Good colimits, weak factorization systems and model categories

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Definition 1. Given a class $\mathcal{X}$ of morphisms in a category $\mathcal{K}$, we define $\mathcal{X}$-*cellular* morphisms as transfinite compositions of pushouts of morphisms from $\mathcal{X}$ and $\mathcal{X}$-*cofibrations* as retracts of $\mathcal{X}$-cellular morphisms.

We use the notation $\mathrm{cell}(\mathcal{X})$ and $\mathrm{cof}(\mathcal{X})$ for the classes of $\mathcal{X}$-cellular morphisms and $\mathcal{X}$-cofibrations. $\mathrm{Po}(\mathcal{X})$ denotes the class of pushouts of morphisms from $\mathcal{X}$.

If $\mathcal{K}$ is a locally presentable category and $\mathcal{X}$ a set then $(\mathrm{cof}(\mathcal{X}), \mathcal{X}^{\square})$ is a functorial weak factorization system where $\mathcal{X}^{\square}$ consists of morphisms having the right lifting property for each morphism in $\mathcal{X}$.

In this case we say that $(\mathcal{K}, \mathrm{cof}(\mathcal{X}))$ is a *combinatorial* category.

An object K is called $\mathcal{X}$-*cofibrant* if the unique morphism $0 \to K$ from the initial object is an $\mathcal{X}$-cofibration.

Let $\mathcal{K}$ be a Grothendieck category and $\mathcal{S}$ a class of objects in $\mathcal{K}$. An $\mathcal{S}$-*monomorphism* is defined as a monomorphism with the cokernel in $\mathcal{S}$.

If $(\mathcal{A}, \mathcal{B})$ is a small generated cotorsion pair then $(\mathcal{K}, \mathcal{A}\text{-Mono})$ is a combinatorial category.

Cofibrant objects are precisely objects from $\mathcal{A}$.

If $\mathcal{S} \subseteq \mathcal{K}$ is closed under retracts then $(\mathcal{K}, \mathcal{S}\text{-Mono})$ is a combinatorial category iff $\mathcal{S}$ is deconstructible (Saorín, Šťovíček).

A functor $F : \mathcal{K} \to \mathcal{L}$ between combinatorial categories is called *combinatorial* if it preserves colimits and cofibrations.

Any combinatorial functor is a left adjoint.

COMB will denote the 2-category of combinatorial categories, combinatorial functors and natural transformations.

COMB is legitimate but not locally small and is equipped with the 2-functor $U : \mathbf{COMB} \to \mathbf{LOC}$ to the 2-category **LOC** of locally presentable categories, colimit preserving functors and natural transformations.

Let **CMOD** denote the 2-category of combinatorial model categories, left Quillen functors and natural transformations.

CMOD is legitimate but not locally small and is equipped with the 2-functors $V_1, V_2 : \mathbf{CMOD} \to \mathbf{COMB}$, $V_1(\mathcal{K}) = (\mathcal{K}, \mathcal{C})$ and $V_2(\mathcal{K}) = (\mathcal{K}, \mathcal{C}_0)$ where $\mathcal{C}$ is the class of cofibrations and $\mathcal{C}_0$ is the class of trivial cofibrations.

A poset P is good if it is well-founded and has the least element $\perp$.

Well-ordeder sets and shape posets for pushouts are good.

An element $x \in P$ is *isolated* if there is a top element x^- strictly below x.

A non-isolated element distinct from $\perp$ is called *limit*.

A *good* diagram $D : P \to \mathcal{K}$ is such that Dx is a colimit of the restriction of D on elements strictly below x for each limit x.

The *composition* of D is the component $\delta_\perp$ of the colimit cocone.

Links of D are morphisms $D(x^- \to x)$ for x isolated.

Proposition 1. (Lurie) Let $\mathcal{X}$ be a class of morphisms in a cocomplete category $\mathcal{K}$. Then the composition of a good diagram with links in $\mathrm{Po}(\mathcal{X})$ belongs to $\mathrm{cell}(\mathcal{X})$.

Corollary 1. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$. Then the composition of a good diagram with links in $\mathcal{L}$ belongs to $\mathcal{L}$.

A good poset is κ-*good* if all its initial segments $\downarrow x$ have cardinality $< \kappa$.

Theorem 1. Let $\mathcal{K}$ be a cocomplete category, κ a regular cardinal and $\mathcal{X}$ a class of morphisms with κ-presentable domains. Then every $\mathcal{X}$-cellular morphism is a composition of a κ-good κ-directed diagram with links in $\mathrm{Po}(\mathcal{X})$.

This result may be called a *fat small object argument* because it replaces a thin transfinite composition containing large objects by a fat good composition of small objects.

Corollary 2. Let $\mathcal{K}$ be a cocomplete category, κ a regular cardinal and $\mathcal{X}$ a class of morphisms between κ-presentable objects. Then any $\mathcal{X}$-cofibrant object is a κ-filtered colimit of κ-presentable $\mathcal{X}$-cofibrant objects.

For $\mathcal{X}$-cellular objects, the claim follows from Theorem 1. The extension to $\mathcal{X}$-cofibrant objects uses [Makkai, Paré].

If $(\mathcal{A}, \mathcal{B})$ is a finitely generated cotorsion pair of R-modules then each object from $\mathcal{A}$ is a filtered colimit of finitely presentable objects from $\mathcal{A}$ (Angereli-Hügel, Trlifaj).

Theorem 2. Let $\mathcal{K}$ be a locally κ-presentable category, κ an uncountable regular cardinal and $\mathcal{X}$ a class of morphisms between κ-presentable objects. Then any $\mathcal{X}$-cofibrant object is a κ-good, κ-directed colimit of κ-presentable $\mathcal{X}$-cofibrant objects.

In this situation, we say that $(\mathcal{K}, \mathsf{cof}(\mathcal{X}))$ is κ-combinatorial.

$\mathsf{cof}_\kappa(\mathcal{X})$ will denote $\mathcal{X}$-cofibrations between κ-presentable objects.

Theorem 2 follows from Theorem 1 by using the result of Lurie:

$$\mathsf{cof}(\mathcal{X}) = \mathsf{cell}(\mathsf{cof}_\kappa(\mathcal{X}))$$

Let $\mathcal{P}$ be the class of projective R-modules. $\mathcal{P}$-monomorphisms are cofibrantly generated by $0 \to R$. Cofibrant objects are precisely projective modules. Following Theorem 2, every projective module is an $\aleph_1$-good $\aleph_1$-directed colimit of $\aleph_1$-presentable projective modules where all links are $\mathcal{P}$-monomorphisms. This implies the theorem of Kaplansky: every projective module is a coproduct of countably generated projective modules. This also shows that Theorem 2 is not valid for $\aleph_0$.

The 2-category **LOC** has all PIE-limits, i.e., products, inserters and equifiers (Makkai, Paré). Consequently, its has all lax limits and all pseudolimits. All these limits are calculated in **CAT**.

Theorem 3. **COMB** has PIE-limits preserved by U.

Corollary 3. (Lurie) Let $\mathcal{K}$ be a combinatorial category and $\mathcal{C}$ a small category. Then $\mathcal{K}^{\mathcal{C}}$ is combinatorial (with respect to pointwise cofibrant morphisms).

Corollary 4. (Šťovíček) Let $\mathcal{C}$ be a deconstructible class in a Grothendieck category $\mathcal{K}$. Then $\mathbf{C}(\mathcal{C})$ is deconstructible in $\mathbf{C}(\mathcal{K})$.

$\mathbf{C}(\mathcal{K})$ denotes the category of complexes in $\mathcal{K}$.

While Šťovíček uses generalized Hill lemma, we replace it by good colimits.

Corollary 5. Let $T : \mathcal{K} \to \mathcal{K}$ be a colimit preserving monad on a combinatorial category $\mathcal{K}$. Then $\mathrm{Alg}(T)$ is combinatorial.

Corollary 6. (Becker) Let $T : \mathcal{K} \to \mathcal{K}$ be a colimit preserving monad on a Grothendieck category $\mathcal{K}$ and $\mathcal{C}$ a deconstructible class in $\mathcal{K}$. Then the class of T-algebras with the underlying object in $\mathcal{C}$ is deconstructible.

Theorem 4.(Barwick) **CMOD** has lax limits preserved by $V_1, V_2 : \mathbf{CMOD} \to \mathbf{COMB}$.

Problem. Does **CMOD** have PIE-limits? Equivalently, does **CMOD** have pseudopullbacks?

Proposition 2. V_2 preserves PIE-limits (pseudopullbacks).

V_2 has a left adjoint sending $(\mathcal{K}, \mathcal{X})$ to $(\mathcal{K}, \mathcal{X}, \mathcal{X})$.

We have an example of a pseudopullback in **CMOD** which is not preserved by V_1.

Let $\mathcal{K}$ be the standard model category of simplicial sets. Let $t : 0 \to 1$ and $\mathcal{L}$ be the model structure on simplicial sets where $\mathrm{cof}(\{t\})$ is the class of cofibrations and any morphism is a weak equivalence. Let $\mathcal{K}_t$ be the trivial model structure in $\mathcal{K}$ (any morphism is a trivial cofibration) and $\mathcal{K}_d$ be the discrete model structure (any morphism is a trivial fibration). Then the intersection of $\mathcal{K}$ and $\mathcal{L}$ over $\mathcal{K}_t$ is $\mathcal{K}_d$. Since $V_1(\mathcal{L}) \subseteq V_1(\mathcal{K})$, this intersection is not preserved by V_1.