ${}^b a$ for any $a \in A$ and $b \in B$, satisfying the axiom $\alpha({}^b a) = b\alpha(a)b^{-1}$. An arrow $(f_0, f_1)\colon (\alpha, A, B) \to (\alpha', A', B')$ in PX-Mod is a pair of homomorphisms of groups making the square

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\
A' & \xrightarrow{\alpha} & B'
\end{array}$$

commute and such that ${}^{f_0(b)}f_1(a) = f_1({}^b a)$. A precrossed module is a *crossed module* [21] when, furthermore, ${}^{\alpha(a)}a' = aa'a^{-1}$. We write X-Mod for the full subcategory of PX-Mod whose objects are crossed modules.

It is well-known [4] that the category X-Mod is equivalent to the category $Grpd(Grp)$ of internal groupoids in the category of groups, whereas the category PX-Mod is equivalent to the category $RG(Grp)$ of internal reflexive graphs in groups. The category X-Mod can then be seen as a subvariety of the variety PX-Mod. The same is true, much more generally, for the so-called "internal crossed modules" in any semi-abelian variety [13].

3. Distributive varieties

In this section we use the previous results to characterize distributive and arithmetical varieties.

We begin by recalling a useful result relating the congruence distributivity to a property of the commutator:

3.1. Theorem. [6] *For a modular variety $\mathcal{V}$ the following properties are equivalent:*

(1) *$\mathcal{V}$ is distributive;*
(2) *$[R, S] = R \wedge S$ for any pair of congruences R and S on any X in $\mathcal{V}$.*

Let $RR(\mathcal{V})$ denote the category of reflexive relations in $\mathcal{V}$, and let $Eq(\mathcal{V})$ be the category of equivalence relations in $\mathcal{V}$ (=congruences). Since the commutator of two congruences is always contained in the intersection $[R, S] \leq R \wedge S$, the category $RR(\mathcal{V})$ is a (full) subcategory of $RG^{+}(\mathcal{V})$. On the other hand, the category $Eq(\mathcal{V})$ is a full subcategory of $Grpd(\mathcal{V})$. Then the following result holds (the equivalence of the conditions 1. and 3. is known, and it was proved in [19]):

3.2. Proposition. *For a modular variety $\mathcal{V}$ the following conditions are equivalent:*

(1) *$\mathcal{V}$ is distributive;*
(2) *$RR(\mathcal{V}) \simeq Rg^{+}(\mathcal{V})$;*
(3) *$Eq(\mathcal{V}) \simeq Grpd(\mathcal{V})$.*

Proof. 1. $\Rightarrow$ 2. follows from Theorem 3.1.

2. $\Rightarrow$ 3. Any internal groupoid is a special reflexive graph, hence a reflexive relation, and then an equivalence relation.

3. $\Rightarrow$ 1. It suffices to prove that for any congruence relation T one has that $[T, T] = T$. Indeed, from this fact it will follow that for any R and S on X

$$R \wedge S = [R \wedge S, R \wedge S] \leq [R, S] \leq R \wedge S$$

thus $\mathcal{V}$ is distributive by Theorem 3.1.

For this, let us consider a congruence T on X and the canonical quotient $\frac{X}{[T,T]}$:

$$[T, T] \rightrightarrows X \xrightarrow{q} \frac{X}{[T,T]}$$

Since $[T, T] \leq T$, there is an arrow $g \colon \frac{X}{[T,T]} \to \frac{X}{T}$ such that $g \circ q = f$, where f is the quotient $X \to \frac{X}{T}$. Moreover, the direct image $q(T)$ of T along q is a congruence, and the universal property of the commutator gives $[q(T), q(T)] = \Delta$. This implies that there is a *connector* [3] on $q(T)$ and $q(T)$, namely an arrow $p \colon q(T) \times_{\frac{X}{[T,T]}} q(T) \to \frac{X}{[T,T]}$ such that

(1) $(x, p(x, y, z)) \in q(T)$ and $(z, p(x, y, z) \in q(T)$
(2) $p(x, y, y) = x$ and $p(x, x, y) = y$
(3) $p(x, y, p(z, u, v) = p(p(x, y, z), u, v)$

One can then costruct the canonical groupoid associated with this connector, which is defined as follows: the underlying reflexive graph is given by

$$q(T) \times_{\frac{X}{[T,T]}} q(T) \underset{p_3}{\overset{p_1}{\underset{\longleftarrow\epsilon}{\rightrightarrows}}} \frac{X}{[T,T]}$$

where $p_1(x, y, z) = x$, $p_3(x, y, z) = z$, $\epsilon(x) = (x, x, x)$. For any composable pair of triples $((x, y, z), (z, u, v))$ in $R[p_1]_{q(T)\times_{\frac{X}{[T,T]}} q(T)} R[p_3]$ one then defines

$$m((x, y, z), (z, u, v)) = (x, p(y, z, u), v)$$

and

$$i(x, y, z) = (z, p(z, y, x), x).$$

One can then check that in this way the reflexive graph just defined is equipped with a structure of internal groupoid. By assumption it is then an equivalence relation, so that the pair of arrows p_1 and p_3 are jointly monic. Now, for any (x, y) in $q(T)$ the elements (x, y, y) and (x, x, y) are both in $q(T) \times_{\frac{X}{[T,T]}} q(T)$. Since the pair of arrows p_1 and p_3 are jointly monic, it follows that $x = y$, $q(T) = \Delta$ and $[T, T] = T$, as desired. □

4. Arithmetical varieties

A variety is arithmetical if it is both Maltsev and distributive. This is the case exactly when there is a ternary term $p(x, y, z)$ satisfying the axioms $p(x, y, y) = x$, $p(x, x, y) = y$ and $p(x, y, x) = x$ (the Pixley axiom). Among the examples of such varieties let us mention Heyting algebras, boolean algebras and von Neumann regular algebras.

4.1. Proposition. *Let $\mathcal{V}$ be a modular variety. Then the following conditions are equivalent:*

(1) *$\mathcal{V}$ is arithmetical;*
(2) *$Eq(\mathcal{V})$ is a subvariety of $RG(\mathcal{V})$.*

Proof. 1. ⇒ 2. By Proposition 3.2 one knows that $Eq(\mathcal{V}) \simeq Grpd(\mathcal{V})$. The result then follows from Corollary 2.4.

2. ⇒ 1. Let us consider any reflexive relation

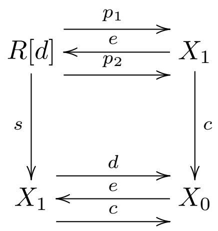
$$R \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} X$$

in $\mathcal{V}$. Since it is a subobject of the equivalence relation

$$X \times X \underset{p_2}{\overset{p_1}{\underset{\longleftarrow e}{\rightrightarrows}}} X$$

it is itself an equivalence relation, and $\mathcal{V}$ is Maltsev.

On the other hand, any internal groupoid X is a quotient of an equivalence relation canonically built from X



where s is the arrow (internally) sending a pair of arrows (α, β) with the same domain to the composite $s(\alpha, \beta) = \beta \circ \alpha^{-1}$. Now, $Eq(\mathcal{V})$ is closed in $RG(\mathcal{V})$ under quotients, hence X is an equivalence relation and the forgetful functor $W: Eq(\mathcal{V}) \to Grpd(\mathcal{V})$ is surjective on objects, hence an equivalence. By Proposition 3.2 the variety $\mathcal{V}$ is distributive, and this completes the proof. □

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