Generalized purity, definability and Brown representability

J. Rosický

Some Trends in Algebra, Prague 2009

Definition. A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-filtered colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered colimit of objects from $\mathcal{A}$.

An object A is λ-presentable if its hom-functor

$$\hom(A, -) : \mathcal{K} \to \mathbf{Set}$$

preserves λ-filtered colimits.

A category is *accessible* if it is λ-accessible for some regular cardinal λ. A cocomplete λ-accessible category is called *locally λ-presentable.* A category is *locally presentable* is it is locally λ-presentable for some regular cardinal λ.

Grothendieck categories are locally presentable.

Let $\mathcal{K}$ be a locally λ-presentable category. A λ-*pure monomorphism* is a morphism $f : K \to L$ such that given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ K & \xrightarrow[\;f\;]{} & L \end{array}$$

with A and B λ-presentable, then u factorizes through g, i.e., $u = tg$ for some $t : B \to K$.

In R-**Mod**, $\aleph_0$-pure monomorphisms coincide with usual pure monomorphisms.

In a locally λ-presentable category $\mathcal{K}$, λ-pure monomorphisms are precisely λ-directed colimits (in $\mathcal{K}^{\to}$) of split monomorphisms.

An object K is λ-*pure injective* if it is injective w.r.t. λ-pure monomorphisms.

A category $\mathcal{K}$ *has enough λ-pure injectives* if every object K of $\mathcal{K}$ admits a λ-pure monomorphism into a λ-pure injective object.

Every finitely accessible additive category with products has enough $\aleph_0$-pure injectives. This is not true without additivity – the category **Gra** of oriented multigraphs does not have enough pure injectives (F. Borceux, J.R. 2007).

Question 1. Let $\mathcal{K}$ is a locally presentable additive category. Does there exist a regular cardinal λ such that $\mathcal{K}$ has enough λ-pure injectives?

The positive answer would follow from the existence of enough injectives in every locally presentable additive category. But there are locally presentable additive categories which do not have enough injectives (A. Neeman 2001). The category **Gra** does not have enough λ-pure injectives for any regular cardinal λ.

Definition 1. A full subcategory $\mathcal{L}$ of a locally λ-presentable category $\mathcal{K}$ will be called λ-*definable* if it is closed under products, λ-filtered colimits and λ-pure subobjects.

$\aleph_0$-definable subcategories in R-**Mod** are usual definable subcategories.

J. Adámek, F. Borceux, J.R. (2002):

Theorem 1. A full subcategory $\mathcal{L}$ of a locally λ-presentable category $\mathcal{K}$ is λ-definable iff it is an injectivity class w.r.t. morphisms between λ-presentable objects.

For $\aleph_0$, the result follows from the compactness theorem: an $\aleph_0$-definable subcategory is closed under ultraproducts and elementary subobjects. Thus it is axiomatizable in the first-order logic and the rest easily follows.

Corollary. A locally λ-presentable category has only a set of λ-definable subcategories.

Theorem 2. In a locally λ-presentable category, every full subcategory closed under products and λ-pure subobjects is weakly reflective.

In R-**Mod**, weakly reflective full subcategories are also called preenveloping. For $\aleph_0$, this result was proved in the additive case by S. Crivei, M. Prest and B. Torrecillas (and by J. Rada and M. Saorín for modules).

Corollary. A full subcategory of a locally λ-presentable category closed under limits and λ-filtered colimits is reflective.

Any full subcategory of a locally λ-presentable category closed under limits and λ-filtered colimits is closed under λ-pure subobjects.

In a locally λ-presentable category $\mathcal{K}$, a λ-*pure epimorphism* is a morphism f such that each λ-presentable object is projective w.r.t. f.

In a locally λ-presentable category $\mathcal{K}$, λ-pure epimorphisms are precisely λ-directed colimits (in $\mathcal{K}^{\rightarrow}$) of split epimorphisms.

Given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in an additive locally λ-presentable category, f is a λ-pure monomorphism iff g is an λ-pure epimorphism.

An object is λ-*pure projective* if it is projective w.r.t. λ-pure epimorphisms.

A category $\mathcal{K}$ *has enough λ-pure projectives* if every object K of $\mathcal{K}$ admits a λ-pure epimorphism from a λ-pure projective object.

Proposition 1. Every locally λ-presentable category has enough λ-pure projectives.

Given an object K, we consider the coproduct $\coprod A$ indexed by all morphisms $f : A \to K$ with A λ-presentable. This coproduct is λ-pure projective and the induced morphism $\coprod A \to K$ is a λ-pure epimorphism.

Corollary. In a locally λ-presentable category, λ-pure projectives are precisely retracts of coproducts of λ-presentable objects.

In a locally λ-presentable additive category $\mathcal{K}$, each object K admits a λ-*pure projective resolution*, i.e., an exact sequence

$$\dots \to P_n \xrightarrow{p_n} P_{n-1} \to \dots \to P_0 \xrightarrow{p_0} K \to 0$$

such that P_n is λ-pure projective and $\ker p_n$ is a λ-pure monomorphism for each n.

Definition 2. We say that the λ-*pure projective dimension* of $\mathcal{K}$ is $\leq n$ if each K has a λ-pure projective resolution

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \to K \to 0$$

We write $\lambda\text{-ppd}\,\mathcal{K} \leq n$.

Question 2. Let $\mathcal{K}$ be a Grothendieck category. Does there exist a regular cardinal λ such that λ-ppd $\mathcal{K} \leq 1$?

If $\mathcal{K}$ is locally finitely presentable Grothendieck category having $\leq \aleph_0$ morphisms between (representative) finitely presentable objects then $\aleph_0\text{-ppd}\,\mathcal{K} \leq 1$ (D. Simson 1977).

Let $\mathcal{K}$ be a category with coproducts and λ a cardinal. An object A of $\mathcal{K}$ is called λ-*small* if for every morphism $f : A \to \coprod_{i \in I} L_i$ there is a subset J of I of cardinality less than λ such that f factorizes as

$$A \to \coprod_{j \in J} L_j \to \coprod_{i \in I} L_i$$

where the second morphism is the subcoproduct injection.

Every λ-presentable object is λ-small.

A. Neeman (2001):

Consider classes $\mathcal{S}$ of λ-small objects of $\mathcal{K}$ such for every morphism $f : S \to \coprod_{i \in I} L_i$ with $S \in \mathcal{S}$ there are morphisms $g_i : S_i \to L_i$ where $S_i \in \mathcal{S}$ for each $i \in I$ such that f factorizes through $\coprod_{i \in I} g_i : \coprod_{i \in I} S_i \to \coprod_{i \in I} L_i$. Since these classes are closed under unions, there is the greatest class $\mathcal{S}$ with this property. Its objects will be called λ-*compact*.

Let $\mathcal{K}$ be a category with a zero object. We say that a set $\mathcal{G}$ of objects *weakly generates* $\mathcal{K}$ if whenever $\hom(G, K) = \{0\}$ for each $G \in \mathcal{G}$ then $K = 0$.

A. Neeman (2001):

Let $\mathcal{K}$ be a category with coproducts and a zero objects. $\mathcal{K}$ is called *well λ-generated* if it has a weakly generating set of λ-compact objects. It is called *well generated* if it is well λ-generated for some cardinal λ.

Every locally λ-presentable category is well λ-generated.

J.R. (2005):

Theorem 3. Let $\mathcal{K}$ be a λ-combinatorial pointed model category. Then $\operatorname{Ho}\mathcal{K}$ is well λ-generated.

The existence of a weakly generating set in $\operatorname{Ho}\mathcal{K}$ is due to M. Hovey (1999). The result is important for stable model categories whose homotopy categories are triangulated.

A model category is λ-*combinatorial* if it is locally λ-presentable and the both weak factorization systems (cofibration, trivial fibrations) and (trivial cofibrations, fibrations) are cofibrantly generated by morphisms between λ-presentable objects. A model category is *combinatorial* if it is λ-combinatorial for some regular cardinal λ.

Nearly all important model categories are either combinatorial or Quillen equivalent to a combinatorial one. E.g., **SSet**, **Sp** and $\mathbf{Ch}(R)$ are $\aleph_0$-combinatorial. The last two model categories are stable.

Let $\mathcal{K}$ be a well λ-generated category. A λ-*pure monomorphism* is a morphism $f : K \to L$ such that given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ K & \xrightarrow[\;f\;]{} & L \end{array}$$

with A and B λ-compact, then u factorizes through g, i.e., $u = tg$ for some $t : B \to K$.

A λ-*pure epimorphism* is a morphism f such that each λ-compact object is projective w.r.t. f.

A morphism $f : K \to L$ is called λ-*phantom* if $fg = 0$ for each morphism $g : A \to K$ with A λ-compact.

Proposition 2. Let $\mathcal{K}$ be a well λ-generated triangulated category. For an exact triangle

$$K \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} \Sigma K$$

the following are equivalent

(1) f is a λ-phantom,

(2) g is a λ-pure monomorphism,

(3) h is a λ-pure epimorphism.

For $\aleph_0$, this result is due to H. Krause (2000). He used different (but equivalent) definition of a $\aleph_0$-pure monomorphism.

In a well λ-generated triangulated category, λ-pure projective objects are defined in a usual way. They are precisely direct summands of coproducts of λ-compact objects. Every well λ-generated triangulated category has enough λ-pure projectives.

Definition 3. We say that a well λ-generated triangulated category has the λ-*pure global dimension less or equal to 1* if each object K admits an exact triangle

$$Q \xrightarrow{f} P \xrightarrow{g} K \xrightarrow{h} \Sigma Q$$

where P and Q are λ-pure projective and g is a λ-pure epimorphism.

A. Neeman (2008):

Question 3. Let $\mathcal{K}$ be a well generated triangulated category. Does there exist a cardinal λ such that λ-ppd $\mathcal{K} \leq 1$?

A positive answer to this question implies the positive answer to Question 2.

Let $\mathcal{K}$ be a well λ-generated triangulated category and $\mathcal{K}_\lambda$ denote its (representative) small full subcategory of λ-compact objects. The *canonical functor*

$$E_\lambda : \mathcal{K} \to \mathbf{Ab}^{\mathcal{K}_\lambda^{\mathrm{op}}}$$

assigns to each object K the restriction

$$E_\lambda(K) = \hom(-, K)/\mathcal{K}_\lambda^{\mathrm{op}}$$

of its hom-functor $\hom(-, K) : \mathcal{K} \to \mathbf{Ab}$ to $\mathcal{K}_\lambda^{\mathrm{op}}$.

A morphism f is

λ-pure mono iff $E_\lambda f$ is mono,

λ-pure epi iff $E_\lambda f$ is epi,

λ-phantom iff $E_\lambda f = 0$.

This is an original definition of H. Krause of $\aleph_0$-pure monomorphisms in well $\aleph_0$-generated triangulated categories.

Theorem 4. Let $\mathcal{K}$ be a well λ-generated triangulated category. Then the following conditions are equivalent:

(1) $\lambda\text{-ppd}\,\mathcal{K} \leq 1$,

(2) E_λ is full,

(3) each K is a weak colimit of its canonical diagram w.r.t. $\mathcal{K}_\lambda$.