Algebras over variable theories

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Abstract

We will show that algebras over variable theories form a variety. This variety interprets the type *Self* of object oriented programming proposed by Abadi and Cardelli in [1].

1 Introduction

It is well-known that modules over variable rings form a variety. Recall that a module over a variable ring is a pair (R, A) where R is a ring and A an R-module. The resulting variety is 2-sorted, with sorts *ring* and *module*. Among operations, it has $a : ring \times module \to module$ giving the action of a ring over a module. Our aim is to show that there is a general result of this kind: algebras A over variable theories T form a variety. However, as in the case of modules, one has to fix the sorts of theories T and the resulting variety has more sorts. The reason is that algebras over arbitrarily variable theories cannot form a variety – one would need a proper class of sorts to describe this. What makes the case of modules simpler is that all variable operations (i.e., operations given by elements of a ring) are unary.

We will also show that, by iterating the procedure of taking algebras over variable theories, we arrive at a situation that one does not need more sorts to describe the resulting variety $\mathcal{V}$ of algebras over variable theories. It means that, for the resulting set Z of sorts, algebras over variable Z-sorted theories form a Z-sorted variety. This is our main result which was motivated by Abadi and Cardelli [1]. In that monograph, in connection with object oriented programming, there is introduced the type *Self*. Our variety $\mathcal{V}$ interprets this type in the sense that we take algebraic theories as types and algebras as "objects". Our fixing of sorts corresponds to a notion of subtype which is a necessary aspect of the type *Self*. The use of algebraic theories as types was advocated by Gray [3].

2 Algebraic theories

We are going to work with the usual finitary, many-sorted algebraic theories (Z, Σ, E). Here Z is a set of sorts, Σ a Z-sorted signature and E a set of equations.

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Elements of Σ are called *operation symbols*. Each operation symbol σ is equipped with an *arity*, i.e., with a finite word $(s_1, \dots, s_n)$ in the alphabet Z (determining the sorts of variables of σ) and an element $s \in Z$ (determining the sort of the value of σ). Notation:

$$\sigma : s_1 \times s_2 \times \cdots \times s_n \to s\,.$$

We choose a "standard" Z-sorted set V of variables, which means that V_s is denumerable for each sort $s \in Z$. The set P_Σ of *terms*

$$\tau : s_1 \times s_2 \times \cdots \times s_n \to s$$

over V is defined in the usual way. An *equation* is a pair (τ_1, τ_2) of terms of the same value sort.

A (Z, Σ, E)-algebra is defined in the usual way as a Z-sorted set $A = (A_s)_{s \in Z}$ together with operations

$$\sigma_A : A_{s_1} \times A_{s_2} \times \cdots \times A_{s_n} \to A_s$$

for each $\sigma \in \Sigma$ of arity $\sigma : s_1 \times s_2 \times \cdots \times s_n \to s$ satisfying all equations from E. We have homomorphisms $f : A \to B$ of (Z, Σ, E)-algebras and the category $\mathrm{ALG}(Z, \Sigma, E)$ of (Z, Σ, E)-algebras. This category corresponds to the variety determined by (Z, Σ, E). We get the forgetful functor

$$U_{(Z,\Sigma,E)} : \mathrm{ALG}(Z, \Sigma, E) \to \mathrm{SET}^Z$$

to the category of Z-sorted sets.

We will need the concept of a morphism of theories

$$F : (Z, \Sigma, E) \to (Z', \Sigma', E')\,.$$

It assigns to each sort $s \in Z$ a sort $F(s) \in Z'$ and to each operation $\sigma \in \Sigma$ a term $F(\sigma) \in P_{\Sigma'}$ such that whenever

$$\sigma : s_1 \times s_2 \times \cdots \times s_n \to s$$

then

$$F(\sigma) : F(s_1) \times F(s_2) \times \cdots \times F(s_n) \to F(s)\,.$$

Moreover, for each equation (τ_1, τ_2) from E, the equation $(F(\tau_1), F(\tau_2))$ is a consequence of E'.

Any theory morphism $F : (Z, \Sigma, E) \to (Z', \Sigma', E')$ induces the functor

$$\mathrm{ALG}(F) : \mathrm{ALG}(Z', \Sigma', E') \to \mathrm{ALG}(Z, \Sigma, E)$$

of the corresponding categories of algebras as follows

$$\mathrm{ALG}(F)(A)_s = A_{F(s)}$$

and

$$\sigma_{\mathrm{ALG}(F)(A)} = F(\sigma)_A \,.$$

If both the theories (Z, Σ, E) and (Z', Σ', E') are Z-sorted (i.e., $Z = Z'$) and F is the identity map on sorts then the diagram

$$\begin{tikzcd}
\mathrm{ALG}(Z,\Sigma',E') \arrow[rr, "\mathrm{ALG}(F)"] \arrow[dr, "U_{(Z,\Sigma',E')}"'] & & \mathrm{ALG}(Z,\Sigma,E) \arrow[dl, "U_{(Z,\Sigma,E)}"] \\
& \mathrm{SET}^Z &
\end{tikzcd}$$

commutes. We will call such morphisms of theories concrete.

A variety can be presented by an algebraic theory in many ways. There is a canonical one which takes all terms as operation symbols. We are going to work with such canonical theories only and, moreover, we will present them as categories. This approach was proposed by F. W. Lawvere in [4, 5] in the one-sorted case. The many-sorted case is completely analogous (see [2]).

Definition 2.1. A small category T equipped with a choice of finite products is said to be an *(algebraic) theory.*

We will denote by $ob(T)$ the set of objects of T. The choice of finite products means that for any objects $t_1, \dots, t_n$ of T, where $n = 0, 1, \dots$, there is given an object $t_1 \times \cdots \times t_n$ of T together with projections

$$p_i : t_1 \times \cdots \times t_n \to t_i, \qquad i = 1, \dots, n,$$

such that for any object t of T and any finite family of morphisms

$$f_i : t \to t_i, \qquad i = 1, \dots, n,$$

there is a unique morphism $f : t \to t_1 \times \cdots \times t_n$ such that $f_i = p_i f$ for $i = 1, \dots, n$. In particular, when $n = 0$ we get the terminal object $\mathbf{1}$ of T.

It is easy to assign an algebraic theory T (in the sense of 2.1) to every equationally presented algebraic theory (Z, Σ, E) – it suffices to take for T the dual of the category of free (Z, Σ, E)-algebras on finitely many generators (see [2], 3.16). On the other hand, having an algebraic theory T, we can reconstruct its equationally presented algebraic theory (Z, Σ, E) as follows: $Z = ob(T)$, morphisms $f : t_1 \times \cdots \times t_n \to t$ of T are operation symbols and equations are given by equalities of morphisms valid in the category T.

The category SET of sets and functions is considered to be equipped with a choice of finite products: $t_1 \times \cdots \times t_n$ is the usual cartesian product of the sets $t_1, \dots, t_n$ and the projections are the usual projections from the cartesian product. However, SET is not a theory because this category is not small.

For an algebraic theory T, T-algebras and their homomorphisms are defined as follows.

Definition 2.2. For a theory T, a functor $A : T \to \mathrm{SET}$ preserving the chosen finite products is called an *algebra* of T. A natural transformation $\varphi : A \to A'$ is called a *homomorphism* of the algebras A, A' of T. The category of algebras of T and their homomorphisms is denoted by $\mathrm{ALG}(T)$.

Of course, our category of algebras is isomorphic with the previous one given by the equational theory corresponding to T. For each algebra $A : T \to \mathrm{SET}$, $\langle A(t)\rangle_{t\in T}$ is the underlying many-sorted set and $A(h) : A(t_1) \times \cdots \times A(t_n) \to A(t)$, for $h : t_1 \times \cdots \times t_n \to t$ are the operations.

We get the forgetful functors $U_T : \mathrm{ALG}(T) \to \mathrm{SET}^{ob(T)}$ to the category of sets sorted by means of objects of T.

Definition 2.3. A functor $F : T \to T'$ between theories which preserves the chosen finite products is said to be a *theory morphism*. The category of theories and theory morphisms is denoted by $\mathcal{T}$.

Preserving the chosen finite products means that

$$F(t_1 \times \cdots \times t_n) = F(t_1) \times \cdots \times F(t_n)$$

on the chosen product objects of T and T', and also that the chosen projections are exactly preserved by F.

Our definition corresponds to the equational one explained above.

Any theory morphism $F : T \to T'$ induces the functor $\mathrm{ALG}(F) : \mathrm{ALG}(T') \to \mathrm{ALG}(T)$ of the corresponding varieties as follows

$$\mathrm{ALG}(F)(A) = A \cdot F \text{ and } \mathrm{ALG}(F)(\varphi) = \varphi F.$$

If F is the identity on objects then the diagram

$$\begin{tikzcd}
\mathrm{ALG}(T') \arrow[rr, "\mathrm{ALG}(F)"] \arrow[dr, "U_{T'}"'] & & \mathrm{ALG}(T) \arrow[dl, "U_T"] \\
& \mathrm{SET}^{ob(T)} &
\end{tikzcd}$$

commutes. We will call such theory morphisms *concrete*.

3 Algebras over variable theories

For each theory T, we consider the category ALG_T whose objects are pairs (F, A) where $F : T \to S$ is a concrete theory morphism and $A : S \to \mathrm{SET}$ an algebra of S. Morphisms $(F_1, A_1) \to (F_2, A_2)$ are pairs (H, φ) where $H : S_1 \to S_2$ is a concrete theory morphism such that $H \cdot F_1 = F_2$ and $\varphi : A_1 \to A_2 \cdot H$ is a homomorphism of S_1-algebras. This can be pictured as follows

$$\begin{tikzcd}
T \arrow[r, "F_1"] \arrow[dr, "F_2"'] & S_1 \arrow[r, "A_1"] \arrow[d, "H"'] \arrow[d, phantom, shift left=3ex, "\Downarrow\varphi"] & \mathrm{SET} \\
& S_2 \arrow[ur, "A_2"'] &
\end{tikzcd}$$

Example 3.1. The category SET of sets is a variety given by the one-sorted equational theory with no operation symbols. The corresponding algebraic theory T_0 is the dual of the full subcategory of SET consisting of finite cardinals. The cardinal 1 corresponds to the only sort in the equational presentation, the cardinal n gives the sort of n-tuples of elements. It is easy to see that concrete theory morphisms $T_0 \to S$ precisely correspond to one-sorted algebraic theories S. Hence ALG_{T_0} is the category of algebras of one-sorted algebraic theories.

We are going to find a theory αT whose category of algebras is isomorphic to ALG_T. For instance, in Example 3.1, we get a (many-sorted) theory whose algebras are precisely algebras of one-sorted algebraic theories.

We will construct αT equationally, i.e., by giving its equational presentation $\bar{\alpha}T$.

The set of sorts for $\bar{\alpha}T$ is the disjoint union

$$(\mathrm{ob}(T) \times \mathrm{ob}(T)) \cup \mathrm{ob}(T)\,.$$

We will denote the elements of $ob(T) \times ob(T)$ as (s,t) and the elements of $ob(T)$ as $\tilde{t}$ (which corresponds to $t \in ob(T)$). The subset of sorts $\mathrm{ob}(T)$ will provide the embedding of T into αT and the subset $\mathrm{ob}(T) \times \mathrm{ob}(T)$ will be used to encode concrete theory morphisms $T \to S$ into αT-algebras.

Our goal is to associate to each $\bar{\alpha}T$-algebra M a pair (F,A) of ALG_T, $F : T \to S$, such that two properties hold: (1) The set $S(t_1,t_2) = \hom_S(t_1,t_2)$ of morphisms $t_1 \to t_2$ in S is the underlying set $M(t_1,t_2)$ for the sort (t_1,t_2) of $\bar{\alpha}T$. (2) The underlying set $A(t)$ of S-algebra A for the sort t is the underlying set $M(\tilde{t})$ for the sort $\tilde{t}$ of $\bar{\alpha}T$.

Therefore, the presentation $\bar{\alpha}T$ has the following operations:

(1) $f^* : \mathbf{1} \to (s,t)$ for each $f : s \to t$ of T,

(2) $m_{r,s,t} : (r,s) \times (s,t) \to (r,t)$ for each triple $r,s,t \in \mathrm{ob}(T)$,

(3) $a_{s,t} : (s,t) \times \tilde{s} \to \tilde{t}$ for each pair $s,t \in \mathrm{ob}(T)$,

(4) $\pi_{u,s,t} : (u,s) \times (u,t) \to (u, s \times t)$ for each $r,s,t \in ob(T)$,

(5) $\tilde{\pi}_{s,t} : \tilde{s} \times \tilde{t} \to \widetilde{s \times t}$ for each $s,t \in ob(T)$.

The role of these four kinds of operations is: (1) makes F a concrete theory morphism $T \to S$ by taking morphisms of T as nullary operations in the added sorts; (2) is to provide the composition of morphisms in S; (3) is to make A into an S-algebra by showing how the operations of S are to act, (4) guarantees that $(u, s) \times (u, t) = (u, s \times t)$ and (5) that $\tilde{s} \times \tilde{t} = \widetilde{s \times t}$.

There are some obvious equations which have to be satisfied to complete the goal. Once these are stated we will have given a set of sorts, a set of operation symbols, and a set of equations, completing the equational presentation of the variety ALG_T and hence of the corresponding theory αT.

To begin, the composition has to be associative and $id_s^* : \mathbf{1} \to (s, s)$ for $s \in \mathrm{ob}(T)$ has to act as the identity for composition,

$$\begin{aligned} m_{r,s,u}(x, m_{s,t,u}(y, z)) &= m_{r,t,u}(m_{r,s,t}(x, y), z)\,, \\ m_{s,s,t}(id_s^*, y) &= y\,, \\ m_{s,t,t}(y, id_t^*) &= y\,. \end{aligned}$$

Second, the passage $(\ )^*$ has to preserve the compositions as well,

$$m_{r,s,t}(f^*, g^*) = (f \cdot g)^*.$$

Third, the passage $(\ )^*$ has to preserve finite products,

$$\begin{aligned} m_{u,s\times t,s}(\pi_{u,s,t}(x, y), p_1^*) &= x\,, \\ m_{u,s\times t,s}(\pi_{u,s,t}(x, y), p_2^*) &= y\,, \\ \pi_{u,s,t}(m_{u,s\times t,s}(z, p_1^*), m_{u,s\times t,s}(z, p_2^*)) &= z\,. \end{aligned}$$

Fourth, where for each morphism $f : s \to t$ of T, $\tilde{f} : \tilde{s} \to \tilde{t}$ denotes the operation given by

$$a_{s,t}(f^*, x) = \tilde{f}(x)\,,$$

we have

$$\begin{aligned} \tilde{p}_1 \cdot \tilde{\pi}_{s,t}(x, y) &= x\,, \\ \tilde{p}_2 \cdot \tilde{\pi}_{s,t}(x, y) &= y\,, \\ \tilde{\pi}_{s,t}(\tilde{p}_1(z), \tilde{p}_2(z)) &= z\,. \end{aligned}$$

Finally, the application $a_{s,t}$ has to provide the above-mentioned S-algebra. This is accomplished via the equations

$$\begin{aligned} a_{s,s}(id_s^*, x) &= x\,, \\ a_{s,t}(x, a_{r,s}(y, z)) &= a_{r,t}(m_{r,s,t}(y, x), z)\,, \end{aligned}$$

which are the usual equations relating application to identities and composition.

The verification that the category of $\bar{\alpha}T$-algebras is isomorphic to ALG_T is straightforward.

Proposition 3.2. *For each T in $\mathcal{T}$, there is αT in $\mathcal{T}$ such that* $\mathrm{ALG}(\alpha T)$ *is isomorphic to* ALG_T*.*

Proof. Above, we have described the equational presentation $\bar{\alpha}T$ of a finitary many-sorted variety such that ALG_T is isomorphic to this variety. Then the theory αT is the dual of the category of free $\bar{\alpha}T$-algebras over finitely many variables. As we have explained before, $\mathrm{ALG}_T \cong \mathrm{ALG}(\alpha T)$. □

Remark 3.3. We obtain a theory embedding $J_T : T \to \alpha T$ which is, on the level of equational presentations, determined by

$$t \longmapsto \tilde{t} \text{ and } f \longmapsto \tilde{f}.$$

In the language of theories, $\tilde{t}$ denotes the free algebra over a one-element many-sorted set having its only element of the sort $\tilde{t}$ and $\tilde{f} : \tilde{t} \to \tilde{s}$ is the algebra homomorphism induced by $f : s \to t$.

4 Main result

The assignment α is functorial – it yields a functor $\alpha : \mathcal{T} \to \mathcal{T}$ as follows. Let $G : T_1 \to T_2$ be a theory morphism. Then

$$\alpha(G)(s,t) = (Gs, Gt)$$

and

$$\alpha(G)(\tilde{s}) = \widetilde{G(s)}\,.$$

The action on morphisms is clear.

$G : T_1 \to T_2$ also induces the functor

$$\mathrm{ALG}_G : \mathrm{ALG}_{T_2} \to \mathrm{ALG}_{T_1}$$

by means of

$$\mathrm{ALG}_G(F, A) = (FG, A)$$

and

$$\mathrm{ALG}_G(H, \varphi) = (H, \varphi)\,.$$

Lemma 4.1. *For each theory morphism $G : T_1 \to T_2$, the square*

$$\begin{tikzcd}
\mathrm{ALG}_{T_2} \arrow[r, "\mathrm{ALG}_G"] \arrow[d, "\cong"'] & \mathrm{ALG}_{T_1} \arrow[d, "\cong"] \\
\mathrm{ALG}(\alpha T_2) \arrow[r, "\mathrm{ALG}(\alpha G)"'] & \mathrm{ALG}(\alpha T_1)
\end{tikzcd}$$

commutes.

Proof is a straightforward verification.

We can now iterate the construction αT to get a least fixed point ζT. There is the theory embedding $J_T : T \to \alpha T$ discussed in the previous remark. We may apply the construction again, now to αT, to obtain $\alpha^2 T = \alpha\alpha T$ with the theory embedding $\alpha J_T : \alpha T \to \alpha^2 T$. This begins a sequence of theory embeddings

$$T \xrightarrow{J_T} \alpha T \xrightarrow{\alpha J_T} \alpha^2 T \xrightarrow{\alpha^2 J_T} \cdots .$$

The colimit of this sequence, which exists because the category of small categories equipped with a choice of finite products is cocomplete, will be denoted by ζT and the components of the colimit cocone will be denoted by $I_n : \alpha^n T \to \zeta T$.

Theorem 4.2. *For each T in $\mathcal{T}$, $\mathrm{ALG}(\zeta T)$ is isomorphic to $\mathrm{ALG}_{\zeta T}$.*

Proof. Applying the functor $\mathrm{ALG}(-)$ to the colimit cocone $I_n : \alpha^n T \to \zeta T$, we obtain a cone of functors,

$$\mathrm{ALG}(I_n) : \mathrm{ALG}(\zeta T) \to \mathrm{ALG}(\alpha^n T)$$

which is evidently a limit cone. Hence

$$\mathrm{ALG}(\zeta T) \cong \lim_n \ \mathrm{ALG}(\alpha^n T) .$$

Analogously, applying the functor ALG_-, we get a cone

$$\mathrm{ALG}_{I_n} : \mathrm{ALG}_{\zeta T} \to \mathrm{ALG}_{\alpha^n T}$$

which is a limit cone too. Hence

$$\mathrm{ALG}_{\zeta T} \cong \lim_n \ \mathrm{ALG}_{\alpha^n T} \ .$$

Since, by Lemma 4.1, the squares

$$\begin{tikzcd}
\mathrm{ALG}_{\alpha^n T} \arrow[r, "\mathrm{ALG}_{\alpha^{n-1} J_T}"] \arrow[d, "\cong"'] & \mathrm{ALG}_{\alpha^{n-1} T} \arrow[d, "\cong"] \\
\mathrm{ALG}(\alpha^{n+1} T) \arrow[r, "\mathrm{ALG}(\alpha^n J_T)"'] & \mathrm{ALG}(\alpha^n T)
\end{tikzcd}$$

commute, we get $\mathrm{ALG}_{\zeta T} \cong \mathrm{ALG}(\zeta T)$ $\square$

Remark 4.3. The theorem just proved provides an interpretation of the type *Self* proposed by Abadi and Cardelli [1]. In object oriented programming one works with types, objects and methods. We interpret algebraic theories as types and algebras as objects. Hence any object has a type and algebraic operations present in an algebra are understood as methods available in objects. Further, we interpret concrete theory morphisms $F : T \to S$ as making S a subtype of T. This means that methods h available in T are replaced (overriden) by methods $F(h)$ in S; moreover, S can have more methods than T. The crucial subsumption property (see [1]) is then satisfied: any S-algebra $A : S \to \mathrm{SET}$ of a subtype induces a T-algebra $A \cdot F : T \to \mathrm{SET}$. A typical feature of object oriented programming is that objects are organized in a hierarchy following their types. Moreover, a method available in an object A need not to be stored in A but it may be inherited from an object B being lower in the hierarchy. In our interpretation, B is a reduct of the algebra A to a smaller theory.

It is clear from our description that, in object oriented programming, one works with objects of variable types but one has to know the type of an object. This is the role of the type Self whose informal semantics is a collection of pairs (C, c) with C a subtype of *Self* and c an object of that subtype. Abadi and Cardelli define *Self* in the second order lambda calculus. Our algebraic theory ζT captures the just mentioned semantics of *Self*: ζT-algebras are pairs (F, A) where F is a subtype of ζT and A an algebra of this subtype. Hence, in our interpretation, any type can be extended to a type *Self*. Further work is required to integrate our algebraic interpretation of *Self* with other aspects of the semantics given in [1].

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