On abstract data types presented by multiequations

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Abstract

Equational presentation of abstract data types is generalized to presentation by multiequations, i.e., exclusive-or's of equations, in order to capture parametric data types such as array or set. Multiinitial-algebra semantics for such data types is introduced. Classes of algebras described by multiequations are characterized.

Introduction

A number of important data types, e.g., stack, list, tree, have an abstract specifications using equational logic: for a given many-sorted signature Σ one considers a set E of equations, and the loose semantics of the data type is then the whole variety of Σ-algebras satisfying E, whereas the initial semantics works with the initial algebras of that variety. There are, however, basic data types which cannot be presented by equations. For example, arrays (or, more mathematically, functions of a finite graph): an argument showing that the

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parametric abstract data type array lies outside the scope of equational logic has been presented by Löwe and Wolter in [LW]. These authors proposed a formalism of abstract data types using a part of the first-order logic which includes the example of arrays.

The present paper has been initiated by our observation that a natural approach to generalizing equational logic also encompasses arrays, as well as a number of interesting algebraic structures such as fields: we use, instead of single equations $\alpha = \beta$, exclusive-or's of equations,

$$\bigtriangledown_{i \in I} (\alpha_i = \beta_i)\,.$$

The meaning of this formula, called *multiequation*, is quite intuitive: for every interpretation of variables there exists a unique index $i \in I$ such that the equality $\alpha_i = \beta_i$ (in the given interpretation) holds.

We first illustrate the expressive power of multiequations on concrete examples of algebras and data types. Then we discuss a generalization of the initial-algebra semantics (used for equational specifications) to multiinitial-algebra semantics applicable to presentations by multiequations, and a generalization of the free-algebra semantics (used for parametric equational specifications) to a multifree-algebra semantics of parametric multiequational specifications.

Multiequational specifications do not, in general, have initial algebras. The loose semantics works here of course: any algebra of the given specification can be considered as a "correct implementation" of the data type – this holds much more generally for all first-order specifications, see [W]. However, multiequational specifications do have something better to offer: multiinitial families of algebras. Like in loose semantics, in the multiinitial semantics the abstract data type is presented by non-isomorphic algebras; however, these are very much restricted. For example, multiinitial families are (like initial algebras) unique up to isomorphism.

Analogously for parametric multiequational specifications: instead of a free functor as semantics we have to work with a free multifunctor semantics.

The approach of Löwe and Wolter in [LW] which has inspired our paper is based on Gentzen formulas, and quasi-free semantics. It is our opinion that multifree semantics is more natural because it is closer to free semantics, and is conceptually simpler. What these two approaches have in common is that in important examples of parametric specifications that cannot be handled by equations (array, set) the following type of "weak" or "unique" persistence takes place: for every parameter algebra there exists a unique member of the multifree family which is carried by an isomorphism. We prove that this makes parameter passing "weakly correct": both for actual parameter protection, and for passing compatibility we find a member of the corresponding multiinitial family which satisfies the required isomorphism property.

The last topic we consider is a generalization of the Birkhoff Variety Theorem to the classes of algebras specified by multiequations, called multivarieties. We are led to consider three variants of the formulas above:

$$\bigtriangledown_{i \in I} (\alpha_i = \beta_i)$$

will be called a *multiequation* if only a finite number of variables appear in it (no restriction on the cardinality of I). Our examples will suggest that it may be more convenient to allow exclusive-or's of conjunctions of equations, but our Lemma 1 will show that if we restrict ourselves to finite conjunctions (such formulas will be called *generalized multiequations*), then nothing really new is added: if a class of structures can be presented by generalized multiequations, then it can be presented by multiequations, modulo a canonical extension of the original signature. After having established some closure properties of (generalized) multivarieties, an example will prove the failure of the "expected" Birkhoff-type theorem. The situation is not improved if we require I to be finite also. However a complete solution is provided for infinitary multiequations, where the only finiteness condition is on the number of variables involved (conjunctions and or's may act on infinite sets): the classes of models of infinitary multivarieties are characterized as those closed under subalgebras, connected limits, directed unions, and so called *consistent quotients*.

One topic *not* addressed in our paper is the logic for multiequations. We have not found any adequate restriction of the first-order logic to treat multiequations. This is a topic of future research. But in any case, we see little hope that a deduction calculus for multiequations could be found which would be anywhere close to equational logic.

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I. Examples

We work with many-sorted algebras, i.e., for a given set $SORTS$, a signature Σ is given by a set $OPNS$ (of operations symbols σ) together with a specification of *arity*

$$\sigma : s_1 \times \cdots \times s_n \to s$$

where $s_1, \ldots, s_n, s$ are sorts. An *algebra* A consists of underlying sets A_s for each sort $s \in SORTS$ and operations $\sigma_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for each $\sigma : s_1 \times \cdots \times s_n \to s$ in $OPNS$. A *homomorphism* from A to an algebra B is a many-sorted function, i.e., a collection of functions $f_s : A_s \to B_s$

($s \in SORTS$) which commutes with the operations in the usual sense, i.e.,

$$f_s\Big(\sigma_A(x_1, \ldots, x_n)\Big) = \sigma_B\Big(f_{s_1}(x_1), \ldots, f_{s_n}(x_n)\Big)\,.$$

Given a many-sorted set, i.e., a collection of sets $X = (X_s)_{s \in SORTS}$ of variables, we denote by $T_\Sigma(X)$ the absolutely free algebra (as a term algebra) over X. For every algebra A and every many-sorted function $f : X \to A$ the computation of terms in A yields the unique homomorphism

$$f^\# : T_\Sigma(X) \to A$$

extending f.

Notation. The logical operation "exclusive-or" is denoted by $\bigtriangledown$. That is, $\alpha \bigtriangledown \beta$ is a shorthand for

$$(\alpha \vee \beta) \wedge \neg\, (\alpha \wedge \beta)\,.$$

Analogously, $\underset{i \in I}{\bigtriangledown} \alpha_i$ is a shorthand for

$$\left(\bigvee_{i \in I} \alpha_i \right) \wedge \bigwedge_{\substack{i,j \in I \\ i \neq j}} \neg\, (\alpha_i \wedge \alpha_j)\ .$$

Definition. By a *multiequation* we mean an exclusive-or of equations over a finite set of variables.

In more detail, let $X = (X_s)_{s \in SORTS}$ be a finite many-sorted set of variables, i.e., such that the disjoint union of all X_s is finite. Then a pair (α, β) of elements of $T_\Sigma(X)$ of the same sort is called equation (usual notation: $\alpha = \beta$); a multiequation, then, has the form

$$\underset{i \in I}{\bigtriangledown} (\alpha_i = \beta_i)$$

where each $\alpha_i = \beta_i$ is an equation over X (where I is an arbitrary set). A Σ-algebra A *satisfies* the multiequation provided that for every interpretation $f : X \to A$ of variables precisely one of the given equations hold, i.e., there exists precisely one $i \in I$ with $f^\#(\alpha_i) = f^\#(\beta_i)$.

Definition. Given a set E of multiequations we denote by $\mathbf{Alg}\,(\Sigma, E)$ the class of all Σ-algebras satisfying each multiequation in E. Such classes are called *multivarieties*.

Remark 1. We consider $\mathbf{Alg}\,(\Sigma, E)$ as a full subcategory of the category $\mathbf{Alg}\,\Sigma$ of all Σ-algebras and all homomorphisms.

Example 1. $BOOL$

$$\begin{array}{ll} SORTS & bool \\ OPNS & T, F : \to bool \\ AXIOM & (x = T) \triangledown (x = F) \end{array}$$

The only model is $\{T, F\}$ where $T \neq F$.

Example 2. INJ

The following is a multiequational presentation of sets with injective functions as morphisms:

$INJ = BOOL +$

$$\begin{array}{ll} SORTS & set \\ OPNS & equal : set \times set \to bool \\ AXIOMS & (x = x') \triangledown \big(equal\ (x, x') = F\big). \end{array}$$

Every set X gives rise to an algebra $\bar{X}$ with set-sort X and bool-sort $\bar{X}_{\mathrm{bool}} = \{T, F\}$ in which "equal" is the equality predicate. And conversely, every model of INJ has the form $\bar{X}$ for a unique set X.

Example 3. Fields. Let (Σ, E) be the usual one-sorted equational presentation of unitary rings. Let Σ' be the signature Σ enriched by a unary operation i (the inverse of x, if $x \neq 0$, and 1 if $x = 0$). Let E' be the set E enriched by the equation $i(0) = 1$ and the following multiequation:

$$(x = 0) \triangledown (x \cdot i(x) = 1)\,.$$

Then (Σ', E') is a presentation of fields. In fact, a (Σ', E')-algebra A is a unitary ring in which every element $x \neq 0$ has the inverse $i(x)$ (and, moreover, $i(0) = 1$), thus A is a field with $i(x) = x^{-1}$ for $x \neq 0$. Conversely, every field can be considered as a (Σ', E')-algebra in this sense.

Example 4. Negated equations. The formula $\neg(p = q)$ is equivalent to the multiequation

$$(x = x) \triangledown (p = q)\,.$$

Remark 2. It is often handy to work with conjunctions of equations rather than individual equations. We prove below (Lemma 1) that this does not influence the concept of multiequational presentation.

More precisely, instead of multiequations we often work with exclusive or's of finite conjunctions of equations, i.e., with expressions of the following form:

$$\bigtriangledown_{i \in I} (\alpha_{i_1} = \beta_{i_1} \wedge \alpha_{i_2} = \beta_{i_2} \wedge \cdots \wedge \alpha_{i_{k(i)}} = \beta_{i_{k(i)}}).$$

We will call such expressions *generalized multiequations*. An algebra A *satisfies* the generalized multiequation if for every interpretation $f : X \to A$ of variables precisely one index $i \in I$ has the property that $f^{\#}(\alpha_{ij}) = f^{\#}(\beta_{ij})$ for all $j = 1, \ldots, k(i)$.

Example 5. Linearly ordered sets. We denote by LIN the following presentation of linearly ordered sets by generalized multiequations (with strictly increasing maps as morphisms).

$LIN = INJ +$

$$\begin{array}{lll}
OPNS: & \sqcup : set \times set & \to set \\
AXIOMS: & \text{Idempotence:} & x \sqcup x = x \\
 & \text{Symmetry:} & x \sqcup y = y \sqcup x \\
 & \text{Associativity:} & x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \\
 & \text{Linearity:} & (x = y) \ \triangledown \ \big((x \sqcup y = y) \wedge (equal\,(x,y) = F)\big) \\
 & & \quad \triangledown \ \big((x \sqcup y = x) \wedge (equal\,(x,y) = F)\big).
\end{array}$$

We can re-write this (in fact any: see Lemma 1 below) presentation using only non-generalized multiequations as follows: we introduce (a) a new sort $[set, bool]$ and (b) new operations

$$\begin{aligned}
tuple : \ & set \times bool \ \to \ [set, bool] \\
\pi^1 : \ & [set, bool] \ \to \ set \\
\pi^2 : \ & [set, bool] \ \to \ bool
\end{aligned}$$

and

$$(\sqcup, equal) : set \times set \ \to \ [set, bool]\,.$$

We add the following equations (to guarantee that (a) the underlying set of sort $[set,\ bool]$ is the product of the underlying sets of set and $bool$ and (b) the last operation is just the tupling of $\sqcup$ and $equal$):

$$\begin{aligned}
& tuple\ (\pi^1 z,\ \pi^2 z) = z, \\
& \pi^1\ tuple\ (x, b) = x, \\
& \pi^2\ tuple\ (x, b) = b, \\
& \pi^1(\sqcup, equal)(x, y) = x \sqcup y, \\
& \pi^2(\sqcup, equal)(x, y) = equal(x, y).
\end{aligned}$$

The linearity is then expressed by the following multiequation:

$$(x = y) \bigtriangledown \ ((\sqcup, equal)(x, y) = tuple(y, F)) \ \bigtriangledown \ ((\sqcup, equal)(x, y) = tuple(x, F)) \ .$$

Example 6. Arrays. Given a pair of sets D (domain, or index set) and C (codomain, or attribute set), an array is a function f from a finite subset of D to a finite subset of C. Or, equivalently, a function

$$f : D \to C_\perp \ \text{ where } \ C_\perp = C + \{\perp\}$$

of finite support, i.e., such that $f(x) = \perp$ for all but finitely many elements $x \in D$.

Abstractly, arrays can be built up from the empty array Ω by the operation *chg* (change-of-value) which assigns to any triple (x, y, f), where $x \in D$, $y \in C$ and f is an array, the array f updated in x to the value y. That operation, together with the operation val which gives the value of an array f at a given $x \in D$, constitute the basic abstract operations on arrays.

We thus consider the following specification (following [LW]):

$ARRAY(D, C) \ =$

$SORTS$	$dom, cod, array$			
$OPNS$	chg	:	$dom \times cod \times array \to array$	
	val	:	$array \times dom \to cod$	
	Ω	:	$\to array$	
	$\perp$	:	$\to cod$	
	d	:	$\to dom$	for each $d \in D$
	c	:	$\to cod$	for each $c \in C$

$AXIOMS$

(1) $val\big(chg(x, y, f), x\big) = y;$

(2) $chg(x, \perp, \Omega) = \Omega;$

(3) $chg\big(x, y, chg(x, y', f)\big) = chg(x, y, f);$

(4) if $x \neq x'$ then $chg\big(x, y, chg(x', y', f)\big) = chg\big(x', y', chg(x, y, f)\big).$

(Here x, x' are variables of sort *dom*; whereas y, y' are variables of sort *cod*, and f is a variable of sort array.) The last axiom is non-equational. In fact, as shown in [LW], no equational presentation (even with hidden operations) exists for the corresponding parametric data type (where the parameter specification has sorts *dom* and *cod* and the only operation $\perp : \to cod$). We can, however, specify arrays by multiequations. This is based on the following

Observation. The axiom (4) above can be replaced by the following axiom

$$\Big[(x = x') \wedge (y \neq y')\Big] \\ \qquad\qquad \triangledown \ \Big[chg\Big(x, y, chg(x', y', f)\Big) = chg\Big(x', y', chg(x, y, f)\Big)\Big]. \quad (4^*)$$

In fact, every algebra A satisfying (1)–(3) and (4^*) obviously satisfies also (4): given elements $x, x' \in A_{cod}$ with $x \neq x'$ then the right-hand bracket of (4^*) holds because the left-hand one does not.

Conversely, if A satisfies (1)–(4), then it also satisfies (4^*): consider elements $x, x' \in A_{cod}$, $y, y' \in A_{dom}$ and $f \in A_{array}$. The right-hand bracket of (4^*) holds whenever either $x \neq x'$ (due to (4)) or $x = x'$ and $y = y'$. That is, whenever the left-hand one does not hold. Conversely, assume that $x = x'$ and $y \neq y'$, then the right-hand bracket of (4^*) does not hold: see (3) and (1). This proves that (4^*) can replace (4).

Thus, let us denote by

$$ARRAY(D, C)^*$$

the above specification with the last (non-equational) axiom removed. We define the following generalized multiequational presentation:

$MARRAY(D, C) = ARRAY(D, C)^* +$

$SORTS$	$bool$
$OPNS$	$equal : cod \times cod \to bool$
	$T: \quad \to bool$
	$F: \quad \to bool$
$AXIOMS$	$(y = y') \ \triangledown \ \Big(equal(y, y') = F\Big);$
	$(z = T) \ \triangledown \ (z = F);$
	$\Big[(x = x') \wedge \Big(equal(y, y') = F\Big)\Big] \ \triangledown$
	$\qquad \triangledown \ \Big[chg\Big(x, y, chg(x', y', f)\Big) = chg\Big(x', y', chg(x, y, f)\Big)\Big].$

(Here x, x' are variables of sort *dom*; y, y' variables of sort *cod*; z a variable of sort *bool*; and f a variable of sort *array*.)

Then $ARRAY(D, C)$ specifies the same class of algebras as $MARRAY(D, C)$. More precisely, for every algebra A of the specification $ARRAY(D, C)$ we have an algebra $\bar{A}$ of the specification $MARRAY(D, C)$ which extends A by $\bar{A}_{bool} = \{T, F\}$ and with $equal, T, F$ having the obvious interpretation (see

Example 2). Then every algebra of the specification $MARRAY(D,C)$ has the form $\bar{A}$ for a unique algebra A of the specification $ARRAY(D,C)$.

The homomorphisms of the new specification $MARRAY(D,C)$ are precisely those homomorphisms of $ARRAY(D,C)$ whose codomain-component is injective (so that the equality operation *equal* is preserved). These are also the homomorphisms considered in [LW] for arrays.

Example 7. Sets. For a given "universe" set U consider subsets of U with the basic operations

$$\begin{aligned}
&union, && (s,s') \mapsto s \cup s' \\
&make, && x \mapsto \{x\} \\
&element, && (s,x) \mapsto T, \text{ if } x \in s, \text{ or } F, \text{ else.}
\end{aligned}$$

The axioms we put on these operations state (a) that union forms a semilattice with the bottom element $\emptyset$ and (b) the operation *element* has the usual properties:

$$\begin{aligned}
&(a1) && s \cup s = s; \\
&(a2) && s \cup s' = s' \cup s; \\
&(a3) && s \cup (s' \cup s'') = (s \cup s') \cup s''; \\
&(a4) && s \cup \emptyset = s;
\end{aligned}$$

and

$$\begin{aligned}
&(b1) && element\,(\emptyset, x) = F; \\
&(b2) && element\,\bigl(s \cup make(x),\, x\bigr) = T; \\
&(b3) && \text{if } \ x \neq x' \ \text{ then } element\,\bigl(s \cup make\,(x),\, x'\bigr) \;=\; element\,(s, x').
\end{aligned}$$

As in Example 6, it is not difficult to show that, in the presence of the other axioms, (b3) can be replaced by the following generalized multiequation:

$$\Bigl[(x = x') \wedge \Bigl(element\,(s,x) = F\Bigr)\Bigr] \\ \qquad\qquad \nabla \; \Bigl[element\,\bigl(s \cup make\,(x),\, x'\bigr) \;=\; element\,(s,x')\Bigr].$$

Thus, we are led to the following generalized multiequational presentation:

$SET(U)$ =

$SORTS \qquad set, data, bool$

$$
\begin{array}{llll}
OPNS & \cup & : & set \times set \to set \\
& make & : & data \to set \\
& element & : & set \times data \to bool \\
& \emptyset & : & \to set \\
& x & : & \to data \quad \text{for each } \; x \in U \\
& T & : & \to bool \\
& F & : & \to bool
\end{array}
$$

$AXIOMS$ (a1)–(a4), (b1), and (b2) above;

$$(z = T) \bigtriangledown (z = F);$$

$$\Big[(x = x') \wedge element\,(s, x) = F\Big] \bigtriangledown$$

$$\bigtriangledown \Big[element\,\Big(s \cup make\,(x),\, x'\Big) = element\,(s, x')\Big].$$

(Here s, s' and s'' are variables of sort set, and x and x' variables of sort $element$.)

Example 8. Prime-cycle algebras. Consider unary algebras on one operation α which forms a single cycle of prime length. This can be axiomatized by the following generalized multiequation

$$\bigtriangledown_{p \text{ a prime}} \;\bigtriangledown_{i=0,1,\ldots,p-1} \Big[\Big(\alpha^p(x) = x\Big) \wedge \alpha^i(x) = y\Big].$$

Remark 3. The above example shows that we cannot restrict multiequations to *finite* exclusive-or's of equations without changing the concept of multivariety: the class of all unary algebras forming a prime cycle is not axiomatizable by any theory of finitary first-order logic - see [AR_2].

We now present the proof, promised in Remark 2, that generalized multiequations do not expand the concept of multivariety. The idea of the proof is well illustrated by Example 5 above.

Lemma 1. Every class of Σ-algebras presented by a set E of generalized multiequations is equivalent to a multivariety: there exists a signature Σ', extending Σ, and a set E' of multiequations such that $\mathbf{Alg}\,(\Sigma, E) \cong \mathbf{Alg}\,(\Sigma', E')$.

Proof. Let Σ be an S-sorted signature. The new set of sorts will be S^+, the set of all nonempty words $s_1 \ldots s_n$ over S. The new set Σ' of S^+-sorted operations consists of

(a) unary operations $\pi^i_{s_1 \ldots s_n} : s_1 s_2 \ldots s_n \to s_i$ for all $s_1 \ldots s_n \in S^+$ and

$i = 1, \ldots, n$ (e.g., $\pi^1_s : s \to s$),

(b) n-ary operations $tuple_{s_1 \ldots s_n} : s_1 \times s_2 \times \cdots \times s_n \to s_1 s_2 \ldots s_n$ for all $s_1 \ldots s_n \in S^+$

and

(c) operations $[\alpha_1, \ldots, \alpha_n]$ where every $\alpha_i : s_{i1} \times \cdots \times s_{im_i} \to t_i$ is a Σ-operation and the Σ'-arity is as follows:

$$[\alpha_1, \ldots, \alpha_n] : s_{11} \times \cdots \times s_{1m_1} \times s_{21} \times \cdots \times s_{2m_2} \times \cdots \times s_{nm_n} \to t_1 \ldots t_n.$$

Given variables x_i of sort s_i $(i = 1, \ldots, n)$ for Σ, we create a new variable $[x_1, \ldots, x_n]$ of sort $s_1 \ldots s_n$ for Σ'. We then assign to every n-tuple $\tau_1, \ldots, \tau_n$ of Σ-terms $\tau_i : s_{i1} \times \cdots \times s_{im_i} \to t_i$ a Σ'-term

$$[\tau_1, \ldots, \tau_n] : s_{11} \times \cdots \times s_{nm_n} \to t_1 \ldots t_n$$

by induction on the maximum k of the complexity of $t_1, \ldots, t_n$ as follows:

$k = 0$: if each τ_i is a variable, then so is $[\tau_1, \ldots, \tau_n]$.

$k + 1$: each τ_i is either a variable x of sort s_i, in which case we "substitute" it by $\alpha_i(\tau_{i1})$ where $\tau_{i1} = x$ and $\alpha_i = \pi^1_{s_i}$, or $\tau_i = \alpha_i\ (\tau_{i1}, \ldots, \tau_{im_i})$ for some Σ-operation $\alpha_i : s_{i1} \times \cdots \times s_{im_i} \to t_i$ and some terms τ_{ij} of complexity $\leq k$ and output sort s_{ij}, then we put

$$[\tau_1, \ldots, \tau_n] = [\alpha_1, \ldots, \alpha_n](\tau_{11}, \ldots, \tau_{1m_1}, \tau_{21}, \ldots, \tau_{2m_2}, \ldots, \tau_{nm_n}).$$

This allows us to assign to each finite conjunction $(\tau_1 = \sigma_1) \wedge \cdots \wedge (\tau_k = \sigma_k)$ of Σ-equations a Σ'-equation $[\tau_1, \ldots, \tau_k] = [\sigma_1, \ldots, \sigma_k]$. Thus, from the set E of generalized multiequations we obtain a set $[E]$ of multiequations over Σ' and we add to them the following equations:

$$\begin{array}{ll} tuple_{s_1 \ldots s_n}(\pi^1_{s_1 \ldots s_n} z, \ldots, \pi^n_{s_1 \ldots s_n} z) = z & \\ \pi^i_{s_1 \ldots s_n} tuple_{s_1 \ldots s_n}(x_1, \ldots, x_n) = x_i & \quad \text{for} \quad s_1 \ldots s_n \in S^+ \end{array}$$

and

$$\pi^i_{t_1 \ldots t_n}[\alpha_1, \ldots, \alpha_n] = \alpha_i$$

for Σ-terms α_i with sorts t_i.

The resulting multivariety over Σ' is obviously isomorphic to $\mathbf{Alg}\,(\Sigma, E)$. $\square$

II. Multiinitial-Algebra Semantics

Definition. A set $\{A_i;\ i \in I\}$ of objects of a category is said to be *multiinitial* provided that for every object B of that category there exists a unique morphism $f : A_i \to B$ for a unique index $i \in I$. In other words, the set $\bigcup\limits_{i \in B} \hom(A_i, B)$ has a unique element.

A multiinitial set is a special case of a general concept of multicolimit (see [D_1]).

Remark 4. If A is an initial object, then $\{A\}$ is a multiinitial set. It is well known that initial objects are unique up-to isomorphism. Multiinitial sets are also essentially unique in the following strong sense:

(1) For every pair $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ of multiinitial sets there exists a (unique) bijection $f : I \to J$ such that $A_i \cong B_{f(i)}$ for all $i \in I$.

(2) If $\{A_i\}_{i \in I}$ is a multiinitial set and $\{B_j\}_{j \in J}$ is a set of objects such that for some bijection $f : I \to J$ we have $A_i \cong B_{f(i)}$ for all $i \in I$, then $\{B_j\}_{j \in J}$ is also multiinitial.

Proof of (1). For each $i \in I$ the multiinitiality of $\{B_j\}_{j \in J}$ guarantees that there exists a unique $f(i) \in J$ and a unique morphism $r_i : B_{f(i)} \to A_i$. By symmetry, for each $j \in J$ there exists a unique $g(j) \in I$ and a unique morphism $s_j : A_{g(j)} \to B_j$. The morphism

$$A_{g\left(f(i)\right)} \xrightarrow{s_{f(i)}} B_{f(i)} \xrightarrow{r_i} A_i$$

guarantees $g\big(f(i)\big) = i$ and $r_i s_{f(i)} = id$ (because the multiinitiality of $\{A_i\}_{i \in I}$ implies that the only morphism from $A_{i'}$ to A_i, $i' \in I$, is id_{A_i}, in which case $i = i'$). Analogously, $f\big(g(j)\big) = j$ and $s_j r_{g(j)} = id$. Consequently, g is the inverse mapping of f, and $r_i = s^{-1}_{f(i)}$, thus, $A_i \cong B_{f(i)}$.

The proof of (2) is trivial. $\square$

Example 9. (a) $FIELD$ (see Example 3). Homomorphisms are here precisely the monomorphisms of rings with unit. For each field F of characteristic p (where p is a prime or ∞) we have a unique homomorphism from Z_p to F, where Z_∞ denotes the field of rational numbers, and for each prime p, Z_p is the Galois field of order p. Thus, the following set

$$\{Z_p\,;\ \ p \ \text{ a prime or } \ p = \infty\}$$

is multiinitial in $FIELD$.

(b) *Arrays* (see Example 6). We describe a multiinitial family in $MARRAY(D, C)$. Denote by

$$\mathscr{R}$$

the set of all pairs $R = (R_{dom}, R_{cod})$ of equivalence relations R_{dom} on the set D and R_{cod} on the set $C_\perp$ subject to the following condition:

$$\text{if } R_{cod} = (C_\perp \times C_\perp) \quad \text{then} \quad R_{dom} = \triangle_D$$

(where, for every set X, we denote by $\triangle_X$ the least and by $X \times X$ the largest equivalence relations). Given $R \in \mathscr{R}$, define an algebra A_R of $MARRAY(D, C)$ by the following rule; we denote by $[z]$ the equivalence class of an element z.

Sorts of A_R : $(A_R)_{dom} = D/R_{dom}$;

$(A_R)_{cod} = C_\perp/R_{cod}$;

$(A_R)_{array}$ is the set of all functions $f : D/R_{dom} \to C_\perp/R_{cod}$ of finite support (i.e., with value $[\perp]$ for all but finitely many elements of D/R_{dom} ;

$(A_R)_{bool} = \{T, F\}$.

Operations of A_R : $chg_{A_R}(f, [x], [y])$ has value $[y]$ at $[x]$, otherwise agrees with f ;

$val_{A_R}(f, [x]) = f([x])$;

Ω_{A_R} is the constant function with value $[\perp]$;

$\perp_{A_R} = [\perp]$;

$d_{A_R} = [d]$;

$c_{A_R} = [c]$;

$equal_{A_R}$ is the equality predicate.

It is easy to verify that A_R satisfies all the multiequations of $MARRAY(D, C)$. We are going to verify that the collection of those algebras is multiinitial.

Let A be an algebra of the multiequational specification $MARRAY(D, C)$. We will prove that there exists a unique $R \in \mathscr{R}$ for which a homomorphism $h : A_R \to A$ can be found, and that h is also unique.

(I) Assume $c_A = \perp_A$ for all $c \in C_\perp$.

Since every homomorphism $h : A_R \to A$ has the sort h_{cod} one-to-one, it follows that $R_{cod} = C_\perp \times C_\perp$. By the definition of $\mathscr{R}$, the unique candidate is $R = (\triangle_D, C_\perp \times C_\perp)$. It is easy to verify that the unique homomorphism

$h : A_R \to A$ is defined by

$$\begin{array}{ll} h_{dom} : D \to A_{dom}\,, & d \mapsto d_A\,; \\ h_{cod} : \{[\bot]\} \to A_{cod}\,, & [\bot] \mapsto \bot_A\,; \\ h_{array} : \{\Omega_{A_R}\} \to A_{array}\,, & \Omega_{A_R} \mapsto \Omega_A\,; \\ h_{bool} = \mathrm{id}_{\{T,F\}}\ . & \end{array}$$

(II) Assume $\widehat{c}_A \neq \bot_A$ for some $\widehat{c} \in C$.

Given a homomorphism $h : A_R \to A$, since h_{cod} is necessarily one-to-one, we know that the equivalence R_{cod} is defined by

$$cR_{cod}c' \quad \text{iff} \quad c_A = c'_A \qquad (c, c' \in C_\bot)\,. \tag{i}$$

Let us verify that the equivalence R_{dom} is defined by

$$dR_{dom}d' \quad \text{iff} \quad d_A = d'_A \qquad (d, d' \in D)\,. \tag{ii}$$

In fact, if $[\quad]$ denotes equivalence classes (w.r.t R_{cod} or R_{dom} in the given pair R, for which $h : A_R \to A$ exists) then in A_R we have

$$[d] \neq [d'] \Rightarrow chg_{A_R}([\widehat{c}], [d], \Omega_{A_R}) \neq chg_{A_R}([\widehat{c}], [d'], \Omega_{A_R})$$

therefore, in A we have

$$[d] \neq [d'] \Rightarrow chg_A(\widehat{c}_A, d_A, \Omega_A) \neq chg_A(\widehat{c}_A, d'_A, \Omega_A)$$

and this proves

$$[d] \neq [d'] \Rightarrow d_A \neq d'_A\,.$$

The reverse implication is clear due to $h_{dom}([d]) = d_A$. Thus (ii) holds.

For the pair $R = (R_{cod}, R_{dom})$ given by (i) and (ii) above we have a unique homomorphism $h : A_R \to A$. In fact, put

$h_{dom} = d_A$,
$h_{cod} = c_A$,
$h_{bool} = \mathrm{id}_{\{T,F\}}$,
$h_{array}(f)$ represents $h_{cod}f$ (for all $f : D/R_{dom} \to C_\bot/R_{cod}$ with finite support) i.e., is the unique element φ of A_{array} with $val_A(\varphi, d_A) = h_{cod}f([d])$ for all $d \in D$ and $val_A(\varphi, z) = \bot_A$ for all $z \in A_{dom} - im(h_{dom})$.

It is easy to verify that h is indeed a homomorphism, and the uniqueness of h follows from the fact that every $f \in (A_R)_{array}$ is generated from Ω_{A_R} by finitely many applications of chg_{A_R}.

(c) Sets (see Example 7). Given an equivalence relation R on the set U, we define an algebra A_R as follows:

$(A_R)_{data} = U/R\,;$

$(A_R)_{set} \;=$ the set of all finite subsets of $U/R\,;$

$(A_R)_{bool} \;= \{T, F\}$

with the obvious operations $\cup_{A_R}$ (union), $make_{A_R} : [x] \mapsto \{[x]\}$, $element_{A_R}$ and the obvious constants. These algebras form a multiinitial family: the proof is analogous to that in (b), since the equality relation on the sort data can be derived from the basic operations by

$$equal(x, x') = element\Big(make(x), x'\Big)\,.$$

Theorem 1. *For every set E of generalized multiequations the multivariety* $\mathbf{Alg}\,(\Sigma, E)$ *has a multiinitial set, and is closed in* $\mathbf{Alg}\,\Sigma$ *under subalgebras and connected limits.*

Proof. (1) $\mathbf{Alg}\,(\Sigma, E)$ is closed under subalgebras: given an algebra A in $\mathbf{Alg}\,(\Sigma, E)$ then every subalgebra A' satisfies all generalized multiequations which A satisfies, thus, A' lies in $\mathbf{Alg}\,(\Sigma, E)$.

(2) $\mathbf{Alg}\,(\Sigma, E)$ is closed under connected limits. That is, given a connected diagram $D : \mathscr{D} \to \mathbf{Alg}\,\Sigma$ with a limit $(A \xrightarrow{\pi_d} Dd)_{d \in \mathscr{D}}$ then we have to show that A satisfies any generalized multiequation $\underset{i \in I}{\bigtriangledown} \bigwedge_{j=1}^{k(i)} (\alpha_{ij} = \beta_{ij})$ that each Dd satisfies. Let $f : X \to A$ be an interpretation of the variables of the given multiequation. For each $d \in D$, since Dd satisfies the multiequation, there exists $i_d \in I$ such that

$$(\pi_d \cdot f)^{\#}(\alpha_{i_d j}) = (\pi_d \cdot f)^{\#}(\beta_{i_d j}) \;\text{ for all }\; j = 1, \ldots, k(i_d)\,.$$

We claim that i_d is independent of d. Since the diagram D is connected, it is sufficient to verify that for each morphism $r : d \to d'$ in $\mathscr{D}$ we have $i_d = i_{d'}$. In fact, since Dr is a Σ-homomorphism, we have

$$Dr \cdot (\pi_d \cdot f)^{\#} = (\pi_{d'} \cdot f)^{\#}$$

(this follows from the fact that both sides are Σ-homomorphisms and due to $Dr \cdot \pi_d = \pi_{d'}$ they have the same value on all variables). Consequently, in Dd' the equalities $\alpha_{i_d j} = \beta_{i_d j}$ are fulfilled under the interpretation $(\pi_{d'} \cdot f)^{\#}$ of variables - but since for Dd' there is a unique index in I with that property, we conclude $i_d = i_{d'}$.

Thus, we have an index i $(= i_d)$ in I such that

$$(\pi_d \cdot f)^{\#}(\alpha_{ij}) = (\pi_d \cdot f)^{\#}(\beta_{ij}) \ \text{ for all } \ d \in D, \quad j = 1, \ldots, k(i).$$

From the properties of limits we conclude that in A we have

$$f^{\#}(\alpha_{ij}) = f^{\#}(\beta_{ij}) \ \text{ for all } \ j = 1, \ldots, k(i).$$

The index i with the last property is unique, since π_d is a homomorphism, thus, $(\pi_d \cdot f)^{\#} = \pi_d \cdot f^{\#}$, and for each Dd the index in I is unique.

(3) $\mathbf{Alg}\,(\Sigma, E)$ has a multiinitial family. In fact, let C be an initial Σ-algebra, and let $\mathscr{R}$ be the (possibly empty) set of all congruence relations R on C with

$$C/R \in \mathbf{Alg}\,(\Sigma, E).$$

The set $\mathscr{R}$ is partially ordered by inclusion (among subsets of $C \times C$) and we denote by $\mathscr{R}^*$ the set of all minimal elements of $\mathscr{R}$. Then the set C/R, $R \in \mathscr{R}^*$, is multiinitial in $\mathbf{Alg}\,(\Sigma, E)$.

In fact, given an algebra $A \in \mathbf{Alg}\,(\Sigma, E)$, denote by A_0 the smallest subalgebra of A (generated by all constant operations in A). By (1), we have $A_0 \in \mathbf{Alg}\,(\Sigma, E)$. Since A_0 has no proper subalgebra, it is isomorphic to C/R_0 for some congruence R_0; then $A_0 \in \mathbf{Alg}\,(\Sigma, E)$ implies $R_0 \in \mathscr{R}$. Let $\{R_t; t \in T\}$ be the set of all elements of $\mathscr{R}$ contained in R_0. For each t we have a canonical homomorphism

$$h_t : C/R_t \to C/R_0$$

and we form a multiple pullback of those homomorphisms in $\mathbf{Alg}\,\Sigma$. This can be described as the collection of canonical homomorphisms

$$k_t : C/\bar{R} \to C/R_t \ \text{ where } \ \bar{R} = \bigcap_{t \in T} R_t\,.$$

Due to (2) (and since multiple pullbacks are connected limits) we conclude that $C/\bar{R}$ lies in $\mathbf{Alg}\,(\Sigma, E)$, i.e., that $\bar{R} \in \mathscr{R}$. The construction of $\bar{R}$ clearly implies that $\bar{R}$ is minimal in $\mathscr{R}$, thus, $\bar{R} \in \mathscr{R}^*$. And we have a homomorphism

$$C/\bar{R} \xrightarrow{k_t} C/R_t \cong A_0 \hookrightarrow A$$

for some $t \in T$. It is obvious that this is the unique homomorphism from $C/\bar{R}$ to A. It remains to verify that also $\bar{R}$ is unique: suppose

$$f : C/R \to A, \quad R \in \mathscr{R}^*$$

is a homomorphism, then we prove $R = \bar{R}$. The image of f is (like C/R_0) an algebra with no proper subalgebra, in other words, f maps C/R onto $A_0 \cong C/R_0$, which implies $\bar{R} \subseteq R$. However, since $R \in \mathscr{R}^*$ (i.e. R is minimal in $\mathscr{R}$), we conclude $R = \bar{R}$. $\square$

Remark 5. Every multivariety $\mathbf{Alg}\,(\Sigma, E)$ is, moreover, *multireflective* in $\mathbf{Alg}\,\Sigma$. This means that every Σ-algebra A has a collection of homomorphisms

$$f_t : A \to A_t \quad (t \in T) \;\text{ with }\; A_t \in \mathbf{Alg}\,(\Sigma, E)$$

with the following universal property: given a homomorphism $h : A \to B$ with $B \in \mathbf{Alg}\,(\Sigma, E)$ there exists a unique $t \in T$ such that h factors through f_t (i.e., $h = h' \cdot f_t$ for some homomorphism $h' : A_t \to B$) and the factorization (i.e., h') is also unique.

The proof is quite analogous to that in Theorem 1: instead of C/R for all minimal congruences R on I with $C/R \in \mathbf{Alg}\,(\Sigma, E)$ we simply consider all the canonical homomorphisms $f_R : A \to A/R$ where R is a minimal congruence on A with $A/R \in \mathbf{Alg}\,(\Sigma, E)$.

Remark 6. Due to Theorem 1, multiequational specifications always allow us to work with *multiinitial-algebra semantics* as the semantics presented by all algebras isomorphic to some member of the (essentially unique) multi-initial set. This is less satisfactory than the initial-algebra semantics which works with just one algebra and its isomorphic copies. On the other hand, the multiinitial-algebra semantics is much more "sharp" than the loose semantics, working with the whole class $\mathbf{Alg}\,(\Sigma, E)$.

In the example of arrays, the multiinitial semantics allows for an arbitrary fixed identification of the domain-elements, and then provides the "expected" abstract data type with the modified domain D/R and the codomain $C_\perp/R$. Among all those possibilities, a striking one consists in identifying nothing, i.e., with R just the diagonal relation. This has inspired Löwe and Wolter in [LW] to look for a *choice* of a concrete member of the multiinitial family. Unfortunately, the results presented in [LW] are very technical, and moreover, lead to a restriction of homomorphisms to "compatible" ones, which we find rather unnatural. It seems to us that the whole multiinitial set should be used. See the next part for the appropriate generalization to parametric data types.

III. Multifree-Algebra Semantics for Parametric Data types

In the main example of our paper, arrays, we have treated above the elements of the index and attribute sets as constants. This has led to the multiinitial-algebra semantics. A much more natural approach is to view those sets as parameters. We use the notation of [EM] for parametric specifications, and we show how to generalize multiinitial-algebra semantics to multifree-algebra semantics to cope with parameters.

Recall that a *parametric specification* is a pair of specifications $(PAR, SPEC)$

such that $SPEC$ (the body) is an extension of PAR (the parameter). Thus, more precisely, a multiequational parametric specification is a pair $(PAR, SPEC)$ such that

$$PAR = (S, \Sigma, E)\,,$$

where Σ is an S-sorted signature and E is a set of generalized multiequations, and

$$SPEC = (S', \Sigma', E')$$

where Σ' is an S'-sorted signature with

$$S \subseteq S' \quad \text{and} \quad \Sigma \subseteq \Sigma'$$

(and for each $\sigma \in \Sigma$ the arity of σ in Σ is the same as the arity of the same element $\sigma \in \Sigma'$) and E' is a set of generalized multiequations with

$$E \subseteq E'.$$

Example 10. Arrays. The parametric specification

$$MARRAY$$

(which summarizes the ideas of Example 6 with parameters D and C) has the following parameter specification

$$
\begin{array}{l}
PAR = \\
\quad \begin{array}{ll}
SORTS & dom, cod \\
OPNS & \perp\ :\to cod
\end{array}
\end{array}
$$

and the following body specification

$$MARRAY\ = PAR+$$

$$
\begin{array}{llcll}
SORTS & array, bool & & & \\
OPNS & chg & : & dom \times cod \times array & \to array \\
 & val & : & array \times dom & \to cod \\
 & \Omega & : & & \to array \\
 & T, F & : & & \to bool \\
 & equal & : & cod \times cod & \to bool
\end{array}
$$

$$
\begin{aligned}
AXIOMS\quad & val\big(chg(x,y,f),x\big)=y;\\
& chg(x,\perp,\Omega)=\Omega;\\
& chg\big(x,y,chg(x,y',f)\big)=chg(x,y,f);\\
& (y=y')\ \triangledown\ \big(equal(y,y')=F\big);\\
& (z=T)\ \triangledown\ (z=F);\\
& \big[(x=x')\wedge\big(equal(y,y')=F\big)\big]\ \triangledown\\
& \qquad \big[chg\big(x,y,chg(x',y',f)\big)=chg\big(x',y',chg(x,y,f)\big)\big].
\end{aligned}
$$

Remark 7. Denote by

$$U:\mathbf{Alg}\,(SPEC)\to\mathbf{Alg}\,(PAR)$$

the forgetful functor which to every algebra A of the body-specification assigns its reduct $U(A)$, i.e., the derived Σ-algebra which (due to $E\subseteq E'$) is an algebra of the parameter-specification.

If U is a right adjoint, i.e., if there exists a free $SPEC$-algebra $F(A)$ on each parameter algebra A, then we can work with the free-algebra semantics: the abstract data type with parameter A is represented by the class of free algebras on A, i.e., all Σ'-algebras isomorphic to $F(A)$. Whenever $SPEC$ is an equational presentation, free-algebra semantics is available, see [EM].

For multiequational specifications we have to work with the corresponding multi-concept: a functor $U:\mathscr{K}\to\mathscr{L}$ is called a *right multiadjoint* provided that for each object A of $\mathscr{L}$ the comma-category $A\downarrow U$ has a multiinitial set. That is, there exists a set of arrows $f_i:A\to UK_i\ \ (i\in I)$ such that for every arrow $f:A\to UK$ in $\mathscr{L}$ there exists a unique $i\in I$ for which f factors as $f=Uh\cdot f_i$ for some $h:K_i\to K$ in $\mathscr{K}$, and, moreover, h is also unique with this property.

If a given parametric specification has the property that the functor

$$U:\mathbf{Alg}\,(SPEC)\to\mathbf{Alg}\,(PAR)$$

is a right multiadjoint, we say that the specification has a *multifree-algebra semantics*. The abstract data type on a parameter A is then represented by the collection of $SPEC$-algebras $K_i\ \ (i\in I)$, where $f_i:A\to UK_i\ \ (i\in I)$ is a multiinitial set for $A\downarrow U$. By Remark 4 that collection is abstract, i.e., determined up-to isomorphism.

Example 10 continued. The parametric specification $MARRAY$ has multi-free-algebra semantics: given a parameter D (domain) and $C_\perp$ (codomain) (with constant $\perp$), forming an algebra of PAR, then a multifree algebra on that parameter is the collection of algebras A_R described in Example 9. (More precisely, the natural reducts of those algebras, since here we have no constants of sort *dom*, and only one constant, $\perp$, of sort *cod*.)

Example 11. Parametric specification SET (see Example 7). Here we put
$PAR =$
$SORTS \quad data$
and the body-specification is
$SET \ = \ PAR+$

$SORTS$	$set, bool$
$OPNS$	see Example 7, except there are no constants of sort data
$AXIOMS$	see Example 7

This parametric specification has multifree-algebra semantics given, for each parameter U, by the algebras A_R of Example 9 (c) (naturally reduced since here we have no constant operations of sort data).

Theorem 2. *For every multiequational parametric specification a multifree-algebra semantics exists. That is, the forgetful functor* $U : \mathbf{Alg}\,(SPEC) \to \mathbf{Alg}\,(PAR)$ *is a right multiadjoint.*

Proof. It is well-known that the forgetful functor

$$U_0 : \mathbf{Alg}\,\Sigma' \to \mathbf{Alg}\,\Sigma$$

assigning to every Σ'-algebra its Σ-reduct, is a right adjoint. Let $F_0 : \mathbf{Alg}\,\Sigma \to \mathbf{Alg}\,\Sigma'$ denote a corresponding left adjoint with the front adjunction $\eta_0 : 1 \to U_0F_0$. For every algebra A of the parameter specification PAR denote by $\mathscr{R}$ the set of all Σ'-congruences R on the algebra $F_0(A)$ such that the quotient algebra $F_0(A)/R$ lies in $\mathbf{Alg}\,(SPEC)$. Denote by

$$f_R : A \to U\Big(F_0(A)/R\Big)$$

the composite of $(\eta_0)_A : A \to U_0F_0(A)$ and U_0h_R, where $h_R : F_0(A) \to F_0(A)/R$ is the canonical Σ'-homomorphism. The collection of all $f_R : A \to U \Big(F_0(A)/R\Big)$, where R is a minimal member of $\mathscr{R}$ (under the set-theoretical order on $\mathscr{R}$) is multiinitial in the comma-category $A \downarrow U$. The proof is analogous to that of Theorem 1. Let K be a $SPEC$-algebra and let $f : A \to UK$ be a Σ-homomorphism. We have the unique Σ'-homomorphism $f^\# : F_0(A) \to K$ with $Uf^\#$ extending f, and the Σ'-subalgebra $K_0 = f^\#\Big[F_0(A)\Big]$ of K is a member of $\mathbf{Alg}\,(SPEC)$ because $\mathbf{Alg}\,(SPEC)$ is closed under subalgebras in $\mathbf{Alg}\,\Sigma'$. On the other hand, $K_0 = f^\#\Big[F_0(A)\Big]$ is a quotient algebra of $F_0(A)$, thus, it is isomorphic to $F_0(A)/R$ for a congruence R; we have $R \in \mathscr{R}$. Next we form a multiple pullback of the canonical homomorphisms $h_t : F_0(A)/R_t \to F_0(A)/R$ where $\{R_t; t \in T\}$ is the collection of all members of $\mathscr{R}$ contained in R. This can be described as the collection of canonical homomorphisms $k_t : F_0(A)/\bar{R} \to F_0(A)/R_t$, where $\bar{R} = \bigcap R_t$, and we have $\bar{R}$ as a minimal element of $\mathscr{R}$, since $SPEC$ is closed under pullbacks in $\mathbf{Alg}(\Sigma')$. It is easy to see that $f^{\#}$ factors through the canonical morphism $k_{\bar{R}} : F_0(A) \to F_0(A)/\bar{R}$ in $\mathbf{Alg}\,\Sigma'$, therefore, f factors through $f_R = U_0 k_{\bar{R}} \cdot (\eta_0)_A$. The argument about unicity of factorization is analogous to that in Theorem 1. $\square$

Remark 8. Semantics of parametric data types. Recall that in the equational case, where the forgetful functor $U : \mathbf{Alg}\,(SPEC) \to \mathbf{Alg}\,(PAR)$ has a left adjoint, any left adjoint

$$F : \mathbf{Alg}\,(PAR) \to \mathbf{Alg}\,(SPEC)$$

(i.e., any free functor) of U is a representative of the semantics. In the multiequational case, instead of a free functor we have a free *multifunctor* (assigning to every PAR-algebra A a multiinitial family of the comma category $A \downarrow U$).

Multifunctors from $\mathscr{K}$ to $\mathscr{L}$ are just functors from $\mathscr{K}$ to a free product completion

$$\Pi\,\mathscr{L}$$

of $\mathscr{L}$ which can be described as follows.

Objects of $\Pi\,\mathscr{L}$ are families $(L_i)_{i\in I}$ of objects of $\mathscr{L}$.

Morphisms f from $(L_i)_{i\in I}$ to $(L'_j)_{j\in J}$ are pairs consisting of a function $\hat{f} : J \to I$ and a collection $(L_{\hat{f}(j)} \xrightarrow{f_j} L'_j)_{j\in J}$ of morphisms of $\mathscr{L}$

Composition of $f : (L_i)_{i\in I} \to (L'_j)_{j\in J}$ and $g : (L'_j)_{j\in J} \to (L''_k)_{k\in K}$ is defined by the function $\hat{f}\hat{g} : K \to I$ and the collection of composites

$$\left(L_{\hat{f}(\hat{g}(k))} \xrightarrow{f_{\hat{g}(k)}} L'_{\hat{g}(k)} \xrightarrow{g_k} L''_k\right)_{k\in K}$$

Note the obvious embedding of $\mathscr{L}$ into $\Pi\,\mathscr{L}$ given by $L \mapsto (L)$.

Thus a *multifunctor* from $\mathscr{K}$ to $\mathscr{L}$, denoted by

$$F : \mathscr{K} \multimap \mathscr{L}$$

is just another name for a functor

$$F : \mathscr{K} \to \Pi\,\mathscr{L}\,.$$

Consequently, the concept of natural isomorphism for multifunctors is that of the corresponding functors. Also composing functors and multifunctors is clear: Given a functor $P : \mathscr{K}_0 \to \mathscr{K}$, the composite $F \cdot P : \mathscr{K}_0 \multimap \mathscr{L}$ is just the composite functor $\mathscr{K}_0 \xrightarrow{P} \mathscr{K} \xrightarrow{F} \Pi\mathscr{L}$. And given a functor $Q : \mathscr{L} \to \mathscr{L}_0$ we denote by $\Pi Q : \Pi\mathscr{L} \to \Pi\mathscr{L}_0$ the free extension to the product-preserving functor (given by $(L_i)_{i\in I} \mapsto (QL_i)_{i\in I}$) and define $Q \cdot F : \mathscr{K} \multimap \mathscr{L}_0$ as the composite functor $\mathscr{K} \xrightarrow{F} \Pi\mathscr{L} \xrightarrow{\Pi Q} \Pi\mathscr{L}_0$. (It is clear that Π is a part of a monad over **CAT** whose Kleisli category is that of all categories and multifunctors.)

Example 12. For every parametric multiequational specification $(PAR, SPEC)$ we have a *free multifunctor*

$$F : \mathbf{Alg}\,(PAR) \multimap \mathbf{Alg}\,(SPEC)$$

together with a (universal) natural transformation

$$\eta : \mathrm{Id}_{\mathbf{Alg}\,(PAR)}) \to U \cdot F$$

defined as follows:

On objects A we choose a multifree family on A consisting of $SPEC$-algebras B_t $(t \in T)$ and PAR-homomorphisms $\eta_t^A : A \to U(B_t)$ $(t \in T)$, and define

$$\begin{aligned} &FA = (B_t)_{t\in T} \\ &\eta^A : (A) \to (U(B_t))_{t\in T} \quad \text{given by } \eta_t^A (t \in T) \end{aligned}$$

On morphisms $f : A \to A'$ where $\eta_s^{A'} \to U(B'_s)$, $s \in S$ is a chosen multifree family on A', we define

$$\widehat{f} : S \to T \quad \text{and} \quad f_s : B_{\widehat{f}(s)} \to B'_s \qquad (s \in S)$$

by the commutativity of the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta^A_{\widehat{f}(s)}"'] & A' \arrow[d, "\eta_s^{A'}"] \\ U(B_s) \arrow[r, "U(f_s)"'] & U(B'_s) \end{tikzcd}$$

where we put $F(f) = (\widehat{f}, (f_s))$. Then F is determined uniquely up-to a natural isomorphism. Following the tradition of equational specifications we define the *semantics* of $(PAR, SPEC)$ to be the class of all free multifunctors $\mathbf{Alg}\,(PAR) \multimap \mathbf{Alg}\,(SPEC)$.

IV. Correctness and Persistency of Multifree Semantics

In the present section we generalize the well-known result concerning equational specifications, see [EM], that correctness is equivalent to persistence.

Remark 9. Category of signatures. (a) Recall from [EM] the concept of the category **SIG** of signatures: its objects are pairs (S, Σ) where S is a set and Σ is an S-sorted signature. Morphisms $h : (S, \Sigma) \to (S', \Sigma')$ are pairs $h = (h_{\mathrm{so}}, h_{\mathrm{op}})$ of functions

$$h_{\mathrm{so}} : S \to S' \quad \text{and} \quad h_{\mathrm{op}} : \Sigma \to \Sigma'$$

preserving arities: to an operation symbol of arity $s_1 \times \cdots \times s_n \to s$ the function h_{op} assigns an operation symbol of arity $h_{\mathrm{so}}(s_1) \times \cdots \times h_{\mathrm{so}}(s_n) \to h_{\mathrm{so}}(s)$.

(b) We have a natural contravariant functor

$$\mathbf{Alg} : \mathbf{SIG} \to \mathbf{CAT}$$

assigning to every signature (S, Σ) the category $\mathbf{Alg}(S, \Sigma)$ of all algebras of that signature. To every signature morphism $h : (S, \Sigma) \to (S', \Sigma')$ it assigns the following functor

$$V_h : \mathbf{Alg}(S', \Sigma') \to \mathbf{Alg}(S, \Sigma) :$$

on objects A': define $V_h(A')$ to be the Σ-algebra with sorts $A'_{h_{so}(s)}$ for all $s \in S$, and operations $h_{\mathrm{op}}(\sigma)_{A'}$, for all $\sigma \in \Sigma$;

on morphisms $f : A' \to B'$: the s-sort of the corresponding Σ-homomorphism is $f_{h_{so}(s)}$ $(s \in S)$.

(c) Every signature homomorphism $h : (S, \Sigma) \to (S', \Sigma')$ leads to a translation of multiequations as follows: given an S-sorted set X of variables, denote by X' the corresponding S'-sorted set:

$$X'_{s'} = \coprod_{t \in h_{\mathrm{so}}^{-1}(s')} X_t \qquad (s' \in S').$$

If $T_\Sigma(X)$ and $T_{\Sigma'}(X')$ denote the corresponding absolutely free algebras (of terms), denote by

$$h^{\#} : T_\Sigma(X) \to V_h(T_{\Sigma'}(X'))$$

the unique Σ-homomorphism with $h^{\#}(x) = x$ for all variables $x \in X$. That is, $h^{\#}$ translates Σ-terms of sort $h_{\mathrm{so}}^{-1}(s)$ into Σ'-terms of sort s' by the recursive rule

$$h^{\#}(\sigma(\tau_1, \ldots, \tau_n)) = h_{\mathrm{op}}(\sigma)(h^{\#}\tau_1, \ldots, h^{\#}\tau_n)$$

for all $\sigma \in \Sigma$ of outcome sort s, and all terms $\tau_1, \ldots, \tau_n \in T_\Sigma(X)$ with appropriate sorts. For every multiequation

$$e = \mathop{\nabla}_{i\in I}(\alpha_i = \beta_i) \quad \text{in variables } X \text{ over } (S,\Sigma)$$

we obtain a "translated" multiequation as follows

$$h^{\#}(e) = \mathop{\nabla}_{i\in I}(h^{\#}\alpha_i = h^{\#}\beta_i) \quad \text{in variables } X' \text{ over } (S',\Sigma')\,.$$

Lemma 2. *Let $h : (S,\Sigma) \to (S',\Sigma')$ be a signature morphism and A' an algebra of signature (S',Σ'). Then $V_h(A')$ satisfies a multiequation e iff A' satisfies the translated multiequation $h^{\#}(e)$.*

Proof. (1) Let $V_h(A')$ satisfy $e = \nabla(\alpha_i = \beta_i)$. For every interpretation f' of variables X' in A' (i.e., mappings $f'_{s'} : X'_{s'} \to A'_{s'},\ s' \in S'$) we define an interpretation f of X in $V_h(A')$ by $f_s(x) = f'_{h_{\mathrm{so}}(s)}(x)$. The unique extension $\widehat{f} : T_\Sigma(X) \to V_h(A')$ of f to a Σ-homomorphism is the composite of $h^{\#}$ with the V_h-image of $\widehat{f'} : T_{\Sigma'}(X') \to A'$; i.e., for every Σ-term α we have

$$\widehat{f}(\alpha) = \widehat{f}'(h^{\#}\alpha)\,.$$

Since $V_h(A')$ satisfies e, there exists a unique i with $\widehat{f}(\alpha_i) = \widehat{f}(\beta_i)$, i.e., a unique i with $\widehat{f}'(h^{\#}\alpha_i) = \widehat{f}'(h^{\#}\beta_i)$. This proves that A' satisfies $h^{\#}(e)$.

(2) Let A' satisfy $h^{\#}(e)$. For every interpretation f of variables X in $V_h(A')$, we define an interpretation f' of X' in A' as follows:

$$f'_{s'} : \coprod X_t \to A_{s'} \quad \text{has components } f_t \text{ for all } s' = h_{\mathrm{so}}(t).$$

We get, again, $\widehat{f} = V_h(\widehat{f'}) \cdot h^{\#}$. If A' satisfies $h^{\#}e$, there is a unique i with $\widehat{f}'(h^{\#}\alpha_i) = \widehat{f}'(h^{\#}\beta_i)$ – thus a unique i with $\widehat{f}(\alpha_i) = \widehat{f}(\beta_i)$; therefore, $V_h(A')$ satifies e. $\square$

Remark 10. Category of specifications. We now introduce the category **SPEC** of multiequational specifications. Objects are triples

$$SPEC = (S,\Sigma,E)$$

where (S,Σ) is a signature, and E a set of multiequations in that signature. Morphisms

$$h : SPEC \to SPEC' = (S',\Sigma',E')$$

are signature morphisms $h : (S,\Sigma) \to (S',\Sigma')$ such that

$$e \in E \Rightarrow h^{\#}(e) \in E'\,.$$

The above lemma proves that the functor $V_h : \mathbf{Alg}\,(S',\Sigma') \to \mathbf{Alg}\,(S,\Sigma)$ sends algebras satisfying E' to algebras satisfying E. Thus, we can form the following contravariant functor

$$\mathbf{Alg} : \mathbf{SPEC} \to \mathbf{CAT}$$

to every specification (S, Σ, E) it assigns the category $\mathbf{Alg}\,(S, \Sigma, E)$ of all algebras of that specification. And to every specification morphism $h : SPEC \to SPEC'$ it assigns the domain-codomain restriction of the above functor V_h to the (equally denoted) functor

$$V_h : \mathbf{Alg}\,(SPEC') \to \mathbf{Alg}\,(SPEC)\,.$$

Example 13. Every parametric specification $(PAR, SPEC)$ defines a specification morphism

$$p : PAR \to SPEC$$

both components of which are the inclusion maps. Then

$$V_p : \mathbf{Alg}\,(SPEC) \to \mathbf{Alg}\,(PAR)$$

is the above forgetful functor U, see Remark 7.

Definition. A *parameter passing diagram* with formal parameter PAR and actual parameter $\overline{PAR}$ is a span in $\mathbf{SPEC}$

$$\begin{tikzcd} PAR \arrow[r, "p"] \arrow[d, "h"'] & SPEC \\ \overline{PAR} & \end{tikzcd}$$

consisting of a parametric specification $p : PAR \to SPEC$ and an arbitrary specification morphism $h : PAR \to \overline{PAR}$. We assume that at least one algebra of the specification $\overline{PAR}$ exists.

Remark 11. The last assumption is not needed in case of equational specifications of course. But in case of multiequational specifications it excludes unwanted examples such as $(x = x) \triangledown (x = x)$.

It is easy to verify that $\mathbf{SPEC}$ is a cocomplete category. We will only need a pushout of the parameter passing diagrams:

$$\begin{tikzcd} PAR \arrow[r, "p"] \arrow[d, "h"'] & SPEC \arrow[d, "\bar{h}"] \\ \overline{PAR} \arrow[r, "\bar{p}"'] & \overline{SPEC} \end{tikzcd}$$

It can be constructed by simply forming the corresponding two pushouts in $\mathbf{Set}$: let $\overline{PAR} = (\overline{S_0}, \overline{\Sigma_0}, \overline{E_0})$, then form pushouts

$$\begin{tikzcd} S_0 \arrow[r, "p_{so}"] \arrow[d, "h_{so}"'] & S \arrow[d, "\bar{h}_{so}"] \\ \overline{S}_0 \arrow[r, "\bar{p}_{so}"'] & \overline{S} \end{tikzcd} \qquad \begin{tikzcd} \Sigma_0 \arrow[r, "p_{\rm op}"] \arrow[d, "h_{\rm op}"'] & \Sigma \arrow[d, "\bar{h}_{\rm op}"] \\ \overline{\Sigma}_0 \arrow[r, "\bar{p}_{\rm op}"'] & \overline{\Sigma} \end{tikzcd}$$

and put $\overline{SPEC} = (\overline{S}, \overline{\Sigma}, \overline{p}^{\#}[\overline{E}_0]) \cup \overline{h}^{\#}[E])$. We can clearly choose these pushouts so that $\overline{p}_{\rm so}$ and $\overline{p}_{\rm op}$ are inclusion maps. The arities of operations in Σ are determined by those of $\overline{\Sigma}_0$ and Σ.

The above functor $\mathbf{Alg} : \mathbf{SPEC} \to \mathbf{CAT}^{\rm op}$ preserves the pushout we have just described (i.e., $\mathbf{Alg}$ maps that pushout to a pullback in $\mathbf{CAT}$). This is known as *amalgamation*:

$$\begin{tikzcd} \mathbf{Alg}\,(PAR) & \mathbf{Alg}\,(SPEC) \arrow[l, "V_p"'] \\ \mathbf{Alg}\,(\overline{PAR}) \arrow[u, "V_h"] & \mathbf{Alg}\,(\overline{SPEC}) \arrow[l, "V_{\bar{p}}"] \arrow[u, "V_{\bar{h}}"'] \end{tikzcd}$$

That is, given algebras $\overline{A}$ of $\overline{PAR}$ and B of $SPEC$ with $V_h(\overline{A}) = V_p(B) = A$, there exists a unique algebra of $\overline{SPEC}$, denoted by $\overline{A} +_A B$, satisfying

$$\overline{A} = V_{\bar{p}}(\overline{A} +_A B) \quad \text{and} \quad B = V_{\bar{h}}(\overline{A} +_A B)\,.$$

In fact: define the $\overline{S}_0$-sorts and $\overline{\Sigma}_0$-operations of $\overline{A} +_A B$ to be those of $\overline{A}$, and the S-sorts and Σ-operations to be those of B (which leads to no contradiction due to $V_h(\overline{A}) = V_p(B)$), then use Lemma 2 to conclude that $\overline{A} +_A B$ satisfies all multiequations of $\overline{E} = \bar{p}^{\#}[\overline{E}_0] \cup \bar{h}^{\#}[E]$. Analogously, given homomorphisms f of $\overline{PAR}$ and g of $SPEC$ with $V_h(f) = V_p(g)$ there exists a unique morphism k of $\overline{SPEC}$ with $f = V_{\bar{p}}(k)$ and $g = V_{\bar{h}}(k)$.

Example 14. Passing to comma categories. For every PAR-algebra K denote by $K \downarrow \mathbf{Alg}\,(PAR)$ the comma-category (of all (A, a) where $a : K \to A$ is a homomorphism of PAR). Let $h : PAR \to \overline{PAR}$ be the extension of PAR by (a) a constant $\widehat{x}$ of sort s for every element $x \in K_s$ and (b) new equations $\widehat{\sigma}(\widehat{x}_1, \ldots \widehat{x}_n) = \widehat{x}$ for every $\sigma \in \Sigma_0$ and every instance of $\sigma_K(x_1, \ldots, x_n) = x$ in K. Then, obviously, $\mathbf{Alg}\,(\overline{PAR})$ can be identified with the above comma-category, and

$$V_h : K \downarrow \mathbf{Alg}\,(PAR) \to \mathbf{Alg}\,(PAR)\,, \qquad (A, a) \mapsto A$$

is the usual forgetful funtor. Furthermore, $\overline{SPEC}$ is obtained from $SPEC$ by the analogous extension, thus, $\mathbf{Alg}\,(\overline{SPEC})$ is the comma-category $K \downarrow V_p$ of all pairs (B, b) where B is a $SPEC$-algebra, and $b : K \to V_p(B)$ is a PAR-homomorphism. Again,

$$V_{\bar{h}} : K \downarrow V_p \to \mathbf{Alg}\,(SPEC)\,, \qquad (B, a) \mapsto B$$

is the usual forgetful functor. And so is

$$V_{\bar{p}} : K \downarrow V_p \to K \downarrow \mathbf{Alg}\,(PAR)\,, \qquad (B,a) \mapsto (UB, a).$$

Given a free multifunctor

$$F : \mathbf{Alg}\,(PAR) \multimap \mathbf{Alg}\,(SPEC)$$

we obtain a derived multifunctor

$$\overline{F} : K \downarrow \mathbf{Alg}\,(PAR) \multimap K \downarrow V_p$$

in the obvious way: given an object (A,a) of $K \downarrow \mathbf{Alg}\,(PAR)$, let FA be the set of $SPEC$-algebras B_t and PAR-homomorphisms $\eta_t : A \to V_p(B_t)$, $t \in T$, then $\overline{F}(A,a)$ is the set of $\overline{SPEC}$-algebras $(B_t, \eta_t a)$ together with $\overline{PAR}$-homomorphisms $\eta_t : (A,a) \to (V_p(B_t), \eta_t a)$, $t \in T$. Analogously on morphisms. It is easy to see that this multifunctor $\overline{F}$ is a free multifunctor of the parametric specification $(\overline{PAR}, \overline{SPEC})$. Observe that the following square

$$\begin{tikzcd}
\mathbf{Alg}\,(PAR) \arrow[r, "F"] & \mathbf{Alg}\,(SPEC) \\
\mathbf{Alg}\,(\overline{PAR}) \arrow[u, "V_h"] \arrow[r, "\overline{F}"'] & \mathbf{Alg}\,(\overline{SPEC}) \arrow[u, "V_{\bar{h}}"']
\end{tikzcd}$$

commutes in **CAT** (recall that multifunctors are just functors into the free-product-completions $\Pi(-)$).

Remark 12. Correctness of parametric specifications. Recall that an equational parametric specification is called correct iff it has, in every parameter passing diagram, the following properties:

(1) Actual parameter protection: $V_{\bar{p}} : \mathbf{Alg}\,(\overline{SPEC}) \to \mathbf{Alg}\,(\overline{PAR})$ preserves initial algebras

and

(2) Passing compatibility: for an initial $\overline{SPEC}$-algebra I the $SPEC$-algebras $V_{\bar{h}}(I)$ and $FV_hV_{\bar{p}}(I)$ are isomorphic.

How do we extend this to the multiequational setting? There are two obvious strategies: a maximalist would request (1) and (2) to hold for every member of a multiinitial family, whereas a minimalist would be happy if some member had the corresponding properties:

Definition. A multiequational parametric specification $(PAR, SPEC)$ is called *correct* if for every parameter passing diagram the following hold:

(1) Actual parameter protection: $V_{\bar{p}}$ maps a multiinitial family of $\mathbf{Alg}\,(\overline{SPEC})$ onto a multiinitial family of $\mathbf{Alg}\,(\overline{PAR})$;

and

(2) Passing compatibility: for every member I of a multiinitial family of $\overline{SPEC}$-algebras $V_{\bar{h}}(I)$ is isomorphic to every member of $FV_hV_{\bar{p}}(I)$.

It is called *weakly correct* if for every parameter passing diagram the following hold:

(1*) Weak actual parameter protection: $V_{\bar{p}}$ maps some member of a multiinitial family of $\mathbf{Alg}\,(\overline{SPEC})$ to a member of a multiinitial family of $\mathbf{Alg}\,(\overline{PAR})$;

and

(2*) Weak passing compatibility: for some member I of a multiinitial family of $\overline{SPEC}$-algebras $V_{\bar{h}}(I)$ is isomorphic to some member of $FV_hV_{\bar{p}}(I)$.

Remark 13. Let us apply these conditions to Example 14 (in which (2) is automatic due to $FV_h = V_{\bar{h}}\overline{F}$): It is obvious that

$$\mathbf{Alg}\,(\overline{PAR}) = K \downarrow \mathbf{Alg}\,(PAR)$$

has an initial object, viz, $K \xrightarrow{\mathrm{id}} K$. And if $\eta_t : K \to V_p(B_t)$, $t \in T$, is a multifree family on K, then

$$\mathbf{Alg}\,(\overline{SPEC}) = K \downarrow V_p$$

has the multiinitial family (B_t, η_t), $t \in T$. Thus, (1) states that each η_t is an isomorphism, whereas (1*) states that some η_t is an isomorphism. In the examples $MARRAY$ and SET above, the latter is true and the former is not. This indicates that weak correctness is the notion we really want to study here. We present now a necessary and sufficient condition for it.

Definition. A multiequational parametric specification is called *weakly persistent* provided that for every PAR-algebra K there exists a member of a multifree family on K, which is an isomorphism of PAR-algebras.

Theorem 3. *A multiequational parametric specification is weakly correct iff it is weakly persistent.*

Proof. (1) Weak correctness implies weak persistence: this follows from Example 14 above. In fact, actual parameter protection here is sufficient. Observe that the category

$$\mathbf{Alg}\,(\overline{PAR}) = K \downarrow (PAR)$$

has an initial object, $K \xrightarrow{\mathrm{id}} K$. Further observe that if $\eta_t : K \to V_p(B_t)$, $t \in T$, is a multifree family on K in $(PAR, SPEC)$, then the category

$$\mathbf{Alg}\,(\overline{SPEC}) = K \downarrow V_p$$

has the multiinitial family (B_t, η_t), $t \in T$. Thus, by weak actual protection some η_t is an isomorphism.

(2) Weak persistence implies weak correctness. We prove that for any parameter passing diagram and its pushout

$$\begin{tikzcd} PAR \arrow[r, "p"] \arrow[d, "h"'] & SPEC \arrow[d, "\bar{h}"] \\ \overline{PAR} \arrow[r, "\bar{p}"'] & \overline{SPEC} \end{tikzcd}$$

with p weakly persistent, the parametric specification $(\overline{PAR}, \overline{SPEC})$ is weakly persistent. Then we derive weak correctness easily.

Given a $\overline{PAR}$-algebra $\overline{K}$ put $K = V_h(\overline{K})$ and form the corresponding multifree families

$$\overline{\eta}_t : \overline{K} \to V_{\bar{p}}(\overline{B}_t) \qquad (t \in T) \text{ on } \overline{K} \text{ in } (\overline{PAR}, \overline{SPEC})$$

and

$$\eta_s : K \to V_p(C_s) \quad (s \in S) \text{ on } K \text{ in } (PAR, SPEC)\,.$$

By hypothesis there exists $s \in S$ such that η_s is an isomorphism of PAR-algebras. Without loss of generality, we can assume that

$$K = V_p(C_s) \quad \text{and} \quad \eta_s = \mathrm{id}_K\ .$$

(In fact, for the isomorphism η_s there clearly exists an isomorphism $r : C'_s \to C_s$ of $SPEC$-algebras with $K = V_p(C'_s)$ and $\eta_s = V_p(r)$. Now substitute, in the given multifree family on K, the algebra C_s by C'_s, and the homomorphism η_s by $V_p(r^{-1}) \cdot \eta_s = \mathrm{id}$; this results in a multifree family on K again). By the above mentioned amalgamation (Remark 11), we have a unique $\overline{SPEC}$-algebra $\overline{C}_s = \overline{K} +_K C_s$ with $V_{\bar{p}}(\overline{C}_s) = \overline{K}$ and $V_{\bar{h}}(\overline{C}_s) = C_s$. Now for the homomorphism

$$\mathrm{id}_{\overline{K}} : \overline{K} \to V_{\bar{p}}(\overline{C}_s)$$

there exists a unique $t \in T$ and a unique homomorphism

$$u : \overline{B}_t \to \overline{C}_s \quad \text{in} \quad \overline{SPEC}$$

with

$$\mathrm{id}_{\overline{K}} = V_{\bar{p}}(u) \cdot \overline{\eta}_t \,. \tag{1}$$

Furthermore, for the homomorphism

$$V_h(\overline{\eta}_t) : K \to V_h V_{\bar{p}}(\overline{B}_t) = V_p V_{\bar{h}}(\overline{B}_t)$$

there exists a unique $s_0 \in S$ and a unique homomorphism

$$v : C_{s_0} \to V_{\bar{h}}(\overline{B}_t)$$

with

$$V_h(\overline{\eta}_t) = V_p(v) \cdot \eta_{s_0} \,. \tag{2}$$

Let us prove that

$$s = s_0 \quad \text{and} \quad V_{\bar{h}}(u) \cdot v = \mathrm{id}_{C_s} \,. \tag{3}$$

In view of the definition of multifreeness, for (3) it is only necessary to prove that the triangle

$$\begin{tikzcd}
 & K \arrow[dl, "\eta_{s_0}"'] \arrow[d, dashed, "V_h(\overline{\eta}_t)"] \arrow[dr, "\eta_s = \mathrm{id}"] & \\
V_p(C_{s_0}) \arrow[r, "V_p(v)"'] & V_p V_{\bar{h}}(\overline{B}_t) \arrow[r, "V_p V_{\bar{h}}(u)"'] & V_p(C_s)
\end{tikzcd}$$

commutes: see (1) and (2). Thus, (2) is actually the equality

$$V_h(\overline{\eta}_t) = V_p(v)$$

which, by amalgamation again, yields a unique $\overline{SPEC}$-homomorphism $w : \overline{C}_s \to \overline{B}_t$ with

$$V_{\bar{p}}(w) = \overline{\eta}_t \quad \text{and} \quad V_{\bar{h}}(w) = V_p(v) \,. \tag{4}$$

Let us verify that w is inverse to u: the equality $wu = \mathrm{id}_{\overline{B}_t}$ follows from the definition of multifreeness and the following equality

$$V_{\bar{p}}(wu) \cdot \overline{\eta}_t = V_{\bar{p}}(w) = \overline{\eta_t}$$

(see (1) and (4)). The equality $uw = \mathrm{id}$ follows from amalgamation since we have both

$$V_{\bar{p}}(uw) = V_{\bar{p}}(u)\overline{\eta}_t = \mathrm{id}_{\overline{C_s}}$$

by (1) and (4), and

$$V_{\bar{h}}(uw) = \mathrm{id}_{\overline{B}_t}$$

which follows from multifreeness via

$$\begin{aligned} V_p\left[V_{\bar{h}}(uw)\right]\cdot\overline{\eta}_t &= \left[V_h V_{\bar{p}}(uw)\right]\cdot\overline{\eta}_t \\ &= V_h\left[V_{\bar{p}}(u)\cdot\overline{\eta}_t\right]\cdot\overline{\eta}_t && \text{by (4)} \\ &= \overline{\eta}_t && \text{by (1).} \end{aligned}$$

Thus, we have proved that u is an isomorphism, and by (1)

$$\overline{\eta}_t = V_{\bar{p}}(u^{-1})$$

is an isomorphism. Thus, $(\overline{PAR}, \overline{SPEC})$ is weakly persistent.

It follows that the parametric specification $(PAR, SPEC)$ is weakly correct:

(1*) Weak actual parameter protection: choose a multiinitial family $(\overline{K}^i)_{i\in I}$ of $\overline{PAR}$-algebras, and for each i let $\eta^i_t : \overline{K}^i \to V_{\bar{p}}(\overline{B}^i_t)$, $t \in T_i$ be a multifree family on $\overline{K}^i$. Then $(\overline{B}^i_t)_{i\in I, t\in T_i}$ is a multiinitial family of $\overline{SPEC}$-algebras. Now the set I is certainly nonempty (because we assume that some $\overline{PAR}$-algebra exists by definition of parameter passing diagrams). Choose any $i \in I$ and find, by the above, some t such that $\overline{\eta}^i_t$ is an isomorphism.

(2*) Weak passing compatibility: for the same choice of i and t as above we have

$$V_{\bar{h}}(\overline{B}^i_t) \cong C_s \qquad \text{(for } \overline{K} = \overline{K}^i \text{ and } C_s \text{ above)}$$

via the above isomorphism $u\eta^i_t : B_t \to \overline{C}_s$, since $V(\overline{C}_s) = C_s$. And

$$V_h V_{\bar{p}}(\overline{B}^i_t) = V_h(\overline{K}) = K$$

thus C_s is a member of the family $FV_hV_{\bar{p}}(\overline{B}^i_t)$. $\square$

Corollary. *The parametric specifications*

$$MARRAY \quad \text{(Example 10)}$$

and

$$SET \quad \text{(Example 11)}$$

are weakly correct.

V. A Birkhoff-type theorem

Recall that varieties, i.e., equational classes of algebras, have a concrete characterization (Birkhoff's Variety Theorem), stating that a class of Σ-algebras is a variety iff it is an HSP-class in $\mathbf{Alg}\Sigma$ (i.e., is closed under quotients, subalgebras and products). Besides, varieties are closed under directed colimits in $\mathbf{Alg}\Sigma$. Now multivarieties have closure properties analogous to varieties: instead of closure under quotients we have closure under consistent quotients (defined below), and instead of closure under limits (products) we have closure under connected limits. Example 15 below will show that these closure conditions are not quite sufficient to characterize multivarieties. However a slight adjustment will lead to Birkhoff-type theorems for natural generalizations of multivarieties.

Definition. A class $\mathcal{K}$ of Σ-algebras is said to be closed under *consistent quotients* provided that for each algebra $A \in \mathcal{K}$ and each congruence $\sim$ on A the quotient $A/\!\sim$ lies in $\mathcal{K}$ whenever $A/\!\sim$ itself has a quotient in $\mathcal{K}$.

Proposition 1. *For every set E of generalized multiequations the class* $\mathbf{Alg}\,(\Sigma, E)$ *is closed in* $\mathbf{Alg}\,\Sigma$ *under*

(a) *subalgebras*
(b) *connected limits*
(c) *directed colimits*

and

(d) *consistent quotients.*

Proof. We prove that given a generalized multiequation

$$\bigtriangledown_{i\in I} \bigwedge_{j=1}^{n_i} p_{ij} = q_{ij} \tag{1}$$

over variables from $X = \{x_1, \ldots, x_k\}$, then the class of all models of (1) is closed under the constructions (a) – (d). For (a) and (b) see Theorem 1.

(c) Let $A = \operatorname*{colim}_{t\in T} A_t$ be a directed colimit in $\mathbf{Alg}\,\Sigma$ and let A_t fulfill (1) for all $t \in T$. Since X is finite, every homomorphism $h : T_\Sigma(X) \to A$ factors through some of the colimit maps $a_t : A_t \to A$, i.e.,

$$h = a_t h' \ \text{ for some homomorphism } \ h' : T_\Sigma(X) \to A_t\,.$$

Since A_t fulfills (1), there is $i \in I$ with $h'(p_{ij}) = h'(q_{ij})$ for all j, thus, $h(p_{ij}) = h(q_{ij})$ for all $j \leq n_i$. Such i is unique: assume that $i^* \in I$ also fulfills $h(p_{i^*j}) = h(q_{i^*j})$ for all $j \leq n_{i^*}$. Since a_t then merges $h'(p_{i^*j})$ and $h'(q_{i^*j})$ for all $j \leq n_{i^*}$ and the above colimit is directed, there exists $s \geq t$ such that the connecting homomorphism $A_t \to A_s$ also merges $h'(p_{i^*j})$ with $h'(q_{i^*j})$ for all $j \leq n_{i^*}$. Since A_s is a model of (1), this implies $i = i^*$. This proves that A is a model of (1).

(d) We have to prove that given epimorphisms

$$A \xrightarrow{a} B \xrightarrow{b} C$$

in $\mathbf{Alg}\,\Sigma$, then B is a model of (1) whenever A and C are models of (1). Since $T_\Sigma X$ is a projective object of $\mathbf{Alg}\,\Sigma$, every interpretation $h : T_\Sigma(X) \to B$ factors through a:

$$h = ah' .$$

Since A is a model of (1), this implies that there exists $i \in I$ with $h(p_{ij}) = h(q_{ij})$ for all $j \leq n_i$. Since C is a model of (1), such i is unique: if also $h(p_{i^*j}) = h(q_{i^*j})$ for all $j \leq n_{i^*}$, then $bh(p_{ij}) = bh(q_{ij})$ and $bh(p_{i^*j}) = bh(q_{i^*j})$ imply $i = i^*$. $\square$

Remark 14. In our definition above we allow infinitary exclusive or's of finite conjunctions of equations. The *finitary* generalized multiequations, i.e., generalized multiequations using finitely many exclusive or's, correspond to multivarieties closed under ultraproducts:

Proposition 2. *Let E be a set of generalized multiequations over Σ. Then $\mathbf{Alg}\,(\Sigma, E)$ is presentable by finitary generalized multiequations over Σ iff it is closed under ultraproducts in $\mathbf{Alg}\,\Sigma$.*

Proof. It is only necessary to prove that if $\mathbf{Alg}\,(\Sigma, E)$ is closed under ultraproducts, then

$$\mathbf{Alg}\,(\Sigma, E) = \mathbf{Alg}\,(\Sigma, E')$$

where E' is the set of all finitary generalized multiequations holding in $\mathbf{Alg}\,(\Sigma, E)$. Assuming the contrary, there exists a generalized multiequation $\underset{i \in I}{\bigtriangledown} \varphi_i$ in E (i.e., the φ_i's are finite conjunctions of equations) such that none of the generalized multiequations $\underset{i \in J}{\bigtriangledown} \varphi_i$, $J \subseteq I$ finite, holds for all models of E. There clearly exists an infinite subset $K \subseteq I$ such that for any $k \in K$ there is a model A_k of E and an interpretation of variables $h_k : T_\Sigma X \to A_k$ such that $A_k \models \varphi_k[h_k]$. Let $\mathcal{U}$ be a free ultrafilter on K and consider the ultraproduct $A = \prod_{\mathcal{U}} A_k$. Let $h : T_\Sigma X \to A$ be the interpretation of variables induced by h_k, $k \in K$. Since $A \in \mathbf{Alg}\,(\Sigma, E)$, there is an $i \in I$ such that $A \models \varphi_i[h]$. Thus there is $U \in \mathcal{U}$ such that $A_k \models \varphi_i[h_k]$ for any $k \in U$. Since $A_k \models \varphi_k[h_k]$, we get $k = i$, i.e., $U \subseteq \{i\}$, which is impossible. $\square$

Example 15. A class of Σ-algebras, closed under subalgebras, connected limits, directed colimits, consistent quotients and ultraproducts in $\mathbf{Alg}\,\Sigma$, which, however, cannot be described by generalized multiequations in $\mathbf{Alg}\,\Sigma$.

This example uses ideas of the paper [H] of the second author. The signature used is very simple: it is the one-sorted signature of nullary operations a_n, b $(n \in \omega)$. The axioms are the following first-order axioms $\varphi_{n,m}$ $(n, m \in \omega,\ n \neq$

m) and $\varphi_n (n \in \omega)$:

$$\begin{aligned} \varphi_{n,m}: &\quad (a_n = b) \vee (a_m = b) \vee \neg(a_n = a_m) \\ \varphi_0: &\quad (a_0 = b) \vee (a_1 = b) \\ \varphi_{n+1}: &\quad \text{is obtained form } \varphi_n \text{ by substituting any conjunct of the form} \end{aligned}$$

$$(a_i = b) \vee (a_j = b) \tag{1}$$

of φ_n by the following 3 conjuncts:

$$\begin{array}{r} (a_i = b) \vee (a_{i'} = b) \\ (a_j = b) \vee (a_{j'} = b) \\ \neg(a_{i'} = b) \vee \neg(a_{j'} = b) \end{array} \tag{2}$$

where i' and j' are chosen to be distinct and larger than any k such that φ_n contains a_k. For example, φ_1 is the sentence

$$\Big[(a_0 = b) \vee (a_{i'} = b)\Big] \wedge \Big[(a_1 = b) \vee (a_{j'} = b)\Big] \wedge \Big[\neg(a_{i'} = b) \vee \neg(a_{j'} = b)\Big]$$

for i', $j' > 1$, $i' \neq j'$.

Denote by T the set $\{\varphi_{n,m};\ n, m \in \omega,\ n \neq m\} \cup \{\varphi_n; n \in \omega\}$ and consider the class $\mathbf{Mod}\, T$ of all models of T.

Proof that $\mathbf{Mod}\, T$ cannot be presented by generalized multiequations. Otherwise, following Proposition 2,

$$\mathbf{Mod}\, T = \mathbf{Alg}\,(\Sigma, E)$$

for some set E of generalized multiequations in the finitary logic $L_{\omega\omega}(\Sigma)$. We derive a contradiction.

Denote by ψ_n $(n > 0)$ the sentence obtained from φ_n by substituting the conjunction of the formulas of (2) by the conjunction of the following formulas

$$\begin{array}{l} (a_{i'} = b) \to (a_j = b) \wedge \neg\,(a_{j'} = b) \\ (a_{j'} = b) \to (a_i = b) \wedge \neg\,(a_{i'} = b) \end{array} \tag{3}$$

Then, obviously, $\varphi_n \vdash \psi_n$; consequently, for

$$T_k = \{\varphi_0, \ldots, \varphi_{k-1}\} \cup \{\psi_n\}_{1 \leq n < \omega} \cup \{\varphi_{n,m};\ n, m \in \omega,\ n \neq m\}$$

we have

$$\mathbf{Mod}\, T = \mathbf{Mod} \bigcup_{k<\omega} T_k\,.$$

Since $\varphi_0 \in T$, we have $E \vdash \varphi_0$ and, by the Compactness Theorem, it follows that there exists a finite set $E_0 \subseteq E$ with $E_0 \vdash \varphi_0$. Next, we have $\bigcup\limits_{k<\omega} T_k \vdash E$ and, by the Compactness Theorem again, there exists k such that

$$T_k \vdash E_0 \,.$$

We assume that k is the least such index. Then $k > 0$ because otherwise we would have $T_0 \vdash E_0 \vdash \varphi_0$, however, $T_0 \nvdash \varphi_0$ (any algebra in which all the constants are pairwise distinct is a model of $T_0 = \{\psi_n\} \cup \{\varphi_{n,m}\}$). Thus, we have

$$T_{k-1} \nvdash E_0 \,.$$

Let A be a Σ-algebra such that

$$A \models T_{k-1} \quad \text{and} \quad A \not\models E_0 \,.$$

The desired contradiction is obtained by finding a pullback in the category $\mathbf{Mod}\, T_k$ such that A is a subalgebra of the domain-object P of the pullback: since $T_k \vdash E_0$, we conclude that $A \models E_0$ because $\mathbf{Alg}\,(\Sigma, E_0)$ is closed under pullbacks and subalgebras (Proposition 1).

I. Assume $k > 1$.

Since $A \models T_{k-1}$ but $A \not\models T_k$, we have $A \not\models \varphi_{k-1}$ (and $A \models \varphi_{k-2}$). Let (i_r, j_r), $r = 1, \ldots, m$ be all the pairs (i,j) of indices such that A satisfies the conjunct (1) of the formula φ_{k-2} but fails to satisfy the substituted conjunction of the three formulas of (2). For each $r = 1, \ldots, m$ we thus have

$$A \models (a_{i_r} = b) \,\vee\, (a_{j_r} = b) \tag{4}$$

and we prove now that

$$A \models \Big[\neg(a_{i_r} = b) \wedge \neg(a_{i'_r} = b)\Big] \vee \Big[\neg(a_{j_r} = b) \wedge \neg(a_{j'_r} = b)\Big] \,. \tag{5}$$

In fact, since $A \models T_{k-1}$ implies $A \models \psi_{k-1}$, we have, clearly, $A \not\models (a_{i'_r} = b) \wedge (a_{j'_r} = b)$. Thus, A satisfies the third formula of (2); consequently, it fails to satisfy the first or the second one – thus (4) implies (5). Without loss of generality we assume that our notation of indices is such that, in (5), the first disjunct is true:

$$A \models \neg(a_{i_r} = b) \wedge \neg(a_{i'_r} = b) \;\; \text{for} \;\; r = 1, \ldots, m \,. \tag{6}$$

Now (6) and $A \models \psi_{k-1}$ imply

$$A \models \neg(a_{j'_r} = b) \;\; \text{for} \;\; r = 1, \ldots, m \tag{7}$$

and (4) implies

$$A \models a_{j_r} = b\,. \tag{8}$$

Let

$$B_1 = A\big/\!\sim$$

be the quotient of A under the equivalence whose only non-singleton equivalence class is $M_1 = \{b, a_{i_1}, \dots, a_{i_m}\}$. Then due to the formulas $\varphi_{n,m}$, for each $(i,j) = (i_r, j_r)$ the algebra B_1 satisfies the formulas of (2), $r = 1, \dots, m$, consequently, $B_1 \models \varphi_{k-1}$. Thus, B_1 lies in $\mathbf{Mod}\, T_k$. Analogously, let

$$B_2 = A\big/\!\approx$$

be the quotient of A under the equivalence whose only non-singleton equivalence class is $M_2 = \{b, a_{i'_1}, \dots, a_{i'_m}\}$, then $B_2 \in \mathbf{Mod}\, T_k$. Finally, the quotient B_3 of A under the join of $\sim$ and $\approx$ (having one class $M_1 \cup M_2$) lies in $\mathbf{Mod}\, T_k$ and the canonical quotient maps $b_i : B_i \to B_3$ have the following pullback

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B_2 \arrow[d, "b_2"] \\
B_1 \arrow[r, "b_1"'] & B_3
\end{tikzcd}$$

P is the subalgebra of $B_1 \times B_2$ of all pairs $(x, y) \in A \times A$ with $x \sim y$ or $x \approx y$. The homomorphism $h : A \to P$ defined by $h(x) = (x, x)$ is a monomorphism. This proves $A \in \mathbf{Mod}\, T_k$, a contradiction.

II. Suppose $k = 1$.

This is analogous to Case I: here the algebra A does not satisfy φ_0, and in the construction of the pullback we put

$$M_1 = \{b, a_0\} \text{ and } M_2 = \{b, a_1\}\,.$$

Proof of the closure properties. It is easy to see that $\mathbf{Mod}\, T$ is closed under subalgebras, directed colimits and (since we are working in finitary logic) ultraproducts.

Connected limits. Since $\mathbf{Mod}\, T$ is closed under subalgebras, it is sufficient to prove closure under multiple pullbacks. Let $a_s : A \to B_s \quad (s \in S)$ form a multiple pullback of homomorphisms $b_s : B_s \to B \quad (s \in S)$ of $\mathbf{Mod}\, T$. Obviously, $A \models \varphi_{n,m}$. We prove

$$A \models \varphi_n \quad \text{for each} \quad n \geq 1$$

(a) A satisfies each conjunct of φ_n of the form $(a_i = b) \vee (a_j = b)$. Assuming the contrary, we derive a contradiction. There exist s, $s' \in S$ with

$$B_s \not\models (a_i = b) \ \text{ and } \ B_{s'} \not\models (a_j = b)\,.$$

Since $B_s \models \varphi_{n+1}$, we conclude $B_s \models (a_{i'} = b)$ and, due to the homomorphism b_s, then $B \models (a_{i'} = b)$. Analogously, $B \models (a_{j'} = b)$. This contradicts to $B \models \varphi_{n+1}$.

(b) A satisfies each conjunct of φ_n of the form $\neg(a_i = b) \vee \neg(a_j = b)$. In fact, if $A \models (a_i = b) \wedge (a_j = b)$, then $B \models (a_i = b) \wedge (a_j = b)$, in contradiction to $B \models \varphi_n$.

Consistent quotients. Given homomorphisms $f : A \to B$ and $g : B \to C$ then $A, C \in \mathbf{Mod}\, T$ implies $B \in \mathbf{Mod}\, T$: it is clear that $B \models \varphi_{n,m}$; to prove $B \models \varphi_n$, for positive conjuncts of φ_n use f, for the negative ones use g. $\square$

Remark 15. Example 15 illustrates the difficulties in trying to characterize multivarieties in the direction of Birkhoff's Variety Theorem. There is a slightly weaker concept, however, for which we will obtain such a theorem. Instead of exclusive-or's of *finite* conjunctions of equations we work with exclusive-or's of arbitrary conjunctions.

Definition. By an *infinitary multiequation* is meant an exclusive-or of conjunctions of equations over a finite set X of variables, i.e., an expression

$$\mathop{\triangledown}_{i\in I} \bigwedge_{j\in J_i} (\alpha_{ij} = \beta_{ij})$$

where α_{ij} and β_{ij} are terms in $T_\Sigma(X)$, X finite.

Theorem 4. *Let Σ be a finitary signature. A class of Σ-algebras can be presented by infinitary multiequations iff it is closed under*

(a) *subalgebras,*
(b) *connected limits,*
(c) *directed unions*

and

(d) *consistent quotients.*

Proof. I. Necessity. (a) & (b) The closedness under subalgebras and connected limits is proved precisely as in Theorem 1 above.

(c) Given an infinitary multiequation

$$\bigtriangledown_{i\in I}\bigwedge_{j\in J_i}(\alpha_{ij}=\beta_{ij}) \qquad \text{(in variables } X=\{x_1,\dots,x_n\}\text{)}$$

then the models of that multiequation form a class of algebras closed under directed unions: let $A=\bigcup\limits_{t\in T}A_t$ be an algebra which is a directed union of subalgebras A_t satisfying the above infinitary multiequation. Then for every interpretation $f:X\to A$ of variables the image of f lies in A_{t_0} for some $t_0\in T$ (since the image is finite and T is directed). Then there exists $i\in I$ such that all the equations $\alpha_{ij}=\beta_{ij}$, $j\in J_i$, are fulfilled in A_{t_0} (hence, in A) under the interpretation f. The uniqueness of such i in A_{t_0} implies the uniqueness in A.

(d) The proof is the same as in Proposition 1.

II. Sufficiency. Let $\mathcal{K}$ be a class of Σ-algebras closed under (a)–(d) in $\mathbf{Alg}\,\Sigma$. Then $\mathcal{K}$ is multireflective in $\mathbf{Alg}\,\Sigma$ - the proof is the same as in Remark 5. For every finite set X (of variables) consider a multireflection

$$f_t: T_\Sigma(X)\to A_t \qquad (t\in T)$$

of the absolutely free Σ-algebra over X. Since $\mathcal{K}$ is closed under subalgebras, each f_t is surjective, thus, without loss of generality we can assume that a congruence $\sim_t$ on $T_\Sigma(X)$ is given such that $A_t=T_\Sigma(X)/\sim_t$ and f_t is the canonical homomorphism. Consider the following infinitary multiequation

$$\bigtriangledown_{t\in T}\bigwedge_{\substack{\alpha,\beta\in T_\Sigma(X)\\ \alpha\sim_t\beta}}(\alpha=\beta). \tag{1}$$

An algebra B satisfies (1) iff for every homomorphism

$$h: T_\Sigma(X)\to B$$

there exists a unique $t\in T$ such that $h(\alpha)=h(\beta)$ for all $\alpha\sim_t\beta$ - in other words, such that h factors through f_t:

$$\begin{tikzcd} T_\Sigma(X) \arrow[r, "h"] \arrow[d, "f_t"'] & B \\ A_t \arrow[ur, dashed] & \end{tikzcd}$$

Furthermore, given a finite set X and a congruence $\approx$ on $T_\Sigma(X)$ such that

$$T_\Sigma(X)/{\sim}\notin\mathcal{K} \ \text{ for all congruences } \sim \text{ containing } \approx, \tag{2}$$

we consider the following infinitary multiequation:

$$(x = x) \bigtriangledown \bigwedge_{\substack{\alpha,\beta \in T_\Sigma(X) \\ \alpha \approx \beta}} (\alpha = \beta)\,. \tag{3}$$

An algebra B satisfies (3) iff for every homomorphism

$$h : T_\Sigma(X) \to B$$

there exist $\alpha, \beta \in T_\Sigma(X)$ with $\alpha \approx \beta$ and $h(\alpha) \neq h(\beta)$, i.e.,

$$h \text{ does not factor through the canonical map } k = T_\Sigma(X) \to T_\Sigma(X)/\approx\ . \tag{4}$$

We claim that the collection E of all infinitary multiequations

(1), indexed by all finite sets X

and

(3), indexed by all congruences $\approx$ satisfying (2)

presents $\mathcal{K}$. That is: a Σ-algebra B lies in $\mathcal{K}$ iff it satisfies (1) and (3).

In fact, if $B \in \mathcal{K}$ then B satisfies (1) because every homomorphism $h : T_\Sigma(X) \to B$ factors through precisely one member of a multireflection of $T_\Sigma(X)$ in $\mathcal{K}$. And it satisfies (3) because otherwise there would exist a homomorphism f from $T_\Sigma(X)/\approx$ into B - however, this contradicts (2) because the congruence $\sim$ which is the kernel of fh contains $\approx$ and the algebra $T_\Sigma(X)/\sim$ (isomorphic to a subalgebra of $B \in \mathcal{K}$) lies in $\mathcal{K}$.

Conversely, let B satisfy (1) and (3). Then we prove that $B \in \mathcal{K}$. Since $\mathcal{K}$ is closed under directed unions as well as under subalgebras, we can assume, without loss of generality, that B is finitely generated. Thus, we can assume $B = T_\Sigma(X)/\approx$ for some finite set X and a congruence $\approx$. That congruence does not have property (2) because if it had, B would satisfy the corresponding multiequation (3) - this is not the case, consider the canonical homomorphism

$$h : T_\Sigma(X) \to B\,.$$

Next, since B satisfies (1), h factors through f_t for some $t \in T$, i.e.,

$$h = kf_t, \quad k : A_t \to B\,.$$

Observe that since h is surjective, so is k. Now $\mathcal{K}$ is closed under consistent quotients, thus, to prove that $B \in \mathcal{K}$, we just observe that B is a quotient of

$A_t \in \mathcal{K}$, and B has a quotient lying in $\mathcal{K}$ (because the congruence $\approx$ does not satisfy (2)). $\square$

Remark 16. Coming back to generalized multiequations, the above proof can be easily modified to yield a characterization of multivarieties in the context of infinitary operations. More precisely, given a cardinal λ and a λ-ary many-sorted signature Σ, by a generalized multiequation we understand an exclusive-or of conjunctions of less-than-λ equations over less-than-λ variables

$$\bigtriangledown_{i\in I} \bigwedge\nolimits_{j\in J_i} {}^{\alpha_{ij}=\beta_{ij}}$$

where

$$\operatorname{card} J_i < \lambda \quad \text{for all} \quad i \in I$$

and

$$\alpha_{ij},\, \beta_{ij} \in T_\Sigma(X) \quad \text{with card} \quad X < \lambda\,.$$

A class of Σ-algebras presented by generalized multiequations is called a multivariety (of λ-ary algebras).

Multivarieties are precisely the classes $\mathcal{K}$ of Σ-algebras closed in $\mathbf{Alg}\,\Sigma$ under

(a) subalgebras,
(b) connected limits,
(c) λ-directed unions for some cardinal λ

and

(d) consistent quotients.

The proof is analogous to that of Theorem 4: one chooses a regular cardinal λ such that

(i) Σ is a λ-ary signature

and

(ii) $\mathcal{K}$ is closed under λ-directed unions in $\mathbf{Alg}\,\Sigma$.

Then one forms multireflections of all absolutely free algebras on less than λ generators and forms multiequations (1) above. And given $T_\Sigma(X)/\approx$, where card $X < \lambda$, such that this algebra has no quotient in $\mathcal{K}$, one forms multiequations (3). This, then, is a presentation of $\mathcal{K}$.

VI. Conclusions and Future Research

Multiequational specifications of abstract data types, i.e. specifications by means of disjoint disjunctions (exclusive or's) of equations, can be used to specify important data types beyond the reach of equational specifications. For example, arrays and sets as parametric data types. The multiinitial-algebra semantics which in this case substitutes the well known initial-algebra semantics presents abstract data types by means of collections of non-isomorphic algebras – nevertheless, this semantics has features resembling initial-algebra semantics: in every correct implementation of the data type we find a unique instance of a unique member of the multiinitial family. Analogously, the multi-free-algebra semantics of parametric data types has features resembling the free semantics in the equational case. In the parameter passing situation, the concepts of correctness and persistence have a natural counterpart of weak correctness (where the requested isomorphisms exist for at least one member of the multiinitial family) and weak persistence. The latter means that every formal parameter allows at least one member of its multiinitial family carried by an isomorphism. In fact, in both of the examples that have inspired our paper, array and set, formal parameters allow a unique such member of their multiinitial families. It would be interesting to study this "unique semantics" generally, in particular, to find the corresponding syntactic property.

Another topic of interest not studied in the present paper is multiequational logic. At the present moment we do not known whether any complete logic for multiequations exists or not.

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