PURE MORPHISMS IN PRO-CATEGORIES

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ABSTRACT. Pure epimorphisms in categories pro-$\mathcal{C}$, which essentially are just inverse limits of split epimorphisms in $\mathcal{C}$, were recently studied by J. Dydak and F. R. R. del Portal in connection with Borsuk's problem of descending chains of retracts of ANRs. We prove that pure epimorphisms are regular epimorphisms whenever $\mathcal{C}$ has weak finite limits, or pullbacks, or copowers. This improves results of the above paper, and results of the present authors on pure subobjects in accessible categories. We also turn to pure monomorphisms in pro-$\mathcal{C}$, essentially just inverse limits of split monomorphisms in $\mathcal{C}$, and prove that they are regular monomorphisms whenever $\mathcal{C}$ has finite products or pushouts.

1. Introduction

There are two independent sources of the concepts of purity for monomorphisms and epimorphisms: one are the pro-categories

$$\text{pro-}\mathcal{C} = \text{free completion of } \mathcal{C} \text{ under inverse limits}$$

introduced by M. Artin and B. Mazur [5], and used in shape theory and proper homotopy theory. Here, pure monomorphisms were recently introduced by J. Dydak and F. R. R. del Portal [9] in connection with a famous open problem of K. Borsuk [6] on descending chains of retracts. And the other source are the ind-categories

$$\text{ind-}\mathcal{C} = \text{ free completion of } \mathcal{C} \text{ under directed colimits}$$

introduced by M. Artin, A. Grothendieck and J. L. Verdier [4]. Here one often concentrates on small categories $\mathcal{C}$: then the corresponding ind-categories are precisely the *finitely accessible categories* of M. Makkai and R. Paré, see [12] or [2]. The motivation of the theory of accessible categories is model-theoretical, and pure subobjects have a strong

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motivation in model theory. The classical concept is that of a pure submodule of a module B: it is a submodule $h : A \hookrightarrow B$ such that for every module C the morphism $h \otimes C : A \otimes C \to B \otimes C$ is a monomorphism. This is equivalent to the following property:

(1.1) every pp-formula satisfied by B is satisfied by A.

Recall that pp-formulas (positive-primitive formulas) are the formulas

$$(\exists x_1, \dots, x_n)(\psi_1 \wedge \cdots \wedge \psi_k) \quad \text{with } \psi_i \text{ atomic.}$$

This inspired model theorists to introduce pure submodels $A \hookrightarrow B$ by the property (1.1). A much more elegant categorical definition was presented by S. Fakir [10] in 1975; the present authors simplified it in [2] to the following

Definition. By a *pure monomorphism* in a category $\mathcal{K}$ is meant a morphism $h : A \to B$ such that given a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

with X and Y finitely presentable, then f factorizes through u:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright", near start] & Y \arrow[d, "v"] \arrow[dl] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

Recall that an object is finitely presentable iff its hom-functor preserves directed colimits; in categories $\mathcal{K} = \text{ind-}\mathcal{C}$ the finitely presentable objects are precisely the split subobjects of objects in $\mathcal{C}$. Therefore, pure monomorphisms in ind-$\mathcal{C}$ can be defined as above, where X and Y range through $\mathcal{C}$. Now in categories of structures, pure submodels (in the above categorical sense) are precisely those which fulfil (1.1), see [2], 5.34.

Although the definition of pure monomorphism h does not involve h being a monomorphism, in ind-categories this actually follows, see [2], 2.29. However, h need not be a regular monomorphism, not even a strong monomorphism. In fact, we present below an example of a finitely complete, countable category $\mathcal{C}$ such that ind-$\mathcal{C}$ contains a pure monomorphism which is not strong. This improves an example of [1] which was based on the same idea, but was not correct. On the other hand, the implication

$$\text{pure monomorphism} \quad \Rightarrow \quad \text{regular monomorphism}$$

holds in all categories ind-$\mathcal{C}$ such that

(a) $\mathcal{C}$ has weak finite colimits, or
(b) $\mathcal{C}$ has pushouts, see [1], or
(c) ind-$\mathcal{C}$ has products, see [2] (Proposition and Remark 2.31).

The first case (a) is one of the major results of the present paper. This has been known for small categories $\mathcal{C}$ because weak finite limits imply that ind-$\mathcal{C}$ has products, see [7], which reduces (a) to (c). However, for large categories we cannot conclude that ind-$\mathcal{C}$ has products: for example ind-*Ord* does not have a terminal object although *Ord* has (finite) colimits. The case (c) improves somewhat the recent result of J. Dydak and F. R. R. Portal [9] who assume in their Theorem 3.15 that $\mathcal{C}$ has products – from that it follows that ind-$\mathcal{C}$ has products, see [4].

The concept of a pure epimorphism is classical in modules; in more general situations it was independently introduced by P. Rothmaler [13] (who did not give it any name, and did not study it any further), the above authors of [9] (who used the unfortunate name “strong” instead of “pure”) and the present authors [3]. The formulation of [13] is model-theoretic, the following formulation was used in [9]; the formulation in [3] is formally different, but this was proved to be equivalent in 3.9 of [9].

Definition. By a *pure epimorphism* is a category $\mathcal{K}$ is meant a morphism $h : A \to B$ such that for every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

with X and Y finitely presentable v factorizes through h:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \arrow[dl] \arrow[d, phantom, shift right=3ex, "\circlearrowright"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

This is, of course, not a categorical dual. Thus, it does not surprise that the concept does not behave dually. For example, every pure epimorphism in an ind-category is strong, see [9], 3.13; in fact, it is "almost" a regular epimorphism: we prove below that h is a collective coequalizer of a set of parallel pairs. Consequently, the implication

$$\text{pure epimorphism} \quad \Rightarrow \quad \text{regular epimorphism}$$

holds in ind-$\mathcal{C}$ whenever $\mathcal{C}$ has finite coproducts.

What *is* surprising is how many dual results actually hold. For example, pure monomorphisms of ind-$\mathcal{C}$ can "almost" be defined as precisely the directed colimits of split monomorphisms: it suffices to assume that $\mathcal{C}$ have weak pushouts. "Dually", pure epimorphisms of ind-$\mathcal{C}$ can "almost" be defined as precisely the directed colimits of split epimorphisms: it suffices that $\mathcal{C}$ have weak pullbacks. Also, of the above cases (a)–(c) for the implication pure $\Rightarrow$ regular, two can be "dualized": In categories ind-$\mathcal{C}$ we have

$$\text{pure epimorphism} \quad \Rightarrow \quad \text{regular epimorphism}$$

whenever

(a) $\mathcal{C}$ has pullbacks, see [3]

or

(b) ind-$\mathcal{C}$ has coproducts, see Corollary 3.6 below.

However, we do not know whether weak finite limits in $\mathcal{C}$ are sufficient. Finally, in abelian ind-categories there is an even stronger connection, (see [3]): pure monomorphisms are precisely the kernels of pure epimorphisms, and pure epimorphisms are precisely the cokernels of pure monomorphisms.

Let us come back to the honest-to-goodness duality: in the title of our paper we speak about pro-categories. For writing the proofs (which often refer to previous results, all about ind-categories) we decided to formulate the body of our paper in ind-categories. We never assume, for results in $\mathcal{K} = \text{ind-}\mathcal{C}$, that $\mathcal{C}$ be small. Thus all our results easily dualize to pro-categories. For the convenience of the reader we explicitly state these results (and open problems) in the summary.

As another convenience, let us mention here explicitly the usual hierarchy of monomorphisms $m : A \to B$ in categories:

(1) m is called *split* if there exists $e : B \to A$ with $em = \text{id}$;

(2) m is called *regular* if it is an equalizer of a parallel pair. This is weaker than split (if m is split, then it is an equalizer of id_B and me);

(3) m is called a *strong monomorphism* if it is a monomorphism and in every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

where f is an epimorphism, there exists a diagonal

$$\begin{tikzcd}[row sep=large, column sep=large] X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \arrow[dl] \arrow[l, phantom, shift left=6, "\circlearrowright"] \\ A \arrow[r, "m"'] & B \arrow[u, phantom, shift left=6, "\circlearrowright"] \end{tikzcd}$$

making both triangles commutative. This is weaker than regular: if m is an equalizer of $p, q : B \to C$ then $pvf = qvf$ implies $pv = qv$, thus, $v = md$ for some $d : Y \to A$. The equality $v = df$ follows since m is a monomorphism.

Let us remark that by [2], 2.30 pure monomorphisms in ind-$\mathcal{C}$ are closed under directed colimits (in the arrow-category of ind-$\mathcal{C}$). Since every split monomorphism is clearly pure, a directed colimit of split monomorphism is always pure.

2. Pure Monomorphisms

Remark 2.1. Our first result uses the weakest assumption: we work with categories having *weak pushouts*, i.e., every span has a commutative square which has the factorization property of pushouts except that the factorization morphisms need not be unique. (Analogously one defines *weak colimits* in general.)

Lemma 2.2. *If $\mathcal{C}$ has weak pushouts, then every pure monomorphism in ind-$\mathcal{C}$ is a directed colimit of split monomorphisms in $\mathcal{C}$.*

Proof. Given a pure monomorphism $h : A \to B$ in ind-$\mathcal{C}$, express it as a directed colimit of morphisms in $\mathcal{C}$:

$$\begin{tikzcd} A_i \arrow[r, "h_i"] \arrow[d, "a_i"'] & B_i \arrow[d, "b_i"] \\ A \arrow[r, "h"'] & B \end{tikzcd} \qquad (i \in I) \tag{2.0}$$

where i ranges through a directed poset I, and, for $i \leq j$ in I, the connecting morphisms are denoted by $a_{ij} : A_i \to A_j$ and $b_{ij} : B_i \to B_j$, respectively. For each $i \in I$ choose a morphism

$$d_i : B_i \to A \quad \text{with} \quad d_i h_i = u_i$$

and (since $B_i \in \mathcal{C}$) choose $i^* \geq i$ such that $d_i = a_{i^*} d_i$ for some $d_i^* : B_i \to D_{i^*}$. Then a_{i^*} merges $d_i^* h_i$ and a_{ii^*}:

$$a_{i^*}(d_i^* h_i) = d_i h_i = a_i = a_{i^*} a_{ii^*} \,.$$

This implies that there exists $i' \geq i^*$ such that $a_{i^* i'}$ merges $d_i^* h_i$ and a_{ii^*}, i.e., such that for $d_i' = a_{i' i^*} d_i^* : B_i \to A_{i'}$ we have a commutative triangle

$$\begin{tikzcd} A_i \arrow[r, "h_i"] \arrow[d, "a_{ii'}"'] & B_i \arrow[dl, "d_i'"] \\ A_{i'} & \end{tikzcd}$$

Form a weak pushout (P_i, p_i, q_i) of $a_{ii'}$ and h_i:

$$\begin{tikzcd} A_i \arrow[rr, "h_i"] \arrow[dd, "a_{ii'}"'] & & B_i \arrow[dl, "q_i"'] \arrow[dd, "b_{ii'}"] \\ & P_i \arrow[dr, "r_i"] & \\ A_{i'} \arrow[ur, "p_i"] \arrow[rr, "h_{i'}"'] & & B_{i'} \end{tikzcd}$$

Since $h_{i'} a_{ii'} = b_{ii'} h_i$, we obtain $r_i : P_i \to B_i$ with

(2.1) $$r_i q_i = b_{ii'} \quad \text{and} \quad r_i p_i = h_{i'} \,.$$

Since $a_{ii'} = d_i' h_i$, we further obtain $s_i : P_i \to A_{i'}$ with

(2.2) $$d_i' = s_i q_i \quad \text{and} \quad \mathrm{id}_{A_{i'}} = s_i p_i \,.$$

Let $J \subseteq I$ be a cofinal subset such that for $j, k \in J$ we have

$$j < k \quad \text{implies} \quad j' \leq k \,.$$

Define a diagram of objects P_j ($j \in J$) and connecting morphisms

$$P_j \xrightarrow{r_j} B_{j'} \xrightarrow{b_{j'k}} B_k \xrightarrow{q_k} P_k \quad \text{for all } j < k \text{ in } J.$$

This is well defined since, given $j < k < l$ in J the connecting morphisms compose due to $r_k q_k = b_{kk'}$, see (2.1):

$$\begin{tikzcd} P_j \arrow[r, "r_j"'] & B_{j'} \arrow[r, "b_{j'k}"'] \arrow[rrrr, bend left, "b_{j'l}"] & B_k \arrow[r, "q_k"'] \arrow[rr, bend left, "b_{kk'}"] & P_k \arrow[r, "r_k"'] & B_{k'} \arrow[r, "b_{k'l}"'] & B_l \arrow[r, "q_l"'] & B_l \end{tikzcd}$$

The morphisms

$$P_j \xrightarrow{r_j} B_{j'} \xrightarrow{b_{j'}} B \qquad (j \in J) \tag{2.3}$$

form a colimit of this diagram. This follows easily from (2.1) and the fact that J is a cofinal in I: given $i \in I$ find $j \in J$ with $i \leq j$, then the colimit morphism b_i (of the diagram of which we know that $B = \underset{i \in I}{colim} B_i$) factorizes through the morphism (2.3):

$$b_i = b_{j'} b_{jj'} b_{ij} = b_{j'} r_j (q_j b_{ij}) \, .$$

Consequently, the colimit cocone (2.0) yields a colimit cocone

$$\begin{tikzcd} A_{j'} \arrow[r, "p_j"] \arrow[d, "a_j"'] & P_j \arrow[d, "b_{j'} r_j"] \\ A \arrow[r, "h"'] & B \end{tikzcd} \qquad (j \in J)$$

where the morphisms p_j are split monomorphisms by (2.2). $\square$

Theorem 2.3. *In categories ind-$\mathcal{C}$ the implication*

pure monomorphism $\Rightarrow$ *regular monomorphism*

holds whenever

(a) *$\mathcal{C}$ has weak finite colimits, or*
(b) *$\mathcal{C}$ has pushouts, or*
(c) *ind-$\mathcal{C}$ has powers (in particular: whenever $\mathcal{C}$ has powers).*

Remark 2.4. (1) We only need to prove the implications in case (a): for case (b) the proof in [1] of the same result (see Proposition 2 there) is formulated for the case $\mathcal{C}$ is small, but that assumption is not needed at all. For the case (c) the proof in [2] of the same result (see Proposition 2.31 there) is formulated for the case $\mathcal{C}$ is small and ind-$\mathcal{C}$ has limits. But the assumption that $\mathcal{C}$ small is not needed at all, and no limits except powers are used in the proof. The parenthetic comment in (c) is based on the result of Artin-Grothendieck-Verdier [4] that

$$\mathcal{C} \text{ has products} \Rightarrow \text{ind-}\mathcal{C} \quad \text{has products.}$$

(2) The outstanding case (a) is known if $\mathcal{C}$ is small: then ind-$\mathcal{C}$ has products iff $\mathcal{C}$ has weak finite colimits, see [7]. Thus, for small categories (a) follows from (c). But for large $\mathcal{C}$ we have not found any proof simpler than the following one. Our proof is based on the construction of an exact completion $\mathcal{C}_{ex}$ of a category $\mathcal{C}$ with weak finite limits presented by A. Carboni and E. Vitale [8]; a similar technique is used by H. Hu in [11] where another proof of (a) for small $\mathcal{C}$ is given.

Proof of (a) in 2.3 (1) Exact completions. Recall that a category is called *exact* if it has

(i) finite limits;
(ii) (regular epi-mono)-factorizations of morphisms;
(iii) regular epimorphisms stable under pushout; and
(iv) effective equivalence relations.

The conditions (i)–(iii) are standard, we do not explain (iv) since it plays no role in our proof. Recall further that a functor $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ has weak finite limits and $\mathcal{B}$ has finite limits, is called *left covering* if for every finite diagram D in $\mathcal{A}$ there exists a weak limit L in $\mathcal{A}$ such that the unique factorization morphism $L \to \lim FD$ is a strong epimorphism. And F is called *exact* if it preserves finite limits and regular epimorphisms.

By Theorem 29 of [8] for every category $\mathcal{C}$ with weak finite limits there exists a left covering full embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{ex}$ into an exact category with the following universal property: every left covering functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{A}$ is exact, has a unique extension to an exact functor $\mathcal{C}_{ex} \to \mathcal{A}$.

We are going to use the dual concepts: coexact (category and functor), right covering (dual to left covering) and the coexact completion

$$\mathcal{C} \hookrightarrow \mathcal{C}_{coex}.$$

(2) For small categories $\mathcal{C}$ the category $\mathcal{C}_{coex}$ is also small. Since $\mathcal{C}_{coex}$ has finite colimits, ind-$\mathcal{C}_{coex}$ can be described as the full subcategory of the functor category $[\mathcal{C}^{\mathrm{op}}_{coex}, \mathbf{Set}]$ consisting of all functors preserving finite limits. It then follows from Theorem 2.7 in [6] that every object of ind-$\mathcal{C}_{coex}$ is a regular subobject of an object in ind-$\mathcal{C}$. (We do not know whether this property holds for large categories $\mathcal{C}$ in general, which would simplify our proof substantially.)

(3) Let $\mathcal{C}$ be a (not necessarily small) category with weak finite colimits. Given a pure monomorphism f in ind-$\mathcal{C}$, we will prove that f is a regular monomorphism.

Let $U : \mathcal{C}^* \hookrightarrow \mathcal{C}$ be a small, full subcategory of $\mathcal{C}$ such that

(a) every finite diagram in $\mathcal{C}^*$ has a weak limit in $\mathcal{C}$ lying in $\mathcal{C}^*$

and

(b) ind-$\mathcal{C}^*$ contains f, more precisely for

$$\widehat{U} = \text{ind-}U : \text{ind-}\mathcal{C}^* \to \text{ind-}\mathcal{C}$$

there is f_0 in ind-$\mathcal{C}^*$ with $f = \widehat{U} f_0$.

We are going to work with the coexact completions of $\mathcal{C}^*$ and $\mathcal{C}$, notation:

$$G^* : \mathcal{C}^* \hookrightarrow \mathcal{C}^*_{coex} \quad \text{and} \quad G : \mathcal{C} \hookrightarrow \mathcal{C}_{coex} \, .$$

Due to (a) above, the functor $GU : \mathcal{C}^* \hookrightarrow \mathcal{C}_{coex}$ is right covering, thus, it extends to a coexact functor

$$V : \mathcal{C}^*_{coex} \to \mathcal{C}_{coex} \quad \text{with } VG^* = GU \, .$$

We obtain a commutative square

$$\begin{tikzcd}
\text{ind-}\mathcal{C}^* \arrow[r, "\widehat{U}"] \arrow[d, "\widehat{G^*}"'] & \text{ind-}\mathcal{C} \arrow[d, "\widehat{G}"] \\
\text{ind-}\mathcal{C}^*_{coex} \arrow[r, "\widehat{V}"'] & \text{ind-}\mathcal{C}_{coex}
\end{tikzcd}$$

where $\widehat{(-)}$ abbreviates ind-$(-)$.

(4) We prove next that $\widehat{V}$ preserves regular monomorphisms. Let $m : A \to B$ be a regular monomorphism in ind-$\mathcal{C}^*_{coex}$. Express m as a directed colimit of morphisms $f_i : A_i \to B_i$ $(i \in I)$ is $\mathcal{C}^*_{coex}$ with colimit cocones (a_i), (b_i):

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[d, "a_i"'] & B_i \arrow[d, "b_i"] \\
A \arrow[r, "m"'] & B
\end{tikzcd}$$

Factorize $f_i = m_i e_i$ where $e_i : A_i \to C_i$ is an epimorphism and $m_i : C_i \to B_i$ a regular monomorphism in the (coexact!) category $\mathcal{C}^*_{coex}$. This gives rise to a directed diagram $(C_i)_{i \in I}$ (with connecting morphisms given by the obvious diagonal fill-in), together with natural transformations (e_i) and (m_i):

$$\begin{tikzcd}
A_i \arrow[r, "e_i"] \arrow[dd, "a_i"'] & C_i \arrow[r, "m_i"] \arrow[d, "c_i"] & B_i \arrow[dd, "b_i"] \\
 & C \arrow[dr, dashed] & \\
A \arrow[ur, "e"] \arrow[rr, "m"'] & & B
\end{tikzcd}$$

Form a colimit $C = colim\, C_i$ (with cocone (c_i)) and $e = colim\, e_i$. Since e_i are epimorphisms, so is e. And from $m = \underset{i\in I}{colim} f_i$ it easily follows that m factors through e (via $\underset{i\in I}{colim}\, m_i$). Since m is a regular monomorphism, this proves that e is an isomorphism. Consequently,

$$m = \underset{i\in I}{colim}\, m_i\,.$$

The functor $\widehat{V} = \text{ind-}V$ preserves this colimit:

$$\widehat{V} m = \underset{i\in I}{colim}\, V m_i\,.$$

Since V is coexact, each $V m_i$ is a regular monomorphism. And this implies that $V m_i$ is an equalizer of its cokernel pair

$$p_i, q_i : V B_i \longrightarrow D_i \qquad (i \in I) \quad \text{in} \quad \mathcal{C}_{coex}\,.$$

This defines an obvious directed diagram $(D_i)_{i\in I}$ in $\mathcal{C}_{coex}$ with natural transformations (p_i) and (q_i). Let $D = colim\, D_i$, $p = colim\, p_i$ and $q = colim\, q_i$ be colimits in ind-$\mathcal{C}_{coex}$. Then

$$\widehat{V} m \quad \text{is an equalizer of } p \text{ and } q$$

because equalizers commute with directed colimits in ind-$\mathcal{C}_{coex}$.

(5) The proof that f is a regular monomorphism.

The morphism $f_0 : A \to B$ with $f = \widehat{U} f_0$ is, obviously, a pure monomorphism in ind-$\mathcal{C}^*$. The functor $\widehat{G}^*$ preserves pure monomorphisms see [2], 2.38, therefore $\widehat{G}^* f_0$ is a pure monomorphism in ind-$\mathcal{C}^*_{coex}$. The category $\mathcal{C}^*_{coex}$ has pushouts, thus, by case (b) of our theorem (see Remark 2.4) all pure monomorphisms are regular. Consequently, if

$$g, h : \widehat{G}^* B \to X$$

is a cokernel pair of $\widehat{G}^* f_0$ in ind-$\mathcal{C}^*_{coex}$, then it follows that $\widehat{G}^* f_0$ is an equalizer of $\widehat{G}^* g$, $\widehat{G}^* h$ in ind-$\mathcal{C}^*_{coex}$. By applying (2) to the small category $\mathcal{C}^*$, we obtain an object $C \in$ ind-$\mathcal{C}^*$ and a regular monomorphism $t : X \to \widehat{G}^* C$ in ind-$\mathcal{C}^*$. Since V is coexact, $\widehat{V}$ preserves colimits, thus,

$\widehat{V}\widehat{G}^*(f_0)$ has a cokernel pair $\widehat{V}g$, $\widehat{V}h$ in ind-$\mathcal{C}_{coex}$. Since $\widehat{V}\widehat{G}^*(f_0)$ is, by (4), a regular monomorphism, this implies that $\widehat{V}\widehat{G}(f_0)$ is an equalizer of $\widehat{V}g$ and $\widehat{V}h$. Since $\widehat{V}t$ is also a regular monomorphism, by (4), the composite morphisms $\widehat{V}t\widehat{V}g$ and $\widehat{V}t\widehat{V}h$ have the same equalizer $\widehat{V}\widehat{G}(f_0)$ in ind-$\mathcal{C}^*_{coex}$:

$$\begin{tikzcd} \widehat{G}\widehat{U}A = \widehat{V}\widehat{G}^*A \arrow[r, "\widehat{V}\widehat{G}f_0", "=\widehat{G}\widehat{U}f_0"'] & \begin{matrix}\widehat{V}\widehat{G}^*B \\ \parallel \\ \widehat{G}\widehat{U}B\end{matrix} \arrow[r, shift left, "\widehat{V}(tg)"] \arrow[r, shift right, "\widehat{V}(th)"'] & \widehat{V}\widehat{G}^*C = \widehat{G}\widehat{U}C \end{tikzcd}$$

We know that G is a full embedding, thus, so is $\widehat{G} = \text{ind-}G$. Consequently, there exists a pair

$$\widehat{U}B \overset{g_0}{\underset{h_0}{\rightrightarrows}} \widehat{C} \quad \text{in ind-}\mathcal{C}$$

with $\widehat{G}g_0 = \widehat{V}(tg)$ and $\widehat{G}h_0 = \widehat{V}(th)$. Then since $\widehat{G}\widehat{U}f_0$ is an equalizer of $\widehat{G}g_0$ and $\widehat{G}h_0$, we conclude the $\widehat{U}f_0$ is an equalizer of g_0, h_0 proving that

$$f = \widehat{U}f_0$$

is a regular monomorphism.

Example 2.5. A category $\mathcal{C}$ such that ind-$\mathcal{C}$ contains a pure monomorphism m which is
(i) an epimorphism but not an isomorphism (thus, m is not a strong monomorphism), and
(ii) not a colimit of split monomorphisms.

The category $\mathcal{C}$ is obtained from the following poset

$$\begin{tikzcd} & \vdots \\ \vdots & B_2 \arrow[u, phantom] \\ A_2 \arrow[u, phantom] \arrow[ur, "m_2"'] & \\ & B_1 \arrow[uu, "b_1"'] \\ A_1 \arrow[uu, "a_1"] \arrow[ur, "m_1"'] & \\ & B_0 \arrow[uu, "b_0"'] \\ A_0 \arrow[uu, "a_0"] \arrow[ur, "m_0"'] & \end{tikzcd}$$

by freely adding morphisms

$$e_i : B_i \to A_{i+1} \quad \text{with} \quad e_i m_i = a_{i,i+1} \quad \text{for} \quad i = 0, 1, 2, \dots$$

Form colimits $A = \operatorname*{\mathit{colim}}_{i \in \mathbb{N}} A_i$ and $B = \operatorname*{\mathit{colim}}_{i \in \mathbb{N}} B_i$ in ind-$\mathcal{C}$. Then the morphism

$$m = \mathit{colim}\ m_i : A \to B$$

satisfies (i) and (ii).

Proof. We prove some auxilliary facts first. Put

$$b_{ij} = b_{j-1} \dots b_i : B_i \to B_j,$$

$(i \leq j)$ and $\mathcal{B} = \{b_{ij}\}_{i \leq j}$; analogously a_{ij} and $\mathcal{A} = \{a_{ij}\}_{i \leq j}$.

(1) The category $\mathcal{C}$ has precisely one morphism from A_i to A_j, $i \leq j$, viz, a_{ij}. In fact, the "additional" morphisms $e_0, e_1, e_2, \dots$ add no new morphisms between the objects A_i due to $e_j b_{ij} m_i = e_j m_j a_{ij} = a_{i,j+1}$. Moreover, each a_{ij} is an epimorphism.

(2) Each B_i has just one endomorphism, id and, denoting

$$c_i = m_{i+1} e_i : B_i \to B_{i+1},$$

the morphisms from B_i to B_j, $i < j$, are precisely all composites of $b_i, b_{i+1}, \dots, b_{j-1}$ (each member precisely once) and $c_i, c_{i+1}, \dots, c_{j-1}$ (each member at most once).

(3) Given morphisms

$$f : A_{j+1} \to X \quad \text{and} \quad g : B_{i+1} \to X \quad (i \leq j)$$

in $\mathcal{C}$, we always have

$$f e_j b_{ij} \neq g b_j b_{ij}\,.$$

In fact, no reduction of the formal composite $f e_j b_{j-1} \dots b_i$ on the left-hand side produces a b_j on the position of e_j.

(4) Each b_i is an epimorphism. In fact, let $f, g : B_{i+1} \to X$ be morphisms in $\mathcal{C}$ with $f b_i = g b_i$. If f lies in $\mathcal{B}$, then (3) implies that g lies in $\mathcal{B}$ – then $f = g$ since $\mathcal{B}$ is a poset. If f does not lie in $\mathcal{B}$, it has the form $f = f' e_j b_{ij}$ for some $f' : A_{j+1} \to X$. It follows from (3) that g has the analogous form, $g = g e_j b_{ij}$ for some $g' : A_{j+1} \to X$. Finally, the equation

$$f' e_j b_{ij} = g' e_j b_{ij}$$

implies $f' e_j = g' e_j$ because, again, no reduction of one side gets rid of the part b_{ij}. Since e_j is an epimorphism (recall that $a_{j,j+1} = e_j m_j$ is an epimorphism), we conclude $f' = g'$. Consequently, $f = g$.

(5) Let $\mathcal{Y}$ be a proper chain

$$Y_0 \xrightarrow{y_0} Y_1 \xrightarrow{y_1} Y_2 \xrightarrow{y_2} \dots$$

in $\mathcal{C}$, i.e., a chain which is non-constant if finitely many members are removed. And let

$$\begin{tikzcd}
\vdots & \vdots \\
B_2 \arrow[u] \arrow[r, "t_2"] & Y_2 \arrow[u] \\
B_1 \arrow[u, "b_1"] \arrow[r, "t_1"] & Y_1 \arrow[u, "y_1"'] \\
B_0 \arrow[u, "b_0"] \arrow[r, "t_0"'] & Y_0 \arrow[u, "y_0"']
\end{tikzcd}$$

be a natural transformation. We prove that all y_i and all t_i ($i = 0, 1, 2, \dots$) are members of the chain $\mathcal{B}$.

(5a) We prove that $b_i \in \mathcal{B}$. In fact, let us assume that some y_i does not lie in $\mathcal{B}$. Then neither does $t_{i+1}b_i$ ($= y_i t_i$) which implies that $t_{i+1}b_i$ has factor e_j for some $j \geq i$:

$$t_{i+1}b_i = f e_j b_{j-1} \dots b_i.$$

Since b_i is an epimorphism, this gives

$$t_{i+1} = f e_j \widehat{b} \quad \text{for} \quad \widehat{b} = b_{j-1} \dots b_{i+1}\,.$$

The naturality implies

$$\widehat{y} t_{i+1} = t_{j+1} b_j \widehat{b} \quad \text{for} \quad \widehat{y} = y_j \dots y_{i+1}$$

in other words,

$$\widehat{y} f e_j \widehat{b} = t_{j+1} \widehat{b}_j \widehat{b}\,.$$

Since $\widehat{b}$ is an epimorphism, we get

$$\widehat{y} f e_j = t_{j+1} \widehat{b}_j\,.$$

This is a contradiction: a morphism starting with e_j is never equal to one starting with b_j.

This proves that $y_i \in \mathcal{B}$ for all i.

(5b) We prove that all t_i's lie in $\mathcal{B}$. We know that $Y_0 = B_k$ for some k, and that t_0 is a composite of $b_0, \dots, b_{k-1}$ and $c_0, \dots, c_k$. From

$$t_{k+1} b_{0k} = y_0 t_0$$

it follows, since $y_0 \in \mathcal{B}$, that all factors c_i of t_0 are also factors of $t_{k+1} : B_k \to B_{k+m}$ – however, such t_{k+1} can only contain c_i with $i > k$ as factors. Consequently, t_{k+1} contains none of $c_0, \dots, c_k$, which proves that $t_0 \in \mathcal{B}$. Analogously with $t_1, t_2, \dots$

(6) The object

$$B = \underset{i\in\mathbb{N}}{colim}\, B_i \quad \text{of} \quad \text{ind-}\mathcal{C}$$

has the property that the only morphism of ind-$\mathcal{C}$ with domain B is id_B. More precisely, observe that $\mathcal{C}$ is skeletal (i.e., no distinct objects are isomorphic) and form a skeletal completion ind-$\mathcal{C}$, then B has the above property.

In fact, for every morphism $f : B \to Y$ of ind-$\mathcal{C}$ which does not lie in $\mathcal{C}$ we express Y as a colimit of a proper chain $\mathcal{Y}$ (since $\mathcal{C}$ is countable, such a chain exists by [2]). Then f is a colimit of a natural transformation $t : \mathcal{B} \to \mathcal{Y}$. By (5), we have $colim\, \mathcal{Y} = B$ (because $\mathcal{B}$ is cofinal in $\mathcal{Y}$) and since the components of t are also members of $\mathcal{B}$, clearly $colim\, t = \mathrm{id}_B$.

(7) $m : A \to B$ is an epimorphism, but not an isomorphism: this is a corollary of (6).

(8) m is a pure monomorphism: this easily follows from $m = colim\ m_i$ and the fact that in every commutative square

$$\begin{tikzcd} A_i \arrow[r, "m_i"] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright", near start] & B_i \arrow[d, "v"] \arrow[dl] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

in ind-$\mathcal{C}$ the morphism u (which, necessarily, is simply a_i) factorizes through m_i, viz, $a_i = a_{i+1}e_i m_i$.

(9) m is not a directed colimit of split monomorphisms in $\mathcal{C}$: in fact, the only split monomorphisms in $\mathcal{C}$ are the identity morphisms. $\square$

Example 2.6. A finitely complete category $\overline{\mathcal{C}}$ such that ind-$\overline{\mathcal{C}}$ contains a pure monomorphism which is not strong: let $E : \mathcal{C} \to \overline{\mathcal{C}}$ be a free completion of $\mathcal{C}$ in Example 2.5 under finite limits. It is easy to see that also

$$\text{ind-}E : \text{ind-}\mathcal{C} \to \text{ind-}\overline{\mathcal{C}}$$

is a free completion under finite limits. This functor ind-E clearly preserves pure monomorphisms, thus, $\overline{m} = \text{ind-}E(m)$ is a pure monomorphism in ind-$\overline{\mathcal{C}}$. And free limit completions always preserve all existing colimits, thus, they preserve epimorphisms. Consequently, $\overline{m}$ is an epimorphism but not an isomorphism in ind-$\overline{\mathcal{C}}$.

3. Pure Epimorphisms

Lemma 3.1. *If $\mathcal{C}$ has weak pullbacks, then every pure epimorphism in ind-$\mathcal{C}$ is a directed colimit of split epimorphisms.*

Proof. Given a pure epimorphism $h : A \to B$, express it as a colimit as in (2.0), find d_i with

$$b_i = hd_i$$

and precisely as in the proof of Lemma 2.2 find $i' \geq i$ and $d'_i : B_i \to A_{i'}$ such that the triangle

$$\begin{tikzcd}
 & B_i \arrow[dl, "d'_i"'] \arrow[d, "b_{ii'}"] \\
A_{i'} \arrow[r, "h_{i'}"'] & B_{i'}
\end{tikzcd}$$

commutes. Form a weak pullback (P_i, p_i, q_i) of $h_{i'}$ and $b_{ii'}$:

$$\begin{tikzcd}
A_i \arrow[rr, "h_i"] \arrow[dr, "r_i"] \arrow[dd, "a_{ii'}"'] & & B_i \arrow[dd, "b_ii'"] \\
 & P_i \arrow[ur, "q_i"'] \arrow[dl, "p_i"] & \\
A_{i'} \arrow[rr, "h_{i'}"'] & & B_{i'}
\end{tikzcd}$$

We have $r_i : A_i \to P_i$ with

$$a_{ii'} = p_i r_i \quad \text{and} \quad h_i = q_i r_i\,. \tag{3.1}$$

and $s_i : B_i \to P_i$ with

$$d'_i = p_i s_i \quad \text{and} \quad \mathrm{id}_{B_i} = q_i s_i\,. \tag{3.2}$$

We again choose $J \subseteq I$ cofinal with $j < k$ in J implying $j' \leq k$. We obtain a diagram with objects P_j $(j \in J)$ and connecting morphisms

$$P_j \xrightarrow{p_j} A_{j'} \xrightarrow{a_{j'k}} A_k \xrightarrow{r_k} P_k \qquad (j, k \in J, j < k)$$

whose colimit is A with the colimit cocone $a_{j'}p_j = P_j \to A$ $(j \in J)$. Consequently, h is a colimit

$$\begin{tikzcd}
P_j \arrow[r, "q_j"] \arrow[d, "a_{j'}p_j"'] & A_j \arrow[d, "b_j"] \\
A \arrow[r, "h"'] & B
\end{tikzcd} \qquad (j \in J)$$

of split epimorphisms: see (3.2) $\square$

Definition 3.2. By a *collective coequalizer* of a small set of parallel pairs $p_i, q_i : P_i \to A$ $(i \in I)$ with a joint codomain is meant a morphism $e : A \to B$ forming a colimit of the diagram. (That is, e coequalizes each pair and is universal.)

A morphism is called a *collective regular epimorphism* if it is a collective coequalizer of some set.

Remark 3.3. (1) In categories with coproducts, this is equivalent to being a regular epi: consider the induced parallel pair $\coprod\limits_{i \in I} P_i \to A$.

(2) Every collective regular epimorphism is a strong epimorphism; this is easy to prove: see the analogous argument for "regular $\Rightarrow$ strong" at the end of the Introduction.

Theorem 3.4. *In every ind-category we have*

$$\textit{pure epimorphism} \quad \Rightarrow \quad \textit{collective regular epimorphism.}$$

Proof. Let $e : A \to B$ be a pure epimorphism in ind-$\mathcal{C}$. Express e as a directed colimit of morphisms $e_i : A_i \to B_i$ $(i \in I)$ in $\mathcal{C}$ with connecting morphisms $a_{ij} : A_i \to A_j$ $(i \le j)$ and colimit cocone $a_i : A_i \to A$ $(i \in I)$; analogously for (B_i):

$$\begin{tikzcd}
A_i \arrow[r, "e_i"] \arrow[d, "a_i"'] & B_i \arrow[d, "b_i"] \arrow[dl, dashed, "d_i"'] \\
A \arrow[r, "e"'] & B
\end{tikzcd}$$

Choose

$$d_i : B_i \to A \quad \text{with} \quad b_i = ed_i \qquad (i \in I)\,.$$

We will prove that the following collection of parallel pairs

$$a_i, d_ie_i : A_i \to A \quad \text{and} \quad d_i, d_jb_{ij} : B_i \to A_j \quad (\text{for } i, j \in I, i \le j)$$

has the desired property. In fact:

(1) e coequalizes all of these pairs:

$$ea_i = b_ie_i = ed_ie_i \qquad (i \in I)$$

as well as

$$ed_i = b_i = b_jb_{ij} = ed_jb_ij \qquad (i \le j)\,.$$

(2) Let $f : A \to D$ coequalize all the above pairs. Then the morphisms

$$fd_i : B_i \to D \qquad (i \in I)$$

form a cocone of the diagram (B_i), thus, there is a unique $g : B \to D$ with

$$gb_i = fd_i \qquad (i \in I)\,.$$

It follows that

$$f = ge$$

because for each $a_i : A_i \to A$ we have

$$\begin{aligned} fa_i &= fd_ie_i && (f \text{ coequalizes } a_i, d_ie_i) \\ &= gb_ie_i && (\text{definitions of } g) \\ &= gea_i. \end{aligned}$$

The uniqueness of g is obvious: from the equality $f = ge$ we obtain $gb_i = ged_i = fd_i$, the original definition of g. □

Corollary 3.5. *Every pure epimorphism in an ind-category is strong.*

This has been proved in a different manner in [9].

Corollary 3.6. *If ind-$\mathcal{C}$ has coproducts (e.g., whenever $\mathcal{C}$ has finite coproducts), all pure epimorphisms in ind-$\mathcal{C}$ are regular.*

Remark 3.7. We have not been able to find an example of an ind-category in which a pure epimorphism would exist that is not regular.

4. Summary and Open Problems

4.1. In the present paper we investigated pure monomorphisms and pure epimorphisms in pro-categories. We observed that, in general, a pure epimorphism in pro-$\mathcal{C}$ can be a monomorphism without being an isomorphism (see Example 2.6). Here $\mathcal{C}$ can even posses finite colimits – but not arbitrary colimits:

Theorem 4.2 (See 2.3)**.** *If $\mathcal{C}$ is a category with*
(a) *weak finite limits, or*
(b) *pullbacks, or*
(c) *copowers*
then every pure epimorphism in pro-$\mathcal{C}$ is regular.

Open Problem 4.3. Is there a category $\mathcal{C}$ with countable copowers such that pro-$\mathcal{C}$ has pure epimorphisms which are not strong? Is there such $\mathcal{C}$ with weak pullbacks?

The last question is relevant in connection to

Lemma 4.4 (See 2.2)**.** *If $\mathcal{C}$ has weak pullbacks, then every pure epimorphism in pro-$\mathcal{C}$ is a directed colimit (in the arrow category of pro-$\mathcal{C}$) of split epimorphisms in $\mathcal{C}$.*

Remark 4.5. We observed that there is a certain, rather surprising, duality between properties of pure monomorphisms and pure epimorphisms in pro-categories. However, the concepts are indeed not dual, e.g., the above example of a pure epimorphism that is not strong contrasts sharply with the following

Theorem 4.6 (See 3.4)**.** *Every pure monomorphism in a pro-category is a collective regular monomorphism (i.e., an equalizer of a set of parallel pairs).*

This strengthens the result of [9] that pure monomorphisms are strong.

Open Problem 4.7. Is every pure monomorphism in a pro-category regular? Or at least in any pro-$\mathcal{C}$ such that $\mathcal{C}$ has weak pushouts?

The last question is relevant in connection to

Lemma 4.8 (See 3.1)**.** *If $\mathcal{C}$ has weak pushouts, then every pure epimorphism in pro-$\mathcal{C}$ is a directed colimit of split epimorphism in $\mathcal{C}$.*

Theorem 4.9. *If $\mathcal{C}$ is a category with*
(a) *finite products, or*
(b) *pushouts,*
then every pure epimorphism in pro-$\mathcal{C}$ is regular.

In fact, in (a) the weaker assumption that pro-$\mathcal{C}$ have products is sufficient: see 3.6.

The case (b) is proved (in dual form) in [3]. The assumption, made in that paper, that $\mathcal{C}$ be small is not used in the proof.

Open Problem 4.10. In comparison to 4.2 the last case is missing here: in fact, we do not know whether, for the regularity of pure epimorphisms in pro-$\mathcal{C}$, weak finite colimits in $\mathcal{C}$ are sufficient.

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